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ABSTRACT

I will study the existence of multi-vortex solutions of the Ginzburg-Landau equations with an external potential on $\mathbb{R}^2$. These equations model the equilibrium states of superconductors: the external potential, W, represents doped impurities or defects of the superconductor. I will show that if the critical points of the potential are spaced widely enough and if the potential W is "strong enough," then there exists a multi-vortex (perturbed) solution with each vortex centered near each critical point of W.

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1. INTRODUCTION

A predominant factor that allows one to label a material as a superconductor is the material's ability to undergo the Meissner-Ochsenfeld Effect (MOE). Superconductors possess what is called a critical temperature, T_c, where if cooled below this temperature then the material goes into its superconducting state. Suppose an external magnetic field, $\overrightarrow{B}_{ext}$, is applied to a superconductor at $T > T_c$ then the $\overrightarrow{B}_{ext}$ freely passes through the material. Now, when $\overrightarrow{B}_{ext}$ is applied to a superconductor at $T < T_c$, then the applied external field is expelled from the material, this phenomena is in fact the MOE.

Two common superconductors are type-I and type-II superconductors, the latter being the focus of this thesis. The type-I superconductor receives its name for its single critical field, H_c. If a type-I superconductor is cooled to its superconducting state and subject to an external magnetic field, then the superconductor will expel the applied field as long as the applied field does not exceed H_c. Once the applied field overcomes H_c, the superconductor loses its superconductivity. A type-II superconductor is a superconductor that has two critical fields, H_{c_1} and H_{c_2}, where $H_{c_1} < H_{c_2}$. Similarly to the type-I superconductors, when in its superconducting state, the type-II superconductor expels all of the external magnetic field when $|\overrightarrow{B}_{ext}| < H_{c_1}$ and loses its superconductivity when $|\overrightarrow{B}_{ext}| > H_{c_2}$ [JA].

When a type-II superconductor is placed in an external magnetic field of magnitude which falls between the two critical field strengths ($H_{c_1} < |\overrightarrow{B}_{ext}| < H_{c_2}$), then the external field penetrates the superconductor through *magnetic vortices*. Magnetic vortices are tubular lines of magnetic flux surrounded by supercurrent, composed of Cooper pairs. These Cooper pairs consist of two coupled electrons. The coupling arises from lattice deformations/electron-phonon interactions. An electron in a metal behaves as a free particle; as it nears a lattice ion, the said ion along with those next to it are drawn toward the electron. This results in a surplus of positive charge near the electron, enough charge such that the Coulomb repulsion from a neighbouring free electron is overcome [PR]. The electron-electron attraction is strongest when their spins are opposite. Electrons are fermions, particles of half-integer spin [O]. When they are coupled to form Cooper pairs, their spins add together and the electron pair forms a boson. A consequence of this phenomena is that the Cooper pair wave function, $\Psi(\vec{r}_1, \vec{r}_2) = Ae^{i(\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2)}$ is symmetric under any interchanging of electrons. In addition, these wave functions are solutions to the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m_e}\vec{\nabla}_1^2 - \frac{\hbar}{2m_e}\vec{\nabla}_2^2 + V(\vec{r}_1, \vec{r}_2)\right]\Psi(\vec{r}_1, \vec{r}_2) = E\Psi(\vec{r}_1, \vec{r}_2)$$

where $V(\vec{r}_1, \vec{r}_2) < 0$ denotes the attraction between the electrons [PR].

Superconductors are used to create large magnetic fields on the order of 10T. Hindering the production of large magnetic fields is the movement of magnetic vortices that dissipate energy, a phenomenon that will be explained below.

Magnetic vortices are not at rest but maneuver about the superconductor due to Lorentz forces between the superconducting electrons and the induced magnetic field. This movement of vortices dissipates energy. In order to minimize the energy loss, the superconductor is doped with impurities. When a vortex encounters an impurity, the impurity acts as a snare and pins down the vortex [T]. The pinning phenomena leads to minimum energy dissipation. This has mathematically (both analytically and numerically) been shown in [ST], [ASaSe], [CDG], [CR] when a vortex is in the presence of an impurity. The objective of this thesis is to demonstrate mathematically what has been observed in the laboratory; namely, multiple magnetic vortices are pinned to multiple impurities within a superconductor.

1.1 The Ginzburg-Landau Equations

The conventional theory of superconductivity on the macroscopic scale is governed by the static equations of Ginzburg and Landau (GL) (in terms of natural units):

$$-\Delta_{\vec{A}}\psi + \lambda(|\psi|^2 - 1)\psi = 0, \tag{1.1.1}$$

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{A} - Im(\bar{\psi}\vec{\nabla}_{\vec{A}}\psi) = 0. \tag{1.1.2}$$

Following the convention, $\triangle_{\vec{A}} = \vec{\nabla}_{\vec{A}} \cdot \vec{\nabla}_{\vec{A}}$ where $\vec{\nabla}_{\vec{A}} = \vec{\nabla} - i\vec{A}$ is the convariant gradient. Here $\lambda > 0$ is the Ginzburg-Landau parameter. The Ginzburg-Landau parameter has the properties, $\lambda > \frac{1}{2}$ and $\lambda < \frac{1}{2}$ that model type I and II superconductors, respectively. The pairs $(\psi, \vec{A})$ satisfying the GL equations are known as stationary states, where $\vec{A} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the magnetic vector potential and $\psi : \mathbb{R}^2 \to \mathbb{C}$ is the order parameter such that $|\psi|^2$ is the density of the superconducting electrons or Cooper pairs [TT]. As we are working in two dimensions it should be noted that given a vector $\vec{A}$, *curl* $\vec{A}$ is $\vec{\nabla} \times A = \partial_1 A_2 - \partial_2 A_1$, a scalar and for a scalar ξ, $\vec{\nabla} \times \xi$ is the vector $(-\partial_2\xi, \partial_1\xi)$. We denote $\bar{\psi}$ as the complex conjugate of the order parameter, ψ and $\vec{B} = curl\ \vec{A}$ as the induced magnetic field. The GL equations are Euler-Lagrange equations of the Ginzburg-Landau energy functional

$$\mathcal{E}(\psi, \vec{A}) = \frac{1}{2}\int_{\mathbb{R}^2}\left[|\vec{\nabla}_{\vec{A}}\psi|^2 + (\vec{\nabla} \times \vec{A}) + \frac{\lambda}{2}(|\psi|^2 - 1)^2\right], \tag{1.1.3}$$

where the solution pairs to the GL equations are the critical points of (1.1.3), i.e., $\mathcal{E}'(\psi, \vec{A}) = 0$, where $\mathcal{E}'(\psi, \vec{A})$ denotes the variational/Fréchet derivative with respect to $(\psi, \vec{A})$,

$$\mathcal{E}'(\psi, \vec{A}) = \begin{pmatrix} \partial_{\chi}\mathcal{E}(\psi, \vec{A}) \\ \partial_{\vec{A}}\mathcal{E}(\psi, \vec{A}) \end{pmatrix}.$$

Here, $\vec{\nabla} \times \vec{A}$ is the induced magnetic field corresponding to the magnetic vector potential $\vec{A}$.

The GL equations on the plane, models superconductors which are spatially homogeneous in one direction while neglecting boundary effects. In addition, they model the equilibrium states of $U(1)$-Higgs model of particle physics[JT].

Finite energy states of the system are categorized by their winding number/vorticity/degree of the vector field ψ at infinity. Here the vorticity is defined as

$$deg(\psi) := deg\left(\frac{\psi}{|\psi|}|_{|x|=R}\right) \in \mathbb{Z} \tag{1.1.4}$$

for sufficiently large values of R. In addition, associated with each finite energy state is a quantization of magnetic flux in such a manner that

$$\int_{\mathbb{R}^2} B\ dx = 2\pi deg(\psi). \tag{1.1.5}$$

To demonstrate the quantization of flux,Φ, we write the supercurrent, $\vec{j} = Im(\bar{\psi}\vec{\nabla}_{\vec{A}}\psi)$ and make the substitution $\psi = |\psi|e^{i\phi}$. After taking the covariant derivative of ψ and factoring out the term $|\psi|^2$, we obtain

$$\vec{j} = \left(\vec{\nabla}\phi - \vec{A}\right)|\psi|^2,$$

where ϕ is the phase of the order parameter. It should be noted that the units here are dimensionless, in order to maintain consistency throughout the thesis. Assuming the $\vec{j}$ vanishes on some closed loop, $\partial\Omega$, enclosing an area Ω, we have

$$\oint_{\partial\Omega} \vec{j} \cdot \vec{dl} = \oint_{\partial\Omega} \left(\vec{\nabla}\phi - \vec{A}\right)|\psi|^2 \cdot \vec{dl} = 0.$$

Making use of Stokes theorem we have

$$\Phi = \oint_{\partial\Omega} \vec{\nabla}\phi \cdot \vec{dl} = \oint_{\partial\Omega} \vec{A} \cdot \vec{dl} = \int_{\Omega} \vec{B} \cdot \vec{dS}.$$

Noting that the phase, ϕ, of the order parameter changes by $2\pi n$ for each circulation of the loop, we are left with

$$\Phi = \oint_{\partial\Omega} \vec{\nabla}\phi \cdot \vec{dl} = 2\pi n,$$

where $n \in \mathbb{Z}$ [AM].

Solutions to the GL equations that are trivial are of the form $(\psi, \vec{A}) = (1, 0)$ and $(\psi, \vec{A}) = (0, B_0)$, the purely superconducting and normal state, respectively. The simplest non-trivial solutions to the GL equations are radially symmetric and are of the form

$$\psi^{(n)}(x) = f_n(r)e^{in\theta}, \qquad \vec{A}^{(n)}(x) = a_n(r)\vec{\nabla}(n\theta). \tag{1.1.6}$$

The pair $(\psi^{(n)}, \vec{A}^{(n)})$ are called n-vortices, with $deg(\psi)_n = n$ and (r, θ) denoting the standard polar coordinates of $x \in \mathbb{R}^2$. Their existence as been shown in [P]&[BC]. As $r \to \infty$, both $f_n(r)$ and $a_n(r)$ converge exponentially towards 1 at the rate

$$\begin{aligned} f_n &\to 1 + O(e^{-m_\lambda r}) \\ a_n &\to 1 + O(e^{-r}) \end{aligned}$$

as $r \to \infty$, where

$$m_\lambda = min(\sqrt{2\lambda}, 2).$$

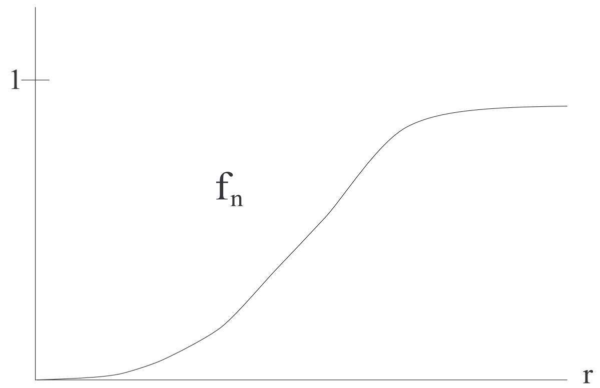

Fig. 1.1.1: Plot of f_n

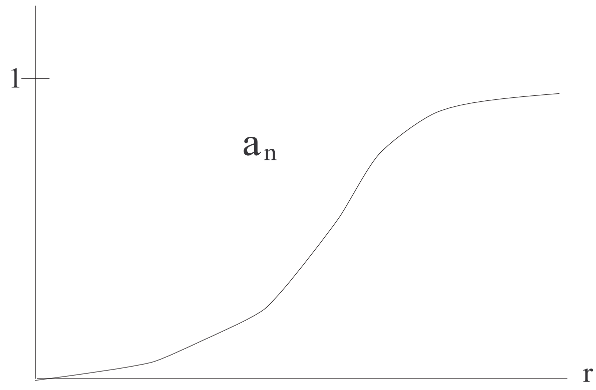

Fig. 1.1.2: Plot of a_n

Now we define the notation of "big-o" and "little-o". We say that $g(r) = O(h(r))$, (big-o) as $r \to \infty$ if $|\frac{g(r)}{h(r)}| \le M$, where M is a constant. If however $|\frac{g(r)}{h(r)}| \to 0$ as $r \to \infty$, then $g(r) = o(h(r))$, (little-o). Near the origin, $f_n(r) \propto r^n$ while $a_n(r) \propto r^2$. From [GS1] we know that $n = \pm 1$ vortices are stable for $\lambda > \frac{1}{2}$ and any integer degree vortex is stable for $\lambda < \frac{1}{2}$.

In addition, we have the following asymptotic of the field components as established in [P] as $r \to \infty$:

$$\begin{aligned}
\vec{j}^{(n)}(r) &= n\beta_n K_1(r)[1 + o(e^{-m_\lambda |x|})\hat{x}^\perp] \\
B^{(n)}(r) &= n\beta_n K_1(r)\left[1 - \frac{1}{2r} + O\left(\frac{1}{r^2}\right)\right] \\
|1 - f_n(r)| &\le ce^{-m_\lambda r} \\
|f_n'(r)| &\le c' e^{-m_\lambda r} \qquad (1.1.7)
\end{aligned}$$

Here, $\hat{x}^\perp = \frac{1}{r}(-x_2, x_1)^T$, $\vec{j}^{(n)} = Im(\psi^{(n)}(\vec{\nabla}_{\vec{A}}\psi)^{(n)})$ represents the supercurrent of the n-vortex, $\beta_n > 0$ is a constant, $K_1(r)$ is the standard first order modified Bessel function of the second kind, and c, $c' > 0$ are constants. Recall that in this thesis we are working in two dimensions, so $B = \vec{\nabla} \times \vec{A}$ is a scalar quantity.

The solutions to the GL equations possess two symmetries: one translational and one gauge. If $(\psi(x), \vec{A}(x))$ is a solution to the GL equations then so are $(\psi(x - z), \vec{A}(x - z))$ where $z \in \mathbb{R}^2$ and $(e^{i\gamma(x)}\psi(x), \vec{A}(x) + \vec{\nabla}\gamma(x))$ for any function $\gamma : \mathbb{R}^2 \to \mathbb{R}$. These two symmetries allow for the single vortex and multi-vortex pinning problems (the latter is the topic of this thesis) to be solved using Lyapunov-Schmidt reduction (see Appendix A). Rotational symmetry also exists, however it plays no role in our analysis.

2. PROBLEM AND RESULTS

2.1 Pinning Single Vortex

In this section, we go over the primary existence theorem (Theorem 2.1) from [ST] describing the pinning phenomena of a single magnetic vortex. We will modify this theorem to accommodate the multi-vortex pinning scenario. In particular, we state the theorem in terms of the potential W instead of W_{eff} and generalize the potential, W, to have a "strength" of ϵ^p as opposed to ϵ as is found in [ST]. The analysis in this thesis is based on the work of [ST] and will make use of Lyapunov-Schmidt reduction[Mc].

Suppose a superconductor is doped with impurities. Then (1.1.1) and (1.1.2) must be altered accordingly. The modified GL equations will then take the form GL+W

$$-\Delta_{\vec{A}}\psi + \lambda(|\psi|^2 - 1)\psi + W(x)\psi = 0 \tag{2.1.1}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} + Im(\overline{\psi}\vec{\nabla}_{\vec{A}}\psi) = 0, \tag{2.1.2}$$

where $W(x)$ represents the potential of the impurity. This model has been analyzed frequently by both physicists and applied mathematicians e.g., [ASaSe], [CDG], [CR], [ST]. The model relevant to this thesis varies slightly from those studied in [ASaSe], [ABP], [CDG] and [CR] since our analysis is subject to the *entire space* $\mathbb{R}^2$ and does not take into account any applied magnetic field. It should be noted that in this thesis, there is *no* consideration given to the point-vortex limit, $\lambda \to \infty$. The results are valid for all $\lambda > \frac{1}{2}$ (and in fact $\lambda \leq \frac{1}{2}$).

It has been shown [ST] that a magnetic vortex is pinned through a solution to the GL equations with external potential $W(x)$.

The following assumptions are needed before proceeding with the solution to the pinning of a magnetic vortex. Fix $p > 0$.

(A) The impurity potential $W(x)$ must be small, $W(x) = O(\epsilon^p)$ in $L^2(\mathbb{R}^2)$

(B) The potential is a shallow bump, $|\partial_x^\alpha W(x)| \leq C\delta^{|\alpha|+1}\epsilon^p$, for $0 \leq |\alpha| \leq 3$, where α denotes the multi-index $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$, i.e., $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$.

Furthermore as stated above, the pinning of vortices minimizes energy dissipation. Therefore, the solution to the system constructed from (2.1.1) and

(2.1.2) must also be critical points of the modified Ginzburg-Landau energy functional

$$\mathcal{E}_W(\psi, \vec{A}) = \frac{1}{2}\int_{\mathbb{R}^2}\left[|\vec{\nabla}_A\psi|^2 + (\vec{\nabla}\times A)^2 + \frac{\lambda}{2}(|\psi|^2-1)^2 + W(x)(|\psi|^2-1)\right]dx. \tag{2.1.3}$$

We will use the convention, $\mathcal{E}_W = \mathcal{E}_0 = \mathcal{E}$ when there is an absence of potential, $W = 0$.

It is also assumed that the potential $W(x)$ has a non-degenerate critical point located at $z_0 \in \mathbb{R}^2$ i.e.,

$$Hessian(W(z_0)) = Hess(W(z_0)) = \begin{pmatrix} \partial^2_{x_1}W(z_0) & \partial^2_{x_1x_2}W(z_0) \\ \partial^2_{x_2x_1}W(z_0) & \partial^2_{x_2}W(z_0) \end{pmatrix}$$

is invertible.

More precisely, we assume there is a $z_0 \in \mathbb{R}^2$ such that

(C) $\vec{\nabla}W(z_0) = 0$, (z_0 is a critical point of the potential W);

(D) $z_0 \in \Omega_{\epsilon\delta}$, where $\Omega_{\epsilon\delta} = \{x \in \mathbb{R}^2 | \ If\ |W'(x)| \ll \epsilon^p\delta^2,\ then\ W''(x)\ is\ invertible\ with\ ||Hess(W(x))|| \leq c(\epsilon^p\delta^3)^{-1}\}$.

In addition, we assume that $\epsilon^p \ll \delta^4$ and $\delta \ll 1$. The solutions to (2.1.1) and (2.1.2) that optimize (2.1.3) are pairs $(\psi_\epsilon, \vec{A}_\epsilon)$ of the form

$$\psi_\epsilon(x) = e^{i\chi(x)}\psi^{(\pm1)}(x - z_\epsilon) + \xi(x) \tag{2.1.4}$$
$$\vec{A}_\epsilon(x) = \vec{A}^{(\pm1)}(x - z_\epsilon) + \vec{\nabla}\chi + \vec{\beta}(x). \tag{2.1.5}$$

Here, $z_\epsilon = z_0 + O(max(\delta, \frac{\epsilon^p}{\delta^3}))$ denotes the position of the ±1 degree pinned vortex with z_0 as the critical point of the impurity potential $W(x)$. The terms $\xi(x)$ and $\vec{\beta}(x)$ represent corrections of $O(\epsilon^p)$ in $H^2(\mathbb{R}^2) = \{\xi \in L^2(\mathbb{R}^2) \mid \partial^\alpha_\xi \in L^2 for\ |\alpha| \leq 2\}$ to the solution pair, and $\chi : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary gauge function.

We denote a *pinned fundamental vortex* as $(\psi_\epsilon, \vec{A}_\epsilon)$. We say the fundamental vortex is pinned as it is localized near z_0 and is a small perturbation to the solution pair $(\psi, \vec{A})$ of the GL equations. Solutions to equations (1.1.1) and (1.1.2) state that a vortex is free to move about the superconductor by translational symmetry and absence of a potential. Once an impurity is added to the superconductor (denoted by the potential $W(x)$), this symmetry is broken since the vortex interacts with the potential. The vortex will "gravitate" towards and becomes pinned next to the impurity.

Examples of potentials satisfying the conditions (A) through (D) are given by $W(x) := \epsilon\delta V(\delta(x - z_0))$ for $V \in L^2(\mathbb{R}^2) \cap C^3(\mathbb{R}^2)$ with a non-degenerate critical point at the origin, with ϵ and δ sufficiently small.

We make note that the analysis in this thesis is centred around that found in [ST], which obtained (2.1.4) and (2.1.5) for a single vortex.

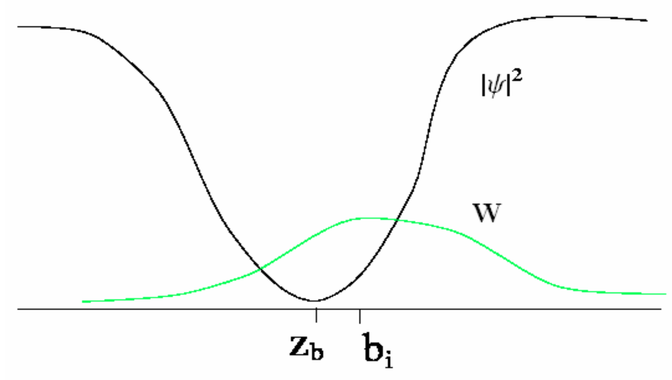

Fig. 2.1.1: $|\psi|^2$ (black) with critical point z_b is pinned near the critical point, b_i of the potential, W (green).

2.2 The Multi-Vortex Pinning Problem and Results

In this section, we state the main problem and results.

As stated above, it has been shown that a *single* vortex gets pinned to a *single* critical point of the potential representing an impurity. The objective of this project is to show that multiple vortices get pinned down when the superconductor is doped with numerous impurities. One method to demonstrate pinning of multi-vortices subject to a multi-well potential $W(x)$ is to show that there exists a solution to (2.1.1) and (2.1.2). Vortex pinning is dependent on the potential. The number of vortices pinned directly corresponds to the number of critical points the potential has.

