# ARGUESIAN LATTICES OF ORDER 3

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Douglas Pickering

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## Contents

## Abstract

Since the mid 19th century it has been known that every Desarguean projective plane is coordinatizable over a division ring. This coordinatization procedure was used by von Neumann [9] to show that every complemented modular lattice with spanning n-frame $(n \geq 4)$ is isomorphic to the lattice of finitely generated submodules of a regular ring. In 1958, Jónsson introduced the Arguesian identity and extended von Neumann's result to every complemented Arguesian lattice with spanning 3-frame. It was further noted by Freese [3] and Artmann [1] that to obtain the ring, von Neumann's proof did not require complementation. In this thesis, we follow the method of von Neumann to show:

__Theorem.__ If $L$ is an Arguesian lattice with spanning 3-frame, $(a_1, a_2, a_3, c_{12}, c_{13}, c_{23})$ and $L_{12} = \{x \in L | x + a_2 = a_1 + a_2, x \cdot a_2 = 0\}$, then $L_{12}$ is an associative ring with unit with respect to the von Staudt operations of addition and multiplication.

I wish to acknowledge with gratitude the assistance and inspiration provided by my supervisor Professor Alan Day. I would also like to thank the Department of Mathematical Sciences at Lakehead University, and Dr. Day particularly, for their patience and financial assistance.

# Chapter 1

## Introduction

The classical theorem concerning the relationship between projective planes, lattices and rings is that every Desarguean projective plane is isomorphic to $L({}_K K^3)$ - the lattice of subspaces of the vector space $K^3$ over a division ring $K$. Projective planes are a special class of projective spaces of dimension $n$, a projective plane being just a projective space of projective dimension 2. For projective spaces of dimension $n \geq 3$, the Desarguean condition is automatically fulfilled and for any projective space of dimension $n \geq 3$, $P$ is isomorphic to $L({}_K K^{n+1})$.

In his lectures on continuous geometry, von Neumann [9] proved that every complemented modular lattice with spanning n-frame $n \geq 4$, is isomorphic to the lattice of finitely generated submodules of a regular ring. Because of the existence of non-Desarguean projective planes, this result cannot hold for $n = 3$. Jónsson, in [5], introduced the Arguesian identity, a lattice theoretic equivalent of Desargues Law in geometry, and later, in [7], extended von Neumann's result to every complemented Arguesian lattice with spanning 3-frame.

Independently Freese [3], and in a geometric way Artmann have shown [1], that to obtain the ring for $n \geq 4$, only modularity and the existence of an n-frame are required. Complementation need not be assumed. Our goal in this thesis is to show that an auxiliary ring can be obtained from any Arguesian lattice with spanning 3-frame, or that in the case $n = 3$, complementation is not required.

The method we follow is close to that of von Neumann [9] where the "points" on a "line" are used as ring elements. Given an Arguesian lattice with spanning 3-frame, we define $D$ to be all complements of $a_2$ in the interval $[0, a_1 + a_2]$ where $a_1, a_2$ are members of the spanning frame. Binary operations (on $D$) of multiplication and addition are then defined following which we prove the properties necessary to make $D$ an associative ring with unit. Several different definitions of addition and multiplication have previously been used, those of von Staudt [10], von Neumann, and Young [11]. In fact, as we will show later, these are equivalent and will be used interchangeably.

Throughout this paper the following notations will be used unless otherwise indicated.

$L$ will represent a lattice with spanning 3-diamond.

$+$ will be the lattice join symbol.

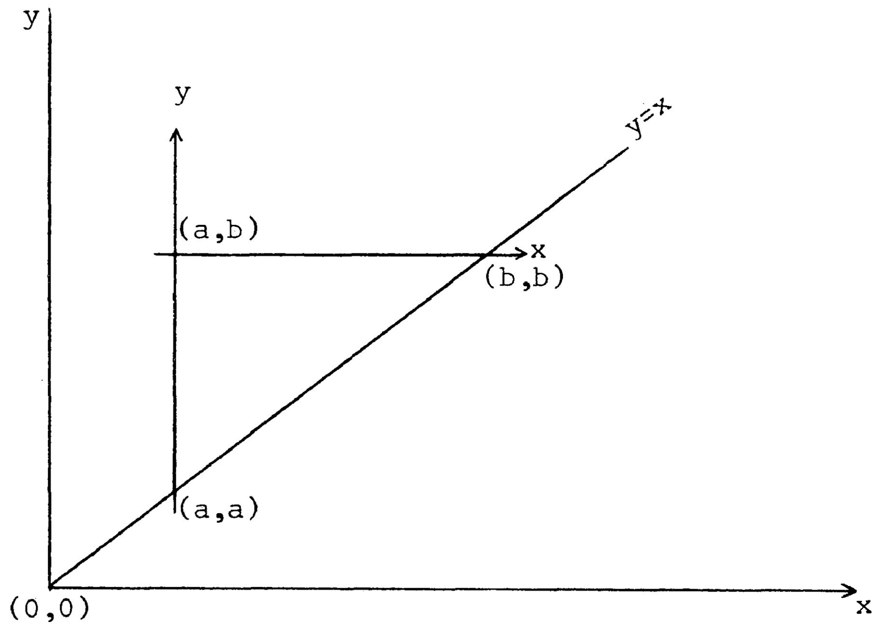
$\cdot$ will indicate lattice meet and will be omitted where convenient.

The order of operations will be $\cdot$ followed by $+$ so that $a + b \cdot c$ means $a + (b \cdot c)$. If $L$ is a lattice then $[a, b] = \{x \in L: a \leq x \leq b\}$.

For the remainder of this chapter we consider the coordinatization of two examples of Arguesian lattices with spanning 3-frames. The first is the lattice arising from any non-degenerate Desarguean projective plane, and the second comes from any associative ring with unit.

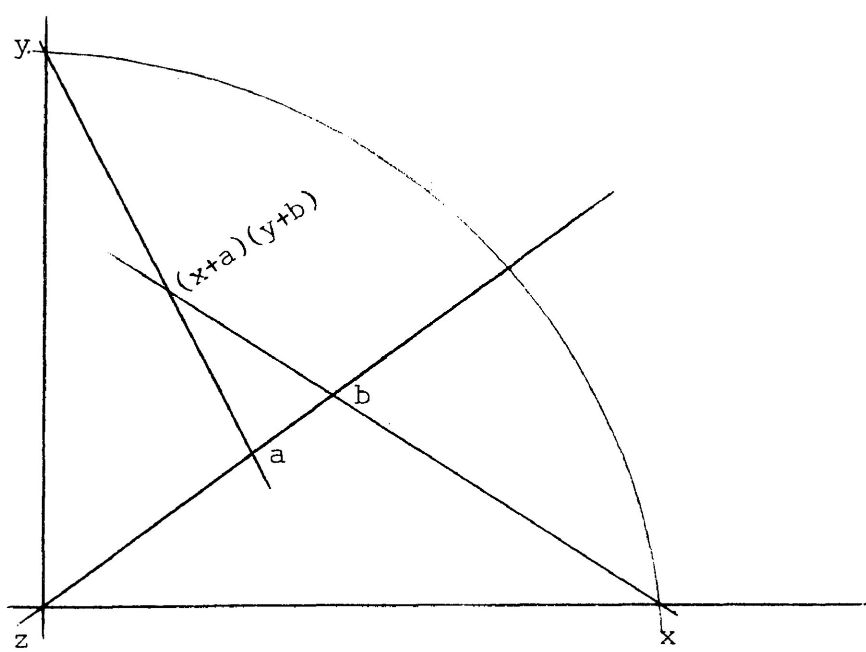
As is well known, a projective plane can be considered as a modular lattice where the join of two distinct points is the (unique) line containing the points and the meet of two lines is their unique intersection point, all other meets and joins being trivial. If the plane is non-degenerate, then there exist four points in general position, i.e. no three of which are on the same line. Lattice theoretically, this corresponds precisely to the idea of a 3-diamond (to be defined in the next chapter).

One possible way to coordinatize a projective plane is accomplished by fixing three distinct lines and defining operations of addition and multiplication on all points on one line distinct from the intersection point of two of the given lines. These operations are then extended to the entire plane. Looking at the affine plane $\mathbb{R} \times \mathbb{R}$ for a moment we have an x-axis, y axis, the points $(0,0)$, $(1, 1)$ and the diagonal $y = x$. Any point in the plane is determined uniquely by some $(a, a)$ and $(b, b)$ on the line $y = x$.

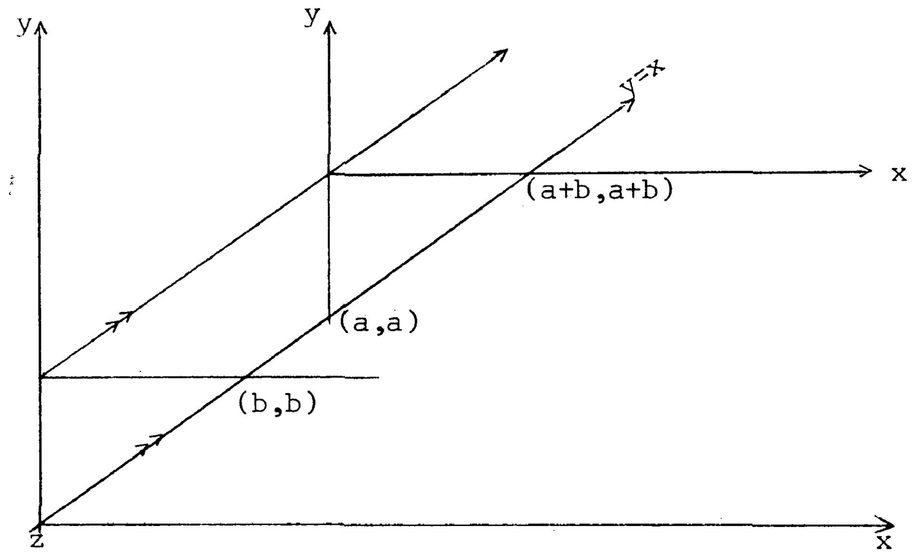



By adding the "line at infinity" we produce a projective plane where (if we call $(0, 0) = z$ and $(1, 1) = t$ ) the points $z, t, x, y$ are four points in general position.

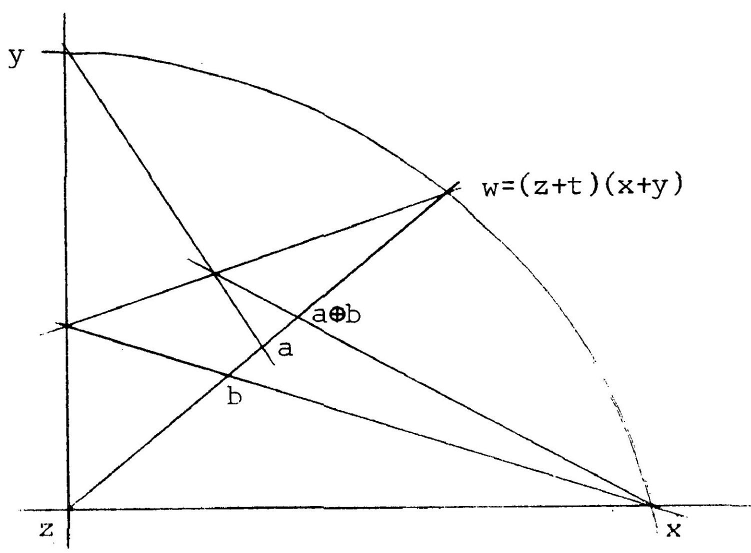
Now any point not on the line at infinity of the projective plane is given by $(x + b)(y + a)$ for some $a, b$ on the line $z + t$. [$a, b$ are distinct from the point $(z + t)(x + y)$].



Back in the affine plane, we can define addition of points on the diagonal according to the following construction.

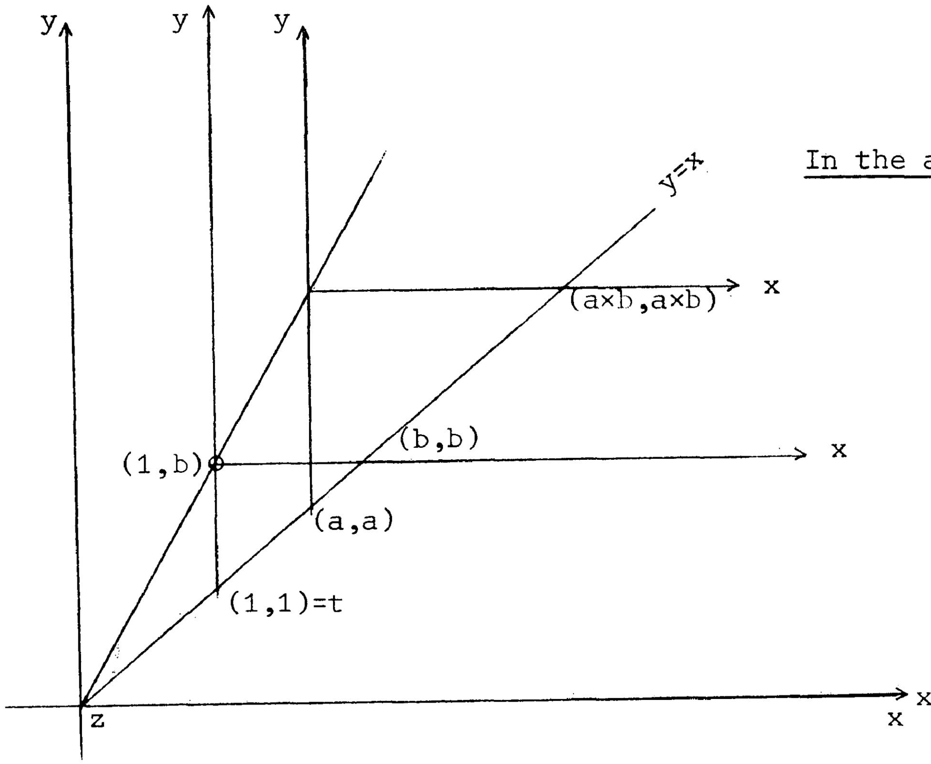



Translating this into the projective plane we obtain:

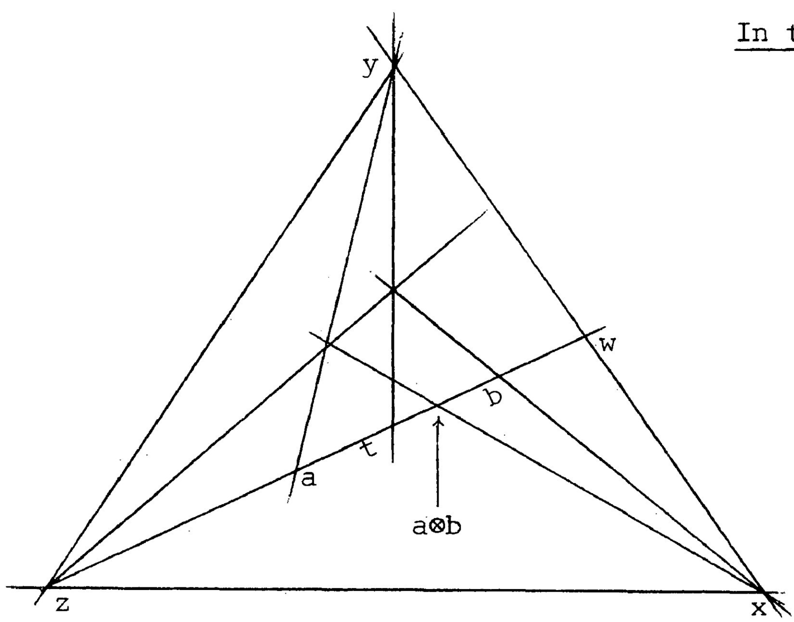



The point $t$ is not used in the construction of $a \oplus b$ . It is rather obvious though that it is needed in the following possible definitions of multiplication.

In the affine plane




In the projective plane




Writing out the expressions for the constructions above we obtain the following two definitions for addition and multiplication (of points on the line $z + t$ distinct from $w = (z + t)(x + y)$ where $(z, t, x, y)$ is a chosen quadrangle).

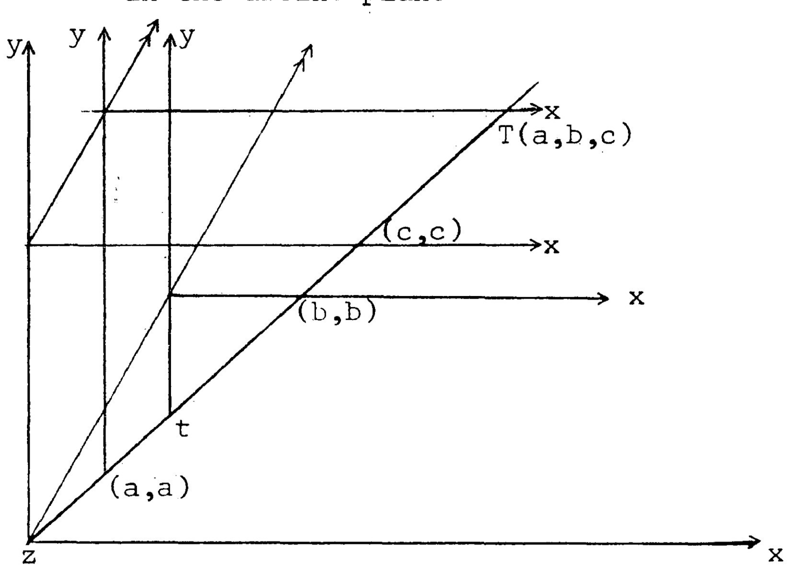
$$a \oplus b = (z + t)(x + (y + a)(w + (z + y)(x + b)))$$

$$a \otimes b = (z + t)(x + (y + a)(z + (y + t)(x + b))) \ .$$

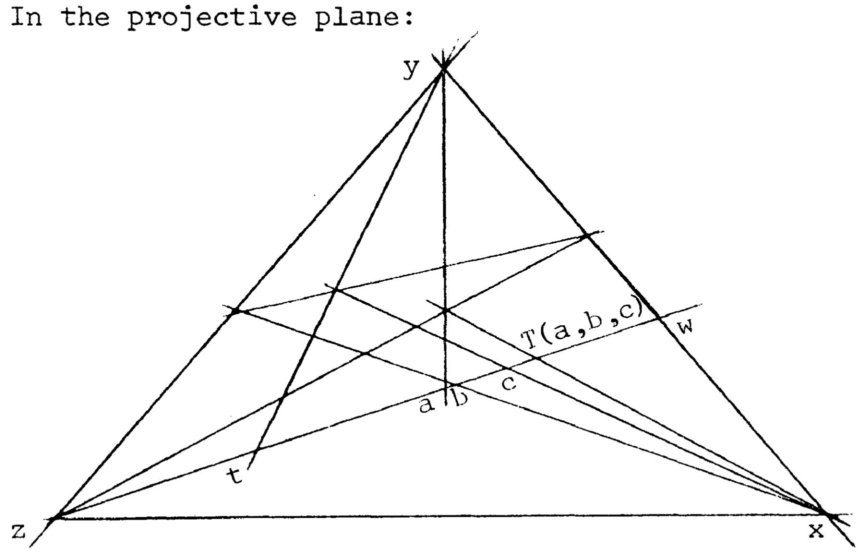
These definitions are those used by von Staudt [10], and are the ones chosen to coordinatize an arbitrary Arguesian lattice in the subsequent chapters.

