Probabilistic Reliability Analysis of Slopes with Truncated Random Variables from Censored Samples

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Abstract

Probabilistic reliability analysis and design have been a research topic of intense interest in geotechnical engineering in recent decades due to the inherent uncertainty in soil property data obtained from field or laboratory testing. Determination of the underlying probability distributions of soil properties and corresponding parameters from observed data is a critical initial step because subsequent risk and reliability analyses depend upon these evaluations. Conventionally, the choice of a probability distribution is dictated by subjective familiarity with a classical distribution, such as Normal or Lognormal. Furthermore, only censored/truncated samples can sometimes be obtained due to technical and environmental limitations.

This research presents an objective and unbiased method to estimate truncated probability distributions of soil parameters using the MaxEnt method constrained by moments of censored/truncated samples and the Akaike information criterion (AIC). This method is described as "objective" because it relies solely on data and constraints rather than subjective choices, and "unbiased" because it avoids assumptions that could skew the distribution, thereby providing a more accurate and representative model of the soil parameters' probability distribution. The probability distribution is based on the concept of MaxEnt and is free from the assumptions of classical distributions. A first-order reliability method (FORM) is presented based on truncated MaxEnt distributions. The new method is applied to the probabilistic reliability analysis and design of Nipigon River slopes including parameters such as unit weight, friction angle, and cohesion of various soil types, to perform the probabilistic reliability analysis. The accuracy of these parameter estimations, drawn from field measurements and reported studies, is crucial for the reliability assessments conducted in this research.

Keywords: Reliability analyses; Maximum Entropy Principle; MaxEnt; Truncated random variable; Concerned sample; FORM.

Dedicated to

Vali & Mahoor

Who are my motivation for progress and hope for the future.

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Contents

Abstract III
Acknowledgment VI
Contents VII
List of Tables IX
List of Figures XI
Abbreviations XIII
Chapter 1: Introduction 1
1.1 Literature Review 2
1.1.1 Probabilistic Slope Stability 4
1.1.2 Target Reliability Indices 5
1.1.3 Reliability Judgment 6
1.2 Censored Samples and Truncated Random Variables 7
1.2.1 Left, Right, or Interval Censoring 13
1.2.2 Type I, Type II, or Random Censoring 14
1.3 Reliability Analysis with Truncated Random Variables 15
1.4 Thesis Outline 17
Chapter 2: Truncated Functions from Censored Samples 18
2.1 Maximum Entropy Principle 18
2.2 Extended Maximum Entropy Principle 21
2.2.1 Maximum Entropy Function from Truncated Sample 22
2.2.2. Optimal Order 23
2.3 Examples of Soil Parameters 24
2.3.1 Nipigon River Landslide in Canada (Case Study) 24
2.3.2 Frequency Histogram 27
2.3.3 Normal and Lognormal Distributions 27
2.3.4 Maximum Entropy Distribution 29
2.3.5 Truncated Normal and Lognormal Distribution 31
2.3.6 Truncated Maximum Entropy Functions from Censored Samples 37
2.3.7 Statistical Goodness-of-fit Test 45
2.3.8 Truncated Statistical Goodness-of-fit Test 48
2.4 Summary 52

Chapter 3: Reliability Method with Truncated Random Variables 54
3.1 First-Order Reliability Method (FORM) 54
3.1.1. First-Order Reliability Method with Truncated Random Variables 55
3.2 Illustrative Examples 57
3.2.1. First-Order Reliability Method with Untruncated Random Variables 57
3.2.2. First-Order Reliability Method with Truncated Random Variables 61
3.3 Summary 70
Chapter 4: Reliability Analysis of Slopes in Nipigon River Landslide 72
4.1 Nipigon River Landslide 72
4.1.1. General Geology 73
4.2 Reliability Analysis of Slopes with Truncated Random Variables in the Case Study 75
4.2.1. Limit State Function 77
4.2.2. Determination of Distributions and Parameters from Observed Data 80
4.2.3. Reliability Analysis of Slopes with Random Variables (Truncated and Untruncated) 83
4.3. Discussions on the Effects of Various Parameters on Slope Reliability 93
4.4. Reliability-Based Design on Slope Stability 94
4.5 Summary 97
Chapter 5: Conclusions and Recommendations 99
5.1 Conclusions 99
5.2 Recommendations 101
Appendix 1 103
Appendix 2 109
Appendix 3 140
Bibliography 141

List of Tables

Table 1.1: Target reliability indices (Engineering and Design Report, 1995).....6
Table 1.2: Probability of failure criteria of slope (Santamarina, Altschaeffl, & Chameau, 1992) 7
Table 1.3: Comparison of censored, truncated, and sample-selected cases (Richard Breen, 1996). 9
Table 2.1. Undrained soil shear strength (kPa) was measured at the Nipigon River Landslide area as a censored sample (n = 121). 26
Table. 2.2. Coefficients of Maximum Entropy Distribution of Soil Strength of Frictional Angle (Moment Order K = 1–5). 30
Table 2.3. Comparison of the truncated moments of the Normal and Lognormal Distributions with the untruncated moments. 32
Table 2.4. Truncated censored sample from the Nipigon River Landslide with different percentages on both sides.....37
Table 2.5. Comparison of partial raw moment with untruncated raw moment on both sides of truncation. 38
Table. 2.6. Left Truncated Coefficients of Entropy Distribution of Soil Strength (Moment Order K = 1–5). 38
Table. 2.7. Right Truncated Coefficients of Entropy Distribution of Soil Strength (Moment Order K = 1–5).....42
Table 2.8. Chi-Square Results for Soil Shear Strength with Normal, Lognormal, and Entropy Distributions.....48
Table 2.9. Left Truncated Chi-Square Results for Soil Shear Strength with Normal, Lognormal, and Entropy Distributions. 49
Table 2.10. Right Truncated Chi-Square Results for Soil Shear Strength with Normal, Lognormal, and Entropy Distributions. 50
Table 3.1. Statistics of basic variables in reliability analysis of settlement in consolidated clay. 58
Table 3.2. Untruncated coefficients of Entropy Distribution λK with different orders (1-5). .59
Table. 3.3. Comparison of untruncated moments about the origin and truncated raw moments (K = 1–5).....60
Table 3.4. A comparison of the probability of failure between Normal, Lognormal, and Entropy Distribution methods.....63
Table 3.5. Statistics of basic variables in reliability analysis and constant numbers. 65
Table 3.6. Untruncated coefficients of Entropy Distribution λK with different orders (1-5). ..66
Table. 3.7. Comparison of the untruncated raw moment and truncated moment about the origin on both sides (K = 1–5).....66
Table 3.8. A comparison of the probability of failure between Normal, Lognormal, and Entropy Distribution methods.....67
Table 4.1. The parameters of the Nipigon River Landslide (Dodds. et. al., 1993). 81

Table 4.2. The parameters were obtained from the information on the Nipigon Landslide. ...82
Table 4.3. Stability analyses (Dodds et al., 1993)..82
Table 4.4. Random variables with mean and standard deviation of the Nipigon. 83
Table 4.5. Coefficients of Entropy Distribution (Moment order K=1-3) with untruncated and truncated levels. 86
Table 4.6. The comparison of the reliability methods on the coefficients of embankment as a random variable. 87
Table 4.7. Random variables with mean and standard deviation of the Nipigon. 88
Table 4.8. Coefficients of Entropy Distribution λ (Moment order K=3, optimal order) with untruncated and truncated levels...89
Table 4.9. The comparison of the reliability methods on the coefficients of embankment as a random variable. 90
Table 4.10. Coefficients of Entropy Distribution λ (Moment order K=3, optimal order) with untruncated and truncated levels...91
Table 4.11. The comparison of the reliability methods on the coefficients of embankment as a random variable. 92

List of Figures

Figure 1.1. Full range of y and full range of x (University of Huston. Lectures 8 & 9). 9
Figure 1.2. Indeterminacy of y Value at $y \geq 6$ (University of Huston. Lectures 8 & 9)..........10
Figure 1.3. The PDF of the exchange rate, St, as a mixture of discrete (mass at $St = E$ and continuous (Prob $[St < E]$) distributions (University of Huston. Lectures 8 & 9). 10
Figure 1.4. Indeterminacy of y Value at $y \leq 5$ (University of Huston. Lectures 8 & 9)..........11
Figure 1.5. The PDF of the observable variable, y, as a mixture of discrete (prob. mass at y=5) and continuous (Prob $[y^*>5]$) distributions (University of Huston. Lectures 8 & 9). 11
Figure 1.6. Allocation of complete probability within the censored region to the censoring point, 5 (University of Houston, Lectures 8 & 9). 12
Figure 1.7. Indeterminacy of x (or y) value at y< 3, illustrating truncation from below (University of Huston. Lectures 8 & 9). 12
Figure 1.8. Representation of a censored distribution as a combination of a PMF and a PDF, illustrating data censoring (University of Houston, Lectures 8 & 9)......................................13
Figure 2.1. The location of Nipigon (Art source international). 25
Figure 2.2. Stratigraphic Section (Dodds & Burak, 1993). 26
Figure 2.3. Histogram and Frequency Diagram of the soil's undrained strength of Nipigon..29
Figure 2.4. Maximum Entropy Distribution of the soil's undrained strength of Nipigon. 31
Figure 2.5. Truncated Normal and Lognormal Distribution of soil undrained strength of Nipigon (Left) 34
Figure 2.6. Truncated Normal and Lognormal Distribution of soil undrained strength of Nipigon (Right)..36
Figure 2.7. Frequency histogram and %1 Left truncated MaxEnt (K=1-5). 39
Figure 2.8. Frequency histogram and %5 Left truncated MaxEnt (K=1-5). 40
Figure 2.9. Frequency histogram and %8 Left truncated MaxEnt (K=1-5). 40
Figure 2.10. Frequency histogram and %10 Left truncated MaxEnt (K=1-5). 41
Figure 2.11. Frequency histogram and %1 Right truncated MaxEnt (K=1-5). 43
Figure 2.12. Frequency histogram and %5 Right truncated MaxEnt (K=1-5). 44
Figure 2.13. Frequency histogram and %8 Right truncated MaxEnt (K=1-5). 44
Figure 2.14. Frequency histogram and %10 Right truncated MaxEnt (K=1-5). 45

Figure 3.1. Soil profile for settlement in consolidated clay. 58
Figure 3.2. Untruncated Maximum Entropy Distribution, Normal, and Lognormal density function. 60
Figure 3.3. Harbor breakwater against sliding. 65
Figure 3.4. Untruncated Maximum Entropy Distribution, Normal, and Lognormal Distribution. 67
Figure 4.1. Location of the landslide (Google Maps) 74
Figure 4.2. Notations for Embankment on the Weak Foundation (Low, 1989). 77
Figure 4.3. Stability Factors $N1,$ and $N2,$ for Embankments on Weak Foundations (Low, 1989). 79
Figure 4.4. Coefficient λ of $\tan \varphi m$(Low, 1989). 79
Figure 4.5. Cross section of the Nipigon Landslide. 81
Figure 4.6. Maximum Entropy Distribution, Normal, and Lognormal density functions when all are untruncated. 84
Figure 4.7. Maximum Entropy Distribution, Normal, and Lognormal density functions when all are untruncated. 89
Figure 4.8. Maximum Entropy Distribution, Normal, and Lognormal density functions when all are untruncated. 91

Abbreviations

PDF = Probability Density Function

CDF = Cumulative Density Function

MEF = Maximum Entropy Function

AIC = Akaike's Information Criterion

FORM = First Order Reliability Method

MCS = Monte Carlo Simulation

RSM = Response Surface Method

RBD = Reliability-Based Design

VST = Vane Shear Test

F_s = Factor of Safety

β = Reliability Index

γ = Unit Weight

p_f = Probability of Failure

C = Cohesion

phi (φ) = Friction Angle

S_u = Undrained Shear Strength

μ = Mean, Average value

σ= Standard deviation

COV= Coefficient of Variance

Chapter 1: Introduction

One of the most common problems faced by geotechnical engineers is slope stability assessment. The predictions of slope stability in soil or rock masses are very important for the design of reservoir dams, roads, tunnels, excavations, open pit mines, and other engineering structures. It is the importance of the slope stability problem that has let to alternate methods for evaluating the safety of a slope. The problem of slope stability is related to risk and reliability. Thus, a single factor of safety cannot be relied on for taking safety measures against slope failure. Analyzing the reliability of slopes involves the calculation of the reliability index for a slope, or alternatively, the probability of failure of a slope (Ullah et al., 2020).

Soil properties exhibit considerable spatial variability due to their different physical and chemical characteristics. This spatial variability brings uncertainties to the estimation of soil properties. Phoon and Kulhawy (1999) indicated that these uncertainties mainly occurred from the following three sources: inherent soil variability, measurement errors, and model transformation uncertainties. Consequently, the probabilistic slope stability analysis allows a comprehensive technique to evaluate the probability of slope failure by incorporating slope-specific variabilities and uncertainties. The design enhanced by probabilistic analysis is also more economical and less likely to collapse in the event of a geohazard.

The primary goal of the thesis is to conduct a probabilistic reliability analysis of slopes based on soil data gathered from the Nipigon River Landslide location through on-site soil tests and reports from the site. By relying on the principle of maximum entropy, the probability distribution is formulated without relying on classical distribution assumptions. Subsequently, a first-order reliability method is introduced, grounded in truncated maximum Entropy Distributions. This novel method is then utilized in probabilistic reliability analysis and the design of slopes along the Nipigon River. Additionally, strategies to alleviate and forestall landslides along the Nipigon River are suggested.

1.1 Literature Review

Soil is a basic geomaterial in geotechnical and geological engineering. It has been widely realized that there is inherent spatial variability in the soil parameters for geotechnical engineering design and practice (Zhang et al., 2022). Generally, it is challenging to accurately determine uncertain soil properties because the test data obtained from a specific site may be limited. Based on Vanmarcke (1977), the spatial variability of soil properties is usually characterized by random fields in the analysis and design of the geotechnical structures. The stability of slopes is a complex issue in geotechnical engineering due to the uncertainties and the possibility of multiple failure surfaces (Zhang et al., 2021). More importantly, previous researchers have concluded that if the spatial variability of soil properties is not considered for risk assessment of slope stability, the slope failure probability may be overestimated. Therefore, the probabilistic analysis of slope stability is a rational tool that can take the spatial variability of soil/ rock parameters into account, and then the slope stability can be rationally assessed by failure probability (P_f) or reliability index (β) from a probabilistic perspective (Li et al., 2016).

To obtain P_f, a lot of reliability methods have been developed for slope reliability analysis, such as the first-order reliability method (FORM), Monte Carlo simulation (MCS), and some other advanced methods. Although the direct MCS can provide accurate evaluations of slope stability with the merits of conceptual simplicity and operational convenience for geotechnical engineers, its computation process may be time-consuming. To this end, response surface methods (RSMs) are proposed as an efficient tool to approximate the implicit performance function for improving computational efficiency, which can construct the explicit functions between the slope stability responses and the input random variables (Ji et al., 2012; Wang et al., 2011; Au & Beck, 2001).

In the probabilistic approach, the initial and crucial step involves modeling and understanding uncertainties related to random variables. This matter is significant because the reliability analysis of a structure relies heavily on the accurate characterization of random variables. Expressing uncertainties in these variables commonly involves the use of probability curves, typically in the format of probability distribution curves along with their associated parameters. However, traditional methods for determining these distributions and parameters from sample data face constraints. They are bound by the assumption of specific standard theoretical distributions and are influenced by the size of the available samples (Deng et al., 2004). A more streamlined and convenient method for assessing sample information involves using sample moments. Alternatively, the maximum entropy principle serves as a crucial approach for fitting distributions. This principle relies on Shannon's entropy, a measure of uncertainty widely adopted in various engineering disciplines for estimating distribution functions (Sobczyk & Trcebicki, 1999; Zhang & Gu., 2015; Li & Xu., 2011).

The maximum entropy method (MaxEnt) provides an unbiased estimate of the probability density function (PDF), representing the most probable PDF among all possible sets of density functions under moment constraints. Li et al. (2012) employed MEM to estimate the probability density function and assess slope stability. They utilized a fourth-moment procedure and the maximum entropy principle for reliability analysis of earth slopes. Lindley

(1956) was the first to use information theory to quantify information in Bayesian analysis, while Commenges (2015) demonstrated its application in statistics, particularly in biostatistics. Furthermore, Baker (1990) introduced a procedure for estimating probability density functions based on information theory concepts. This method combined Jayne's maximum entropy formalism with Akaike's information criterion to select the optimal model from a group based on order. Baker validated this approach by applying it to structural live loads, soil parameters, and stochastic foundation design.

Subsequently, the concept of cross entropy was introduced as a refined approach to blend a prior distribution with available data (Deng & Pandey., 2009b, 2000; Sobczyk., 2003). Moreover, Deng & Pandey (2009a) developed a robust quantile function that fits exceptionally well for small sample sizes using maximum entropy. In another work, Deng & Pandey (2008b) devised an estimation method combining Monte Carlo simulations and optimization algorithms to compute fractions of probability-weighted moments for generating the best-fit quantile function. Hence, maximum entropy has been employed in various fields of engineering simultaneously with geotechnical engineering. In the following subsections, the critical aspects of slope stability analysis, including probabilistic slope stability, target reliability indices, and reliability judgment will be represented.

1.1.1 Probabilistic Slope Stability

To assess the reliability or probability of failure of a system, the initial step is to establish a performance function, which outlines the relationship between various parameters, known as variables X_i(Haldar & Mahadevan., 2000). The performance function is defined as:

$$Z = g(X_1, X_2, \dots, X_n), \quad (1.1)$$

The performance function or limit state, which represents the failure surface, is defined as $Z = 0$ (Haldar & Mahadevan, 2000). This limit state is the boundary that separates the safe and unsafe regions, and it plays a critical role in assessing the reliability or probability of failure of the system. It can be an explicit or implicit function of the basic random variables. The failure occurs when $Z < 0$ or $P(R < S)$, where R is resistance and S is load. Therefore, the probability of failure, p_f is given by the integral:

$$P_f = \int \ldots \int_{g()<0} f_X\,(x_1, x_2, \ldots, x_n)\mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n. \quad (1.2)$$

in which $f_X(x_1, x_2, \ldots, x_n)$ is the joint probability density function for the basic random variables $X_1, X_2, \ldots, X_n$ and the integration is performed over the failure region, that is, $g() < 0$. The computation of p_f by Eq. (1.2) is called the full distributional approach and can be the fundamental equation of reliability analysis. In general, the joint probability density function of random variables is practically impossible to obtain. Even if this information is available, evaluating the multiple integral is difficult. Therefore, one approach is to use analytical approximations of this integral that are simpler to compute. To clarify the presentation, these methods can be grouped into two types, namely, first-order reliability methods (FORM) and second-order reliability methods (SORM) (Haldar & Mahadevan., 2000).

1.1.2 Target Reliability Indices

The reliability index, denoted by β, serves as a gauge for assessing the dependability of an engineering system, encompassing both the underlying mechanics of the issue and the uncertainties associated with input variables. Structural engineers devised this index to offer a means of evaluating relative reliability, avoiding the need to assume or ascertain the specific probability distribution required for a precise calculation of failure probability. Formulated concerning the expected value and standard deviation of the performance function, the reliability index enables the comparison of reliability across various structures or performance modes, eliminating the necessity of computing absolute probability values (Engineering and Design Report, 1995).

Reliability indices serve as a comparative assessment of the existing state, offering a qualitative estimation of anticipated performance. Embankments boasting higher reliability indices are anticipated to function effectively, while those with lower indices are likely to exhibit suboptimal performance, posing significant challenges for rehabilitation. In cases where reliability indices are extremely low, the embankment might be categorized as a hazard. The target reliability values outlined in Table 1.1 are generally recommended for use in this context.

Table 1.1: Target reliability indices (Engineering and Design Report, 1995).

Expected Performance Level	Beta	Probability of Unsatisfactory Performance
High	5	0.0000003
Good	4	0.00003
Above average	3	0.001
Below average	2.5	0.006
Poor	2.0	0.023
Unsatisfactory	1.5	0.07
Hazardous	1.0	0.16

1.1.3 Reliability Judgment

The reliability index is a better measure than the factor of safety's probability density function to determine the current and future condition of a slope or structure. It gives a more accurate understanding of the reliability of the slope, and higher values indicate greater reliability, while lower values indicate greater risk. According to Shien (2005), slopes with a low reliability index are considered hazardous. Santamarina et al. (1992) developed a set of criteria to assess the consequences of slope failure, which are presented in Table 1.2.

Table 1.2: Probability of failure criteria of slope (Santamarina, Altschaeffl, & Chameau, 1992)

Conditions	Criteria for Probability of Failure
Temporary structures with low repair cost	0.1
Existing large cuts on interstate highway	0.01
Acceptable in most cases Except if life may be lost	0.001
Acceptable for all slopes	0.0001
Unnecessarily low	0.00001

1.2 Censored Samples and Truncated Random Variables

A sample is considered truncated when certain observations are deliberately excluded. For instance, if data is exclusively available for individuals earning over $20,000 per year when studying income and education, this results in a truncated sample. Conversely, a censored sample occurs when no observations are omitted but specific information within them is withheld. In simpler terms, in a truncated sample, the missing observations are known, whereas in a censored sample, there is only partial information about the observations. Censoring takes place when data points exceeding or falling below a specified threshold are not precisely recorded but are instead reported as being at the threshold (Ao, 2009). In other words, the obvious definitions of them are:

Uncensored values refer to data points that are reported and used as they are without any modification or restrictions.

Censored values, on the other hand, are data points that are reported with certain constraints. They can be categorized into three types:

- Values reported as less than a specific threshold (e.g., < 5 ppb).
- Values reported as greater than a specific threshold (e.g., > 100 days).
- Values reported as an interval, indicating a range within which the value lies (e.g., between 67 and 75 degrees).

Truncated values are data points that are not reported if they exceed a certain limit. In other words, values beyond a predetermined threshold are not included or recorded in the dataset. In practical applications, truncated samples arise from various experimental situations in which sample selection is possible over only a partial range of the variable (Cohen, 2016).

Observed and censored environmental data are more commonly encountered than truncated data. Truncated data is relatively less common but can occur in specific situations. The two main scenarios that result in truncated data are:

- Exceedances: In certain cases, only concentrations or values above a specific reporting limit are recorded or reported. This means that values below the reporting limit are not included in the dataset, resulting in truncated data.
- Below detection-limit values: Sometimes, when values fall below the detection limit, they are incorrectly entered into a computer program as "<5" (indicating a value below 5, for example). However, many software programs interpret this as a character string rather than a numerical value. Consequently, when performing numerical analyses, the software may treat this value as missing or invalid, leading to the inadvertent truncation of the data.

It's important to be aware of these situations to avoid potential issues with truncation and ensure accurate analysis of the data. Moment and maximum likelihood estimators are the principal estimators for calculating estimates of distribution parameters from truncated and censored samples. (Cohen, 2016). Table 1.3 represents a comparison of censored and truncated samples based on Richard Breen's book (1996).

Table 1.3: Comparison of censored, truncated, and sample-selected cases (Richard Breen, 1996).

Sample	Y Variable	X Variable	Example
Censored	y is known exactly only if some criterion is defined. In terms of y, it is met.	x variables are observed for the entire sample, regardless of whether y is observed exactly.	Determinants of income; income is measured exactly only if it is above the poverty line. All other incomes are reported at the poverty line
Truncated	y is known only if some criterion defined in terms of y is met.	x variables are observed only if y is observed.	Donations to political campaigns.

When the full information of x and y variables are available, the diagram can be determined as Figure 1.1.

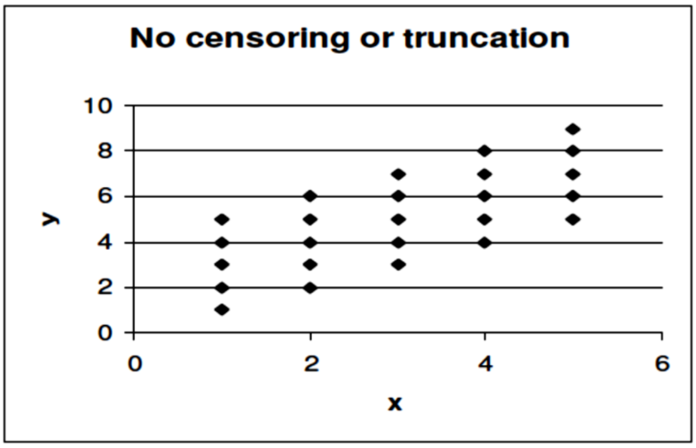


Figure 1.1. Full range of y and a full range of x (University of Huston. Lectures 8 & 9).

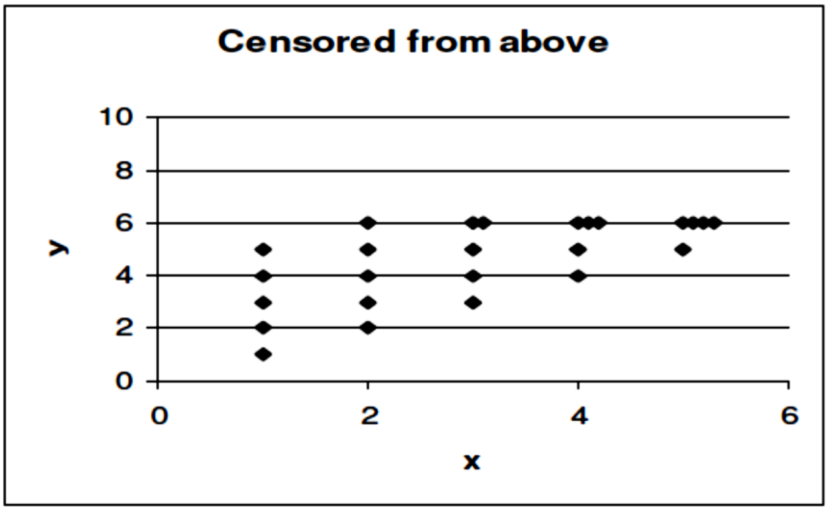


Figure 1.2. Indeterminacy of y Value at $y \geq 6$ (University of Huston. Lectures 8 & 9).

Example: A central bank intervenes if the exchange rate hits the band's upper limit.

=> If $S_t \geq E'$ =>$S_t = E'$

In the Figure 1.2, the information of $y \leq 6$ is available which is similar to Figure 1.3 ($S_t < E$), while the censored sample from below is illustrated in Figures 1.4 and 1.5.

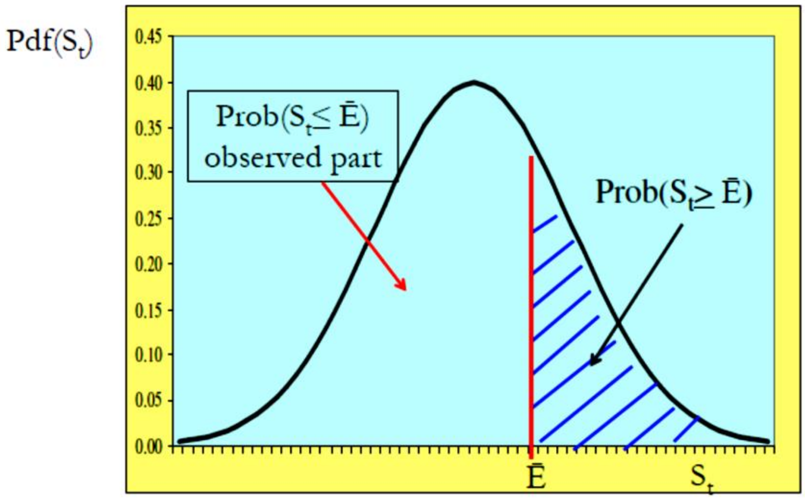


Figure 1.3. The PDF of the exchange rate, S_t, as a mixture of discrete (mass at $S_t = E$ and continuous (Prob [$S_t < E$]) distributions (University of Huston. Lectures 8 & 9).

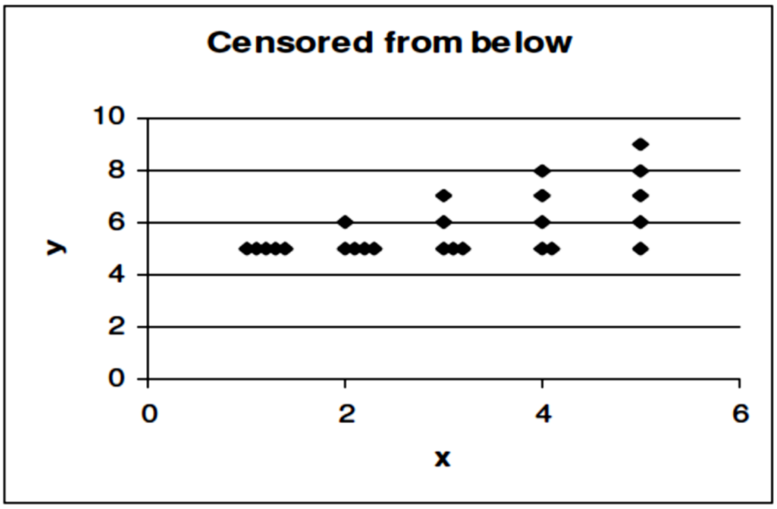


Figure 1.4. Indeterminacy of y Value at $y \leq 5$ (University of Huston. Lectures 8 & 9).

Example: A central bank intervenes if the exchange rate hits the band's lower limit.

=> If $S_t \leq \bar{E}$ => $S_t = \bar{E}$.

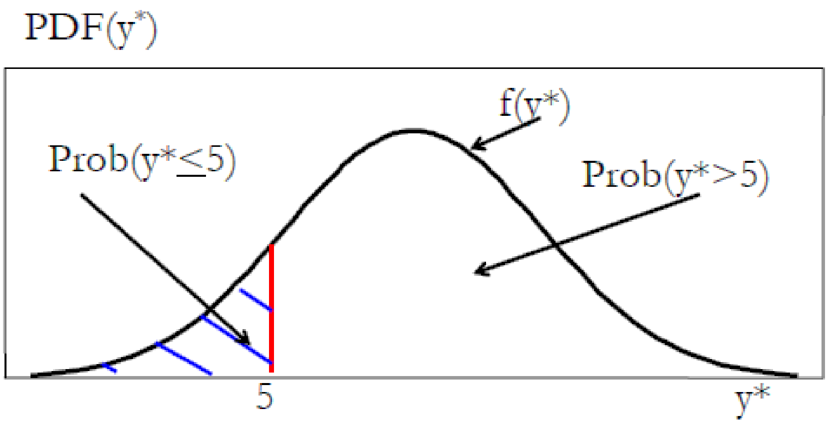


Figure 1.5. The PDF of the observable variable, y, as a mixture of discrete (prob. mass at y=5) and continuous (Prob [y*>5]) distributions (University of Huston. Lectures 8 & 9).

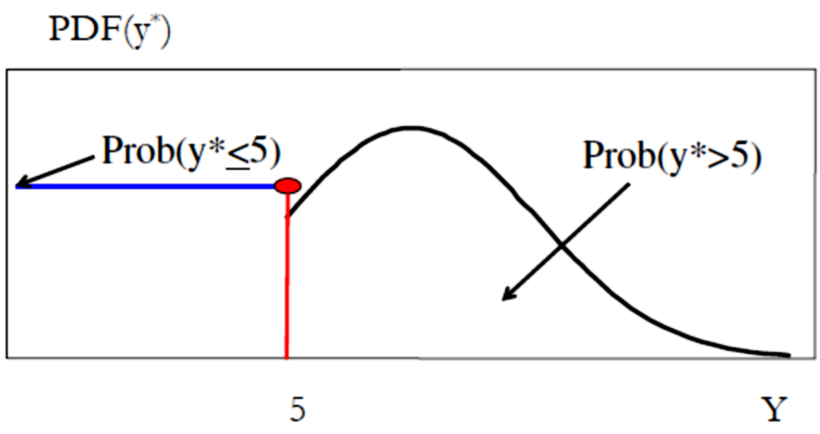


Figure 1.6. Allocation of complete probability within the censored region to the censoring point, 5 (University of Houston, Lectures 8 & 9).

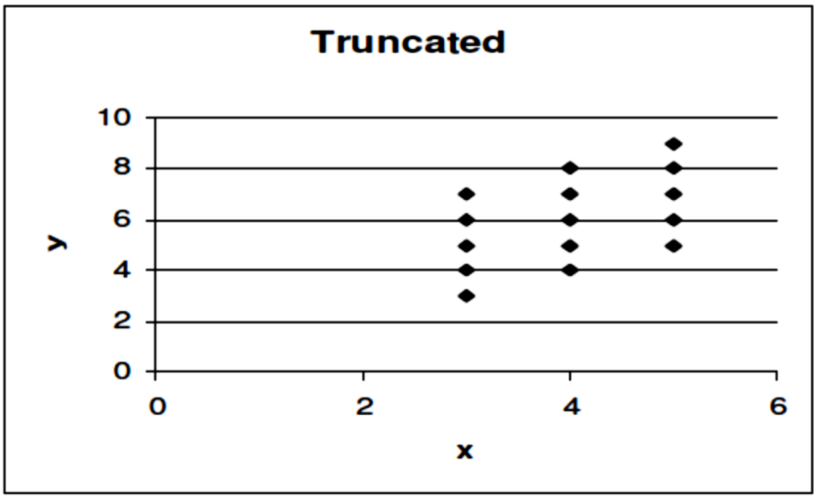


Figure 1.7. Indeterminacy of x (or y) value at $y< 3$, illustrating truncation from below (University of Huston. Lectures 8 & 9).

Figures 1.6, 1.7, and 1.8 illustrate the truncated sample which deliberately removes the information from the left side.

In cases where the income of a family falls below a defined threshold, our comprehension of the family's characteristics becomes unattainable. The cumulative sum of these probabilities equates to 1. The scenario takes on a distinct nature in the context of truncation, necessitating the establishment of a probability density function (PDF) for variable y, which involves the utilization of a conditional PDF.

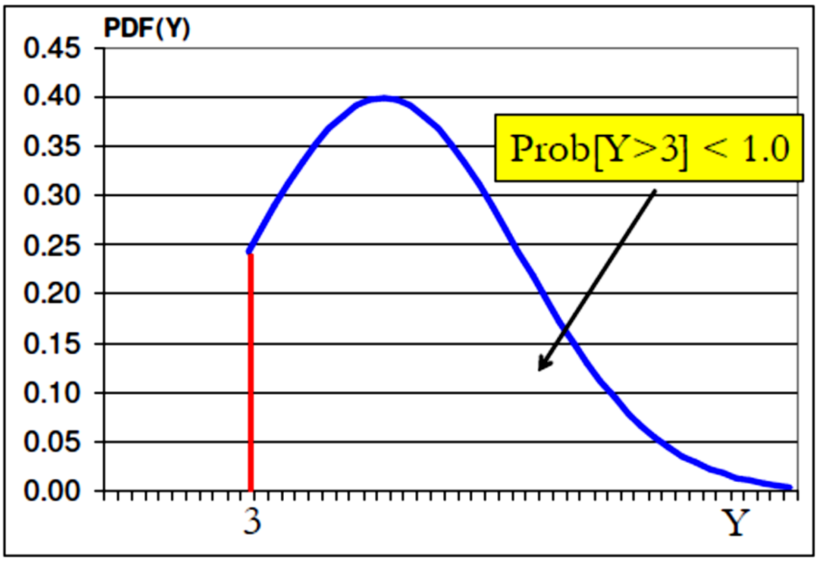


Figure 1.8. Representation of a censored distribution as a combination of a PMF and a PDF, illustrating data censoring (University of Houston, Lectures 8 & 9).

1.2.1 Left, Right, or Interval Censoring

A left-censored value refers to a data point that is known to be less than a specified value. For example, if a value is reported as "< 5 ppm," it means that the actual value is known to be less than 5 parts per million.

A right-censored value, on the other hand, is known to be greater than a specific value. For instance, if a value is reported as "> 100 days," it means that the actual value exceeds 100 days.

An interval-censored value is reported within a specified range or interval. For example, if a value is reported as "5 ppb < $X \leq$ 10 ppb," means that the true value falls within the interval between 5 parts per billion (exclusive) and 10 parts per billion (inclusive).

In some cases, observations of continuous random variables can be considered interval-censored because they are reported with limited decimal places. For instance, a reported value of X = 25 ppb might be interpreted as 24.5 ppb ≤ X < 25.5 ppb, indicating an interval that captures the reported value with greater precision.

When the intervals are small and the data range is limited, it is generally acceptable to treat the values as exactly observed, ignoring the fine-scale interval censoring. However, when the intervals are relatively large compared to the data range, such as having 10 or fewer intervals over the data range, it is better to consider the values as interval-censored. Truncated data can also be referred to as left-truncated, right-truncated, or very rarely, interval-truncated, depending on which portion of the data is excluded or truncated (Cohen, 2016).

1.2.2 Type I, Type II, or Random Censoring

In statistical analysis, different types of censorship can occur depending on the nature of the data and the circumstances of the study. The three main types of censoring are:

1. Type I Censoring: This occurs when the censoring levels (the values at which the observations are censored) are known in advance. The number of censored observations and the number of uncensored observations are random outcomes, even if the total sample size is fixed. Environmental data sets often involve type I censoring.
2. Type II Censoring: In type II censoring, both the sample size and the number of censored observations are fixed in advance. However, the censoring levels themselves

are random outcomes. This type of censoring is commonly encountered in time-to-event studies, where the study is planned to end after a specified number of failures. Type II censoring is sometimes referred to as failure-censored sampling.