First, we construct test functions which describe several vortices linked together with their centers at $z_1, z_2 \dots$ and degrees $n_1, n_2 \dots$. This configuration is described by

$$v_{\underline{z}\chi} = \left(\psi_{\underline{z}\chi}, \vec{A}_{\underline{z}\chi}\right)$$

where

$$\psi_{\underline{z}\chi}(x) = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j) \tag{2.2.1}$$

$$\vec{A}_{\underline{z}\chi}(x) = \sum_{j=1}^{m} \vec{A}^{(n_j)}(x - z_j) + \vec{\nabla}\chi(x), \tag{2.2.2}$$

$\underline{z} = (z_1, z_2 \dots, z_m) \in \mathbb{R}^{2m}$, n_j is the degree of the j-th vortex and $\chi : \mathbb{R}^2 \to \mathbb{R}$ is the gauge transformation. We take our gauge transformation to be of the form

$$\chi = \sum_{j=1}^{m} z_j \cdot \vec{A}^{(n_j)}(x - z_j) + \vec{\nabla}\tilde{\chi} \tag{2.2.3}$$

for some $\tilde{\chi} \in H^2(\mathbb{R}^2;\mathbb{R})$. We take the gauge, χ, to be of the form (2.2.3) in order to ensure that $v_{\underline{z}\chi} \in X^{(n)}$, where

$$X^{(n)} := \{(\psi, \vec{A}) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2 \mid (\psi, \vec{A}) - (\psi^{(n)}, \vec{A}^{(n)}) \in H^1(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)\}$$

is the affine space of degree n configurations. In order for the group action defined by the gauge symmetry to preserve $X^{(n)}$, we choose our gauge transformations to be of the form (2.2.3), [GU].
The pair $(\underline{z}, \chi)$ is called a multi-vortex configuration, and the *inter-vortex separation* is defined as

$$R(\underline{z}) = min_{j \neq k}|z_j - z_k|.$$

For large separations $R(\underline{z})$, equations (2.2.1) and (2.2.2) (to be denoted as $(\psi_{\underline{z}\chi}, \vec{A}_{\underline{z}\chi})$) are in fact approximate solutions to the GL equations, which can be seen in Theorem 3.0.2 (a). After adding the potential $W(x)$ to the GL equations, we have a new system of equations GL+W, (2.1.1) and (2.1.2). For small potentials, $W(x)$, we aim to demonstrate there is a solution to the GL+W equations of the form $v_{\underline{z}\chi} + \eta$, where η is a small perturbation.

In the case of pinning a single vortex, the potential is modeled as a shallow bump, whereas in this study the potential is a function with several shallow bumps. For physical and mathematical significance we make the following assumptions about the potential W in the case of multiple pinning.

Theorem 2.2.1. *For small parameters, $\epsilon > 0$ and $\delta > 0$, assume the external potential $W(x)$ satisfies:*

(A) ***Strength of Potential***
$W(x) = O(\sqrt{\epsilon})$ *in* $L^2(\mathbb{R}^2)$

(B) ***Smallness of Derivative***
$|\partial_x^\alpha W(x)| \leq C\delta^{|\alpha|+1}\sqrt{\epsilon}$ *for* $0 \leq |\alpha| \leq 3$

(C) ***Widely Spaced Critical Points***
The critical points b_i, for $i = 1, \ldots, m$ of the potential $W(x)$ are assumed to have large separations between them, i.e., $\frac{e^{-R(\underline{b})}}{\sqrt{R(\underline{b})}} < \epsilon$, where $R(\underline{b}) = min_{i \neq j}|b_i - b_j| \gg 1$ and $\underline{b} := (b_1, b_2, \ldots, b_m)$.

(D) ***Non-Degeneracy of Critical Points***
The critical points of W, $b_1, b_2, \ldots, b_m$ are non-degenerate, i.e., $\underline{b} \in \Omega_{\epsilon\delta\underline{z}}$ where $\Omega_{\epsilon\delta\underline{z}} = \{\underline{z} = (z_1, z_2, \ldots, z_m) \in \mathbb{R}^{2m} \mid$ if $|W'(z_j)| \ll \sqrt{\epsilon}\delta^2$, then $W''(z_j)$ is invertible with $||W''(z_j)^{-1}|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}$ for all $j = 1, 2, 3, \ldots, m\}$.

Let $\lambda > \frac{1}{2}$ *and* n_i *equal to either* 1 *or* -1 *for* $i = 1, \ldots, m$. *Set* $\epsilon > 0$ *and* $\delta > 0$ *be sufficiently small with* $\delta \ll 1$ *and* $\sqrt{\epsilon} \ll \delta^4$. *Suppose W satisfies conditions (A)* $\rightarrow$ *(D) and that the multi-vortex configurations are widely spaced, i.e.,*

$$\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}} < \epsilon. \tag{2.2.4}$$

With the above assumptions in place, the intention of this thesis is to show that a solution to (2.1.1) and (2.1.2) that optimizes the energy functional (2.1.3) is described by an m-vortex system expressed as the pair $(\psi_{MVP}, \vec{A}_{MVP})$ *of the form*

$$\psi_{MVP}(x) = e^{i\chi(x)} \prod_{i=1}^{m} \psi^{(n_i)}(x - z_i) + \xi(x), \tag{2.2.5}$$

$$\vec{A}_{MVP}(x) = \sum_{i=1}^{m} \vec{A}^{(n_i)}(x - z_i) + \vec{\beta}(x) + \vec{\nabla}\chi(x). \tag{2.2.6}$$

In the above pair, MVP denotes multi-vortex pinning, n_i *is the degree of the* i^{th} *vortex,* z_i *is the centre of the* i^{th} *pinned-vortex and,* $z_i = b_i + O\left(max\left(\frac{\sqrt{\epsilon}}{\delta^3}, \delta\right)\right)$. *The terms* $\xi(x)$ *along with* $\vec{\beta}(x)$ *denote the* $O(\sqrt{\epsilon})$ *corrections (in* $H^2(\mathbb{R}^2;\mathbb{C})$ *and* $H^2(\mathbb{R}^2;\mathbb{R}^2)$*) to the multi-vortex configurations* $\psi_{\underline{z}\chi}$ *and* χ *is an arbitrary gauge function such that* $\chi \in H^2(\mathbb{R}^2;\mathbb{R})$. *In the case of this thesis, the vortices of degree* ± 1 *are the focus of attention.*

Remarks

1) Condition (C) and equation (2.2.4) says that impurities along with the centres of vortices must be spaced wide enough for pinning to occur.

2) Note that since $+1$ or -1 vortices are permitted in the multi-vortex configuration, vortex and anti-vortex pairs pairs are pinned if they are separated widely enough. Numerical confirmations have been provided in [K]. We have yet to learn of any physical experiments demonstrating this phenomena.

3) One can also show multi-vortex pinning for the case where $\lambda \leq \frac{1}{2}$, $n_j =$ *any integer* and when the strength of the potential, W, is of the order ϵ^p, for $0 < p < 1$ (in Theorem 2.2.1, $\sqrt{\epsilon}$ is replaced by ϵ^p throughout Conditions (A) $\rightarrow$ (D)). This thesis is concerned with the special case when $p = \frac{1}{2}$. It is conjectured that for $p > 1$, pinned Abrikosov lattices exist. For future purposes, we call the external potential *strong* if $0 < p < 1$ and *weak* if $p > 1$ [PT].

Computational analyses provided in [ASaSe] and [ADP] have shown that stable configurations correspond to maxima of $W(x)$ in the regime where $\lambda \rightarrow$

∞ for applied external magnetic fields and bounded domains. They use a potential of the form $W(x) = \frac{1}{\epsilon^2}(1-a_\epsilon(x))$ and find that vortex pinning occurs near the minima of $\epsilon \to 0$. Here it should be noted that this thesis is focused on the case where $\lambda > \frac{1}{2}$. Numerical studies, [CDG] and [DGP], have shown that vortices of the same degree are attracted to maxima of $W(x)$. Dynamic stability and instability of pinned vortices has been shown in [GT]. In [SS], they derived the effective dynamics of one vortex subject to a local potential, while [Ti] presents the derivation for the dynamics of both weak and strong external potentials for gradient/dissipative flow.

Similar pinning results have been obtained for solitons via the non-linear Schrödinger equation with an external potential, [FW], [ABC], [Oh1], [Oh2], [Oh3].

The pinning phenomena serve practical purposes especially when seeking to optimize the strength of a generated magnetic field. The medical industry relies heavily on large magnetic fields for imaging purposes through MRI, in addition to obtaining bio-chemical information from a patient via NMR. Large magnetic fields are integral to current physics and material science research. Particle physicists depend on large magnetic fields for their experiments with particle accelerators while material scientists study the properties (whether it be deformation or phase transition) of materials subject to large magnetic fields.

3. OUTLINE OF PROOF

In this section, we outline the proof of Theorem 2.2.1.

We begin with some preliminary *notation*. We denote the real inner product on $L^2(\mathbb{R}^2;\mathbb{C}) \oplus L^2(\mathbb{R}^2;\mathbb{R}^2)$ to be

$$\langle \begin{pmatrix} \xi \\ \alpha \end{pmatrix}, \begin{pmatrix} \rho \\ \beta \end{pmatrix} \rangle := \int_{\mathbb{R}^2} Re(\bar{\xi}\rho) + \alpha \cdot \beta$$

We will denote vector $\mathbf{L}^p$ norms by $||\cdot||_p = ||\cdot||_{\mathbf{L}^p}$ and vector $\mathbf{H}^s$ norms by $||\cdot||_{\mathbf{H}^s}$, e.g., $||\begin{pmatrix} \xi \\ \alpha \end{pmatrix}||^p_{\mathbf{L}^p} = ||\xi||^p_{L^p(\mathbb{R}^2;\mathbb{C})} + ||\alpha||^p_{L^p(\mathbb{R}^2;\mathbb{R}^2)}$ and $||\begin{pmatrix} \xi \\ \alpha \end{pmatrix}||^p_{\mathbf{H}^s} = ||\xi||^2_{H^s(\mathbb{R}^2;\mathbb{C})} + ||\alpha||^2_{H^s(\mathbb{R}^2;\mathbb{R}^2)}$.

Finally, we denote the constants that do not depend on any small parameters by c or C.

Following the convention of [ST], define the vector $u = \begin{pmatrix} \psi \\ \vec{A} \end{pmatrix}$ and a map $F_W : \mathbf{H}^2 \to \mathbf{L}^2$ by

$$F_W \begin{pmatrix} \psi \\ \vec{A} \end{pmatrix} = \begin{pmatrix} -\Delta_{\vec{A}}\psi + \lambda(|\psi|^2 - 1)\psi + W(x)\psi \\ \vec{\nabla} \times \vec{\nabla} \times \vec{A} + Im(\bar{\psi}\vec{\nabla}_{\vec{A}}\psi) \end{pmatrix}.$$

Indeed, for example,

$$\begin{aligned} ||(|\psi|^2 - 1)\psi||_{L^2} &\leq ||(|\psi|^2 - 1)||_{L^2}||\psi||_\infty \\ &< \infty \end{aligned}$$

by equation (1.1.7) and Hölder's inequality.

We look for a solution, u of $F_W(u) = 0$ of the form $u = v_{\underline{z}\chi} + \eta = \begin{pmatrix} \psi_{\underline{z}\chi} + \xi \\ \vec{A}_{\underline{z}\chi} + \vec{\beta} \end{pmatrix}$ where $\eta = (\xi, \vec{\beta})$ is a small perturbation. In order to find a solution, Lyapunov-Schmidt Reduction ([JMST], [Mc]) will be used to decompose the equation into two separate ones: the first dealing with the orthogonal component of $F_W(u) = 0$ while the second concerns the tangential component (orthogonal and tangential to the "almost zero translational and gauge symmetry modes", see below, eq. (3.0.2) for further details). For reasons why we use Lyapunov-Schmidt reduction, refer to Appendix A. Using an Implicit Function Theorem (IFT) type argument (see Appendix A, [JMST] and [Mc]), the problem in the orthogonal direction of $F_W(u) = 0$ is expected to yield a solution. The solution in the tangential direction is considerably much more difficult to achieve. Combining the solutions obtained in the orthogonal and tangential subproblems via

Lyapunov-Schmidt Reduction, a solution is shown to exist. We discuss this in further details below.

We first define a manifold of multi-vortex configurations made up of a collection of widely spaced vortices connected together. More precisely, define our infinite dimensional manifold

$$M_{mv\epsilon} := \{v_{\underline{z}\chi} | (\underline{z}, \chi) \in \Sigma_\epsilon\}$$

parameterized by the widely separated centres of the m vortices and gauge function χ:

$$\Sigma_\epsilon := \{(\underline{z}, \chi) |\ \underline{z} \in \mathbb{R}^{2m}\ with\ \frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}} < \epsilon\ and\ \chi \in \mathbf{H}^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R})\}$$

where

$$\mathbf{H}^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R}) := \{\chi \in H^2(\mathbb{R}^2; \mathbb{R}) |\ \chi - \sum_{j=1}^{m} z_j \cdot \vec{A}^{(n_j)}(x - z_j) \in H^2(\mathbb{R}^2; \mathbb{R})\}.$$

Recall $v_{\underline{z}\chi} = (\psi_{\underline{z}\chi}, \vec{A}_{\underline{z}\chi})$ are test functions defined in (2.2.1) and (2.2.2). The manifold, $M_{mv\epsilon}$, is infinite dimensional since the parametric space included $H^2(\mathbb{R}^2; \mathbb{R})$, which is itself infinite dimensional. The parameter, ϵ, emphasizes that we only consider *widely spaced* multi-vortex configurations.

The tangent space to the point $v_{\underline{z}\chi} \in M_{mv\epsilon}$ is

$$T_{v_{\underline{z}\chi}} M_{mv\epsilon} = span\{\langle\gamma, \partial_\chi\rangle v_{\underline{z}\chi},\ \partial_{z_{jk}} v_{\underline{z}\chi} \mid j = 1, \ldots, m,\ k = 1, 2\ ;\ \gamma \in H^2(\mathbb{R}^2; \mathbb{R})\},$$

where the notation $\langle\gamma, \partial_\chi\rangle$ signifies a variational derivative with respect to χ and evaluated at $\chi = \gamma$. We label z_{jk} as the k-th component of the j-th vortex position, z_j.

Next, we define the *almost zero modes* using equations (2.2.1), (2.2.2) and (2.2.3). Define the gauge tangent *almost zero mode* as

$$G^{\underline{z}\chi}_\gamma := \langle\gamma, \partial_\chi\rangle v_{\underline{z}\chi} = \begin{pmatrix} i\gamma\psi_{\underline{z}\chi} \\ \vec{\nabla}\gamma \end{pmatrix} \tag{3.0.1}$$

and the translational tangent *almost zero mode* as

$$\begin{aligned} T^{\underline{z}\chi}_{jk} &:= \partial_{z_{jk}} v_{\underline{z}\chi} \\ &= -\begin{pmatrix} e^{i\chi} \prod_{l \neq j} \psi^{(n_l)}(x - z_l)[\partial_{x_{jk}} - i(\vec{A}^{(n_j)}(x - z_j))_k]\psi^{(n_j)}(x - z_j) \\ B^{(n_j)}(x - z_j) e^\perp_k \end{pmatrix} \end{aligned} \tag{3.0.2}$$

where $B^{(n_j)} = \vec{\nabla} \times \vec{A}^{(n_j)}$, $e^\perp_1 = (0, 1)$ and $e^\perp_2 = (-1, 0)$. For the calculations of $G^{\underline{z}\chi}_\gamma$ and $T^{\underline{z}\chi}_{jk}$, see Appendix C. These tangent vectors are called *almost zero* *modes* since they *almost* solve $\mathcal{E}_0''(v_{\underline{z}\chi})\eta = 0$, i.e., $\partial_{z_{jk}}\mathcal{E}_0'(v_{\underline{z}\chi}) = \mathcal{E}_0''(v_{\underline{z}\chi})T_{jk}^{\underline{z}\chi} \simeq 0$ and $\partial_\chi \mathcal{E}_0'(v_{\underline{z}\chi}) = \mathcal{E}_0''(v_{\underline{z}\chi})G_\gamma^{\underline{z}\chi} \simeq 0$ (see Theorem 3.0.2 (d) and (e)).

Define the orthogonal projection operator

$$\Pi_{\underline{z}\chi} = L^2 \text{ orthogonal projection onto } span\{T_{jk}^{\underline{z}\chi}, G_\chi^{\underline{z}\chi} | j = 1, \ldots, m,\ k = 1, 2, \chi \in H^2(\mathbb{R}^2; \mathbb{R})\} \tag{3.0.3}$$

along with $\Pi_{\underline{z}\chi}^\perp = 1 - \Pi_{\underline{z}\chi}$ (for explicit expressions, see Section 5).

From (3.0.1), (3.0.2) and (3.0.3), it is clear that

$$T_{v_{\underline{z}\chi}} M_{mv\epsilon} = Ran\, \Pi_{\underline{z}\chi},$$

where *Ran* represents the Range of a linear map.
The proof of Theorem 2.2.1 is composed of the following three components:

(1) **Lyapunov-Schmidt Reduction and Solution in Orthogonal Direction**: We use Lyapunov-Schmidt reduction to break the problem into its tangential and orthogonal components. It will be shown that there exists a solution in the orthogonal direction through Implicit Function theorem (IFT). Specifically, it will be shown that for all widely-spaced multi-vortex configurations, i.e., $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a unique $\eta_{\underline{z}\chi W} \in Ran\left(\Pi_{\underline{z}\chi}^\perp\right)$ with

$$\Pi_{\underline{z}\chi}^\perp F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) = 0. \tag{3.0.4}$$

(2) **Reduced Problem in Tangential Direction**: We substitute $v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}$ for $\eta_{\underline{z}\chi W} \in Ran\left(\Pi_{\underline{z}\chi}^\perp\right)$ in the full energy functional to obtain the reduced energy $\Phi_W(\underline{z}) := \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W})$. We'll show that if W has critical points at $(b_1, b_2, \ldots, b_m) := \underline{b}$, then the reduced energy $\Phi_W : \mathbb{R}^{2m} \to \mathbb{R}$ has a critical point at $\underline{z} = \underline{z}_p$ if and only if

$$\Pi_{\underline{z}_p\chi} F_W(v_{\underline{z}_p\chi} + \eta_{\underline{z}_p\chi W}) = 0. \tag{3.0.5}$$

(3) **Critical Points of Reduced Energy**: We show that the reduced energy, Φ_W attains a critical point at some multi-vortex configuration $\underline{z}_b$. Furthermore we assume that if W attains a local non-degenerate, widely spaced critical points $\underline{b} := (b_1, b_2, \ldots, b_m)$, satisfying conditions (C) and (D) of Theorem (2.2.1), then Φ_W has a local critical point at some multi-vortex configuration $\underline{z}_b$ close to $\underline{b}$.

Demonstrations for parts (1) , (2) and (3) will be given in Sections 4, 5 and 6 respectively.

Now, we put forth results from [GS2] that are integral to the analysis in this thesis.

Theorem 3.0.2. *For* $(\underline{z}, \chi) \in \Sigma_\epsilon$,

(a) ***Almost Solution***
$||\mathcal{E}_0'(v_{\underline{z}\chi})||_{\mathbf{H}^1} = O\left(\epsilon \; log^{\frac{1}{4}}\left(\frac{1}{\epsilon}\right)\right)$

(b) ***Almost Orthogonality of Translational Tangent Vectors***
$|\langle T_{ij}^{\underline{z}\chi}, T_{lm}^{\underline{z}\chi}\rangle| = \gamma_{(n_i)}\delta_{il}\delta_{jm} + O\left(\epsilon \; log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right)$

$$\gamma_{(n_i)} = \frac{1}{2}||\vec{\nabla}_{\vec{A}^{(n_i)}}\psi^{(n_i)}||^2 + ||curl \; \vec{A}^{(n_i)}||^2 \tag{3.0.6}$$

(c) ***Almost Orthogonality of Translational and Gauge Tangent Vectors***
$|\langle T_{jk}^{\underline{z}\chi}, G_{\gamma}^{\underline{z}\chi}\rangle| \le c\epsilon \; log^{\frac{1}{4}}\left(\frac{1}{\epsilon}\right)||\gamma||_{L^2}$

(d) ***Almost Zero Mode of Translational Tangent Vectors***
Define

$$L_{\underline{z}\chi} := \mathcal{E}_0''(v_{\underline{z}\chi}),$$

where the Hessian of the GL energy functional at $\begin{pmatrix} \psi \\ \vec{A} \end{pmatrix}$ *is defined as* $\left[\mathcal{E}_0''\begin{pmatrix} \psi \\ \vec{A} \end{pmatrix}\right]\begin{pmatrix} \xi \\ \vec{B} \end{pmatrix} := \frac{\partial}{\partial\zeta}\mathcal{E}'(\psi+\zeta\xi, \vec{A}+\zeta\vec{B})|_{\zeta=0}$ *(for explicit calculation and expression, see Appendix C).*

Then $||L_{\underline{z}\chi}T_{jk}^{\underline{z}\chi}||_{\mathbf{L}^2} \le c\epsilon \; log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)$.

(e) ***Almost Zero Mode of Gauge Tangent Vectors***
$||L_{\underline{z}\chi}G_{\gamma}^{\underline{z}\chi}||_{\mathbf{L}^2} \le c\epsilon \; log^{\frac{1}{4}}\left(\frac{1}{\epsilon}\right)||\gamma||_{L^2}$.

(f) ***Almost Zero Mode***
$||L_{\underline{z}\chi}\Pi_{\underline{z}\chi}|| \le c\epsilon \; log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)$.

(g) ***Coercivity of Hessian***
For $\eta \in [Ran(\Pi_{\underline{z}\chi})]^\perp$, $\langle \eta, L_{\underline{z}\chi}\eta\rangle \ge c||\eta||_{\mathbf{H}^1}^2$.

(h) ***Invertibility of Hessian***
For $\eta \in [Ran(\Pi_{\underline{z}\chi})]^\perp$, $||L_{\underline{z}\chi}\eta||_{\mathbf{L}^2} \ge c||\eta||_{\mathbf{H}^2}$.

Corollary 3.0.2.1, Lemma 4.0.1 and Theorem 3.0.2 (d) are necessary for step (1), while Theorem (3.0.2) (b) and (d) are needed for step (2).

Proof. See Appendix B, Lemma B.0.4 for the proof of (a). Parts (b) to (g) have already been proven in [GS2]. More specifically, see equations (46) and (47) of [GS2] for (b) and (c), Lemma 2 for (d), (e) and the definition of $\Pi_{\underline{z}\chi}$, as shown in equation (3.0.6). Part (f) follows from (d) and (e), while (g) is demonstrated in Lemma 3 of [GS2]. Part (h) follows from Appendix D. □

Corollary 3.0.2.1. *For* $(\underline{z}, \chi) \in \Sigma_\epsilon$, $||\mathcal{E}_W'(v_{\underline{z}\chi})||_{\mathbf{L}^2} = ||F_W(v_{\underline{z}\chi})||_{\mathbf{L}^2} \leq \kappa\sqrt{\epsilon}$, *where*

$$\kappa := sup_\epsilon \frac{1}{\sqrt{\epsilon}} ||F_W(v_{\underline{z}\chi})||_{\mathbf{L}^2} \tag{3.0.7}$$

exists and is finite.

Proof. Since $\mathcal{E}_W'(v_{\underline{z}\chi}) = \mathcal{E}_0'(v_{\underline{z}\chi}) + \begin{pmatrix} W\psi_{\underline{z}\chi} \\ 0 \end{pmatrix}$, we have

$$\begin{aligned} ||\mathcal{E}_W'(v_{\underline{z}\chi})||_{\mathbf{L}^2} &\leq ||\mathcal{E}_0'(v_{\underline{z}\chi})||_{\mathbf{L}^2} + ||W\psi_{\underline{z}\chi}||_{\mathbf{L}^2} \\ &\leq c\epsilon\, log^{\frac{1}{4}}\left(\frac{1}{\epsilon}\right) + ||W||_{\mathbf{L}^2}||\psi_{\underline{z}\chi}||_\infty \\ &\leq c\epsilon\, log^{\frac{1}{4}}\left(\frac{1}{\epsilon}\right) + c\sqrt{\epsilon} \\ &\leq \kappa\sqrt{\epsilon}, \end{aligned}$$

where we applied condition (A) of Theorem 2.1.1 on to the potential W.

