# Investigating Children's Development of Multiplicative Reasoning

by

Melanie Jenny Biesenthal

Lakehead University, Thunder Bay, Ontario

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**Abstract**

This study investigated how an instructional focus on the meaning of the numbers and the development of Schwartz's (1988, 1996) notion of "intensive quantity" in multiplicative situations, impacted students' development of multiplicative reasoning, with particular attention to the role of simultaneity. This qualitative case study involved 21 grade 3 students from a classroom in Northwestern Ontario, representing one of seven classrooms participating in a larger study. Using a robust dataset including pre- and post-assessment mathematical interviews, video-recorded classroom sessions, and field notes, I analyzed student development over a four-week, 20 lesson mathematical intervention. Students were divided into three multiplicative reasoning categories: Pre-, Beginning, and Developing, based on their readiness at the outset of the intervention. Strategy Development Maps were created to examine student outcomes for seven assessment items. This study demonstrated that students, even those initially reliant on single-counting, can effectively approach or be on a pathway to develop multiplicative reasoning through the construction of the intensive quantity. The research highlights how understanding the meaning of the referents and the intensive quantity serve as a vital bridge, enabling early learners to transition toward more sophisticated mathematical reasoning. These findings are presented in a hypothetical progression, *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning*, providing indicators of how students' progress in their learning. The sample size limits the generalizability of these findings.

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# Chapter 1: Introduction

In elementary school, children learn the four basic arithmetic operations: addition, subtraction, multiplication, and division. Educators, based on their own understanding, may teach multiplication and division as extensions of addition and subtraction respectively. Researchers and mathematics educators argue, however, that this instructional approach does not provide children with a conceptual understanding of the fundamental differences between the operations (Hulbert et al., 2017, p. 2; Nunes & Bryant, 1996, p. 142; Schwartz, 1988, p. 46). Usiskin and Bell (1983) suggest that by defining multiplication as repeated addition, textbooks neglect "the important and unique contributions of this operation" (p. 227). Because multiplication entails distinct relationships that do not exist in addition (Schwartz, 1988), children must move beyond repeated addition to develop true multiplicative reasoning.

Drawing on Piaget's (1983/1987) theory, Clark and Kamii (1996) distinguish multiplication from addition by the increased complexity in the "number of levels of abstraction and the numbers of inclusion relationships the child has to make simultaneously" (p. 43). Given this complexity, multiplicative reasoning can be defined as the ability to work with a quantity that has a functional relationship (i.e., the intensive quantity) to solve problems in various multiplicative situations, including multiplication and division (Schwartz, 1988, 1996; Vergnaud, 1983, 1988, 1994, 1997). The development of multiplicative reasoning is foundational for children's understanding of fractions, ratios, and proportional reasoning (Siemon et al., 2005, p. 282; Vergnaud, 1988, p. 141; Wilkins & Ulrich, 2017, p. 8).

Multiplicative reasoning, however, is challenging to develop. Because early multiplication and division problems can often be solved through counting and additive strategies, children may arrive at correct answers without thinking multiplicatively (Ulrich, 2016,

p. 39; Wilkins & Ulrich, 2017, p. 13). This lack of conceptual development hinders the understanding of foundational mathematical concepts, ultimately limiting a student's access to complex number relationships (Sowder et al., 1988; Zwanch & Wilkins, 2021, p. 166).

Given that mathematics serves as a "gatekeeper" for future success (Byun et al., 2015, p. 439), students who lack this foundation face significant obstacles throughout their academic careers (Meece et al., 1982, p. 324). This raises a critical question: what strategies and key components can educators employ to ensure students develop a strong foundation in multiplicative reasoning?

**Purpose of this Study**

The purpose of this qualitative research case study was to investigate whether children could create a strong foundation for development of multiplicative reasoning when they learn early multiplication and division with whole numbers as a *functional relationship* through the creation of the *intensive quantity*. There has been little research based on Schwartz's theory (1988) of multiplicative reasoning. Schwartz (1988, p. 42) argues that children will develop multiplicative reasoning if they are guided to construct multiplicative relationships when they first start to learn multiplication in school.

**Research Question and Sub Question**

I undertook this study to answer the following research question and sub question:

*(1) How does an instructional focus on the meaning of the referents and the development of the intensive quantity in multiplicative situations affect children's early development of multiplicative reasoning?*

*(1a) What is the role of simultaneity in the development of multiplicative reasoning in grade 3 students?*

## Significance of the Study

Researchers and theorists have long investigated and continue to investigate how children develop multiplicative reasoning (see for example Clark & Kamii, 1996; Greer, 1992; Hulbert et al., 2017; Nunes & Bryant, 2022; Schwartz, 1988, 1996; Steffe, 1988, 1992, 1994; Supovitz et al., 2021; Vergnaud, 1983, 1988, 1994, 1997; Zwanch & Wilkins, 2021). My research contributes to this literature examining how third-grade students construct multiplicative reasoning, specifically through their nascent understanding of the intensive quantity in multiplication and division.