3. Random Censoring: Random censoring occurs when both the number of censored observations and the censoring levels are random outcomes. This type of censoring is frequently observed in medical time-to-event studies. An example of random censorship is when a subject in a study moves away from the study area before the event of interest occurs. In this case, the outcome for the subject can be represented as a pair of random variables, (X, C), where X is the random time of the event, and C is the random time until the subject moves away. If $X < C$, then X is an observed value. If $X > C$, then X is right censored at C, indicating that the event did not occur before the subject moved away.

Understanding the type of censoring present in a dataset is crucial for appropriate statistical analysis and interpretation of results (Cohen, 2016).

1.3 Reliability Analysis with Truncated Random Variables

Utilizing ideal probability distributions to address uncertain parameters inevitably introduces a significant margin of error in reliability analysis. Enhancing precision in reliability analysis suggests a more rational and promising approach by employing truncated distributions to handle uncertain parameters within the structure. The convenience offered by ideal probability distributions may make them appealing for reliability analysis, and for some practical applications, such an approach may indeed be sufficiently accurate. Consequently, there has been limited discourse in the current literature regarding reliability analysis concerning truncated distributions (Jiang et al., 2012). In engineering applications, truncated random variables are frequently used. These variables' probability distributions are obtained by limiting the domain of other probability distributions. For instance, design engineers usually establish tolerances for their design variables, including dimension variables that define a part's geometry. To satisfy tolerance specifications, factories implement quality controls (Du & Hu, 2012).

Sun and He (1997) derived doubly truncated probability density functions (PDFs) for various commonly utilized random distributions and provided an analytical framework to compute reliability for linear limit-state functions. He and Wang (2006) utilized a singly truncated distribution to characterize resistance force and developed a method to calculate the structural reliability index. Xu and Chen (2006) introduced a Monte Carlo simulation method for structural reliability analysis utilizing truncated distributions. Melchers et al. (2003), proposed a modified FORM algorithm to address structural reliability problems involving discontinuous and truncated PDFs. Sweet and Tu (2006) constructed a statistical model using truncated distributions, offering a novel approach to evaluating tolerances between a bore and a shaft.

The structure of the thesis has been organized as thesis outline:

1.4 Thesis Outline

Chapter 1: Introduction to slope stability and reliability methods with truncated random variables and censored samples.

Chapter 2: Truncated maximum entropy for uncertainty modeling with censored samples, including examples of soil parameters.

Chapter 3: First-order reliability method (FORM) with truncated random variables and illustrative examples.

Chapter 4: Reliability analysis of Nipigon River Landslide using FORM with truncated random variables, along with a sensitivity analysis to evaluate the effects of various parameters on slope reliability.

Chapter 5: Conclusions and recommendations for future research directions.

Chapter 2: Truncated Functions from Censored Samples

Conventionally, when selecting a probability distribution for a random variable, professionals often rely on their subjective familiarity with classical distributions like Normal or Lognormal Distributions. This decision is typically based on professional experience or a simple comparison, lacking a strong theoretical foundation and potentially leading to arbitrary choices. The parameters of a distribution are usually estimated using the method of moments or maximum likelihood, with higher-order sample moments rarely taken into consideration. An alternative technique for probabilistic modeling is based on MaxEnt (Deng, 2022). MaxEnt, which is based on Shannon's measure of uncertainty, has been used for diverse science and engineering problems (Lee & Lee, 2002). MaxEnt provides an objective probability distribution derived from observed data. This principle has been applied in various applications, such as slope stability analysis of earth dams, determination of sampling locations, and numerous other fields in science and engineering (Deng, 2022).

The maximum entropy principle is commonly employed to estimate the probability density function when moment constraints are specified. The density function is subsequently integrated to determine the cumulative distribution function (Deng et al., 2012).

2.1 Maximum Entropy Principle

Entropy is defined as a measure of uncertainty involved in a random variable. The maximum entropy method (MEM) involves the selection of a probability density function (PDF) that maximizes Shannon's entropy while being constrained by the known information regarding moments. Studies have demonstrated the MEM to be a rational approach for choosing the PDF with minimal bias, containing the least amount of spurious information and aligning well with the available sample data (Shore & Johnson, 1980). MaxEnt is a logical method for selecting a probability distribution from the myriads of potential distributions, aiming to minimize extraneous information. This principle asserts that the least biased estimation of a probability distribution occurs when entropy is maximized, given constraints based on the existing information, such as the moments of a random variable. The resulting distribution is termed the most unbiased because its derivation systematically maximizes the uncertainty surrounding the unknown data. Consistency represents a foundational prerequisite in mathematical analysis. In essence, if a quantity can be determined through multiple approaches, the outcomes yielded by these distinct methods must align or be identical. To guarantee this consistency, the inference method must adhere to fundamental conditions known as consistency axioms (Deng & Pandey., 2009).

For a continuous random variable x with the density function $f(x)$, Shannon's entropy is defined as:

$$H[f(x)] = -\int_{R} f(x)\ln f(x)\mathrm{d}x, \tag{2.1}$$

Where H is the entropy of X, a continuous random variable on the domain R; $f(x)$ is the PDF. The most accurate representation of the current state of knowledge can be achieved by optimizing Eq. (2.1) while considering certain limitations or conditions.

$$\int_{R} f(x)\mathrm{d}x = 1,\ \int_{R} x^k f(x)\mathrm{d}x = \mu_k,\ k = 1,2,\dots,K, \tag{2.2}$$

where μ_k is the k-th raw moment (moment about the origin), and K is the highest order of moments. The probability density function (PDF) can provide the most accurate representation of the current knowledge regarding the random variable. The initial equation within Eq. (2.2) corresponds to the normalization requirement, and the second equation pertains to the constraint involving sample raw moments. To tackle the maximization problem outlined in Eqs. (2.1) and (2.2), can be addressed through the utilization of an augmented Lagrangian function.

$$\bar{H} = \int_R f(x)\ln[f(x)]\mathrm{d}x + (\lambda_0 - 1)\int_R f(x)dx + \sum_{k=1}^{K}\lambda_k\left[\int_R x^k f(x)dx - \mu_k\right], \tag{2.3}$$

where $\bar{H}$ is the Lagrangian and λ_k is an unknown Lagrangian multiplier. The term $(\lambda_0 - 1)$ is used for ease of calculation instead of λ_0. The Lagrangian maximization yields:

$$\frac{\partial \bar{H}}{\partial f(x)} = 0. \tag{2.4}$$

Substitution of Eq. (2.3) into Eq. (2.4) results in:

$$f(x) \approx f_k(x|\lambda) = \exp[-\sum_{k=0}^{K}\lambda_k x^k], \tag{2.5}$$

where $\lambda_k, k = 1,2,\dots,K$ is the Lagrangian multiplier and is also called the coefficients of the PDF $f(x)$, and $f_k(x|\lambda)$ is the k-th order maximum entropy PDF, which is an approximation of the true PDF $f(x)$ in the random space. $f_k(x|\lambda)$ is also simply called the K-th order Entropy Distribution or maximum Entropy Distribution. Substitution of Eq. (2.5) into the term $\ln[f(x)]$ of Eq. (2.1) and considering the second equation of Eq. (2.2) yield an approximate of $H[f(x)]$,

$$\hat{H}[f(x)] = \sum_{k=0}^{K}\lambda_k\mu_k. \tag{2.6}$$

In practice, moments are usually estimated from a sample of observed values of the random variable. Integral raw moments can be estimated by

$$\mu_k \approx m_k = \frac{1}{n}\sum_{i=1}^{n}(x_i)^k, \tag{2.7}$$

And

$$m_1 = m_X = \frac{1}{n}\sum_{i=1}^{n}x_i. \tag{2.8}$$

where m_k is the k-th integral order sample raw moment, n is the sample size, and x_i is the i −th sample value of a random variable X. Both m_1 and m_X are the sample means of X. Substituting Eq. (2.5) into the first equation of Eq. (2.2) gives the first coefficient:

$$\lambda_0 = -\ln\left\{\int_R \exp[-\sum_{k=0}^{K} \lambda_k x^k]\,\mathrm{d}x\right\}. \quad (2.9)$$

The MATLAB software is utilized to solve the system of nonlinear equations described in Eqs. (2.9). Eq. (2.5) is a family of exponential PDFs generated from MaxEnt that is only based on the sample information in terms of raw moments. The PDF is distribution-free because no classical distributions are assumed a priori during the derivation. These exponential functions have been extensively studied due to their important statistical properties (Deng., 2021).

2.2 Extended Maximum Entropy Principle

The entropy functions for a continuous distribution with a density function f are given:

$$H^p[f(x)] = -\int_{-\infty}^{+\infty} f(x)\ln f(x)\mathrm{d}x, \quad (2.10)$$

where X represents a continuous random variable, $f(x)$ is the PDF; and H is the entropy of a random variable X with $f(x)$. The probability distribution that best represents the current state of knowledge may be obtained by maximizing Eq. (2.10) with constraints of:

$$\mathrm{H} = -\int_{-\infty}^{+\infty} f(x)\ln f(x)\mathrm{d}x = \text{maximum}. \quad (2.11)$$

subjected to some known moment constraints or equations of moments:

$$\int_{-\infty}^{+\infty} x^i f(x)\mathrm{d}x = \mu_i,\ (i = 0,1,2,\ldots,N). \quad (2.12)$$

where N is the number of moments to be used and μ_i is the i-th moment of the origin which can be determined originally from the sample of data. Based on Eq. (2.5), the method of

Lagrange multiplier is $f(x) = \exp[-\sum_{k=0}^{K} \lambda_k x^k]$, where $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_N)$ are unknown parameters. These unknown parameters can be calculated from the following nonlinear equation.

$$G_i(\lambda) = \int_{-\infty}^{+\infty} x^i \exp[-\sum_{k=0}^{K} \lambda_k x^k] \mathrm{d}x, \quad i = 0,1, \dots, N. \quad (2.13)$$

2.2.1 Maximum Entropy Function from Truncated Sample

When only incomplete information is available, it is possible to define partial entropy for a continuous distribution as follows:

$$\mathrm{H}^{p}[f(x)] = -\int_{x_0}^{+\infty} f(x) \ln f(x) \mathrm{d}x, \quad (2.14)$$

where $\mathrm{H}^{p}[f(x)]$ is the partial entropy for continuous probability distribution, which is defined on a finite interval $[x_0, +\infty]$, x_0 refers to the threshold where truncation occurs (Deng et al., 2012).

Furthermore, the information is derived from a truncated data set.

$$\int_{x_0}^{+\infty} x^j f(x) \mathrm{d}x = a^j, \quad j = 0,1, \dots, m. \quad (2.15)$$

where the subscript x_0 represents a point of left truncation, which is known and fixed, and a^j's are the known values, usually estimated from the samples. In the case of a^j, considering Wang (1990), given a complete sample $x_1 < x_2 < \cdots < x_n$, $a^j \approx \frac{1}{n} \sum_{i=1}^{n} x_i^j$, where:

$$x_i = \begin{cases} 0 & x_i < x_0 \\ x_i & x_i > x_0 \end{cases}.$$

In other words, instead of the samples that are truncated, zero is placed. Therefore, the total number of samples (n) is fixed, and moments are calculated. The extended entropy is called the tail entropy with cut-off x_0. The cut-off value is appropriately specified to improve the accuracy of tail probability estimation. Depending on the choice of the normed linear space, the entropy is defined for various distributions, including continuous or discrete distributions, and even for nonnegative functions that are not probability densities or mass functions. When x_0 is taken to be larger than the lower bound of the domain of samples, the EMEM model is called a tail entropy model. Otherwise, the model is called a full entropy model (Lee & Lee, 2002).

Based on Eq. (2.13), the system of nonlinear equations is solved by MATLAB software, in which algorithms have been incorporated into specially designed functions. By adjusting parameters, one can obtain unknown parameter λ_i and accurately if a^j and i-th order are known.

2.2.2. Optimal Order

The optimal order of the MaxEnt can be determined by the AIC (Akaike Information Criterion). Let $f\ (x)$ be the unknown but true distribution and $f_K\ (x|\lambda)$ be k -th order PDF from a sample. The difference between $f_K\ (x|\lambda)$ and $f\ (x)$ can be measured by Kullback-Leibler (KL) entropy.

$$KL[f(x), f_K(x,\lambda)] = \int_R f(x)\ln\frac{f(x)}{f_K(x,\lambda)}dx = \mathrm{C} - \mathrm{L}(\lambda, K), \tag{2.16}$$

$$C = \int_R f(x)\ln f(x)dx, L(\lambda, K) = \int_R f(x)\ln f_K(x,\lambda)dx. \tag{2.17}$$

Here, C is not related to $f_K\ (x|\lambda)$, so when the KL entropy is minimized with respect to λ , C is regarded as a constant. $\mathrm{L}(\lambda, K)$ is the expectation of $\ln f_K\ (x|\lambda)$; therefore, a natural estimate $\hat{L}(\lambda, K)$ of $\mathrm{L}(\lambda, K)$ can be obtained from the sample, x_i $(i\ =\ 1,2,\dots,N)$:

$$\hat{L}(\lambda, K) = \frac{1}{N}\sum_{i=1}^{N}\ln f_K(x_i,\lambda), \widehat{kL}(\lambda, K) = C - \hat{L}(\lambda, K), \tag{2.18}$$

where $\widehat{kL}(\lambda, K)$ is a sample estimate of the KL entropy. If K is given, minimization of $\widehat{kL}(\lambda, K)$ will result in the best choice of λ,

$$min_{\lambda}\{\widehat{kL}(\lambda, K)\} = C + min_{\lambda}\{-\hat{L}(\lambda, K)\} = C + min_{\lambda}\left\{-\frac{1}{N}\sum_{i=1}^{N} \ln f_K(x_i, \lambda)\right\}. \quad (2.19)$$

However, AIC suggested that the term $-\hat{L}(\lambda, K)$ is a biased likelihood function. One of the unbiased estimates of $\hat{L}(\lambda, K)$is given as:

$$\hat{\Gamma}(\lambda, K) = -\hat{L}(\lambda, K) + \frac{K}{N}. \quad (2.20)$$

This equation can be expanded by considering the above equations.

$$\hat{\Gamma}(\lambda, K) = \sum_{k=0}^{K} \lambda_k \left[\frac{1}{N}\sum_{i=1}^{N}(x_i)^k\right] + \frac{K}{N} = \sum_{k=0}^{K} \lambda_k \mu_k + \frac{K}{N} = \hat{H}[f(x)] + \frac{K}{N}. \quad (2.21)$$

The maximum entropy PDF can be derived based on sample moments with a specified order of K. Given a series of K, there must exist a number, K, to minimize $\hat{\Gamma}(\lambda, K)$ in Eq. (2.21), which means to minimize $\widehat{KL}[f(x), f_K(x, \lambda)]$ in Eq. (2.13). This K is the optimal order of the Entropy Distribution.

2.3 Examples of Soil Parameters

2.3.1 Nipigon River Landslide in Canada (Case Study)

On April 23, 1990, an early morning landslide occurred on the Nipigon River, situated north of the town of Nipigon in Ontario, Canada. The slide covered an extensive area, reaching almost 350 meters inland, and had a maximum width of approximately 290 meters (Figure 2.1).

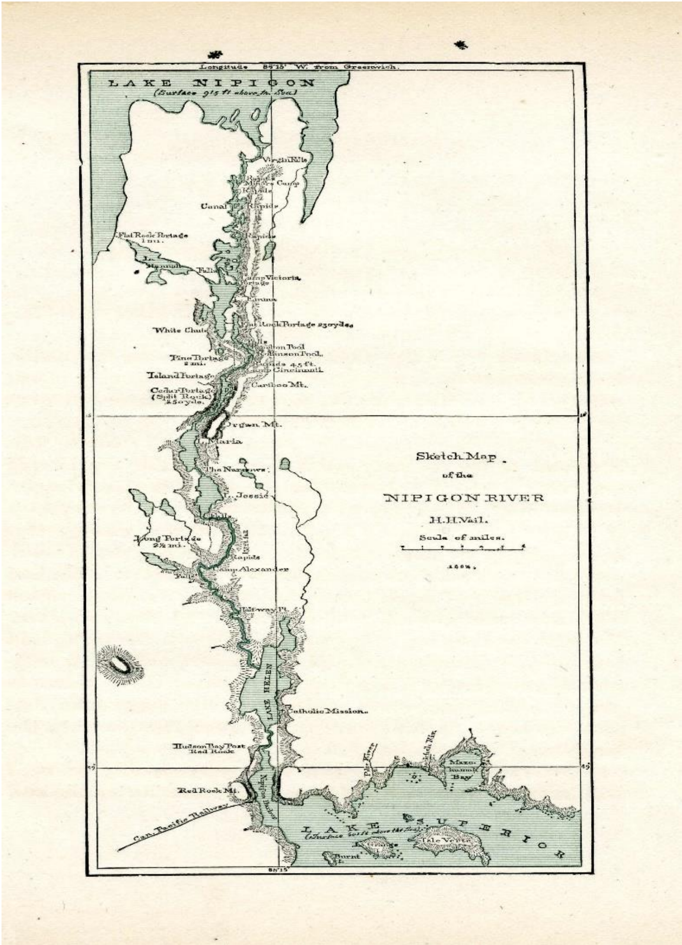


Figure 2.1. The location of Nipigon (Art source international).

Soil samples were taken using the vane shear test (VST) outlined in ASTM D2573-08, which is one of the most commonly used methods to measure the undrained shear strength of soils. In situ testing is needed because evaluation of short-term slope stability requires the shear strength of undrained soils. The fieldwork included drilling, soil sampling, and in-situ soil testing at various boreholes (Figure 2.2) to obtain 121 values of the undrained shear strength (Table 2.1). It assumes that, by applying a 121-censored sample from a real-world case study, all steps for illustrating truncated effects will be explained further.

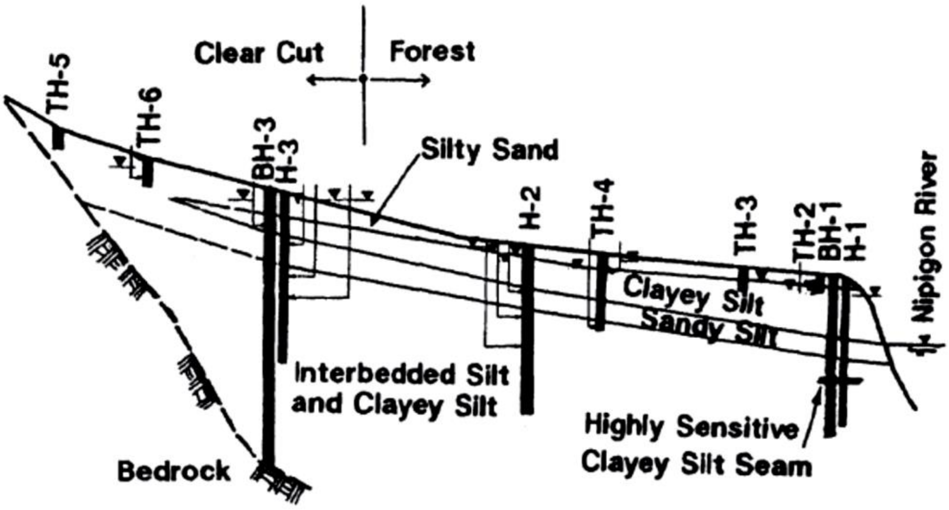


Figure 2.2. Stratigraphic Section (Dodds & Burak, 1993).

Table 2.1. Undrained soil shear strength (kPa) measured at the Nipigon River Landslide area as censored sample (n = 121).

60.80	23.75	39.90	44.65	57.00	33.25	19.00	53.20	65.55	54.15
56.05	32.30	18.05	84.55	52.25	63.65	27.55	46.55	58.90	47.50
52.25	61.75	25.65	39.90	44.65	57.00	33.25	20.90	55.10	95.00
41.80	56.05	33.25	19.00	85.50	53.20	64.60	28.50	51.30	39.90
80.75	52.25	61.70	26.60	39.90	45.60	57.00	33.25	104.5	20.90
38.95	42.75	57.00	33.25	19.00	87.40	95.00	64.60	41.80	29.45
64.60	54.15	95.00	22.80	33.25	57.95	49.40	40.85	23.75	30.40
66.50	66.50	66.50	66.50	67.45	67.45	68.40	68.40	68.40	35.15
57.95	49.40	40.85	29.45	64.60	55.10	96.90	22.80	32.30	34.20
18.05	81.70	52.25	62.70	26.60	39.90	39.90	57.95	35.15	33.25
68.40	68.40	68.40	71.25	71.25	74.10	77.90	77.90	80.75	65.55
36.10	37.05	37.05	38.00	38.00	38.00	38.00	38.00	38.00	104.50
114.00									

2.3.2 Frequency Histogram

To establish the underlying distribution of the soil undrained shear strength in the Nipigon River area, a frequency histogram is constructed using the following steps:

- Arrange the observed sample data in ascending numerical order and identify both the maximum and minimum values within the dataset.
- Divide the observed sample data into m equal intervals, where m is determined based on the specific analysis requirements or statistical considerations.
- Calculate the number of data points that fall into each interval and determine the frequency of data points in each interval.

$$h_i = \frac{n_i}{n\left(\frac{x_{\max}-x_{\min}}{m}\right)}, \tag{2.22}$$

where n_i and h_i are the number and frequency of observations in the i-th interval, respectively; n is the sample size; $x_{\max}$ and $x_{\min}$ are the maximum and minimum undrained shear strength values, respectively; and m is the number of intervals as determined and rounded using Eq. (2.23).

$$m = 1 + 3.3\log n = 7.87 \approx 8, \tag{2.23}$$

The frequency of observations in each interval (h_i) is plotted against the random variable, X, (i.e., undrained shear strength). The area under the histogram equals unity, which corresponds to the area under a PDF and permits the comparison of a fitted PDF to the frequency histogram.

2.3.3 Normal and Lognormal Distributions

Normal and Lognormal Distributions are commonly selected to model soil data due to their widespread use in statistics. The distribution parameters for these models are typically estimated using the method of moments. In this method, the sample mean and sample standard deviation are key statistics used to estimate the distribution parameters.

$$\mu_x = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 51.6847, \tag{2.24}$$

$$\sigma_x = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = 21.1043, \tag{2.25}$$

The Normal Distribution is then obtained as follows:

$$f_{1(x)} = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{x_i-\mu_x}{\sigma_x})^2\right] = \frac{1}{21.1043\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{x_i-51.68}{21.1043})^2\right], \tag{2.26}$$

where $f_1(x)$ is the Normal PDF, and X is the random variable, i.e., the soil's undrained shear strength. For Lognormal Distributions, the relationships between parameters λ and ζ, respectively, and the mean and standard deviation are:

$$\zeta_x = \sqrt{1 + (\frac{\mu_x}{\sigma_x})^2} = 0.3927 \tag{2.27}$$

$$\lambda_x = \ln\mu_x - \frac{1}{2}\zeta_x^2 = 3.8681, \tag{2.28}$$

$$f_2(x) = \frac{1}{\zeta_x x_i\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{\ln x_i-\lambda_x}{\zeta_x})^2\right] = \frac{1}{0.3927 x_i\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{\ln x_i-3.8681}{0.3927})^2\right]. \tag{2.29}$$

As a result, the classical distribution including Normal and Lognormal are illustrated in to the Figure 2.3 which are matched to the Histogram.

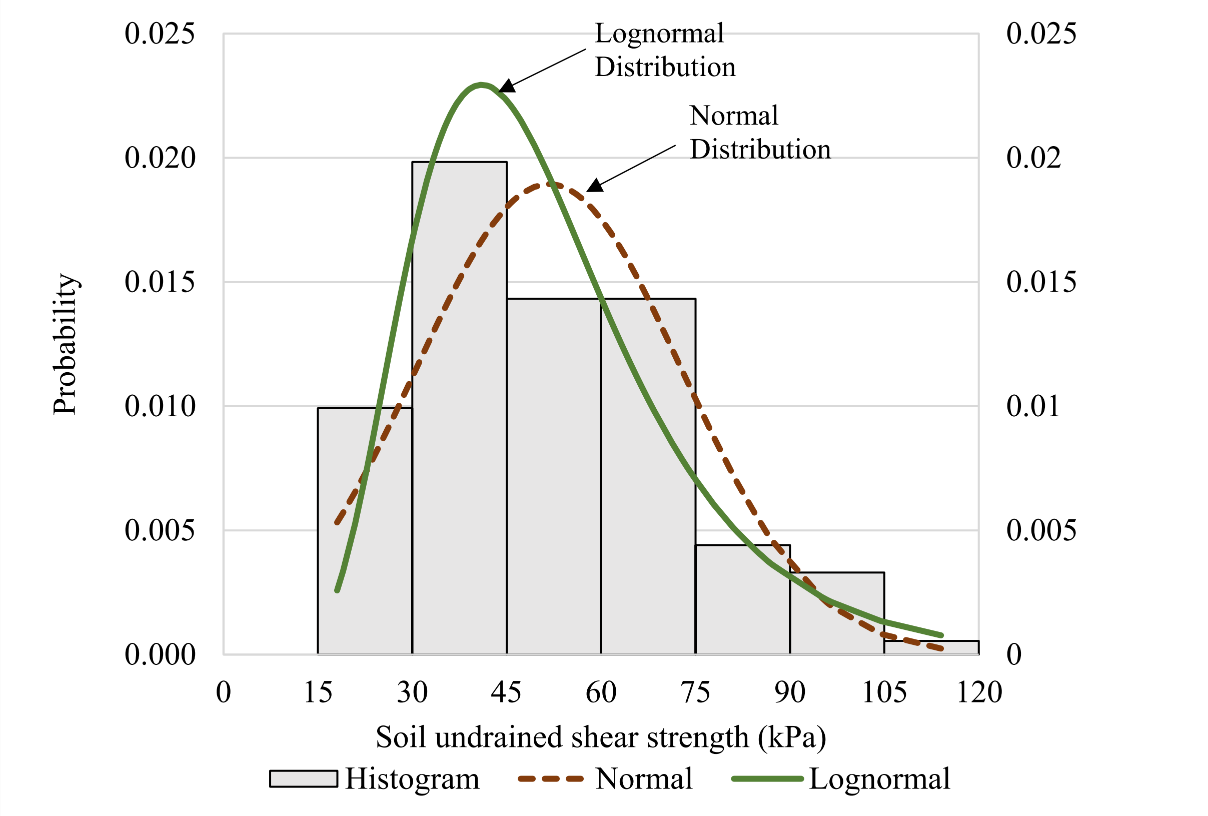


Figure 2.3. Histogram and Frequency Diagram of the soil's undrained strength of Nipigon.

2.3.4 Maximum Entropy Distribution

Following the procedure, one can derive Entropy Distribution functions and make a comparison to classical probabilistic distributions. To obtain Entropy Distributions, using the MATLAB program, coefficients (λ_k) are calculated with ten significant digits after the decimal point for the purpose of accuracy, as reported in Table 2.2. Furthermore, the untruncated moments of the samples are represented in comparison with truncated moments in Table 2.5. It needs to be mentioned that the first moment is the mean and the second is the standard deviation.

Table. 2.2. Coefficients of Maximum Entropy Distribution of Soil Strength of Frictional Angle (Moment Order K = 1–5).

	K_1	K_2	K_3	K_4	K_5
λ_0	4.3025175209	6.76676375933	8.0251219826	10.1642206239	12.3225781591
λ_1	0.0097005504	-0.1086951923	-0.1994930606	-0.40676363622	-0.6697176535
λ_2		0.0010624097	0.0028588028	0.00920423775	0.0202935404
λ_3			-1.029055691E-05	-8.597442532E-05	-2.939282972E-04
λ_4				3.047046072E-07	2.075370022E-06
λ_5					-5.566090498E-09
$\hat{\Gamma}(\lambda, K)$	-0.055	-0.395	-0.409	-0.416(Optimal)	-0.413

The table lists the unbiased KL entropy calculated from Eq. (2.22) for various orders of moment constraints. The fourth-order Entropy Distribution had the minimum unbiased KL entropy, $\hat{\Gamma}(\lambda, K)$, and therefore is the most unbiased, optimal probabilistic model for the soil property, therefore, $f_3(x)$, is the maximum entropy PDF for the random variable x:

$$f_3(x) = \exp\left(-10.16360285 + 0.40671402\, x - 0.0092029036\, x^2 + 8.595990827 \times 10^{-5} x^3 - 3.04649777 \times 10^{-7}\, x^4\right). \quad (2.30)$$

The fitted distributions in Figure 2.4 shows the Entropy Distribution (K=1-5). The Figure shows that K=4 is the most consistent with the histogram.

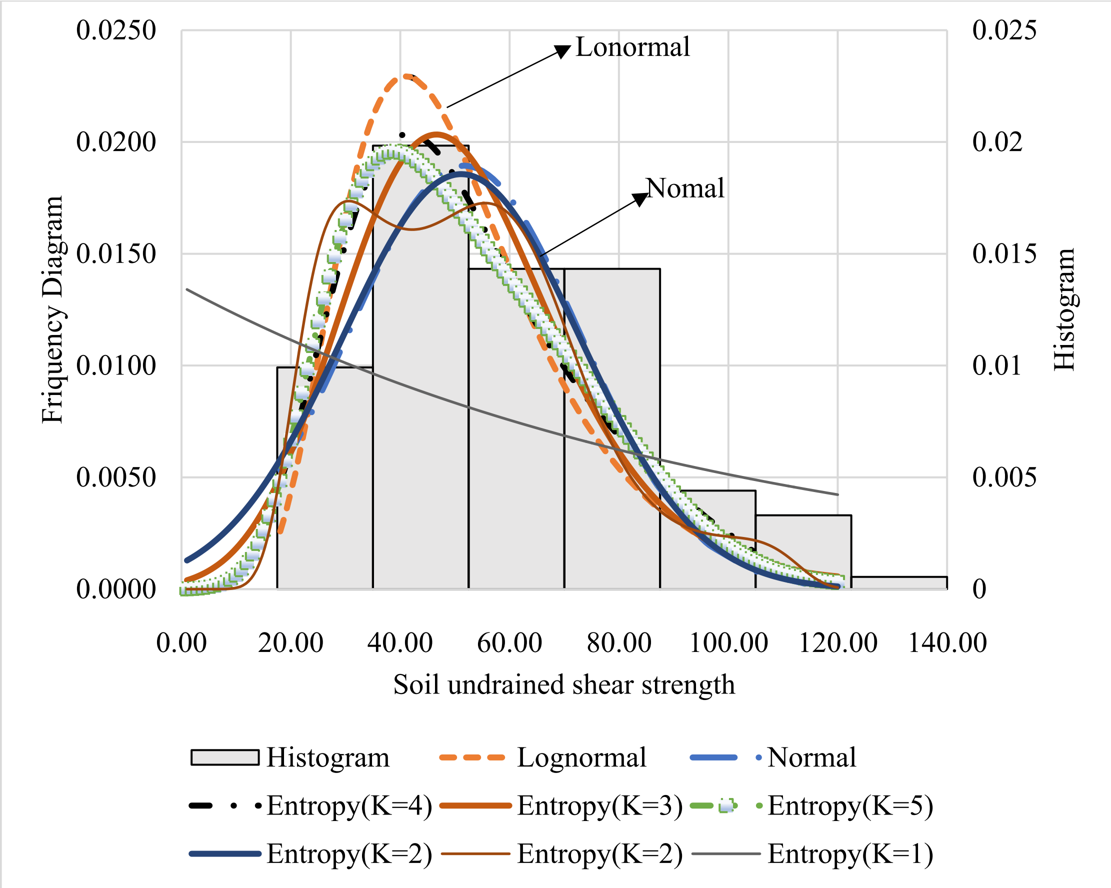


Figure 2.4. Maximum Entropy Distribution of the soil's undrained strength of Nipigon.

2.3.5 Truncated Normal and Lognormal Distribution

The truncated normal probability density function is informally described in two steps. Initially, performing a standard normal probability density function by defining parameters μ and σ, and subsequently, applying a range of truncation (a, b).

Depending on the truncation range, there are four cases:

- The nontruncated case: $a = -\infty,\ b = +\infty$;
- The lower truncated case: $-\infty < a,\ b = +\infty$;
- The upper truncated case: $-\infty = a,\ b < +\infty$;
- The doubly truncated case: $-\infty < a,\ b < +\infty$.

To estimate the truncated μ and σ, according to Cohen's (2003) theory, the equations are:

$$\hat{\sigma}^2 = \sigma^2 + \lambda(h, \hat{\alpha})(\bar{x} - T)^2, \tag{2.31}$$

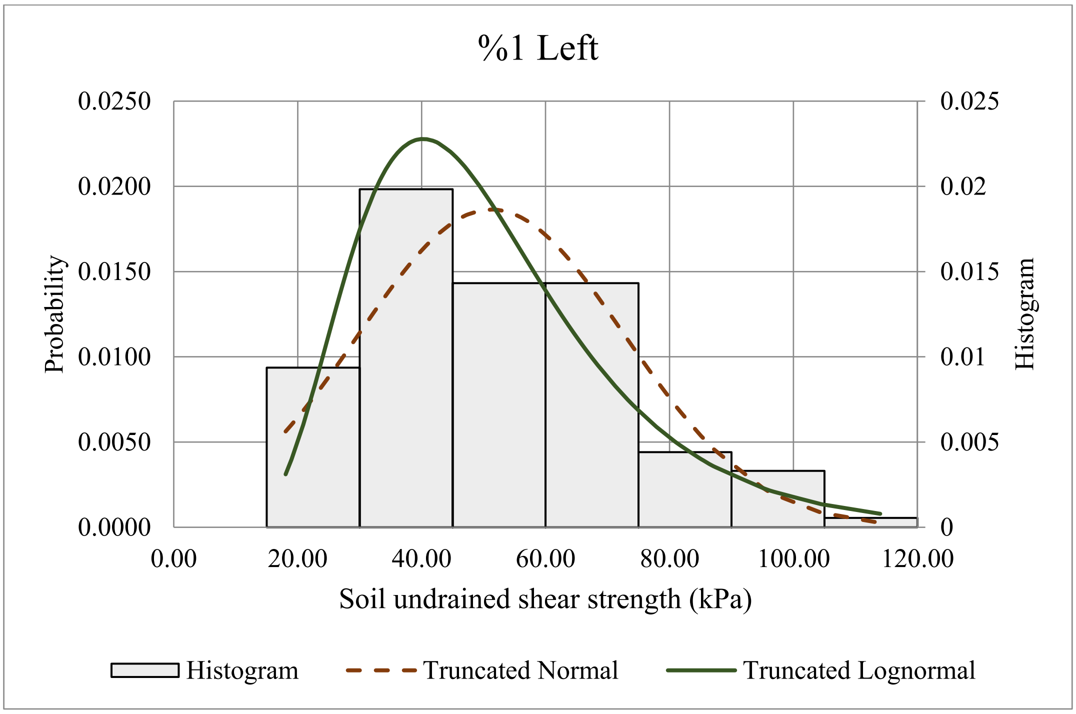
$$\hat{\mu} = \bar{x} - \lambda(h, \hat{\alpha})(\bar{x} - T). \tag{2.32}$$

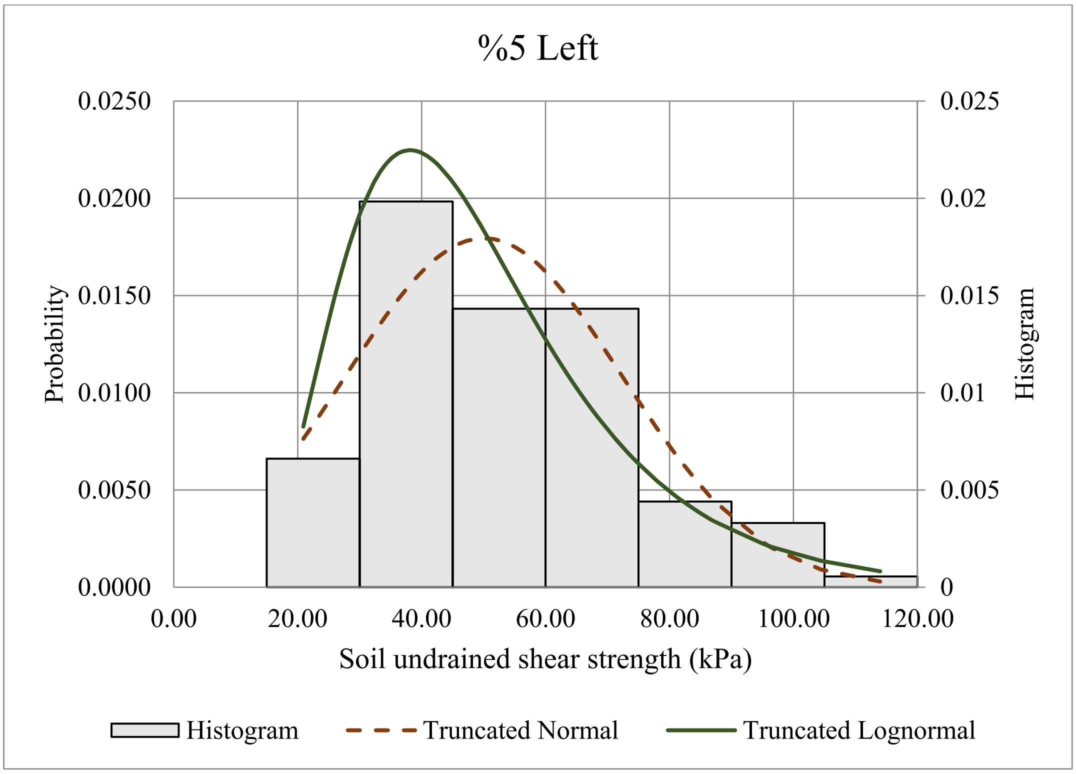
where $\hat{\alpha} = \frac{\sigma^2}{(\bar{x}-T)^2}$ and $h = \frac{c}{N}$, c is the number of missing data and N is the total amount of data. T is also a truncated point. Tables of the auxiliary estimating function $\lambda(h, \hat{\alpha})$ are included in Table 2.A, Appendix 1. As a result, the truncated μ and σ in both sides (left and right) from 1-10 percent are listed in Table 2.3.