It is also possible to combine both addition and multiplication into a single operation as is done in the ternary ring operator T(a, b, c) described below.

In the affine plane:




In the projective plane:



Looking closely at the above diagram, one might notice that $T(a, t, b)$ is just $a \oplus b$ and $T(a, b, z)$ is $a \otimes b$ . In fact, in an Arguesian lattice, $T(a, b, c)$ is $(a \otimes b) \oplus c$ .

Since the late 19th century, it has been known that if $P$ is a Desarguean projective plane and $(z, t, x, y)$ are four points in general position, then $D$ -the set of all points on the line $z + t$ , [excluding $(z + t)(x + y)$] , together with the operations $\oplus$ , $\otimes$ , forms an associative ring with unit. Historically, the method of proof is similar to the proofs given in the later chapters of this paper. However one need only consider the following cases, making many results easier to obtain.

(1) $a \in D \Rightarrow a + z = z + t$ or $a = z$

(2) $a \in D \Rightarrow a + t = z + t$ or $a = t$

Because of the way addition and multiplication are defined, when $z = a$ or $t = a$ , properties of addition and multiplication become simple to calculate, so one is left with the case $z + a = z + t = a + t$ .

We now turn our attention to an example of an Arguesian lattice with spanning 3-frame obtainable from any associative ring with unit $R$ .

The lattice of left sub-modules of $R^3$ , $L({}_R R^3)$ , is Arguesian. The proof follows closely the proof that the lattice of subspaces of a vector space is Arguesian. We know also that $R^3$ has the usual basis $e_1 = (1, 0, 0)$ $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ . If we then define $z, t, x, y$ submodules of $R^3$ as

$z = \langle e_1 \rangle$ , the submodule generated by $e_1$

$t = \langle e_1 - e_2 \rangle$ , the submodule generated by $e_1 - e_2$

$x = \langle e_2 - e_3 \rangle$ , the submodule generated by $e_2 - e_3$

$y = \langle e_3 \rangle$ , the submodule generated by $e_3$

then $(z, t, x, y)$ is a spanning 3-diamond in $L({}_R R^3)$ .

As in the case of a projective plane, we wish to coordinatize all complements of $(z + t)(x + y)$ on the "line" $z + t$ . If $a$ is a complement of $(z + t)(x + y)$ in the interval $[0, z + t]$ then the submodule $a$ is $\langle e_1 - re_2 \rangle$ for some $r \in R$ .

Addition and multiplication defined on $D$ must somehow reflect addition and multiplication in the original ring $R$ , so that the corresponding properties carry through into the "new" ring. By using the same definitions of multiplication and addition as in the projective plane example, we see that that is what indeed happens. If $a, b \in D$ and $a = \langle e_1 - re_2 \rangle$ $b = \langle e_1 - se_2 \rangle$ then $a \oplus b$ is $\langle e_1 - (r + s)e_2 \rangle$ . Similarly, $a \otimes b$ is $\langle e_1 - rse_2 \rangle$ . It is then clear why any properties of addition [multiplication] that hold in $R$ also hold in $(D, \oplus, \otimes)$ . In $D$ , commutativity, associativity, inverses and distributivity follow from the corresponding properties in $R$ . Thus $(D, \oplus, \otimes)$ is an associative ring with unit isomorphic to $R$ .

# Chapter 2

## Preliminaries

In this chapter we formally introduce the basic definitions and the most frequently used preliminary results. The first half covers the ideas of spanning 3-diamond and ternary ring operator. The second part is devoted to the Arguesian identity and its equivalent, the **Desarguean** implication.

Definition 2.1 Suppose $L$ is a $\{0, 1\}$ modular lattice. A spanning 3-diamond in $L$ is a quadruple $(a_1, a_2, a_3, a_4)$ of elements in $L$ such that the following conditions are satisfied for $i \neq j \neq k \neq i$, $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$.

(1) $a_i(a_j + a_k) = 0$

(2) $a_i + a_j + a_k = 1$ .

Throughout the remainder of this text we will use $(z, t, x, y)$ to represent **a** spanning 3-diamond. Let $w = (z + t)(x + y)$ and $D = \{a \in L: w \cdot a = 0 , w + a = z + t\}$ so that $D$ is the set of complements of $w$ in the interval $[0, z + t]$ . Furthermore, let $v = (z + y)(x + t)$ and $u = (z + x)(y + t)$ . Under the projective isomorphism $[0, z + t] \underset{\wedge}{\overset{x}{=}} [0, z + y]$ , $w$ is mapped to $y$ and so the image (under the isomorphism) of any complement of $w$ in the interval $[0, z + t]$ is a complement of $y$ in the interval $[0, z + y]$ . That is: if $a \in D$ and we define $a_0 = (z + y)(x + a)$ then $y + a_0 = z + y$ and $y \cdot a_0 = 0$ . In a similar way, for any $a \in D$ we define:

$a_2 = (z + x)(y + a)$, the image under $[0, z + t] \overset{y}{\underset{\wedge}{=}} [0, z + x]$ of $a$.

$a_\infty = (x + y)(z + (y + t)(x + a))$, the image of $a$ under the projectivity

$$[0, z + t] \overset{x}{\underset{\wedge}{=}} [0, y + t] \overset{z}{\underset{\wedge}{=}} [0, x + y] .$$

$a_1 = (y + t)(x + a)$, the image of $a$ under the projectivity

$$[0, z + t] \overset{x}{\underset{\wedge}{=}} [0, y + t] .$$

So by tracing $w$ through the above projectives, we see that $a_2$ is a complement of $x$ in $[0, z + x]$, $a_\infty$ is a complement of $y$ in the interval $[0, x + y]$, and $a_1$ is a complement of $y$ in the interval $[0, y + t]$.

See Figure 2.4.

<u>Definition 2.2</u> For $a, b, c \in D$ define the following.

(1) $T(a, b, c) \equiv (z + t)(x + (y + a)(c_0 + b_\infty))$ see Figure 2.5

(2) $a \otimes b \equiv T(a, b, z) = (z + t)(x + (y + a)(z + b_\infty))$

(3) $a \oplus b = T(a, t, b) = (z + t)(x + (y + a)(w + b_0))$

Our goal is to coordinatize $D$ so we hope if $a, b, c$ are in $D$ that $T(a, b, c)$ is also.

<u>Lemma 2.3.</u> If $L$ is modular and $a, b, c \in D$ then $T(a, b, c) \in D$.

<u>Proof.</u> We must calculate $w + T(a, b, c)$ and $w \cdot T(a, b, c)$.

$$\begin{aligned} w + T(a, b, c) &= (z + t)(w + x + (y + a)(c_0 + b_\infty)) \\ &= (z + t)(x + y + (y + a)(c_0 + b_\infty)) \\ &= (z + t)(x + (y + a)(y + c_0 + b_\infty)) \\ &= (z + t)(x + y + a) \\ &= z + t \end{aligned}$$

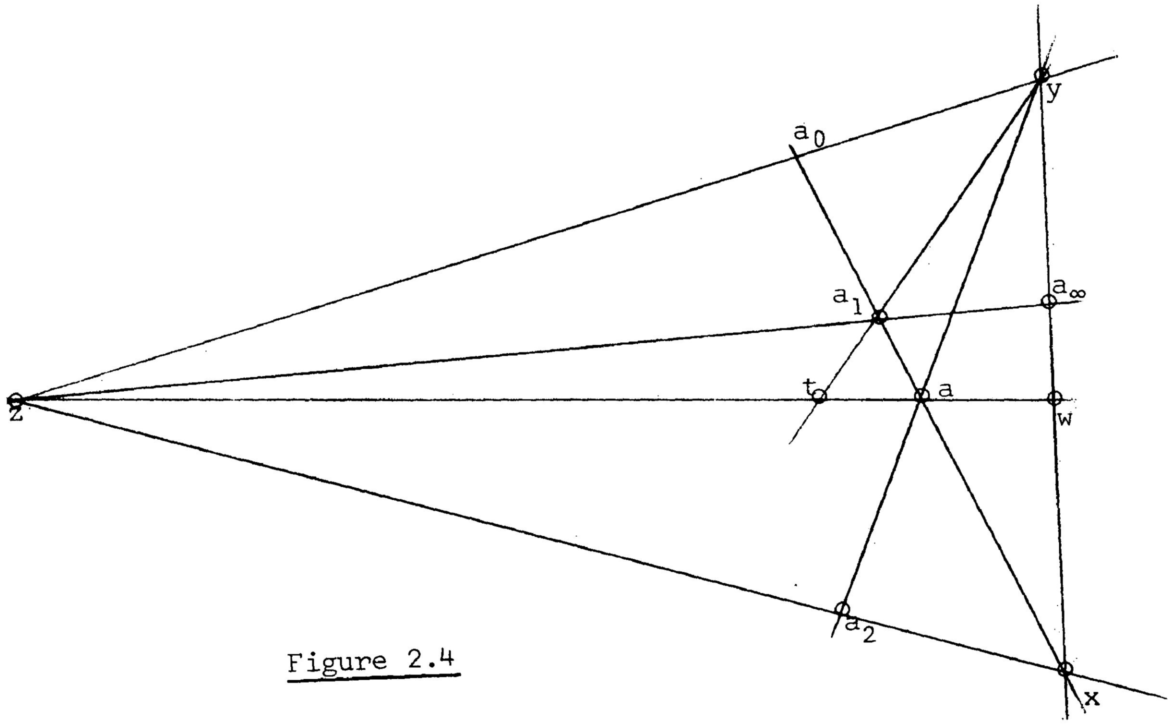



Figure 2.4

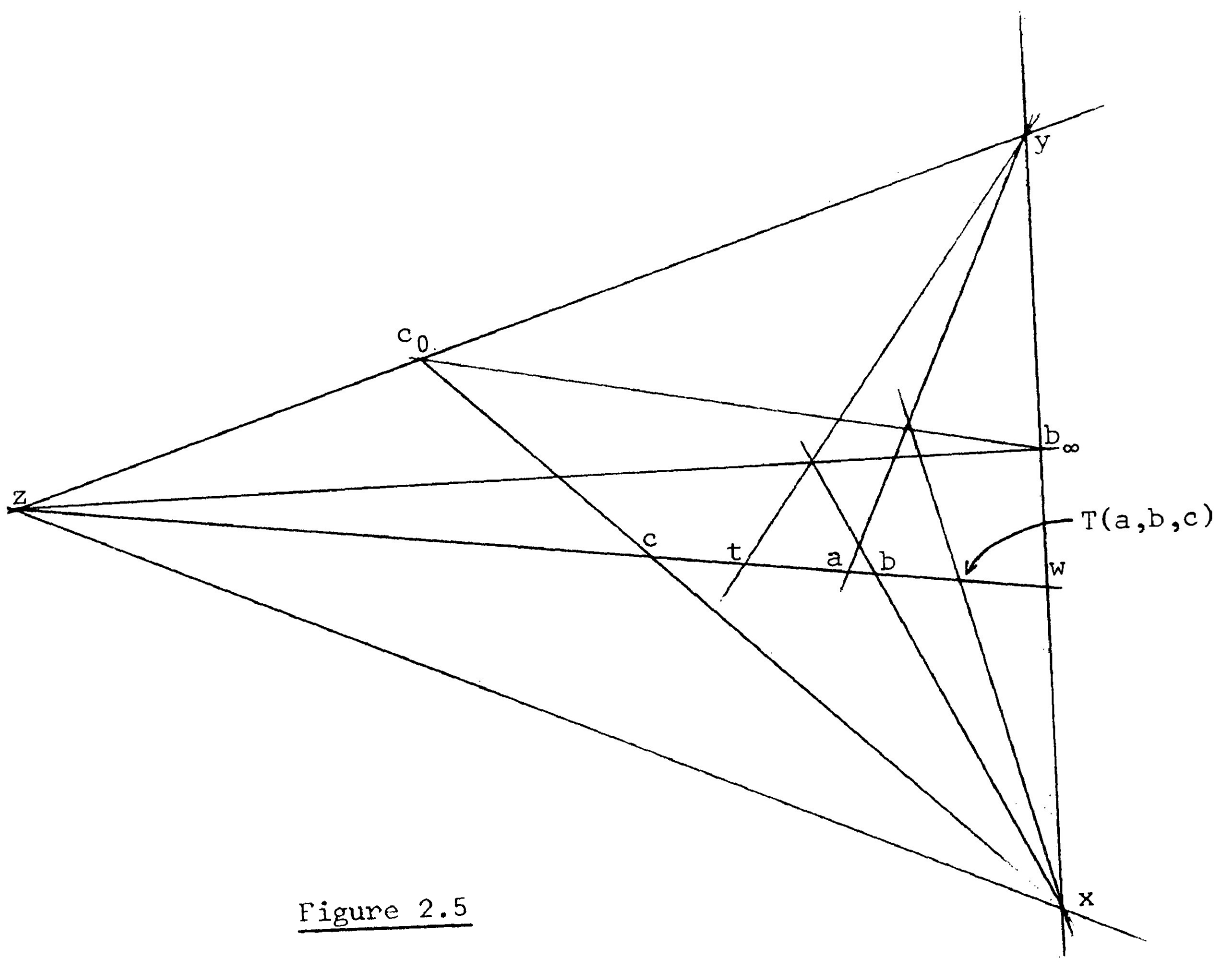



Figure 2.5

$$\begin{aligned} w \cdot T(a, b, c) &= w(x + (y + a)(c_0 + b_\infty)) \\ &= w((y + a)(c_0 + b_\infty) + (x + t)(x + y)) \\ &= w(x + t + (x + y)(y + a)(c_0 + b_\infty)) \\ &= w(x + t) \\ &= 0 \end{aligned}$$

Therefore $T(a, b, c) \in D$ hence $a \oplus b \in D$ and $a \otimes b \in D$ .

∎

The definitions of $\oplus$ and $\otimes$ given above are due to von Staudt [10]. In [ 9 ], von Neumann introduced the $\boxplus$ and $\boxtimes$ defined below.

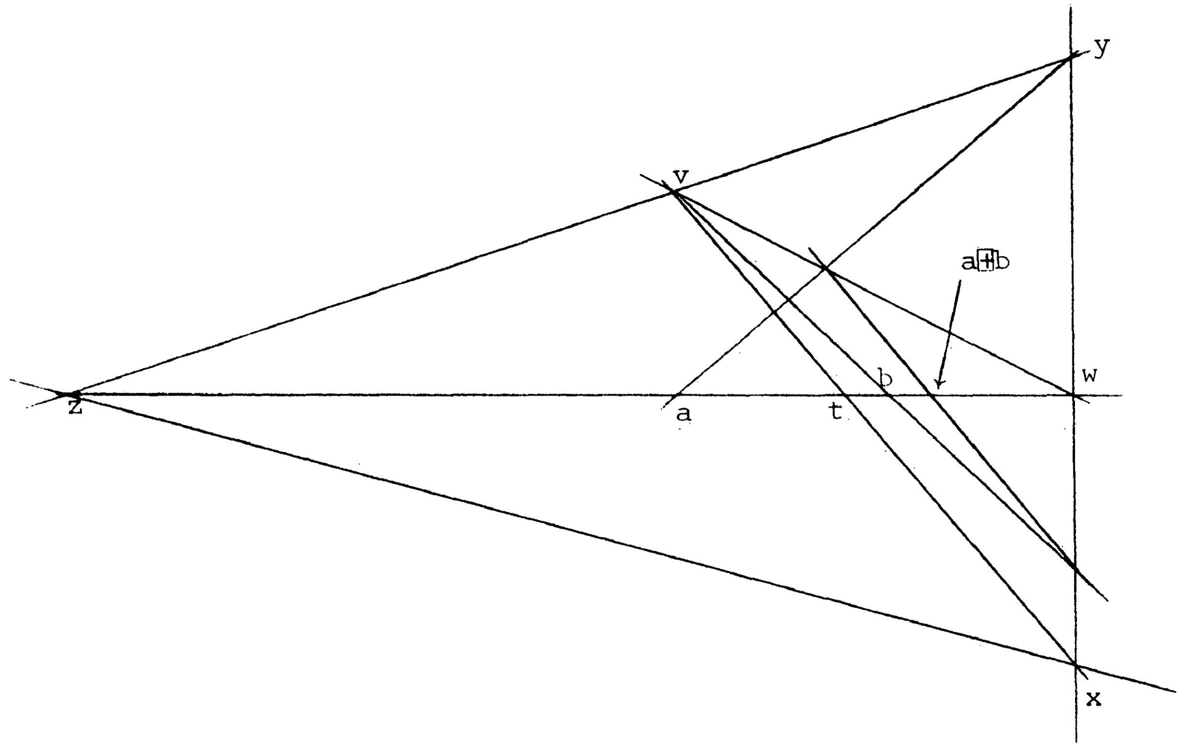
<u>Definition 2.6</u> In a lattice $L$ with spanning 3-diamond $(z, t,x,y)$ define $a \boxplus b$ and $a \boxtimes b$ as (Figure 2.9)

$$\begin{aligned} a \boxplus b &= (z + t)((x + y)(v + b) + (w + v)(y + a)) \\ a \boxtimes b &= (z + t)(a_0 + (x + y)(v + b)) . \end{aligned}$$

<u>Lemma 2.7.</u> If $L$ is modular and $a, b \in D$ then $a \boxplus b \in D$ and $a \boxtimes b \in D$ .

The proof of the above lemma is a straightforward calculation similar to those done in the proof that $T(a, b, c) \in D$ . We will now show some simple properties of $\oplus$ , $\otimes$ in a modular lattice with $a, b \in D$ .