## Definitions

Each of the following terms will be defined within and used throughout the dissertation. They are summarized here, in Table 1 for easy reference.

**Table 1**

*Definition of Key Terms*

| Term | Definition | Reference(s) |
|---|---|---|
| adjectival quantity | the number is used as an adjective to describe the noun with every adjectival quantity made up of a number and a referent; e.g., five books | Schwartz, 1996 |
| additive reasoning | understanding how to join, separate, or compare quantities where the referents are typically the same | Carpenter et al., 2015; Hulbert et al., 2017; Schwartz, 1996 |
| composite unit | a unit that is made up of individual units; e.g., five ones or one five | Lamon, 1994; Steffe, 1994 |
| continuous quantity | a quantity that uses a mass noun with the structure {(magnitude , unit), attribute}; e.g., {(4, millilitres), water} | Schwartz, 1996 |
| count noun | a referent that can use the word ‘one’ in front of it; e.g., one book | Schwartz, 1996 |

| Term | Definition | Reference(s) |
|---|---|---|
| developmental progression of thinking and learning | a pathway that describes how children's thinking and learning may evolve in their construction of mathematics concepts moving from least sophisticated to a comprehensive understanding | Clements & Sarama, 2004<br>Lawson, 2015 |
| discrete quantity | a quantity that uses a count noun with the structure {cardinality of set, definition of set}; e.g., {5, books} | Schwartz, 1996 |
| extensive quantity | a quantity that can be either discrete or continuous | Schwartz, 1988, 1996 |
| functional relationship | describes the constant multiplicative relationship of two quantities such as a rate; e.g., kilometres per hour | Vergnaud, 1994<br>Nune & Bryant, 1996 |
| hypothetical learning trajectory | the iterative process of designing, implementing, and refining a set of instructional tasks to support the development of a particular mathematics concept | Clements & Sarama, 2004<br>Simon, 1995<br>Gravemeijer, 1999 |
| intensive quantity | a quantity that is created through the invariant relationship between two extensive quantities which are either discrete, continuous, or both; e.g., 3 litres per tank | Schwartz, 1988; 1996<br>Nunes & Bryant, 1996 |
| implicit theory | the theory that there is an intuitive model for a mathematical operation. For example, the intuitive model for multiplication is repeated addition | Park & Nunes, 2001 |
| iterations | the number of times the intensive quantity is repeated; e.g., 5 candies per bag, 3 bags, the iterations are 3 times | Schwartz, 1988 |
| mass noun | a referent which often refers to a measurement where 'one' cannot be used; e.g., one water | Schwartz, 1996 |
| multiplication | a word problem in which the composite unit/intensive quantity (e.g., | Carpenter et al., 2015 |

| Term | Definition | Reference(s) |
|---|---|---|
| | 3 cookies per bag) and the iterations are known (e.g., 5 bags) and the total amount is unknown (e.g., how many cookies?) | |
| multiplicative reasoning | the ability to work with a quantity that has a functional relationship (i.e., the intensive quantity) to solve problems in various multiplicative situations, including multiplication and division | Vergnaud, 1983, 1988, 1994, 1997; Schwartz, 1988, 1996 |
| nominal quantity | a bare number without a referent; e.g., 4 | Schwartz, 1996 |
| partitive division | a word problem in which the total amount (e.g., 15 cookies) and the number of iterations (e.g., 5 bags) are known, and the composite unit/intensive quantity is unknown (e.g., how many cookies per bag) | Carpenter et al., 2015; Schwartz, 1988 |
| problem situations | a written context that supports the development of a mathematics concept and is not necessarily tied to a single operation. Problem situations can also be referred to as types or structures | Carpenter et al., 2015; Schwartz, 1988; Vergnaud, 1997 |
| quotative (or measurement) division | a word problem in which the total amount (e.g., 15 cookies) and the composite unit/intensive quantity (e.g., 3 cookies per bag) are known and the iterations are unknown (e.g., 5 bags) | Carpenter et al., 2015; Schwartz, 1988 |
| quantity | a noun used to describe 'how many' that includes both a description of the amount with a referent; e.g., 4 candies; a few tablespoons | Schwartz, 1996 |
| referent | the meaning of the number; e.g., 4 candies, the referent is candies | Schwartz, 1996 |
| schema of correspondence theory | the theory that identifies the distinct relationship that exists in multiplication that does not exist in addition. Multiplication is a | Park & Nunes, 2001 |

| Term | Definition | Reference(s) |
|---|---|---|
| | relationship between two quantities such as ratio and rate | |
| simultaneity in early multiplicative situations | the simultaneous interactions of the intensive quantity and the iterations (i.e., the ability to keep track of the intensive quantity and the iterations at the same time) when solving multiplicative situations including multiplication, quotative, and partitive division, involving whole numbers and discrete quantities (e.g., keeping track of 2 candies per bag and the number of bags at the same time) | Ulrich, 2015, 2016<br>Lawson, 2015<br>Schwartz, 1988, 1996 |
| simultaneity within the intensive quantity | the ability to understand and work with the relationship in the construction between two extensive quantities that exists in the intensive quantity; e.g., candies per bag | Clark & Kamii, 1996<br>Schwartz, 1988, 1996 |
| unit | an individual count | Ulrich, 2015; Steffe, 1994 |

**Limitations**

This study possesses four main limitations, all of which pertain to generalizability. First, the small sample size suggests that further analysis will be necessary to determine whether these findings extend to other classrooms that were part of the larger project. Secondly, the sample is not representative of the broader Ontario population, as the diverse range of backgrounds and contexts vary between Northern to Southern regions of the province.