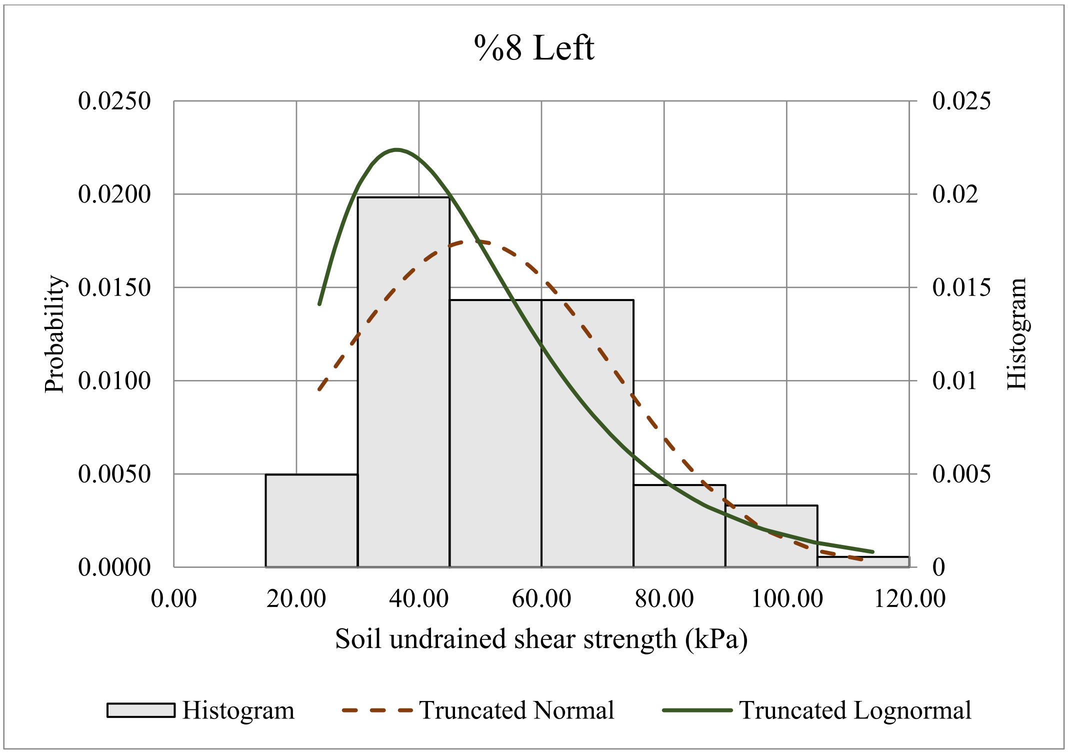
Table 2.3. Comparison of the truncated moments of the Normal and Lognormal Distributions with the untruncated moments.

	μ	σ	ζ	λ
%1 Left	51.25189	21.44269	0.4016	3.8561
%5 Left	50.03112	22.27429	0.4252	3.8222
%8 Left	48.85987	22.97635	0.4470	3.7891
%10 Left	48.2417	23.16151	0.4554	3.7725
Original	51.68	21.1043	0.3927	3.8681
%1 Right	52.36677	22.12666	0.4053	3.8761
%5 Right	53.77582	23.15561	0.4124	3.8998
%8 Right	54.81072	23.50305	0.4108	3.9195
%10 Right	55.36637	23.50647	0.4071	3.9311

In this case, the truncated Normal and Lognormal Distributions are illustrated in Figures 2.5 and 2.6, with different truncations, on the left and right sides, respectively.







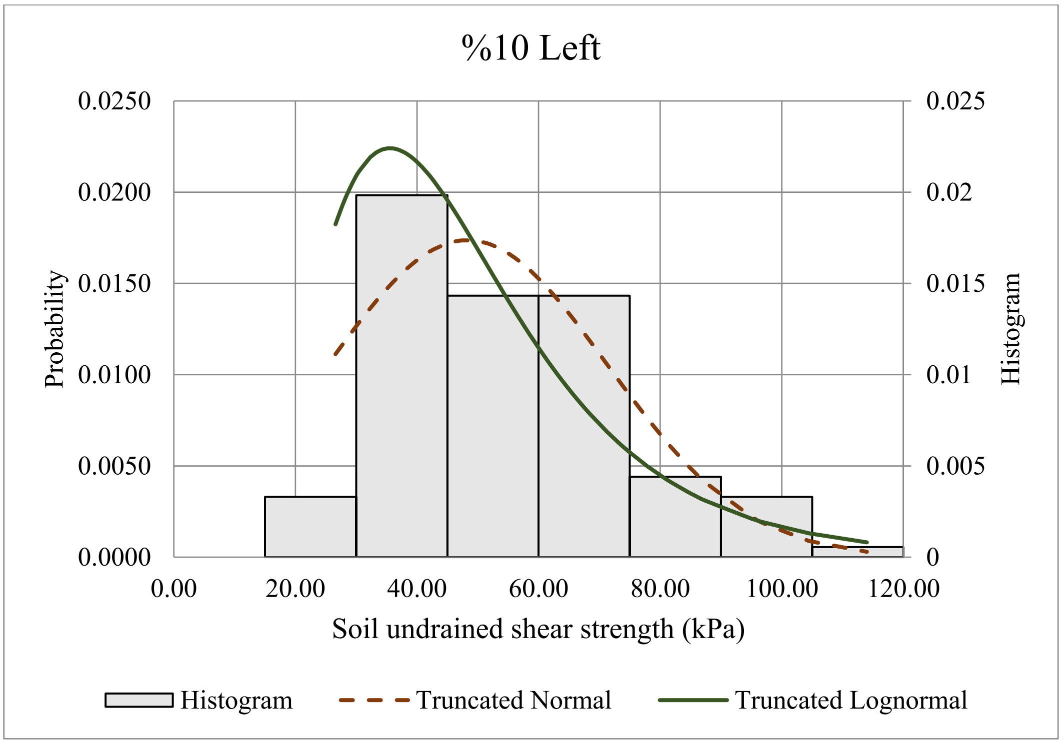
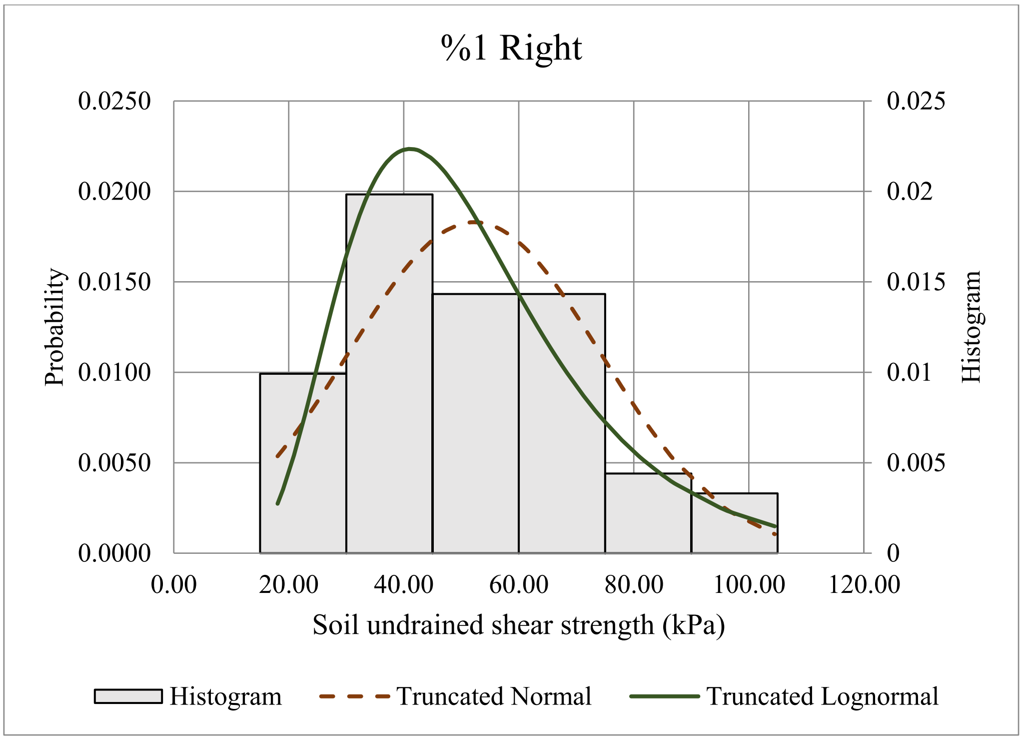
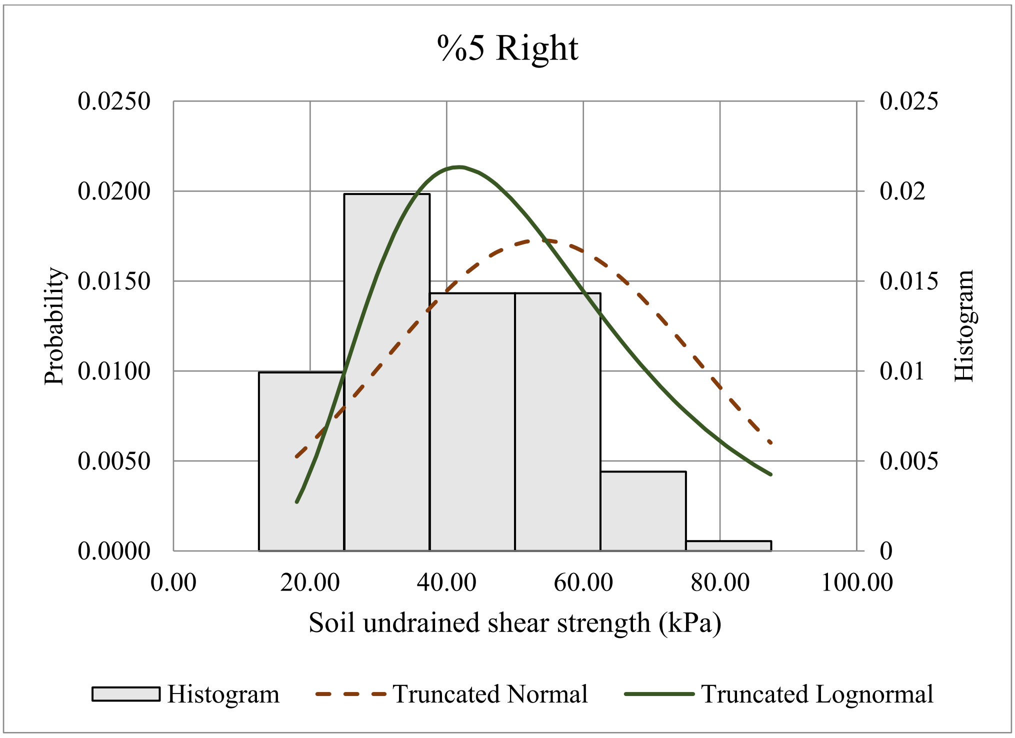
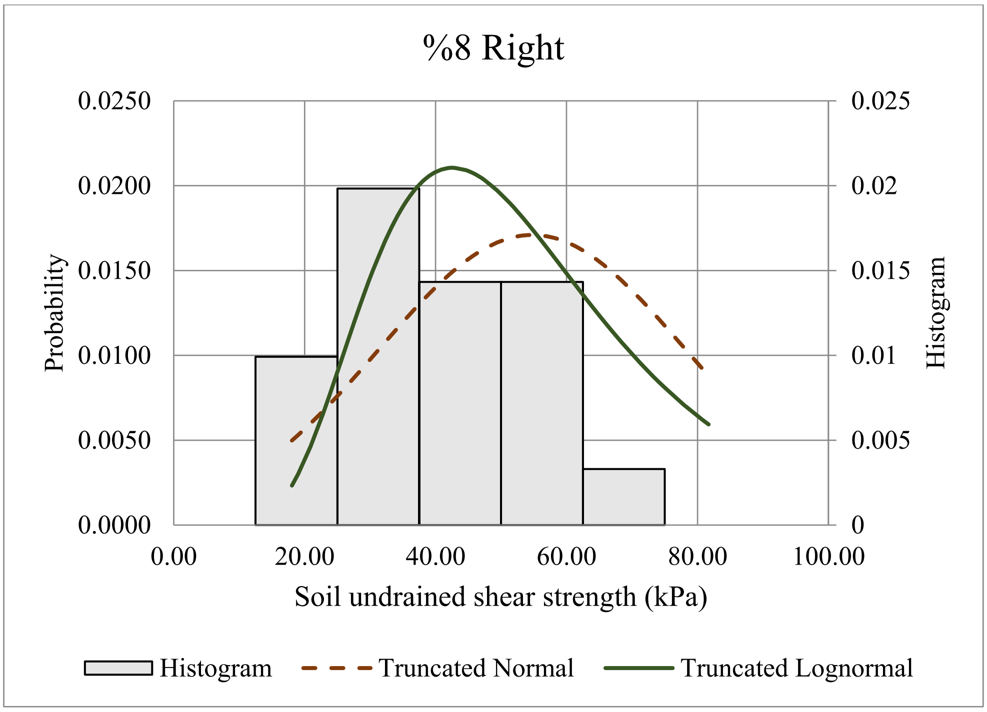


Figure 2.5. Truncated Normal and Lognormal Distribution of soil undrained strength of Nipigon (Left)


%1 Right
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
Probability
0.025
0.02
0.015
0.01
0.005
0
Histogram
0.00
20.00
40.00
60.00
80.00
100.00
120.00
Soil undrained shear strength (kPa)
Histogram
Truncated Normal
Truncated Lognormal


%5 Right
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
Probability
0.025
0.02
0.015
0.01
0.005
0
Histogram
0.00
20.00
40.00
60.00
80.00
100.00
Soil undrained shear strength (kPa)
Histogram
Truncated Normal
Truncated Lognormal



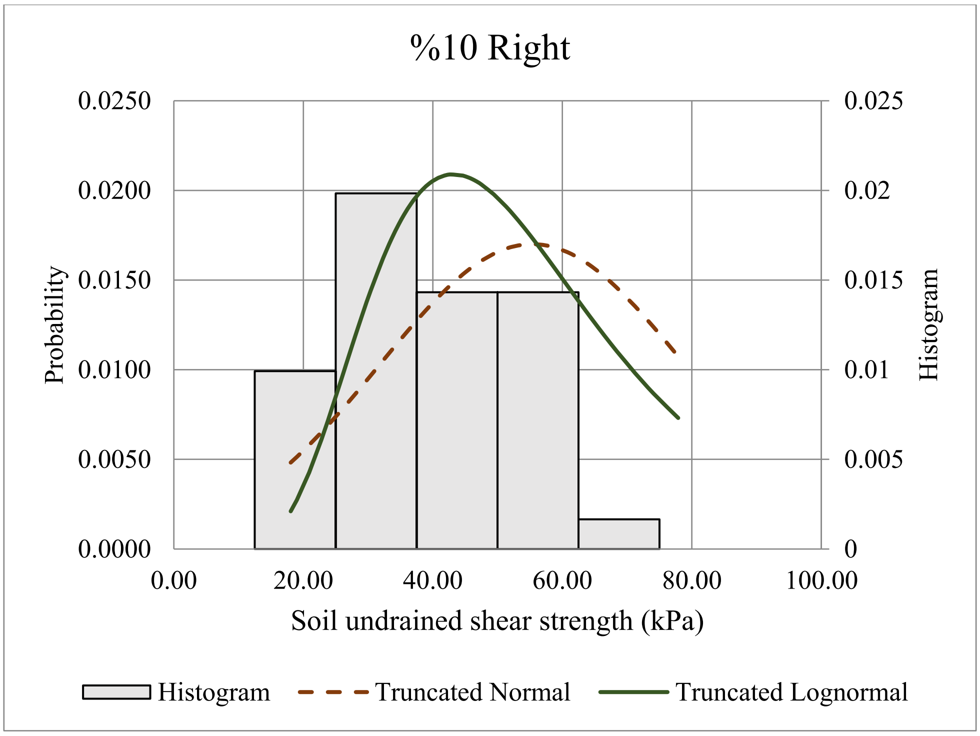


Figure 2.6. Truncated Normal and Lognormal Distribution of soil undrained strength of Nipigon (Right)

2.3.6 Truncated Maximum Entropy Functions from Censored Samples

In this section, the maximum likelihood method was employed for fitting distribution functions to censored samples, so for K=1 to 5, the data are truncated %1, %5, %8, and %10 from left, right, which are plotted in the pictures. The truncated samples on both sides are illustrated in Table 2.4. Based on the definition of the partial maximum entropy function, in order to calculate the coefficient of entropy, at the first step, the raw moment or moment about the origin is estimated and represented in Table 2.5, which is the comparison of an untruncated moment with a truncated moment. Moreover, by truncating the data, coefficients (λ_k) will be changed and calculated by the MATLAB program (Table 2.6).

Table 2.4. Truncated censored sample from the Nipigon River Landslide with different percentages on both sides.

Left truncated				Right truncated			
10%	8%	5%	1%	1%	5%	8%	10%
18.05	18.05	18.05	18.05	114.00	95.00	85.50	80.75
18.05	18.05	18.05			95.00	87.40	81.70
19.00	19.00	19.00			96.90	95.00	84.55
19.00	19.00	19.00			104.50	95.00	85.50
19.00	19.00	19.00			104.50	95.00	87.40
20.90	20.90	20.90			114.00	96.90	95.00
20.90	20.90					104.50	95.00
22.80	22.80					104.50	95.00
22.80	22.80					114.00	96.90
23.75							104.50
23.75							104.50
25.65							114.00

Table 2.5. Comparison of partial raw moment with untruncated raw moment on both sides of truncation.

Order (K)	Left truncated				Origin	Right truncated			
	10%	8%	5%	1%		1%	5%	8%	10%
1	0.299248	0.293716	0.28784	0.28281	0.39757	0.30298	0.31096	0.33385	0.33661
2	0.009397	0.008892	0.00840	0.00751	0.02614	0.00836	0.00976	0.01217	0.01350
3	-0.000098	-0.000001	0.00011	0.00032	0.00243	0.00029	-0.00017	-0.00048	-0.00070
4	0.000276	0.000254	0.00023	0.00017	0.00200	0.00019	0.00026	0.00039	0.00046
5	-0.000003	0.000001	0.00001	0.00002	0.00050	0.00002	-0.00001	-0.00004	-0.00006

Table. 2.6. Left Truncated Coefficients of Entropy Distribution of Soil Strength (Moment Order K = 1–5).

%1 Truncated Left					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.5831879021E+000	6.8794663541E+00	7.4760051251E+00	8.6092092665E+00	8.2961289639E+00
λ_1	1.2054404684E-02	-1.1257199913E-01	-1.5974680344E-01	-2.7605258501E-01	-2.2497826604E-01
λ_2		1.0930929496E-03	2.0662848404E-03	5.6176649233E-03	3.0981562306E-03
λ_3			-5.7047774474E-06	-4.6711922748E-05	5.3107367871E-06
λ_4				1.5683352160E-07	-3.1509605683E-07
λ_5					1.5494258794E-09
$\hat{\Gamma}(\lambda, K)$	-0.293	-1.011	-1.019	-1.021(Optimal)	-1.013

%5 Truncated Left					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.6439267745E+00	6.4942533517E+00	6.6161878035E+00	6.5807305753E+00	6.7425338062E+00
λ_1	1.1562148300E-02	-9.6472193060E-02	-1.0878690582E-01	-1.0086325456E-01	-8.5604168922E-02
λ_2		9.5191107093E-04	1.2359273663E-03	9.2065956835E-04	-8.3527276142E-04
λ_3			-1.7837465958E-06	2.4051206123E-06	4.9905284510E-05
λ_4				-1.7477102505E-08	-5.0311333355E-07
λ_5					1.6774810361E-09
$\hat{\Gamma}(\lambda, K)$	-0.278	-0.955(Optimal)	-0.948	-0.941	-0.937

%8 Truncated Left					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.70497824E+00	6.29562227E+00	6.27346426E+00	6.27691128E+00	6.36271879E+00
λ_1	1.10303932E-02	-8.80149873E-02	-8.52427869E-02	-7.70135336E-02	-7.34246163E-02
λ_2		8.78148012E-04	8.09503541E-04	3.92004998E-04	-1.79717623E-04
λ_3			4.47466065E-07	6.68724292E-06	2.21350596E-05
λ_4				-2.83104525E-08	-1.78908291E-07
λ_5					4.83995588E-10
$\hat{\Gamma}(\lambda, K)$	-0.262	-0.927(Optimal)	-0.919	-0.912	-0.905

%10 Truncated Left					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.76222494E+00	6.11831159E+00	6.05524307E+00	6.09957461E+00	6.09515960E+00
λ_1	1.05420110E-02	-8.03608463E-02	-6.97865519E-02	-6.46615501E-02	-6.47714327E-02
λ_2		8.11750394E-04	5.30692438E-04	1.62025809E-04	1.84026023E-04
λ_3			1.89201144E-06	8.15496871E-06	7.59698885E-06
λ_4				-3.08722716E-08	-2.60399902E-08
λ_5					-1.29259473E-11
$\hat{\Gamma}(\lambda, K)$	-0.246	-0.899(Optimal)	-0.893	-0.886	-0.878

Figure 2.7 to Figure 2.10 illustrates left truncated MaxEnt in different levels (%1-10).

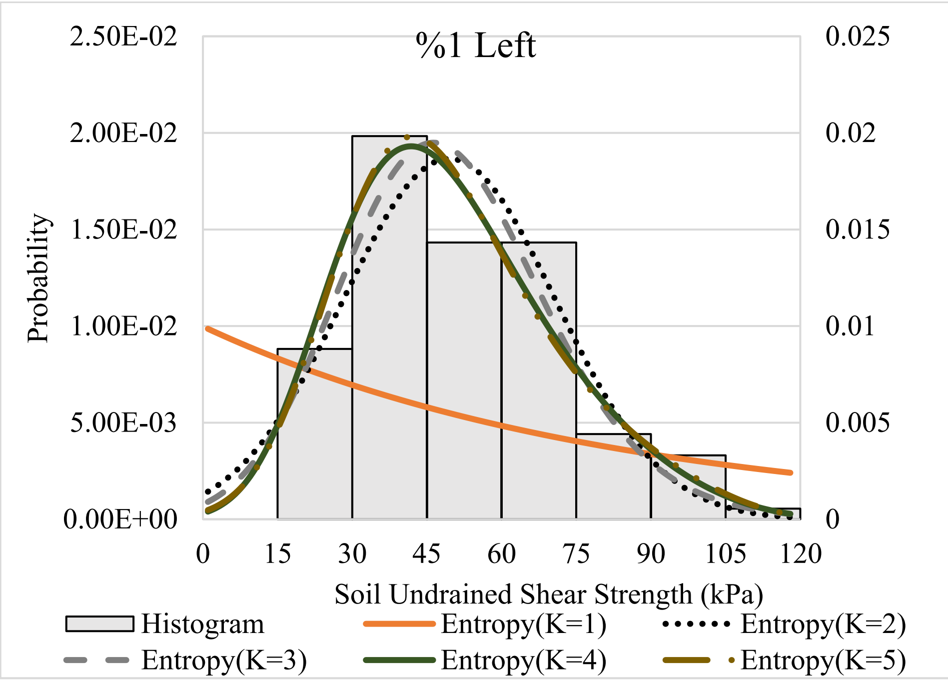


Figure 2.7. Frequency histogram and %1 Left truncated MaxEnt (K=1-5).

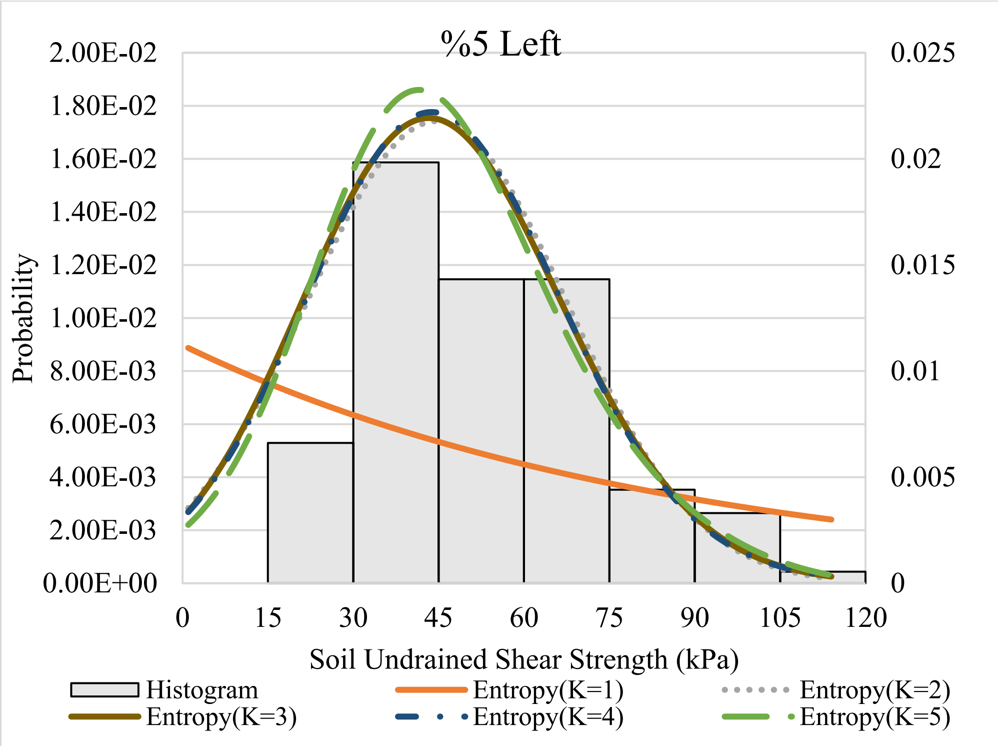


Figure 2.8. Frequency histogram and %5 Left truncated MaxEnt (*K*=1-5).

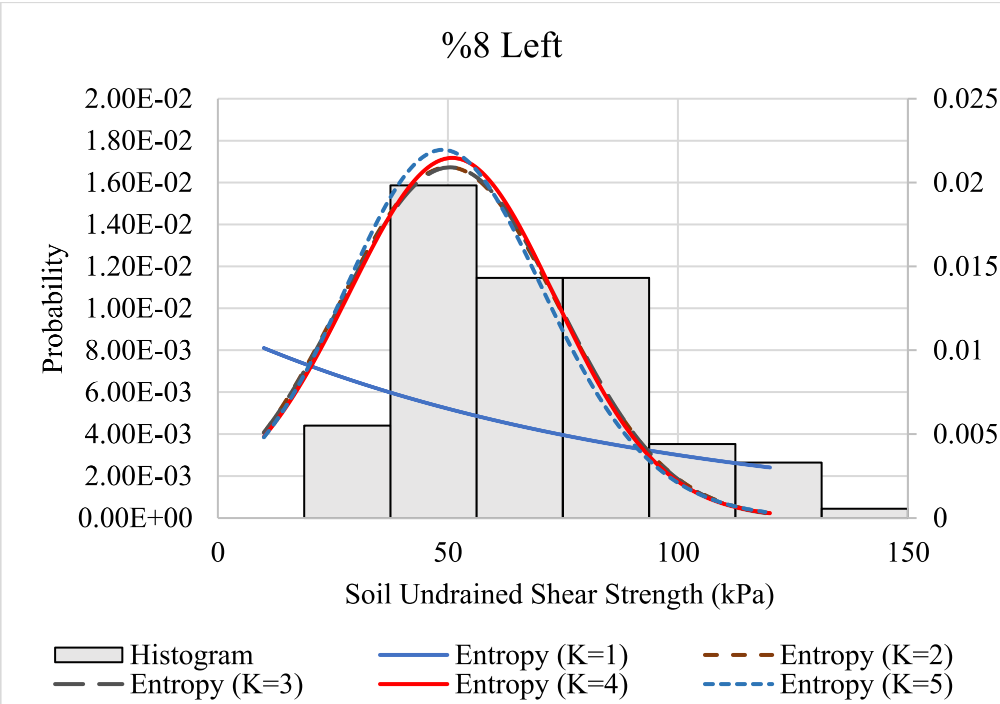


Figure 2.9. Frequency histogram and %8 Left truncated MaxEnt (*K*=1-5).

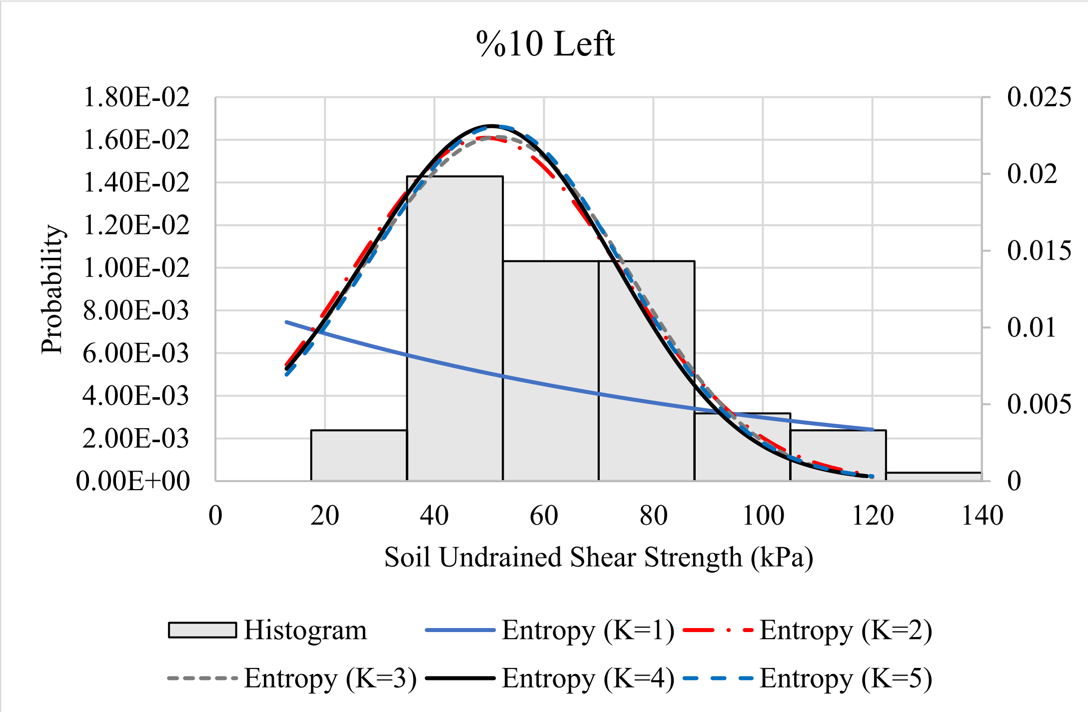


Figure 2.10. Frequency histogram and %10 Left truncated MaxEnt (K=1-5).

Based on the presented figures, increasing the truncation percentage from 1 to 10 percent causes the diagrams of the different order moments (K=2-5) to converge and closely align with the histogram. This result is observed particularly on the right side of the samples, as illustrated in Figures 2.11 to 2.14, and detailed in Table 2.7.

Table. 2.7. Right Truncated Coefficients of Entropy Distribution of Soil Strength (Moment Order K = 1–5).

%1 Truncated Right					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.5943289033E+00	6.9215038954E+00	7.4744857629E+00	8.5609113073E+00	8.2512672132E+00
λ_1	1.1549435836E-02	-1.1704032880E-01	-1.6227862056E-01	-2.8475636357E-01	-2.3031604307E-01
λ_2		1.1540812170E-03	2.1186404224E-03	6.1875795602E-03	3.3289356980E-03
λ_3			-5.8468541130E-06	-5.6942721466E-05	5.6995653832E-06
λ_4				2.1318210207E-07	-3.8995333632E-07
λ_5					2.1042152518E-09
$\hat{\Gamma}(\lambda, K)$	-0.237	-0.957	-0.960	-0.962(Optimal)	-0.954

%5 Truncated Right					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.5543644782E+00	6.4890701662E+00	6.3004694808E+00	6.2961381302E+00	6.3708545237E+00
λ_1	1.1984477456E-02	-1.0918369564E-01	-8.5023983098E-02	-1.0361607395E-01	-9.8898479600E-02
λ_2		1.1719358058E-03	5.1895234257E-04	1.6602311937E-03	6.7989455037E-04
λ_3			4.7408267241E-06	-1.6151524307E-05	2.0061042252E-05
λ_4				1.1751070868E-07	-3.7400427702E-07
λ_5					2.2466249839E-09
$\hat{\Gamma}(\lambda, K)$	-0.216	-0.880	-0.876	-0.870(Optimal)	-0.862

%8 Truncated Right					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.55223579E+00	6.22762595E+00	5.99019549E+00	5.85444997E+00	5.92246578E+00
λ_1	1.14398162E-02	-1.02661777E-01	-6.01500239E-02	-8.71492095E-02	-8.63814373E-02
λ_2		1.15770126E-03	-1.35312107E-04	2.14623515E-03	1.39429282E-03
λ_3			1.01534421E-05	-3.83957371E-05	-4.34336532E-06
λ_4				3.06250398E-07	-2.23270754E-07
λ_5					2.71862077E-09
$\hat{\Gamma}(\lambda, K)$	-0.163	-0.770	-0.779	-0.780(Optimal)	-0.773

%10 Truncated Right					
	K_1	K_2	K_3	K_4	K_5
λ_0	4.52912448E+00	5.97584806E+00	5.79186168E+00	5.52241268E+00	5.53577291E+00
λ_1	1.16961101E-02	-9.52173579E-02	-4.68933026E-02	-7.66081098E-02	-7.67322920E-02
λ_2		1.12685755E-03	-4.88761543E-04	2.75310307E-03	2.60075721E-03
λ_3			1.34390063E-05	-6.28395135E-05	-5.50350321E-05
λ_4				5.18235670E-07	3.86596278E-07
λ_5					7.24455954E-10
$\hat{\Gamma}(\lambda, K)$	-0.157	-0.719	-0.739	-0.752(Optimal)	-0.744

Figure 2.11 to Figure 2.14 illustrates right truncated MaxEnt in different levels (%1-10).

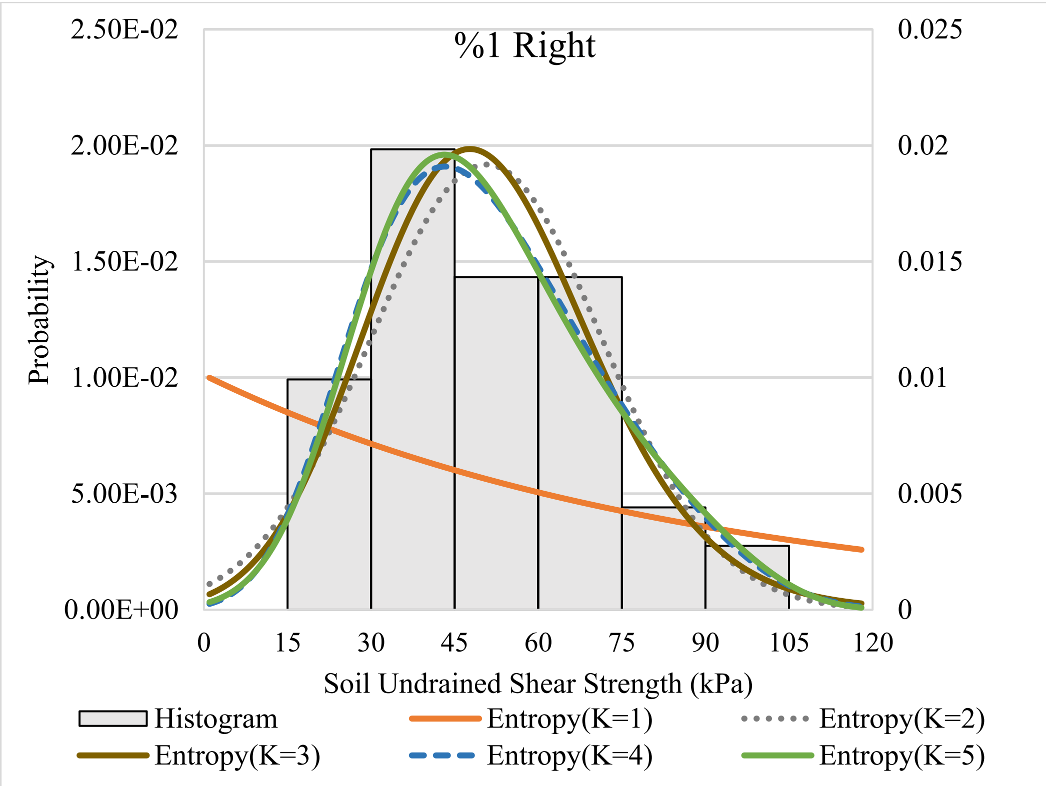


Figure 2.11. Frequency histogram and %1 Right truncated MaxEnt (K=1-5).