<u>Lemma 2.8.</u> If $L$ is modular and $a, b \in D$ then the following properties hold for $\oplus$, $\otimes$



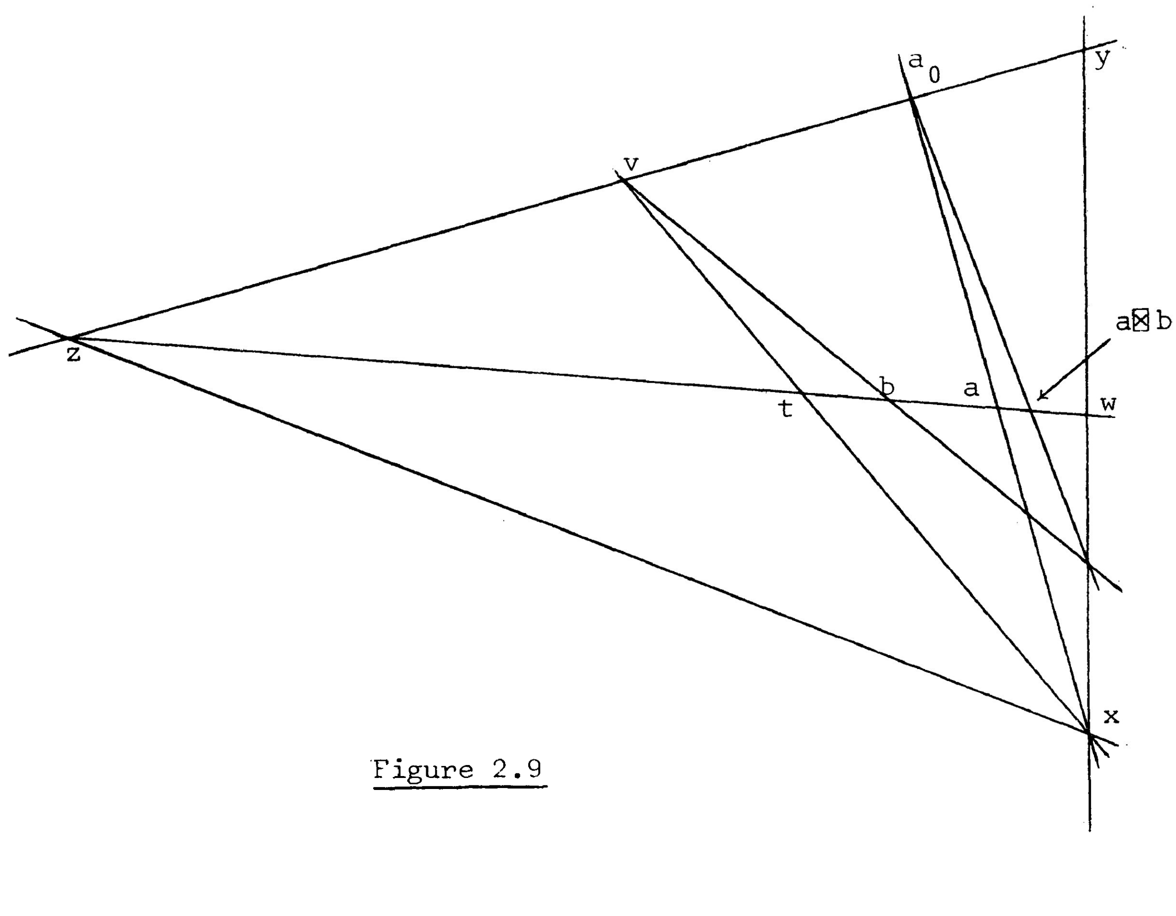




Figure 2.9

(1) $z \oplus a = a = a \oplus z$

(2) $z \otimes a = z = z \otimes z$

(3) $t \otimes a = a = a \otimes t$

(4) $a \otimes b \leq z + b$

(5) $a + (a \oplus t) = z + t$ and $a \cdot (a \oplus t) = 0$

Proof. For (1) we calculate

$$\begin{aligned} z \oplus a &= (z + t)(x + (y + z)(w + a_0)) \\ &= (z + t)(x + a_0) \\ &= (z + t)(x + a) \\ &= a \end{aligned}$$

$$\begin{aligned} a \oplus z &= (z + t)(x + (y + a)(w + z)) \\ &= (z + t)(x + a) \\ &= a \end{aligned}$$

for (2)

$$\begin{aligned} z \otimes a &= (z + t)(x + (y + z)(z + (y + t)(x + a)) \\ &= (z + t)(x + z + (z + y)(y + t)(x + a)) \\ &= (z + t)(x + z + y(x + a) \\ &= (z + t)(z + x) \\ &= z \end{aligned}$$

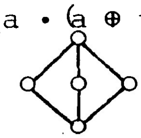
$$\begin{aligned} a \otimes z &= (z + t)(x + (y + a)(z + (y + t)(x + z)) \\ &= (z + t)(x + (y + a)(z + x)) \\ &= (z + t)(z + x) \\ &= z \end{aligned}$$

for (3)
$$\begin{aligned} a \otimes t &= (z + w)(x + (y + a)(z + (y + t)(x + t)) \\ &= (z + w)(x + (y + a)(z + t))) \\ &= (z + w)(y + a) \\ &= a \end{aligned}$$

$$\begin{aligned} t \otimes a &= (z + w)(x + (y + t)(z + (y + t)(x + a)) \\ &= (z + w)(x + (y + t)(x + a)) \\ &= (z + w)(x + a) \\ &= a\ . \end{aligned}$$

In order to prove (4), that $a \otimes b \leq z + b$ we show $z + a \otimes b \leq z + b$ .

$$\begin{aligned} z + a \otimes b &= (z + w)(z + x + (y + a)(z + (y + t)(x + b)) \\ &= (z + w)(x + (z + y + a)(z + (y + t)(x + b)) \\ &= (z + w)(x + z + (z + y + a)(y + t)(x + b)) \\ &= (z + w)(z + (x + b)(x + (z + y + a)(y + t)) \\ &= z + (z + w)(x + b)(x + (z + y + a)(y + t)) \\ &= z + b(x + (z + y + a)(y + t)) \leq z + b \end{aligned}$$

Part (5), $a + (a \oplus t) = z + t$ and $a \cdot (a \oplus t) = 0$ is equivalent to showing $a, a \oplus t, w$ form an $M_3$  in $[0, z + t]$ .

$$\begin{aligned} a + (a \oplus t) &= (z + t)(x + a + (y + a)(w + v)) \\ &= (z + t)(x + (w + v + a)(y + a)) \\ &= (z + t)(x + y + a) \\ &= z + t \end{aligned}$$

$$\begin{aligned} a \cdot (a \oplus t) &= a(x + (y + a)(w + v)) \\ &= a(w + v + x(y + a)) \\ &= a(w + v) \\ &= 0 . \end{aligned}$$

So we have $a \otimes b \le (z + b)$ and therefore $(z + b)(a \otimes b) = a \otimes b = (z + b)(x + (y + a)(z + (y + t)(x + b)))$ .

The Arguesian identity is due to Jónsson [ 5 ], and is precisely what is required to coordinatize a projective plane.

<u>Definition 2.10</u> A lattice $L$ is called <u>Arguesian</u> if for any set of elements $\{a_0, a_1, a_2, b_0, b_1, b_2\}$ in $L$ and

$c_i$ defined as $c_i = (a_j + a_k)(b_j + b_k)$

for $\{i, j, k\} = \{0, 1, 2\}$ and $y = c_2(c_0 + c_1)$ , the following inequality holds.

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \le a_0(y + a_1) + b_0(y + b_1)$$

An easy corollary to the definition is that every Arguesian lattice is modular - also due to Jónsson [6].

<u>Corollary 2.11</u> Every Arguesian lattice is modular.

<u>Proof</u>. Let $L$ be an Arguesian lattice. We must show that for any $s, t, u \in L$, $s \le u$ implies $s + (t \cdot u) \le (s + t) \cdot u$ since the reverse inequality holds in any lattice. Taking $a_0 = b_2 = t$ , $b_1 = u$, $a_1 = a_2 = b_0 = s$ together with the assumption $s \le u$ , we have that $a_1 + b_1 = b_1$ so

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) = (s + t)(s + t)u = (s + t)u .$$

Similarly, $c_2 = (a_0 + a_1)(b_0 + b_1) = (s + t)(s + u) = u(t + s)$

$c_1 = (a_0 + a_2)(b_0 + b_2) = (s + t)(s + t) = s + t$

$c_0 = (a_1 + a_2)(b_1 + b_2) = s \cdot (t + u) = s$

so that $c_2(c_0 + c_1) = u(s + t)$

hence

$$\begin{aligned} & a_0(a_1 + y) + b_0(b_1 + y) \\ &= t \cdot (s + u(s + t)) + s(u + u(s + t)) \\ &= s \cdot u + t(u \cdot (s + t)) \\ &= s \cdot u + t \cdot u \\ &= s + (t \cdot u) \end{aligned}$$

and it follows that $L$ is modular.

∎

Throughout this text, the Arguesian identity will be used in its equivalent form - the Desarguean implication.

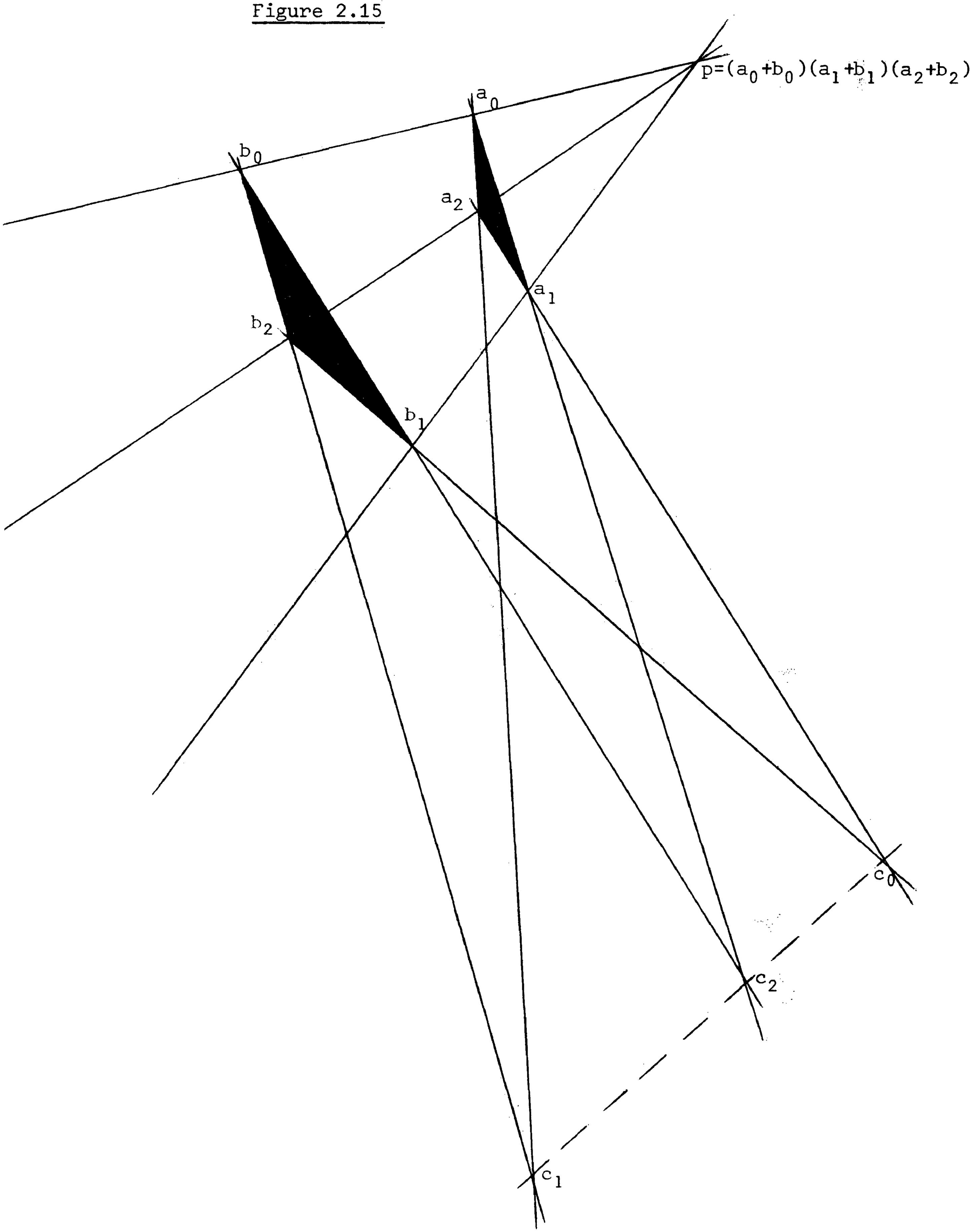
Definition 2.12 Given a lattice $L$, a triangle in $L$ is any triple $\langle a, b, c\rangle$ of elements of $L$.

Definition 2.13 Given two triangles $A = \langle a_0, a_1, a_2\rangle$ $B = \langle b_0, b_1, b_2\rangle$ in a modular lattice $L$, the pair $A, B$ will be called centrally perspective if $(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ and they are called axially perspective in case $c_2 \leq c_0 + c_1$ where $c_i = (a_j + a_k)(b_j + b_k)$ for $\{i, j, k\} = \{0, 1, 2\}$.

Desargues implication can now be stated as:

Definition 2.14 A lattice L is Desarguean if every pair of triangles that are centrally perspective are axially perspective.

Figure 2.15



See Figure 2.15.

We can now prove that any Arguesian lattice is Desarguean.

<u>Theorem 2.16</u> Suppose $L$ is an Arguesian lattice and $A = \langle a_0, a_1, a_2\rangle$ $B = \langle b_0, b_1, b_2\rangle$ are centrally perspective in $L$. Then $A, B$ are axially perspective in $L$.

<u>Proof</u>. Since $A, B$ are centrally perspective $(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ so let

$$c_0 = (a_1 + a_2)(b_1 + b_2) \qquad d = (a_1 + a_2)(c_1 + c_2)$$

$$c_1 = (a_0 + a_2)(b_0 + b_2)$$

$$c_2 = (a_0 + a_1)(b_0 + b_1)$$

Since $L$ is Arguesian, $L$ is modular hence,

$$\begin{aligned}
c_1 + d &= (a_1 + a_2 + c_1)(c_1 + c_2) \\
&= (c_1 + c_2)(a_1 + (a_0 + a_2)(b_0 + b_2 + a_2)) \\
&\geq (c_1 + c_2)(a_1 + (a_0 + a_2)(b_0 + (a_1 + b_1)(a_0 + b_0))) \\
&= (c_1 + c_2)(a_1 + (a_0 + a_2)(a_0 + b_0)(b_0 + a_1 + b_1))) \\
&\geq (c_1 + c_2)(a_1 + a_0(b_0 + a_1 + b_1)) \\
&= (c_1 + c_2)(a_1 + a_0)(b_0 + a_1 + b_1) \\
&\geq (c_1 + c_2)(a_1 + a_0)(b_0 + b_1) \\
&= (c_1 + c_2) \cdot (c_2) \\
&= c_2
\end{aligned}$$

and now in the definition of Arguesian lattice we use the following points $c_1, b_0, a_0, d, b_1, a_1$ so that

$$(c_1 + d)(b_0 + b_1)(a_0 + a_1) \leq c_1(y + b_0) + d(y + b_1)$$

where $y = (c_1 + b_0)(d + b_1)[(a_0 + b_0)(a_1 + b_1) + (c_1 + a_0)(a_1 + d)]$ .

By our previous argument $c_2 = (a_0 + a_1)(b_0 + b_1) \leq c_1 + d$ so the above becomes:

$$c_2 \leq c_1(b_0 + y) + d(y + b_1)$$

We will now show that $d(y + b_1) \leq c_0 + c_1$ , which implies $c_2 \leq c_0 + c_1$ , and completes the proof.

First we show that

$$\begin{aligned}
y &= (d + b_1)(b_0 + a_0 + a_2)(b_0 + b_2)[(a_0 + b_0)(a_1 + b_1) + \\
&\qquad (a_1 + a_2)(a_0 + a_2)(a_0 + b_0 + b_2)(a_1 + c_1 + c_2)] \\
&\leq (b_0 + b_2)[(a_0 + a_2)(a_1 + a_2) + (a_0 + b_0)(a_1 + b_1)] \\
&\leq (b_0 + b_2)[(a_0 + a_2)(a_1 + a_2) + a_2 + b_2] \text{ by central perspectivity} \\
&= (b_0 + b_2)[b_2 + (a_0 + a_2)(a_1 + a_2)] \\
&= b_2 + (a_0 + a_2)(a_1 + a_2)(b_0 + b_2) \\
&= b_2 + c_1(a_1 + a_2) .
\end{aligned}$$

Also
$$\begin{aligned}
d(y + b_1) &\leq (a_1 + a_2)[b_1 + b_2 + c_1(a_1 + a_2)] \\
&= (a_1 + a_2)(b_1 + b_2) + c_1(a_1 + a_2) \\
&\leq c_0 + c_1
\end{aligned}$$

and therefore the triangles are axially perspective. Later in this chapter we will strengthen the statement of central perspectivity slightly so that the Arguesian Identity will be equivalent to a statement of the form "centrally perspective" if and only if "axially perspective". First however we will complete the proof that the Arguesian identity is equivalent to the Desarguean implication.

This proof is due to Gratzer, Jónsson, Lakser [4]. We show first that if L is Desarguean, then L is modular.

<u>Lemma 2.17</u>. If $L$ is Desarguean, then L is modular.

<u>Proof</u>. Suppose $L$ is Desarguean and $a, b, c \in L$. If $a \le c$, we must show $(a + b)c \le a + bc$. Consider the triangles $\langle a, c, a\rangle$ and $\langle a, b, b\rangle$.

$$(a + a)(c + b) = a(c + b) \quad \text{but} \quad a \le c \Rightarrow a(c + b) = a$$

therefore these triangles are centrally perspective, and hence axially perspective.

$$(a + c)(a + b) \le (a + c)(b + b) + (a + a)(a + b)$$

$$\Rightarrow \qquad c(a + b) \le bc + a$$

and so L is modular. ∎

<u>Lemma 2.18</u>. If a lattice $L$ is Desarguean, then $L$ is Arguesian.

<u>Proof</u>. Suppose that $a_0,a_1,a_2,b_0,b_1,b_2 \in L$ and let $p = (a_0 + b_0)(a_1 + b_1) \cdot (a_2 + b_2)$. If we assume further that (1) $p + a_i = p + b_i = a_i + b_i$ for $i = 0, 1, 2$ then the triangles $\langle a_0, a_1, a_2 + a_0a_1\rangle$ and $\langle b_0, b_1, b_2 + b_0(a_1 + b_1)\rangle$ are centrally perspective since

$$(a_0 + b_0)(a_1 + b_1) = (p + a_0)(p + a_1) = p + a_0(p + a_1)$$

$$\begin{aligned} a_2 + b_2 + (a_0a_1 + b_0)(a_1 + b_1) &= a_2 + b_2 + (p + a_1)(a_0a_1 + b_0) \\ &= a_2 + (p + a_1)(p + b_0 + a_0a_1) \\ &= a_2 + (p + a_1)(p + a_0) . \end{aligned}$$

So we have

$$\begin{aligned} c_2 &\le c_1 + (a_1 + a_2)(b_2 + (b_0 + b_1)(a_1 + b_1)) \\ &= c_1 + (a_1 + a_2)(b_1 + b_2 + a_1(b_0 + b_1)) \\ &= c_1 + c_0 + a_1(b_0 + b_1)\ . \end{aligned}$$

Therefore

$$\begin{aligned} c_2 &= c_2(c_0 + c_1 + a_1(b_0 + b_1)) \\ &= c_2(c_0 + c_1) + a_1(b_0 + b_1) \end{aligned}$$

by modularity.