□

Furthermore we need the following Theorem for part (1), which will be proven in Section 4. Denote $B_X(z, r)$ as the open ball in a Banach space X of radius r and centred at z.

Theorem 3.0.3. *Suppose* $W(x)$ *satisfies condition (A) of Theorem 2.2.1 in Section 2.2. There exists an* $\epsilon_0 > 0$ *and* $\delta_0 > 0$ *such that for* ϵ *satisfying* $0 < \sqrt{\epsilon} < \epsilon_0$ *and for all* $(\underline{z}, \chi) \in \Sigma_\epsilon$, *there exists a unique* $\eta_{\underline{z}\chi W} \in [Ran(\Pi_{\underline{z}\chi})]^\perp \cap B_{\mathbf{H}^2}(0, \delta_0)$ *that satisfies* $\Pi^\perp_{\underline{z}\chi W} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) = 0$, *and that* $\eta_{\underline{z}\chi W}$ *satisfies:*

a) $||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} = O(\sqrt{\epsilon})$ *by condition(A) in Theorem 2.1*

b) $||\partial^\alpha_{\underline{z}}\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} = O(\sqrt{\epsilon})$, *for* $0 \leq |\alpha| \leq 3$.

Therefore the size of the perturbation in the solution is directly proportional to the strength of the potential $W(x)$.

For part (2), we need the following Theorem which will be proven in Section 5.

Theorem 3.0.4. *Assume the conditions of Theorem 3.0.3 hold and let* $\eta_{\underline{z}\chi W}$ *be this unique solution of (3.0.4). Then for* ϵ *satisfying* $0 < \sqrt{\epsilon} < \epsilon_0$ *and for* $(\underline{z}, \chi) \in \Sigma_\epsilon$, $\Phi_W(\underline{z}) = \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W})$ *is well defined, that is, for every* $\underline{z}$ *and* χ, *there exists a unique number,* Φ_W. *Moreover,* $\Phi_W(\underline{z})$ *has a critical point at* $\underline{z} = \underline{z}_p \in \{\underline{z} \in \mathbb{R}^{2m} \mid \frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}} < \epsilon\}$ *if and only if (3.0.5) is satisfied at* $\underline{z} = \underline{z}_p$.

The following Theorem corresponds to part (3):

Theorem 3.0.5. *Suppose our potential satisfies the conditions (A) and (B) from Section (2) for* $m = 3$ *and* $\delta \ll 1$ *and* $\sqrt{\epsilon} \ll \delta^4$*. Then, if* W *satisfies the conditions (C) and (D) from Section 2 (specifically if* $b_1, \ldots, b_m$ *are critical points of* $W(x)$ *with* $(b_1, \ldots, b_m) := \underline{b} \in \Omega_{\epsilon\delta\underline{z}}$*), then* Φ_W *has a unique critical point* $\underline{z}_{\underline{b}} \in B_{\mathbb{R}^{2m}}\left(\underline{b}, \frac{c}{\delta}\right)$ *with* $|\underline{z}_{\underline{b}} - \underline{b}| \leq c\, max\left(\frac{\sqrt{\epsilon}}{\delta^3}, \delta\right)$*.*

Physically, Theorem 3.0.5 allows us to model impurities/homogeneities by small, shallow bumps or non-degenerate critical points (as outlined in condition (D)) for the study of magnetic vortex pinning in superconductors.

Proofs for Theorems 3.0.3, 3.0.4 and 3.0.5 are found in Sections 4, 5 and 6.0.3 respectively. Now, we set off to prove the main theorem in this thesis, Theorem 2.2.1.

3.1 Proof of the Main Theorem 2.2.1

Suppose that W satisfies condition (A). By Theorem 3.0.3, there exists an $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for $\epsilon > 0$ satisfying $0 < \sqrt{\epsilon} < \epsilon_0$ and for all $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a unique $\eta_{\underline{z}\chi W} \in [Ran(\Pi_{\underline{z}\chi})]^\perp \cap B_{\mathbf{H}^2}(0, \delta_0)$ such that $\Pi^\perp_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) = 0$. Substituting $v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}$ into the energy functional $\mathcal{E}_W$, we obtain the reduced energy $\Phi_W(\underline{z}) := \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W})$. To solve $\Pi_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) = 0$ we make use of Theorem 3.0.4 to note that the critical points $\underline{z} \in \{\underline{z} \in \mathbb{R}^{2m} \mid \frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}} < \epsilon\}$ of $\Phi_W(\underline{z})$ are solutions of $\Pi_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) = 0$.

Now suppose W satisfies the additional conditions (B), (C) and (D). By Theorem 3.0.5, it is shown that $\Phi_W(\underline{z})$ has critical points $\underline{z}_b \in B_{\mathbb{R}^2m}(\underline{b}, \frac{c}{\delta})$ such that$|\underline{z}_b - \underline{b}| = O\left(max\left(\frac{\sqrt{\epsilon}}{\delta^3}, \delta\right)\right)$. By Theorem 3.0.4 with $\underline{z}_p = \underline{z}_b$, we have that $\Pi_{\underline{z}_b\chi} F_W(v_{\underline{z}_b\chi} + \eta_{\underline{z}_b\chi W}) = 0$ and therefore $F_W(v_{\underline{z}_b\chi} + \eta_{\underline{z}_b\chi W}) = 0$. By condition a) from Theorem 3.0.3 and noting the separation between $\underline{z}_b := (z_1, \ldots, z_m)$ and $\underline{b} = (b_1, \ldots, b_m)$, we have

$$\begin{pmatrix} \psi_{MVP} \\ \vec{A}_{MVP} \end{pmatrix} = v_{\underline{z}_b\chi} + \eta_{\underline{z}_b\chi W} = \begin{pmatrix} e^{i\chi(x)} \prod_{i=1}^n \psi^{(n_i)}(x - z_i) + \xi(x) \\ \sum_{i=1}^n \vec{A}^{(n_i)}(x - z_i) + \vec{\beta(x)} + \vec{\nabla}\chi(x) \end{pmatrix},$$

with $(\xi(x), \vec{\beta}(x)) = O(\sqrt{\epsilon})$ in $\mathbf{H}^2$, $z_i = b_i + O(max(\delta, \frac{\sqrt{\epsilon}}{\delta^3}))$ and some arbitrary function $\chi \in H^2(\mathbb{R}^2; \mathbb{R})$. Therefore, Theorem 2.2.1 is proven.

It should be noted that since $|\underline{z}_b - \underline{b}| = O\left(max\left(\delta, \frac{\sqrt{\epsilon}}{\delta^3}\right)\right)$, then we can say $R(\underline{z}_b) \leq R(\underline{b}) + c\ max\left(\frac{\sqrt{\epsilon}}{\delta^3}, \delta\right)$. Therefore, by equation (2.2.4), $\delta \ll 1$ and $\sqrt{\epsilon} \ll \delta^4$, we have $\frac{e^{-R(\underline{z}_b)}}{\sqrt{R(\underline{z}_b)}} < \epsilon$. Thus the Theorems 3.0.3, 3.0.4 and 3.0.5 used to prove Theorem 2.2.1 all hold since $(\underline{z}_b, \chi) \in \Sigma_\epsilon$.

4. SOLUTION IN ORTHOGONAL DIRECTION

This section is devoted to proving Theorem 3.0.4.

Since solving the orthogonal equation for (3.0.4) is less difficult than solving its tangential counterpart, it will be dealt with first. In order to demonstrate the pinning phenomenon, the Implicit Function Theorem is applied to the orthogonal component of (3.0.4).

Expanding $F_W(v_{\underline{z}\chi}+\eta)$ as a Taylor series around $v_{\underline{z}\chi}$ and applying $\Pi^{\perp}_{\underline{z}\chi}$ yields

$$\Pi^{\perp}_{\underline{z}\chi}F_W(v_{\underline{z}\chi}+\eta)=\Pi^{\perp}_{\underline{z}\chi}F_W(v_{\underline{z}\chi})+\Pi^{\perp}_{\underline{z}\chi}F'_W(v_{\underline{z}\chi})\eta+\Pi^{\perp}_{\underline{z}\chi}N_W(v_{\underline{z}\chi},\eta), \quad (4.0.1)$$

where for $\eta=\begin{pmatrix}\xi\\ \vec{B}\end{pmatrix}$, $N_W(v_{\underline{z}\chi},\eta)$ is defined by the relation (4.0.1) and is given by

$$N_W(v_{\underline{z}\chi},\eta)=\begin{pmatrix}\frac{\lambda}{2}(2\psi_{\underline{z}\chi}\bar{\xi}+\bar{\psi_{\underline{z}\chi}}\xi+|\xi|^2)\xi+||\vec{B}||^2(\psi_{\underline{z}\chi}+\xi)+[i(\vec{\nabla}\cdot\vec{B}+\vec{B}\cdot\vec{\nabla})+2\vec{A}\cdot\vec{B}]\xi\\ \vec{B}(2Re(\bar{\psi_{\underline{z}\chi}}\xi)+|\xi|^2)-Im(\bar{\xi}\vec{\nabla}_{\vec{A}_{\underline{z}\chi}}\xi)\end{pmatrix}.$$

We now need the following Lemma which comes as a consequence of Theorem 3.0.2 (h). Define

$$L_{\underline{z}\chi W}:=\Pi^{\perp}_{\underline{z}\chi}F'_W(v_{\underline{z}\chi})\restriction_{Ran(\Pi^{\perp}_{\underline{z}\chi})\cap H^2}$$

where "$\restriction$" denotes the restriction of the operator on the space $Ran(\Pi^{\perp}_{\underline{z}\chi})\cap H^2$.

Lemma 4.0.1. *There exists a constant c_1 such that for ϵ satisfying $0<\sqrt{\epsilon}<c_1$, $L_{\underline{z}\chi W}$ is invertible on $[Ran(\Pi_{\underline{z}\chi})]^{\perp}$, i.e.,*

$$||L_{\underline{z}\chi W}\eta||_{\mathbf{L}^2}\geq\beta||\eta||_{\mathbf{L}^2}$$

for η in $[Ran(\Pi_{\underline{z}\chi})]^{\perp}$ and some constant $\beta>0$.

Proof. From Theorem 3.0.2 part (h), we have for $\eta\in[Ran(\Pi_{\underline{z}\chi})]^{\perp}$,

$$||L_{\underline{z}\chi}\eta||_{\mathbf{L}^2}\geq c||\eta||_{\mathbf{H}^2}. \quad (4.0.2)$$

Now as $L_{\underline{z}\chi W}=L_{\underline{z}\chi}+\begin{pmatrix}W & 0\\ 0 & 0\end{pmatrix}$ with $\eta=\begin{pmatrix}\xi\\ \vec{B}\end{pmatrix}$ we have

$$\begin{aligned}||L_{\underline{z}\chi W}\eta||_{\mathbf{L}^2} &\geq ||L_{\underline{z}\chi}\eta||_{\mathbf{L}^2} - ||\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}\eta||_{L^2} \\ &\geq c||\eta||_{\mathbf{H}^2} - ||W\xi||_{L^2}, \qquad (4.0.3)\end{aligned}$$

where we have used (4.0.2). By condition (A) on the potential W, we have

$$\begin{aligned}||W\xi||_{L^2} &\leq ||W||_{L^2}||\xi||_\infty \\ &\leq \tilde{c}\sqrt{\epsilon}||\xi||_{\mathbf{H}^2} \\ &\leq \tilde{c}\sqrt{\epsilon}||\eta||_{\mathbf{H}^2},\end{aligned}$$

and hence by choosing ϵ to be arbitrarily small, we are left with

$$\begin{aligned}||L_{\underline{z}\chi W}\eta||_{L^2} &\geq (c-\tilde{c}\sqrt{\epsilon})||\eta||_{\mathbf{H}^2} \\ &\geq c_1||\eta||_{\mathbf{H}^2}\end{aligned}$$

by (4.0.3). □

Setting $\Pi^{\perp}_{\underline{z}\chi}F_W(v_{\underline{z}\chi}+\eta)=0$ in (4.0.1) and solving for η yields

$$\eta = -L^{-1}_{\underline{z}\chi W}[\Pi^{\perp}_{\underline{z}\chi}F_W(v_{\underline{z}\chi}) + \Pi^{\perp}_{\underline{z}\chi}N_W(v_{\underline{z}\chi},\eta)] \qquad (4.0.4)$$

where $L_{\underline{z}\chi W} := \Pi^{\perp}_{\underline{z}\chi}F'_W(v_{\underline{z}\chi}) \restriction_{Ran(\Pi_{\underline{z}\chi})^{\perp}}$ is invertible by Lemma 4.0.1.

In order to employ an IFT argument, we make use of the following Lemma regarding the bound on the non-linearity/remainder term, $N_W(v_{\underline{z}\chi},\eta)$.

Lemma 4.0.2. *There exists constants c_2, c_3, c_4 that are independent of $\underline{z}, \chi, \epsilon$ such that for $\eta \in \mathbf{H}^2$ with $||\eta||_{\mathbf{H}^2} \leq c_2$,*

$$||N_W(v_{\underline{z}\chi},\eta)||_{\mathbf{L}^2} \leq c_3||\eta||^2_{\mathbf{H}^2} \qquad (4.0.5)$$

and

$$||\partial_\eta N_W(v_{\underline{z}\chi},\eta)||_{\mathbf{H}^2\to\mathbf{L}^2} \leq c_4||\eta||_{\mathbf{H}^2} \qquad (4.0.6)$$

Proof. This Lemma follows from a form of the Sobolev Embedding Theorem ($||\phi||_p \leq c_p||\phi||_{H^1}$ for $2 \leq p \leq \infty$ in $\mathbb{R}^2$, see [Mc] and [JMST]) and Mean Value Theorem outlined in [ST]. □

Two points need to be demonstrated in order to properly solve this sub-problem. First, since $L_{\underline{z}\chi W}$ is invertible on $Ran(\Pi^{\perp}_{\underline{z}\chi})$, define a map $S_{\underline{z}\chi W}$: $\mathbf{H}^2 \to Ran(\Pi_{\underline{z}\chi})^{\perp}$ by

$$S_{\underline{z}\chi W}(\eta) = -L^{-1}_{\underline{z}\chi W}[\Pi^{\perp}_{\underline{z}\chi}F_W(v_{\underline{z}\chi}) + N_W(v_{\underline{z}\chi},\eta)].$$

We have to show that $S_{\underline{z}\chi W}$ has a fixed point, i.e., there exists an η in some ball such that $S_{\underline{z}\chi W}(\eta) = \eta$. Showing that $S_{\underline{z}\chi W}$ has a fixed point is equivalent to demonstrating that there exists an η that satisfies (4.0.4), i.e., there exists an η such that $\Pi^{\perp}_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta) = 0$. In order to demonstrate that $S_{\underline{z}\chi W}$ has a fixed point we have to show two things:

1. There exists an $\epsilon_0 > 0$ and a $\delta_0 > 0$ such that $S_{\underline{z}\chi W}$ maps the ball of radius δ_0, $B^{\perp}_{\delta_0} = B_{\mathbf{H}^2}(0, \delta_0) \cap Ran(\Pi^{\perp}_{\underline{z}\chi})$ to itself.

2. $S_{\underline{z}\chi W}(\eta)$ is a strict contraction in $B^{\perp}_{\delta_0}$, i.e., $||S_{\underline{z}\chi W}(\eta) - S_{\underline{z}\chi W}(\eta')||_{\mathbf{H}^2} \leq c||\eta - \eta'||_{\mathbf{H}^2}$ for some $0 < c < 1$.

Let $\beta, c_1, c_2, c_3, c_4, \kappa$ be constants defined in Lemmas 4.0.1 and 4.0.2, along with Corollary 3.0.2.1. Define $\delta_0 := min\left(c_2, \frac{\beta}{2c_3}, \frac{\beta}{2c_4}\right)$ and $\epsilon_0 := min\left(\frac{\delta_0\beta}{2\kappa}, c_1\right)$. Finally we let $\eta \in B^{\perp}_{\delta_0}$ and ϵ satisfy $0 < \sqrt{\epsilon} < \epsilon_0$. Taking the $\mathbf{H}^2$ norm of $S_{\underline{z}\chi W}(\eta)$ and using Lemmas 4.0.1, 4.0.2 and Corollary 3.0.2.1 we obtain

$$\begin{aligned}
||S_{\underline{z}\chi W}(\eta)||_{\mathbf{H}^2} &\leq \frac{1}{\beta}||N^{\perp}_{\underline{z}\chi W}(\eta) + F^{\perp}_{\underline{z}\chi W}||_{\mathbf{L}^2} \\
&\leq \frac{1}{\beta}\left(c_3||\eta||^2_{\mathbf{H}^2} + ||F^{\perp}_{\underline{z}\chi W}||_{\mathbf{L}^2}\right) \\
&\leq \frac{1}{\beta}\left(c_3||\eta||^2_{\mathbf{H}^2} + \kappa\sqrt{\epsilon}\right).
\end{aligned}$$

Substitute in the relation $||\eta||_{H^2} \leq \delta_0$ and use the definition of δ_0 and ϵ_0 to obtain

$$||S_{\underline{z}\chi W}(\eta)||_{\mathbf{H}^2} \leq \frac{1}{\beta}\left(c_3\frac{\delta_0\beta}{2c_3} + \frac{\kappa\delta_0\beta}{2\kappa}\right) = \delta_0.$$

Therefore $S_{\underline{z}\chi W}(\eta)$ is in $B^{\perp}_{\delta_0}$, which confirms 1. Now to show 2: $S_{\underline{z}\chi W}(\eta)$ is a strict contraction. First subtract $S_{\underline{z}\chi W}(\eta')$ from $S_{\underline{z}\chi W}(\eta)$ for η, $\eta' \in B^{\perp}_{\delta_0}$ and then take the norm

$$\begin{aligned}
||S_{\underline{z}\chi W}(\eta) - S_{\underline{z}\chi W}(\eta')||_{\mathbf{H}^2} &\leq \frac{1}{\beta}||N^{\perp}_{\underline{z}\chi W}(\eta) + F^{\perp}_{\underline{z}\chi W} - N^{\perp}_{\underline{z}\chi W}(\eta') - F^{\perp}_{\underline{z}\chi W}||_{\mathbf{L}^2} \\
&\leq \frac{1}{\beta}||N^{\perp}_{\underline{z}\chi W}(\eta) - N^{\perp}_{\underline{z}\chi W}(\eta')||_{\mathbf{H}^2} \\
&\leq \frac{1}{\beta}[c_4\delta_0||\eta - \eta'||_{\mathbf{H}^2}] \\
&\leq \frac{1}{2}||\eta - \eta'||_{\mathbf{H}^2}
\end{aligned}$$

which demonstrates the map as a strict contraction in $B^{\perp}_{\delta_0}$.

Furthermore it must be shown that $||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} \leq D\sqrt{\epsilon}$, as in Theorem 3.0.3 a). First start with $\eta_{\underline{z}\chi W} = S_{\underline{z}\chi W}(\eta_{\underline{z}\chi W})$ and add zero.

$$\eta_{\underline{z}\chi W} = S_{\underline{z}\chi W}(\eta_{\underline{z}\chi W}) - S_{\underline{z}\chi W}(0) + S_{\underline{z}\chi W}(0).$$

Taking the norm of $\eta_{\underline{z}\chi W}$ we have

$$||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} \leq ||S_{\underline{z}\chi W}(0)||_{\mathbf{H}^2} + ||S_{\underline{z}\chi W}(\eta_{\underline{z}\chi W}) - S_{\underline{z}\chi W}(0)||_{\mathbf{H}^2}$$

and noting that

$$||S_{\underline{z}\chi W}(0)||_{\mathbf{H}^2} = ||-L^{-1}_{\underline{z}\chi W}[N^{\perp}_{\underline{z}\chi W}(0)+F^{\perp}_{\underline{z}\chi W}]||_{\mathbf{H}^2} = ||-L^{-1}_{\underline{z}\chi W}F^{\perp}_{\underline{z}\chi W}||_{\mathbf{H}^2} \leq \frac{1}{\beta}||F_W(v_{\underline{z}\chi})||_{\mathbf{H}^2}$$

along with

$$||S_{\underline{z}\chi W}(\eta) - S_{\underline{z}\chi W}(0)||_{\mathbf{H}^2} \leq \frac{1}{2}||w||_{\mathbf{H}^2},$$

we obtain

$$||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} \leq \frac{1}{\beta}||F_W(v_{\underline{z}\chi})||_{\mathbf{L}^2} + \frac{1}{2}||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2}.$$

From here we insert the relation $||F_W(v_{\underline{z}\chi})||_{\mathbf{H}^2} \leq \kappa\sqrt{\epsilon}$ from Corollary 3.0.2.1 to get

$$\frac{1}{2}||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} \leq \frac{1}{\beta}||F_W(v_{\underline{z}\chi})||_{\mathbf{L}^2} \leq \frac{\kappa\sqrt{\epsilon}}{\beta},$$

and therefore $||\eta_{\underline{z}\chi W}||_{\mathbf{H}^2} \leq \frac{2}{\beta}\kappa\sqrt{\epsilon}$ which satisfies the desired condition $||\eta_{\underline{z}\gamma W}||_{\mathbf{H}^2} \leq D\sqrt{\epsilon}$.

Now we prove part b) of Theorem 3.0.3. Define the function

$$F^{\perp}(\underline{z}, \eta) = \Pi^{\perp}_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta). \tag{4.0.7}$$

By Theorem 3.0.3, for $(\underline{z}, \chi) \in \Sigma_\epsilon$ and for ϵ satisfying $0 < \sqrt{\epsilon} < \epsilon_0$, we have $F^{\perp}(\underline{z}, \eta(\underline{z})) = 0$. Through Implicit Differentiation we have

$$\partial_{z_{jk}}\eta(\underline{z}) = -\partial_\eta F^{\perp}(\underline{z}, \eta(\underline{z}))^{-1}\partial_{z_{jk}}F^{\perp}(\underline{z}, \eta(\underline{z})). \tag{4.0.8}$$

Now we intend to prove that

$$||\partial_{z_{jk}}\eta(\underline{z})||_{\mathbf{H}^2} \leq c\sqrt{\epsilon}. \tag{4.0.9}$$

First, note that

$$||\partial_\eta F^{\perp}(\underline{z}, \eta(\underline{z}))^{-1}|| \leq c,$$

through Chain Rule and (4.0.7), we have

$$\partial_{z_{jk}} F^{\perp}(\underline{z}, \eta(\underline{z})) = \left(\partial_{z_{jk}} \Pi^{\perp}_{\underline{z}\chi}\right) F_W(v_{\underline{z}\chi} + \eta) + \Pi^{\perp}_{\underline{z}\chi} \left(\partial_{z_{jk}} F_W(v_{\underline{z}\chi} + \eta)\right). \quad (4.0.10)$$

By Proposition 5.1.1 and equations (5.1.12), (5.1.14) and (5.1.16), we have $||\Pi^{\perp}_{\underline{z}\chi}|| \leq c$ and $||\partial_{z_{jk}} \Pi^{\perp}_{\underline{z}\chi}|| \leq c$. Taylor expanding $F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W})$ about η and taking the $\mathbf{L}^2$ norm we have

$$\begin{aligned} ||F_W(v_{\underline{z}\chi} + \eta)||_{\mathbf{L}^2} &\leq ||F_W(v_{\underline{z}\chi}) + F'_W(v_{\underline{z}\chi})\eta + N_W(v_{\underline{z}\chi}, \eta)||_{\mathbf{L}^2} \\ &\leq ||F_W(v_{\underline{z}\chi})||_{\mathbf{L}^2} + C||\eta||_{\mathbf{H}^2} + c||\eta||^2_{\mathbf{H}^2} \quad (4.0.11) \\ &= O(\sqrt{\epsilon}) \end{aligned}$$

by $||\eta||_{\mathbf{L}^2} = O(\sqrt{\epsilon})$ from part (a) of Theorem 3.0.3, Corollary 3.0.2.1, the fact that F_W is C^1 (make note that F_W is C^s, where s is a natural number that signifies F_W has s *continuous* derivatives) and by equation (4.0.5).