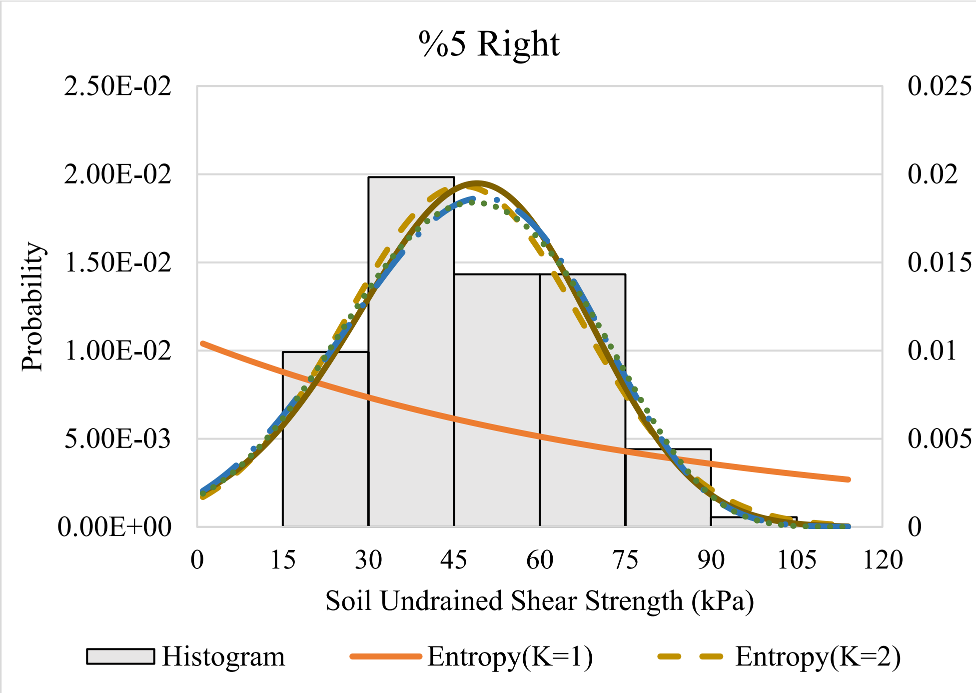


Figure 2.12. Frequency histogram and %5 Right truncated MaxEnt (K=1-5).

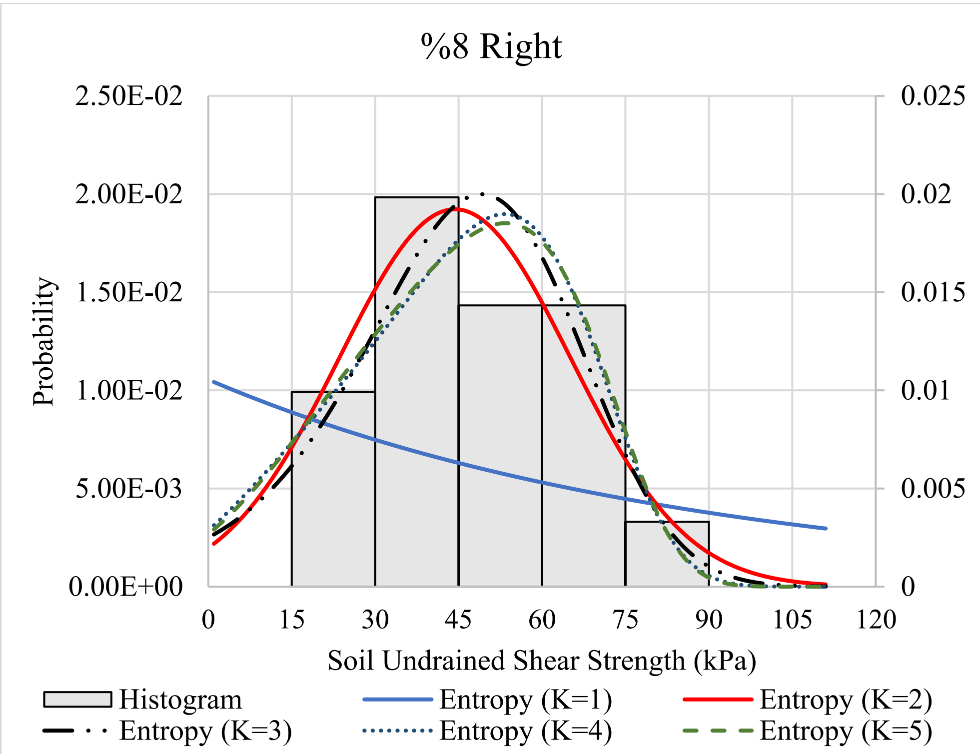


Figure 2.13. Frequency histogram and %8 Right truncated MaxEnt (K=1-5).

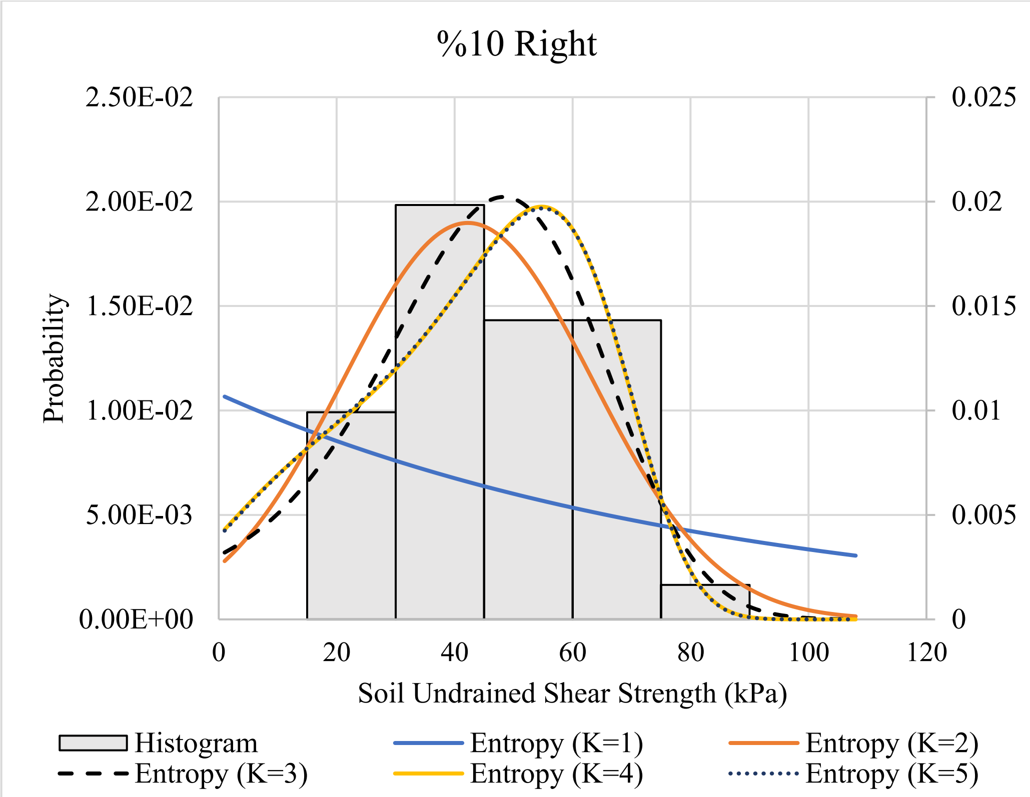


Figure 2.14. Frequency histogram and %10 Right truncated MaxEnt (K=1-5).

The figures demonstrate that when data truncation is applied from the right side, unlike left truncation, although the graphs were very close at first, they moved away from each other as the truncation increased. Moreover, the unbiased KL entropy is listed in the tables. In this case, the optimal order for the right truncated is $K = 4$, and for the left truncated except %1 left truncated is $K = 2$.

2.3.7 Statistical Goodness-of-fit Test

Determining the inherent distribution of a random variable through probability theory can be a challenging task, especially when the construction of the probability theory is required. Additionally, even after plotting, one must assess whether the relationship between the random variable and its CDF closely resembles a linear pattern. Achieving a perfectly linear relationship is quite uncommon. To address these challenges, more decisive and less cumbersome statistical tests for goodness-of-fit are available to identify the underlying distribution. Two commonly employed tests for this purpose are the chi-square (χ^2) and the Kolmogorov-Smirnov (K-S) tests (Haldar & Mahadevan, 2000). The chi-square test assesses the error between the observed and assumed PDF of the distribution, while the K-S test examines the error between the observed and assumed CDF of the distribution. To assess how well the fundamental maximum Entropy Distribution aligns with the actual data, a statistical test must be conducted. The non-parametric Chi-square (χ^2) goodness-of-fit test is employed to gauge whether the observed data significantly deviates from the expected values generated by an underlying probabilistic distribution. This assessment is based on evaluating the disparity between the observed and expected PDF (Deng, 2021).

The initial step involves partitioning the domain of the sample data into m intervals. There exists an empirical relationship that defines the connection between m and the sample size, denoted as n (Eq. 2.23).

Then the test statistic is defined as:

$$\chi^2 = \sum_{i=1}^{m}(c_i) = \sum_{i=1}^{m}\left[\frac{(n_i-e_i)^2}{e_i}\right], c_i = \frac{(n_i-e_i)^2}{e_i}, \tag{2.33}$$

In this context, "n_i" represents the expected frequency, and "e_i" represents the observed frequency for the interval "i" based on the assumed PDF. The assumed distribution will be considered acceptable at the chosen significance level if the following condition is met:

$$\chi^2 < c_{1-\alpha,f}, f = m - 1 - k. \tag{2.34}$$

Where α is a specified significance level, k is the number of distribution parameters, m is the number of intervals, and $c_{1-\alpha,f}$ is the value of the χ^2 distribution with f degrees of freedom at the CDF of the nonconfidence level $(1 - \alpha)$. A significance level of α = 5% implies that for five out of a total of 100 different samples, the assumed distribution cannot be an acceptable model. The common significance levels lie between 1% and 10%. Therefore, χ^2 is only used for comparison: the smaller the value of χ^2, the better the probability distribution. Furthermore, e_i and n_i are replaced by the probability for interval i due to the large sample size.

In order to perform the statistical goodness-of-fit test (χ^2) in the present example of Nipigon, 121 observations were categorized into eight intervals. In each of these intervals, the number of observations was recorded under "n_i" in Table 2.6. "e_i " and "n_i" represent the observed and expected frequencies, respectively, for each interval based on the assumed PDF. "c_i" is defined in Eq. (2.34).

The theoretical frequencies for each interval are then computed for the Normal, Lognormal, and Entropy Distributions. The degrees of freedom for this test are given as "f = 8 - 1 - 2 = 5." With a chosen significance level of 5% (α = 0.05), the critical value "$c_{0.95,5}$" is determined to be 11.07. This critical value is used to assess the goodness of fit of the assumed distribution to the observed data (Haldar & Mahadevan, 2000). Among the distributions considered, the critical value "$c_{0.95,5}$" is smaller than the critical values for both the Normal Distribution and the Entropy Distribution with K = 2, as listed in Table 2.8. Therefore, these two distributions (Normal and Entropy with K = 2) are not deemed acceptable at the 5% significance level. In contrast, the Lognormal Distribution and the other three Entropy Distributions with 3–5 order moments are considered acceptable.

Table 2.8. Chi-Square Results for Soil Shear Strength with Normal, Lognormal, and Entropy Distributions.

Soil shear strength	Observation	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
≤ 15	0	0.186	0.186	4.953	4.953	4.933	4.933	2.886	2.886	1.654	1.654	1.224	1.224
15-30	18	13.973	1.161	17.220	0.035	14.709	0.736	14.949	0.623	16.582	0.121	17.859	0.001
30-45	36	38.836	0.207	26.979	3.016	27.686	2.497	32.188	0.452	34.883	0.036	34.356	0.079
45-60	26	30.118	0.563	33.994	1.880	32.916	1.453	34.237	1.982	30.937	0.788	29.423	0.398
60-75	26	18.863	2.701	25.800	0.002	24.722	0.066	21.464	0.959	19.425	2.225	20.561	1.438
75-90	8	8.636	0.047	11.994	1.330	11.727	1.184	9.549	0.251	11.022	0.829	11.768	1.207
90-105	6	3.745	1.358	3.470	1.844	3.511	1.764	3.668	1.482	5.120	0.151	4.440	0.548
> 105	1	0.508	0.477	0.524	0.433	0.663	0.171	1.496	0.164	1.288	0.064	1.110	0.011
Total	121	114.865	6.699	124.932	13.492	120.867	12.806	120.437	8.798	120.911	5.869	120.743	4.907
$c_{0.95,5} = 11.07$		Acceptable		Unacceptable		Unacceptable		Acceptable		Acceptable		Acceptable	

2.3.8 Truncated Statistical Goodness-of-fit Test.

In this stage, left and right truncations were implemented on the Chi-Square to facilitate a comparison with the diagrams. Specifically, the results of the Chi-Square were truncated at percentages of 1%, 5%, 8%, and 10% from both the left and right sides, as indicated in Tables 2.9 and 2.10. The outcomes demonstrated consistency with the diagrams, particularly noteworthy in the context of right truncation, where this consistency is unmistakably evident, affirming the appropriateness of the goodness-of-fit assessment. It is mentioned that the last results of truncation are illustrated in the tables below, and the total information are in Table 2.B, Appendix 1.

Table 2.9. Left Truncated Chi-Square Results for Soil Shear Strength with Normal, Lognormal, and Entropy Distributions.

%1 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	119	117.440	6.602	118.120	13.099	117.686	12.308	118.035	9.296	118.670	6.742	118.662	6.867
$c_{0.95,5} = 11.07$		Acceptable		Unacceptable		Unacceptable		Acceptable		Acceptable		Acceptable	
%5 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	115	113.810	5.899	114.398	13.274	111.867	12.178	112.181	11.418	111.987	11.452	111.799	9.266
$c_{0.95,5} = 11.07$		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Acceptable	
%8 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	112	111.014	5.825	111.548	14.347	106.004	14.401	105.903	14.537	105.697	14.108	105.801	13.013
$c_{0.95,5} = 11.07$		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	

%10 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	109	108.170	6.333	108.661	16.567	101.468	17.628	101.056	17.933	101.055	17.053	101.045	17.115
$c_{0.95,5} = 11.07$		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	

Table 2.10. Right Truncated Chi-Square Results for Soil Shear Strength with Normal, Lognormal, and Entropy Distributions.

%1 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	119	117.617	8.823	118.237	12.967	118.046	12.163	118.260	9.783	118.854	7.278	118.834	7.458
$c_{0.95,5} = 11.07$		Acceptable		Unacceptable		Unacceptable		Acceptable		Acceptable		Acceptable	
%5 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	115	114.125	11.521	114.539	9.050	113.566	10.870	112.992	10.927	113.337	11.033	113.283	10.160
$c_{0.95,5} = 11.07$		Unacceptable		Acceptable		Acceptable		Acceptable		Acceptable		Acceptable	
%8 Right Truncated													
S o :	o b					Maximum Entropy Distribution							

		Lognormal Distribution		Normal Distribution		K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	112	111.374	14.424	111.667	10.265	109.142	15.158	107.729	13.603	108.376	15.062	108.369	14.163
$c_{0.95,5} = 11.07$		Unacceptable		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	
%10 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa	n_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i	e_i	c_i
total	109	108.541	17.494	108.741	11.456	105.569	20.656	103.570	16.045	104.399	19.038	104.403	18.869
$c_{0.95,5} = 11.07$		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	

2.4 Summary

The chapter explores the conventional methods used in selecting probability distributions for random variables and introduces an alternative approach based on MaxEnt. Traditionally, professionals often rely on their subjective familiarity with classical distributions such as Normal or Lognormal Distributions, leading to arbitrary choices lacking strong theoretical foundations. Parameters of these distributions are typically estimated using methods like moments or maximum likelihood, with higher-order sample moments seldom considered.

The MaxEnt offers an objective approach to probabilistic modeling, maximizing Shannon's entropy while adhering to known moment constraints. It has found widespread application in diverse scientific and engineering fields, providing rational and unbiased probability distributions derived from observed data. The method involves the optimization of an augmented Lagrangian function, resulting in maximum entropy probability density functions (PDFs) approximating the true PDF. In practice, moments are estimated from observed sample data, and the Lagrangian multipliers are calculated using iterative methods like MATLAB. The resulting Entropy Distribution functions can be compared to classical distributions like Normal and Lognormal Distributions.

The chapter also discusses truncated distributions arising from censored samples, where values beyond certain thresholds are not included in the dataset. Truncated maximum entropy functions are derived to account for this, with coefficients adjusted based on the degree of truncation.

A case study of the Nipigon River Landslide in Canada illustrates the application of Maximum Entropy Distribution in determining the soil's undrained shear strength. Various statistical tests, including the chi-square goodness-of-fit test, are employed to assess the compatibility of assumed distributions with observed data, with a focus on the impact of truncation on the analysis.

Overall, the chapter highlights the utility of the maximum entropy principle in probabilistic modeling, providing a rigorous and objective approach to selecting probability distributions based on observed data while addressing challenges such as data truncation.

Chapter 3: Reliability Method with Truncated Random Variables

In many engineering applications, the probability distributions of some random variables are truncated; these truncated distributions result from restricting the domain of other probability distributions. Truncated distributions apply to the situation when the range of a random variable is bounded from below and/or above for some reason. This is a common situation in reliability applications (Zhang & Xie., 2010). Therefore, truncated random variables are frequently encountered in engineering applications. They arise when the domain of other probability distributions is restricted or limited. In other words, the probability distributions of truncated random variables are derived by imposing constraints on the original probability distributions, thereby restricting the range of possible values. If the first-order reliability method (FORM) is directly used, the truncated random variables will be transformed into unbounded standard normal distributions (Du & Hu., 2012).

3.1 First-Order Reliability Method (FORM)

The First Order Reliability Method (FORM) is indeed a widely used approximate analytical method for estimating the structural failure probability in engineering reliability problems. It is often employed when conducting reliability analyses to determine the probability of failure of a system or structure (Zhou et al., 2020). Its accuracy is generally dependent on three parameters, i.e., the curvature radius at the design point, and the number of random variables. and the first-order reliability index (Zhao & Ono., 1999).

FORM achieves a balance between accuracy and computational efficiency by approximating the limit state function (the function that defines the boundary between the failure and non-failure regions) using a first-order Taylor series expansion. By linearizing the limit state function, FORM transforms the problem into a standard normal space, where the failure probability can be readily estimated. The method assumes that the limit state function can be represented as a hyperplane in the standard normal space, allowing for a simpler calculation of the failure probability. While FORM is an approximate method, it is generally accurate enough for many engineering applications and provides significant computational advantages over more computationally intensive methods, such as Monte Carlo simulation.

3.1.1. First-Order Reliability Method with Truncated Random Variables

Implementing bounds on variables is a common practice in quality control programs to ensure the reliability and quality of product components. These bounds serve as constraints that influence various aspects of the product's design, including cost, manufacturability, and reliability. Therefore, it is important to assess the sensitivity of the product's reliability to the imposed bounds. Truncated distributions can also arise naturally when modeling nonparametric distributions (Millwater & Feng., 2011). The First Order Reliability Method (FORM) is a well-established technique for assessing the likelihood of structural failure, particularly with random variables characterized by entirely continuous probability density functions. Nevertheless, challenges may emerge in achieving convergence with the standard iterative algorithm when dealing with a discontinuous or truncated probability density function for one or more random variables. The limited prevalence of such distributions in typical engineering scenarios has resulted in scant discussion on this matter in the literature (Melchers et al., 2003).

The major task of reliability analysis is to compute the probability of failure which is follows:

$$p_f = Pr\{g(X) < q\}. \tag{3.1}$$

where $g(X)$ is a limit-state function, X is a vector of random variables, and q is a limit state. The precise calculation of the probability of failure (p_f) is computationally demanding, necessitating the use of approximation methods (Zhang & Du., 2010).

Assuming there is a response variable Z, its computational evaluation can be performed using a limit-state function defined as follows:

$$Z = g(X, \acute{Y}), \tag{3.2}$$

where X and $\acute{Y}$ are vectors of untruncated random variables and truncated random variables, respectively. If a failure occurs when $g() < 0$, the probability of failure is calculated by:

$$p_f = \Pr\{g(X, \acute{Y}) < 0\}. \tag{3.3}$$

where $\Pr\{.\}$ stands for the probability. To compute this probability, the First Order Reliability Method (FORM) transforms the random variables $X = (X_1, X_2, \dots, X_n)$ and $\acute{Y} = (\acute{Y}_1, \acute{Y}_2, \dots, \acute{Y}_m)$ into independent variables that follow standard normal distributions.

When employing the First Order Reliability Method (FORM), a truncated random variable is converted into an unbounded random variable by mapping it to a standard normal distribution, which ranges from $-\infty$ to $+\infty$. However, this transformation can introduce numerical challenges and lead to a loss of accuracy (Du & Hu, 2012).

For a truncated random variable, suppose that Y_j $(j = 1,2, \dots, m)$ be a continuous random variable and denote its probability density function (PDF) and cumulative distribution function (CDF) as $f_{Y_j}(y)$ and $F_{Y_j}(y)$, respectively. If restricting the domain of Y_j by $a_j \le Y_j \le b_j$, it then has a new random variable, denoted by $\acute{Y}_j$, which is a truncated random variable. It also calls Y_j the original variable of $\acute{Y}_j$.

Denote the PDF and CDF of $\acute{Y}_j$ by $f_{\acute{Y}_j}(y)$ and $F_{\acute{Y}_j}(y)$, respectively. $f_{\acute{Y}_j}(y)$ is given by:

$f_{\acute{Y}_j}(y) = p_{Y_j} f_{Y_j}(y)$ where $a_j \leq y \leq b_j$, (3.4)

where p_{Y_j} is a constant. Integrating the above PDF from a_j to b_j yields 1.0, and therefore,

$p_{Y_j} = 1/[F_{Y_j}(b_j) - F_{Y_j}(a_j)]$, (3.5)

where $F_{Y_j}(a_j)$ and $F_{Y_j}(b_j)$ are the CDFs of Y_j at a_j and b_j, respectively. Integrating the PDF, we obtain the CDF of $\acute{Y}_j$:

$F_{\acute{Y}_j}(y) = p_{Y_j}[F_{Y_j}(y) - F_{Y_j}(a_j)]$ where $a_j \leq y \leq b_j$. (3.6)

Du and Hu (2012) represented that the direct use of FORM may produce a large error, so it has been modified to accommodate truncated random variables. The major modification is to transform truncated random variables into truncated standard normal variables. Since the linearized limit-state function is not normally distributed with the truncated random variables, using the reliability index to directly calculate the probability of failure is no longer feasible.

3.2 Illustrative Examples

3.2.1. First-Order Reliability Method with Untruncated Random Variables

Example 1: As an illustration, the MaxEnt from integral moments is applied to the reliability analysis of settlement in consolidated clay. Settlement at Point A, as illustrated in the picture at the structure's base, is predominantly attributed to the consolidation of the clay layer. In the case of a normally loaded clay, the settlement can be expressed as follows:

$S = N \frac{C_c}{1+e_v} H \log_{10} \frac{P_0+\Delta P}{P_0}$, (3.7)

Where S represents the settlement of the structure at Point A, N denotes the model error arising from the lack of homogeneity and nonuniform thickness of the clay layer, Cc is the compression index of the clay, e_v is the void ratio of the clay before loading, p_0 is the original effective pressure at Point B (mid-height of the clay layer) before loading, H is the thickness of the clay layer, and Δp is the increase in pressure at Point B caused by the construction of the structure (refer to Table 3.1). The criterion for satisfactory structural performance is a settlement of less than 2.5 inches. Therefore, the performance function is expressed as follows:

$$g(X) = 2.5 - S = 2.5 - N\frac{C_c}{1+e_v}H\log_{10}\frac{P_0+\Delta P}{P_0}. \quad (3.8)$$

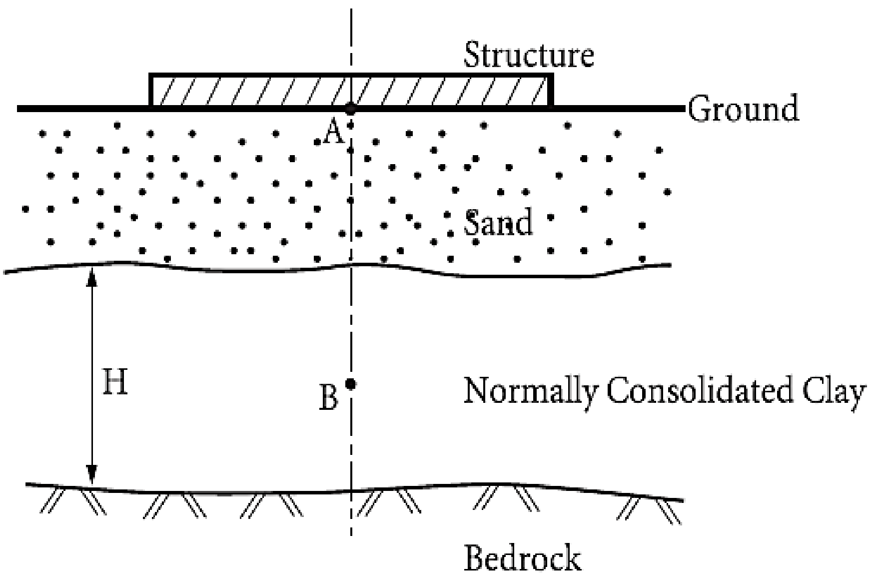


Figure 3.1. Soil profile for settlement in consolidated clay.

Table 3.1. Statistics of basic variables in reliability analysis of settlement in consolidated clay.

Random variable	Mean (μ_{Xi})	Standard deviation (σ_{Xi})	Coefficient of variation
N	1.0	0.10	0.10
C_c	0.396	0.099	0.25
e_v	1.19	0.1785	0.15
H	168 in.	8.4	0.05
p_o	3.72 ksf	0.186	0.05
Δp	0.50 ksf	0.10	0.20

Assume that all variables, except for the second random variable (compression index of the clay, C_c), follow Normal Distributions with mean (μ_{Xi}) and standard deviation (σ_{Xi}) values specified in Table 3.1. The probability density distribution of C_c is unknown and requires inference from a sample of 100 random numbers. For illustrative purposes, the sample is generated using the Monte Carlo Simulation (MCS) technique from a Normal Distribution with a mean of 0.396 and a standard deviation of 0.099. The procedure outlined is then applied to determine a MaxEnt for the sample, yielding the following result:

$$f_{E(C_c)} = \exp\left(-6.68972796625439 + 44.4802804949409(C_c) - 72.2310687577704(C_c)^2 + 26.8665426766954(C_c)^3\right), \quad (3.9)$$

where $f_{E(C_c)}$ is the optimal MaxEnt for Cc.

Table 3.2 represents the coefficients of Entropy Distribution when it is untruncated and estimated by MATLAB program. Furthermore, the comparison of untruncated moments about the origin and truncated raw moments is illustrated in Table 3.3 considering the definition of the partial raw moment and partial maximum entropy.

Table 3.2. Untruncated coefficients of Entropy Distribution λ_K with different orders (1-5).

	K_1	K_2	K_3	K_4	K_5
λ_0	-7.88418301E-01	5.08867771E+00	6.68972797E+00	6.51871275E+00	4.68334698E+00
λ_1	2.10700849E+00	-3.10500884E+01	-4.44802805E+01	-4.26985717E+01	-1.38206028E+01
λ_2		3.80418304E+01	7.22310688E+01	6.59737313E+01	-9.23769865E+01
λ_3			-2.68665427E+01	-1.79666020E+01	3.73071294E+02
λ_4				-4.35737933E+00	-4.47628934E+02
λ_5					1.86853079E+02
$\hat{\Gamma}(\lambda, K)$	-0.324	-1.130	-1.134(Optimal)	-1.120	-1.111

Table. 3.3. Comparison of untruncated moments about the origin and truncated raw moments (K = 1–5).

Order (K)	Left truncated				Origin	Right truncated			
	10%	8%	5%	1%		1%	5%	8%	10%
1	0.372582	0.370777	0.37009	0.346385	0.272021	0.337383	0.380063	0.38416	0.378437
2	0.008103	0.007355	0.006138	0.004951	0.005823	0.006044	0.010412	0.0134	0.014844
3	-0.0007	-0.00057	-0.00035	8.14E-06	0.000117	-0.00011	-0.00118	-0.00181	-0.00207
4	0.000276	0.000245	0.000184	9.77E-05	0.000113	0.000139	0.000482	0.000726	0.000818
5	-4.02E-05	-3.54E-05	-2.4E-05	-2.47E-06	6.72E-06	-1.10E-05	-0.00012	-0.00018	-0.0002

The Normal and Lognormal Distributions can also fit the same sample, as follows:

$$f_{N(C_c)} = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] = \frac{1}{0.115156\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{C_c-408185}{0.115156}\right)^2\right], \tag{3.10}$$

$$f_{L(C_c)} = \frac{1}{\zeta_x x\sqrt{2\pi})} \exp\left[-\frac{1}{2}\left(\frac{\ln x-\lambda_x}{\zeta_x}\right)^2\right] = \frac{1}{0.276732x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln C_c+0.93432}{0.276732}\right)^2\right]. \tag{3.11}$$

where $f_{N(C_c)}$ and $f_{L(C_c)}$ are the Normal and Lognormal PDF for C_c, respectively. A comparison of the optimal MaxEnt, Normal, and Lognormal Distributions is shown in Figure 3.2.

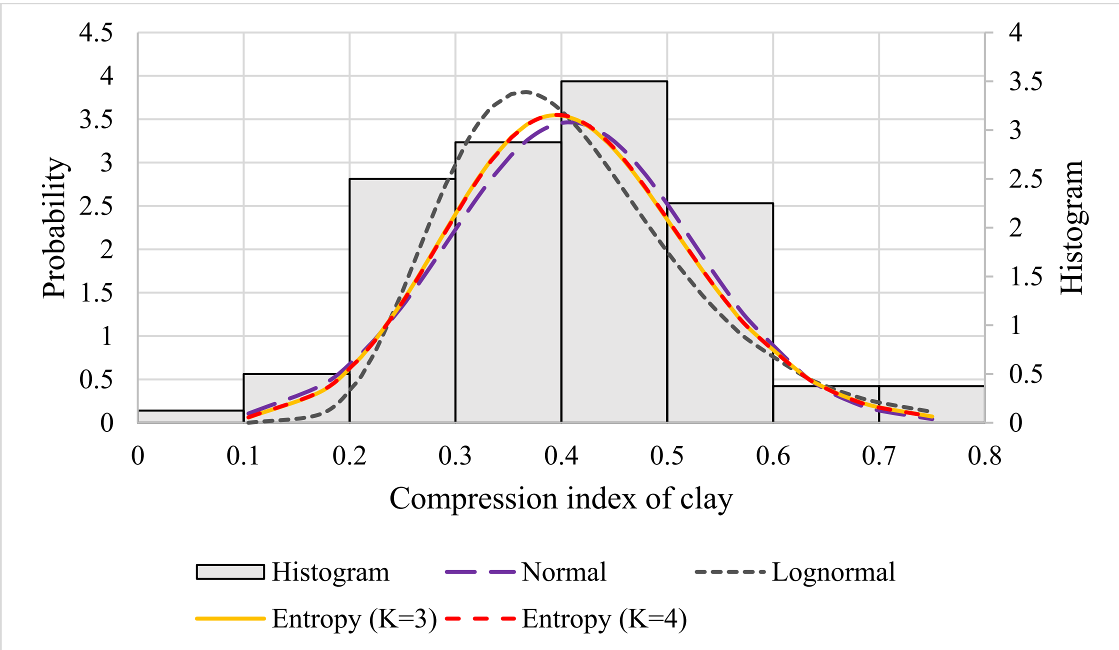


Figure 3.2. Untruncated Maximum Entropy Distribution, Normal, and Lognormal density function.

Reliability analyses are based on the Hasofer-Lind method. Because $f_{L(C_c)}$ and $f_{E(C_c)}$ represent nonnormal distributions, Rackwitz and Fiessler's two-parameter equivalent normal transformation is employed for analysis:

$$\sigma_{X_i}^N = \frac{\phi\{\Phi^{-1}[F_{X_i}(x_i^*)]\}}{f_{X_i}(x_i^*)},$$

$$\mu_{X_i}^N = x_i^* - \Phi^{-1}[F_{X_i}(x_i^*)]\sigma_{X_i}^N, \qquad (3.12)$$

In the given context, Φ and ϕ represent the CDF and PDF of the standard normal variate, respectively. Additionally, $F_{X_i}(x_i^*)$ and $f_{X_i}(x_i^*)$ stand for the CDF and PDF of the original nonnormal variate x_i^*, respectively. Here, x_i^* denotes the checking point of X_i on the failure surface. Furthermore, $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ represent the mean and standard deviation of the equivalent normal variable at the checking point x_i^*, respectively.

3.2.2. First-Order Reliability Method with Truncated Random Variables

The calculation steps of the Hasofer-Lind reliability method involving optimal MaxEnt for non-correlated random variables will be repeated with truncated random variables the same as with untruncated random variables. In this case, Step 8 is related to the truncation:

1. Formulate the relevant limit-state equation that describes the performance or failure criteria of the system. Define the limit state function that represents the relationship between the random variables and the failure state. Determine the probability distributions, specifically the quantile functions, for all basic random variables X_i (where X_i=1,2,…, M), where M is the total number of random variables involved in the analysis. These probability distributions provide information about the likelihood of different values for each random variable.

2. Assume the initial design points $x^* = x_i^*$ (where i=1,2,.., n) are typically set to the mean values of the respective random variables.
3. For each non-normal random variable X_i, calculate the mean ($\mu_{X_i}^N$ and $\sigma_{X_i}^N$) at the design point of the equivalent normal distribution using Eq. (3.12).
4. Compute the direction cosines (α_{X_i}) at the design point, as specified in the methodology. The direction cosines help determine the orientation of the failure surface in the multidimensional space.

$$\alpha_{X_i} = -\frac{\sum_{j=1}^{M}(\rho X_i X_j \frac{\partial g_X(X^*)}{\partial X_i}\sigma_{X_j})}{\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{M}(\rho X_i X_j \frac{\partial g_X(X^*)\partial g_X(X^*)}{\partial X_i \quad \partial X_j}\sigma_{X_i}\sigma_{X_j})}}, i = 1,2,\dots,N. \tag{3.13}$$

5. Compute the reliability index β as

$$\beta = \frac{g_x(x^*)+\sum_{j=1}^{M}\left[\frac{\partial g_X(X^*)}{\partial X_i}\left(\mu_{X_i}-x_i^*\right)\right]}{\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{M}(\rho X_i X_j \frac{\partial g_X(X^*)\partial g_X(X^*)}{\partial X_i \quad \partial X_j}\sigma_{X_i}\sigma_{X_j})}}. \tag{3.14}$$

6. Determine the new design point $x^* = x_i^*(i = 1,2,\dots,N)$ using

$$x_i^* = \mu_{X_i} + \beta\alpha_{X_i}\sigma_{X_i}, i = 1,2,\dots,N. \tag{3.15}$$

7. Discontinuous probability density functions in the present context are those in which the probability density changes rapidly locally and, in the limit, has a step change. In the truncated FORM algorithm, when $x_1 \leq x_{d1}$,the algorithm attempts to set a new trial design point with coordinates involving $x_1 \leq x_{d1}$, therefore, the value is instead set to $x_1 + \delta_{x1}$, where δ_{x1} is some arbitrarily small quantity. When this strategy is adopted, the design point coordinate of x_1 at iteration 2 is replaced by a value marginally higher than x_1 (Melchers et al., 2003). The estimated reliability index and failure probability are shown in Table 3.A, Appendix 2.

8. Repeat Steps (2)–(7) until the reliability index β converges to a predetermined tolerance level.

Moreover, from different Entropy Distribution methods, the parameters λ and moment from the samples will be determined in different order with different truncations, which are listed in Tables 3.B and 3.C, Appendix 2. Therefore, the comparison of the probability of failure among different methods is illustrated in Table 3.4.

Table 3.4. A comparison of the probability of failure between Normal, Lognormal and Entropy Distribution methods.

Normal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.1499	0.1463	0.1412	0.1357	0.1354	0.1256	0.0992	0.0891	0.0830
Lognormal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.1803	0.1745	0.1659	0.1526	0.1480	0.1429	0.1276	0.1201	0.1155

Maximum Entropy Distribution									
	Left (Percent)				Untruncated	Right (Percent)			
	10	8	5	1		1	5	8	10
K_1	0.4667	0.4649	0.4624	0.4585	0.4572	0.4549	0.4440	0.4383	0.4359
K_2	0.1475	0.1440	0.1392	0.1344	0.1347	0.1245	0.0975	0.0875	0.0814
K_3	0.1432	0.1395	0.1362	0.1303	0.1313	0.1236	0.1000	0.0895	0.0830
K_4	0.1356	0.1336	0.1308	0.1285	0.1297	0.1230	0.1036	0.0923	0.0853
K_5	0.1368	0.1349	0.1322	0.1290	0.1280	0.1231	0.1092	0.0975	0.0901

As a result, it is seen that the results of reliability analysis from optimal MaxEnt are very close to those of the Normal and Lognormal Distribution methods. The table depicts a comparative analysis of the MaxEnt method with various truncation levels alongside the Normal and Lognormal Distribution methods. Each method's probability of failure is assessed across different truncation scenarios, including 1%, 5%, 8%, and 10% on both the left and right sides. The Normal distribution reveals decreasing probabilities of failure across the specified truncation levels. Similarly, the Lognormal Distribution exhibits a decreasing trend in failure probabilities.