Also

$$a_0 \le a_0 + a_1 \le a_1 + (a_0 + a_1)(b_0 + b_1)$$

since

$$a_1 + b_0 + b_1 = p + b_1 + b_0 = p + b_1 + a_0$$

hence (2) $a_0 \le a_1 + c_2 = a_1 + c_2(c_0 + c_1)$ and we obtain

(3) $$\begin{aligned} (a_0 + b_0)(a_1 + b_1)(a_2 + b_2) &\le b_0 + a_0 \\ &= b_0 + a_0(a_1 + c_2(c_0 + c_1)) \end{aligned}$$

by (2) above.

Suppose now that $a_0, a_1, a_2, b_0, b_1, b_2$ are arbitrary and do not necessarily satisfy (1) above. We then define $a_i' = a_i(p + b_i)$ $b_i' = (p + a_i)$ and claim that $a_0', a_1', a_2', b_0', b_1', b_2'$ satisfy (1). Clearly $p + a_i' = p + b_i' = (p + a_i)(p + b_i)$ , and

$$\begin{aligned} a_i' + b_i' &= a_i(p + b_i) + b_i(p + a_i) \\ &= (a_i + b_i)(p + a_i)(p + b_i) \\ &= (p + a_i)(p + b_i)\ . \end{aligned}$$

But we also have that $p' = p$ where

$$\begin{aligned} p' &= (a_0' + b_0')(a_1' + b_1')(a_2' + b_2') \\ &= (p + a_0)(p + b_0)(p + a_1)(p + b_1)(p + a_2)(p + b_2) \\ &\geq p \ . \end{aligned}$$

But $a_i' \leq a_i$ and $b_i' \leq b_i$ for $i = 0, 1, 2$ so we also have $p' \leq p$.

By (3) $\quad p' = p \leq b_0' + a_0'(a_1' + c_2'(c_0' + c_1'))$

$$\leq b_0 + a_0(a_1 + c_2(c_0 + c_1))$$

since every $a_i' \leq a_i$ and $b_i' \leq b_i$.

∎

The lattice identity (3) is equivalent to the Arguesian identity in Definition 2.10.

Axially perspectivity does not imply central perspectivity in an Arguesian lattice. We can take for example $a_0 = a_1 \neq b_0 = b_1$ in a projective plane with $a_2 = b_2$ then $c_2 = 0 \leq c_0 + c_1$ but $(a_0 + b_0)(a_1 + b_1) \geq a_0$ which is not less than or equal to $a_2$ unless $a_0 = a_2$. We have however the following lemma, see Jónsson and Monk [8].

Lemma 2.19. If $L$ is Desarguean and $a_0, a_1, a_2, b_0, b_1, b_2 \in L$ then the triangles $A = \langle a_0, a_1, a_2\rangle$ $B = \langle b_0, b_1, b_2\rangle$ are axially perspective if and only if

(1) $(a_0 + b_0)(a_1 + b_1) \leq (a_0 + a_2)(a_1 + a_2) + (b_0 + b_2)(b_1 + b_2)$.

Proof. Define $c_i = (a_j + a_k)(b_j + b_k)$ for $\{i, j, k\} = \{0, 1, 2\}$ and suppose A, B are axially perspective. That is:

$$(a_0 + a_1)(b_0 + b_1) \leq c_0 + c_1 \ .$$

Then the triangles $A' = \langle a_0, b_0, c_1\rangle$ and $B' = \langle a_1, b_1, c_0\rangle$ are centrally perspective and therefore axially perspective since $L$ was assumed to be Desarguean. Axial perspectivity gives (1).

$$\begin{aligned}(a_0 + b_0)(a_1 + b_1) &\le (a_0 + c_1)(a_1 + c_0) + (b_0 + c_1)(b_1 + c_0) \\ &\le (a_0 + a_2)(a_1 + a_2) + (b_0 + b_2)(b_1 + b_2) .\end{aligned}$$

To prove the converse let $a_2' = (a_0 + a_2)(a_1 + a_2)$ and $b_2' = (b_0 + b_2)(b_1 + b_2)$, so that $a_i + a_2' = a_i + a_2$ and $b_i + b_2' = b_i + b_2$ for $i = 1, 2$. Then $A'' = \langle a_0, a_1, a_2'\rangle$, $B'' = \langle b_0, b_1, b_2'\rangle$ are centrally perspective by assumption, and for $i = 1, 2$,

$$c_i = (a_i + a_2)(b_i + b_2) = (a_i + a_2')(b_i + b_2') .$$

∎

<u>Corollary 2.20</u> If $L$ is an Arguesian lattice and $A = \langle a_0, a_1, a_2\rangle$ $B = \langle b_0, b_1, b_2\rangle$ are two triangles such that $(a_0 + a_2)(a_1 + a_2) = a_2$ and $(b_0 + b_2)(b_1 + b_2) = b_2$ then $A$ and $B$ are centrally perspective if and only if $A$ and $B$ are axially perspective.

The above lemma will be used extensively throughout the remainder of this text and so we will call a triangle $A$ <u>NORMAL</u> if it satisfies the condition

$$a_2 = (a_0 + a_2)(a_1 + a_2) \quad \text{where } A = \langle a_0, a_1, a_2\rangle .$$

Throughout this text we use the following identities valid in a modular lattice

(1) $a \cdot (b + c(a + d)) = a \cdot (c + b(a + d))$

(2) comparable complements are equal.

We close this chapter with two proofs that require that $L$ be Arguesian. The first proof concerns the additive inverse of an element $a \in D$, the second shows $a \oplus b = a \boxplus b$.

Given $a, b \in D$ we perform the operation $\oplus$ and obtain $a \oplus b \in D$. Suppose though that we are given some $a \in D$ and wish to find an $\bar{a} \in D$ such that $a \oplus \bar{a} = z$. Assume for the moment that $a \oplus \bar{a} = z$ and calculate:

$$z = a \oplus \bar{a}$$

$$\text{iff} \quad z + x = x + (y + a)(w + \bar{a}_0)$$

$$\text{iff} \quad (y + a)(z + x) = (y + a)(w + \bar{a}_0)$$

$$\text{iff} \quad w + (y + a)(z + x) = w + \bar{a}_0$$

$$\text{iff} \quad (z + y)(w + (z + x)(y + a)) = (z + y)(x + \bar{a})$$

$$\text{iff} \quad (z + w)(x + (z + y)(w + (z + x)(y + a))) = \bar{a}$$

Similarly, if we suppose that $\underline{a} \oplus a = z$ for some $\underline{a}$ then we obtain $\underline{a} = (z + w)(y + (z + x)(w + (z + y)(x + a)))$. In fact it is easy to show that $a \in D$ implies $\bar{a}$, $\underline{a}$ are also in $D$. (See Figure 2.22)

Lemma 2.21. If $a \in D$ and $L$ is modular then

$$\bar{a} = (z + t)(x + (z + y)(w + (z + x)(y + a))) \quad \text{and}$$

$$\underline{a} = (z + t)(y + (z + x)(w + (z + y)(x + a)))$$

are also in $D$.

Proof. We will calculate $w + \underline{a}$, the other calculations being similar.

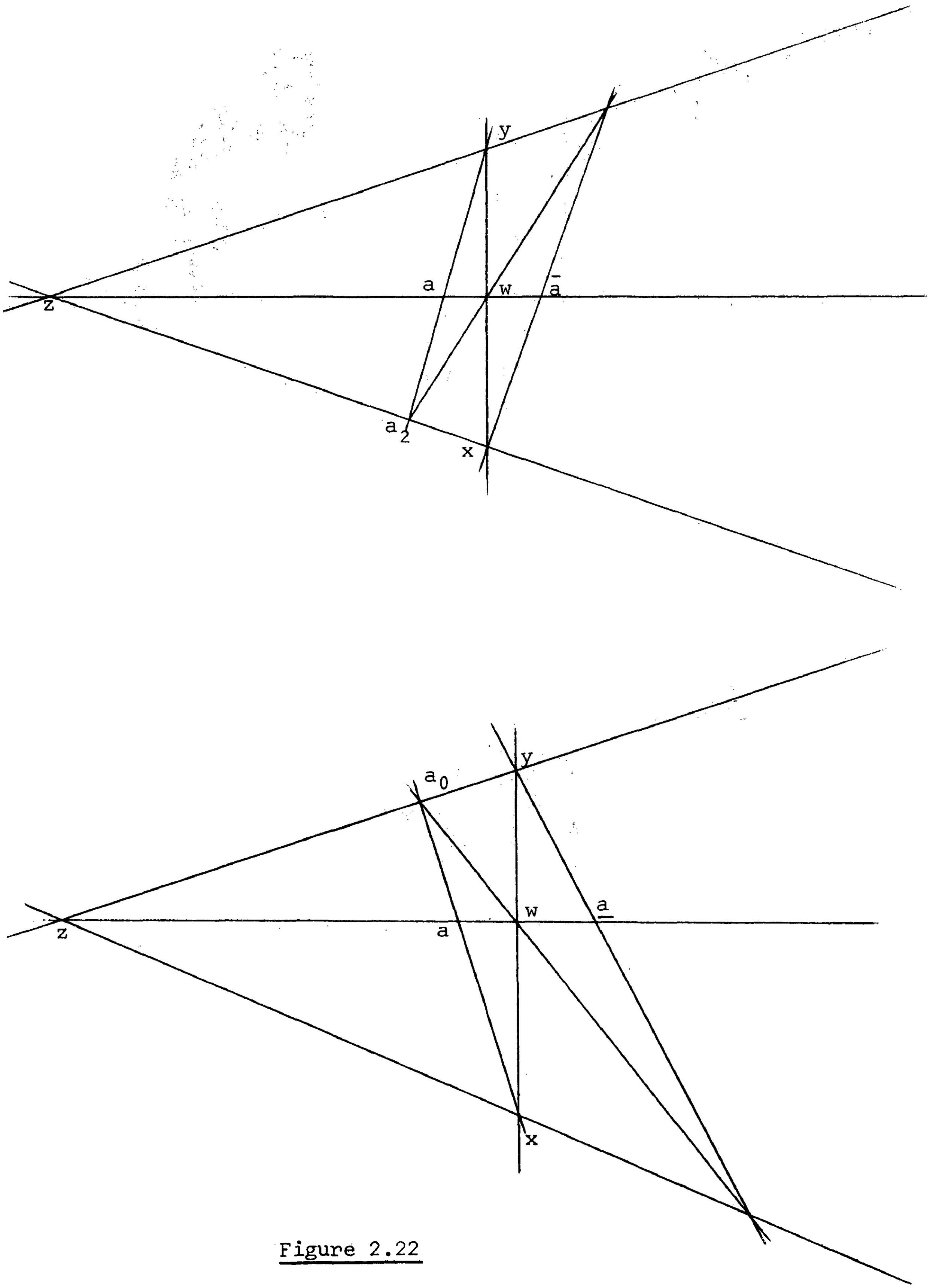



Figure 2.22

$$
\begin{aligned}
w + \underline{a} &= (z + w)(y + w + (z + x)(w + a_0)) \\
&= (z + w)(y + w + a_0) \\
&= (z + w)(w + z + y) \\
&= z + w
\end{aligned}
$$

∎

We will now show that $\bar{a} = \underline{a}$ .

Lemma 2.23. If $L$ is Arguesian and $a \in D$ then

$$
\begin{aligned}
\underline{a} &= (z + t)(y + (z + x)(w + a_0)) \\
&= (z + t)(x + (z + y)(w + (z + x)(y + a))) \\
&= \bar{a} \ .
\end{aligned}
$$

Proof. Consider the triangles

$A = \langle x, y, a\rangle$ and

$B = \langle (z + x)(w + a_0), (z + y)(w + a_2), w\rangle$ .

Central perspectivity is equivalent to

$$(z + x)(z + y) \leq w + a$$

which is clearly true if $a \in D$ .

We have therefore that

$$
\begin{aligned}
(1)\ (x + y)((z + x)(w + a_0) + (z + y)(w + a_2)) &\leq a_2 + (x + a)(w + a_0) \\
&= a_0 + a_2 \ .
\end{aligned}
$$

We now take the triangles $A_1, B_1$ whose statement of central perspectivity is (1) above.

$A_1 = \langle x, (z + y)(w + a_2), a_0\rangle$

$B_1 = \langle y, (z + x)(w + a_0), a_2\rangle$ .

Axial perspectivity of $A_1$ and $B_1$ yields the inequality

$$(x + (z + y)(w + (z + x)(y + a)))(y + (z + x)(w + a_0)) \le$$

$$(z + y)((w + a_0) + (z + x)(y + a))(z + x)(w + a_0 + (z + x)(y + a))$$

$$+ (x + a)(y + a)$$

which is $\le z + a \le z + t$ .

So we have

$$(x + (z + y)(w + (z + x)(y + a)))(y + (z + x)(w + a_0))) \le$$

$$(z + t)(y + (z + x)(w + a_0))$$

which implies [by joining both sides with $x$ then meeting with $z + t$] that

$$(z + t)(x + (z + y)(w + (z + x)(y + a))) = \bar{a}$$

$$\le \underline{a} = (z + t)(y + (z + x)(w + a_0)) .$$

Since both $\underline{a}$, $\bar{a}$ are complements of $w$ , they are equal. ∎

<u>Lemma 2.24</u>. If $L$ is Arguesian and $a, b \in D$ then $a \oplus b = a \boxplus b$ . (See Figure 2.25)

<u>Proof</u>. Consider the triangles

$$A = \langle x, (y + a)(w + b_0), y \rangle$$

$$B = \langle b, w, v \rangle .$$

Central perspectivity follows from the inequalities $b_0 \le x + b$ , $w(x + b) = 0$ and $y + v = z + y$ .

Axial perspectivity implies

$$(w + b)(x + (y + a)(w + b_0)) \le (x + y)(v + b) + (w + v)(y + a) .$$

The left side is $a \oplus b$ , and meeting both sides with $z + t$ we obtain $a \oplus b \le a \boxplus b$ .

∎

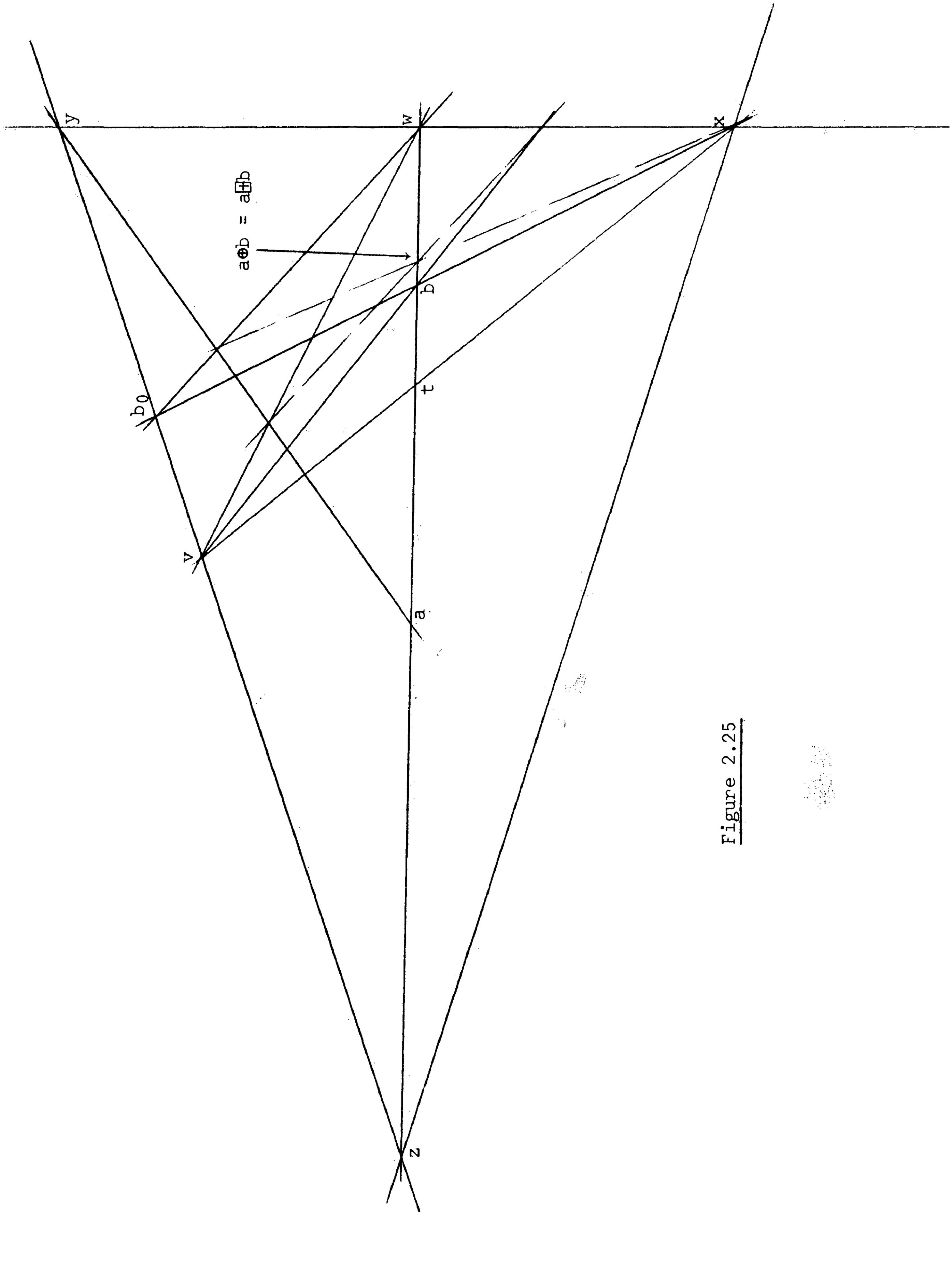



Figure 2.25

## Chapter 3

### Distributivity and the associativity of multiplication

In the preceding chapters different forms of multiplication and addition were discussed. In this chapter we will use these properties to prove: the associativity of multiplication in section 2, left and right distributivity of multiplication over addition in section 1.

### SECTION 1

Our goal in this section is to show the following two equalities for $a, b, c$ in $D$.

$$(1)\quad a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

$$(2)\quad (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)\ .$$

The proofs given are straightforward applications of the Desarguean implication. Both use the ternary operator $T(a, b, c)$ defined in 2.2.

Lemma 3.1. If $L$ is Arguesian with spanning 3-diamond $(z, t, x, y)$ and $a, b, c \in D$ then $T(a, b, c) = (a \otimes b) \oplus c$. Geometrically this says that the ternary ring is linear.

Proof. Following the format of most results in this paper, we take the triangles

$$A = \langle y, c_0, (y + a)(c_0 + b_\infty)\rangle$$

$$B = \langle a \otimes b, w, x\rangle\ .$$

These triangles are normal since for $A$ we have

$(y + (y + a)(c_0 + b_\infty))(c_0 + (y + a)(c_0 + b_\infty)) \le (y + a)(c_0 + b_\infty)$

and for $B$, $(x + w)(x + a \otimes b) = x$.