We now move on to show that

$$||\partial_{z_{jk}} F_W(v_{\underline{z}\chi} + \eta)|| \leq c\sqrt{\epsilon}. \quad (4.0.12)$$

Using

$$F'_0(v_{\underline{z}\chi}) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} - F'_W(v_{\underline{z}\chi}) = 0,$$

since

$$F'_W = \mathcal{E}''_W(v_{\underline{z}\chi}) = \mathcal{E}''_0(v_{\underline{z}\chi}) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$$

(for explicit computations of the Hessian, $\mathcal{E}''_0(v_{\underline{z}\chi})$, see equation (C.0.1) from Appendix C) we obtain

$$\begin{aligned} ||\partial_{z_{jk}} F_W(v_{\underline{z}\chi} + \eta)|| &= ||F'_W(v_{\underline{z}\chi} + \eta)\partial_{z_{jk}} v_{\underline{z}\chi}||_{\mathbf{L}^2} \\ &= \left|\left| \left[F'_W(v_{\underline{z}\chi} + \eta) + F'_0(v_{\underline{z}\chi}) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} - F'_W(v_{\underline{z}\chi})\right] \partial_{z_{jk}} v_{\underline{z}\chi}\right|\right|_{\mathbf{L}^2} \\ &\leq || \left[F'_W(v_{\underline{z}\chi} + \eta) - F'_W(v_{\underline{z}\chi})\right] \partial_{z_{jk}} v_{\underline{z}\chi}||_{\mathbf{L}^2} + \left|\left| \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \partial_{z_{jk}} v_{\underline{z}\chi}\right|\right|_{\mathbf{L}^2} \\ &+ O\left(\epsilon \, log^{\frac{1}{2}} \left(\frac{1}{\epsilon}\right)\right), \end{aligned}$$

as we used part (d) of Theorem 3.0.2.
Furthermore we Taylor expand $F'_W(v_{\underline{z}\chi} + \eta)$ around η and rearrange it such that we have

$$F'_W(v_{\underline{z}\chi} + \eta) - F'_W(v_{\underline{z}\chi}) = F''_W(v_{\underline{z}\chi})\eta + O\left(||\eta||^2\right).$$

We are now able to write

$$\begin{aligned}
||\partial_{z_{jk}} F_W(v_{\underline{z}\chi} + \eta)|| &\leq || \left[F''_W(v_{\underline{z}\chi})\eta + O\left(||\eta||^2\right)\right] \partial_{z_{\underline{z}\chi}} v_{\underline{z}\chi}||_{\mathbf{L}^2} + || \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \partial_{z_{jk}} v_{\underline{z}\chi}||_{\mathbf{L}^2} \\
&+ O\left(\epsilon \, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right) \\
&\leq ||F''_W(v_{\underline{z}\chi})\eta \partial_{z_{jk}} v_{\underline{z}\chi}||_{\mathbf{L}^2} + ||O\left(||\eta||^2\right)||\partial_{z_{jk}} v_{\underline{z}\chi}||_{\infty} \\
&+ || \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \partial_{z_{jk}} v_{\underline{z}\chi}||_{\mathbf{L}^2} + O\left(\epsilon \, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right) \\
&= c\left(||\partial_{z_{jk}} v_{\underline{z}\chi}||_{\infty}\right) ||\eta||_{\mathbf{H}^2} + ||W||_{\mathbf{L}^2} \, ||\partial_{z_{jk}} v_{\underline{z}\chi}||_{\infty} + O\left(\epsilon \, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right) \\
&= O(\sqrt{\epsilon})
\end{aligned}$$

since F_W is C^2, $||\eta||_{\mathbf{H}^2} = O(\sqrt{\epsilon})$, $W = O(\sqrt{\epsilon})$ in $\mathbf{L}^2$, $||\partial_{z_{jk}} v_{\underline{z}\chi}||_{\infty} \leq \infty$ and $\epsilon \, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right) \ll \sqrt{\epsilon}$.

Thus from (4.0.9), (4.0.10), (4.0.11), (4.0.12) we arrive at

$$||\partial_{z_{jk}} \eta(\underline{z})||_{\mathbf{H}^2} \leq c\sqrt{\epsilon}$$

and so part (b) of Theorem 3.0.3 is proven (the cases for $2 \leq |\alpha| \leq 3$ are proven in a similar manner).

5. REDUCED ENERGY PROBLEM AND SOLUTION IN TANGENTIAL DIRECTION

In this section, we look to provide a proof for Theorem 3.0.4 and show that the critical points of the reduced energy are equivalent to those of the full energy. More precisely, if (3.0.4) holds, then the reduced energy $\Phi_W(\underline{z}) := \mathcal{E}_W(v_{\underline{z}\chi W})$ has a critical point at $\underline{z}_p$ if and only if $\mathcal{E}'_W(v_{\underline{z}_p\chi W}) = 0$, where, $v_{\underline{z}\chi W} := v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}$.

5.1 General Argument

Equation (3.0.4) implies that for any $(\underline{z}, \chi) \in \Sigma_\epsilon$, i.e., for widely spaced multi-vortex configurations,

$$\mathcal{E}'_W(v_{\underline{z}\chi W}) \in T_{v_{\underline{z}\chi}} M_{mv\epsilon}. \tag{5.1.1}$$

Since the full energy functional, $\mathcal{E}_W$, is gauge invariant and we can express the reduced energy in terms of $\mathcal{E}$, we have

$$0 = \partial_\chi \Phi_W(\underline{z}) = \partial_\chi \mathcal{E}_W(v_{\underline{z}\chi W}) = \langle \mathcal{E}'_W(v_{\underline{z}\chi W}), \partial_\chi v_{\underline{z}\chi W}\rangle. \tag{5.1.2}$$

We now claim that given (5.1.1) and (5.1.2), we have

$$\partial_{\underline{z}} \Phi_W(\underline{z})|_{\underline{z}=\underline{z}_p} = 0 \;\Leftrightarrow\; \mathcal{E}'_W(v_{\underline{z}\chi W})|_{\underline{z}=\underline{z}_p} = 0 \tag{5.1.3}$$

where $\partial_{\underline{z}} := (\partial_{z_{11}}, \partial_{z_{12}}, \partial_{z_{21}}, \partial_{z_{22}}, \ldots, \partial_{z_{m1}}, \partial_{z_{m2}})$. Make note that our claim (5.1.3) is equivalent to that of Theorem 3.0.4. Demonstrating the $\Leftarrow$ part of (5.1.3) requires little effort. If $\mathcal{E}'_W(v_{\underline{z}\chi W})|_{\underline{z}=\underline{z}_p} = 0$ then

$$\begin{aligned}
\partial_{\underline{z}} \Phi_W(\underline{z})|_{\underline{z}=\underline{z}_p} &= \partial_{\underline{z}} \mathcal{E}_W(v_{\underline{z}\chi W})|_{\underline{z}=\underline{z}_p} \\
&= \langle \partial_{\underline{z}} v_{\underline{z}\chi W}, \mathcal{E}'_W(v_{\underline{z}\chi W})\rangle|_{\underline{z}=\underline{z}_p} \\
&= \langle \partial_{\underline{z}} v_{\underline{z}\chi W}|_{\underline{z}=\underline{z}_p}, \mathcal{E}'_W(v_{\underline{z}\chi W})|_{\underline{z}=\underline{z}_p}\rangle \\
&= 0.
\end{aligned}$$

Now it remains to show the $\Rightarrow$ part of (5.1.3) . Making use of

$$\langle \partial_{\underline{z}} v_{\underline{z}\chi W}, \mathcal{E}'_W(v_{\underline{z}\chi W})\rangle|_{\underline{z}=\underline{z}_p} = \partial_{\underline{z}} \Phi_W(\underline{z})|_{\underline{z}=\underline{z}_p} = 0$$

along with (5.1.2) implies

$$\mathcal{E}'_W(v_{\underline{z}\chi W}) \perp T_{v_{\underline{z}\chi W}} M^W_{mv\epsilon} \tag{5.1.4}$$

where

$$M^W_{mv\epsilon} = \{v_{\underline{z}\chi W} = v_{\underline{z}\chi} + \eta_{\underline{z}\chi W} \mid (\underline{z}, \chi) \in \Sigma_\epsilon\}. \tag{5.1.5}$$

We will implement a geometrical argument to show $\mathcal{E}'_W(v_{\underline{z}\chi W}) = 0$. A more rigorous proof will follow the geometrical argument.

Equation (5.1.1) says that $\mathcal{E}'_W(v_{\underline{z}\chi W})$ lies within the tangent space $T_{v_{\underline{z}\chi}} M_{mv\epsilon}$ whereas (5.1.4) states that $\mathcal{E}'_W(v_{\underline{z}\chi W})$ is orthogonal to $T_{v_{\underline{z}\chi}} M^W_{mv\epsilon}$. It should be noted that both manifolds $M_{mv\epsilon}$ and $M^W_{mv\epsilon}$ are very close to each other and separated by a distance $||\eta_{\underline{z}\chi W}||$. If we reduce the inter-manifold spacing to zero then $M^W_{mv\epsilon}$ approaches $M_{mv\epsilon}$. Therefore we can say that at $\underline{z} = \underline{z}_p$, we have $\mathcal{E}'_W(v_{\underline{z}_p\chi W}) \in T_{v_{\underline{z}_p\chi}} M_{mv\epsilon}$ and $\mathcal{E}'_W(v_{\underline{z}_p\chi W}) \perp T_{v_{\underline{z}_p\chi W}} M_{mv\epsilon}$. So, the only way for $\mathcal{E}'_W(v_{\underline{z}_p\chi W})$ to be both within and orthogonal to the tangent spaces to the manifolds is to be zero.

Here we will rigorously prove that $\mathcal{E}'_W(v_{\underline{z}_p\chi W}) = 0$. First off we define

$$\begin{aligned} f_W &= \mathcal{E}'_W(v_{\underline{z}_p\chi W}) \\ \Pi &= \mathbf{L}^2 - \textit{orthogonal projection onto } T_{v_{\underline{z}_p\chi}} M_{mv\epsilon} \\ \Pi_W &= \mathbf{L}^2 - \textit{orthogonal projection onto } T_{v_{\underline{z}_p\chi W}} M^W_{mv\epsilon}. \end{aligned} \tag{5.1.6}$$

Note that we are working with the orthogonal projections Π and Π_W as they aid us in showing $f_W = 0$.

Now (5.1.1) and (5.1.4) can be written as $\Pi f_W = f_W$ and $\Pi_W f_W = 0$ and thus we can write

$$f_W = \Pi f_W = (\Pi - \Pi_W) f_W. \tag{5.1.7}$$

By Proposition 5.1.1 below, we have

$$\begin{aligned} ||f_W|| &\leq ||\Pi - \Pi_W||\, ||f_W|| \\ &\leq c\sqrt{\epsilon}||f_W||, \end{aligned}$$

which implies that $f_W = 0$. This completes the proof of the sufficient part of (5.1.3). Now we establish Proposition 5.1.1 along with its proof.

Proposition 5.1.1.

a) Π *and* Π_W *are bounded*

b) $||\Pi - \Pi_W|| = O(\sqrt{\epsilon})$

Proof. The proof for Proposition 5.1.1 follows from equation (5.1.12) through (5.1.17) along with Lemmas 5.2.3, 5.2.4 and 5.3.1 □

The remainder of this section is devoted to proving Proposition 5.1.1.

Write

$$v_{\underline{z}\chi W} = v_{\underline{z}\chi} + \eta_{\underline{z}\chi W} := \begin{pmatrix} \psi_{\underline{z}\chi W} \\ \vec{A}_{\underline{z}\chi W} \end{pmatrix}$$

and consider the bases of $T_{v_{\underline{z}\chi}} M_{mv\epsilon}$ and $T_{v_{\underline{z}\chi W}} M^W_{mv\epsilon}$. Recall that $T^{\underline{z}\chi W}_{jk} = \partial_{z_{jk}} v_{\underline{z}\chi W}$ and $T^{\underline{z}\chi}_{jk} = \partial_{z_{jk}} v_{\underline{z}\chi}$ are the $2m$ almost zero translational tangent vectors at $v_{\underline{z}\chi W}$ and $v_{\underline{z}\chi}$. We define $T^{\underline{z}\chi}_r$ and $T^{\underline{z}\chi W}_r$ for $r = 1, 2, \ldots, 2m$ by

$$\bigoplus_{r=1}^{2m} T^{\underline{z}\chi}_r = \bigoplus_{j=1}^{m} \bigoplus_{k=1}^{2} T^{\underline{z}\chi}_{jk} = \bigoplus_{j=1}^{m} \bigoplus_{k=1}^{2} \partial_{z_{jk}} v_{\underline{z}\chi} \tag{5.1.8}$$

$$\bigoplus_{r=1}^{2m} T^{\underline{z}\chi W}_r = \bigoplus_{j=1}^{m} \bigoplus_{k=1}^{2} T^{\underline{z}\chi W}_{jk}, \tag{5.1.9}$$

where $\bigoplus$ denotes the direct sum of two Hilbert Spaces.

For a basis for the space spanned by the almost zero gauge tangent vectors, we write

$$G^{\underline{z}\chi}_{\delta_x} = \begin{pmatrix} i\delta_x \psi_{\underline{z}\chi} \\ \vec{\nabla}\delta_x \end{pmatrix} \tag{5.1.10}$$

$$G^{\underline{z}\chi W}_{\delta_x} = \begin{pmatrix} i\delta_x \psi_{\underline{z}\chi W} \\ \vec{\nabla}\delta_x \end{pmatrix}, \tag{5.1.11}$$

where δ_x denotes the delta function at x. Note that this basis is uncountable. We define the projections,

$$\Pi = \Pi^t + \Pi^g \tag{5.1.12}$$

$$\Pi_W = \Pi^t_W + \Pi^g_W \tag{5.1.13}$$

where, recall, Π and Π_W are defined in equation (5.1.7) and

- ★ $\Pi^t = \mathbf{L}^2$ - orthogonal projection onto $span\{T^{\underline{z}\chi}_r \mid r = 1, 2, \ldots, 2m\}$;
- ★ $\Pi^g = \mathbf{L}^2$ - orthogonal projection onto $span\{G^{\underline{z}\chi}_\gamma \mid \gamma \in H^2(\mathbb{R}^2 : \mathbb{R})\}$;
- ★ $\Pi^t_W = \mathbf{L}^2$ - orthogonal projection onto $span\{T^{\underline{z}\chi W}_r \mid r = 1, 2, \ldots, 2m\}$;
- ★ $\Pi^g_W = \mathbf{L}^2$ - orthogonal projection onto $span\{G^{\underline{z}\chi W}_\gamma \mid \gamma \in H^2(\mathbb{R}^2 : \mathbb{R})\}$.

More explicitly

$$\Pi^t = \sum_{r=1}^{2m}\sum_{s=1}^{2m} |T_r^{\underline{z}\chi}\rangle[\beta^{-1}]_{rs}\langle T_s^{\underline{z}\chi}|, \tag{5.1.14}$$

$$\Pi_W^t = \sum_{r=1}^{2m}\sum_{s=1}^{2m} |T_r^{\underline{z}\chi W}\rangle[\beta_W^{-1}]_{rs}\langle T_s^{\underline{z}\chi W}|, \tag{5.1.15}$$

$$\Pi^g = \int\int |G_{\delta_x}^{\underline{z}\chi}\rangle(-\Delta + |\psi_{\underline{z}\chi}|^2)^{-1}\langle G_{\delta_y}^{\underline{z}\chi}|dxdy, \tag{5.1.16}$$

$$\Pi_W^g = \int\int |G_{\delta_x}^{\underline{z}\chi W}\rangle(-\Delta + |\psi_{\underline{z}\chi W}|^2)^{-1}\langle G_{\delta_y}^{\underline{z}\chi W}|dxdy. \tag{5.1.17}$$

The terms β and β_W are $2m \times 2m$ matrices with elements

$$[\beta]_{rs} = \langle T_r^{\underline{z}\chi}|T_s^{\underline{z}\chi}\rangle = \gamma_{(n_r)}\delta_{rs} + O\left(\epsilon\, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right), \tag{5.1.18}$$

$$[\beta_W]_{rs} = \langle T_r^{\underline{z}\chi W}|T_s^{\underline{z}\chi W}\rangle \tag{5.1.19}$$

where $\gamma_{(n_i)}$ is defined in equation (3.0.6).
Since $T_r^{\underline{z}\chi W} = T_r^{\underline{z}\chi} + \partial_{z_r}\eta_{\underline{z}\chi W}$ where (z_r corresponds to a unique z_{jk} as the correspondence $T_{jk}^{\underline{z}\chi} \Leftrightarrow T_r^{\underline{z}\chi}$) we have $[\beta^W]_{rs} = [\beta]_{rs} + [\nu]_{rs}$ where $[\nu]_{rs} = \langle T_r^{\underline{z}\chi}, \partial_{z_s}\eta\rangle + \langle \partial_{z_r}\eta, T_s^{\underline{z}\chi}\rangle + \langle \partial_{z_r}\eta, \partial_{z_s}\eta\rangle$.
By Theorem 3.0.2 b we have $||\nu|| = O(\sqrt{\epsilon})$, and by Lemma 6.0.2 we can assess the invertibility of β_W (as a sum of an invertible matrix and a small matrix).

We compute the inner products of the almost zero gauge tangent vectors using (5.1.10) and (5.1.11) to arrive at

$$\langle G_{\delta_x}^{\underline{z}\chi}|G_{\delta_y}^{\underline{z}\chi}\rangle = (-\Delta + |\psi_{\underline{z}\chi}|^2)(x,y)$$

$$\langle G_{\delta_x}^{\underline{z}\chi W}|G_{\delta_y}^{\underline{z}\chi W}\rangle = (-\Delta + |\psi_{\underline{z}\chi W}|^2)(x,y).$$

5.2 Boundedness of Π^g and Smallness of $||\Pi_W^g - \Pi^g||$

In this section we intend to demonstrate the boundedness of the operator Π^g and the order of $\Pi_W^g - \Pi^g$. The following proofs follow the method outlined in [ST] where G_{δ_x} and $G_{\delta_x}^{\epsilon}$ are replaced with G_{δ_x} and $G_{\delta_x}^W$. It should be noted that there is a 1-1 correspondence between the operators J on $H^s(\mathbb{R}^2)$ and Π_J^g defined as

$$\Pi_J^g = \int\int |G_{\delta_x}\rangle J(x,y)\langle G_{\delta_y}|dxdy \tag{5.2.1}$$

where $J(x,y)$ is the integral kernel of an operator J. In addition, this correspondence is linear, such that $\Pi_A + \Pi_B = \Pi_{A+B}$ where Π_A and Π_B represent the projection operators (5.1.14), (5.1.15), (5.1.16) and (5.1.17). From the projection operators (5.1.16) and (5.1.17), we write the operator J as $J = (-\Delta_x + |\psi|^2)^{-1}$.