In contrast, the MaxEnt method, categorized by different orders, demonstrates unique patterns of failure probabilities for each truncation level. Notably, the MaxEnt method displays variability in performance across orders, providing a nuanced perspective on reliability compared to the more standard Normal and Lognormal Distributions. This comprehensive overview facilitates a broader understanding of how the MaxEnt method with truncation compares with traditional distribution methods, shedding light on potential variations in reliability assessments under different truncation scenarios.

Example 2: A harbor breakwater is constructed with massive concrete tanks filled with sand (Figure 3.3). It is necessary to evaluate the risk that the breakwater will slide under the lateral pressure of a large wave during a major storm. Stability against sliding exists when the total horizontal destabilizing force Q_h does not exceed the available horizontal frictional resistance $c_f R_v'$ along the base, where c_f is the coefficient of friction. The ultimate limit state of sliding is:

$$g(x) = c_f R_v' - Q_h = X_1(X_2 - a_1 X_3 X_4) - X_3(a_2 X_4^2 + a_3 X_4) = 0. \tag{3.16}$$

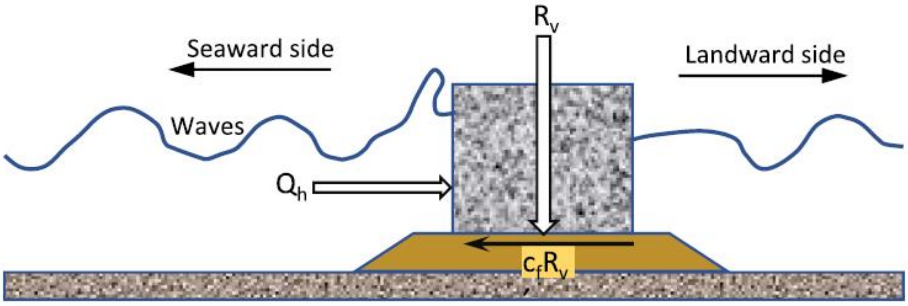


Figure 3.3. Harbor breakwater against sliding.

in which X_1 represents the interface coefficient of friction at the base of concrete tanks, X_2 is the effective vertical dead weight, a_1X_4 is the dynamic uplift force due to breaking waves, X_3 is a correction factor to account for the simplifications adopted to model the dynamic vertical and horizontal wave forces, and $a_2X_4^2 + a_3X_4$ represents the horizontal wave force. The constants a_1, a_2 and a_3 depend on the geometry of the system.

All four random variables are assumed independent and normal variates with mean and standard deviation except X_2 which is unknown and needs to be inferred from a sample of 100 random numbers. Table 3.5 represents the random variable from the limit state function in example 2.

Table 3.5. Statistics of basic variables in reliability analysis and constant numbers.

Random variable	Mean(μ_{Xi})	Standard deviation (σ_{Xi})	Coefficient of variation
X_1	0.64	0.096	0.15
X_2	3400	170	0.05
X_3	1	0.2	0.2
X_4	5.16	0.93	0.18
a_1	a_2	a_3	
70	17	145	

In this case, Untruncated Maximum Entropy Distribution, Normal and Lognormal Distribution can fit the sample as follows:

$$f_{E(X_2)} = \exp\left(-270.053831780354 + 0.187173793122993(X_2) - 0.0000416940759476899(X_2)^2 + 2.78477624659659E-09(X_2)^3\right), \quad (3.17)$$

As a result, Tables 3.6 and 3.7 show the information of untruncated coefficients of entropy and partial raw moment respectively.

Table 3.6. Untruncated coefficients of Entropy Distribution λ_K with different orders (1-5).

	K_1	K_2	K_3	K_4	K_5
λ_0	8.57689145E+00	1.57561045E+02	2.70053655E+02	4.56264557E+02	1.52280333E+01
λ_1	1.11100629E-04	-8.84911941E-02	-1.87173639E-01	-3.74164142E-01	3.38415948E+00
λ_2		1.29338128E-05	4.16940309E-05	1.10028997E-04	-4.12602583E-03
λ_3			2.78477189E-09	-1.34322004E-08	1.88693109E-06
λ_4				5.84302843E-13	-3.83896389E-10
λ_5					2.93140821E-14
$\hat{\Gamma}(\lambda, K)$	-0.022	-2.263	-2.269(Optimal)	-2.255	-2.245

Table. 3.7. Comparison of the untruncated raw moment and truncated moment about the origin on both sides (K = 1–5).

Order (K)	Left truncated 10%	Left truncated 8%	Left truncated 5%	Left truncated 1%	Origin	Right truncated 1%	Right truncated 5%	Right truncated 8%	Right truncated 10%
1	0.562384	0.567156	0.574702	0.580814	0.426727	0.585231	0.593438	0.588338	0.582232
2	0.008768	0.007226	0.00476	0.00125	0.000602	0.001308	0.005395	0.008309	0.010095
3	-0.00209	-0.00179	-0.00124	-0.00029	3.88E-06	-0.00031	-0.00151	-0.00223	-0.0026
4	0.000595	0.00052	0.000368	8.97E-05	1.20E-06	9.88E-05	0.000478	0.000692	0.000793
5	-0.00016	-0.00015	-0.00011	-2.75E-05	2.31E-08	-3.10E-05	-0.00015	-0.00021	-0.00024

$$f_{N(X_2)} = \frac{1}{\sigma_x\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] = \frac{1}{197.6078\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{X_2-3420.925}{197.6078}\right)^2\right], \quad (3.18)$$

$$f_{L(X_2)} = \frac{1}{\zeta_x x\sqrt{2\pi})}\exp\left[-\frac{1}{2}\left(\frac{\ln x-\lambda_x}{\zeta_x}\right)^2\right] = \frac{1}{0.0577163X_2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln X_2-8.136}{0.0577163}\right)^2\right]. \quad (3.19)$$

Reliability analyses are based on the Hasofer-Lind method, and it illustrates in Figure 3.4.

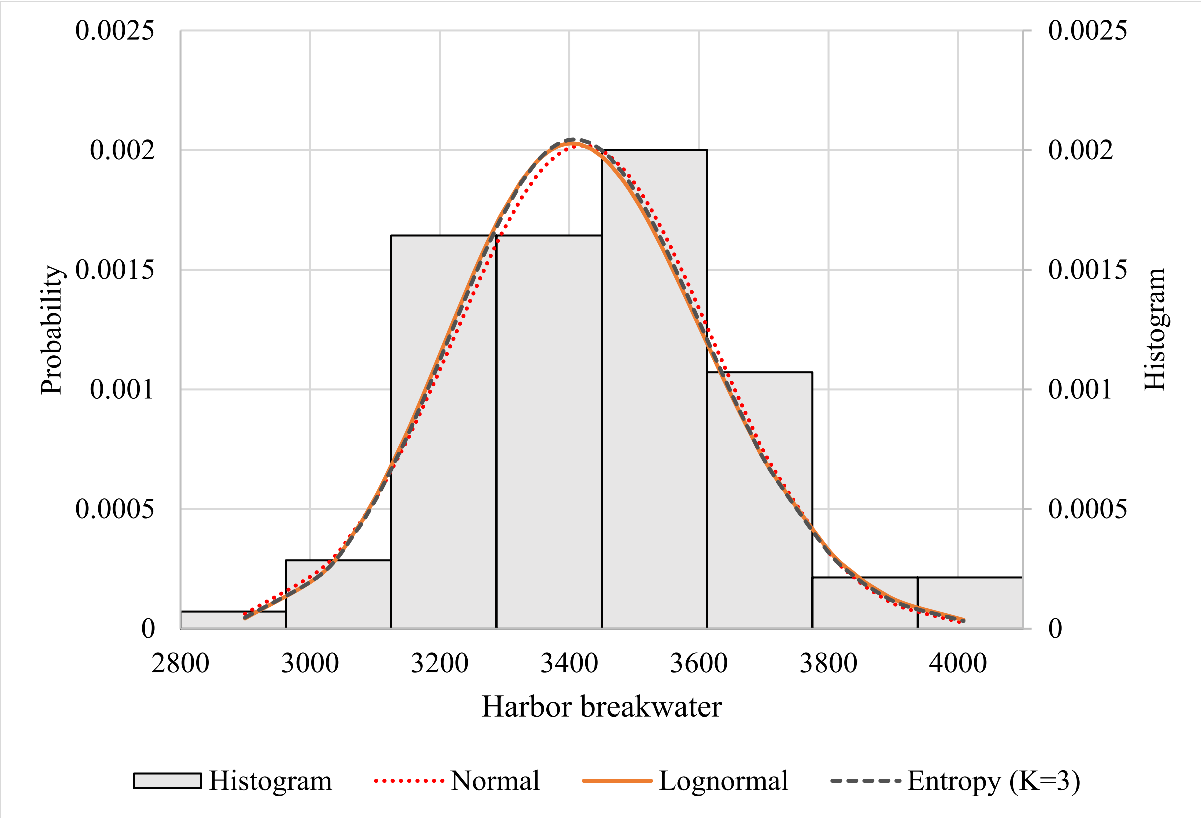


Figure 3.4. Untruncated Maximum Entropy Distribution, Normal, and Lognormal Distribution.

All steps of the Hasofer-Lind reliability method for this example are repeated, such as Example 1, and are listed in Tables 3.D, 3.E, and 3.F, Appendix 2. However, the comparison of probability of failure methods is represented in Table 3.8.

Table 3.8. A comparison of the probability of failure between Normal, Lognormal, and Entropy Distributions methods.

Normal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
4.19E-14	4.16E-14	4.11E-14	4.04E-14	4.02E-14	3.97E-14	3.84E-14	3.77E-14	3.74E-14

Lognormal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
4.21E-14	4.18E-14	4.12E-14	4.04E-14	4.01E-14	3.98E-14	3.87E-14	3.82E-14	3.78E-14

Maximum Entropy Distribution									
	Left (Percent)				Untruncated	Right (Percent)			
	10	8	5	1		1	5	8	10
K_1	4.19E-14	4.16E-14	4.12E-14	4.07E-14	4.06E-14	3.99E-14	3.76E-14	3.74E-14	3.75E-14
K_2	4.16E-14	4.13E-14	4.09E-14	4.04E-14	4.03E-14	3.97E-14	3.82E-14	3.76E-14	3.72E-14
K_3	4.13E-14	4.10E-14	4.06E-14	4.00E-14	3.99E-14	3.95E-14	3.88E-14	3.79E-14	3.69E-14
K_4	4.11E-14	4.08E-14	4.03E-14	3.97E-14	3.96E-14	3.95E-14	3.90E-14	3.87E-14	3.83E-14
K_5	4.11E-14	4.08E-14	4.03E-14	3.97E-14	3.92E-14	3.91E-14	4.01E-14	3.84E-14	3.64E-14

The presented table offers a comparison of failure probabilities among different distribution methods and truncation levels, measured in scientific notation. In the context of the Normal and Lognormal Distributions, the probabilities of failure decrease consistently across the specified truncation levels. The Entropy K order method, with varying orders, demonstrates a distinctive pattern of failure probabilities for each truncation level. The probabilities associated with each order provide insights into the method's performance under different truncation scenarios, showcasing nuanced variations compared to the more standard Normal and Lognormal Distributions. This analysis aids in understanding how the Entropy K order method with truncation compares to traditional distribution methods, offering a detailed perspective on reliability assessments in scenarios with varying levels of truncation. When the probability of failure is extremely low, the modified first-order reliability method based on the Melcher theory is not applicable.

3.3 Summary

This chapter delves into the application of reliability methods with truncated random variables, a common occurrence in engineering scenarios where certain random variables have restricted probability distributions. Truncated distributions arise when the domain of probability distributions is limited, often encountered in reliability applications due to bounded ranges of random variables.

The FORM is an analytical method used for estimating structural failure probability in reliability problems. It approximates the limit-state function through a first-order Taylor series expansion, transforming the problem into a standard normal space for efficient probability estimation. However, challenges arise when dealing with truncated random variables, especially with discontinuous or truncated probability density functions. The method of handling truncated random variables involves transforming them into truncated standard normal variables. This transformation facilitates the calculation of failure probability but may introduce numerical challenges and a loss of accuracy. The chapter outlines the steps involved in this process and discusses modifications to the FORM algorithm to accommodate truncated random variables.

Illustrative examples are provided to demonstrate the application of these methods in practical engineering scenarios, such as settlement analysis in consolidated clay and stability evaluation of harbor breakwaters. The examples showcase how reliability analyses are conducted, considering both untruncated and truncated random variables, and compare the results obtained using different probability distribution methods.

Overall, the chapter offers insights into the challenges and methodologies involved in conducting reliability analyses with truncated random variables, providing valuable information for engineers and researchers in the field of structural reliability. The truncated MaxEnt-based FORM method enhances the accuracy of failure probability estimation by incorporating the truncated characteristics of soil parameter distributions. This method offers a more realistic representation of the actual data distribution, particularly in scenarios where the data deviates from traditional distribution models.

Chapter 4: Reliability Analysis of Slopes in Nipigon River Landslide

The investigation conducted in the preceding chapters has demonstrated that addressing the stabilization of unstable slopes is a crucial geotechnical consideration for ensuring the safety of structures. In the design of slope engineering, a conventional deterministic approach has been employed to reduce costs and improve quality in a rational manner. However, this traditional deterministic design is insufficient for distinguishing the inherent variability and internal scattering of geotechnical variables, as the selection of measures of central tendency relies on field tests and engineering systems (Phoon & Kulhawy, 1999). This chapter presents the results of a reliability analysis of the slopes in the Nipigon River area when the random variables were truncated. It also represents the results of the first-order reliability method (FORM) with the maximum Entropy Distribution.

4.1 Nipigon River Landslide

On April 23, 1990, a significant landslide occurred on the eastern side of the Nipigon River, approximately 8 kilometers downstream from the Alexander Hydroelectric Generating Station dam and to the north of the Town of Nipigon in Ontario, Canada. This landslide involved an estimated 300,000 cubic meters of soil and extended nearly 350 meters inland, with a maximum width of around 290 meters. The soil displaced by the landslide was pushed into the Nipigon River, both upstream and downstream, spanning a distance of 300 meters, and it created multiple islands within the river. This alteration in the river's course redirected the flow of water and subsequently led to erosion on the western bank of the Nipigon River, opposite the site of the landslide. This, in turn, is believed to have triggered several additional landslides further south about a month after the initial landslide event. The 1990 Nipigon River landslide has been identified as one of the most devastating landslides to have occurred in Canada. As a consequence of this event, landslides have become a recurring phenomenon in the Nipigon River region, manifesting in various sizes and with varying degrees of impact (Dodds, Burak & Eigenbrod, 1993).

4.1.1. General Geology

The region under investigation is depicted in Figure 4.1, encompassing the segment of the Nipigon River that extends from its mouth into Lake Helen to an upstream railway bridge beyond the landslide site. The primary geological feature in this area is a glaciolacustrine plain and delta, composed mainly of sands and silts. The topography exhibits relatively low relief, coupled with generally inadequate drainage conditions.

The river valleys' sides, carved deeply into these fine-grained deposits, often display extensive dissection and are prone to rill-and-gully erosion, particularly on recently excavated ditch slopes and highway backslopes. Natural and man-made slopes that are higher and steeper are susceptible to the occurrence of small failures. Analysis of aerial photographs revealed that the river area in question has witnessed instances of bank failures in the past, predating the construction of the upstream hydro dam in 1931. The terrain in this specific section is characterized by high moisture levels and inadequate surface drainage, which may contribute to the inherent instability of the riverbanks in this region (Dodds, Burak & Eigenbrod, 1993).

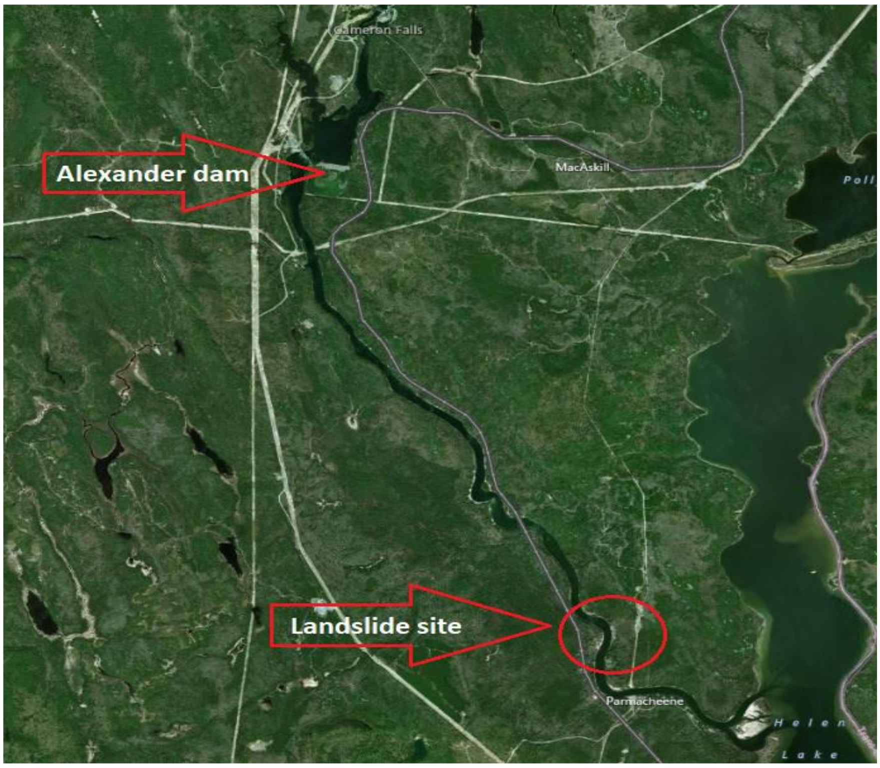


Figure 4.1. Location of the landslide (Google maps)

The characteristics of each of the main soil types encountered are summarized in the following sections:

- Silty Sand: This stratum consists of generally loose silty sand and ranges in thickness from 1 to 3 m.
- Clayey Silt: This section of the stratum has peak undrained shear strengths ranging from 48 to 86 kPa and sensitivities in the order of 2 to 4. Below this, in the softer and more saturated zone, the in-situ strengths range from 19 to 34 kPa, with moderate sensitivities ranging from 1.9 to 2.9. The effective strength parameters of this unit, as determined by consolidated undrained triaxial compression tests, are C' = 12.8 kPa and φ = 30°.
- Sandy Silt: It is generally compact, with SPT values of 18 to 35

blows/ft and moisture contents ranging from 18% to 22%. The stratum appears to be stratified with evidence of interbedded fine sand and silt. This stratum is 3 to 5 m thick.

- Interbedded Silt and Clayey Silt: At greater depths, there exists an interbedded stratum composed of silt and clayey silt. The upper segment of this stratum exhibits stiffness, but as one goes deeper, the silt becomes very soft, with a moisture content of around 33%. The clayey silt, distinguished by a darker gray color, is stiffer than the silt, featuring higher plasticity and a moisture content of approximately 56%. Silt layers are up to 150 mm thick, while clayey silt layers generally range from 25 to 50 mm in thickness. A thin layer of very soft, light gray, clayey silt with exceptionally high sensitivity was encountered at a depth of 12.0 meters. The effective strength parameters of this unit in its upper portion are C' = 5 kPa and φ' = 25° (Dodds, Burak & Eigenbrod, 1993).

4.2 Reliability Analysis of Slopes with Truncated Random Variables in the Case Study

The FORM method is modified to incorporate the non-normal parameters of Entropy Distribution obtained from maximum entropy formalism and will be truncated from 1 to 10 percent on both sides. The steps for the FORM method to evaluate the reliability or safety index are described in Chapter 3, which are:

1. Define the appropriate limit-state equation.
2. Assume the initial values of the design point x^*, i= 1,2,..., N. The initial design point can be considered to be at the mean and standard deviation values of the random variables.

3. Estimate the mean and standard deviation at the design point of the equivalent normal distribution for nonnormal variables.
4. Calculate partial derivatives $(\frac{\partial g}{\partial X_i})^*$, computed at the design point x^*.
5. Measure the direction cosines at the design point as:

$$\alpha_{X_i} = -\frac{\sum_{j=1}^{M}(\rho X_i X_j \frac{\partial g_X(X^*)}{\partial X_i}\sigma_{X_j})}{\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{M}(\rho X_i X_j \frac{\partial g_X(X^*)}{\partial X_i}\frac{\partial g_X(X^*)}{\partial X_j}\sigma_{X_i}\sigma_{X_j})}}, i = 1,2,\dots,N. \quad (4.1)$$

6. Calculate the new values of checking points x_i^*.

$$x_i^* = \mu_{X_i}^N - \alpha_i \beta \sigma_{X_i}^N, \quad (4.2)$$

7. If required, iterate steps 4 to 7 until the direction cosines reach a convergence within the specified tolerance level of 0.005. Upon convergence of the direction cosines, a new checking point can be computed, while maintaining the unknown variables.
8. Adhering to the requirement that limit-state equation must be satisfied at the new checking points, the updated value can be approximated.
9. Continue to repeat Steps 2 through 8 until convergence is achieved within the tolerance level of 0.001.

The algorithm converges quickly in a few cycles, depending on how linear the limit state equation is. A computer program in MATLAB is constructed for the analysis to carry out the computations. The FORM method is required to define the performance function to carry out reliability analysis using the algorithm described above. In this case, the FORM is modified with the maximum Entropy Distribution when the random variables from the Nipigon River Landslide are truncated.

4.2.1. Limit State Function

The factor of safety (F_s), denoting the ratio of shear strength or resisting moment to mobilized shear stress or overturning moment in a potential sliding mass, serves as an indicator of stability. The identification of a critical slip surface associated with the minimum factor of safety is imperative. For simple slopes with regular geometry and $\varphi = 50$ conditions, a closed-form equation can be derived for F_s. Otherwise, numerical computation of the factor of safety is required, using either the limit equilibrium method or slices (Ji & Low, 2012). In carrying out slope reliability analysis, the performance function defining the margin of safety is:

$$g(x) = F_s - 1. \tag{4.3}$$

As a result, when no explicit equation exists for F_s, the performance function $g(x)$ is also implicit.

Hence, the factor of safety, as shown in Figure 4.2, has been derived using a number of approximations. Equations were derived for both the overturning moment (M_0) and the resisting moment (M_R) corresponding to an arbitrary slip circle tangent to a horizontal plane at a depth (D) beneath the top of the clay layer.

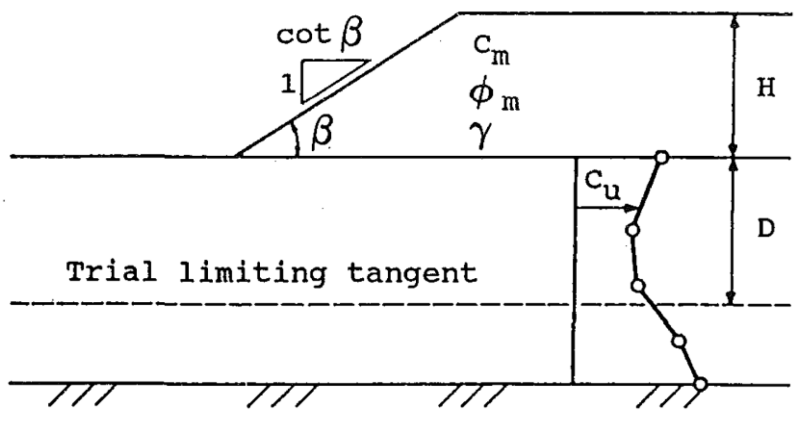


Figure 4.2. Notations for Embankment on the Weak Foundation (Low, 1989).

The critical circle center's coordinates were subsequently determined through partial derivatives of the factor of safety (F_s), representing the ratio of M_R to M_0. These derivatives revealed that the critical circle center lies on a vertical line intersecting the mid-slope of the embankment. Upon substituting the expressions for the critical circle center's coordinates into the equations for M_R and M_0, the final equation for the minimum F_s was established.

Then, the minimum F_s corresponding to a trial limiting tangent at depth D is given by the equation (Low, 1989).

$$(F_s)_D = N_1 \frac{C_A}{\gamma H} + N_2 \left(\frac{C_m}{\gamma H} + \lambda \tan\varphi_m\right). \tag{4.4}$$

Where:

$$N_1 = N_1\left(\frac{D}{H}, \cot\beta\right), \tag{4.5}$$

$$N_2 = N_2\left(\frac{D}{H}, \cot\beta\right), \tag{4.6}$$

$$\lambda = \lambda\left(\frac{D}{H}, \cot\beta\right). \tag{4.7}$$

Additionally, C_A is the average undrained shear strength within the depth D, computed using a derived equation. Moreover, it is possible to find λ with the equation $\lambda = 0.19 + 0.02 \frac{\cot\beta}{\frac{D}{H}}$. (Low & Tang, 1997).

The coefficients N_1, N_2, and λ in Eq. (4.4) can be determined from the charts shown in Figures 4.3 and 4.4. Their values depend on $\frac{D}{H}$ and $\cot\beta$. The symbol λ stands for embankment unit weight in both terms of Eq. (4.4). For base circles, the ratio $\frac{D}{H}$ has to be greater than zero. In Figures 4.3 and 4.4, the curves begin with $\frac{D}{H}$ values of about 0.5. The term C_A in Eq. (4.4) corresponds to the undrained shear strength of the foundation material when the soil is uniform.

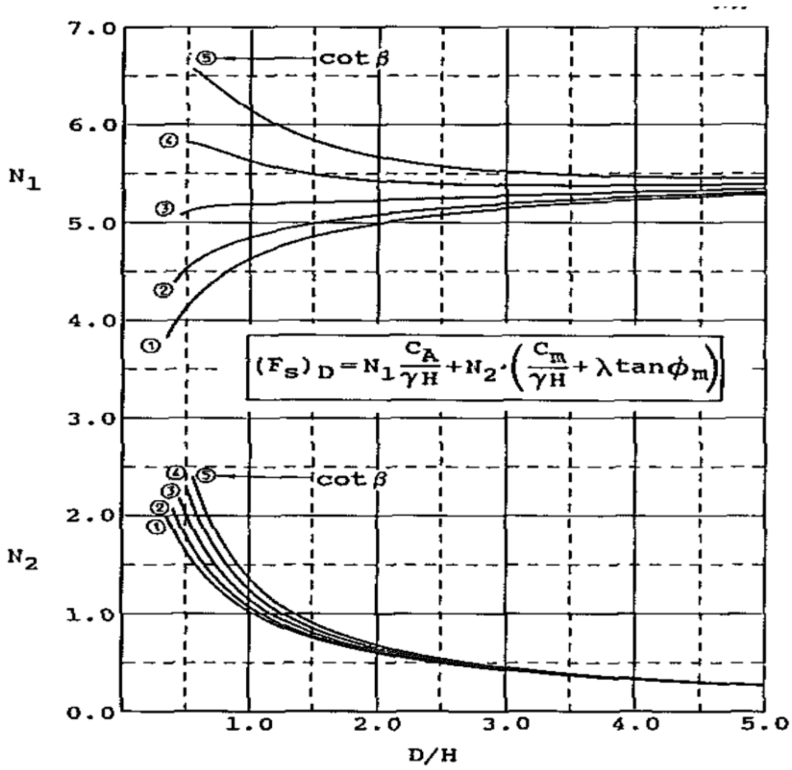


Figure 4.3. Stability Factors N_1, and N_2, for Embankments on Weak Foundations (Low, 1989).

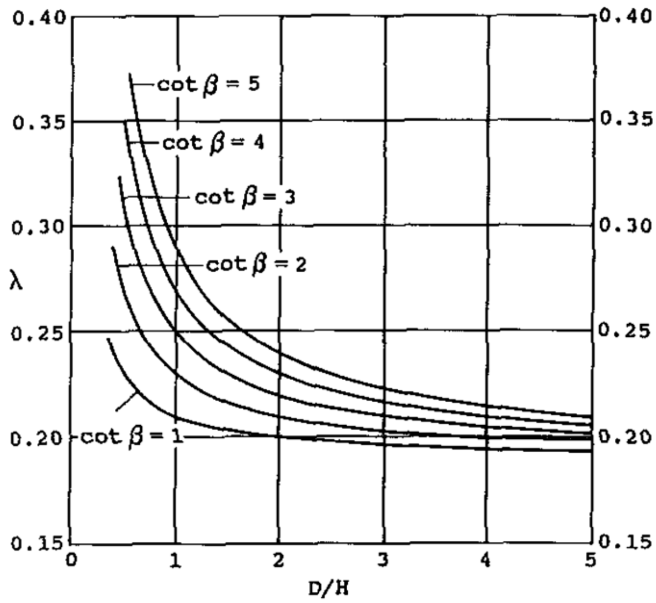


Figure 4.4. Coefficient λ of $\tan \varphi_m$(Low, 1989).

In instances where the undrained shear strength of the foundation rises with depth, it becomes essential to examine multiple tangent depths to identify the one yielding the minimum factor of safety. Each term on the right side of Eq. (4.4). lacks dimensions. The term $\frac{c_A}{\gamma H}$ can be viewed as the normalized foundation strength parameter (S_F), and the term $\frac{c_m}{\gamma H} + \lambda \tan\varphi_m$ as the normalized embankment strength parameter (S_M). Consequently, the factor of safety (FS)D is the summation of two components: S_F and S_M.

$$(F_s)_D = N_1 S_F + N_2 S_M. \tag{4.8}$$

It is noteworthy to emphasize that the unit weight (γ), present in both the normalized foundation strength and normalized embankment strength terms in Eq. (4.4), specifically refers to the unit weight of the embankment material. In the context of undrained conditions within the clay foundation and a horizontally oriented foundation surface, the unit weight of the foundation clay does not impact the factor of safety. Consequently, it is absent from Eq. (4.4) (Low, 1989).

4.2.2. Determination of Distributions and Parameters from Observed Data

To establish a distribution with precision, it is essential to estimate its parameters. Typically, the data available for analysis is employed to estimate both the distribution and its parameters. Therefore, it is crucial to explore existing methodologies for determining the distribution and parameters corresponding to a particular dataset of a random variable. Subsequent assessments of risk and reliability hinge on the accuracy of these estimations (Haldar & Mahadevan, 2000). However, considering the information about the Nipigon River Landslide that was reported by Dodds et al. (1993), the parameters are listed in Table 4.1.

Table 4.1. The parameters of the Nipigon River Landslide (Dodds. et. al., 1993).

Soil Type	Unit Weight $kN/cu.m$	Friction Angle φ'	Cohesion C' kPa
Sandy silt	17.6	30°	0
Clayey silt	19.0	28° (30°)	30 (12.8)
Very soft clayey silt	18.2	28° (25°)	0 (5)

Therefore, according to the information obtained from Table 4.1 and the cross section, which is shown in Figure 4.5. the parameters for obtaining the factor of safety can be calculated, as reported in Table 4.2:

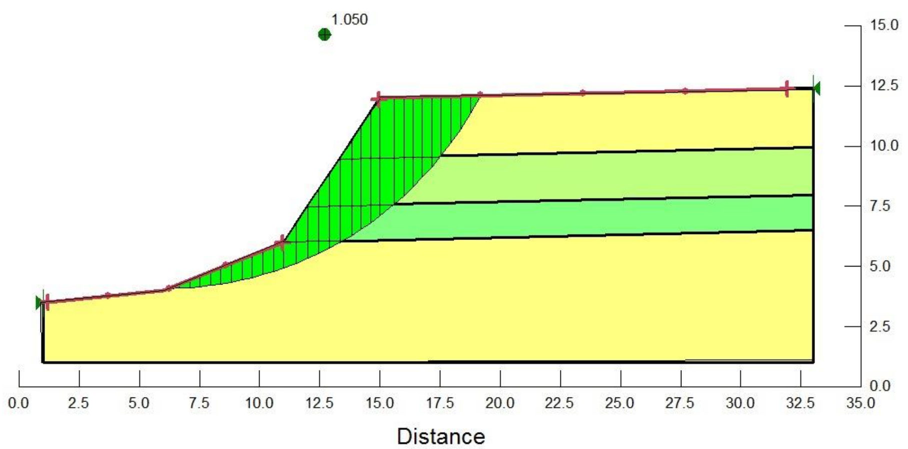


Figure 4.5. Cross section of the Nipigon Landslide.

Table 4.2. The parameters were obtained from the information on the Nipigon Landslide.

depth of trial limiting tangent in the foundation soil		D (m)	3
height of embankment		H (m)	6
D/H			0.5
slope calculations	Vertical value		1.5
	Horizontal Value		1
	Tan of slope angle		1.5
	Slope angle (rad)		0.98
	Slope angle (deg)		56.31
	Cot of slope angle	β	0.67
stability number for the foundation soil		N_1	3.98
stability number for the embankment soil		N_2	1.80
equivalent undrained shear strength in the foundation soil		C_A (kN/m^2)	10
unit weight of embankment soil		γ (kN/m^3)	18
cohesion of embankment soil		C_m (kN/m^2)	32.85
friction angle of embankment soil		φ_m (deg)	28
friction angle of embankment soil		φ_m (rad)	0.48
tan friction angle of embankment soil		Tan φ_m	0.53
coefficient of tan φ_m		λ	0.22
Normalized foundation strength parameter		S_F	0.36
Normalized embankment strength parameter		S_M	0.75
lowest factor of safety corresponding to trial limiting tangent of depth D		$(F_s)_D$	**1.128**

It should be noted that C_m is determined based on the samples from the Nipigon River, as utilized in the second chapter. Therefore, based on this information, the factor of safety is 1.128, consistent with the stability analysis (Table 4.3), reported in Dodds et al.'s (1993) study on the Nipigon.

Table 4.3. Stability analyses (Dodds et al., 1993).

Case No.	Groundwater	River Level	Minimum Factor of Safety
1	low	low	1. 08
2	same as river	high	1. 01
3	high	high	0.93
4	high	low	0.86

The consistency between our calculated factor of safety and the reported values reinforces the reliability of the applied methodologies. The accuracy of parameter estimation plays a pivotal role in subsequent risk and reliability assessments, emphasizing the importance of robust data analysis techniques in geotechnical studies.

4.2.3. Reliability Analysis of Slopes with Random Variables (Truncated and Untruncated)

As a result of the above discussion, the next step involves utilizing both truncated and untruncated Entropy Distributions for the random variables in the FORM method. Therefore, given the information acquired, the limit-state function is:

$$g(x) = F_s - 1 = \left[\frac{6.645}{\gamma} + \left(0.3015\frac{C_m}{\gamma}\right) + 0.398\tan\varphi_m\right] - 1. \quad (4.9)$$

It is evident that there exist three random variables, namely γ (Unit Weight), C_m (Cohesion of embankment), and φ_m (Friction angle). The average values and standard deviations of these random variables are provided in Table 4.4. COV of 0.1 is assumed for these variables.

Table 4.4. Random variables with mean and standard deviation of the Nipigon.

Random variables	Mean (μ)	COV	Standard deviation (σ)
γ (Unit Weight) kN/cu.m	18.20	0.1	1.82
C_m (Cohesion of embankment) kPa	32.85	0.1	3.28
φ_m (Friction angle) (degree)	28.00	0.1	2.80

It is assumed that there are three different assessments with this information:

❖ Assumption 1: Suppose that all variables follow Normal Distributions with the mean (μ) and standard deviation (σ), except the second random variable, i.e., the cohesion of the embankment, C_m. The probability density of C_m is unknown and needs to be inferred from a sample of 100 random numbers. The procedure is then used to determine a MaxEnt for the sample, with the result as:

1- $f_E(C_m) = \exp[-76.7037115471542 + 5.38028718378423C_m - 0.118798274334366{C_m}^2 + 0.000734762234818117{C_m}^3]$, (4.10)

2- $f_N(C_m) = \frac{1}{3.30159\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{C_m - 32.61124}{3.30159}\right)^2\right]$, (4.11)

3- $f_L(C_m) = \frac{1}{0.100983\, C_m \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln C_m - 3.47955}{0.100983}\right)^2\right]$. (4.12)

where $f_E(C_m)$ is the optimal MaxEnt for C_m and $f_N(C_m)$, and $f_L(C_m)$ are the Normal and Lognormal PDFs for C_m, respectively. A comparison of the optimal MaxEnt, Normal, and Lognormal Distributions is shown in Figure 4.6.

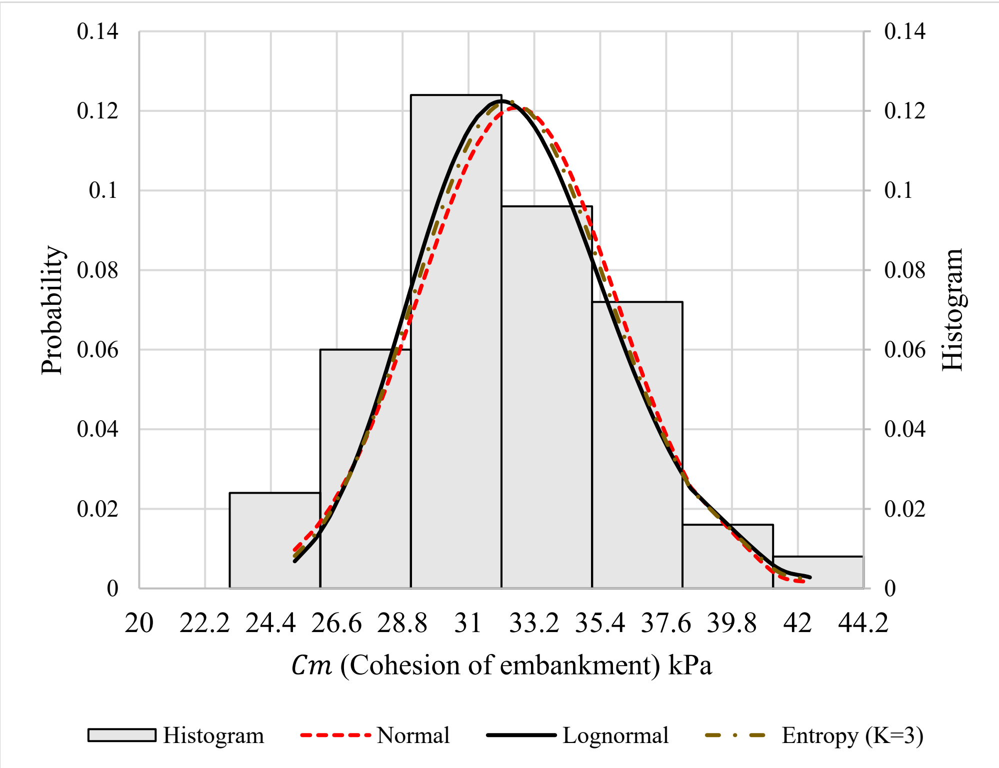


Figure 4.6. Maximum Entropy Distribution, Normal, and Lognormal density functions when all are untruncated.