Their statement of central perspectivity is the following inequality

$$(y + a \otimes b)(w + c_0) \le x + (y + a)(c_0 + b_\infty)$$

which, if proven, would imply $(a \otimes b) \oplus c \le T(a, b, c)$.
If we show axial perspectivity, then we are finished. We do the following calculations.

(1) $(y + c_0)(w + a \otimes b) = (z + y)(z + t) = z$

(2) $(x + w)(c_0 + (y + a)(c_0 + b_\infty)) = (x + y)(z + y + a)(c_0 + b_\infty)$
$$= b_\infty(z + y + a)$$

(3) $(x + a \otimes b)(y + (y + a)(c_0 + b_\infty))$
$$= (x + (y + a)(z + b_\infty))(y + a)$$
$$= (y + a)(z + b_\infty)$$

The join of (2) and (3) above is

$$b_\infty(z + y + a) + (y + a)(z + b_\infty)$$
$$= (y + a + b_\infty(z + y + a))(z + b_\infty)$$
$$= (z + y + a)(z + b_\infty)$$

and clearly

$$z \le (z + y + a)(z + b_\infty).$$

So $A$, $B$ are axially perspective.

∎

In order to prove right distributivity we must show $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ which by the last lemma is equivalent to

$$(a \oplus b) \otimes c = T(a, c, b \otimes c) .$$

<u>Theorem 3.2</u> In an Arguesian lattice with spanning 3-diamond $(z, t, x, y)$ and $a, b, c \in D$ ,

$$(a \oplus b) \otimes c = T(a, c, b \otimes c) .$$

<u>Proof.</u> By simple calculation we have

$$T(a, c, b \otimes c) \leq (a \oplus b) \otimes c$$

iff $(z + t)(x + (y + a)((b \otimes c)_0 + c_\infty)) \leq (z + t)(x + (y + a \oplus b)(z + c_\infty))$

iff $x + (y + a)(c_\infty + (b \otimes c)_0) \leq x + (y + a \oplus b)(z + c_\infty))$

iff (1) $(y + a \oplus b)(x + (y + a)(c_\infty + (b \otimes c)_0)) \leq z + c_\infty$ .

The last inequality is the one we will show. The proof of the theorem requires the use of two sets of triangles where the axial perspectivity of the first set is implied by axial perspectivity of the second. We will begin with those triangles which imply (1) above.

Let $A_1 = \langle x, a \oplus b, z \rangle$

$B_1 = \langle (y + a)(c_\infty + (b \otimes c)_0), y, c_\infty \rangle$ .

These triangles are normal and the central perspectivity statement is the theorem, so we must show axial perspectivity:

$$(x + (a \oplus b))(y + a) \leq (z + (a \oplus b))(x + y) + (z + x)(c_\infty + (b \otimes c)_0)$$

but $\qquad (x + a \oplus b)(y + a) = (y + a)(w + b_0)$

and $\qquad (z + (a \oplus b))(x + y) = w(z + x + (y + a)(w + b_0))$

and therefore the above inequality becomes

$$(y + a)(w + b_0) \le [z + x + (y + a)(w + b_0)][w + (z + x)(c_\infty + (b \otimes c)_0)] \ .$$

Now this holds iff $(y + a)(w + b_0) \le w + (z + x)(c_\infty + (b \otimes c)_0)$

iff $w + b_0 \le w + (z + x)(c_\infty + (b \otimes c)_0)$

iff $b_0 \le w + (z + x)(c_\infty + (b \otimes c)_0)$ .

Consider the triangles

$$A_2 = \langle x, b, z\rangle$$

$$B_2 = \langle (b \otimes c)_0, y, c_\infty\rangle$$

whose axial perspectivity implies our desired conclusion.

Central perspectivity is:

$$(x + (b \otimes c)_0)(y + b) \le z + c_\infty$$

or $$(y + b)(x + (y + b)(z + c_\infty)) \le z + c_\infty$$

or $$(y + b)(z + c_\infty) \le z + c_\infty$$

and thus multiplication is right distributive over addition.

∎

To prove left-distributivity we will compare the following two expressions:

(1) $a \otimes T(t, b, c)$

which by lemma 3.1 is equal to $a \otimes ((t \otimes b) \oplus c) = a \otimes (b \oplus c)$

and (2) $T(a, b, a \otimes c) = (a \otimes b) \oplus (a \otimes c)$ .

Theorem 3.3 In an Arguesian lattice with spanning 3-frame and $a, b, c \in D$ ,

$$a \otimes T(t, b, c) = T(a, b, a \otimes c) \ .$$

Proof. We require

$$(z + w)(x + (y + a)(z + (y + t)(b_\infty + c_0)) =$$
$$= (z + w)(x + (y + a)(b_\infty + (a \otimes c)_0))$$

which will follow from the statement

(1) $$(y + a)(z + (y + t)(b_\infty + c_0)) \leq b_\infty + (a \otimes c)_0 .$$

The triangles $A_1$, $B_1$ given below are normal and the statement of central perspectivity is (1) above.

$$A_1 = \langle (y + a)(x + (a \otimes c)_0), z, (a \otimes c)_0 \rangle$$

$$B_1 = \langle y, (y + t)(b_\infty + c_0), b_\infty \rangle$$

To prove axial perspectivity let

$$p_1 = (y + (y + t)(b_\infty + c_0))(z + (y + a)(x + (a \otimes c)_0))$$

$$p_2 = (z + (a \otimes c)_0)(b_\infty + (y + t)(b_\infty + c_0))$$

$$p_3 = (y + b_\infty)((a \otimes c)_0 + (y + a)(x + (a \otimes c)_0)))$$

Doing some simple calculations we get

$$\begin{aligned} p_1 &= (y + t)(z + (y + a)(x + a \otimes c)) \\ &= (y + t)(z + (y + a)(x + (y + a)(z + c_1))) \\ &= (y + t)(z + c_1)(z + y + a) \\ &= (z + y + a)(y + t)(x + c) = c_1(z + y + a) \end{aligned}$$

$$p_2 = (z + (a \otimes c)_0)(b_\infty + c_0)$$
$$= c_0(z + (a \otimes c)_0)$$
$$= c_0(z + y)(x + z + (y + a)(z + c_1))$$
$$= c_0(x + (z + y + a)(z + c_1))$$

$$p_3 = (x + y)(z + y + a)(x + (a \otimes c)_0))$$
$$= x(z + y + a)$$

$$p_2 + p_3 = x(z + y + a) + c_0(x + (z + y + a)(z + c_1))$$
$$= (x + c)(z + y + a)(x + (z + y + a)(z + c_1))$$

Clearly $p_1 \leq p_2 + p_3$ and therefore multiplication is left distributive over addition. ∎

## SECTION 2

In chapter 2, we gave two definitions of multiplication in a modular lattice with spanning 3-diamond, the $\otimes$ of von Staudt and what we have called the $\boxtimes$ due to von Neumann. By definition, $a \boxtimes b = (z + t)((z + v)(x + a) + (b + v)(x + w))$ and we notice that in the above expression, only $v$ and $x$ are not on the "line" $z + t$.

In this section, using Theorem 3.6 of Jónsson and Monk [8], we show that $a \boxtimes b$ is independent of the $x$ and $v$. That is, $a \boxtimes b = (z + t)((z + \bar{v})(\bar{x} + a) + (b + \bar{v})(\bar{x} + w))$ where $\bar{x}$ is any complement of $w$ in the interval $[0, x + y]$ and $\bar{v}$ is defined by $\bar{x}$. We then convert this new expression into a von Staudt form of multiplication. As a corollary we obtain $a \otimes b = a \boxtimes b$. Our proof of the associativity of multiplication requires the use of both forms of multiplication.

The next theorem (3.6 in [8]) is really the geometric quadrangle property equivalent to the Desarguean implication in projective planes.

<u>Theorem 3.4</u> Suppose $L$ is an Arguesian lattice, $p_1, p_2, p_1', p_2'$, $m, a_1, a_2, a_3, a_4$ are elements of $L$ such that

(1) $a_1, a_2, a_3, a_4 \le m$

(2) $(p_1 + p_2)m = (p_1' + p_2')m$

(3) $p_1m = p_2m = p_1'm = p_2'm = 0$

(4) $(p_1 + p_1')m = (p_2 + p_2')m = (p_1 + p_1')(p_2 + p_2') = k$

and $q_1 = (p_1 + a_1)(p_2 + a_2) \qquad q_2 = (p_1 + a_3)(p_2 + a_4)$

$q_1' = (p_1' + a_1)(p_2' + a_2) \qquad q_2' = (p_1' + a_3)(p_2' + a_4)$ .

Then $(q_1 + q_2)m \le (q_1' + q_2')m + k \cdot a_1 + k \cdot a_3$ . We also note that in case $k \cdot a_1 + k \cdot a_3 = 0$ or $k \cdot a_2 + k \cdot a_4 = 0$ that $m(q_1 + q_2) = m(q_1' + q_2')$ . (See Figure 3.5)

<u>Proof</u>. If we take the triangles $A = \langle a_1, p_1, p_1' \rangle$ $B = \langle a_2, p_2, p_2' \rangle$ then $(a_1 + a_2)(p_1 + p_2) \le m(p_1 + p_2) \le p_1' + p_2'$ by (2). So we have

$$(a_1 + p_1)(a_2 + p_2) = q_1 \le (a_1 + p_1')(a_2 + p_2') + (p_1 + p_1')(p_2 + p_2')$$
$$= q_1' + k .$$

Similarly, the triangles $A_1 = \langle a_3, p_1, p_1' \rangle$ and $B_1 = \langle a_4, p_2, p_2' \rangle$ imply that $q_2 \le q_2' + k$ .

$$m(q_1 + q_2) \le m(p_1 + a_1 + a_3) = a_1 + a_3$$

So $m(q_1 + q_2) = (a_1 + a_3)(q_1 + q_2)$ .

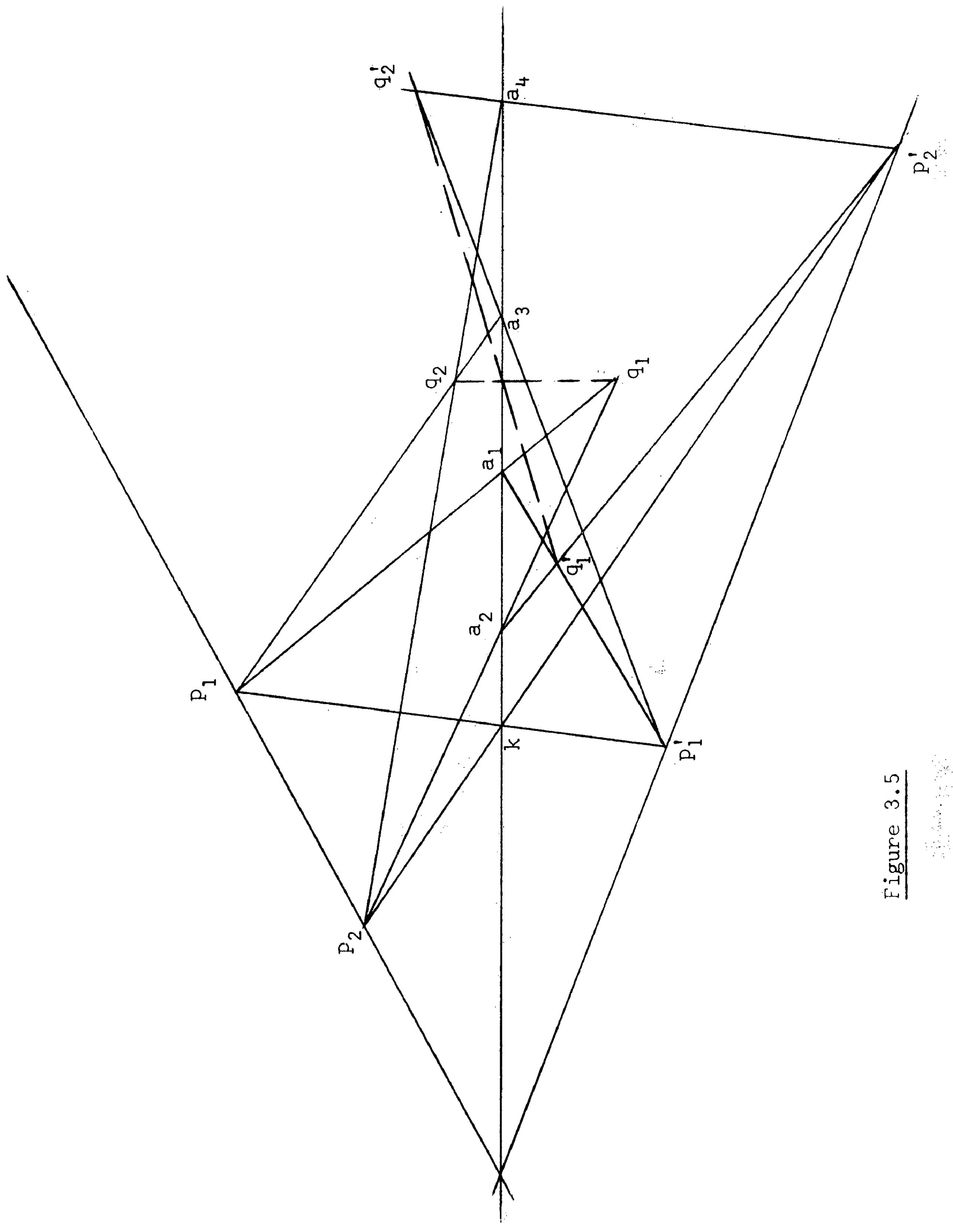



Figure 3.5

We now take the triangles $A_2 = \langle a_1, a_3, p_1' \rangle$ and $B_2 = \langle q_1, q_2, k \rangle$, and claim that axial perspectivity implies our desired conclusion.

$$(a_1 + a_3)(q_1 + q_2) = m(q_1 + q_2)$$

$$\begin{aligned}(a_1 + p_1')(k + q_1) &\le (a_1 + p_1')(k + q_1') = q_1' + k(a_1 + p_1') \\ &= q_1' + ka_1\end{aligned}$$

and $$\begin{aligned}(a_3 + p_1')(k + q_2) &\le (a_3 + p_1')(k + q_2') = q_2' + k(a_3 + p_1') \\ &= q_2' + ka_3 .\end{aligned}$$

If we can show central perspectivity then

$$m(q_1 + q_2) \le m(q_1' + q_2' + ka_1 + ka_3) = m(q_1' + q_2') + ka_1 + ka_3 .$$

By lemma 2.19, chapter 2, the result follows if

$$(a_1 + q_1)(a_3 + q_2) \le (a_1 + p_1')(a_3 + p_1') + (k + q_1)(k + q_2) .$$

Now

$$\begin{aligned}
(a_1 + q_1)(a_3 + q_2) &= (p_1 + a_1)(p_2 + a_2 + a_1)(p_1 + a_3) \\
&= (p_1 + a_1a_3)(p_2 + a_2 + a_1) \\
&= a_1a_3 + p_1(p_2 + a_1 + a_2) \\
&= a_1a_3 + p_1(p_1 + p_2)(p_2 + a_1 + a_2) \\
&= a_1a_3 + p_1(p_2 + (a_1 + a_2)(p_1 + p_2)) \\
&\le a_1a_3 + p_1(p_2 + m(p_1 + p_2)) \\
&\le a_1a_3 + (p_1 + p_1')(p_1' + p_2' + p_2) \\
&= a_1a_3 + p_1' + k \\
&\le a_1a_3 + p_1' + k + q_1(k + q_2) \\
&\le (a_1 + p_1')(a_3 + p_1') + (k + q_1)(k + q_2)
\end{aligned}$$

and the theorem is proven.

∎

<u>Theorem 3.6</u> In an Arguesian lattice $L$ with spanning 3-diamond $(z, t, x, y)$, $a, b \in D$,

$$a \boxtimes b = (z + t)((z + \hat{v})(\hat{x} + a) + (b + \hat{v})(\hat{x} + w))$$

if $\hat{x} + w = x + y$, $\hat{x} \cdot w = 0$, and $\hat{v}$ is defined as $(w + v)(t + \hat{x})$.

<u>Proof</u>. In the previous theorem, we let

$$m = z + t$$
$$p_1 = x$$
$$p_2 = v$$
$$p_1' = \hat{x}$$
$$p_2' = \hat{v}$$
$$a_1 = a$$
$$a_2 = z$$
$$a_3 = w$$
$$a_4 = b$$

Then $m(p_1 + p_2) = (z + t)(x + v) = t$

and $m(p_1' + p_2') = (z + t)(t + \hat{x}) = t$.

Also $(p_1 + p_1')m = (z + t)(x + \hat{x}) = w(x + \hat{x})$

$$(p_2 + p_2')m = (z + t)(w + v)(t + \hat{x} + v) = w(t + \hat{x} + x) = w(x + \hat{x})$$

$$(p_1 + p_1')(p_2 + p_2') = (x + \hat{x})(w + v)(x + \hat{x} + t) = w(x + \hat{x})$$

and $v \cdot (z + t) = \hat{x}(z + t) = \hat{v}(z + t) = x(z + t) = 0$.

So by theorem 3.4,

$$(z + t)((x + a)(v + z) + (x + w)(v + b)) \quad [= a \boxtimes b]$$
$$= (z + t)((\hat{x} + a)(\hat{v} + z) + (\hat{x} + w)(\hat{v} + b)) .$$

∎

Now consider the triangles

$$A = \langle z, (z + \hat{v})(\hat{x} + a), (\hat{x} + a)(z + (\hat{x} + t)(\hat{y} + b))\rangle$$

$$B = \langle b, (\hat{x} + \hat{y})(b + \hat{v}), \hat{y}\rangle$$

where $\hat{y} = (z + \hat{v})(x + y) = (x + y)(z + (w + v)(\hat{x} + t))$ . By the definition of $\hat{y}$ , we have $\hat{x} + \hat{y} = x + y$ and $w \cdot \hat{y} = 0$ . $A, B$ are normal triangles whose central perspectivity implies $(z + b)(a \boxtimes b) \leq (z + t)(\hat{y} + (\hat{x} + a)(z + (\hat{x} + t)(\hat{y} + b)))$ . But $a \boxtimes b \leq z + b$ as can be seen by calculating $z + a \boxtimes b$ , so the left side of the last inequality is precisely $a \boxtimes b$ . Axial perspectivity requires

$$\begin{aligned}(z + \hat{v})(b + \hat{v})(z + \hat{x} + a) \leq\ & (\hat{x} + \hat{y})(\hat{x} + a)(\hat{y} + \hat{v} + b) \\ & \cdot ((z + \hat{v})(z + \hat{x} + a) + (\hat{x} + t)(\hat{y} + b)) \\ & + (\hat{y} + b)(z + (\hat{x} + t)(\hat{y} + b))(z + \hat{x} + a) .\end{aligned}$$

On the left we have $(\hat{v} + b \cdot z)(z + \hat{x} + a)$ and on the right,

$$\begin{aligned}&\hat{x}(z + \hat{v} + b)(z + \hat{v}(z + \hat{x} + a) + (\hat{x} + t)(\hat{y} + b)) \\ &\qquad + (z \cdot b + (\hat{x} + t)(\hat{y} + b)))(z + \hat{x} + a) \\ &= \hat{x}(z + \hat{v}(z + \hat{x} + a) + (\hat{x} + t)(\hat{y} + b)) \\ &\qquad + (z \cdot b + (\hat{x} + t)(\hat{y} + b))(z + \hat{x} + a) \\ &= [(z + \hat{v}(z + \hat{x} + a) + (\hat{x} + t)(\hat{y} + b))](\hat{x} + t + z \cdot b)(z + \hat{x} + a)\end{aligned}$$

which is clearly greater than or equal to $(\hat{v} + z \cdot b)(z + \hat{x} + a)$ .