Lemma 5.2.1. *If* $f = \begin{pmatrix} \xi \\ \vec{B} \end{pmatrix}$ *then*

$$\Pi_J^g f = \begin{pmatrix} i\psi J \left[Im(\bar{\psi}\xi) - \vec{\nabla} \cdot \vec{B} \right] \\ \vec{\nabla}[J \ Im(\bar{\psi}\xi) - \vec{\nabla} \cdot \vec{B}] \end{pmatrix}. \tag{5.2.2}$$

Proof.

$$\Pi_J^g f = \int\int |G_{\delta_x}\rangle J(x,y) \langle G_{\delta_y}|f\rangle dxdy$$

$$\begin{aligned} \langle G_{\delta_y}|f\rangle &= \int (i\delta_y \psi_{\underline{z}\chi}(z), \vec{\nabla}\delta_y) \begin{pmatrix} \xi(z) \\ \vec{B}(z) \end{pmatrix} dz \\ &= \int Re[-i\delta(z-y)\overline{\psi}_{\underline{z}\chi}(z)\xi(z)] - \vec{\nabla}\delta(z-y) \cdot \vec{B}(z) dz \end{aligned}$$

following integration by parts we are left with

$$\langle G_{\delta_y}|f\rangle = Im[\overline{\psi}_{\underline{z}\chi}(y)\xi(y)] - \vec{\nabla} \cdot \vec{B}(y).$$

$$\begin{aligned} (\Pi_J^g f)(z) &= \int\int \begin{pmatrix} i\delta_x \psi_{\underline{z}\chi} \\ \vec{\nabla}\delta_x \end{pmatrix} J(x,y) \left(Im[\overline{\psi}_{\underline{z}\chi}(y)\xi(y) - \vec{\nabla} \cdot \vec{B}(y)] \right) dxdy \\ &= \int\int \begin{pmatrix} i\delta(z-x)\psi_{\underline{z}\chi} J(x,y) \left(Im[\overline{\psi}_{\underline{z}\chi}(y)\xi(y) - \vec{\nabla} \cdot \vec{B}(y)] \right) \\ \vec{\nabla}\delta(z-x) J(x,y) \left(Im[\overline{\psi}_{\underline{z}\chi}(y)\xi(y)] - \vec{\nabla} \cdot \vec{B}(y) \right) \end{pmatrix} dxdy \\ &= \begin{pmatrix} i\psi_{\underline{z}\chi}(z)[J \ Im(\overline{\psi}_{\underline{z}\chi}(z)\xi(z)) - \vec{\nabla} \cdot \vec{B}(z)] \\ \vec{\nabla}\delta[J \ Im(\overline{\psi}_{\underline{z}\chi}(z)\xi(z)) - \vec{\nabla} \cdot \vec{B}(z)] \end{pmatrix} \end{aligned}$$

□

Lemma 5.2.2. *Let* $J : H^{s-1} \to H^{s+1}$ *be a bounded operator. Then the operator* $\Pi_J^g : H^s \to H^s$ *is bounded with*

$$||\Pi_J^g||_{H^s \to H^s} \leq C||J||_{H^{s-1} \to H^{s+1}} \tag{5.2.3}$$

Proof. The proof follows from equation (5.2.2), $||\partial_x^{|\alpha|}\psi_{\underline{z}\chi}||_\infty < \infty$ for $|\alpha| \leq s$ and the boundedness of the operator J for H^{s-1} to H^{s+1}. □

Lemma 5.2.3. *Fix $\underline{z} \in \mathbb{R}^{2m}$, $\chi \in H^1$, while the operators $A := -\Delta + |\psi_{\underline{z}\chi}|^2$ and $A_W := -\Delta + |\psi_{\underline{z}\chi W}|^2 : H^1 \to H^{-1}$ are invertible. Operators $J = A^{-1}$ and $J_W = A_W^{-1}$ satisfy the conditions of Lemma 5.2.2 with $s = 0$. Therefore, the operators $\Pi^g = \Pi^g_{A^{-1}}$ and $\Pi^g_W = \Pi^g_{A_W^{-1}}$ are bounded in $\mathbf{L}^2$ by $C||A_W^{-1}||_{H^1 \to H^{-1}}$.*

Proof. The proof for this lemma is sufficiently difficult, the interested reader can refer to Lemma 5.2 of [ST]. □

Lemma 5.2.4. $||\Pi^g_W - \Pi^g|| = O(\sqrt{\epsilon})$

Proof. Following the convention of Lemma 5.2.1 we have

$$\begin{aligned}(\Pi^g_W - \Pi^g)f &= \int\int |G^W_{\delta_x}\rangle J_W(x,y)\langle G^W_{\delta_y}|f\rangle dxdy \\ &- \int\int |G_{\delta_x}\rangle J(x,y)\langle G_{\delta_y}|f\rangle dxdy \end{aligned} \tag{5.2.4}$$

We adjust (5.2.4) such that

$$\begin{aligned}((\Pi^g_W - \Pi^g)f)(z) &= \int\int |G^W_{\delta_x} - G_{\delta_x}\rangle J_W(x,y)\langle G^W_{\delta_y}|f\rangle dxdy \\ &+ \int\int |G^W_{\delta_x}\rangle (J_W - J)(x,y)\langle G^W_{\delta_y}|f\rangle dxdy \\ &+ \int\int |G_{\delta_x}\rangle J(x,y)\langle G^W_{\delta_y} - G_{\delta_y}|f\rangle dxdy \\ &= \begin{pmatrix} i(\psi_W(z) - \psi(z))J_W[Im(\overline{\psi_W(z)}\xi(z)) + \vec{\nabla}\cdot\vec{B}] \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} i\psi(J_W(|\psi|^2 - |\psi_W|^2)J)[Im(\overline{\psi_W(z)}\xi(z)) + \vec{\nabla}\cdot\vec{B}(z)] \\ \vec{\nabla}(J_W(|\psi|^2 - |\psi_W|^2)J)[Im(\overline{\psi_W(z)}\xi(z)) + \vec{\nabla}\cdot\vec{B}(z)] \end{pmatrix} \\ &+ \begin{pmatrix} i\psi J[Im((\overline{\psi_W(z) - \psi(z)})\xi(z))] \\ \vec{\nabla}J[Im((\overline{\psi_W(z) - \psi(z)})\xi(z))] \end{pmatrix}\end{aligned}$$

Noting that both $|\psi(z)|$ and $|\psi_W(z)|$ have an upper bound of 1 and Theorem 3.0.3 (a) we have $|||\psi_W(z)|^2 - |\psi(z)|^2||_\infty \leq 2||\psi_W(z) - \psi(z)||_\infty \leq 2||\eta||_\infty \leq 2||\eta||_{\mathbf{H}^2} \leq c\sqrt{\epsilon}$. We make use of the bounds on J and J_W from Lemmas 5.2.2 and 5.2.3. From here it follows that $||(\Pi^g_W - \Pi^g)f|| \leq c\sqrt{\epsilon}||f||$. □

5.3 Boundedness of Π^t and Smallness of $||\Pi^t_W - \Pi^t||$

Lemma 5.3.1. *The operators Π^t_W and Π^t are bounded and the difference between operators Π^t and Π^t_W is such that $||\Pi^t_W - \Pi^t|| = O(\sqrt{\epsilon})$.*

Proof. Noting that Π^t and Π^t_W are $2m \times 2m$ matrices with finite entries given by (5.1.18) and (5.1.19) which are indeed bounded. Since $T^W_j = T_j + \partial_j \eta$, we have by equation (5.1.15),

$$\begin{aligned}
\Pi^t_W - \Pi^t &= \sum_{ij} |T^W_i\rangle[\beta^{-1}_W]_{ij}\langle T^W_j| - \sum_{ij} |T_i\rangle[\beta^{-1}]_{ij}\langle T_j| \\
&= \sum_{ij}\left[(|T_i\rangle + |\partial_i\eta\rangle)[\beta^{-1}_W]_{ij}(\langle T_j| + \langle\partial_j\eta|) - \sum_{ij}|T_i\rangle[\beta^{-1}]_{ij}\langle T_j|\right] \\
&= \sum_{ij}[|T_i\rangle[\beta^{-1}_W]_{ij}\langle T_j| + |T_i\rangle[\beta^{-1}_W]_{ij}\langle\partial_j\eta| + |\partial_i\eta\rangle[\beta^{-1}_W]_{ij}\langle T_j| \\
&+ |\partial_i\eta\rangle[\beta^{-1}_W]_{ij}\langle\partial_j\eta|] - \sum_{ij}|T_i\rangle[\beta^{-1}]_{ij}\langle T_j| \\
&= \sum_{ij}[|T_i\rangle[\beta^{-1}_W - \beta^{-1}]_{ij}\langle T_j| + |T_i\rangle[\beta^{-1}_W]_{ij}\langle\partial_j\eta| + |\partial_i\eta\rangle[\beta^{-1}_W]_{ij}\langle T_j| \\
&+ |\partial_i\eta\rangle[\beta^{-1}_W]_{ij}\langle\partial_j\eta|] \\
&= \sum_{ij}[|T_i\rangle[\beta^{-1} - \beta^{-1}_W]_{ij}\langle T_j| - |T_i\rangle[\beta^{-1}_W]_{ij}\langle\partial_j\eta| - |\partial_i\eta\rangle[\beta^{-1}_W]_{ij}\langle T_j| \\
&- |\partial_i\eta\rangle[\beta^{-1}_W]_{ij}\langle\partial_j\eta|]. \qquad (5.3.1)
\end{aligned}$$

We now show that each term in equation (5.3.1) is of the order, $O(\sqrt{\epsilon})$. By condition a) from Theorem 3.0.3 we have $||\partial_{z_i}\eta|| = O(\sqrt{\epsilon})$. Now we demonstrate that $||(\beta^{-1} - \beta^{-1}_W)_{ij}|| = O(\sqrt{\epsilon})$. For convenience rewrite the difference of $\beta^{-1} - \beta^{-1}_W$ in the following manner:

$$||\beta^{-1} - \beta^{-1}_W|| = ||\beta^{-1}_W(\beta - \beta_W)\beta^{-1}||.$$

Recall that $(\beta_W)_{ij} = \gamma_{(n_i)}\delta_{ij} + O\left(\epsilon\, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right) + (\nu)_{ij}$ while $(\beta)_{ij} = \gamma_{(n_i)}\delta_{ij} + O\left(\epsilon\, log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right)$. Subtracting $\beta - \beta_W$ leaves $(\nu)_{ij}$ and by Theorem (3.0.3), $||\nu|| = O(\sqrt{\epsilon})$. Noting that β is invertible, we can apply Lemma (6.0.2) and have β_W being invertible as well. Now, β and β_W are $2m \times 2m$ matrices. For sufficiently small values of $\epsilon > 0$, β and β_W are essentially diagonal matrices, in the sense that they have constants along the diagonal and almost zero off-diagonal elements (see equations (5.1.18) and (5.1.19)). Therefore, we can say β^{-1} and β^{-1}_W are bounded by some constant C. This results with

$$\begin{aligned}
||\beta^{-1} - \beta^{-1}_W|| &= ||\beta^{-1}_W(\beta - \beta_W)\beta^{-1}|| \\
&\leq ||\beta^{-1}||O(\sqrt{\epsilon})||\beta^{-1}_W||| \\
&= O(\sqrt{\epsilon})
\end{aligned}$$

□

6. CRITICAL POINT OF REDUCED ENERGY

In this section, we prove Theorem 3.0.5 and analyze the relationship between the critical points of W and the reduced energy $\Phi_W(\underline{z})$.

Define $v_{z_j\chi} = \begin{pmatrix} e^{i\chi(x)}\psi^{(n_j)}(x-z_j) \\ \vec{A}^{(n_j)}(x-z_j)+\vec{\nabla}\chi \end{pmatrix}$ and $E^{(n_j)} = \mathcal{E}_0(v_{z_j\chi})$ is the energy of a lone and independent vortex $(\psi^{(n_j)}, \vec{A}^{(n_j)})$. The *inter-vortex/interaction energy* and *external potential energy* are defined as

$$V_{int}(\underline{z}) := \mathcal{E}_0(v_{\underline{z}\chi}) - \sum_{j=1}^{m} E^{(n_j)} \tag{6.0.1}$$

and

$$W_{ext,full}(\underline{z}) := \mathcal{E}_W(v_{\underline{z}\chi}) - \mathcal{E}_0(v_{\underline{z}\chi}) = \frac{1}{2}\int_{\mathbb{R}^2} W(x)(|\psi_{\underline{z}\chi}|^2-1)dx, \tag{6.0.2}$$

respectively. We denote $\vec{\nabla}_{z_l}V_{int}(\underline{z})$ and $\vec{\nabla}_{z_l}W_{ext,full}(\underline{z})$ as the *inter-vortex/interaction force* and *external potential force*.

For a sufficiently small $\epsilon > 0$, $(\underline{z}, \chi) \in \Sigma_\epsilon$ and W satisfying the conditions (A) and (B) (see Section 2), we will show that the inter-vortex/interaction forces are in fact *weaker* than its respective external counterpart. This property allows for the pinning phenomena.

We begin with the following proposition which is integral to this thesis.

Proposition 6.0.1. *For $(\underline{z}, \chi) \in \Sigma_\epsilon$, the reduced energy $\Phi_W(z)$ can be written as,*

(a)

$$\Phi_W(\underline{z}) = C + V_{int}(\underline{z}) + W_{ext}(\underline{z}) + W_{ext,Rem} + R_W(\underline{z}) \tag{6.0.3}$$

where $V_{int}(\underline{z})$ is defined by equation (6.0.1);

$$C := \sum_{j=1}^{m} E^{(n_j)} \text{ and } E^{(n_j)} = \mathcal{E}_0(v_{z_j\chi}) \text{ are constant,} \tag{6.0.4}$$

$$W_{ext}(\underline{z}) := \sum_{j=1}^{m} W_{ext,j}(z_j) \tag{6.0.5}$$

and

$$\begin{aligned}
W_{ext,j}(z_j) &:= \frac{1}{2}\int_{\mathbb{R}^2} W(x)(|\psi^{(n_j)}(x-z_j)|^2-1)dx, && (6.0.6)\\
W_{ext,Rem}(\underline{z}) &= \frac{1}{2}\int_{\mathbb{R}^2}\sum_{j\neq k} W(x)(f_j^2-1)(f_k^2-1) && (6.0.7)\\
&+ \frac{1}{2}\int_{\mathbb{R}^2}\sum_{j\neq k\neq l} W(x)(f_j^2-1)(f_k^2-1)(f_l^2-1)+\dots\\
R_W(\underline{z}) &:= \mathcal{E}_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi W})-\mathcal{E}_W(v_{\underline{z}\chi}). && (6.0.8)
\end{aligned}$$

(b)

$$\begin{aligned}
\partial_{\underline{z}}^{|\alpha|}V_{int}(\underline{z}) &= O(\epsilon)\ for\ 0\le|\alpha|\le 1 && (6.0.9)\\
&= O\left(\epsilon\ log^{\frac{1}{2}}(\frac{1}{\epsilon})\right)\ for\ 2\le|\alpha|\le 3 && (6.0.10)\\
\partial_{\underline{z}}^{|\alpha|}W_{ext}(\underline{z}) &= O\left(\sqrt{\epsilon}\delta^{|\alpha|+1}\right) && (6.0.11)\\
\partial_{\underline{z}}^{|\alpha|}W_{ext,Rem}(\underline{z}) &= O\left(\sqrt{\epsilon}\delta^{|\alpha|+1}\left[\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right]\right)\\
&= O\left(\sqrt{\epsilon}\delta^{|\alpha|+1}\epsilon\right)=o(\epsilon) && (6.0.12)\\
\partial_{\underline{z}}^{|\alpha|}R_W(\underline{z}) &= O(\epsilon) && (6.0.13)
\end{aligned}$$

Here, $f_j = |\psi^{(n_j)}|$, and the ϵ and δ relationship in (6.0.11) and (6.0.12) comes from condition (B) in Theorem 2.2.1. The first two terms in (6.0.3) come from the potential due to intervortex forces while the potential due to external force is responsible for the latter three.

Proof. For part (a), we have by definition of $\Phi_W(\underline{z})$ and Taylor expansion of $\mathcal{E}_W(v_{\underline{z}\chi W}+\eta_{\underline{z}\chi W})$,

$$\Phi_W(\underline{z}) = \mathcal{E}_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi W}) = \mathcal{E}_W(v_{\underline{z}\chi})+R_W(\underline{z}),$$

where $R_W(\underline{z})$ is defined by this relation, and hence we have (6.0.8).
However

$$\begin{aligned}
\mathcal{E}_W(v_{\underline{z}\chi}) &= \mathcal{E}_0(v_{\underline{z}\chi})+\frac{1}{2}\int_{\mathbb{R}^2}W(x)(|\psi_{\underline{z}\chi}|^2-1)dx\\
&= C+V_{int}(\underline{z})+\frac{1}{2}\int_{\mathbb{R}^2}W(x)(|\psi_{\underline{z}\chi}|^2-1)dx
\end{aligned}$$

by definition of $V_{int}(\underline{z})$ in (6.0.1). From

$$\left(\prod_{j=1}^{m} f_j^2\right) - 1 = \sum_{j=1}^{m}(f_j^2 - 1) + \sum_{j\neq l}^{m}(f_j^2 - 1)(f_l^2 - 1) + \dots$$

where we use the notation $\sum_{j\neq l}$ to denote $\sum_{j=1}^{m}\sum_{l\neq j}^{m}$, we can write

$$\frac{1}{2}\int_{\mathbb{R}^2} W(x)(|\psi_{\underline{z}\chi}(x)|^2 - 1)dx = W_{ext}(\underline{z}) + W_{ext,Rem}(\underline{z})$$

which shows (6.0.6) and (6.0.7). It should be noted that $W_{ext,full}(\underline{z}) = W_{ext}(\underline{z}) + W_{ext,Rem}(\underline{z})$, where $W_{ext}(\underline{z})$ is labeled as the *effective external potential.*

Now to prove part (b). Equation (6.0.9) with $|\alpha| = 0, 1$ follows from the asymptotic expressions

$$\begin{aligned} V_{int}(\underline{z}) &= \frac{1}{2}\sum_{l\neq k} c_l n_l n_k \frac{e^{-|z_l - z_k|}}{\sqrt{|z_l - z_k|}} \int_{\mathbb{R}^2} e^{x\cdot(z_l - z_k)/|z_l - z_k|} \\ &\times \frac{1}{r}[2(1-a)ff' + a(1-f^2)]dx \\ &+ o\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right),\ as\ R(\underline{z}) \to \infty \end{aligned} \tag{6.0.14}$$

and

$$\vec{\nabla}_{z_l} V_{int}(\underline{z}) = \sum_{j\neq l} n_j n_l c_{jl} \frac{e^{-|z_j - z_j|}}{\sqrt{|z_j - z_l|}} \frac{z_j - z_l}{|z_j - z_l|} + o\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right)$$

as $R(\underline{z}) \to \infty$ and $c_{jl} > 0$ is a constant. Recall here that n_j is the degree of the j-th vortex and f is the profile of a degree ± 1 vortex (see equation (1.1.6)). These expressions have been confirmed by Lemmas 7 and 11 in [GS2].

In order to demonstrate (6.0.10) for $|\alpha| = 2$, we use (6.0.1) to arrive at

$$\begin{aligned} \partial_{z_{jk}} V_{int}(\underline{z}) &= \partial_{z_{jk}}\left(\mathcal{E}_0(v_{\underline{z}\chi}) - \sum_{j=1}^{m} E^{(n_j)}\right) \\ &= \langle \mathcal{E}_0'(v_{\underline{z}\chi}), T_{jk}^{\underline{z}\chi}\rangle. \end{aligned}$$

Going one step further we obtain

$$\begin{aligned} |\partial_{z_{pq}}\partial_{z_{jk}} V_{int}(\underline{z})| &= |\langle \mathcal{E}_0''(v_{\underline{z}\chi})T_{pq}^{\underline{z}\chi}, T_{jk}^{\underline{z}\chi}\rangle + \langle \mathcal{E}_0'(v_{\underline{z}\chi}), \partial_{z_{pq}} T_{jk}^{\underline{z}\chi}\rangle| \\ &\leq ||\mathcal{E}_0''(v_{\underline{z}\chi})T_{pq}^{\underline{z}\chi}||_{L^2}||T_{jk}^{\underline{z}\chi}||_{L^2} + ||\mathcal{E}_0'(v_{\underline{z}\chi})||_{L^2}||\partial_{z_{pq}} T_{jk}^{\underline{z}\chi}||_{L^2} \\ &\leq c\epsilon\ log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right) + c\epsilon\ log^{\frac{1}{4}}\left(\frac{1}{\epsilon}\right) \\ &= O\left(\epsilon\ log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right) \end{aligned}$$

by Theorem 3.0.2 (a), (d) and since $||T_{jk}^{\underline{z}\chi}||_{L^2}$ along with $||\partial_{z_{pq}} T_{jk}^{\underline{z}\chi}||_{L^2} < \infty$. The same procedure is repeated for $|\alpha| = 3$.

For (6.0.13) with $|\alpha| = 1$, we use $R_W(\underline{z}) = \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) - \mathcal{E}_W(v_{\underline{z}\chi})$ and have

$$\begin{aligned} \partial_{\underline{z}} R_W(\underline{z}) &= \langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} v_{\underline{z}\chi} + \partial_{\underline{z}} \eta_{\underline{z}\chi W} \rangle - \langle \mathcal{E}'_W(v_{\underline{z}\chi}), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle \\ &= \langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle + \langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} \eta_{\underline{z}\chi W} \rangle \\ &- \langle \mathcal{E}'_W(v_{\underline{z}\chi}), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle \end{aligned}$$

Now apply Taylor expansion to $\langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} v_{\underline{z}\chi} + \partial_{\underline{z}} \eta_{\underline{z}\chi W} \rangle$.

$$\begin{aligned} \partial_{\underline{z}} R_W(\underline{z}) &= \langle \mathcal{E}'_W(v_{\underline{z}\chi}) + \mathcal{E}''_W(v_{\underline{z}\chi})_W \eta_{\underline{z}\chi W} + O(||\eta_{\underline{z}\chi W}||^2), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle \\ &+ \langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} \eta_{\underline{z}\chi W} \rangle - \langle \mathcal{E}'_W(v_{\underline{z}\chi}), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle \\ &= \langle \mathcal{E}''_W(v_{\underline{z}\chi}) \eta_{\underline{z}\chi W}, \partial_{\underline{z}} v_{\underline{z}\chi} \rangle + \langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} \eta_{\underline{z}\chi W} \rangle \\ &+ \langle O(||\eta_{\underline{z}\chi W}||^2), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle \end{aligned}$$

Making use of the self-adjointness of $\mathcal{E}''_W(v_{\underline{z}\chi})$,

$$\begin{aligned} \langle \mathcal{E}''_W(v_{\underline{z}\chi}) \eta_{\underline{z}\chi W}, \partial_{\underline{z}} v_{\underline{z}\chi} \rangle &= \langle \eta_{\underline{z}\chi W}, \mathcal{E}''_W(v_{\underline{z}\chi}) \partial_{\underline{z}} v_{\underline{z}\chi} \rangle \\ &= O(\sqrt{\epsilon}) \cdot O(\sqrt{\epsilon}) \end{aligned}$$

as $\eta_{\underline{z}\chi W} = O(\sqrt{\epsilon})$ by Theorem 3.0.3 part (a) and $\mathcal{E}''_W(v_{\underline{z}\chi}) \partial_{\underline{z}} v_{\underline{z}\chi} = O(\sqrt{\epsilon})$ by Lemma 3.0.2 (d), $W = O(\sqrt{\epsilon})$ by condition (A) along with $\mathcal{E}''_W(v_{\underline{z}\chi}) = \mathcal{E}''_0(v_{\underline{z}\chi}) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$.

The term $\langle O(||\eta_{\underline{z}\chi W}||^2), \partial_{\underline{z}} v_{\underline{z}\chi} \rangle$ is of the order $O(\sqrt{\epsilon}) \cdot O(\sqrt{\epsilon})$ by Theorem 3.0.3 a) while $\langle \mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}), \partial_{\underline{z}} \eta_{\underline{z}\chi W} \rangle = O(\sqrt{\epsilon}) \cdot O(\sqrt{\epsilon})$ as $\mathcal{E}'_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi W}) = O(\sqrt{\epsilon})$ and by (4.0.11) along with $||\partial_{\underline{z}} \eta|| = O(\sqrt{\epsilon})$. Therefore $\partial_{\underline{z}} R_W(\underline{z}) = O(\epsilon)$.

6.0.1 Effective External Potential

Here, we will derive some estimates on the functional $W_{ext} : \mathbb{R}^{2m} \to \mathbb{R}$:

$$W_{ext}(\underline{z}) = \frac{1}{2} \sum_{j=1}^{m} \int_{\mathbb{R}^2} W(x) \left(|\psi^{(n_j)}(x - z_j)|^2 - 1 \right) dx$$

which we call the effective external potential where $\underline{z} = (z_1, z_2, \ldots, z_m)$. Now in the next section, we will demonstrate the existence of a one to one correspondence between critical points of W and the critical points of W_{ext}.

In order to proceed we implement the following Lemmas:

Lemma 6.0.2. *Suppose operators $A : X \to X$ and $B : X \to X$ are bounded operators with A being invertible and B having the property*

$$||B|| < ||A^{-1}||^{-1}. \tag{6.0.15}$$

Then $A + B$ is invertible.

Proof. Rearranging (6.0.15) we have $||A^{-1}B|| < 1$ since this is the case, $A^{-1}B$ can be written in the form of a Neumann Series:

$$\sum_{n=0}^{\infty} \left(-A^{-1}B\right)^n = \left[\mathbb{1} - (-A^{-1}B)\right]^{-1}$$

and so we have the invertibility of $[\mathbb{1} + A^{-1}B]$. Because

$$A + B = A\left[\mathbb{1} + A^{-1}B\right]$$

and the product of two invertible operators is also invertible, we have $A + B$ is invertible.