At the next step, following the discussion in Chapter 3, reliability analysis is conducted utilizing the FORM distribution method. The steps for calculating the FORM method, which includes determining the optimal MaxEnt for noncorrected variables are as follows:

1- Formulate the limit-state function: $g(x) = F_s - 1 = \left[\frac{6.645}{\gamma} + \left(0.3015\frac{C_m}{\gamma}\right) + 0.398\tan\varphi_m\right] - 1$

2- Identify the initial design points x^*.

3- For non-normally distributed random variables, utilize Eqs. 4.13 and 4.14 to calculate $\sigma_{X_i}^N$, $\mu_{X_i}^N$, and $f_{X_2}(x_2^*)$, as follows:

$$\sigma_{X_i}^N = \frac{\phi\left\{\Phi^{-1}[F_{X_i}(x_i^*)\right\}}{f_{x_i}(x_i^*)},$$

$$\mu_{X_i}^N = x_i^* - \Phi^{-1}\left[F_{X_i}(x_i^*)\right]\sigma_{X_i}^N, \tag{4.13}$$

$$f_{X_2}(x_2^*) = f_{C_m}(C_m{}^*), F_{X_2}(x_2^*) = \int_0^{C_m{}^*} f_{C_m}(C_m{}^*)dx. \tag{4.14}$$

4- Calculate the coefficient α_{X_i}

5- Compute the reliability index β.

6- Establish the new design point.

7- Repeat Steps 2-6

8- Apply truncation to the design points.

The resulting coefficients (λ_K) are calculated and listed in Table 4.5, with additional values for K=4-5 provided in Table 4.A, Appendix 3.

Table 4.5. Coefficients of Entropy Distribution (Moment order K=1-3) with untruncated and truncated levels.

K_1

Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
3.81E+00	3.83E+00	3.84E+00	3.85E+00	3.81E+00	3.83E+00	3.92E+00	3.92E+00	3.94E+00
1.67E-02	1.62E-02	1.59E-02	1.57E-02	1.69E-02	1.57E-02	1.15E-02	1.14E-02	1.04E-02

K_2

Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
5.38E+01	6.30E+01	6.87E+01	7.19E+01	5.14E+01	5.53E+01	6.35E+01	6.76E+01	7.08E+01
-3.16E+00	-3.70E+00	-4.03E+00	-4.21E+00	-3.02E+00	-3.28E+00	-3.81E+00	-4.10E+00	-4.31E+00
4.84E-02	5.62E-02	6.08E-02	6.33E-02	4.63E-02	5.04E-02	5.91E-02	6.39E-02	6.74E-02

K_3

Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
8.51E+01	1.03E+02	1.13E+02	1.19E+02	7.67E+01	5.08E+01	1.20E+01	9.55E+00	8.89E+00
-6.06E+00	-7.31E+00	-8.05E+00	-8.44E+00	-5.38E+00	-2.85E+00	1.12E+00	1.50E+00	1.68E+00
1.37E-01	1.65E-01	1.81E-01	1.89E-01	1.19E-01	3.73E-02	-9.71E-02	-1.15E-01	-1.24E-01
-8.95E-04	-1.08E-03	-1.18E-03	-1.24E-03	-7.35E-04	1.35E-04	1.64E-03	1.88E-03	2.03E-03

The calculations in the FORM method for the sample of the Nipigon when the coefficient of embankment is an Entropy Distribution with truncated and untruncated forms are listed in Table 4.6. Moreover, the reliability methods for the Normal and Lognormal Distribution methods are in Table 4.6 and compared with each other.

Table 4.6. The comparison of the reliability methods on the coefficients of embankment as a random variable.

Normal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.56823	0.56881	0.56978	0.57130	0.57172	0.57226	0.57375	0.57468	0.57529
Lognormal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.56818	0.56810	0.56923	0.57126	0.57167	0.57222	0.57334	0.57438	0.57525

Maximum Entropy Distribution									
	Left (Percent)				Untruncated	Right (Percent)			
	10	8	5	1		1	5	8	10
K_1	0.40526	0.40496	0.40446	0.40365	0.40342	0.40719	0.42012	0.42082	0.42382
K_2	0.56860	0.56917	0.57007	0.57137	0.57169	0.57232	0.57391	0.57485	0.57548
K_3	0.56907	0.56966	0.57062	0.57204	0.57230	0.57223	0.57308	0.57405	0.57472
K_4	0.57055	0.57107	0.57171	0.57222	0.57229	0.57230	0.57271	0.57334	0.57371
K_5	0.57059	0.57110	0.57173	0.57223	0.57230	0.57227	0.57223	0.57219	0.57216

The Normal and Lognormal Distributions exhibit certain similarities in their reliability indices, with comparable values across various probability levels. However, the Entropy

Distribution method introduces a distinct perspective, showcasing differences in reliability assessments, particularly in comparison to the more conventional approaches. Moreover, the impact of truncation on the coefficients of embankment is apparent in Table 4.8. It illustrates how applying truncation to the Entropy Distribution in the FORM method influences the reliability indices. This insight is crucial for understanding the robustness and limitations of different reliability modeling techniques in engineering applications.

- ❖ Assumption 2: Assuming a COV of 0.2, the aforementioned steps will be repeated with COV = 0.2, and the corresponding data is listed in Table 4.7.

Table 4.7. Random variables with mean and standard deviation of the Nipigon.

Random variables	Mean (μ)	COV	Standard deviation (σ)
γ (Unit Weight) kN/cu.m	18.20	0.2	3.64
C_m (Cohesion of embankment) kPa	32.85	0.2	6.57
φ_m (Friction angle) (degree)	28.00	0.2	5.60

1- $f_E(C_m) = \exp\left[-16.0296429613816 + 0.898152998001765C_m - 0.0182225414196515{C_m}^2 + 0.00009197918971819{C_m}^3\right],$ (4.15)

2- $f_N(C_m) = \frac{1}{7.64\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{C_m - 33.66}{7.64}\right)^2\right],$ (4.16)

3- $f_L(C_m) = \frac{1}{0.224053\, C_m \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln C_m - 3.91171}{0.224053}\right)^2\right].$ (4.17)

As in the previous section, a comparison of the optimal MaxEnt, Normal, and Lognormal Distributions is shown in Figure 4.7, and the coefficient of Entropy Distribution is listed in Table 4.8 for K=3. The comparison of the reliability methods is also shown in Table 4.9.

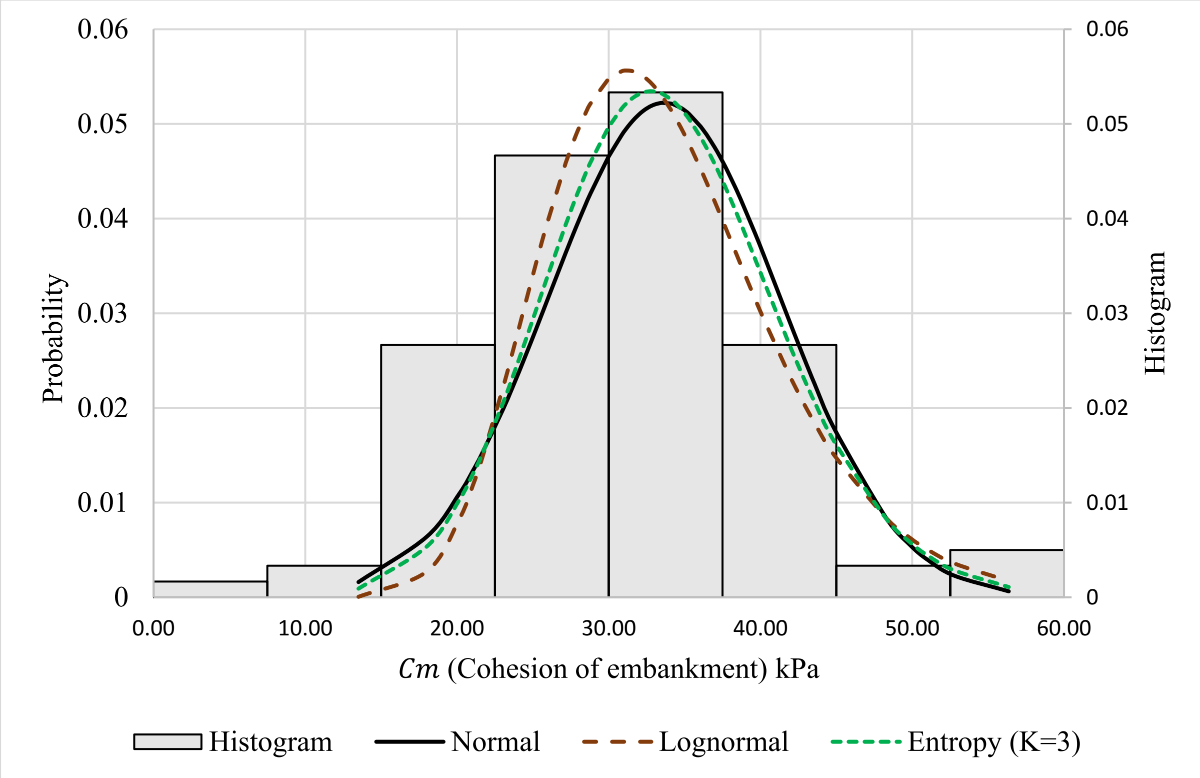


Figure 4.7. Maximum Entropy Distribution, Normal, and Lognormal density functions when all are untruncated.

Table 4.8. Coefficients of Entropy Distribution λ (Moment order K=3, optimal order) with untruncated and truncated levels.

K_3								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
1.78E+01	2.07E+01	2.23E+01	2.34E+01	1.60E+01	1.47E+01	1.06E+01	1.01E+01	9.86E+00
-1.02E+00	-1.21E+00	-1.31E+00	-1.38E+00	-8.98E-01	-7.56E-01	-2.67E-01	-1.98E-01	-1.57E-01
2.11E-02	2.47E-02	2.65E-02	2.77E-02	1.82E-02	1.33E-02	-4.82E-03	-8.11E-03	-1.01E-02
-1.12E-04	-1.32E-04	-1.41E-04	-1.48E-04	-9.20E-05	-3.87E-05	1.75E-04	2.22E-04	2.52E-04

Table 4.9. The comparison of the reliability methods on the coefficients of embankment as a random variable.

Normal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.52911	0.52979	0.53083	0.53248	0.53306	0.53371	0.53573	0.53675	0.53741
Lognormal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.52907	0.52976	0.53080	0.53245	0.53303	0.53368	0.53570	0.53672	0.53738
Maximum Entropy Distribution (K=3)								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.53009	0.53080	0.53186	0.53354	0.53401	0.53408	0.53480	0.53573	0.53635

❖ Assumption 3: In the final stage, it is assumed that all variables conform to Normal Distributions characterized by their mean (μ) and standard deviation (σ), except for the third random variable, namely the friction angle (φ_m). The probability density function of φ_m is unknown and requires estimation from a set of 50 randomly generated numbers. Subsequently, a procedure is employed to establish the MaxEnt for the sample with COV= 0.1, yielding the following outcome:

1- $f_E(\varphi_m) = \exp[-41.9520014726256 + 2.98521637678018\varphi_m - 0.0630817636894855{\varphi_m}^2 + 0.00025659117327659{\varphi_m}^3]$, (4.18)

2- $f_N(\varphi_m) = \frac{1}{3.5350\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\varphi_m - 28.7953}{3.5350}\right)^2\right]$, (4.19)

3- $f_L(\varphi_m) = \frac{1}{0.122307\ \varphi_m \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln \varphi_m - 3.35273}{0.122307}\right)^2\right].$ (4.20)

The MaxEnt for the density function with COV= 0.1, Normal, and Lognormal Distribution are illustrated in Figure 4.8.

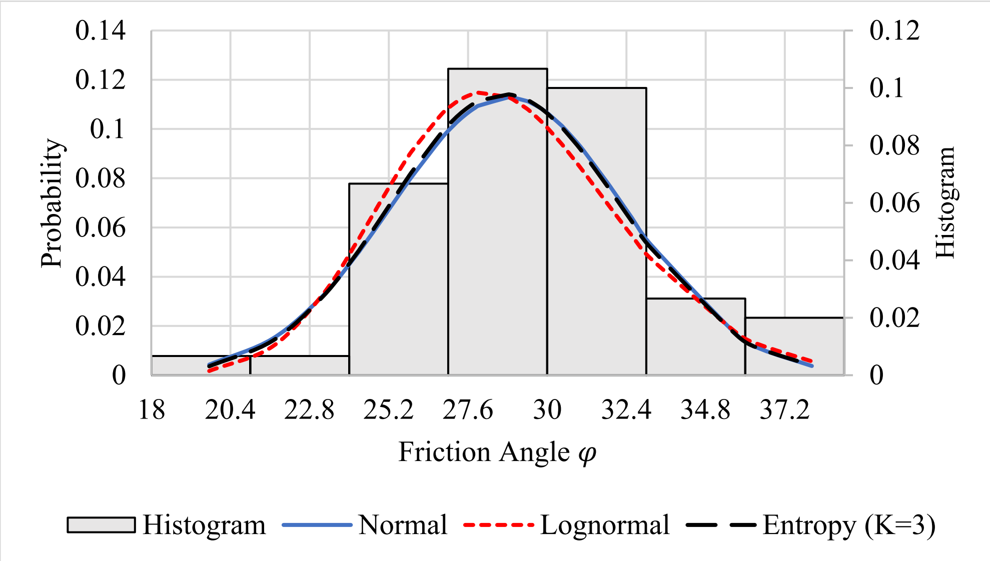


Figure 4.8. Maximum Entropy Distribution, Normal, and Lognormal density functions when all are untruncated.

Similarly, all steps are repeated in this case as in the previous sections. So, Table 4.10 shows coefficients of Entropy Distribution and Table 4.11 also represents the comparison of reliability methods.

Table 4.10. Coefficients of Entropy Distribution λ (Moment order K=3, optimal order) with untruncated and truncated levels.

K_3								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
6.46E+01	7.80E+01	8.20E+01	8.67E+01	4.20E+01	2.04E+01	7.61E+00	7.43E+00	7.34E+00
-5.15E+00	-6.21E+00	-6.52E+00	-6.88E+00	-2.99E+00	-5.05E-01	1.29E+00	1.37E+00	1.44E+00

1.31E-01	1.57E-01	1.65E-01	1.73E-01	6.31E-02	-3.09E-02	-1.10E-01	-1.16E-01	-1.21E-01
-9.58E-04	-1.15E-03	-1.21E-03	-1.27E-03	-2.57E-04	9.14E-04	2.04E-03	2.16E-03	2.24E-03

Table 4.11. The comparison of the reliability methods on the coefficients of embankment as a random variable.

Normal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.14833	0.17577	0.20600	0.30211	0.37683	0.44977	0.61344	0.67341	0.72822
Lognormal Distribution								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.34402	0.36100	0.37792	0.42188	0.45162	0.48226	0.55159	0.56733	0.58224
Maximum Entropy Distribution (K=3)								
Left (Percent)				Untruncated	Right (Percent)			
10	8	5	1		1	5	8	10
0.38267	0.40052	0.41730	0.45864	0.47180	0.47311	0.50841	0.52466	0.54001

By integrating the variability and uncertainty in soil properties using an entropy-based FORM, the proposed measures are more accurately tailored to the real conditions of the Nipigon River slopes. Unlike traditional methods that may misjudge risks by depending on assumed distributions, this approach utilizes data-driven distributions to provide a more precise risk assessment. The results also represent that the probability of failure is sensitive to changing the COV and the random variables.

4.3. Discussions on the Effects of Various Parameters on Slope Reliability

Considering the outcomes of the calculation above, the results of the investigations are as follows:

1. Effect of Changing Random Variables, i.e., Cohesion to Friction Angle: Shifting the focus from cohesion (C_m) to friction angle (φ_m) as the modified random variable introduces unique complexities in slope reliability analysis. The choice of PDF for φ_m, including Normal, Lognormal, and Entropy Distributions, yields varied reliability indices. The results highlight the sensitivity of slope stability to changes in the friction angle and underscore the significance of accurately characterizing its distribution for precise reliability assessments.

2. Impact of COV Increase (from 0.1 to 0.2): The decision to elevate the coefficient of variation (COV) from 0.1 to 0.2 introduces heightened variability in the material properties, particularly in unit weight (γ), cohesion (C_m), and friction angle (φ_m). This deliberate increase in uncertainty has noticeable effects on reliability indices. The sensitivity analysis underscores the importance of understanding how higher variability influences the stability predictions, emphasizing the need for a more robust and adaptable approach to handle increased uncertainties in slope stability assessments.

3. Combined Influence of Altered Random Variable and Increased COV: Examining the combined effects of changing the random variable to friction angle and increasing the COV provides a comprehensive view of the interplay between these factors. The results showcase how alterations in material properties, especially when associated with higher variability, contribute to variations in reliability indices. This combined influence sheds light on the

complex nature of slope stability assessments, emphasizing the need for a nuanced and adaptive approach that considers multiple sources of uncertainty.

In conclusion, the deliberate changes made in the analysis, specifically shifting from cohesion to friction angle as the random variable and increasing the COV, provide valuable insights into the nuanced dynamics of slope reliability. These adjustments highlight the importance of accurate parameter characterization and the adaptability of the reliability analysis methodology to accommodate increased uncertainties for robust slope stability predictions in engineering applications.

4.4. Reliability-Based Design on Slope Stability

The stability design of slopes is closely related to the safety of human life and property and the development of infrastructure industries. Originally, the stability design of an engineered slope was commonly carried out using deterministic methods in which a single factor of safety is quantified. The deterministic methods through trial calculations are easily implemented by practitioners to obtain a feasible design scheme that ensures an allowable factor of safety is met (Dawson et al., 1999). Moreover, slope engineering is fraught with measurement and transformation uncertainties owing to random testing errors, and the imperfection and simplification of geologic or geotechnical models. These uncertainties are rarely accounted for at the same time in the stability designs of slopes (Jiang et al., 2002). There are several factors to consider regarding landslides. Firstly, the variability in soil characteristics and geological data results in unique design scenarios for each case. Secondly, the mechanism of slope stability remains consistent across different types of slopes. Thirdly, the reliability of the stability analysis is a key determinant in slope design. A combination of judgment, experience, intuition, advanced data collection, and analytical techniques contributes to finding the best solutions.

Landslides pose significant challenges for detection and can be costly during construction. It's evident that erosion will become a significant concern in the future, particularly regarding slope stability along riverbanks. Consequently, proactive measures may be necessary to prevent such failures from compromising slope stability. These measures may involve the installation of geo-synthetics, rocks, and other reinforcements along the riverbed, extending hundreds of meters both upstream and downstream. Addressing bottom erosion serves to mitigate the effects of the primary landslide event.

In reliability-based design, the parameters C and φ (often denoted as cohesion and friction angle, respectively) are fundamental properties used to characterize the strength and behavior of soil materials. These parameters have significant impacts on the reliability of geotechnical structures, particularly in slope stability analysis (Low, 2022). Here's how they influence reliability-based design:

1. Cohesion C: Cohesion represents the internal strength of soil particles and is typically associated with cohesive soils such as clay. In the case of stability analysis, cohesion contributes to the resisting forces within the soil mass, particularly in resisting shear along potential failure surfaces. Higher cohesion generally leads to a higher factor of safety against slope failure. In reliability-based design, the uncertainty associated with cohesion (often represented as a probability distribution) directly affects the reliability index. A higher degree of uncertainty in cohesion parameters may decrease the overall reliability of the design.

2. Friction Angle φ: Friction angle characterizes the resistance of soil to shearing and is typically associated with cohesionless soils such as sand and gravel. It influences the shear strength of soil and determines the angle at which soil particles can resist sliding along potential failure surfaces. Higher friction angles generally lead to higher stability and a higher factor of safety. Similar to cohesion, the uncertainty associated with friction angle parameters affects the reliability index. Variability in friction angle due to factors like heterogeneity of soil deposits or measurement errors can influence the reliability of the design.

Therefore, it's essential to characterize their uncertainty through probabilistic models or statistical analyses. Sensitivity analysis techniques are often employed to assess the relative importance of these parameters and understand their impacts on the reliability of geotechnical structures. By considering the variability and uncertainty associated with cohesion and friction angle, engineers can develop more robust and reliable designs for geotechnical applications such as slope stability analysis, foundation design, and earth-retaining structures (Jian et al., 2022).

4.5 Summary

The chapter delves into the critical aspect of reliability analysis in slope engineering, using the Nipigon River Landslide as a case study. The chapter underscores the limitations of conventional deterministic approaches in slope design and highlights the necessity of accounting for inherent variability and uncertainties in geotechnical variables.

The Nipigon River Landslide, which occurred in 1990, serves as the focal point of the study due to its significant impact and recurrence in the region. The chapter provides a detailed overview of the geological features of the area, emphasizing the presence of fine-grained deposits, inadequate drainage conditions, and susceptibility to erosion and slope failures. Understanding the geological context is crucial for assessing slope stability and formulating effective stabilization measures.

The reliability analysis methodology employed in the study integrates the First Order Reliability Method (FORM) with the MaxEnt to account for the variability in random variables. The FORM method, modified to incorporate non-normal parameters obtained from MaxEnt, enables the evaluation of reliability or safety indices for slope stability.

The limit state function, representing the margin of safety, is defined based on the factor of safety (F_S), which denotes the ratio of shear strength or resisting moment to mobilized shear stress or overturning moment in a potential sliding mass. The performance function, implicit in cases where no explicit equation for F_S exists, necessitates numerical computation of the factor of safety using appropriate methodologies, such as the limit equilibrium methods of slices.

The determination of distributions and parameters from observed data is crucial for establishing accurate reliability assessments. The chapter outlines the process of estimating parameters for random variables such as unit weight, cohesion, and friction angle based on available data from the Nipigon River Landslide. The reliability analyses conducted with truncated and untruncated Entropy Distributions provide insights into the sensitivity of slope stability predictions to variations in material properties and uncertainties.

The discussion section elucidates the effects of various parameters on slope reliability, including the impact of changing random variables, increasing coefficients of variation (COV), and the combined influence of altered random variables and COV. The results underscore the importance of accurate parameter characterization and the adaptability of reliability analysis methodologies to accommodate increased uncertainties for robust slope stability predictions.

Furthermore, the chapter explores the significance of reliability-based design in slope engineering, emphasizing the role of parameters such as cohesion and friction angle in influencing the reliability of geotechnical structures. The proactive measures proposed to mitigate landslide risks along riverbanks underscore the importance of integrating reliability analysis into slope design to ensure the safety of human life and infrastructure.

In conclusion, Chapter 4 provides valuable insights into the application of reliability analysis methodologies in slope engineering, using the Nipigon River Landslide as a case study. The comprehensive approach to assessing slope stability and formulating design solutions underscores the importance of accounting for uncertainties and variability in geotechnical variables for effective risk management and infrastructure development.

Chapter 5: Conclusions and Recommendations

5.1 Conclusions

The comprehensive study presented across the provided chapters constitutes a significant contribution to the field of probabilistic modeling, offering insights into the application of the MaxEnt and related methodologies. By exploring MaxEnt's utilization in estimating PDF from observed data, the study advocates for a systematic and objective approach that minimizes bias and spurious information while maximizing alignment with available sample data.

One of the study's key strengths lies in its recognition of real-world complexities, particularly concerning data truncation and censorship. Through an examination of scenarios where data truncation occurs due to exceedances or below detection-limit values, the study underscores the importance of accurately addressing these situations in statistical analysis. By highlighting the principal estimators, including moment and maximum likelihood estimators, the study provides a robust framework for calculating distribution parameters from truncated and censored samples.

Furthermore, the study emphasizes the significance of consistency in probabilistic modeling, asserting that unbiased estimation is paramount for reliable results. By adhering to fundamental conditions such as consistency axioms, the study ensures that the derived probability distributions systematically maximize uncertainty surrounding the unknown data.

The case study of the Nipigon River Landslide in Canada serves as a compelling illustration of the practical application of the discussed methodologies. By applying probabilistic modeling techniques to analyze real-world phenomena, the study demonstrates the value of these approaches in understanding complex events and informing decision-making processes. The truncated MaxEnt-based FORM method provides a more accurate estimation of the probability of failure by accounting for the truncated nature of the soil parameter distributions. This approach considers the actual data distribution more realistically, especially in cases where the data does not fit the classical distribution models well. The reliability analysis using truncated distributions allowed for a more site-specific assessment of slope stability, leading to tailored mitigation measures. Unlike general methods that might apply uniform safety factors, this approach identifies critical areas with greater precision, leading to more focused and effective interventions. By incorporating the variability and uncertainty in soil properties through the entropy-based method, the proposed measures are better suited to the actual conditions of the Nipigon River slopes. This contrasts with traditional methods that might underestimate or overestimate risks by relying on assumed distributions rather than data-driven distributions.

In conclusion, the study advocates for the widespread adoption of the maximum entropy principle and related methodologies in probabilistic modeling. By offering a systematic, objective, and robust framework for estimating probability distributions from observed data while addressing challenges such as data truncation and censorship, the study contributes to advancing the field of probabilistic modeling in various scientific and engineering domains. Its emphasis on consistency, unbiased estimation, and real-world applicability positions it as a valuable resource for researchers and practitioners alike.

5.2 Recommendations

1. **Exploration of Advanced Techniques:** Consider exploring advanced techniques for probabilistic modeling beyond the maximum entropy principle. This could involve incorporating Bayesian methods, machine learning algorithms, or other advanced statistical techniques to further improve the accuracy and robustness of probability distribution estimations.
2. **Integration of Data Fusion Approaches:** Investigate the integration of data fusion approaches to combine information from multiple sources, such as satellite imagery, sensor data, and historical records. Data fusion techniques can help improve the reliability and comprehensiveness of probabilistic models by leveraging diverse data sources.
3. **Further Case Studies:** Advocate for additional case studies, similar to the Nipigon River Landslide presented in Chapter 4, to validate the efficacy and applicability of the proposed methods across different geological and environmental settings.
4. **Research on Truncation Effects:** Encourage further research on the effects of truncation, specifically focusing on refining the precision of reliability analysis when truncated random variables are present. Investigate methods to minimize errors associated with the conventional first-order reliability method (FORM) when dealing with truncated distributions.
5. **Continuous Validation and Updating:** Emphasize the need for continuous validation and updating of the proposed target reliability indices as more data becomes available.

This iterative process will contribute to the refinement and improvement of the reliability assessment framework.

Appendix 1

Chapter 2:

Table 2.A. Auxiliary estimation function $\lambda(h, \alpha)$, for singly censored samples from the Normal Distribution:

α \ h	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.15
.00	.01010	.02040	.03090	.04161	.05251	.06363	.07495	.08649	.09824	.11020	.17342
.01	.01020	.02059	.03118	.04197	.05297	.06417	.07557	.08719	.09902	.11106	.17465
.02	.01029	.02077	.03145	.04233	.05341	.06469	.07618	.08787	.09978	.11190	.17586
.03	.01038	.02095	.03172	.04268	.05384	.06520	.07677	.08854	.10052	.11272	.17704
.04	.01047	.02113	.03197	.04302	.05426	.06570	.07734	.08919	.10125	.11352	.17821
.05	.01055	.02129	.03223	.04335	.05467	.06619	.07791	.08983	.10197	.11431	.17935
.06	.01064	.02146	.03247	.04367	.05507	.06667	.07846	.09046	.10267	.11508	.18047
.07	.01072	.02162	.03271	.04399	.05546	.06713	.07900	.09107	.10335	.11584	.18157
.08	.01080	.02178	.03294	.04430	.05585	.06759	.07953	.09168	.10403	.11659	.18266
.09	.01087	.02193	.03317	.04460	.05623	.06804	.08006	.09227	.10469	.11732	.18373
.10	.01095	.02208	.03340	.04490	.05660	.06848	.08057	.09285	.10534	.11804	.18479
.11	.01102	.02223	.03362	.04519	.05696	.06892	.08107	.09343	.10598	.11875	.18583
.12	.01110	.02238	.03384	.04548	.05732	.06934	.08157	.09399	.10661	.11944	.18685
.13	.01117	.02252	.03405	.04577	.05767	.06976	.08205	.09454	.10723	.12013	.18786
.14	.01124	.02266	.03426	.04604	.05802	.07018	.08254	.09509	.10785	.12081	.18886
.15	.01131	.02280	.03447	.04632	.05836	.07059	.08301	.09563	.10845	.12148	.18985
.16	.01138	.02293	.03467	.04659	.05869	.07099	.08348	.09616	.10905	.12214	.19082
.17	.01145	.02307	.03487	.04685	.05902	.07138	.08394	.09668	.10963	.12279	.19178
.18	.01151	.02320	.03507	.04712	.05935	.07177	.08439	.09720	.11021	.12343	.19273
.19	.01158	.02333	.03526	.04737	.05967	.07216	.08484	.09771	.11079	.12407	.19367
.20	.01164	.02346	.03545	.04763	.05999	.07254	.08528	.09822	.11135	.12469	.19460
.21	.01171	.02359	.03564	.04788	.06030	.07291	.08572	.09871	.11191	.12531	.19552
.22	.01177	.02371	.03583	.04813	.06061	.07329	.08615	.09921	.11246	.12592	.19643
.23	.01183	.02383	.03601	.04838	.06092	.07365	.08657	.09969	.11301	.12653	.19733
.24	.01189	.02396	.03620	.04862	.06122	.07401	.08700	.10017	.11355	.12713	.19822
.25	.01195	.02408	.03638	.04886	.06152	.07437	.08741	.10065	.11408	.12772	.19910
.26	.01201	.02420	.03656	.04909	.06182	.07473	.08783	.10112	.11461	.12831	.19997
.27	.01207	.02431	.03673	.04933	.06211	.07508	.08823	.10158	.11513	.12889	.20083
.28	.01213	.02443	.03691	.04956	.06240	.07542	.08864	.10205	.11565	.12946	.20169
.29	.01219	.02454	.03708	.04979	.06269	.07577	.08904	.10250	.11616	.13003	.20254
.30	.01224	.02466	.03725	.05002	.06297	.07611	.08943	.10295	.11667	.13059	.20338
.31	.01230	.02477	.03742	.05024	.06325	.07644	.08982	.10340	.11717	.13115	.20421
.32	.01236	.02488	.03758	.05047	.06353	.07678	.09021	.10384	.11767	.13170	.20503
.33	.01241	.02499	.03775	.05069	.06380	.07711	.09060	.10428	.11816	.13225	.20585
.34	.01247	.02510	.03791	.05090	.06408	.07743	.09098	.10472	.11865	.13279	.20666
.35	.01252	.02521	.03808	.05112	.06435	.07776	.09136	.10515	.11914	.13333	.20747
.36	.01257	.02532	.03824	.05133	.06461	.07808	.09173	.10557	.11962	.13386	.20826
.37	.01263	.02542	.03840	.05155	.06488	.07839	.09210	.10600	.12009	.13439	.20906
.38	.01268	.02553	.03855	.05176	.06514	.07871	.09247	.10642	.12057	.13491	.20984
.39	.01273	.02563	.03871	.05197	.06540	.07902	.09283	.10683	.12103	.13543	.21062
.40	.01278	.02574	.03887	.05217	.06566	.07933	.09319	.10725	.12150	.13595	.21139
.41	.01284	.02584	.03902	.05238	.06592	.07964	.09355	.10766	.12196	.13646	.21216
.42	.01289	.02594	.03917	.05258	.06617	.07994	.09391	.10806	.12242	.13697	.21292
.43	.01294	.02604	.03932	.05278	.06642	.08025	.09426	.10847	.12287	.13747	.21368
.44	.01299	.02614	.03947	.05298	.06667	.08055	.09461	.10887	.12332	.13797	.21443
.45	.01304	.02624	.03962	.05318	.06692	.08085	.09496	.10926	.12377	.13847	.21517
.46	.01309	.02634	.03977	.05338	.06717	.08114	.09530	.10966	.12421	.13896	.21591
.47	.01313	.02644	.03992	.05357	.06741	.08143	.09565	.11005	.12465	.13945	.21665
.48	.01318	.02654	.04006	.05377	.06765	.08173	.09598	.11044	.12509	.13994	.21738
.49	.01323	.02663	.04021	.05396	.06790	.08201	.09632	.11082	.12552	.14042	.21810

α \ h	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.15
.50	.01328	.02673	.04035	.05415	.06813	.08230	.09666	.11121	.12595	.14090	.21882
.51	.01333	.02682	.04049	.05434	.06837	.08259	.09699	.11159	.12638	.14138	.21954
.52	.01337	.02692	.04064	.05453	.06861	.08287	.09732	.11196	.12681	.14185	.22025
.53	.01342	.02701	.04078	.05472	.06884	.08315	.09765	.11234	.12723	.14232	.22095
.54	.01347	.02710	.04092	.05490	.06907	.08343	.09797	.11271	.12765	.14278	.22166
.55	.01351	.02720	.04105	.05509	.06931	.08371	.09830	.11308	.12806	.14325	.22235
.56	.01356	.02729	.04119	.05527	.06954	.08398	.09862	.11345	.12848	.14371	.22305
.57	.01360	.02738	.04133	.05546	.06976	.08426	.09894	.11382	.12889	.14417	.22374
.58	.01365	.02747	.04146	.05564	.06999	.08453	.09926	.11418	.12930	.14462	.22442
.59	.01369	.02756	.04160	.05582	.07022	.08480	.09957	.11454	.12970	.14507	.22510
.60	.01374	.02765	.04173	.05600	.07044	.08507	.09989	.11490	.13011	.14552	.22578
.61	.01378	.02774	.04187	.05617	.07066	.08534	.10020	.11526	.13051	.14597	.22645
.62	.01383	.02783	.04200	.05635	.07088	.08560	.10051	.11561	.13091	.14641	.22712
.63	.01387	.02791	.04213	.05653	.07110	.08586	.10082	.11596	.13131	.14685	.22779
.64	.01391	.02800	.04226	.05670	.07132	.08613	.10112	.11631	.13170	.14729	.22845
.65	.01396	.02809	.04239	.05687	.07154	.08639	.10143	.11666	.13209	.14773	.22910
.66	.01400	.02817	.04252	.05705	.07175	.08665	.10173	.11701	.13248	.14816	.22976
.67	.01404	.02826	.04265	.05722	.07197	.08690	.10203	.11735	.13287	.14859	.23041
.68	.01409	.02834	.04278	.05739	.07218	.08716	.10233	.11769	.13326	.14902	.23106
.69	.01413	.02843	.04290	.05756	.07239	.08742	.10263	.11804	.13364	.14945	.23170
.70	.01417	.02851	.04303	.05773	.07260	.08767	.10292	.11837	.13402	.14987	.23234
.71	.01421	.02860	.04316	.05789	.07281	.08792	.10322	.11871	.13440	.15030	.23298
.72	.01425	.02868	.04328	.05806	.07302	.08817	.10351	.11905	.13478	.15072	.23361
.73	.01430	.02876	.04341	.05823	.07323	.08842	.10380	.11938	.13515	.15113	.23425
.74	.01434	.02885	.04353	.05839	.07344	.08867	.10409	.11971	.13553	.15155	.23487
.75	.01438	.02893	.04365	.05856	.07364	.08892	.10438	.12004	.13590	.15196	.23550
.76	.01442	.02901	.04377	.05872	.07385	.08916	.10467	.12037	.13627	.15237	.23612
.77	.01446	.02909	.04390	.05888	.07405	.08941	.10495	.12070	.13664	.15278	.23674
.78	.01450	.02917	.04402	.05904	.07425	.08965	.10524	.12102	.13700	.15319	.23735
.79	.01454	.02925	.04414	.05920	.07445	.08989	.10552	.12134	.13737	.15360	.23797
.80	.01458	.02933	.04426	.05936	.07465	.09013	.10580	.12167	.13773	.15400	.23858
.81	.01462	.02941	.04438	.05952	.07485	.09037	.10608	.12199	.13809	.15440	.23918
.82	.01466	.02949	.04450	.05968	.07505	.09061	.10636	.12231	.13845	.15480	.23979
.83	.01470	.02957	.04461	.05984	.07525	.09085	.10664	.12262	.13881	.15520	.24039
.84	.01474	.02965	.04473	.06000	.07545	.09108	.10691	.12294	.13916	.15559	.24099
.85	.01478	.02972	.04485	.06015	.07564	.09132	.10719	.12325	.13952	.15599	.24158
.90	.01497	.03011	.04542	.06092	.07661	.09248	.10854	.12480	.14126	.15793	.24452
.95	.01515	.03048	.04599	.06168	.07755	.09361	.10987	.12632	.14297	.15983	.24740
1.00	.01534	.03085	.04654	.06241	.07847	.09472	.11116	.12780	.14465	.16170	.25022
1.50	.01699	.03417	.05153	.06908	.08682	.10476	.12290	.14125	.15981	.17858	.27585
2.00	.01842	.03703	.05583	.07483	.09403	.11343	.13304	.15287	.17291	.19318	.29806
2.50	.01969	.03958	.05967	.07996	.10046	.12117	.14210	.16325	.18463	.20624	.31794
3.00	.02085	.04191	.06317	.08464	.10633	.12823	.15037	.17273	.19532	.21816	.33611
3.50	.02192	.04406	.06641	.08897	.11176	.13477	.15802	.18150	.20522	.22919	.35294
4.00	.02293	.04607	.06943	.09302	.11684	.14088	.16517	.18970	.21448	.23951	.36870
5.00	.02477	.04977	.07499	.10046	.12616	.15211	.17832	.20478	.23150	.25849	.39766
6.00	.02644	.05312	.08004	.10721	.13464	.16232	.19026	.21847	.24696	.27573	.42400
7.00	.02798	.05622	.08470	.11345	.14245	.17173	.20128	.23111	.26123	.29165	.44832
8.00	.02942	.05910	.08905	.11926	.14975	.18052	.21157	.24291	.27455	.30650	.47103
10.00	.03206	.06440	.09701	.12992	.16312	.19662	.23042	.26454	.29897	.33373	.51265

Table 2.B. The results of the Truncated Chi-square.