So we have the following lemma.

Lemma 3.7. If $L$ is Arguesian with $a, b \in D$ and $\hat{y} = (x + y)(z + (w + v)(\hat{x} + t))$ where $\hat{x}$ is any complement of $w$ in the interval $[0, x + y]$, then

$$a \boxtimes b = (z + t)(\hat{y} + (\hat{x} + a)(z + (\hat{x} + t)(\hat{y} + b))) .$$

The next lemma shows that the $\hat{y}$ used above need not depend upon the $\hat{x}$ chosen.

Lemma 3.8. If $L$ is Arguesian and $\bar{x}, \bar{y}$ are such that $(w, \bar{x}, \bar{y})$ form an $M_3$ with top $x + y$ and bottom $0$, then (Figure 3.9)

$$a \boxtimes b = (z + t)(\bar{y} + (\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b))) .$$

Proof. Using lemma 3.7, we let $\hat{x} = \bar{x}$, $\hat{y} = (x + y)(z + (w + v)(x + t))$, and take the following triangles.

$$A = \langle z, (\bar{x} + a)(z + (\bar{x} + t)(\hat{y} + b)), (\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b))\rangle$$

$$B = \langle b, \hat{y}, \bar{y}\rangle$$

where $\bar{x}, \bar{y}$ satisfy the conditions of the lemma.

Axial perspectivity of $A$, $B$ above is the inequality

$$(z + \bar{x} + a)(\hat{y} + b)(z + (\bar{x} + t)(\hat{y} + b)) \le$$
$$\le (z + \bar{x} + a)(\bar{y} + b)(z + (\bar{x} + t)(\bar{y} + b))$$
$$+ (\bar{y} + \hat{y})(\bar{x} + a)(z + (\bar{x} + t)(\hat{y} + b) + (\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b)))$$

or

$$(z + \bar{x} + a)(zb + \bar{x} + t)(\hat{y} + b) \le$$
$$\le (z + \bar{x} + a)(\bar{y} + b)(zb + \bar{x} + t)$$
$$+ \bar{x}(\bar{y} + \hat{y})((\bar{x} + t)(\hat{y} + b) + (z + \bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b)))$$
$$= (z + \bar{x} + a)(\bar{y} + b)(zb + \bar{x} + t)$$
$$+ \bar{x}(\bar{y} + \hat{y})(\hat{y} + b + (z + \bar{x} + a)(\bar{y} + b)(\bar{x} + t))$$
$$= (z + \bar{x} + a)(zb + \bar{x} + t)[\bar{y} + b + \bar{x}(\bar{y} + \hat{y})$$
$$(\hat{y} + b + (z + \bar{x} + a)(\bar{y} + b)(\bar{x} + t))]$$
$$= (z + \bar{x} + a)(zb + \bar{x} + t)[\bar{y} + b + (y + b)$$
$$(\bar{x}(\bar{y} + \hat{y}) + (z + \bar{x} + a)(\bar{y} + b)(\bar{x} + t))]$$
$$= (z + \bar{x} + a)(zb + \bar{x} + t)[\bar{y} + b + (\hat{y} + b)(\bar{y} + \hat{y} + b + (z + \bar{x} + a)(\bar{x} + t))$$
$$= (z + \bar{x} + a)(zb + \bar{x} + t)(\bar{y} + \hat{y} + b) .$$

But $A$, $B$ are normal and therefore centrally perspective which implies

$$(z + b)(\hat{y} + (\bar{x} + a)(z + (\bar{x} + t)(\hat{y} + b))) \le \bar{y} + (\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b))$$

or equivalently

$$(z + b)(a \boxtimes b) \le (z + t)(\bar{y} + (\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b)))$$

and again the left side is equal to $(z + t)(a \boxtimes b) = a \boxtimes b$ .

∎

To prove the following corollary, we need only let $\bar{y} = x$ and $\bar{x} = y$ in lemma 3.8.

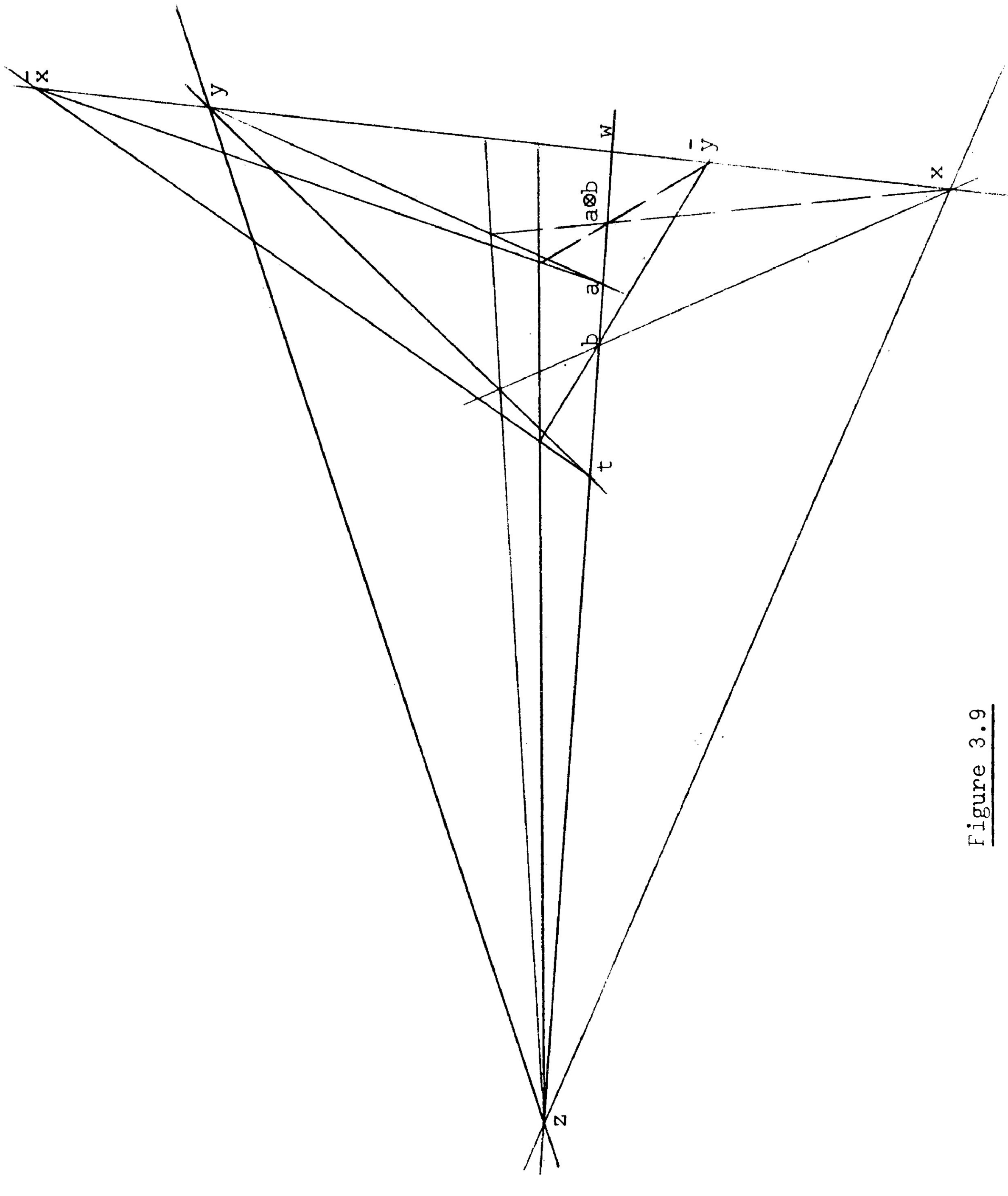



Figure 3.9

Corollary 3.10 If $L$ is Arguesian and $a, b \in D$ then $a \otimes b = a \boxtimes b$.

The projectivity $[0, z + t] \overset{v}{\bar{\wedge}} [0, x + y]$ determines a bijection between D and complements of w in the interval $[0, x + y]$. So any $c \in D$ determines a $\bar{y}$ (as in lemma 3.8) and every $\bar{x}$ determines a $c' \in D$.

We now proceed to the proof that multiplication is associative.

Theorem 3.11 If $L$ is Arguesian and $a, b, c \in D$ then $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

Proof. By Corollary 3.10, we can write $(a \otimes b) \otimes c$ as

$$(a \otimes b) \otimes c = (z + t)((x + y)(v + c) + (z + y)(x + (y + a)(z + (y + t)(x + b)))$$

and by lemma 3.8,

$$a \otimes (b \otimes c) = (z + t)(\bar{y} + (\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b \otimes c))$$

where $\bar{y} = (x + y)(v + c)$ and $\bar{x} = (x + y)(v + (c \oplus t))$. Now

$$b \otimes c = (z + t)((x + y)(v + c) + (z + y)(x + b))$$
$$= (z + t)(\bar{y} + b_0)$$

so $\bar{y} + b \otimes c = \bar{y} + b_0$.

If we show (1) $(\bar{x} + a)(z + (\bar{x} + t)(\bar{y} + b_0)) \leq \bar{y} + (z + y)(x + (y + a)(z + (y + t)(x + b)))$ then we will have associativity.

Consider the triangles

$$A = \langle a,\ z,\ (z + y)(x + a \otimes b)\rangle$$

$$B = \langle \bar{x},\ (\bar{x} + t)(\bar{y} + b_0),\ \bar{y}\rangle$$

$B$ is clearly normal but it is not so obvious that $A$ is normal.

Calculating

$$\begin{aligned}
&(a + (z + y)(x + a \otimes b))(z + (z + y)(x + a \otimes b)) \\
= \ &(z + y)(z + x + a \otimes b)(a + (z + y)(x + a \otimes b)) \\
= \ &(z + x + a \otimes b)(z + y)(x + az + a \otimes b)) \\
= \ &(z + x + a \otimes b)(z + y)(x + az + (y + a)(z + (y + t)(x + b)) \\
= \ &(z + y)(x + a \otimes b) \quad \text{since} \quad a \cdot z \le y + a \quad \text{and} \\
&\qquad\qquad\qquad\qquad\qquad\quad az \le z + (y + t)(x + b)
\end{aligned}$$

and so $A$ is normal.

Central perspectivity of $A, B$ is (1) above, Axial perspectivity is the following inequality.

$$(2) \quad (z + a)(\bar{x} + t) \ \le \ (x + y)(a + (z + y)(x + a \otimes b)) \ + \\ (\bar{y} + b_0)(z + y)(z + x + a \otimes b) \ .$$

We now take the triangles

$$A_1 = \langle z,\ a,\ (z + y)(x + a \otimes b)\rangle \quad \text{and}$$

$$B_1 = \langle (x + b)(y + t),\ y,\ x\rangle$$

which are centrally perspective directly from the definition of $a \otimes b$ , hence axially perspective, that is:

$$(3) \quad (y + t)(z + a) \le (x + b)(z + y)(z + x + a \otimes b) + (x + y)(a + (z + y)(x + a \otimes b))$$

but $(y + t)(z + a) = t\cdot(z + a) = (\bar{x} + t)(z + a)$ and

$$(x + b)(z + y)(z + x + a \otimes b) = b_0(z + x + a \otimes b) \le (b_0 + \bar{y}(z + y))(z + x + a \otimes b)$$

and so (2) follows from (3). (1) follows from (2) and therefore multiplication is associative.

∎

# Chapter 4

## Addition: Commutativity and Associativity

In this chapter we prove that the operation of addition is both commutative and associative. Section 1 deals with commutativity and is required in section 2 to prove associativity. If one could prove that addition is associative then commutativity (of addition) is an easy corollary to the distributive laws, but as yet we cannot prove associativity of addition without first proving commutativity. In chapter 3, multiplication was shown to be defined by an expression that is independent of the $\bar{x}$ and $\bar{y}$ chosen so long as they satisfied certain conditions $(M_3(\bar{x}, \bar{y}, w))$. If one could show that addition can be defined similarly, then the proof of associativity of addition is much simpler and follows closely that of multiplication in chapter 3, section 2. In fact, it is true that addition can be defined by any $\bar{x}, \bar{y}$ such that $M_3(w, \bar{x}, \bar{y})$, but the current proof of this requires associativity (of addition).

### SECTION 1

The proof of commutativity of addition is long and involves many semmingly unrelated results, so we give here a short outline of the definitions and lemmata used in the proof.

If $a \oplus b$ is the von Staudt addition defined in definition 2.2, then one of the proofs given in geometry books (see e.g. [ 2]) is sufficient to prove that $a \oplus t = t \oplus a$ in an Arguesian lattice with spanning 3-diamond $(z, t, x, y)$. This is the content of lemma 4.2. Definition 4.3 is a definition of addition that is similar in form to the von Staudt $\oplus$ except the frame members $x$ and $y$ have been interchanged wherever they occur. We call this addition $\mathbin{\triangle\mkern-14mu+}$ and in lemma 4.7 show that $a \oplus t = t \mathbin{\triangle\mkern-14mu+} a$. Lemma 4.8 gives us that for any $a, b \in D$, $a \oplus b = a \mathbin{\triangle\mkern-14mu+} b$. Two technical lemmata are then required before the proof that addition is commutative.

<u>Lemma 4.1</u>. If $L$ is Arguesian with spanning 3-diamond $(z, t, x, y)$ then the following inequality holds for any $a \in D$.

$$(1) \quad (y + a)(w + (z + y)(x + a)) \leq [z + (y + t)(w + v)][w + (z + y)(x + a)]$$

<u>Proof</u>. Consider the triangles

$$A = \langle z, t, (y + t)(w + v)\rangle \qquad B = \langle (z + y)(x + a), x, w\rangle .$$

We know $v(z + x + a) \leq w + v$ so that

$$(z + y)(z + x + a)(x + t) \leq w + v$$

and hence $A, B$ are centrally perspective, and therefore axially perspective since $L$ is assumed to be Arguesian. Thus

$$(z + t)(x + a) \leq (y + t)(x + y) + [z + (y + t)(w + v)][w + (z + y)(x + a)]$$

or $\qquad a \leq y + [z + (y + t)(w + v)][w + (z + y)(x + a)]$

iff $\quad (y + a)(w + (z + y)(x + a)) \leq [z + (y + t)(w + v)][w + (z + y)(x + a)]$.

<u>Lemma 4.2</u>. If $L$ is Arguesian with spanning 3-diamond $(z, t, x, y)$ and $a \in D$ then $a \oplus t = t \oplus a$.

Proof. Take the triangles given by

$$A = \langle y,\ (y + a)(w + a_0),\ w\rangle$$

$$B = \langle b_0,\ (y + b)(w + b_0),\ b\rangle\ .$$

For central perspectivity, we need

$$(y + b_0)[(y + b)(w + b_0) + (y + a)(w + a_0)] \leq w + b$$

and by the preceding lemma, the left side of the above expression is less than or equal to the following.

$$(z + y)[(w + b_0)(z + (y + t)(w + v)) + (w + a_0)(z + (w + v)(y + t))]$$
$$= (z + y)[z + (y + t)(w + v)][w + b_0 + (w + a_0)(z + (w + v)(y + t))]$$
$$\leq z + y(z + (y + t)(w + v)) = z$$

Since $b \in D$, $w + b = z + w$ and so $A$, $B$ are centrally perspective. As a consequence we have the following inequality:

$$(2)\quad (y + a)(w + b_0)(y + b + b_0) \leq (x + y)(b + b_0) + (y + b)(w + b + b_0)(w + a_0)$$
$$\leq x + (y + b)(w + a_0)\ .$$

If $a \leq z + b$ then (2) becomes

$$(y + a)(w + b_0) \leq x + (y + b)(w + a_0)$$

which implies $a \oplus b \leq b \oplus a$ and therefore $a \oplus t = t \oplus a$ .

Definition 4.3 In a modular lattice $L$ with spanning 3-diamond $(z, t, x, y)$ and $a, b \in D$ define

$$a \mathbin{\triangle\!\!\!\!\!+} b \quad = \quad (z + t)(y + (x + a)(w + (z + x)(y + b))) \ .$$

The properties we have proven for $a \oplus b$ also hold for $a \mathbin{\triangle\!\!\!\!\!+} b$ since $(z, t, x, y)$ is a spanning 3-diamond if and only if $(z, t, y, x)$ is also. (Figure 4.5)

So we know the following:

(1) $a, b \in D \Rightarrow a \mathbin{\triangle\!\!\!\!\!+} b \in D$

(2) $a \mathbin{\triangle\!\!\!\!\!+} t \ = \ t \mathbin{\triangle\!\!\!\!\!+} a$ if $a \in D$ and $L$ is Arguesian.

Recall in chapter two, we denoted $(z + x)(y + a)$ by $a_2$ and noticed that $x + a_2 \ = \ z + x$, $y + a_2 \ = \ y + a$ . We require some properties of $a_0 + a_2 \ = \ (z + x)(y + a) \ + \ (z + y)(x + a)$ before returning to addition.

Lemma 4.4. In an Arguesian lattice $L$ with spanning 3-diamond $(z, t, x, y)$ and $a, b \in D$ , (Figure 4.6)

(1) $(x + y)(a_0 + a_2) \ \leq \ v \ + \ u$

(2) $(x + y)(b_0 + b_2) \ = \ (x + y)[(y + a)(w + b_0) \ + \ (x + a)(w + b_2)]$ .