□

Lemma 6.0.3. *Suppose $\delta \ll 1$ and $\underline{z} \in \Omega_{\epsilon\delta\underline{z}}$. Then for $n = 1, 2, 3, \ldots$*

$$W_{ext}^{(n)} = O(\sqrt{\epsilon}\delta^{n+1}) \tag{6.0.16}$$

where $W_{ext}^{(n)}(\underline{z}) = \partial_{\underline{z}}^{(n)} W_{ext}(\underline{z})$ as n denotes the order of the derivative. In addition, if $|W_{ext}'(\underline{z})| \ll \sqrt{\epsilon}\delta^2$, then $W_{ext}''(\underline{z})$ is invertible with

$$||W_{ext}''(\underline{z})^{-1}|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}. \tag{6.0.17}$$

Proof. First we shift the potential and fix the vortex in $W_{ext}(\underline{z})$, i.e., $W_{ext}(\underline{z}) = \frac{1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^2} W(x + z_j)(|\psi^{n_j}(x)|^2 - 1)$. Now we compute $W_{ext}'(\underline{z})$:

$$\begin{aligned} W_{ext}'(\underline{z}) &= \begin{pmatrix} \vec{\nabla}_{z_1} W_{ext}(\underline{z}) \\ \vec{\nabla}_{z_2} W_{ext}(\underline{z}) \\ \vdots \\ \vec{\nabla}_{z_m} W_{ext}(\underline{z}) \end{pmatrix} \\ &= \begin{pmatrix} \int_{\mathbb{R}^2} \vec{\nabla}_{z_1} W(x + z_1)\left(|\psi^{(n_1)}(x)|^2 - 1\right) dx \\ \int_{\mathbb{R}^2} \vec{\nabla}_{z_2} W(x + z_2)\left(|\psi^{(n_2)}(x)|^2 - 1\right) dx \\ \vdots \\ \int_{\mathbb{R}^2} \vec{\nabla}_{z_m} W(x + z_m)\left(|\psi^{(n_m)}(x)|^2 - 1\right) dx \end{pmatrix} \end{aligned}$$

$$W'_{ext}(\underline{z}) = \begin{pmatrix} \int_{\mathbb{R}^2} \left(\vec{\nabla} W(z_1) + W''(z_1)x + O(\sqrt{\epsilon}\delta^4 x^2)\right)(|\psi^{(n_1)}(x)|^2 - 1) \\ \int_{\mathbb{R}^2} \left(\vec{\nabla} W(z_2) + W''(z_2)x + O(\sqrt{\epsilon}\delta^4 x^2)\right)(|\psi^{(n_2)}(x)|^2 - 1) \\ \vdots \\ \int_{\mathbb{R}^2} \left(\vec{\nabla} W(z_m) + W''(z_m)x + O(\sqrt{\epsilon}\delta^4 x^2)\right)(|\psi^{(n_m)}(x)|^2 - 1) \end{pmatrix}$$
$$= \begin{pmatrix} \int_{\mathbb{R}^2} \left(\vec{\nabla} W(z_1) + O(\sqrt{\epsilon}\delta^4 x^2)\right)(|\psi^{(n_1)}(x)|^2 - 1) \\ \int_{\mathbb{R}^2} \left(\vec{\nabla} W(z_2) + O(\sqrt{\epsilon}\delta^4 x^2)\right)(|\psi^{(n_2)}(x)|^2 - 1) \\ \vdots \\ \int_{\mathbb{R}^2} \left(\vec{\nabla} W(z_m) + O(\sqrt{\epsilon}\delta^4 x^2)\right)(|\psi^{(n_m)}(x)|^2 - 1) \end{pmatrix} \tag{6.0.18}$$

as $\int_{\mathbb{R}^2} x(|\psi|^2 - 1)dx = 0$ due to the polar symmetry of the vortex. From condition (B) of Theorem 2.1.1, $|\partial_x W| \leq c\delta^2\sqrt{\epsilon}$ and

$$\int_{\mathbb{R}^2} \left(1 - |\psi^{(n_j)}(x)|^2\right) dx = O\left(m_\lambda^{-2}\right)$$

we arrive at $W'_{ext}(\underline{z}) = O\left(\frac{\sqrt{\epsilon}\delta^2}{m_\lambda^2}\right)$, where recall, $m_\lambda = min(\sqrt{2\lambda}, 2)$. This method holds and can be repeated for the higher derivatives of W_{ext}, and hence validates (6.0.16).

To demonstrate invertibility of $W''_{ext}(\underline{z})$ we first compute the $2m \times 2m$ matrix $W''_{ext}(\underline{z})$ from (6.0.18):

$$W''_{ext}(\underline{z}) = diag_{j=1,\ldots,m}\left(\frac{1}{2}\int_{\mathbb{R}^2} Hess_{z_j}(W(z_j))\right) + O(\sqrt{\epsilon}\delta^4 x)\left(|\psi_j(x)|^2 - 1\right) dx \tag{6.0.19}$$

where $Hess_{z_j}(W(z_j))$ is the Hessian of W with respect to z_j.

If $||W'_{ext}(\underline{z})|| \ll \sqrt{\epsilon}\delta^2$, then by (6.0.18), $|\vec{\nabla} W(z_j)| \ll c\sqrt{\epsilon}\delta^2$ (since $\delta^2 \ll 1$) for all $j = 1, 2, \ldots, m$. Since $\underline{z} \in \Omega_{\epsilon\delta\underline{z}}$, then $W''(z_j)$ is invertible with the bound $||W''(z_j)^{-1}|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}$ for all $j = 1, 2, \ldots, m$. Thus by (6.0.19), $\delta \ll 1$ and Lemma 6.0.5., $W''_{ext}(\underline{z})$ is invertible and bounded by $||(W''_{ext}(\underline{z}))^{-1}|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}$.

□

6.0.2 Relationship Between W and W_{ext}

Here in this section we state and prove the following proposition relating critical point of W and W_{ext}.

Proposition 6.0.2. *Suppose our potential W satisfies the conditions (A) and (B) from Section 2 with $\delta \ll 1$. Then W has critical points $b_1, b_2. \dots, b_m$ and $\underline{b} = (b_1, b_2, \dots, b_m) \in \Omega_{\epsilon\delta\underline{z}}$ if and only if $W_{ext}(\underline{z})$ has a critical point at $\underline{z}_0 \in \Omega_{\epsilon\delta\underline{z}}$ with $|\underline{b} - \underline{z}_0| \leq c\delta$.*

Proof. Suppose W has critical points at $b_1, \dots, b_m$ and $\underline{b} = (b_1, \dots, b_m) \in \Omega_{\epsilon\delta\underline{z}}$. Then $\vec{\nabla} W(b_j) = 0$ for $j = 1, \dots, m$ and (6.0.18) imply that

$$W'_{ext}(\underline{b}) = O\left(\sqrt{\epsilon}\delta^4\right). \tag{6.0.20}$$

Since $\underline{b} \in \Omega_{\epsilon\delta\underline{z}}$ and $\delta \ll 1$, then (6.0.20) and Lemma 6.0.3 implies that

$$||W''_{ext}(\underline{b})^{-1}|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}. \tag{6.0.21}$$

By (6.0.20), (6.0.21) along with (6.0.16) for $n = 3$ and an Implicit Function Theorem argument as outlined in Appendix E, $W_{ext}(\underline{z})$ has a unique critical point at $z_0 \in B_{\mathbb{R}^{2m}}(\underline{b}, \alpha)$ where $\alpha = O(\frac{1}{\delta})$ and $|z_0 - \underline{b}| = O(\delta)$.

□

Now we demonstrate that $\underline{z}_0 := (z_{01}, \dots, z_{0j}) \in \Omega_{\epsilon\delta\underline{z}}$. We first suppose that $|W'(z_{0_j})| \ll \sqrt{\epsilon}\delta^2$ and intend to show that $W''(z_{0_j})$ is invertible with the bound

$$||W''(\underline{z}_{0_j})|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}. \tag{6.0.22}$$

Taylor expanding $W''(z_{0_j})$ about b_j, we have by condition (B) on W:

$$W''(z_{0_j}) = W''(b_j) + O(\sqrt{\epsilon}\delta^4 |z_{0_j} - b_j|).$$

Recall, $|\underline{b} - \underline{z}_0| < c\delta$ and since $\underline{b} \in \Omega_{\epsilon\delta z}$, then by Lemma 6.0.2 we conclude that $W''(z_{0_j})$ is invertible and bounded by (6.0.22).
For the proof of the necessary part of Proposition 6.0.2 refer to Appendix E.

6.0.3 Proof of Theorem 3.0.5

Proof. Suppose W satisfies condition (C) and (D); in particular W has critical points at $b_1, b_2, \dots, b_m$ with $\underline{b} = (b_1, b_2, \dots, b_m) \in \Omega_{\epsilon\delta\underline{z}}$. We look to identify critical points of $W_{ext}(\underline{z})$. By Proposition 6.0.2 we find that $W_{ext}(\underline{z})$ indeed has a critical point at $\underline{z}_0 \in \Omega_{\epsilon\delta\underline{z}}$ with

$$\underline{z}_0 \in B_{\mathbb{R}^{2m}}(\underline{b}, \alpha) \text{ where } \alpha = O\left(\frac{1}{\delta}\right) \text{ and } |\underline{z}_0 - \underline{b}| = O(\delta). \tag{6.0.23}$$

Now we make use of Implicit Function Theorem to show that Φ_W has a critical point. Taking a derivative of $\Phi_W(\underline{z})$ with respect to $\underline{z}$ in (6.0.3) and evaluating for $\underline{z} = \underline{z}_0$ we have

$$\begin{aligned}\Phi'_W(\underline{z}_0) &= V'_{int}(\underline{z}_0) + W'_{ext}(\underline{z}_0) + W'_{ext,Rem}(\underline{z}_0) + R'_W(\underline{z}_0) \\ &= O(\epsilon) \qquad (6.0.24)\end{aligned}$$

since $W'_{ext}(\underline{z}_0) = 0$ at $\underline{z} = \underline{z}_0$, and by (6.0.9) to (6.0.13) for $|\alpha| = 1$.

Prior to the use of the Implicit Function Theorem, $\Phi''_W(\underline{z})$ must be shown to be invertible. Taking two derivatives of $\Phi_W(\underline{z})$, and evaluating at $\underline{z}_0$, we have

$$\begin{aligned}\Phi''_W(\underline{z}_0) &= V''_{int}(\underline{z}_0) + W''_{ext}(\underline{z}_0) + W''_{ext,Rem}(\underline{z}_0) + R''_W(\underline{z}_0) \\ &= W''_{ext}(\underline{z}_0) + O(\epsilon) \qquad (6.0.25)\end{aligned}$$

by (6.0.10) to (6.0.13) for $|\alpha| = 2$, and since $\sqrt{\epsilon} \ll \delta^3$ (as $\sqrt{\epsilon} \ll \delta^4$ and $\delta \ll 1$). Noting that $W'_{ext}(\underline{z}_0) = 0$, $\sqrt{\epsilon} \ll \delta^4$, and $\underline{z}_0 \in \Omega_{\epsilon\delta\underline{z}}$, we see that by Lemma 6.0.3 that $W''_{ext}(\underline{z}_0)$ is invertible with bound given by (6.0.17). Therefore, by (6.0.25) and $\sqrt{\epsilon} \ll \delta^3$, $\Phi''_W(\underline{z})$ is invertible and has the bound

$$||\Phi''_W(\underline{z})|| \leq c(\sqrt{\epsilon}\delta^3)^{-1}. \qquad (6.0.26)$$

Likewise for $\Phi'''_W(\underline{z}_0)$, we use $\sqrt{\epsilon} \ll \delta^4$ to obtain the bound on the third derivative of the reduced energy

$$\Phi'''_W(\underline{z}_0) = O(\sqrt{\epsilon}\delta^4). \qquad (6.0.27)$$

Following from equations (6.0.24), (6.0.26), (6.0.27) and the Implicit Function Theorem, we then find that Φ_W has a critical point $\underline{z}_b$ such that

$$\underline{z}_b \in B_{\mathbb{R}^{2m}}(\underline{z}_0, \alpha) \text{ where } \alpha = O\left(\frac{1}{\delta}\right) \text{ and } |\underline{z}_b - \underline{z}_0| = O\left(\frac{\sqrt{\epsilon}}{\delta^3}\right). \qquad (6.0.28)$$

Finally, using equations (6.0.23) and (6.0.28), we have shown that if W has critical points at $\underline{b} \in \Omega_{\epsilon\delta\underline{z}}$ then Φ_W has a critical point at $\underline{z}_b$ where $\underline{z}_b \in B_{\mathbb{R}^{2m}}(\underline{b}, \frac{c}{\delta})$ and $|\underline{z}_b - \underline{b}| \leq O\left(max\left(\frac{\sqrt{\epsilon}}{\delta^3}, \delta\right)\right)$.

□

APPENDIX

A. AN OUTLINE OF IFT ARGUMENT AND MOTIVATION TO USE LYAPUNOV-SCHMIDT REDUCTION

Here in this Appendix A we outline the IFT argument and go over the Lyapunov-Schmidt reduction method and its importance to this thesis.

Let X, Y, Z be Banach spaces and $F : X \times Y \to Z$ a map. We want to solve $F(x, y) = 0$ for $y = G(x)$ such that $F(x, G(x)) = 0$. Without loss of generality, we solve $F(x, y) = 0$ for y near $(x, y) = (0, 0)$. Expanding F in y around 0, we have

$$F(x, y) = F(x, 0) + D_y F(x, 0)y + R(x, y) \tag{A.0.1}$$

where $R(x, y) = o(||y||)$ and $D_y F(x, y)$ is the partial Fréchet derivative with respect to y. Setting $F(x, y) = 0$ and solving for y, we obtain

$$y = -D_y F^{-1}(x, 0) \left[F(x, 0) + R(x, y)\right]. \tag{A.0.2}$$

The invertibility of $D_y F(x, 0)$ here is necessary for this IFT argument. For *every fixed* x, we define the map $H_X : Y \to Y$ by

$$H_x(y) := -D_y F^{-1}(x, 0) \left[F(x, 0) + R(x, y)\right]. \tag{A.0.3}$$

Now we can write equation (A.0.2) as the *fixed point equation* $H_x(y) = y$. Therefore, finding $y = G(x)$ in (A.0.2) is equivalent to obtaining fixed points of the map H_x defined by (A.0.3). In order to show the H_x has a fixed point, we must show that there exists an $\epsilon > 0$ and a $\delta > 0$ such that for all $x \in B_x(0, \delta)$:

i) $H_x : B_Y(0, \epsilon) \to B_Y(0, \epsilon)$

ii) $||D_Y H(y)|| \leq \frac{1}{2}$, for all $y \in B_Y(0, \epsilon)$, i.e., H_x is a strict contraction.

Prior to solving the problem concerning the pinning of multiple vortices, the case of pinning of a single vortex is investigated. The objective is to solve the equation $F_W(u) = 0$, with $u = v_{z\gamma} + w$, where

$$v_{z\gamma} = \begin{pmatrix} \psi_{z\gamma} \\ \vec{A}_{z\gamma} \end{pmatrix} = \begin{pmatrix} e^{i\gamma}\psi^{(\pm 1)}(x - z) \\ \vec{A}^{(\pm 1)}(x - z) + \vec{\nabla}\gamma \end{pmatrix}.$$

First expand $F_W(v_{z\gamma} + w) = 0$ as a Taylor Series about $v_{z\gamma}$.

$$F_W(v_{z\gamma} + w) = F_W(v_{z\gamma}) + wF'_W(v_{z\gamma}) + N_W(v_{z\gamma}, w) = 0 \tag{A.0.4}$$

Arrange the expression so w is on the L.H.S.

$$F_W'(v_{z\gamma})w = -[F_W(v_{z\gamma}) + N_W(v_{z\gamma})] \tag{A.0.5}$$

We cannot proceed any further to solve for w as $F_W'(v_{z\gamma})$ is not invertible. This claim of non-invertibility arises from the following: since $F_0(v_{z\chi}) = 0$, then taking a derivative with respect to z and γ, we have

$$F_0'(v_{z\gamma})\partial_z v_{z\gamma} = 0$$

and

$$F_0'(v_{z\gamma})\partial_\gamma v_{z\gamma} = 0.$$

Therefore, $F_0'(v_{z\gamma})$ has a non-zero kernel and thus not invertible. In addition, since

$$F_W(v) = F_0(v_{z\gamma}) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} v_{z\gamma},$$

and $F_0(v_{z\gamma}) = 0$, then

$$F_W(v_{z\gamma}) = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} v_{z\gamma}.$$

Using Chain Rule, take a derivative with respect to both z and γ

$$F_W'(v_{z\gamma})\partial_z v_{z\gamma} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \partial_z v_{z\gamma} \tag{A.0.6}$$

$$F_W'(v_{z\gamma})\partial_\gamma v_{z\gamma} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \partial_\gamma v_{z\gamma} \tag{A.0.7}$$

where the R.H.S. of both (A.0.6) and (A.0.7) are of $O(\epsilon)$. The order of ϵ is arbitrarily small so the operator $F_W'(v_{z\gamma})$ is said to have "almost" zero vectors and thus cannot be inverted readily since its kernel is non-zero.

To remedy the situation with the invertibility of $F_W'(v_{z\gamma})$, (A.0.4) is decomposed into orthogonal and tangential components. More precisely, we project equation (A.0.4) onto the tangent space generated by $\partial_z v_{z\gamma}$ and $\partial_\gamma v_{z\gamma}$ (the translational and gauge symmetries) and its orthogonal complement. The operator $F_W'(v_{z\gamma})$ is invertible on the orthogonal complement and hence, w can be solved for in (A.0.5). Hence $F_W'(v_{z\gamma})$ is invertible on the orthogonal complement [ST], and the pinning of a single vortex subject to a potential can be carried out.

B. THE ALMOST SOLUTION PROPERTY AND INTER-VORTEX ENERGY

Presented here are two Lemma's from [GS2] which are needed in order to help demonstrate the phenomena of multi-vortex pinning. The following lemmas have been proved in [GS2], however, we prove them here as well for the reader's convenience.

Lemma B.0.4. *(Almost Solution Property) For large values of $R(\underline{z})$,*

$$||\mathcal{E}'{}_{GL}(v_{\underline{z}\chi})||_{\mathbf{L}^2} \leq c\frac{e^{-R(\underline{z})}}{R^{\frac{1}{4}}(\underline{z})}$$

Proof. First we examine the scalar component of $\mathcal{E}'_{GL}(v_{\underline{z}\chi})$,

$$[\mathcal{E}'_{GL}(v_{\underline{z}\chi})]_\psi = -\Delta_{\vec{A}_{\underline{z}\chi}}\psi_{\underline{z}\chi} + \lambda(|\psi_{\underline{z}\chi}|^2 - 1)\psi_{\underline{z}\chi}.$$

Compute the $\mathbf{L}^2$ norm of $[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi$ where

$$E_\psi^{(\underline{z}\chi)} := -e^{i\chi(x)}\sum_{j\neq k}\left(\prod_{l\neq j,k}\psi_l\right)(\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)})\cdot(\vec{\nabla}_{\vec{A}^{(n_k)}}\psi^{(n_k)}).$$

Here we have

$$\begin{aligned}
[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi &= -\Delta_{\vec{A}_{\underline{z}\chi}}\psi_{\underline{z}\chi} + \lambda(|\psi_{\underline{z}\chi}|^2 - 1)\psi_{\underline{z}\chi} - E_\psi^{(\underline{z}\chi)} \\
&= -e^{i\chi(x)}[\sum_j\left(\prod_{k\neq j}\psi^{(n_k)}\right)(\Delta_{\vec{A}^{(n_j)}}\psi^{(n_j)}) \\
&+ \sum_{j\neq k}\left(\prod_{l\neq j,k}\psi^{(n_l)}\right)\left(\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right)\cdot\left(\vec{\nabla}_{\vec{A}^{(n_k)}}\psi^{(n_k)}\right)] \\
&+ \lambda(|\psi_{\underline{z}\chi}|^2 - 1)\psi_{\underline{z}\chi} - E_\psi^{(\underline{z}\chi)}.
\end{aligned}$$

After some cancelations we have

$$[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi = -e^{i\chi(x)}\sum_j\left(\prod_{k\neq j}\psi^{(n_k)}\right)(\Delta_{\vec{A}^{(n_j)}}\psi^{(n_j)}) + \lambda(|\psi_{\underline{z}\chi}|^2 - 1)\psi_{\underline{z}\chi}.$$

Now add zero to the expression

$$\begin{aligned}[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi &= -e^{i\chi(x)}[\sum_j \left(\prod_{k\neq j}\psi^{(n_k)}\right)(\Delta_{\vec{A}^{(n_j)}}\psi^{(n_j)}) + \lambda(|\psi^{(n_j)}|^2-1)\psi^{(n_j)} \\ &- \lambda(|\psi^{(n_j)}|^2-1)\psi^{(n_j)}] + \lambda(|\psi_{\underline{z}\chi}|^2-1)\psi_{\underline{z}\chi}\end{aligned}$$

and using the fact that $\psi^{(n_j)}$ satisfies equation (1.1.1) for all j, we get

$$[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi = e^{i\chi(x)}\sum_j\left(\prod_{k\neq j}\psi^{(n_k)}\right)\left[-\lambda(|\psi^{(n_j)}|^2-1)\psi^{(n_j)}\right] + \lambda(|\psi_{\underline{z}\chi}|^2-1)\psi_{\underline{z}\chi}.$$

Re-write $\psi_{\underline{z}\chi}$ as a product to obtain

$$\begin{aligned}[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi &= -e^{i\chi(x)}\sum_j\lambda\left(\prod_{k\neq j}\psi^{(n_k)}\right)(|\psi^{(n_j)}|^2-1)\psi^{(n_j)} \\ &+ \lambda e^{i\chi(x)}\left(|\prod_j\psi^{(n_j)}|^2-1\right)\left(\prod_j\psi^{(n_j)}\right) \\ &= -\lambda e^{i\chi(x)}\left(\prod_j\psi^{(n_j)}\right)\left[\sum_j(|\psi^{(n_j)}|^2-1) - \left(|\prod_j\psi^{(n_j)}|^2-1\right)\right] \\ &= \lambda\left(\prod_j\psi^{(n_k)}\right)\left[\left(\prod_j f_k^2-1\right)-\sum_j(f_j^2-1)\right].\end{aligned}$$

Take the norm of $[\mathcal{E}_{GL}(v_{\underline{z}\chi})]_\psi - [E_\psi^{(\underline{z}\chi)}]_\psi$ and using the last two bounds of equation (1.1.7), we obtain

$$|\left(\left(\prod_j f_j^2\right)-1\right)-\sum_j(f_j^2-1)| \leq c\sum_{j\neq k}e^{-m_\lambda(|x-z_j|+|x-z_k|)}.$$