%1 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.467	0.467	4.036	4.036	4.191	4.191	3.385	3.385	2.611	2.611	2.621	2.621
15-30	16	16.665	0.027	13.432	0.491	13.926	0.309	14.587	0.137	16.298	0.005	15.798	0.003
30-45	36	37.132	0.035	27.169	2.871	26.971	3.023	29.715	1.329	31.962	0.510	32.442	0.390
45-60	26	30.916	0.782	33.421	1.648	32.570	1.325	33.199	1.561	30.977	0.800	31.389	0.925
60-75	26	17.698	3.895	25.009	0.039	24.528	0.088	22.480	0.551	20.334	1.579	19.816	1.930
75-90	8	8.704	0.057	11.380	1.004	11.517	1.074	10.220	0.482	10.704	0.683	10.486	0.589
90-105	6	4.029	0.964	3.146	2.588	3.369	2.054	3.468	1.849	4.508	0.494	4.765	0.320
> 105	1	1.829	0.376	0.528	0.422	0.613	0.244	0.979	0.000	1.276	0.060	1.347	0.089
total	119	117.440	6.602	118.120	13.099	117.686	12.308	118.035	9.296	118.670	6.742	118.662	6.867
$C_{0.95,5}$=11.07													
Chi-Square Test		Acceptable		Unacceptable		Unacceptable		Acceptable		Acceptable		Acceptable	

%5 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.185	0.185	3.165	3.165	4.081	4.081	4.015	4.015	3.939	3.939	3.085	3.085
15-30	12	12.454	0.017	11.726	0.006	14.611	0.467	15.084	0.631	14.729	0.505	14.325	0.377
30-45	36	35.700	0.003	25.668	4.159	25.478	4.345	26.298	3.580	26.212	3.655	27.865	2.375
45-60	26	32.493	1.297	33.239	1.577	29.382	0.389	29.346	0.382	29.768	0.477	30.360	0.626
60-75	26	18.674	2.874	25.471	0.011	22.411	0.575	21.650	0.874	21.884	0.774	20.537	1.453
75-90	8	8.835	0.079	11.546	1.089	11.304	0.966	10.907	0.775	10.758	0.707	10.215	0.480
90-105	6	3.846	1.206	3.093	2.732	3.769	1.320	3.877	1.163	3.723	1.393	4.157	0.817
> 105	1	1.623	0.239	0.489	0.534	0.830	0.035	1.005	0.000	0.974	0.001	1.255	0.052
total	115	113.810	5.899	114.398	13.274	111.867	12.178	112.181	11.418	111.987	11.452	111.799	9.266
$C_{0.95,5}$=11.07													
Chi-Square Test		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Acceptable	

%8 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.089	0.089	2.616	2.616	3.230	3.230	3.221	3.221	3.058	3.058	2.860	2.860
15-30	9	9.831	0.070	10.512	0.217	14.677	2.196	14.526	2.102	14.008	1.790	13.845	1.695
30-45	36	34.135	0.102	24.454	5.451	24.200	5.753	24.005	5.994	24.091	5.887	24.803	5.055
45-60	26	33.430	1.651	32.973	1.475	27.221	0.055	27.257	0.058	27.969	0.139	28.393	0.202
60-75	26	19.373	2.267	25.780	0.002	20.889	1.251	21.093	1.141	21.325	1.025	20.813	1.293
75-90	8	8.938	0.098	11.683	1.161	10.934	0.787	11.033	0.834	10.700	0.681	10.324	0.523
90-105	6	3.727	1.386	3.066	2.809	3.903	1.127	3.867	1.176	3.647	1.517	3.729	1.383
> 105	1	1.491	0.162	0.465	0.616	0.950	0.003	0.900	0.011	0.900	0.011	1.034	0.001
total	112	111.014	5.825	111.548	14.347	106.004	14.401	105.903	14.537	105.697	14.108	105.801	13.013
$C_{0.95,5}$=11.07													
Chi-Square Test		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	

%10 Left Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.042	0.042	2.148	2.148	1.893	1.893	1.828	1.828	1.722	1.722	1.727	1.727
15-30	6	7.662	0.361	9.368	1.211	14.908	5.323	14.150	4.694	13.712	4.337	13.721	4.345
30-45	36	32.244	0.438	23.182	7.088	23.354	6.847	22.560	8.006	22.869	7.540	22.834	7.591
45-60	26	34.132	1.937	32.591	1.333	25.668	0.004	25.922	0.000	26.681	0.017	26.651	0.016
60-75	26	20.036	1.775	26.048	0.000	19.793	1.946	20.726	1.342	20.796	1.302	20.816	1.291
75-90	8	9.049	0.122	11.830	1.240	10.708	0.685	11.137	0.884	10.685	0.675	10.711	0.686
90-105	6	3.626	1.554	3.049	2.855	4.063	0.924	3.885	1.152	3.688	1.449	3.690	1.446
> 105	1	1.380	0.104	0.445	0.691	1.081	0.006	0.849	0.027	0.901	0.011	0.895	0.012
total	109	108.170	6.333	108.661	16.567	101.468	17.628	101.056	17.933	101.055	17.053	101.045	17.115
$C_{0.95,5}$=11.07													
Chi-Square Test		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	

%1 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.631	0.631	4.071	4.071	4.457	4.457	3.633	3.633	2.911	2.911	2.910	2.910
15-30	18	18.880	0.041	14.315	0.949	14.339	0.935	15.083	0.564	17.121	0.045	16.644	0.111
30-45	36	38.227	0.130	29.218	1.574	28.029	2.267	30.585	0.959	31.998	0.500	32.618	0.351
45-60	26	30.037	0.542	34.654	2.161	33.311	1.605	33.521	1.687	30.634	0.701	30.852	0.763
60-75	26	16.593	5.333	23.890	0.186	24.075	0.154	21.989	0.732	20.635	1.395	20.025	1.783
75-90	8	7.977	0.000	9.567	0.257	10.577	0.628	9.585	0.262	10.888	0.766	10.880	0.762
90-105	5	3.637	0.510	2.223	3.469	2.822	1.681	3.093	1.175	3.973	0.265	4.257	0.130
> 105	0	1.635	1.635	0.299	0.299	0.437	0.437	0.771	0.771	0.694	0.694	0.649	0.649
total	119	117.617	8.823	118.237	12.967	118.046	12.163	118.260	9.783	118.854	7.278	118.834	7.458
$C_{0.95,5}$=11.07													
Chi-Square Test		Acceptable		Unacceptable		Unacceptable		Acceptable		Acceptable		Acceptable	

%5 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.540	0.540	3.444	3.444	6.139	6.139	6.282	6.282	6.830	6.830	6.469	6.469
15-30	18	18.986	0.051	14.401	0.900	17.485	0.015	16.222	0.195	16.926	0.068	17.214	0.036
30-45	36	39.127	0.250	31.441	0.661	30.043	1.181	28.789	1.806	27.849	2.386	28.364	2.056
45-60	26	29.448	0.404	35.908	2.734	31.158	0.854	32.222	1.201	31.272	0.889	30.644	0.704
60-75	26	15.206	7.661	21.460	0.960	19.505	2.162	20.915	1.236	21.788	0.814	21.650	0.874
75-90	8	6.774	0.222	6.703	0.251	7.367	0.054	7.246	0.078	7.648	0.016	8.021	0.000
90-105	1	2.856	1.206	1.091	0.008	1.677	0.273	1.232	0.044	0.994	0.000	0.910	0.009
> 105	0	1.187	1.187	0.092	0.092	0.192	0.192	0.084	0.084	0.030	0.030	0.012	0.012
total	115	114.125	11.521	114.539	9.050	113.566	10.870	112.992	10.927	113.337	11.033	113.283	10.160
$C_{0.95,5}$=11.07													
Chi-Square Test		Unacceptable		Acceptable		Acceptable		Acceptable		Acceptable		Acceptable	

%8 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.492	0.492	3.082	3.082	7.242	7.242	6.982	6.982	8.582	8.582	8.304	8.304
15-30	18	19.117	0.065	14.479	0.856	18.909	0.044	15.980	0.255	16.709	0.100	17.134	0.044

30-45	36	39.591	0.326	32.886	0.295	29.962	1.217	28.181	2.169	25.552	4.272	25.731	4.098
45-60	26	28.781	0.269	36.216	2.882	28.827	0.277	31.744	1.039	30.895	0.776	30.252	0.598
60-75	26	14.132	9.967	19.344	2.290	16.839	4.984	19.114	2.481	21.522	0.932	21.817	0.802
75-90	6	5.955	0.000	5.000	0.200	5.969	0.000	5.182	0.129	4.942	0.226	5.009	0.196
90-105	0	2.372	2.372	0.623	0.623	1.282	1.282	0.532	0.532	0.174	0.174	0.122	0.122
> 105	0	0.933	0.933	0.037	0.037	0.112	0.112	0.014	0.014	0.000	0.000	0.000	0.000
total	112	111.374	14.424	111.667	10.265	109.142	15.158	107.729	13.603	108.376	15.062	108.369	14.163
$C_{0.95,5}$=11.07													
Chi-Square Test		Unacceptable		Acceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	
%10 Right Truncated													
Soil shear strength	observed frequency	Lognormal Distribution		Normal Distribution		Maximum Entropy Distribution							
						K=2		K=3		K=4		K=5	
kPa		*ni*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*	*ci*	*ei*
≤ 15	0	0.462	0.462	2.843	2.843	8.476	8.476	7.582	7.582	10.237	10.237	10.188	10.188
15-30	18	19.298	0.087	14.607	0.788	20.178	0.235	16.195	0.201	16.399	0.156	16.504	0.136
30-45	36	39.813	0.365	33.979	0.120	29.530	1.418	28.084	2.231	23.929	6.089	23.928	6.090
45-60	26	27.932	0.134	35.916	2.738	26.577	0.013	30.799	0.748	30.969	0.797	30.843	0.760
60-75	26	13.078	12.769	17.256	4.431	14.708	8.669	16.918	4.875	20.141	1.705	20.239	1.640
75-90	3	5.236	0.955	3.756	0.152	5.002	0.801	3.726	0.142	2.703	0.033	2.684	0.037
90-105	0	1.981	1.981	0.368	0.368	1.044	1.044	0.262	0.262	0.021	0.021	0.018	0.018
> 105	0	0.741	0.741	0.016	0.016	0.055	0.000	0.003	0.003	0.000	0.000	0.000	0.000
total	109	108.541	17.494	108.741	11.456	105.569	20.656	103.570	16.045	104.399	19.038	104.403	18.869
$C_{0.95,5}$=11.07													
Chi-Square Test		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable		Unacceptable	

Appendix 2

Chapter 3:

Table 3.A. Example 1: Iterative calculations for random variable Cc modeled by Normal Distribution.

Normal Distribution						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0	1.0	0.1	0.2710	1.2882
	Cc	0.4082	0.4082	0.1151	0.7641	
	ev	1.1900	1.1900	0.1785	-0.2209	
	H	168.0	168.0	8.4	0.1355	
	P0	3.7200	3.7200	0.1860	-0.1273	
	Δp	0.5	0.5	0.1	0.5093	
2	N	1.0	1.0	0.1	0.3093	1.1019
	Cc	0.5173	0.4082	0.1151	0.7111	
	ev	1.1411	1.1900	0.1785	-0.2665	
	H	169.4	168.0	8.4	0.1585	
	P0	3.6906	3.7200	0.1860	-0.1502	
	Δp	0.6	0.5	0.1	0.5291	
3	N	1.0	1.0	0.1	0.3025	1.1012
	Cc	0.4984	0.4082	0.1151	0.7222	
	ev	1.1376	1.1900	0.1785	-0.2612	
	H	169.5	168.0	8.4	0.1550	
	P0	3.6892	3.7200	0.1860	-0.1471	
	Δp	0.6	0.5	0.1	0.5225	
4	N	1.0	1.0	0.1	0.3030	1.1012
	Cc	0.4997	0.4082	0.1151	0.7210	
	ev	1.1387	1.1900	0.1785	-0.2613	
	H	169.4	168.0	8.4	0.1552	
	P0	3.6899	3.7200	0.1860	-0.1472	
	Δp	0.6	0.5	0.1	0.5238	
5	N	1.0	1.0	0.1	0.3029	1.1012
	Cc	0.4995	0.4082	0.1151	0.7211	
	ev	1.1386	1.1900	0.1785	-0.2613	
	H	169.4	168.0	8.4	0.1552	
	P0	3.6898	3.7200	0.1860	-0.1472	
	Δp	0.6	0.5	0.1	0.5236	
	pF =	0.1354				

Untruncated Maximum Entropy						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0	1.0	0.1	0.2737	1.2882
	Cc	0.4029	0.4029	0.1132	0.7587	
	ev	1.1900	1.1900	0.1785	-0.2231	
	H	168.0	168.0	8.4	0.1368	
	P0	3.7200	3.7200	0.1860	-0.1286	
	Δp	0.5	0.5000	0.1	0.5142	
2	N	1.0353	1.0000	0.1	0.2996	1.1294
	Cc	0.5135	0.3987	0.1214	0.7333	
	ev	1.1387	1.1900	0.1785	-0.2589	
	H	169.4807	168.0000	8.4	0.1537	
	P0	3.6892	3.7200	0.1860	-0.1457	
	Δp	0.5662	0.5000	0.1	0.5105	
3	N	1.0338	1.0000	0.1	0.2957	1.1276
	Cc	0.4992	0.3997	0.1203	0.7364	
	ev	1.1378	1.1900	0.1785	-0.2553	
	H	169.4585	168.0000	8.4000	0.1515	
	P0	3.6894	3.7200	0.1860	-0.1438	
	Δp	0.5577	0.5000	0.1	0.5114	
4	N	1.0333	1.0000	0.1	0.2959	1.1276
	Cc	0.4996	0.3997	0.1203	0.7362	
	ev	1.1386	1.1900	0.1785	-0.2552	
	H	169.4354	168.0000	8.4000	0.1516	
	P0	3.6898	3.7200	0.1860	-0.1438	
	Δp	0.5577	0.5000	0.1	0.5115	
5	N	1.0334	1.0000	0.1	0.2959	1.1276
	Cc	0.4995	0.3997	0.1203	0.7362	
	ev	1.1386	1.1900	0.1785	-0.2552	
	H	169.4359	168.0000	8.4000	0.1516	
	P0	3.6898	3.7200	0.1860	-0.1438	
	Δp	0.5577	0.5000	0.1	0.5115	
	p_F =	0.1313				

Example 1. Iterative calculations for a random variable Cc modeled by Left Truncated Maximum Entropy Distribution.

%1 Left Truncation						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.2807	1.3061
	Cc	0.4051	0.4051	0.1082	0.7441	
	ev	1.1900	1.1900	0.1785	-0.2288	
	H	168.0000	168.0000	8.4000	0.1403	
	P0	3.7200	3.7200	0.1860	-0.1319	
	Δp	0.5000	0.5000	0.1000	0.5274	
2	N	1.0367	1.0000	0.1000	0.3040	1.1390
	Cc	0.5102	0.4005	0.1174	0.7250	
	ev	1.1367	1.1900	0.1785	-0.2633	
	H	169.5397	168.0000	8.4000	0.1561	
	P0	3.6880	3.7200	0.1860	-0.1481	
	Δp	0.5689	0.5000	0.1000	0.5161	
3	N	1.0346	1.0000	0.1000	0.3006	1.1370
	Cc	0.4975	0.4016	0.1162	0.7264	
	ev	1.1365	1.1900	0.1785	-0.2598	
	H	169.4938	168.0000	8.4000	0.1541	
	P0	3.6886	3.7200	0.1860	-0.1463	
	Δp	0.5588	0.5000	0.1000	0.5191	
4	N	1.0342	1.0000	0.1000	0.3007	1.1370
	Cc	0.4975	0.4016	0.1162	0.7264	
	ev	1.1373	1.1900	0.1785	-0.2598	
	H	169.4719	168.0000	8.4000	0.1542	
	P0	3.6891	3.7200	0.1860	-0.1463	
	Δp	0.5590	0.5000	0.1000	0.5189	
5	N	1.0342	1.0000	0.1000	0.3007	1.1370
	Cc	0.4975	0.4016	0.1162	0.7264	
	ev	1.1373	1.1900	0.1785	-0.2598	
	H	169.4723	168.0000	8.4000	0.1542	
	P0	3.6891	3.7200	0.1860	-0.1463	
	Δp	0.5590	0.5000	0.1000	0.5189	
	p_F	0.1303				

%5 Left Truncation						
Iteration	Random variable	Assume design point x*	$\mu^N_{X_i'}$	$\sigma^N_{X_i'}$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.2920	1.2957
	Cc	0.4139	0.4139	0.1005	0.7189	
	ev	1.1900	1.1900	0.1785	-0.2380	
	H	168.0000	168.0000	8.4000	0.1460	
	P0	3.7200	3.7200	0.1860	-0.1372	
	Δp	0.5000	0.5000	0.1000	0.5488	
2	N	1.0378	1.0000	0.1000	0.3150	1.1210
	Cc	0.5075	0.4103	0.1089	0.7014	
	ev	1.1349	1.1900	0.1785	-0.2734	
	H	169.5892	168.0000	8.4000	0.1619	
	P0	3.6869	3.7200	0.1860	-0.1536	
	Δp	0.5711	0.5000	0.1000	0.5332	
3	N	1.0353	1.0000	0.1000	0.3117	1.1188
	Cc	0.4959	0.4111	0.1079	0.7018	
	ev	1.1353	1.1900	0.1785	-0.2697	
	H	169.5249	168.0000	8.4000	0.1599	
	P0	3.6880	3.7200	0.1860	-0.1518	
	Δp	0.5598	0.5000	0.1000	0.5376	
4	N	1.0349	1.0000	0.1000	0.3118	1.1188
	Cc	0.4958	0.4111	0.1078	0.7020	
	ev	1.1361	1.1900	0.1785	-0.2697	
	H	169.5026	168.0000	8.4000	0.1599	
	P0	3.6884	3.7200	0.1860	-0.1518	
	Δp	0.5601	0.5000	0.1000	0.5373	
5	N	1.0349	1.0000	0.1000	0.3118	1.1188
	Cc	0.4958	0.4111	0.1078	0.7020	
	ev	1.1361	1.1900	0.1785	-0.2697	
	H	169.5026	168.0000	8.4000	0.1599	
	P0	3.6884	3.7200	0.1860	-0.1518	
	Δp	0.5601	0.5000	0.1000	0.5373	
	p_F	0.1362				

%8 Left Truncation						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.2970	1.2772
	Cc	0.4195	0.4195	0.0972	0.7072	
	ev	1.1900	1.1900	0.1785	-0.2421	
	H	168.0000	168.0000	8.4000	0.1485	
	P0	3.7200	3.7200	0.1860	-0.1395	
	Δp	0.5000	0.5000	0.1000	0.5581	
2	N	1.0379	1.0000	0.1000	0.3201	1.0997
	Cc	0.5073	0.4163	0.1053	0.6896	
	ev	1.1348	1.1900	0.1785	-0.2778	
	H	169.5934	168.0000	8.4000	0.1646	
	P0	3.6869	3.7200	0.1860	-0.1561	
	Δp	0.5713	0.5000	0.1000	0.5416	
3	N	1.0352	1.0000	0.1000	0.3168	1.0975
	Cc	0.4962	0.4171	0.1043	0.6896	
	ev	1.1355	1.1900	0.1785	-0.2741	
	H	169.5202	168.0000	8.4000	0.1625	
	P0	3.6881	3.7200	0.1860	-0.1542	
	Δp	0.5596	0.5000	0.1000	0.5466	
4	N	1.0348	1.0000	0.1000	0.3170	1.0975
	Cc	0.4961	0.4171	0.1043	0.6898	
	ev	1.1363	1.1900	0.1785	-0.2741	
	H	169.4980	168.0000	8.4000	0.1625	
	P0	3.6885	3.7200	0.1860	-0.1542	
	Δp	0.5600	0.5000	0.1000	0.5462	
5	N	1.0348	1.0000	0.1000	0.3170	1.0975
	Cc	0.4961	0.4171	0.1043	0.6898	
	ev	1.1363	1.1900	0.1785	-0.2741	
	H	169.4985	168.0000	8.4000	0.1625	
	P0	3.6885	3.7200	0.1860	-0.1542	
	Δp	0.5599	0.5000	0.1000	0.5463	
	p_F	0.1395				

%10 Left Truncation						
Iteration	Random variable	Assume design point x*	$\mu^N_{X'_i}$	$\sigma^N_{X'_i}$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.3002	1.2640
	Cc	0.4232	0.4232	0.0951	0.6995	
	ev	1.1900	1.1900	0.1785	-0.2447	
	H	168.0000	168.0000	8.4000	0.1501	
	P0	3.7200	3.7200	0.1860	-0.1410	
	Δp	0.5000	0.5000	0.1000	0.5641	
2	N	1.0380	1.0000	0.1000	0.3234	1.0847
	Cc	0.5073	0.4203	0.1030	0.6817	
	ev	1.1348	1.1900	0.1785	-0.2807	
	H	169.5939	168.0000	8.4000	0.1663	
	P0	3.6868	3.7200	0.1860	-0.1577	
	Δp	0.5713	0.5000	0.1000	0.5472	
3	N	1.0351	1.0000	0.1000	0.3201	1.0824
	Cc	0.4965	0.4210	0.1021	0.6815	
	ev	1.1357	1.1900	0.1785	-0.2769	
	H	169.5148	168.0000	8.4000	0.1642	
	P0	3.6882	3.7200	0.1860	-0.1558	
	Δp	0.5594	0.5000	0.1000	0.5524	
4	N	1.0346	1.0000	0.1000	0.3203	1.0824
	Cc	0.4964	0.4210	0.1021	0.6817	
	ev	1.1365	1.1900	0.1785	-0.2769	
	H	169.4928	168.0000	8.4000	0.1642	
	P0	3.6886	3.7200	0.1860	-0.1558	
	Δp	0.5598	0.5000	0.1000	0.5520	
5	N	1.0347	1.0000	0.1000	0.3203	1.0824
	Cc	0.4964	0.4210	0.1021	0.6817	
	ev	1.1365	1.1900	0.1785	-0.2769	
	H	169.4932	168.0000	8.4000	0.1642	
	P0	3.6886	3.7200	0.1860	-0.1558	
	Δp	0.5598	0.5000	0.1000	0.5521	
	p_F	0.1432				

%1 Right Truncation						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.2785	1.3129
	Cc	0.4026	0.4026	0.1097	0.7487	
	ev	1.1900	1.1900	0.1785	-0.2270	
	H	168.0000	168.0000	8.4000	0.1393	
	P0	3.7200	3.7200	0.1860	-0.1308	
	Δp	0.5000	0.5000	0.1000	0.5233	
2	N	1.0366	1.0000	0.1000	0.3115	1.1606
	Cc	0.5105	0.4015	0.1120	0.7085	
	ev	1.1368	1.1900	0.1785	-0.2697	
	H	169.5357	168.0000	8.4000	0.1600	
	P0	3.6881	3.7200	0.1860	-0.1517	
	Δp	0.5687	0.5000	0.1000	0.5289	
3	N	1.0361	1.0000	0.1000	0.3055	1.1592
	Cc	0.4936	0.4018	0.1117	0.7165	
	ev	1.1341	1.1900	0.1785	-0.2648	
	H	169.5595	168.0000	8.4000	0.1568	
	P0	3.6873	3.7200	0.1860	-0.1489	
	Δp	0.5614	0.5000	0.1000	0.5258	
4	N	1.0354	1.0000	0.1000	0.3059	1.1592
	Cc	0.4945	0.4017	0.1117	0.7157	
	ev	1.1352	1.1900	0.1785	-0.2648	
	H	169.5272	168.0000	8.4000	0.1570	
	P0	3.6879	3.7200	0.1860	-0.1490	
	Δp	0.5610	0.5000	0.1000	0.5266	
5	N	1.0355	1.0000	0.1000	0.3059	1.1592
	Cc	0.4944	0.4018	0.1117	0.7158	
	ev	1.1352	1.1900	0.1785	-0.2648	
	H	169.5285	168.0000	8.4000	0.1570	
	P0	3.6879	3.7200	0.1860	-0.1490	
	Δp	0.5610	0.5000	0.1000	0.5265	
	p_F	0.1236				

%5 Right Truncation						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.2946	1.4056
	Cc	0.4003	0.4003	0.0988	0.7129	
	ev	1.1900	1.1900	0.1785	-0.2401	
	H	168.0000	168.0000	8.4000	0.1473	
	P0	3.7200	3.7200	0.1860	-0.1384	
	Δp	0.5000	0.5000	0.1000	0.5536	
2	N	1.0414	1.0000	0.1000	0.3371	1.2556
	Cc	0.4993	0.4037	0.0925	0.6502	
	ev	1.1298	1.1900	0.1785	-0.2942	
	H	169.7393	168.0000	8.4000	0.1737	
	P0	3.6838	3.7200	0.1860	-0.1649	
	Δp	0.5778	0.5000	0.1000	0.5654	
3	N	1.0423	1.0000	0.1000	0.3275	1.2555
	Cc	0.4791	0.4025	0.0938	0.6680	
	ev	1.1241	1.1900	0.1785	-0.2869	
	H	169.8324	168.0000	8.4000	0.1688	
	P0	3.6815	3.7200	0.1860	-0.1606	
	Δp	0.5710	0.5000	0.1000	0.5567	
4	N	1.0411	1.0000	0.1000	0.3284	1.2556
	Cc	0.4811	0.4026	0.0936	0.6655	
	ev	1.1257	1.1900	0.1785	-0.2871	
	H	169.7806	168.0000	8.4000	0.1692	
	P0	3.6825	3.7200	0.1860	-0.1609	
	Δp	0.5699	0.5000	0.1000	0.5588	
5	N	1.0412	1.0000	0.1000	0.3283	1.2556
	Cc	0.4808	0.4026	0.0937	0.6659	
	ev	1.1256	1.1900	0.1785	-0.2871	
	H	169.7842	168.0000	8.4000	0.1691	
	P0	3.6824	3.7200	0.1860	-0.1608	
	Δp	0.5702	0.5000	0.1000	0.5584	
6	N	1.0412	1.0000	0.1000	0.3283	1.2556
	Cc	0.4809	0.4026	0.0937	0.6658	
	ev	1.1257	1.1900	0.1785	-0.2871	
	H	169.7839	168.0000	8.4000	0.1691	
	P0	3.6824	3.7200	0.1860	-0.1608	
	Δp	0.5701	0.5000	0.1000	0.5585	
	p_F	0.100				

%8 Right Truncation						
Iteration	Random variable	Assume design point x*	$\mu^N_{X'_i}$	$\sigma^N_{X'_i}$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.4172	2.0216
	Cc	0.3973	0.3973	0.0114	0.1169	
	ev	1.1900	1.1900	0.1785	-0.3401	
	H	168.0000	168.0000	8.4000	0.2086	
	P0	3.7200	3.7200	0.1860	-0.1960	
	Δp	0.5000	0.5000	0.1000	0.7840	
2	N	1.0843	1.0000	0.1000	0.4451	1.8577
	Cc	0.4000	0.3907	0.0211	0.2549	
	ev	1.0673	1.1900	0.1785	-0.4167	
	H	171.5426	168.0000	8.4000	0.2363	
	P0	3.6463	3.7200	0.1860	-0.2268	
	Δp	0.6585	0.5000	0.1000	0.6753	
3	N	1.0827	1.0000	0.1000	0.4346	1.8597
	Cc	0.4007	0.3910	0.0204	0.2390	
	ev	1.0518	1.1900	0.1785	-0.4093	
	H	171.6876	168.0000	8.4000	0.2302	
	P0	3.6416	3.7200	0.1860	-0.2223	
	Δp	0.6254	0.5000	0.1000	0.6957	
4	N	1.0808	1.0000	0.1000	0.4358	1.8598
	Cc	0.4001	0.3907	0.0210	0.2473	
	ev	1.0541	1.1900	0.1785	-0.4094	
	H	171.5963	168.0000	8.4000	0.2306	
	P0	3.6431	3.7200	0.1860	-0.2223	
	Δp	0.6294	0.5000	0.1000	0.6919	
5	N	1.0811	1.0000	0.1000	0.4360	1.8599
	Cc	0.4004	0.3909	0.0207	0.2432	
	ev	1.0541	1.1900	0.1785	-0.4096	
	H	171.6026	168.0000	8.4000	0.2307	
	P0	3.6431	3.7200	0.1860	-0.2225	
	Δp	0.6287	0.5000	0.1000	0.6931	
6	N	1.0811	1.0000	0.1000	0.4358	1.8599
	Cc	0.4002	0.3908	0.0208	0.2453	
	ev	1.0540	1.1900	0.1785	-0.4094	
	H	171.6043	168.0000	8.4000	0.2306	
	P0	3.6430	3.7200	0.1860	-0.2224	
	Δp	0.6289	0.5000	0.1000	0.6926	
7	N	1.0811	1.0000	0.1000	0.4359	1.8599
	Cc	0.4003	0.3908	0.0208	0.2443	
	ev	1.0541	1.1900	0.1785	-0.4095	

	H	171.6030	168.0000	8.4000	0.2307	
	P0	3.6431	3.7200	0.1860	-0.2224	
	Δp	0.6288	0.5000	0.1000	0.6928	
8	N	1.0811	1.0000	0.1000	0.4359	1.8599
	Cc	0.4003	0.3908	0.0208	0.2448	
	ev	1.0541	1.1900	0.1785	-0.4095	
	H	171.6037	168.0000	8.4000	0.2306	
	P0	3.6431	3.7200	0.1860	-0.2224	
	Δp	0.6288	0.5000	0.1000	0.6927	
9	N	1.0811	1.0000	0.1000	0.4359	1.8599
	Cc	0.4003	0.3908	0.0208	0.2445	
	ev	1.0541	1.1900	0.1785	-0.4095	
	H	171.6033	168.0000	8.4000	0.2307	
	P0	3.6431	3.7200	0.1860	-0.2224	
	Δp	0.6288	0.5000	0.1000	0.6927	
	p_F	0.0895				

%10 Right Truncation						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	N	1.0000	1.0000	0.1000	0.3118	1.5572
	Cc	0.3912	0.3912	0.0877	0.6702	
	ev	1.1900	1.1900	0.1785	-0.2542	
	H	168.0000	168.0000	8.4000	0.1559	
	P0	3.7200	3.7200	0.1860	-0.1465	
	Δp	0.5000	0.5000	0.1000	0.5859	
2	N	1.0486	1.0000	0.1000	0.3518	1.3921
	Cc	0.4827	0.3954	0.0807	0.6168	
	ev	1.1194	1.1900	0.1785	-0.3107	
	H	170.0394	168.0000	8.4000	0.1822	
	P0	3.6776	3.7200	0.1860	-0.1733	
	Δp	0.5912	0.5000	0.1000	0.5797	
3	N	1.0490	1.0000	0.1000	0.3418	1.3916
	Cc	0.4647	0.3940	0.0822	0.6339	
	ev	1.1128	1.1900	0.1785	-0.3029	
	H	170.1311	168.0000	8.4000	0.1770	
	P0	3.6751	3.7200	0.1860	-0.1688	
	Δp	0.5807	0.5000	0.1000	0.5742	
4	N	1.0476	1.0000	0.1000	0.3428	1.3916
	Cc	0.4665	0.3941	0.0820	0.6314	
	ev	1.1148	1.1900	0.1785	-0.3031	
	H	170.0692	168.0000	8.4000	0.1774	
	P0	3.6763	3.7200	0.1860	-0.1690	
	Δp	0.5799	0.5000	0.1000	0.5761	
5	N	1.0477	1.0000	0.1000	0.3427	1.3916
	Cc	0.4662	0.3941	0.0820	0.6319	
	ev	1.1147	1.1900	0.1785	-0.3031	
	H	170.0736	168.0000	8.4000	0.1773	
	P0	3.6763	3.7200	0.1860	-0.1690	
	Δp	0.5802	0.5000	0.1000	0.5757	
6	N	1.0477	1.0000	0.1000	0.3427	1.3916
	Cc	0.4663	0.3941	0.0820	0.6318	
	ev	1.1147	1.1900	0.1785	-0.3031	
	H	170.0731	168.0000	8.4000	0.1773	
	P0	3.6763	3.7200	0.1860	-0.1690	
	Δp	0.5801	0.5000	0.1000	0.5758	
	p_F	0.083				

Table 3.B. Coefficient of the Maximum Entropy Distribution using in the example 1.

K_1								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
-0.778404	-0.749130	-0.731139	-0.718913	-0.788418	-0.769821	-0.706438	-0.693523	-0.702181
2.082256	2.012027	1.969315	1.940976	2.107008	2.032291	1.762191	1.686817	1.700383

K_2								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
5.582489	6.713147	7.356204	7.803763	5.088678	5.494886	6.649174	6.957922	7.149358
-33.374380	-38.349097	-41.010648	-42.837545	-31.050088	-33.530745	-40.881391	-43.230250	-44.736967
40.581050	45.664129	48.209801	49.937326	38.041830	41.428649	51.894067	55.652836	58.124032

K_3								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
7.864423	9.820368	10.886885	11.634396	6.689728	6.090382	4.209206	4.001518	3.887562
-51.782615	-62.045875	-67.283424	-70.906614	-44.480280	-38.665173	-18.295484	-15.396300	-13.705394
86.121298	101.907571	109.416260	114.556015	72.231069	54.892992	-11.816105	-24.348961	-32.110035
-35.009327	-41.917344	-44.973672	-47.046215	-26.866543	-10.929935	55.866950	71.591867	81.767812

K_4								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
10.906107	21.501835	26.036691	29.507469	6.518713	5.978947	9.716999	9.918530	9.905525
-82.139673	-170.026363	-203.082368	-228.207786	-42.698572	-37.287927	-96.834994	-102.366188	-103.922440
190.595480	452.596714	539.095003	604.601826	65.973731	49.170908	363.450945	402.955671	419.030541
-183.165429	-517.732144	-614.848747	-688.309235	-17.966602	-1.231919	-677.417655	-786.372913	-839.442957
73.473587	228.189678	267.730870	297.677792	-4.357379	-5.742151	503.459044	605.261110	660.758612

K_5								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
11.889351	26.352911	31.838948	36.025604	4.683347	5.253176	3.348344	3.245043	3.265396
-95.217888	-227.313517	-270.070626	-302.483511	-13.820603	-24.407479	38.261035	43.542686	43.631371
255.014041	709.707908	834.669693	929.109158	-92.376986	-29.747835	-625.227904	-694.904784	-708.418510
-331.335333	-1067.475501	-1239.132437	-1368.841036	373.071294	216.193343	2587.755228	2938.343936	3044.016446
233.739071	789.849919	900.330062	984.000077	-447.628934	-281.326724	-4487.391054	-5242.436050	-5529.218661
-65.596232	-220.046116	-246.655089	-266.862501	186.853079	130.445970	2870.463673	3454.927835	3713.390393

Table 3.C. Truncated moment from samples used in example 1.

K_1							
Left				Right			
1%	5%	8%	10%	1%	5%	8%	10%
0.346385	0.37009	0.370777	0.372582	0.337383	0.380063	0.38416	0.378437
K_2							
Left				Right			
1%	5%	8%	10%	1%	5%	8%	10%
0.346385	0.37009	0.370777	0.367777	0.33739	0.380071	0.384168	0.378445
0.004951	0.006138	0.007355	0.008227	0.006043	0.010412	0.013399	0.014844
K_3							
Left				Right			
1%	5%	8%	10%	1%	5%	8%	10%
0.346385	0.37009	0.370777	0.372582	0.337383	0.380063	0.38416	0.378437
0.004951	0.006138	0.007355	0.008103	0.006044	0.010412	0.0134	0.014844
8.14E-06	-0.00035	-0.00057	-0.0007	-0.00011	-0.00118	-0.00181	-0.00207
K_4							
Left				Right			
1%	5%	8%	10%	1%	5%	8%	10%
0.346385	0.37009	0.370777	0.372582	0.337383	0.380063	0.38416	0.378437
0.004951	0.006138	0.007355	0.008103	0.006044	0.010412	0.0134	0.014844
8.14E-06	-0.00035	-0.00057	-0.0007	-0.00011	-0.00118	-0.00181	-0.00207
9.77E-05	0.000184	0.000245	0.000276	0.000139	0.000482	0.000726	0.000818
K_5							
Left				Right			
1%	5%	8%	10%	1%	5%	8%	10%
0.346385	0.37009	0.370777	0.372582	0.337383	0.380063	0.38416	0.378437
0.004951	0.006138	0.007355	0.008103	0.006044	0.010412	0.0134	0.014844
8.14E-06	-0.00035	-0.00057	-0.0007	-0.00011	-0.00118	-0.00181	-0.00207
9.77E-05	0.000184	0.000245	0.000276	0.000139	0.000482	0.000726	0.000818
-2.47E-06	-2.4E-05	-3.54E-05	-4.02E-05	-1.10E-05	-0.00012	-0.00018	-0.0002

Table 3.D. Example 2. Iterative calculations for a random variable X_2 modeled by Normal Distribution.

Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	X1	0.6400	0.6400	0.0960	-0.8966	6.8792
	X2	3420.9245	3420.9245	197.6078	-0.3860	
	X3	1.0000	1.0000	0.2000	-0.0393	
	X4	5.1600	5.1600	0.9300	0.2136	
2	X1	0.0479	0.6400	0.0960	-0.9452	7.6993
	X2	2896.1699	3420.9245	197.6078	-0.0378	
	X3	0.9459	1.0000	0.2000	-0.1601	
	X4	6.5264	5.1600	0.9300	0.2821	
3	X1	-0.0586	0.6400	0.0960	-0.9649	7.4877
	X2	3363.3777	3420.9245	197.6078	0.0390	
	X3	0.7535	1.0000	0.2000	-0.1308	
	X4	7.1799	5.1600	0.9300	0.2242	
4	X1	-0.0536	0.6400	0.0960	-0.9687	7.4824
	X2	3478.6409	3420.9245	197.6078	0.0345	
	X3	0.8041	1.0000	0.2000	-0.1509	
	X4	6.7213	5.1600	0.9300	0.1942	
5	X1	-0.0558	0.6400	0.0960	-0.9716	7.4760
	X2	3471.9097	3420.9245	197.6078	0.0358	
	X3	0.7742	1.0000	0.2000	-0.1615	
	X4	6.5111	5.1600	0.9300	0.1693	
6	X1	-0.0573	0.6400	0.0960	-0.9732	7.4726
	X2	3473.7919	3420.9245	197.6078	0.0366	
	X3	0.7585	1.0000	0.2000	-0.1691	
	X4	6.3371	5.1600	0.9300	0.1515	
7	X1	-0.0581	0.6400	0.0960	-0.9742	7.4709
	X2	3474.9511	3420.9245	197.6078	0.0370	
	X3	0.7473	1.0000	0.2000	-0.1739	
	X4	6.2127	5.1600	0.9300	0.1392	
8	X1	-0.0587	0.6400	0.0960	-0.9747	7.4702
	X2	3475.5725	3420.9245	197.6078	0.0373	
	X3	0.7401	1.0000	0.2000	-0.1771	
	X4	6.1270	5.1600	0.9300	0.1310	
9	X1	-0.0590	0.6400	0.0960	-0.9751	7.4699
	X2	3475.9589	3420.9245	197.6078	0.0374	
	X3	0.7354	1.0000	0.2000	-0.1791	
	X4	6.0703	5.1600	0.9300	0.1257	

10	X1	-0.0592	0.6400	0.0960	-0.9752	7.4698
	X2	3476.1927	3420.9245	197.6078	0.0375	
	X3	0.7325	1.0000	0.2000	-0.1803	
	X4	6.0335	5.1600	0.9300	0.1224	
11	X1	-0.0594	0.6400	0.0960	-0.9754	7.4697
	X2	3476.3385	3420.9245	197.6078	0.0376	
	X3	0.7306	1.0000	0.2000	-0.1811	
	X4	6.0100	5.1600	0.9300	0.1202	
12	X1	-0.0594	0.6400	0.0960	-0.9754	7.4697
	X2	3476.4292	3420.9245	197.6078	0.0376	
	X3	0.7295	1.0000	0.2000	-0.1815	
	X4	5.9952	5.1600	0.9300	0.1189	
13	X1	-0.0595	0.6400	0.0960	-0.9755	7.4697
	X2	3476.4858	3420.9245	197.6078	0.0377	
	X3	0.7288	1.0000	0.2000	-0.1818	
	X4	5.9859	5.1600	0.9300	0.1181	
14	X1	-0.0595	0.6400	0.0960	-0.9755	7.4697
	X2	3476.5211	3420.9245	197.6078	0.0377	
	X3	0.7283	1.0000	0.2000	-0.1820	
	X4	5.9801	5.1600	0.9300	0.1175	
15	X1	-0.0595	0.6400	0.0960	-0.9755	7.4697
	X2	3476.5430	3420.9245	197.6078	0.0377	
	X3	0.7281	1.0000	0.2000	-0.1822	
	X4	5.9764	5.1600	0.9300	0.1172	
	Pf	4.0186E-14				

Iterative calculations for a random variable X_2 modeled by Untruncated Maximum Entropy Distribution.

Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	**New β**
1	X1	0.6400	0.6400	0.0960	-0.8980	6.8808
	X2	3416.0165	3416.0165	195.4553	-0.3824	
	X3	1.0000	1.0000	0.2000	-0.0394	
	X4	5.1600	5.1600	0.9300	0.2139	
2	X1	0.0468	0.6400	0.0960	-0.9449	7.7232
	X2	2901.6868	3594.9116	263.5361	-0.0492	
	X3	0.9458	1.0000	0.2000	-0.1599	
	X4	6.5289	5.1600	0.9300	0.2814	
3	X1	-0.0606	0.6400	0.0960	-0.9677	7.4905
	X2	3494.8288	3415.4082	198.2983	0.0388	
	X3	0.7529	1.0000	0.2000	-0.1262	
	X4	7.1811	5.1600	0.9300	0.2150	
4	X1	-0.0558	0.6400	0.0960	-0.9687	7.4818
	X2	3473.1110	3415.6990	197.4571	0.0359	
	X3	0.8110	1.0000	0.2000	-0.1551	
	X4	6.6576	5.1600	0.9300	0.1904	
5	X1	-0.0558	0.6400	0.0960	-0.9720	7.4762
	X2	3468.7939	3415.7457	197.2902	0.0357	
	X3	0.7679	1.0000	0.2000	-0.1627	
	X4	6.4851	5.1600	0.9300	0.1659	
6	X1	-0.0576	0.6400	0.0960	-0.9733	7.4732
	X2	3468.4144	3415.7496	197.2755	0.0368	
	X3	0.7567	1.0000	0.2000	-0.1704	
	X4	6.3135	5.1600	0.9300	0.1494	
7	X1	-0.0583	0.6400	0.0960	-0.9742	7.4718
	X2	3469.9437	3415.7336	197.3346	0.0371	
	X3	0.7454	1.0000	0.2000	-0.1747	
	X4	6.1984	5.1600	0.9300	0.1379	
8	X1	-0.0588	0.6400	0.0960	-0.9747	7.4712
	X2	3470.4131	3415.7286	197.3528	0.0374	
	X3	0.7389	1.0000	0.2000	-0.1777	
	X4	6.1180	5.1600	0.9300	0.1303	
9	X1	-0.0591	0.6400	0.0960	-0.9750	7.4709
	X2	3470.8007	3415.7244	197.3677	0.0375	
	X3	0.7345	1.0000	0.2000	-0.1795	
	X4	6.0652	5.1600	0.9300	0.1254	

10	X1	-0.0593	0.6400	0.0960	-0.9752	7.4708
	X2	3471.0152	3415.7221	197.3760	0.0376	
	X3	0.7318	1.0000	0.2000	-0.1807	
	X4	6.0310	5.1600	0.9300	0.1222	
11	X1	-0.0594	0.6400	0.0960	-0.9753	7.4707
	X2	3471.1540	3415.7206	197.3814	0.0377	
	X3	0.7301	1.0000	0.2000	-0.1814	
	X4	6.0092	5.1600	0.9300	0.1202	
12	X1	-0.0595	0.6400	0.0960	-0.9754	7.4707
	X2	3471.2395	3415.7197	197.3847	0.0377	
	X3	0.7290	1.0000	0.2000	-0.1818	
	X4	5.9954	5.1600	0.9300	0.1190	
13	X1	-0.0595	0.6400	0.0960	-0.9754	7.4707
	X2	3471.2931	3415.7191	197.3868	0.0377	
	X3	0.7283	1.0000	0.2000	-0.1821	
	X4	5.9867	5.1600	0.9300	0.1182	
14	X1	-0.0596	0.6400	0.0960	-0.9754	7.4707
	X2	3471.3265	3415.7187	197.3881	0.0377	
	X3	0.7279	1.0000	0.2000	-0.1823	
	X4	5.9813	5.1600	0.9300	0.1177	
15	X1	-0.0596	0.6400	0.0960	-0.9755	7.4707
	X2	3471.3473	3415.7185	197.3889	0.0377	
	X3	0.7276	1.0000	0.2000	-0.1824	
	X4	5.9779	5.1600	0.9300	0.1174	
	Pf	3.99E-14				

Iterative calculations for a random variable X_2 modeled by Left Truncated Maximum Entropy Distribution.

%1 left						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	X1	0.6400	0.6400	0.0960	-0.9026	6.9259
	X2	3421.1033	3421.1033	188.5631	-0.3708	
	X3	1.0000	1.0000	0.2000	-0.0396	
	X4	5.1600	5.1600	0.9300	0.2150	
2	X1	0.0399	0.6400	0.0960	-0.9463	7.6972
	X2	2936.8073	3634.1595	276.5955	-0.0434	
	X3	0.9452	1.0000	0.2000	-0.1595	
	X4	6.5449	5.1600	0.9300	0.2779	
3	X1	-0.0592	0.6400	0.0960	-0.9688	7.4898
	X2	3541.7194	3419.7924	192.7224	0.0364	
	X3	0.7545	1.0000	0.2000	-0.1259	
	X4	7.1497	5.1600	0.9300	0.2102	
4	X1	-0.0566	0.6400	0.0960	-0.9690	7.4803
	X2	3472.3428	3420.8619	190.3475	0.0351	
	X3	0.8114	1.0000	0.2000	-0.1570	
	X4	6.6245	5.1600	0.9300	0.1876	
5	X1	-0.0558	0.6400	0.0960	-0.9723	7.4751
	X2	3470.8751	3420.8755	190.2963	0.0344	
	X3	0.7651	1.0000	0.2000	-0.1635	
	X4	6.4653	5.1600	0.9300	0.1636	
6	X1	-0.0577	0.6400	0.0960	-0.9735	7.4724
	X2	3469.8634	3420.8847	190.2611	0.0355	
	X3	0.7556	1.0000	0.2000	-0.1710	
	X4	6.2971	5.1600	0.9300	0.1478	
7	X1	-0.0583	0.6400	0.0960	-0.9744	7.4711
	X2	3471.3636	3420.8710	190.3134	0.0358	
	X3	0.7445	1.0000	0.2000	-0.1751	
	X4	6.1871	5.1600	0.9300	0.1368	
8	X1	-0.0588	0.6400	0.0960	-0.9748	7.4705
	X2	3471.7404	3420.8675	190.3265	0.0360	
	X3	0.7384	1.0000	0.2000	-0.1779	
	X4	6.1102	5.1600	0.9300	0.1295	
9	X1	-0.0591	0.6400	0.0960	-0.9751	7.4703
	X2	3472.0883	3420.8643	190.3386	0.0362	

	X3	0.7342	1.0000	0.2000	-0.1796	
	X4	6.0600	5.1600	0.9300	0.1249	
10	X1	-0.0593	0.6400	0.0960	-0.9753	7.4702
	X2	3472.2755	3420.8625	190.3452	0.0362	
	X3	0.7316	1.0000	0.2000	-0.1807	
	X4	6.0274	5.1600	0.9300	0.1219	
11	X1	-0.0594	0.6400	0.0960	-0.9754	7.4701
	X2	3472.3979	3420.8613	190.3494	0.0363	
	X3	0.7300	1.0000	0.2000	-0.1814	
	X4	6.0068	5.1600	0.9300	0.1200	
12	X1	-0.0595	0.6400	0.0960	-0.9754	7.4701
	X2	3472.4730	3420.8606	190.3520	0.0363	
	X3	0.7290	1.0000	0.2000	-0.1818	
	X4	5.9937	5.1600	0.9300	0.1188	
13	X1	-0.0595	0.6400	0.0960	-0.9755	7.4701
	X2	3472.5202	3420.8602	190.3537	0.0364	
	X3	0.7283	1.0000	0.2000	-0.1821	
	X4	5.9855	5.1600	0.9300	0.1181	
14	X1	-0.0596	0.6400	0.0960	-0.9755	7.4701
	X2	3472.5496	3420.8599	190.3547	0.0364	
	X3	0.7280	1.0000	0.2000	-0.1823	
	X4	5.9804	5.1600	0.9300	0.1176	
15	X1	-0.0596	0.6400	0.0960	-0.9755	7.4701
	X2	3472.5679	3420.8597	190.3553	0.0364	
	X3	0.7277	1.0000	0.2000	-0.1824	
	X4	5.9772	5.1600	0.9300	0.1173	
	Pf	4.01E-14				

%5 left						
Iteration	Random variable	Assume design point x*	$\mu^N_{X'_i}$	$\sigma^N_{X'_i}$	α_{X_i}	New β
1	X1	0.6400	0.6400	0.0960	-0.9189	7.1358
	X2	3463.6406	3463.6406	162.7565	-0.3258	
	X3	1.0000	1.0000	0.2000	-0.0403	
	X4	5.1600	5.1600	0.9300	0.2189	
2	X1	0.0106	0.6400	0.0960	-0.9515	7.5804
	X2	3085.2062	3577.1089	217.6691	-0.0086	
	X3	0.9425	1.0000	0.2000	-0.1576	
	X4	6.6126	5.1600	0.9300	0.2642	
3	X1	-0.0524	0.6400	0.0960	-0.9703	7.4794
	X2	3562.9267	3463.0634	166.2541	0.0276	
	X3	0.7611	1.0000	0.2000	-0.1304	
	X4	7.0227	5.1600	0.9300	0.2021	
4	X1	-0.0567	0.6400	0.0960	-0.9704	7.4720
	X2	3497.3927	3463.6893	164.6785	0.0302	
	X3	0.8050	1.0000	0.2000	-0.1585	
	X4	6.5661	5.1600	0.9300	0.1797	
5	X1	-0.0561	0.6400	0.0960	-0.9733	7.4674
	X2	3500.8001	3463.6706	164.7655	0.0297	
	X3	0.7631	1.0000	0.2000	-0.1645	
	X4	6.4090	5.1600	0.9300	0.1572	
6	X1	-0.0577	0.6400	0.0960	-0.9744	7.4650
	X2	3500.1581	3463.6743	164.7492	0.0305	
	X3	0.7544	1.0000	0.2000	-0.1712	
	X4	6.2514	5.1600	0.9300	0.1426	
7	X1	-0.0583	0.6400	0.0960	-0.9751	7.4639
	X2	3501.1367	3463.6687	164.7741	0.0307	
	X3	0.7444	1.0000	0.2000	-0.1749	
	X4	6.1503	5.1600	0.9300	0.1327	
8	X1	-0.0587	0.6400	0.0960	-0.9755	7.4635
	X2	3501.3843	3463.6672	164.7804	0.0308	
	X3	0.7389	1.0000	0.2000	-0.1773	
	X4	6.0813	5.1600	0.9300	0.1263	
9	X1	-0.0590	0.6400	0.0960	-0.9758	7.4633
	X2	3501.6048	3463.6659	164.7861	0.0309	
	X3	0.7353	1.0000	0.2000	-0.1788	
	X4	6.0369	5.1600	0.9300	0.1223	
10	X1	-0.0591	0.6400	0.0960	-0.9759	7.4632
	X2	3501.7252	3463.6652	164.7891	0.0310	

	X3	0.7331	1.0000	0.2000	-0.1798	
	X4	6.0085	5.1600	0.9300	0.1197	
11	X1	-0.0592	0.6400	0.0960	-0.9760	7.4632
	X2	3501.8027	3463.6648	164.7911	0.0310	
	X3	0.7317	1.0000	0.2000	-0.1803	
	X4	5.9908	5.1600	0.9300	0.1181	
12	X1	-0.0593	0.6400	0.0960	-0.9760	7.4632
	X2	3501.8500	3463.6645	164.7923	0.0311	
	X3	0.7308	1.0000	0.2000	-0.1807	
	X4	5.9797	5.1600	0.9300	0.1171	
13	X1	-0.0593	0.6400	0.0960	-0.9761	7.4632
	X2	3501.8794	3463.6643	164.7931	0.0311	
	X3	0.7303	1.0000	0.2000	-0.1809	
	X4	5.9728	5.1600	0.9300	0.1165	
14	X1	-0.0593	0.6400	0.0960	-0.9761	7.4632
	X2	3501.8976	3463.6642	164.7935	0.0311	
	X3	0.7300	1.0000	0.2000	-0.1811	
	X4	5.9685	5.1600	0.9300	0.1161	
15	X1	-0.0593	0.6400	0.0960	-0.9761	7.4632
	X2	3501.9088	3463.6642	164.7938	0.0311	
	X3	0.7298	1.0000	0.2000	-0.1811	
	X4	5.9659	5.1600	0.9300	0.1159	
	Pf	4.08E-14				

%8 left						
Iteration	Random variable	Assume design point x*	$\mu^N_{X'_i}$	$\sigma^N_{X'_i}$	α_{X_i}	New β
1	X1	0.6400	0.6400	0.0960	-0.9217	7.1769
	X2	3473.0929	3473.0929	157.9509	-0.3172	
	X3	1.0000	1.0000	0.2000	-0.0404	
	X4	5.1600	5.1600	0.9300	0.2196	
2	X1	0.0050	0.6400	0.0960	-0.9522	7.5632
	X2	3113.5096	3565.3650	203.7625	-0.0037	
	X3	0.9420	1.0000	0.2000	-0.1572	
	X4	6.6255	5.1600	0.9300	0.2617	
3	X1	-0.0514	0.6400	0.0960	-0.9704	7.4774
	X2	3559.5909	3472.7491	161.2237	0.0263	
	X3	0.7622	1.0000	0.2000	-0.1314	
	X4	7.0009	5.1600	0.9300	0.2011	
4	X1	-0.0566	0.6400	0.0960	-0.9707	7.4702
	X2	3504.4397	3473.2158	159.9390	0.0292	
	X3	0.8035	1.0000	0.2000	-0.1585	
	X4	6.5585	5.1600	0.9300	0.1784	
5	X1	-0.0561	0.6400	0.0960	-0.9735	7.4656
	X2	3508.0669	3473.1974	160.0278	0.0288	
	X3	0.7632	1.0000	0.2000	-0.1645	
	X4	6.3993	5.1600	0.9300	0.1561	
6	X1	-0.0577	0.6400	0.0960	-0.9746	7.4633
	X2	3507.5569	3473.2001	160.0153	0.0295	
	X3	0.7543	1.0000	0.2000	-0.1711	
	X4	6.2436	5.1600	0.9300	0.1417	
7	X1	-0.0583	0.6400	0.0960	-0.9753	7.4623
	X2	3508.4377	3473.1955	160.0368	0.0297	
	X3	0.7446	1.0000	0.2000	-0.1748	
	X4	6.1437	5.1600	0.9300	0.1320	
8	X1	-0.0587	0.6400	0.0960	-0.9757	7.4619
	X2	3508.6707	3473.1942	160.0425	0.0299	
	X3	0.7392	1.0000	0.2000	-0.1771	
	X4	6.0758	5.1600	0.9300	0.1257	
9	X1	-0.0589	0.6400	0.0960	-0.9759	7.4617
	X2	3508.8720	3473.1931	160.0474	0.0300	
	X3	0.7356	1.0000	0.2000	-0.1786	
	X4	6.0323	5.1600	0.9300	0.1217	
10	X1	-0.0591	0.6400	0.0960	-0.9760	7.4616
	X2	3508.9828	3473.1925	160.0501	0.0300	

	X3	0.7335	1.0000	0.2000	-0.1795	
	X4	6.0047	5.1600	0.9300	0.1192	
11	X1	-0.0592	0.6400	0.0960	-0.9761	7.4616
	X2	3509.0536	3473.1921	160.0519	0.0301	
	X3	0.7321	1.0000	0.2000	-0.1801	
	X4	5.9873	5.1600	0.9300	0.1177	
12	X1	-0.0592	0.6400	0.0960	-0.9762	7.4616
	X2	3509.0969	3473.1919	160.0529	0.0301	
	X3	0.7313	1.0000	0.2000	-0.1804	
	X4	5.9766	5.1600	0.9300	0.1167	
13	X1	-0.0593	0.6400	0.0960	-0.9762	7.4616
	X2	3509.1237	3473.1917	160.0536	0.0301	
	X3	0.7308	1.0000	0.2000	-0.1806	
	X4	5.9699	5.1600	0.9300	0.1161	
14	X1	-0.0593	0.6400	0.0960	-0.9762	7.4616
	X2	3509.1402	3473.1916	160.0540	0.0301	
	X3	0.7305	1.0000	0.2000	-0.1807	
	X4	5.9658	5.1600	0.9300	0.1158	
15	X1	-0.0593	0.6400	0.0960	-0.9762	7.4616
	X2	3509.1504	3473.1916	160.0542	0.0301	
	X3	0.7303	1.0000	0.2000	-0.1808	
	X4	5.9632	5.1600	0.9300	0.1155	
	Pf	4.12E-14				

%10 left						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	X1	0.6400	0.6400	0.0960	-0.9153	7.0843
	X2	3451.5292	3451.5292	168.5551	-0.3362	
	X3	1.0000	1.0000	0.2000	-0.0401	
	X4	5.1600	5.1600	0.9300	0.2181	
2	X1	0.0175	0.6400	0.0960	-0.9504	7.6041
	X2	3050.1193	3593.6152	235.2459	-0.0156	
	X3	0.9431	1.0000	0.2000	-0.1580	
	X4	6.5966	5.1600	0.9300	0.2674	
3	X1	-0.0538	0.6400	0.0960	-0.9701	7.4821
	X2	3565.7525	3450.6477	172.3112	0.0294	
	X3	0.7597	1.0000	0.2000	-0.1291	
	X4	7.0509	5.1600	0.9300	0.2035	
4	X1	-0.0568	0.6400	0.0960	-0.9700	7.4742
	X2	3488.4965	3451.4935	170.3679	0.0314	
	X3	0.8069	1.0000	0.2000	-0.1585	
	X4	6.5762	5.1600	0.9300	0.1815	
5	X1	-0.0560	0.6400	0.0960	-0.9731	7.4696
	X2	3491.4310	3451.4756	170.4472	0.0307	
	X3	0.7630	1.0000	0.2000	-0.1644	
	X4	6.4216	5.1600	0.9300	0.1586	
6	X1	-0.0578	0.6400	0.0960	-0.9741	7.4671
	X2	3490.6125	3451.4807	170.4251	0.0316	
	X3	0.7544	1.0000	0.2000	-0.1713	
	X4	6.2616	5.1600	0.9300	0.1439	
7	X1	-0.0583	0.6400	0.0960	-0.9749	7.4660
	X2	3491.7188	3451.4738	170.4550	0.0318	
	X3	0.7442	1.0000	0.2000	-0.1751	
	X4	6.1590	5.1600	0.9300	0.1337	
8	X1	-0.0588	0.6400	0.0960	-0.9753	7.4656
	X2	3491.9853	3451.4721	170.4621	0.0320	
	X3	0.7386	1.0000	0.2000	-0.1776	
	X4	6.0884	5.1600	0.9300	0.1272	
9	X1	-0.0590	0.6400	0.0960	-0.9756	7.4654
	X2	3492.2315	3451.4705	170.4688	0.0321	
	X3	0.7348	1.0000	0.2000	-0.1791	
	X4	6.0428	5.1600	0.9300	0.1230	
10	X1	-0.0592	0.6400	0.0960	-0.9757	7.4653
	X2	3492.3647	3451.4696	170.4724	0.0322	

	X3	0.7325	1.0000	0.2000	-0.1801	
	X4	6.0137	5.1600	0.9300	0.1203	
11	X1	-0.0593	0.6400	0.0960	-0.9758	7.4653
	X2	3492.4509	3451.4691	170.4747	0.0322	
	X3	0.7311	1.0000	0.2000	-0.1807	
	X4	5.9953	5.1600	0.9300	0.1187	
12	X1	-0.0593	0.6400	0.0960	-0.9759	7.4653
	X2	3492.5036	3451.4687	170.4761	0.0323	
	X3	0.7302	1.0000	0.2000	-0.1811	
	X4	5.9838	5.1600	0.9300	0.1176	
13	X1	-0.0594	0.6400	0.0960	-0.9759	7.4653
	X2	3492.5365	3451.4685	170.4770	0.0323	
	X3	0.7297	1.0000	0.2000	-0.1813	
	X4	5.9766	5.1600	0.9300	0.1170	
14	X1	-0.0594	0.6400	0.0960	-0.9759	7.4653
	X2	3492.5568	3451.4684	170.4776	0.0323	
	X3	0.7293	1.0000	0.2000	-0.1814	
	X4	5.9721	5.1600	0.9300	0.1166	
15	X1	-0.0594	0.6400	0.0960	-0.9759	7.4653
	X2	3492.5695	3451.4683	170.4779	0.0323	
	X3	0.7291	1.0000	0.2000	-0.1815	
	X4	5.9694	5.1600	0.9300	0.1163	
	Pf	4.16E-14				

%1 Right						
Iteration	Random variable	Assume design point x*	$\mu_{X_i'}^N$	$\sigma_{X_i'}^N$	α_{X_i}	New β
1	X1	0.0006	0.0006	0.096	-0.9027	6.9077
	X2	3.4113	3.4113	188.3485	-0.3705	
	X3	0.001	0.001	0.2	-0.0396	
	X4	0.0052	0.0052	0.93	0.2151	
2	X1	0	0.0006	0.096	-0.9461	7.6929
	X2	2.9293	3.5572	248.3591	-0.0406	
	X3	0.0009	0.001	0.2	-0.1596	
	X4	0.0065	0.0052	0.93	0.2787	
3	X1	-0.0001	0.0006	0.096	-0.9676	7.4911
	X2	3.4797	3.411	189.829	0.0362	
	X3	0.0008	0.001	0.2	-0.1278	
	X4	0.0072	0.0052	0.93	0.2146	
4	X1	-0.0001	0.0006	0.096	-0.9687	7.4829
	X2	3.4625	3.4111	189.3976	0.0346	
	X3	0.0008	0.001	0.2	-0.1557	
	X4	0.0067	0.0052	0.93	0.1902	
5	X1	-0.0001	0.0006	0.096	-0.9719	7.4774
	X2	3.4602	3.4111	189.3381	0.0344	
	X3	0.0008	0.001	0.2	-0.1632	
	X4	0.0065	0.0052	0.93	0.166	
6	X1	-0.0001	0.0006	0.096	-0.9732	7.4745
	X2	3.4599	3.4111	189.3307	0.0354	
	X3	0.0008	0.001	0.2	-0.1708	
	X4	0.0063	0.0052	0.93	0.1497	
7	X1	-0.0001	0.0006	0.096	-0.9741	7.4731
	X2	3.4613	3.4111	189.3659	0.0357	
	X3	0.0007	0.001	0.2	-0.1751	
	X4	0.0062	0.0052	0.93	0.1383	
8	X1	-0.0001	0.0006	0.096	-0.9746	7.4725
	X2	3.4617	3.4111	189.3767	0.036	
	X3	0.0007	0.001	0.2	-0.1781	
	X4	0.0061	0.0052	0.93	0.1307	
9	X1	-0.0001	0.0006	0.096	-0.9749	7.4722
	X2	3.462	3.4111	189.3856	0.0361	
	X3	0.0007	0.001	0.2	-0.1799	
	X4	0.0061	0.0052	0.93	0.1258	
10	X1	-0.0001	0.0006	0.096	-0.9751	7.4721
	X2	3.4622	3.4111	189.3906	0.0362	

	X3	0.0007	0.001	0.2	-0.181	
	X4	0.006	0.0052	0.93	0.1227	
11	X1	-0.0001	0.0006	0.096	-0.9752	7.4720
	X2	3.4624	3.4111	189.3938	0.0363	
	X3	0.0007	0.001	0.2	-0.1818	
	X4	0.006	0.0052	0.93	0.1207	
12	X1	-0.0001	0.0006	0.096	-0.9753	7.4720
	X2	3.4625	3.4111	189.3958	0.0363	
	X3	0.0007	0.001	0.2	-0.1822	
	X4	0.006	0.0052	0.93	0.1194	
13	X1	-0.0001	0.0006	0.096	-0.9753	7.4720
	X2	3.4625	3.4111	189.397	0.0363	
	X3	0.0007	0.001	0.2	-0.1825	
	X4	0.006	0.0052	0.93	0.1187	
14	X1	-0.0001	0.0006	0.096	-0.9754	7.4720
	X2	3.4625	3.4111	189.3978	0.0363	
	X3	0.0007	0.001	0.2	-0.1827	
	X4	0.006	0.0052	0.93	0.1182	
15	X1	-0.0001	0.0006	0.096	-0.9754	7.4720
	X2	3.4626	3.4111	189.3983	0.0363	
	X3	0.0007	0.001	0.2	-0.1828	
	X4	0.006	0.0052	0.93	0.1179	
	Pf	3.95E-14				

Table 3.E. Coefficient of the Maximum Entropy Distribution using in the example 2.

K_1								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
8.58038E+00	8.59028E+00	8.59649E+00	8.60060E+00	8.57689E+00	8.80021E+00	8.66745E+00	8.68124E+00	8.68060E+00
1.10078E-04	1.07195E-04	1.05393E-04	1.04205E-04	1.11100E-04	1.27214E-05	6.89554E-05	6.18269E-05	6.14231E-05

K_2								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
1.679E+02	1.895E+02	2.007E+02	2.084E+02	1.575E+02	1.702E+02	2.091E+02	2.226E+02	2.313E+02
-9.4451E-02	-1.0663E-01	-1.1285E-01	-1.1709E-01	-8.8491E-02	-9.6120E-02	-1.1960E-01	-1.2789E-01	-1.3332E-01
1.37837E-05	1.54941E-05	1.63540E-05	1.69384E-05	1.2933E-05	1.40732E-05	1.7608E-05	1.8880E-05	1.9717E-05

K_3								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
2.908E+02	3.325E+02	3.538E+02	3.685E+02	2.700E+02	2.588E+02	1.446E+01	1.341E+01	1.235E+01
-2.0172E-01	-2.3051E-01	-2.4508E-01	-2.5503E-01	-1.8717E-01	-1.7420E-01	5.14295E-02	5.96382E-02	6.78470E-02
4.49118E-05	5.11834E-05	5.43173E-05	5.64554E-05	4.16940E-05	3.69518E-05	-3.23002E-05	-3.70247E-05	-4.1749E-05
-3.00151E-09	-3.41765E-09	-3.62361E-09	-3.76392E-09	2.784771E-09	-2.22760E-09	4.837368E-09	5.54203E-09	6.24670E-09

K_4								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
1.817E+03	4.928E+03	5.836E+03	8.213E+03	4.562E+02	2.311E+02	7.334E+02	8.128E+03	8.731E+03
-1.8828E+00	-5.3160E+00	-6.2866E+00	-8.8989E+00	-3.7416E-01	-1.4177E-01	-8.9971E-01	-9.8556E+00	-1.0632E+01
7.3507E-04	2.15286E-03	2.54058E-03	3.61501E-03	1.10028E-04	2.27509E-05	4.34709E-04	4.49504E-03	4.86944E-03
-1.281E-07	-3.87792E-07	-4.56451E-07	-9.07096E-07	-1.34322E-08	5.27696E-10	-9.61551E-08	-9.12882E-07	-9.92825E-07
8.45215E-12	2.624466E-11	3.079240E-11	6.182969E-11	5.84302E-13	-1.99858E-13	8.12463E-12	6.962683E-11	7.601098E-11

K_5								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
2.74E+03	2.74E+03	2.7447E+03	9.6425E+05	1.5228E+01	2.6042E+01	1.5197E+01	1.5207E+01	1.5217E+01
-3.198E+00	-3.198E+00	-3.198E+00	-3.809E+02	3.3841E+00	1.2743E-01	2.2823E+00	2.6496E+00	3.0168E+00
1.47962E-03	1.47962E-03	1.47962E-03	5.30074E-02	-4.1260E-03	-1.1617E-04	-2.7344E-03	-3.1982E-03	-3.6621E-03
-3.38187E-07	-3.38187E-07	-3.38187E-07	-3.29439E-06	1.886931E-06	3.561854E-08	1.229205E-06	1.448447E-06	1.667689E-06
3.799242E-11	3.799242E-11	3.7992423E-11	1.388457E-10	-3.838963E-10	-4.512004E-12	-2.460182E-10	-2.919775E-10	-3.379363E-10
-1.65679E-15	-1.65679E-15	-1.65679E-15	-5.84151E-15	2.93140E-14	2.042070E-16	1.849731E-14	2.210290E-14	2.570849E-14

Table 3.F. Truncated moment from samples using in example 2.

K_1							
%1 Left	%5 Left	%8 Left	%10 Left	%1 Right	%5 Right	%8 Right	%10 Right
0.580814	0.574702	0.567156	0.562384	0.585231	0.593438	0.588338	0.582232
K_2							
%1 Left	%5 Left	%8 Left	%10 Left	%1 Right	%5 Right	%8 Right	%10 Right
0.580814	0.574702	0.567156	0.562384	0.585231	0.593438	0.588338	0.582232
0.00125	0.00476	0.007226	0.008768	0.001308	0.005395	0.008309	0.010095
K_3							
%1 Left	%5 Left	%8 Left	%10 Left	%1 Right	%5 Right	%8 Right	%10 Right
0.580814	0.574702	0.567156	0.562384	0.585231	0.593438	0.588338	0.582232
0.00125	0.00476	0.007226	0.008768	0.001308	0.005395	0.008309	0.010095
-0.00029	-0.00124	-0.00179	-0.00209	-0.00031	-0.00151	-0.00223	-0.0026
K_4							
%1 Left	%5 Left	%8 Left	%10 Left	%1 Right	%5 Right	%8 Right	%10 Right
0.580814	0.574702	0.567156	0.562384	0.585231	0.593438	0.588338	0.582232
0.00125	0.00476	0.007226	0.008768	0.001308	0.005395	0.008309	0.010095
-0.00029	-0.00124	-0.00179	-0.00209	-0.00031	-0.00151	-0.00223	-0.0026
8.97E-05	0.000368	0.00052	0.000595	9.88E-05	0.000478	0.000692	0.000793
K_5							
%1 Left	%5 Left	%8 Left	%10 Left	%1 Right	%5 Right	%8 Right	%10 Right
0.580814	0.574702	0.567156	0.562384	0.585231	0.593438	0.588338	0.582232
0.00125	0.00476	0.007226	0.008768	0.001308	0.005395	0.008309	0.010095
-0.00029	-0.00124	-0.00179	-0.00209	-0.00031	-0.00151	-0.00223	-0.0026
8.97E-05	0.000368	0.00052	0.000595	9.88E-05	0.000478	0.000692	0.000793
-2.75E-05	-0.00011	-0.00015	-0.00016	-3.10E-05	-0.00015	-0.00021	-0.00024

Appendix 3

Chapter 4:
Table 4.A. Coefficients of Entropy Distribution (Moment order K=4-5) with untruncated and truncated levels.

K_4								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
1.15E+02	4.66E+02	7.47E+02	8.58E+02	7.44E+01	2.12E+02	8.28E+02	1.02E+03	1.28E+03
-9.52E+00	-4.88E+01	-7.99E+01	-9.18E+01	-5.10E+00	-2.33E+01	-1.05E+02	-1.32E+02	-1.66E+02
2.85E-01	1.92E+00	3.21E+00	3.68E+00	1.06E-01	9.99E-01	5.03E+00	6.40E+00	8.16E+00
-3.65E-03	-3.39E-02	-5.73E-02	-6.56E-02	-4.84E-04	-1.98E-02	-1.08E-01	-1.39E-01	-1.78E-01
1.90E-05	2.26E-04	3.86E-04	4.40E-04	-1.85E-06	1.54E-04	8.71E-04	1.13E-03	1.46E-03

K_5								
Left (Percent)				Untruncated	Right (Percent)			
1	5	8	10		1	5	8	10
1.49E+02	7.10E+02	1.18E+03	1.35E+03	8.56E+01	3.65E+02	-6.57E+02	-1.68E+03	-2.70E+03
-1.43E+01	-8.40E+01	-1.42E+02	-1.62E+02	-6.63E+00	-4.69E+01	1.35E+02	3.16E+02	4.97E+02
5.53E-01	3.93E+00	6.73E+00	7.67E+00	1.88E-01	2.45E+00	-1.03E+01	-2.31E+01	-3.59E+01
-1.10E-02	-9.09E-02	-1.56E-01	-1.78E-01	-2.59E-03	-6.39E-02	3.82E-01	8.28E-01	1.27E+00
1.18E-04	1.03E-03	1.77E-03	2.00E-03	2.41E-05	8.18E-04	-6.91E-03	-1.46E-02	-2.24E-02
-5.19E-07	-4.48E-06	-7.70E-06	-8.68E-06	-1.21E-07	-3.97E-06	4.91E-05	1.02E-04	1.55E-04

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