Proof. To prove (1) we take triangles

$$A_1 \ = \ \langle a, x, y\rangle \qquad\qquad B_1 \ = \ \langle z, v, u\rangle$$

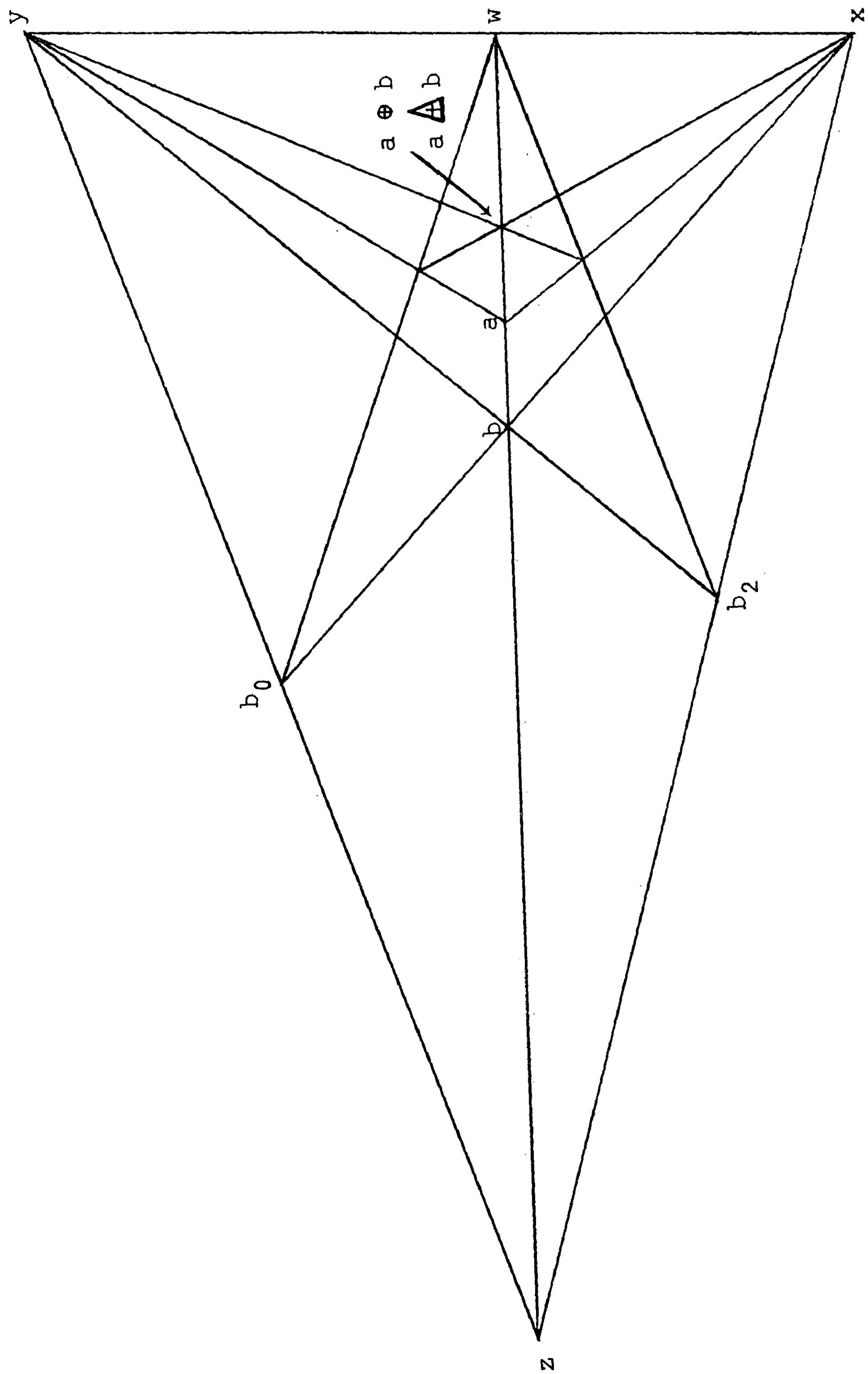



Figure 4.5

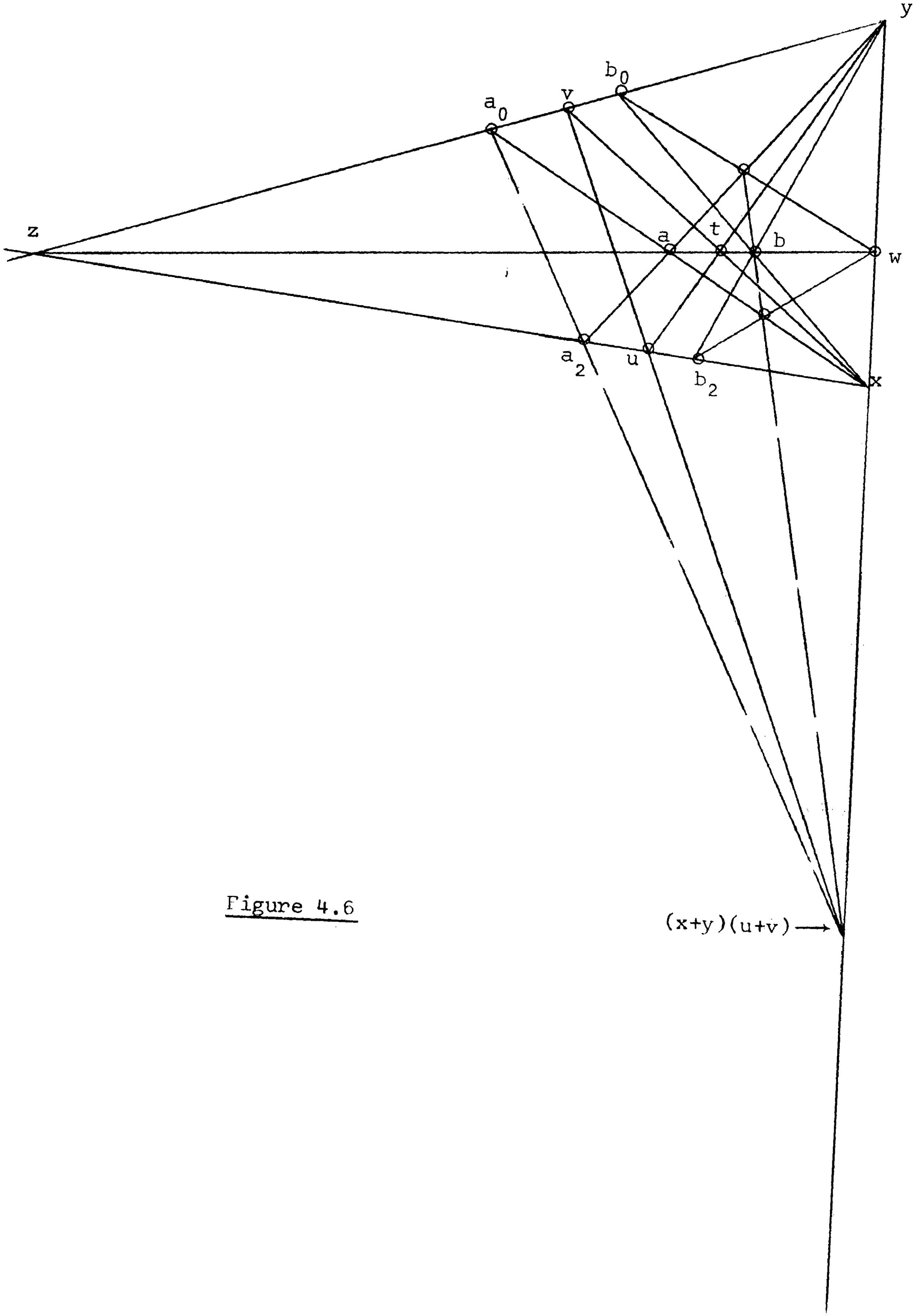



Figure 4.6

$(z + a)(x + v) = t(z + a) \leq y + t = y + u$ , so $A_1, B_1$ are centrally perspective at $t(z + a)$ . By Desargues therefore

$$(x + a)(z + v) \leq (x + y)(v + u) + (y + a)(z + u)$$

or equivalently

$$a_0 \leq a_2 + (x + y)(v + u) ,$$

so that

$$\begin{aligned} (x + y)(a_0 + a_2) &\leq (x + y)(a_2 + (x + y)(v + u)) \\ &= (x + y)(v + u) . \end{aligned}$$

The proof of (2) is similar. The triangles

$$A_2 = \langle y, x, a\rangle \qquad B_2 = \langle b_0, b_2, w\rangle$$

are centrally perspective at $z$ , and so

$$(x + y)(b_0 + b_2) \leq (y + a)(w + b_0) + (x + a)(w + b_2) .$$

To obtain the reverse inequality, we note that the triangles $A_3 = \langle y, a, x\rangle$ , $B_3 = \langle b_0, w, b_2\rangle$ and $A_4 = \langle a, x, y\rangle$ , $B_4 = \langle w, b_2, b_0\rangle$ are also centrally perspective at $z$ . Thus

$$(y + a)(w + b_0) \leq (x + y)(b_0 + b_2) + (x + a)(w + b_2) .$$

<u>Lemma 4.7</u>. If $L$ is Arguesian and $a, b \in D$ then $t \oplus a = t +_{\triangle} a$ .

<u>Proof</u>. We have the following equivalences.

$$(1)\quad [x + (y + t)(w + a_0)][y + (x + t)(w + a_2)] \leq z + t$$

$$\text{iff}\quad [x + (y + t)(w + a_0)][y + (x + t)(w + a_2)] \leq (z + t)(y + (x + t)(w + a_2))$$

$$\text{iff}\quad x + (y + t)(w + a_0) \leq x + t \mathbin{\triangle\!\!\!\!+} a$$

$$\text{iff}\quad (z + t)(x + (y + t)(w + a_0)) \leq (z + t)(x + t \mathbin{\triangle\!\!\!\!+} a)$$

$$\text{iff}\quad t \oplus a \leq t \mathbin{\triangle\!\!\!\!+} a \ .$$

We will show (1) above.

Consider the triangles $A = \langle x, y, z\rangle$ and $B = \langle (y + t)(w + a_0), (x + t)(w + a_2), t\rangle$ Both $A, B$ are normal and the statement of central perspectivity is (1). By lemma 4.4,

$$(x + y)[(y + t)(w + a_0) + (x + t)(w + a_2)] = (x + y)(a_0 + a_2)$$

and also by lemma 4.4,

$$(x + y)(a_0 + a_2) \leq v + u$$

What we require though is that

$$\begin{aligned}(x + y)(a_0 + a_2) &\leq v(z + t + a_2) + u(z + t + a_0)\\ &= v(z + x + a) + u(z + y + a)\\ &= (z + x + a)(u + v)(z + y + a)\ .\end{aligned}$$

But $a_0 \leq (z + x + a)$ , $a_2 \leq z + x + a$ , $a_0 \leq z + y + a$ and $a_2 \leq z + y + a$ , so the proof is complete.

∎

We can now show that $a \oplus b = a \mathbin{\triangle\!\!\!\!+} b$ .

Lemma 4.8. If $L$ is Arguesian and $a, b \in D$ then $a \oplus b = a \mathbin{A\!\!\!\!+} b$ .

Proof. The triangles $A_1 = \langle x, y, a \oplus t \rangle$ and
$B_1 = \langle (y + a)(w + b_0), (x + a)(w + b_2), a \rangle$ are normal and therefore

$$[x + (y + a)(w + b_0)][y + (x + a)(w + b_2)] \leq a + a \oplus t$$

if and only if

$$(x + y)[(y + a)(w + b_0) + (x + a)(w + b_2)] \leq (y + a \oplus t)(x + a)(z + t + b_2) + (x + a \oplus t)(y + a)(z + t + b_0) .$$

We have $[x + (y + a)(w + b_0)][y + (x + a)(w + b_2)] \leq a + a \oplus t = z + t$

iff $y + [x + (y + a)(w + b_0)][y + (x + a)(w + b_2)] \leq y + (z + t)[x + (y + a)(w + b_0)]$

iff $y + (x + a)(w + b_2) \leq y + (z + t)[x + (y + a)(w + b_0)]$

iff $(z + t)(y + (x + a)(w + b_2)) \leq (z + t)(x + (y + a)(w + b_0))$

which is equivalent to the statement of the lemma.

By lemma 4.4, the second is

$$(x + y)(b_0 + b_2) \leq (w + v)(y + a)(z + t + b_0) + (w + u)(x + a)(z + t + b_2)$$

since $(y + a \oplus t)(x + a) = (y + a + t)(x + a) = (w + u)(x + a)$ .

Also notice
$$\begin{aligned}(y + a)(z + t + b_0) &= (y + a)(w + a + b_0)\\ &= a + y(z + b_0)\\ &= a + y(z + x + b) .\end{aligned}$$

Similarly $(x + a)(z + t + b_2) = a + x(z + y + b)$ .

Now take the triangles $A_2 = \langle v, u, w \rangle$ and
$B_2 = \langle y(z + x + b), x(z + y + b), a \rangle$

For central perspectivity we calculate:

$$\begin{aligned}&[v + y(z + x + b)][u + x(z + y + b)]\\ &= (z + t)(z + x)[y + t + x(z + y + b)][x + t + y(z + x + b)]\\ &= z[t + y + w(z + b)][t + x + w(z + b)]\\ &\leq w + a \quad \text{since } a \in D .\end{aligned}$$

Since $\mathcal{L}$ is Desarguean, we have

$$(u + v)(x + y)(z + x + b)(z + y + b) \le (w + u)(a + x(z + y + b)) + (w + v)(a + y(z + x + b))$$

and by lemma 4.4,

$$(x + y)(b_0 + b_2) \le (x + y)(u + v)(z + x + b)(z + y + b) .$$

Thus $a \oplus b = a \mathbin{\triangle\!\!\!\!\!+} b$ .

$\blacksquare$

<u>Lemma 4.9.</u> If $\mathcal{L}$ is Arguesian and $a, b \in D$ then the following are equal. (Figure 4.10)

(1) $(x + y)[b \oplus t + (x + b)(w + a_2)]$

(2) $(x + y)(t + a_2)$

(3) $(x + y)(a + (y + t)(w + a_0))$

(4) $(x + y)[(x + t)(y + a) + (y + t)(w + v)]$

(5) $(x + y)[(x + b)(y + a) + (y + t)(w + b_0)]$

<u>Proof</u>. It is easy to check that all of the above are complements of $x$ in $[0, x + y]$ so we need only show inequalities.

To show (3) is comparable to (4), we take the triangles $A_1 = \langle x, w, v\rangle$ , $B_1 = \langle a, (y + t)(w + a_0), y\rangle$ . Axial perspectivity is precisely what is required. Central perspectivity is $(x + a)(w + a_0) \le z + y$ which is clearly true.

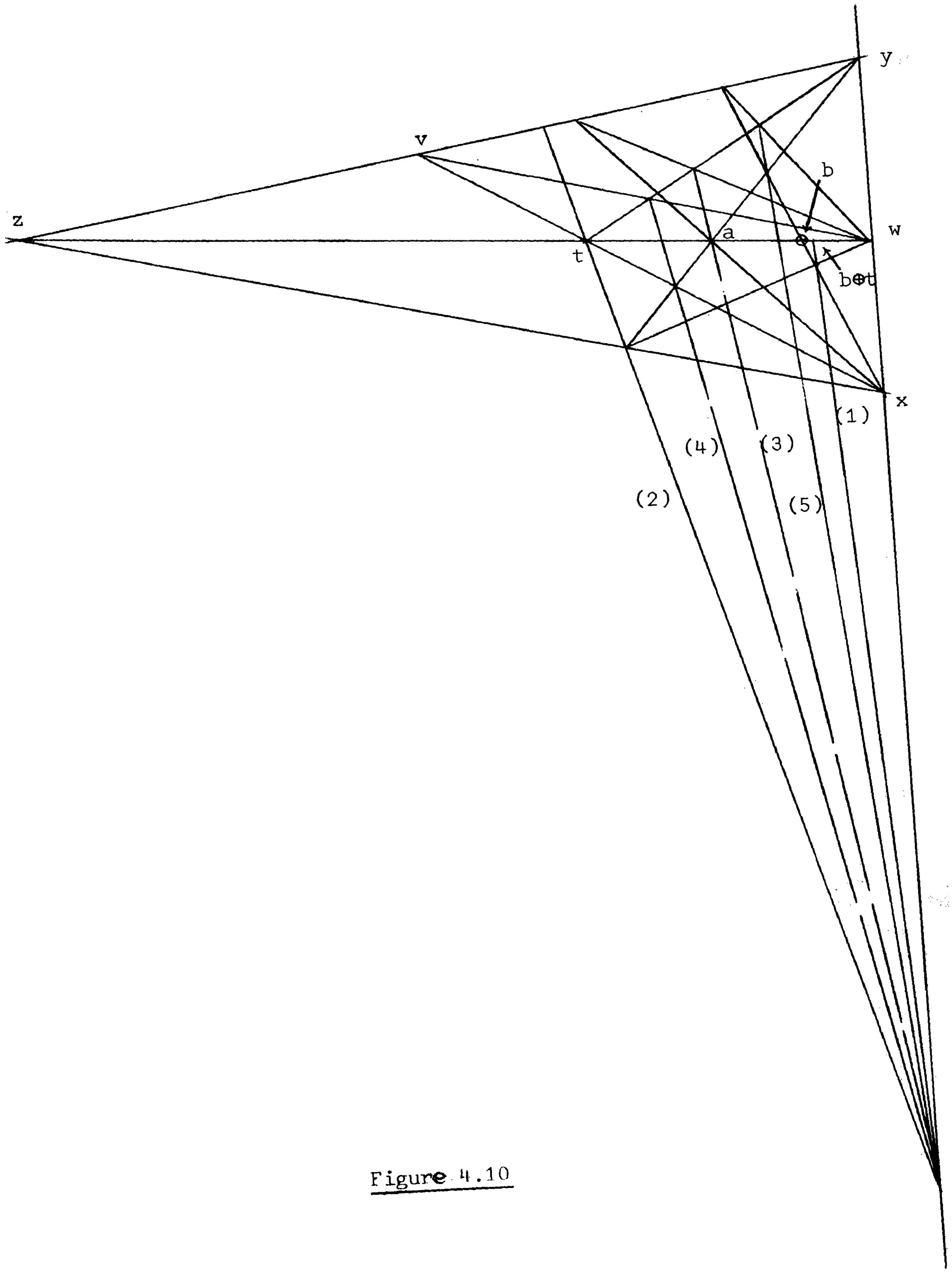



Figure 4.10

That (1) is comparable to (2) follows from the axial perspectivity of the following triangles

$$A_2 = \langle y, x, u\rangle \qquad B_2 = \langle b \oplus t, (x + b)(w + a_2), w\rangle .$$

Central perspectivity is the inequality below.

$$(y + b \oplus t)(x + b) \leq w + u$$

and by previous results we know

$$\begin{aligned}(x + b)(y + b \oplus t) &= (x + b)(y + (u + w)(x + b)) \\ &= (w + u)(x + b) .\end{aligned}$$

We show (2) is equal to (4) by taking the triangles

$$A_3 = \langle x, w, v\rangle \qquad B_3 = \langle a_2, t, y\rangle .$$

$A_3$, $B_3$ are centrally perspective at $z$ therefore they are axially perspective, the desired inequality.

Finally, we show (5) is equal to (3) by considering the triangles

$$A_4 = \langle x, w, b_0\rangle \qquad B_4 = \langle a, (y + t)(w + a_0), y\rangle .$$

$$(x + a)(w + a_0) = a_0 \leq z + y = y + b_0$$

Consequently

$$(x + y)(a + (y + t)(w + a_0)) \leq (w + b_0)(y + t) + (x + b)(y + a) .$$

The final result of this section is commutativity of addition.