Making use of Lemma 12 from [GS2]

$$\int_{\mathbb{R}^2}dx\frac{e^{-\alpha|x|}e^{-\beta|x-a|}}{|x|^\gamma|x-a|^\delta} \leq c\frac{|a|^{-\frac{1}{2}}e^{-\alpha|a|}}{|a|^{\gamma+\delta-2}} \tag{B.0.1}$$

with $\alpha=\beta=2m_\lambda>2$ (since $m_\lambda = min(\sqrt{2\lambda},2)$), $\gamma=\delta=0$ and $a=z_k-z_j$ one obtains

$$\begin{aligned}||[\mathcal{E}_G'L(v_{\underline{z}\chi})]_\psi - [E^{(\underline{z}\chi)}]_\psi||_{\mathbf{L}^2} &\leq [\frac{c^2|z_k-z_j|^{-1}e^{-2m_\lambda|z_k-z_j|}}{|z_k-z_j|^{-4}}]^{\frac{1}{2}} \\ &\leq cR(\underline{z})^{\frac{3}{2}}e^{-m_\lambda R(\underline{z})}.\end{aligned}$$

Now to examine the vector component of $\mathcal{E}'_{\underline{z}\chi}(v_{\underline{z}\chi})$ and $E^{(\underline{z}\chi)}$. As before $[\mathcal{E}'_{\underline{z}\chi}(v_{\underline{z}\chi})]_{\vec{A}} = curlB_{\underline{z}\chi} - \vec{j}_{\underline{z}\chi}$, where $\vec{j}_j = Im(\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)})$.
As $curlB_{\underline{z}\chi} = \sum_j curlB_j$ and

$$
\begin{aligned}
\vec{j}_{\underline{z}\chi} &= Im(\overline{\psi_{\underline{z}\chi}}\vec{\nabla}_{\vec{A}_{\underline{z}\chi}}\psi_{\underline{z}\chi}) \\
&= Im\left(\overline{\prod_j \psi^{(n_j)}}\right)\left[\sum_j\left(\prod_{k\neq j}\psi^{(n_k)}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right)\right] \\
&= Im\sum_j\left(\prod_{k\neq j}|\psi^{(n_k)}|^2\right)\left(\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right).
\end{aligned}
$$

Add zero to the expression in order to obtain

$$
\begin{aligned}
\vec{j}_{\underline{z}\chi} &= Im\left[\sum_j\left(\prod_{k\neq j}|\psi^{(n_k)}|^2 - 1 + 1\right)(\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)})\right] \\
&= Im\left[\sum_j\left(\prod_{k\neq j}|\psi^{(n_k)}|^2 - 1\right)\left(\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right) + \sum_j\left(\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right)\right] \\
&= \sum_j \vec{j}_j + \sum_j\left(\prod_{k\neq j}f_k^2 - 1\right)\vec{j}_j
\end{aligned}
$$

Substituting the above into

$$[\mathcal{E}'_{GL}(v_{\underline{z}\chi})]_{\vec{A}} - E_{\vec{A}}^{(\underline{z},\chi)}$$

with $E_{\vec{A}}^{(\underline{z},\chi)} := \sum_j[\sum_{k\neq j}(1-f_k^2)]\vec{j}_j$ and using the fact that $curlB_j - \vec{j}_j = 0$ (by equation (1.1.2)) we arrive at

$$
\begin{aligned}
[\mathcal{E}'_{GL}(v_{\underline{z}\chi})]_{\vec{A}} - E_{\vec{A}}^{(\underline{z},\chi)} &= \sum_j curlB_j - \sum_j \vec{j}_j - \sum_j\left(\prod_{k\neq j}f_k^2 - 1\right)\vec{j}_j - \sum_j\left[\sum_{k\neq j}(1-f_k^2)\right]\vec{j}_j \\
&= -\sum_j\left(\prod_{k\neq j}f_k^2 - 1\right)\vec{j}_j - \sum_j\left[\sum_{k\neq j}(1-f_k^2)\right]\vec{j}_j \\
&= -\sum_j\left[\left(\prod_{k\neq j}f_k^2 - 1\right) + \left[\sum_{k\neq j}(1-f_k^2)\right]\right]\vec{j}_j.
\end{aligned}
$$

Using (1.1.7), we have $j_j \sim \frac{ce^{-r}}{\sqrt{r}}$ as $r \to \infty$ along with modulus $|[\mathcal{E}'_{GL}(v_{\underline{z}\chi})]_{\vec{A}} - E_{\vec{A}}^{(\underline{z},\chi)}|$ being bounded by

$$c\sum_{j,k,l,\ldots} e^{-m_\lambda(|x-z_j|+|x-z_k|)-|x-z_l|}.$$

Taking the $\mathbf{L}^2$ norm of $[\mathcal{E}'_{GL}(v_{\underline{z}\chi})]_{\vec{A}} - E^{(\underline{z},\chi)}_{\vec{A}}$ via (B.0.1) demonstrates that

$$||[\mathcal{E}'_{GL}(v_{\underline{z}\chi})]_{\vec{A}} - E^{(\underline{z},\chi)}_{\vec{A}}||_{\mathbf{L}^2} \leq cR^{\frac{3}{2}}(\underline{z})e^{-m_\lambda R(\underline{z})}.$$

On to demonstrate

$$||\mathcal{E}'_{GL}(v_{\underline{z},\chi})||_{\mathbf{L}^2} \leq c\frac{e^{-R(\underline{z})}}{R^{\frac{1}{4}}(\underline{z})}.$$

Using triangle inequality

$$||\mathcal{E}'_{GL}(v_{\underline{z}\chi})||_{\mathbf{L}^2} \leq ||\mathcal{E}'_{GL}(v_{\underline{z}\chi}) - E^{(\underline{z}\chi)}||_{\mathbf{L}^2} + ||E^{(\underline{z}\chi)}||_{\mathbf{L}^2},$$

which leaves $||E^{(\underline{z}\chi)}||$ to be determined where

$$E^{\underline{z}\chi} = \begin{pmatrix} -e^{i\chi(x)}\sum_{j\neq k}(\prod_{l\neq j,k}\psi^{(n_l)})(\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)})\cdot(\vec{\nabla}_{\vec{A}^{(n_k)}}\psi^{(n_k)}) \\ \sum_j\left[\sum_{k\neq j}(1-f_k^2)\vec{j}_j\right] \end{pmatrix}.$$

Adjusting the modulus of $E^{(\underline{z}\chi)}$ such that at $r=0$, $|E^{(\underline{z}\chi)}| = 1$,

$$|E^{(\underline{z}\chi)}| \leq c\sum_{k\neq j}\frac{e^{-|x-z_j|}}{\sqrt{(1+|x-z_j|)}}\frac{e^{-|x-z_k|}}{\sqrt{(1+|x-z_k|)}}.$$

Applying (B.0.1) to $|E^{(\underline{z}\chi)}|$ yields

$$||E^{(\underline{z}\chi)}||_{\mathbf{L}^2} \leq c\frac{e^{-R(\underline{z})}}{R^{\frac{1}{4}}(\underline{z})}.$$

Now it can be shown that

$$\begin{aligned} ||\mathcal{E}'_{GL}(v_{\underline{z}\chi})|| &\leq ||\mathcal{E}'_{GL}(v_{\underline{z}\chi}) - E^{(\underline{z}\chi)}||_{\mathbf{L}^2} + ||E^{(\underline{z}\chi)}||_{\mathbf{L}^2} \\ &\leq cR^{\frac{3}{2}(\underline{z})}e^{-m_\lambda R(\underline{z})} + c\frac{e^{-R(\underline{z})}}{R^{\frac{1}{4}}(\underline{z})} \end{aligned}$$

and since $cR^{\frac{3}{2}(\underline{z})}e^{-m_\lambda R(\underline{z})} \ll c\frac{e^{-R(\underline{z})}}{R^{\frac{1}{4}}(\underline{z})}$ it is evident that

$$||E^{(\underline{z}\chi)}||_{\mathbf{L}^2} \leq c\frac{e^{-R(\underline{z})}}{R^{\frac{1}{4}}(\underline{z})}.$$

□

Lemma B.0.5. *(Inter-vortex Energy)*
For $\lambda > \frac{1}{2}$, the inter-vortex energy of a multi-vortex configuration with large vortex separations $R(\underline{z})$ is,

$$W(\underline{z}) = \sum_{j \neq k} n_j n_k c_{jk} \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}} + o\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right)$$

as $R(\underline{z}) \to \infty$ *with* $c_{jk} > 0$ *being constants.*

Proof. To demonstrate the validity of the lemma, one computes the reduced energy

$$W(\underline{z}) := \mathcal{E}(v_{\underline{z}\chi}) - \sum_{j=1}^{m} E^{(n_j)},$$

where $E^{(n_j)}$ is the intrinsic energy associated to the j-th vortex $\psi^{(n_j)}$

$$E^{(n_j)} = \frac{1}{2}\int_{R^2}\left[|\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}|^2 + (\vec{\nabla}\times\vec{A}^{(n_j)})^2 + \frac{\lambda}{4}(|\psi^{(n_j)}|^2 - 1)^2\right].$$

Evaluate each term separately

- Term 1
$$\frac{1}{2}\int_{\mathbb{R}^2}(\vec{\nabla}\times\vec{A}_{\underline{z}\chi})^2 - \frac{1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^2}(\vec{\nabla}_{\times}\vec{A}^{(n_j)})^2$$
- Term 2
$$\frac{1}{2}\int_{\mathbb{R}^2}|\vec{\nabla}_{\vec{A}_{\underline{z}\chi}}\psi_{\underline{z}\chi}|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\sum_{j=1}^{m}|\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}|^2$$
- Term 3
$$\frac{1}{2}\int_{\mathbb{R}^2}\frac{\lambda}{2}\left(|\psi^2_{\underline{z}\chi}| - 1\right)^2 - \frac{1}{2}\int_{\mathbb{R}^2}\frac{\lambda}{2}\sum_{j=1}^{m}\left(|\psi^{(n_j)}|^2 - 1\right)^2$$

Dealing with Term 1 and making use of $B = \sum_j B_j$

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}(\vec{\nabla}\times\vec{A}_{\underline{z}\chi})^2 - \frac{1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^2}(\vec{\nabla}_{\times}\vec{A}^{(n_j)})^2 \\
= \;& \frac{1}{2}\int_{\mathbb{R}^2}B^2 - \frac{1}{2}\int_{\mathbb{R}^2}\sum_j B_j \\
= \;& \frac{1}{2}\int_{\mathbb{R}^2}\left(\sum_j B_j\right)^2 - \frac{1}{2}\int_{\mathbb{R}^2}\sum_j B_j^2 \\
= \;& \frac{1}{2}\int_{\mathbb{R}^2}\sum_j B_j^2 + \frac{1}{2}\int_{\mathbb{R}^2}2\sum_{k\neq j}B_jB_k - \frac{1}{2}\int_{\mathbb{R}^2}\sum_j B_j^2 \\
= \;& \frac{1}{2}\int_{\mathbb{R}^2}\sum_{k\neq j}B_jB_k.
\end{aligned}$$

Now shift attention to Term 2.

$$\frac{1}{2}\int_{\mathbb{R}^2} |\vec{\nabla}_{\vec{A}_{\underline{z}\chi}} \psi_{\underline{z}\chi}|^2 - \frac{1}{2}\int_{\mathbb{R}^2} \sum_{j=1}^{m} |\vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}|^2$$

$$\begin{aligned}
= & \frac{1}{2}\int_{\mathbb{R}^2} |\sum_j \left(\prod_{k\neq j} \psi^{(n_j)}\right) \vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}|^2 - \frac{1}{2}\int_{\mathbb{R}^2} |\vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}|^2 \\
= & \frac{1}{2}\int_{\mathbb{R}^2} \overline{\sum_j \left(\prod_{k\neq j} \vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}\right)} \left[\sum_{k\neq j} \left(\prod_{l\neq k} \psi^{(n_k)} \vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}\right)\right] \\
= & \frac{1}{2}\int_{\mathbb{R}^2} \sum_j \left(|\prod_{k\neq j} \psi^{(n_k)}|^2 - 1\right) |\vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}|^2 \\
+ & \frac{1}{2}\int_{\mathbb{R}^2} \sum_j |\prod_{k\neq j,l} \psi^{(n_k)}|^2 Re\left[\overline{\psi^{(n_j)}} \vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}\right] Re\left[\overline{\psi^{(n_l)}} \vec{\nabla}_{\vec{A}^{n_l}} \psi^{(n_l)}\right] \\
+ & \frac{1}{2}\int_{\mathbb{R}^2} \sum_{j\neq l} \left(|\prod_{k\neq j,l} \psi^{(n_l)}|^2 - 1\right) Im\left[\overline{\psi^{(n_k)}} \vec{\nabla}_{\vec{A}^{(n_k)}} \psi^{(n_k)}\right] Im\left[\overline{\psi^{(n_l)}} \vec{\nabla}_{\vec{A}^{n_l}} \psi^{(n_l)}\right] \\
= & \frac{1}{2}\int_{\mathbb{R}^2} \sum_j \left(\prod_{k\neq j} f_k^2 - 1\right) |\vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}|^2 \\
+ & \frac{1}{2}\int_{\mathbb{R}^2} \sum_j \left(\prod_{k\neq j,l} f_k^2\right) Re\left[\overline{\psi^{(n_j)}} \vec{\nabla}_{\vec{A}^{(n_j)}} \psi^{(n_j)}\right] Re\left[\overline{\psi^{(n_l)}} \vec{\nabla}_{\vec{A}^{n_l}} \psi^{(n_l)}\right] \\
+ & \frac{1}{2}\int_{\mathbb{R}^2} \sum_{j\neq l} \left(\prod_{k\neq j,l} f_k^2 - 1\right) \vec{j}_k \cdot \vec{j}_l
\end{aligned}$$

On to Term 3

$$\frac{\lambda}{4}\int_{\mathbb{R}^2} \left(|\prod_j \psi^{(n_j)}|^2 - 1\right)^2 - \frac{\lambda}{4}\sum_j \int_{\mathbb{R}^2} \left(|\psi^{(n_j)}|^2 - 1\right)^2$$

$$\begin{aligned}
= & \frac{\lambda}{4}\int_{\mathbb{R}^2} [\sum_j (|\psi^{(n_j)}|^2 - 1) + \sum_{k\neq j} (|\psi^{(n_j)}|^2 - 1)(|\psi^{(n_k)}|^2 - 1) \\
+ & \sum_{j\neq k\neq l} (|\psi^{(n_j)}|^2 - 1)(|\psi^{(n_k)}|^2 - 1)(|\psi^{(n_l)}|^2 - 1) + \ldots]^2 - \frac{\lambda}{4}\sum_j \int_{\mathbb{R}^2} (|\psi^{(n_j)}|^2 - 1)^2 \\
= & \frac{\lambda}{4}\int_{\mathbb{R}^2} \left[\sum_{j\neq k} (f_j^2 - 1)(f_k^2 - 1) + \sum_{j\neq k\neq l} (f_j^2 - 1)(f_k^2 - 1)(f_l^2 - 1) + \ldots\right]
\end{aligned}$$

The ellipsis above includes the continuation of the multiplication of binomials pattern in addition to any cross terms as well.

Combining the three terms together

$$W(\underline{z}) := \mathcal{E}(v_{\underline{z}x}) - \sum_j E^{(n_j)}$$

$$\begin{aligned} = \quad & \frac{1}{2}\sum_{k\neq j}\int_{\mathbb{R}^2}[\vec{j}_j\cdot\vec{j}_k + B_j\cdot B_k] + \frac{1}{2}\sum_{j\neq l}\int_{\mathbb{R}^2}\left(\prod_{k\neq j,l} f_k^2 - 1\right)\vec{j}_j\cdot\vec{j}_l + \frac{1}{2}\sum_j\int_{\mathbb{R}^2}\left(\prod_{k\neq j} f_k^2 - 1\right) \\ \times \quad & |\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}|^2 + \frac{1}{2}\sum_j\int_{R^2}\left(\prod_{k\neq j,l} f_k^2\right) Re\left[\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right] Re\left[\overline{\psi^{(n_l)}}\vec{\nabla}_{\vec{A}^{n_l}}\psi^{(n_l)}\right] \\ + \quad & \frac{\lambda}{4}\int_{\mathbb{R}^2}\left[\sum_{j\neq k}(f_j^2-1)(f_k^2-1) + \sum_{j\neq k\neq l}(f_j^2-1)(f_k^2-1)(f_l^2-1) + \ldots\right] \end{aligned}$$

The above expression is reduced into two terms, LO and REM. LO concerns the lowest order of the above terms which happen to be of most importance while REM deals with those that remain and bare little significance. LO and REM are defined as

$$LO := \frac{1}{2}\sum_{l\neq k}\int_{\mathbb{R}^2}[\vec{j}_l\cdot\vec{j}_k + B_l B_k]$$

$$\begin{aligned} REM \quad := \quad & \frac{1}{2}\sum_{j\neq l}\int_{\mathbb{R}^2}\left(\prod_{k\neq j,l} f_k^2 - 1\right)\vec{j}_j\cdot\vec{j}_l + \frac{1}{2}\sum_j\int_{\mathbb{R}^2}\left(\prod_{k\neq j} f_k^2 - 1\right) \\ \times \quad & |\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}|^2 + \frac{1}{2}\sum_j\int_{R^2}\left(\prod_{k\neq j,l} f_k^2\right) Re\left[\overline{\psi^{(n_j)}}\vec{\nabla}_{\vec{A}^{(n_j)}}\psi^{(n_j)}\right] Re\left[\overline{\psi^{(n_l)}}\vec{\nabla}_{\vec{A}^{n_l}}\psi^{(n_l)}\right] \\ + \quad & \frac{\lambda}{4}\int_{\mathbb{R}^2}\left[\sum_{j\neq k}(f_j^2-1)(f_k^2-1) + \sum_{j\neq k\neq l}(f_j^2-1)(f_k^2-1)(f_l^2-1) + \ldots\right] \end{aligned}$$

The terms of REM are bounded by either $e^{-(min(m_\lambda,2)|x-z_j|+m_\lambda|x-z_k)}$ or $e^{-(m_\lambda|x-z_k|+|x-z_j|+|x-z_l|)}$ by equation (1.1.7) and using Lemma 12 of [GS2], $REM \ll \frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}$ and so it is neglected for the remainder of the proof.
Integrating LO by parts leads to

$$\begin{aligned} LO &= \frac{1}{2}\sum_{l\neq k}\int_{R^2}(\partial_k B_l \partial_k B_k + \partial_l B_l \partial_l B_k) + B_l B_k \\ &= \frac{1}{2}\sum_{k\neq l}\int_{R^2} -B_l(\partial_k^2 B_k + \partial_l^2 B_k) + B_l B_k \\ &= \frac{1}{2}\sum_{k\neq l}\int_{R^2} -B_l \Delta B_k + B_l B_k \\ &= \frac{1}{2}\sum_{k\neq l}\int_{R^2} B_l(-\Delta+1)B_k. \end{aligned}$$

Furthermore $(-\Delta+1)B$ can be expanded in terms of $f(r)e^{i\theta}$ and $a(r)\vec{\nabla}(\theta)$,

$$(-\Delta+1)B = \frac{1}{r}[2(1-a)ff' + a'(1-f^2)] \tag{B.0.2}$$

Inserting (B.0.2) into LO and applying the integral

$$\begin{aligned} I(z) &:= \int_{R^2} b(x)e(x-z)dx \\ &= c\frac{e^{-|z|}}{\sqrt{|z|}}\int_{R^2} e^{x\cdot z/|z|}b(x)dx[1+O(1/|z|)] \end{aligned}$$

and setting $e(x-z) = B_l(x-[z_l-z_k])$ and $b(x) = (-\Delta+1)B_k(x)dx$ leads to

$$\begin{aligned} W(\underline{z}) &= LO \\ &= \frac{1}{2}\sum_{l\neq k}\int_{R^2} B_l(x-[z_l-z_k])(-\Delta+1)B_k(x)dx \\ &= \frac{1}{2}\sum_{l\neq k} c_l n_l n_k \int_{R^2} e^{x\cdot(z_l-z_k)/|z_l-z_k|}\frac{1}{r}[2(1-a_k)f_k f_k' + a_k(1-f_k^2)]dx + o\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right), \end{aligned}$$

where $c_l > 0$ are constants, and recall, n_j is the degree of the j-th vortex, and $a_k \equiv a_{n_k}(r)$ in equation (1.1.6).
The integral above produces a constant and so

$$W(\underline{z}) = \sum_{l\neq k} c_{l,k} n_l n_k \frac{e^{-|z_l-z_k|}}{\sqrt{|z_l-z_k|}} + o\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right)$$

which completes the proof. □

C. COMPUTATION OF HESSIAN OF GL ENERGY FUNCTIONAL AND ALMOST ZERO TANGENT VECTORS

In this Appendix, the computation for the Hessian corresponding to the GL energy functional is computed. Also, we compute the "almost zero" tangent vectors. The GL Hessian is the second variational derivative of the energy functional or simply the first order variational derivative of the GL equations (1.1.1) and (1.1.2)

$$\left[\mathcal{E}_0''\begin{pmatrix}\psi\\ \vec{A}\end{pmatrix}\right]\begin{pmatrix}\xi\\ \vec{B}\end{pmatrix} := \frac{\partial}{\partial\zeta}\mathcal{E}'(\psi+\zeta\xi,\vec{A}+\zeta\vec{B})|_{\zeta=0}. \qquad \text{(C.0.1)}$$

This evaluation will be completed in four separate steps to maintain simplicity.