Theorem 4.11 If $L$ is Arguesian with spanning 3-diamond $(z, t, x, y)$ and $a, b \in D$ then $a \oplus b = b \oplus a$ .

Proof.

$$(1) \quad [x + (y + a)(w + b_0)][y + (x + b)(w + a_2)] \leq w + b \oplus t$$

$$\Leftrightarrow \quad [x + (y + a)(w + b_0)][y + (x + b)(w + a_2)] \leq (z + t)(y + (x + b)(w + a_2))$$

$$\Leftrightarrow \quad x + (y + a)(w + b_0) \leq x + b \mathbin{\triangle\mkern-14mu+} a$$

$$\Leftrightarrow \quad (z + t)(x + (y + a)(w + b_0)) \leq b \mathbin{\triangle\mkern-14mu+} a$$

$$\Leftrightarrow \quad a \oplus b \leq b \mathbin{\triangle\mkern-14mu+} a = b \oplus a \quad \text{by lemma 4.8.}$$

We will prove (1) above.

Consider the triangles $A_1 = \langle x, (x + b)(w + a_2), b \oplus t\rangle$ and $B_1 = \langle (y + a)(w + b_0), y, w\rangle$ .

Clearly $B_1$ is normal.

$$\begin{aligned} &(b \oplus t + x)(b \oplus t + (x + b)(w + a_2)) \\ &= b \oplus t + x(w + a_2 + (x + b)\cdot(b \oplus t)) \\ &= b \oplus t + x(w + a_2) = b \oplus t \end{aligned}$$

and therefore $A_1$ is normal.

Central perspectivity is the inequality (1), our desired result. So addition is commutative if and only if

$$(2) \quad (x + b)(y + a) \leq (x + y)[b \oplus t + (x + b)(w + a_2)] + (w + b_0)(x + b \oplus t)$$

$$\text{iff} \quad (x + b)(y + a) \leq (x + y)[b \oplus t + (x + b)(w + a_2)] + (w + b_0)(x + t \oplus b)$$

$$\text{iff} \quad (x + b)(y + a) \leq (x + y)[b \oplus t + (x + b)(w + a_2)] + (y + t)(w + b_0)$$

$$\text{iff (3)} \quad (x + y)[(x + b)(y + a) + (y + t)(w + b_0)] \leq (x + y)[b \oplus t + (x + b)(w + a_2)] .$$

We have shown (3) in the previous lemma, and thus addition is commutative.

SECTION 2

In this section we will show that the operation of addition is associative. Surprisingly, the proof does not use the technique of taking two triangles and considering their central and axial perspectivity. Instead, the proof involves considering the two expressions $(b \oplus c) \oplus a$ and $(b \oplus a) \oplus c$ with respect to a different coordinate frame. We deduce associativity of addition in the frame $(z, t, x, y)$ from commutativity of addition, first in another frame (to show $(b \oplus c) \oplus a = (b \oplus a) \oplus c$), then in the frame $(z, t, x, y)$. The expression used for addition is the one derived from the ternary operator $T(a, b, c)$ introduced in chapter 2, definition 2.2. We know from chapter 3, lemma 3.1, that $T(a, b, c)$ is precisely $(a \otimes b) \oplus c$ and from chapter 2 that $t \otimes a = a$ so that $T(t, a, b)$ is equal to $a \oplus b$.

Theorem 4.12 If $L$ is Arguesian with spanning 3-diamond $(z, t, x, y)$ and $a, b, c \in D$ then

$$(b \oplus c) \oplus a = (b \oplus a) \oplus c .$$

Proof. As mentioned previously, we have the following equivalent definition of $a \oplus b$.

$$a \oplus b = (z + t)(x + (y + t)(a_\infty + b_0))$$

In order to prove the theorem, we must show:

$$(z + t)(x + (y + t)(a_0 + (b \oplus c)_\infty)) = (z + t)(x + (y + t)(c_0 + (b \oplus a)_\infty)$$

or (2) $$(y + t)(a_0 + (b \oplus c)_\infty) = (y + t)(c_0 + (b \oplus a)_\infty)$$

where
$$\begin{aligned}(b \oplus c)_\infty &= (x + y)(z + (y + t)(x + b \oplus c)) \\ &= (x + y)(z + (y + t)(c_0 + b_\infty))\end{aligned}$$

and similarly $(b \oplus a)_\infty = (x + y)(z + (y + t)(a_0 + b_\infty))$ .

Now let $\bar{a} = (y + t)(a_0 + b_\infty)$ and $\bar{c} = (y + t)(c_0 + b_\infty)$

So (2) becomes

$$(y + t)(a_0 + (x + y)(z + \bar{c})) = (y + t)(c_0 + (x + y)(z + \bar{a}))$$

Since $b_\infty$ is a complement of $y$ in $[0, x + y]$ , $a_0 = (z + y)(b_\infty + \bar{a})$ and $c_0 = (z + y)(b_\infty + \bar{c})$ . (2) is therefore equivalent to

(3) $$\begin{aligned}(y + t)((z + y)(b_\infty + \bar{a}) &+ (b_\infty + y)(z + \bar{c})) \\ &= (y + t)((z + y)(b_\infty + \bar{c}) \\ &\qquad + (b_\infty + y)(z + \bar{a}))\end{aligned}$$

We claim that (3) is the von Neumann addition with respect to the coordinate frame $(\bar{z}, \bar{w}, \bar{y}, \bar{x}, \bar{t}, \bar{v})$ defined below. (Figure 4.13)

Let
$$\begin{aligned}\bar{z} &= (y + t)(x + b) \\ \bar{w} &= y \\ \bar{y} &= z \\ \bar{x} &= v \\ \bar{t} &= (y + t)(b_\infty + v) \\ \bar{v} &= b_\infty .\end{aligned}$$

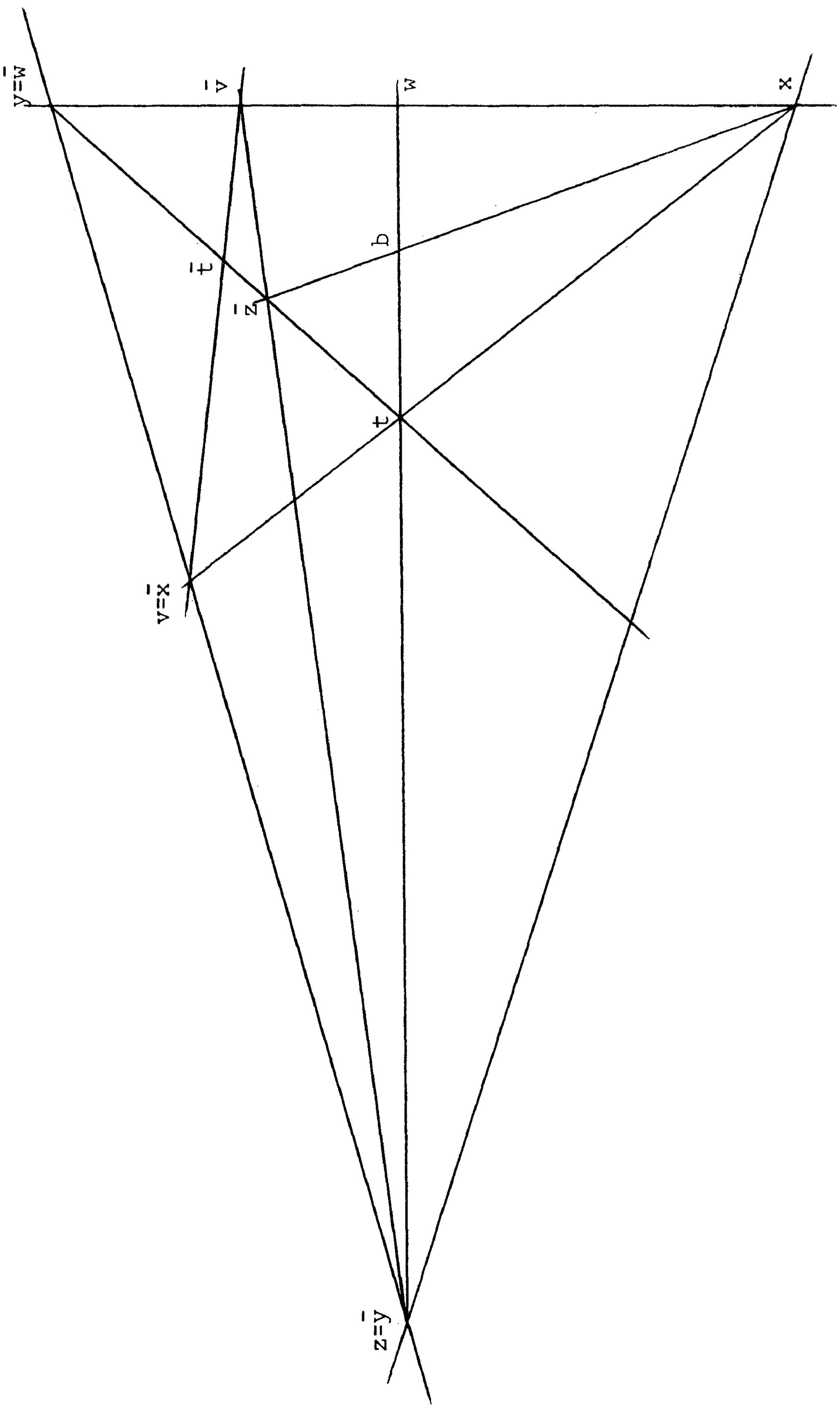



Figure 4.13

It is clear that if $a, b \in D$ then $y + \bar{a} = y + t$ and $y \cdot \bar{a} = 0$ .

$$\bar{z} + \bar{w} = (y + t) = \bar{z} + \bar{t} = \bar{t} + \bar{w}$$

In terms of the new frame, (3) above becomes

$$(\bar{z} + \bar{t})((\bar{w} + \bar{y})(\bar{v} + \bar{a}) + (\bar{w} + \bar{v})(\bar{y} + \bar{c}))$$
$$= (\bar{z} + \bar{t})((\bar{w} + \bar{y})(\bar{v} + \bar{c}) + (\bar{w} + \bar{v})(\bar{y} + \bar{a}))$$

which is precisely commutativity of addition in the frame $(\bar{z}, \bar{w}, \bar{y}, \bar{x}, \bar{t}, \bar{v})$ .

So by commutativity, we have the following equality

$$(b \oplus c) \oplus a = (b \oplus a) \oplus c .$$

Corollary 4.14 If $L$ is Arguesian and $a, b, c \in D$ , then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ .

Proof. By the previous theorem, we know $(b \oplus a) \oplus c = (b \oplus c) \oplus a$ and therefore by commutativity of addition, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ .

At the beginning of this chapter we stated that the associativity of addition is equivalent to the equality of $a \oplus b$ and the expression $(z + t)(\bar{x} + (\bar{y} + a)(w + (z + \bar{y})(\bar{x} + b)))$ for any $\bar{x}, \bar{y} \in [0, x + y]$ such that $\bar{x} + w = \bar{y} + w = x + y = \bar{x} + \bar{y}$ , $\bar{x} \cdot w = \bar{y} \cdot w = \bar{x} \cdot \bar{y} = 0$ . The proof of this follows here.

Lemma 4.15. If $L$ is Arguesian and $a, b \in D$ then

$$(z + t)(\bar{y} + (\bar{x} + a)(w + (z + \bar{x})(\bar{y} + b))) \qquad \text{(Figure 4.16)}$$
$$= (z + t)(\bar{y} + (w + v)(a + (\bar{x} + \bar{y})(z + (w + v)(\bar{y} + b))$$

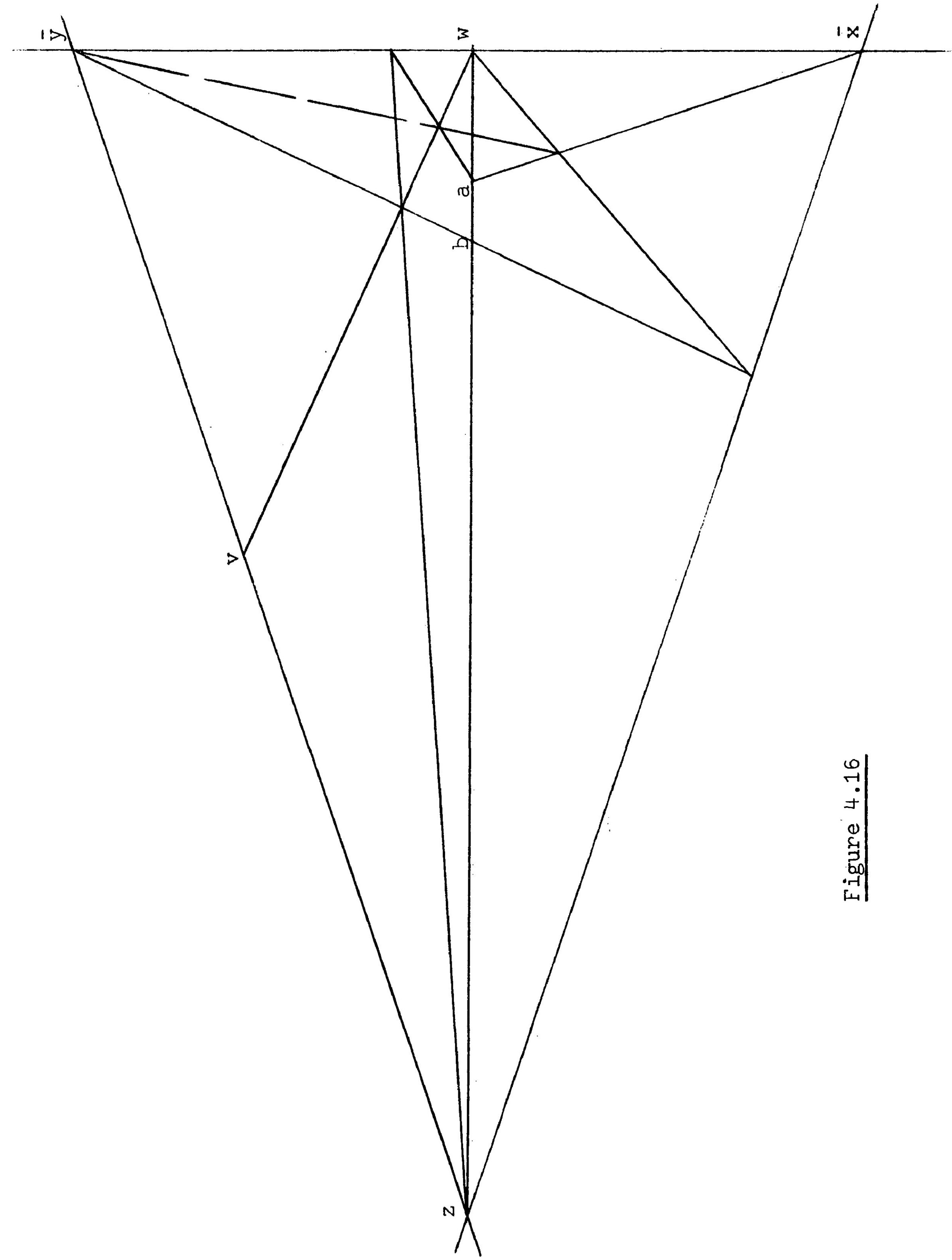



Figure 4.16

<u>Proof</u>. Consider the triangles

$$A = \langle w,\ (w + v)(\bar{y} + b),\ (z + \bar{x})(\bar{y} + b)\rangle$$

$$B = \langle a,\ (\bar{x} + \bar{y})(z + (w + v)(\bar{y} + b)),\ \bar{x}\rangle$$

Central perspectivity requires

$$(w + a)(z + (w + v)(\bar{y} + b)) \ \leq\ z + \bar{x}$$

and since the left side of the above expression is precisely $z$, $A$, $B$ are centrally perspective, hence axially perspective.

$$\begin{aligned}
&(w + v)(a + (\bar{x} + \bar{y})(z + (w + v)(\bar{y} + b))) \\
&\quad \leq\ (\bar{y} + b)(z + \bar{x} + (w + v)(\bar{y} + b))(\bar{x} + \bar{y})(z + \bar{x} + (w + v)(\bar{y} + b)) \\
&\qquad\qquad + (\bar{x} + a)(w + (z + \bar{x})(\bar{y} + b)) \\
&\quad \leq\ \bar{y} + (\bar{x} + a)(w + (z + \bar{x})(\bar{y} + b))
\end{aligned}$$

which implies

$$\begin{aligned}
(1) \qquad &(z + t)(\bar{y} + (w + v)(a + (\bar{x} + \bar{y})(z + (w + v)(\bar{y} + b)))) \\
&\qquad \leq\ (z + t)(\bar{y} + (\bar{x} + a)(w + (z + \bar{x})(\bar{y} + b)))
\end{aligned}$$

Since both sides of the inequality (1) are complements of $w$ in the interval $[0, z + w]$, the proof is complete.

∎

<u>Theorem 4.17</u> If $L$ is Arguesian, $a, b, c \in D$ then addition is associative if and only if

$$a \oplus b \ =\ (z + t)(\bar{y} + (\bar{x} + a)(w + (z + \bar{y})(\bar{x} + b))) .$$

Proof. Let $\bar{y} = (x + y)(z + (w + v)(x + c))$ and

$$\bar{x} = (x + y)(z + (w + v)(x + c \oplus t)))$$

so that $\bar{x}, \bar{y}$ satisfy the conditions

$$\bar{x} + \bar{y} = x + y = \bar{x} + w = \bar{y} + w, \bar{x}w = \bar{y}w = \bar{x}\bar{y} = 0 .$$

We have

$$(a \oplus b) \oplus c = (z + t)(x + (w + v)(\bar{y} + a \oplus b))$$

$$a \oplus (b \oplus c) = (z + t)(x + (w + v)(a + (x + y)(z + (w + v)(\bar{y} + b).$$

Therefore, addition is associative if and only if

$$(w + v)(a + (x + y)(z + (w + v)(\bar{y} + b))) \leq (w + v)(\bar{y} + a \oplus b)$$

iff $(z + t)(\bar{y} + (w + v)(a + (x + y)(z + (w + v)(\bar{y} + b)))) \leq a \oplus b$

iff $a \oplus b = (z + t)(\bar{y} + (w + v)(a + (x + y)(z + (w + v)(\bar{y} + b))))$

$$= (z + t)(\bar{y} + (\bar{x} + a)(w + (z + \bar{x})(\bar{y} + b)))$$

by previous lemma.

∎

## Conclusion

The results in the preceding three chapters give us the following theorem.

Theorem  If $L$ is an Arguesian lattice with spanning 3-diamond $(z, t, x, y)$ the $(D, \oplus, \otimes)$ is an associative ring with unit.

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