Scalar Derivative of: $-\Delta_{\vec{A}}\psi + \frac{\lambda}{2}(|\psi|^2-1)\psi$

$$\begin{aligned}
&= \frac{\partial}{\partial\zeta}[-\Delta_{\vec{A}}(\psi+\zeta\xi) + \frac{\lambda}{2}(|\psi+\zeta\xi|^2-1)(\psi+\zeta\xi)]|_{\zeta=0}\\
&= -\Delta_{\vec{A}}\xi + \frac{\lambda}{2}(\psi\overline{\xi}+\overline{\psi}\xi)\psi + \frac{\lambda}{2}(|\psi|^2-1)\xi\\
&= -\Delta_{\vec{A}}\xi + \frac{\lambda}{2}\psi\overline{\xi} + \frac{\lambda}{2}|\psi|^2\xi + \frac{\lambda}{2}|\psi|^2\xi - \frac{\lambda}{2}\xi\\
&= -\Delta_{\vec{A}}\xi + \lambda(|\psi|^2-\frac{1}{2})\xi + \frac{\lambda}{2}\psi^2\overline{\xi}
\end{aligned}$$

Vector Derivative of $-\Delta_{\vec{A}}\psi + \frac{\lambda}{2}(|\psi|^2-1)\psi$

$$\frac{\partial}{\partial\zeta}[-(\vec{\nabla}-i(\vec{A}+\zeta\vec{B}))\cdot(\vec{\nabla}-i(\vec{A}+\zeta\vec{B}))\psi + \frac{\lambda}{2}(|\psi|^2-1)\psi]|_{\zeta=0}$$

$$\begin{aligned}
&= \frac{\partial}{\partial\zeta}[-(\vec{\nabla}-i(\vec{A}+\zeta\vec{B}))\cdot(\vec{\nabla}-i(\vec{A}+\zeta\vec{B}))\psi]|_{\zeta=0}\\
&= i\vec{B}[\vec{\nabla}-i(\vec{A}+\zeta\vec{B})]\psi + [\vec{\nabla}-i(\vec{A}+\zeta\vec{B})](i\vec{B}\psi)|_{\zeta=0}\\
&= i\vec{B}\vec{\nabla}_{\vec{A}}\psi + i\vec{\nabla}\vec{B}\psi + \vec{A}\vec{B}\psi\\
&= i\vec{B}\vec{\nabla}_{\vec{A}}\psi + i\psi\vec{\nabla}\vec{B} + i[\vec{\nabla}_{\vec{A}}\psi]\vec{B}\\
&= i[2\vec{\nabla}_{\vec{A}}\psi + \psi\vec{\nabla}]\cdot\vec{B}
\end{aligned}$$

Scalar Derivative of $-\vec{\nabla}\times\vec{\nabla}\times\vec{A} - Im(\overline{\psi}\vec{\nabla}_{\vec{A}}\psi)$

$$\frac{\partial}{\partial\zeta}[-\vec{\nabla}\times\vec{\nabla}\times\vec{A} - Im((\overline{\psi+\zeta\xi})\vec{\nabla}_{\vec{A}}(\psi+\eta\xi))]|_{\zeta=0}$$

$$\begin{aligned} &= -Im(\overline{\xi}\vec{\nabla}_{\vec{A}}\psi + \overline{\psi}\vec{\nabla}_{\vec{A}}\xi) \\ &= Im([\overline{\vec{\nabla}_{\vec{A}}\psi} - \overline{\psi}\vec{\nabla}_{\vec{A}}]\xi) \end{aligned}$$

Vector Derivative of $-\vec{\nabla}\times\vec{\nabla}\times\vec{A} - Im(\overline{\psi}\vec{\nabla}_{\vec{A}}\psi)$

$$\frac{\partial}{\partial\zeta}\left[-\vec{\nabla}\times\vec{\nabla}\times(\vec{A}+\zeta\vec{B}) - Im(\overline{\psi}(\vec{\nabla}-i(\vec{A}+\zeta\vec{B}))\psi)\right]|_{\zeta=0}$$

$$\begin{aligned} &= -\vec{\nabla}\times\vec{\nabla}\times\vec{B} - Im(-i\overline{\psi}\vec{B}\psi) \\ &= -\vec{\nabla}\times\vec{\nabla}\times\vec{B} + |\psi|^2\vec{B} \\ &= (\Delta - \vec{\nabla}\vec{\nabla} + |\psi|^2)\cdot\vec{B} \end{aligned}$$

Combining the terms yields the GL Hessian denoted as L

$$\frac{\partial}{\partial\eta}\mathcal{E}'(\psi+\zeta\xi,\vec{A}+\zeta\vec{B})|_{\zeta=0} = Lw = \begin{pmatrix} [-\Delta_{\vec{A}} + \lambda(|\psi|^2-\frac{1}{2})]\xi + \frac{\lambda}{2}\psi^2\overline{\xi} + i[2\vec{\nabla}_{\vec{A}}\psi + \psi\vec{\nabla}]\cdot\vec{B} \\ Im([\overline{\vec{\nabla}_{\vec{A}}\psi} - \overline{\psi}\vec{\nabla}_{\vec{A}}]\xi) + (-\Delta + \vec{\nabla}\vec{\nabla} + |\psi|^2)\cdot\vec{B} \end{pmatrix}$$

where
$$\eta = \begin{pmatrix} \xi \\ \vec{B} \end{pmatrix}.$$

Next, we compute the "almost zero" translational and gauge tangent vectors. We start with the gauge tangent vector. Recall,

$$v_{\underline{z}\chi} = \begin{pmatrix} \psi_{\underline{z}\chi} \\ \vec{A}_{\underline{z}\chi} \end{pmatrix} = \begin{pmatrix} e^{i\chi(x)}\prod_{j=1}^m \psi^{(n_j)}(x-z_j) \\ \sum_{j=1}^m \vec{A}^{(n_j)}(x-z_j) + \vec{\nabla}\chi(x) \end{pmatrix}$$

where

$$\chi(x) = \sum_{j=1}^m z_j\cdot\vec{A}^{(n_j)}(x-z_j) + \tilde{\chi}(x)$$

for some $\tilde{\chi}(x)\in H^2(\mathbb{R}^2;\mathbb{R})$.

We begin with the "almost zero" gauge tangent vector,

$$\partial_\gamma\begin{pmatrix} e^{i\chi(x)}\prod_{j=1}^m \psi^{(n_j)}(x-z_j) \\ \sum_{j=1}^m \vec{A}^{(n_j)}(x-z_j) + \vec{\nabla}\chi(x) \end{pmatrix}.$$

The method follow that from above in the computation of the Hessian.

Scalar Derivative

$$\frac{\partial}{\partial\lambda}[e^{i(\chi(x)+\lambda\gamma(x))}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j)]\ |_{\lambda=0}$$

$$\begin{aligned} &= i\gamma e^{i(\chi(x)+\lambda\gamma(x))}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j)|_{\lambda=0} \\ &= i\gamma(x)e^{i\chi(x)}\psi_{\underline{z}\chi} \end{aligned}$$

Vector Derivative

$$\frac{\partial}{\partial\lambda}[\sum_{j=1}^{m}\vec{A}^{(n_j)}(x-z_j)+\vec{\nabla}(\chi(x)+\lambda\gamma(x))]|_{\lambda=0}$$

$$\begin{aligned} &= \frac{\partial}{\partial\lambda}[\vec{\nabla}(\chi(x)+\lambda\gamma(x))]|_{\lambda=0} \\ &= \vec{\nabla}\gamma(x) \end{aligned}$$

Next, we compute the "almost zero" translational and gauge tangent vectors.

$$\partial_{z_{jk}}\begin{pmatrix} e^{i\chi(x)}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j) \\ \sum_{j=1}^{m}\vec{A}^{(n_j)}(x-z_j)+\vec{\nabla}\chi(x) \end{pmatrix}$$

Scalar Derivative

Here, we will use the notation $\hat{e}_k$ to be the unit vector in the k-th direction with $e_1^{\perp}=\begin{pmatrix}0\\1\end{pmatrix}$ and $e_2^{\perp}=\begin{pmatrix}-1\\0\end{pmatrix}$. Using the fact that our gauge functions, χ, are of the form $\chi=\sum_{j=1}^{m}z_j\cdot\vec{A}^{(n_j)}(x-z_j)+\vec{\nabla}\tilde{\chi}$, (see equation (2.2.3)), we have

$$\begin{aligned} \partial_{z_{jk}}\psi_{\underline{z}\chi} &= \partial_{z_{jk}}e^{i\chi(x)}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j) \\ &= \left(\partial_{z_{jk}}e^{i\chi(x)}\right)\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j)+e^{i\chi(x)}\left(\partial_{z_{jk}}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j)\right) \\ &= \left(\hat{e}_k\cdot[\vec{A}^{(n_j)}(x-z_j)]_k-z_j\cdot(-\hat{e}_k)[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)]_k\right)e^{i\chi(x)}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j) \\ &+ e^{i\chi(x)}\cdot\left(\prod_{l\neq k}\psi^{(n_l)}(x-z_j)\right)\left[-\partial_{x_k}\psi^{(n_j)}(x-z_j)\right] \end{aligned}$$

$$\begin{aligned}
\partial_{z_{jk}}\psi_{\underline{z}\chi} &= -e^{i\chi(x)}\left(\prod_{l\neq k}\psi^{(n_l)}(x-z_j)\right)\left[\partial_{x_k}\psi^{(n_j)}(x-z_j)-i[\vec{A}^{(n_j)}(x-z_j)]_k\psi^{(n_j)}(x-z_j)\right] \\
&+ ie^{i\chi x}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j)\left(-z_j\cdot\hat{e}_k[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)]_k\right) \\
&= [T^{\underline{z}\chi}_{jk}]_\psi - i\psi_{\underline{z}\chi}(x)\phi
\end{aligned}$$

where $\phi = z_j\cdot\hat{e}_k[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)]_k$ is a gauge term.

Vector Derivative

$$\begin{aligned}
\partial_{z_{jk}}\vec{A}_{\underline{z}\chi} &= \partial_{z_{jk}}\left(\sum_{l=1}^{m}\vec{A}^{(n_l)}(x-z_l)+\vec{\nabla}\chi(x)\right) \\
&= -e_k^{\perp}\left[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)\right]_k+\vec{\nabla}[e_k\cdot\vec{A}^{(n_j)} \\
&- z_j\cdot e_k^{\perp}[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)]_k] \\
&= B^{(n_j)}(x-z_j)e_k^{\perp}-\vec{\nabla}\left(z_j\cdot e_k[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)]_k\right) \\
&= B^{(n_j)}(x-z_j)e_k^{\perp}+\vec{\nabla}\phi \\
&= [T^{\underline{z}\chi}_{jk}]_{\vec{A}}+\vec{\nabla}\phi
\end{aligned}$$

Once again, $\phi = z_j\cdot e_k[\partial_{x_k}\vec{A}^{(n_j)}(x-z_j)]_k$ is a gauge term.

D. INVERTIBILITY CALCULATION FOR THEOREM 3.0.2

Here we demonstrate Theorem 3.0.2 part (h).
First we re-write $||L_{\underline{z}\chi}\eta||^2_{L^2}$ as the inner product $\langle L_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\eta\rangle$ by using the positivity of $L_{\underline{z}\chi}$ exemplified in Lemma (3.0.2):

$$\begin{aligned}\langle L_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\eta\rangle &= \langle L^{\frac{1}{2}}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L^{\frac{1}{2}}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle\end{aligned}$$

Invoking the orthogonal projection operators $\Pi_{\underline{z}\chi}$ and $\Pi^{\perp} = 1 - \Pi_{\underline{z}\chi}$, along with $\Pi_{\underline{z}\chi}\Pi^{\perp}_{\underline{z}\chi} = \Pi^{\perp}_{\underline{z}\chi}\Pi_{\underline{z}\chi} = 0$, $\Pi^2_{\underline{z}\chi} = \Pi_{\underline{z}\chi}$ and $\Pi^*_{\underline{z}\chi} = \Pi_{\underline{z}\chi}$ we have

$$\begin{aligned}\langle L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle &= \langle(\Pi_{\underline{z}\chi}+\Pi^{\perp}_{\underline{z}\chi})L^{\frac{1}{2}}_{\underline{z}\chi}\eta, (\Pi_{\underline{z}\chi}+\Pi^{\perp}_{\underline{z}\chi})L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &+ \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle\Pi_{\underline{z}\chi}\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &+ \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi^{\perp}_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle L_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle\end{aligned}$$

Add $\Pi_{\underline{z}\chi} + \Pi^{\perp}_{\underline{z}\chi}$ to $\langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\rangle$

$$\begin{aligned}\langle L_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\eta\rangle &= \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}(\Pi_{\underline{z}\chi}+\Pi^{\perp}_{\underline{z}\chi})L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle L_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi^{\perp}_{\underline{z}\chi}L_{\underline{z}\chi}\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &+ \langle L^{\frac{1}{2}}_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &= \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle + \langle\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\Pi^{\perp}_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\ &+ \langle L_{\underline{z}\chi}\Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\chi, \Pi_{\underline{z}\chi}L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle\end{aligned}$$

Using parts (f) and (g) of Theorem (3.0.2) along with $||a+b|| \geq ||a|| - ||b||$ we have

$$\begin{aligned}
\langle L_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\eta\rangle &\geq |\langle \Pi^{\perp}_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\Pi^{\perp}_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle| - |\langle \Pi^{\perp}_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta, L_{\underline{z}\chi}\Pi_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle \\
&+ \langle L_{\underline{z}\chi}\Pi_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta, \Pi_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta\rangle| \\
&\geq c_1||L^{\frac{1}{2}}\eta||^2_{\mathbf{H}^1} - c_2\epsilon\ log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)||\eta||^2_{\mathbf{H}^1} \\
&\geq c_1'||\eta||^2_{\mathbf{H}^2} - c_2'\epsilon\ log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)||\eta||^2_{\mathbf{H}^2} \\
&= \left(c_1' - c_2'\epsilon\ log^{\frac{1}{2}}\left(\frac{1}{\epsilon}\right)\right)||\eta||^2_{\mathbf{H}^2} \\
&\geq c_3||\eta||^2_{H^2}.
\end{aligned}$$

In the second inequality above, we made use of Theorem 3.0.2 (g) along with $\Pi_{\underline{z}\chi} L^{\frac{1}{2}}_{\underline{z}\chi}\eta \in [Ran(\Pi_{\underline{z}\chi}]^{\perp}$ for the first term, and Theorem 3.0.2 (f) for the second term. Taking the root of each side leaves

$$||L_{\underline{z}\chi}\eta||_{L^2} \geq c||\eta||_{H^2}.$$

E. IMPLICIT FUNCTION THEOREM ARGUMENT FOR PROPOSITION 6.0.2

In this Appendix, we use an Implicit Function Theorem (IFT) argument to prove (6.0.2). We begin by expanding $W'_{ext}(\underline{z})$ around $\underline{b}$ in a Taylor Series,

$$W'_{ext}(\underline{z}) = W'_{ext}(\underline{b}) + W''_{ext}(\underline{b})(\underline{z} - \underline{b}) + N(\underline{z} - \underline{b}),$$

where $N(\underline{z} - \underline{b})$ consists of the remaining higher order terms and is of the order $N(\underline{z} - \underline{b}) = O\left(\delta^4\sqrt{\epsilon}(\underline{z} - \underline{b})^2\right)$. For simplicity rename $\underline{a} = \underline{z} - \underline{b}$. Setting $W'_{ext}(\underline{z}) = 0$ and solving for $\underline{a}$, we arrive at

$$\underline{a} = -W''_{ext}(\underline{b})^{-1}\left[W'_{ext}(\underline{b}) + N(\underline{a})\right]$$

due to the invertibility of $W''_{ext}(\underline{b})$ from (6.0.21). Set

$$H(\underline{a}) = -W''_{ext}(\underline{b})^{-1}\left[W'_{ext}(\underline{b}) + N(\underline{a})\right].$$

We want to show that H has a fixed point. More specifically

a) For every $\alpha > 0$ such that $H : B_{\mathbb{R}^2}(0,\alpha) \to B_{\mathbb{R}^2}(0,\alpha)$

b) H is a strict contraction on $B_{\mathbb{R}^2}(0,\alpha)$.

First we show a) H maps $B_{\mathbb{R}^2}(0,\alpha) \to B_{\mathbb{R}^2}(0,\alpha)$, where $\alpha = O(\frac{1}{\delta})$. Taking the modulus of $|H(\underline{a})|$

$$|H(\underline{a})| \leq ||W''_{ext}(\underline{b})^{-1}|| \left[|W'_{ext}(\underline{b})| + |N(\underline{a})|\right].$$

By (6.0.16), (6.0.20) and (6.0.21) for $n = 3$ we have

$$\begin{aligned} |H(\underline{a})| &\leq ||W''_{ext}(\underline{b})^{-1}|| \ [\ |W'_{ext}(\underline{b})| + |N(\underline{a})|\ ] \\ &\leq \left(\frac{c}{\sqrt{\epsilon}\delta^3}\right) [\ \sqrt{\epsilon}\delta^4 + \sqrt{\epsilon}\delta^4|\underline{a}|^2\ ] \\ &\leq c_1\delta + c_2\delta|\underline{a}|^2. \end{aligned}$$

Setting $|\underline{a}| \leq \alpha$ and allowing for $|H(\underline{a})| \leq \alpha$ we come to

$$c_1\delta + c_2\delta\alpha^2 \leq \alpha$$

and solving for α we arrive at

$$\alpha = \frac{1 \pm \sqrt{1 - 4c_1c_2\delta^2}}{2c_2\delta}$$

with the following bound imposed on δ, $\delta \leq \sqrt{\frac{1}{4c_1c_2}}$, which holds due to our assumption, $\delta \ll 1$, in Theorem 2.2.1. From here it can be seen that $\alpha = O(\frac{1}{\delta})$.

Now we demonstrate that the map H is a strict contraction on $B_{\mathbb{R}^2}(\underline{b}, \alpha)$ where $\alpha = O(\frac{1}{\delta})$.

Replace the term $W'_{ext}(\underline{b}) + N(\underline{a})$ in $H(\underline{a})$ with $W'_{ext}(\underline{z}) - W_{ext}(\underline{b})\underline{a}$

$$H(\underline{a}) = -W''_{ext}(\underline{b})^{-1} \left[W'_{ext}(\underline{z}) - W''_{ext}(\underline{b})\underline{a}\right].$$

Next, differentiate $H(\underline{a})$ with respect to $\underline{z}$:

$$H'(\underline{a}) = -W''_{ext}(\underline{b})^{-1} \left[W''_{ext}(\underline{z}) - W''_{ext}(\underline{b})\right].$$

Taking the norm of the linear map, $H'(a)$, we have

$$||H'(\underline{a})|| \leq || - W''_{ext}(\underline{b})^{-1}|| \; [\; ||W''_{ext}(\underline{z}) - W''_{ext}(\underline{b})|| \;]$$

and by the continuity of W_{ext} exemplified in (6.0.19) the bracketed term in $H(\underline{a})$ can be adjusted such that $||W''_{ext}(\underline{z}) - W''_{ext}(\underline{b})|| \leq \frac{1}{2}|| - W''_{ext}(\underline{b})^{-1}||^{-1}$. This results in

$$\begin{aligned} |H(\underline{a})| &\leq || - W''_{ext}(\underline{b})^{-1}|| \left(\frac{1}{2}|| - W_{ext}(\underline{b})^{-1}||^{-1}\right) \\ &\leq \frac{1}{2} \end{aligned}$$

and $H(\underline{a})$ is a strict contraction. This can be taken one step further with the Mean Value Theorem.

$$\begin{aligned} |\underline{a}| &\leq |H(\underline{a})| \\ &\leq |H(0)| + |H(\underline{a}) - H(0)| \\ &\leq c\delta + \frac{1}{2}|\underline{a}|. \end{aligned}$$

It can then be concluded that $H(\underline{a})$ is a strict contraction with a fixed point $\underline{z}_0$ satisfying $|\underline{z}_0 - \underline{b}| < c\delta$, such that $\underline{z}_0$ is also a critical point of W_{ext}. Now it is necessary to demonstrate that $W(z)$ has critical points at $b_1, \ldots, b_m$ when $W_{ext}(\underline{z})$ has a critical point at $z_0 := (z_{01}, \ldots, z_{0m})$. The same Implicit Function Theorem argument is used as above for this case.

For fixed $j = 1, \ldots, m$, expand $W'(z)$ as a Taylor Series around z_{0j}

$$W'(z) = W'(z_{0j}) + W''(z_{0j})(z - z_{0j})$$

where we set $(z - z_{0j}) = a$, $W'(z) = 0$ and solve for a

$$a = -W''(z_{0j})^{-1} \left[\, W'(z_{0j}) + N(a) \,\right].$$

Define $\bar{H}(a) = -W''(z_{0j})^{-1} [\, W'(z_{0j}) + N(a) \,]$. Here we wish to show that $\bar{H}$ maps $B_{\mathbb{R}^2}(0, \alpha) \to B_{\mathbb{R}^2}(0, \alpha)$ where $\alpha = O(\frac{1}{\delta})$ and $\bar{H}$ is a strict contraction. To show that $\bar{H} : B_{\mathbb{R}^2}(0, \alpha) \to B_{\mathbb{R}^2}(0, \alpha)$, take its modulus

$$|\bar{H}(a)| \leq || - W''(z_{0j})^{-1}|| \, [\, |W'(z_{0j})| + |N(a)| \,]$$

At $\underline{z}_0 := (z_{01}, \ldots, z_{0j})$, $W'_{ext}(\underline{z}_0) = 0$ and so by (6.0.18)

$$W'(z_{0j}) = O(\sqrt{\epsilon}\delta^4), \tag{E.0.1}$$

for all $j = 1, \ldots, m$.
From condition (B) in Theorem 2.2.1

$$W''(z_{0j}) = O(\sqrt{\epsilon}\delta^3)$$

and

$$N(\bar{a}) = O(\sqrt{\epsilon}\delta^4).$$

Since $\delta \ll 1$, (E.0.1) implies that $W'(z_{0j}) \ll \sqrt{\epsilon}\delta^2$ and since $\underline{z}_0 \in \Omega_{\epsilon\delta\underline{z}}$, $W''(z_{0j})$ is invertible and bounded by $||W''(\underline{z}_0)^{-1}|| \leq c(\sqrt{\epsilon}\delta^3)$ for all $j = 1, \ldots, m$, it follows

$$\begin{aligned} |\bar{H}(a)| &\leq || - W''(z_{0j})^{-1}|| \, [\, |W'(z_{0j})| + |N(a)| \,] \\ &\leq c_3\delta + c_4\delta|a|^2. \end{aligned}$$

Similarly to what has previously been shown, we find α to be

$$\alpha = \frac{1 \pm \sqrt{1 - 4c_3c_4\delta^2}}{2c_3\delta},$$

δ with the bound, $\delta \leq \sqrt{\frac{1}{4c_3c_4}}$ and $\alpha = O(\frac{1}{\delta})$.

The final step is to show that $\bar{H}(a)$ is a contraction on $B_{\mathbb{R}^2}(0, \alpha)$, as with the W_{ext} case we write $\bar{H}$ as

$$\bar{H}(a) = -W''(z_{0j})^{-1} \left[W''(z) - W''(z + 0)a\right]$$

and after taking a derivative

$$\bar{H}'(a) = -W''(z_{0j})^{-1} \left[W''(z) - W''(z_{0j})\right].$$

Taking the modulus and adjusting the value of a such that $|W''(z) - W''(z_{0j})| \leq \frac{1}{2}|| - W''(z_{0j})^{-1}||^{-1}$ yields

$$|\bar{H}'(a)| \leq \frac{1}{2}$$

which in fact is a contraction. In addition, by adding zero and applying the Mean Value Theorem to a, we arrive at

$$
\begin{aligned}
|a| &\leq |\bar{H}(0)| + |\bar{H}(a) - \bar{H}(0)| \\
&\leq c_5\delta + \frac{1}{2}|a|
\end{aligned}
$$

which is a strict contraction and thus $\bar{H}(a)$ has a unique fixed point b_j such that $|b_j - z_{0j}| \leq c\delta$ for all $j = 1, \ldots, m$. Therefore, $W'(b_j) = 0$ for all $j = 1, \ldots, m$ with $|b_j - z_{0j}| \leq c\delta$. Demonstrating that $\underline{b} := (b_1, \ldots, b_m) \in \Omega_{\epsilon\delta\underline{z}}$ follows from the sufficient part of proposition

□

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