Third, the lack of a control group limits our ability to isolate the intervention's effects; without a comparison group receiving typical instruction in multiplication and division, it is difficult to distinguish between the study's impact and the natural progression of learning. Additionally, the inclusion of a 'post-post' assessment, at the start of the following academic

year, would have strengthened the findings by demonstrating whether the gains in multiplicative reasoning were sustained over time.

Finally, the fourth limitation concerns the extent to which other educators could replicate this study. The presence of additional adult support, rarely available to a typical classroom teacher, likely influenced outcomes. Furthermore, my unique position as a researcher, combined with the insight gained from four iterative cycles of the study, differs from the experience of a standard classroom teacher. These factors, alongside my personal experiences and my positionality, highlight the complexities inherent in my dual role as both researcher and educator.

**Personal Connections and Positionality**

My career has focused on and continues to focus on mathematics education through a variety of roles and experiences. As a classroom teacher, I began consciously working on changing my practice by learning how to engage children in problem solving and lead mathematical discussions—shifting away from direct instruction of procedures. I believed that students would develop a deeper understanding of mathematics if they were actively involved in the process. I was involved in various initiatives within my school board, leading to my role as a Math Resource Teacher. In this position I supported educators in the teaching and learning of mathematics within their classrooms and in professional development. These experiences enabled my move to the K-12 Coordinator of Mathematics in my board. I also continue to build my skills in promoting change in mathematics education as a contract lecturer at a university teaching the mandatory first year course, *Curriculum Instruction in Mathematics, Primary and Junior,* to undergraduate education students. Throughout, I focused and continue to focus on understanding how the students think, beginning with their understanding and providing opportunities for them to go beyond memorized procedures.

My views on learning, based on personal experiences and experiences I have had in my various roles in education continuously influence my decisions as a researcher and educator. My beliefs about how we learn are rooted in social constructivism (Cobb et al., 1992; Creswell & Creswell, 2018; Davis & Francis, 2022; Santrock et al., 2010; Simon, 1995; Vygotsky, 1978), a wide field to define; therefore, I will briefly delineate social constructivism in relation to the field of mathematics education in order to better identify my stance on effective mathematics instruction.

Creswell and Creswell (2018, p.7), in simple terms, define social constructivism as the learning that happens when individuals are trying to understand the world collectively. The social construction of knowledge happens when children are actively constructing knowledge, and not through unquestioning passive acquisition (Santrock et al., 2010, p. 50). As children are learning mathematics, they are building on what they know and are changing and evolving in their thinking through social interactions as well as time to think on their own (Cobb et al., 1992, pp. 5,6,26).

Vygotsky (1978), identified as a *social constructivist* (Santrock et al., 2010), indicated that children learn when they are in their *zone of proximal development*, which: "is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). This description explicitly identifies the teacher as an essential part of a child's cognitive development.

Simon (1995, p. 121) describes building on and guiding children's thinking through collective mathematical discourse as *inquiry*. An educator needs to understand how the mathematical content knowledge builds while also understanding children's thinking because

when we build from student thinking, it "provides an entry point into striving for equity" (Carpenter et al., 2017, p. 147). Building on student thinking requires leading rich, intentional discussions that support mathematical learning while fostering social interactions. These interactions help students develop ideas and navigate challenges together, ultimately creating a community of learning (Chapin et al., 2009, p.9).

As a researcher and as an educator, creating the conditions for learning mathematics through a constructivist lens, I am aware of my position as a middle-class white settler, cisgender female. I may have conscious and unconscious biases that impact my interactions and decisions that could support or hinder children's learning. Self-reflection is integral. Coupled with this, I need to consistently challenge my own mathematical content knowledge in order to be able to make intentional decisions to guide and support children to build on their mathematical understanding. My worldview, positionality, and knowledge are interconnected, impacting all decisions that I make in the field of research and in education as a practitioner. I delineate my dissertation as follows.

**Plan of the Thesis**

The plan for the thesis chapters is as follows:

- Chapter 2 is a detailed review of the literature including a focus on three influential theorists' and their views on children's development of multiplicative reasoning.
- Chapter 3 outlines the methodology for this qualitative case study, explains the process of data collection, provides an overview of the intervention, and how I analyzed the data.
- Chapter 4 involves the analysis of students' responses before and after the intervention, particularly around seven assessment items. Each assessment item analysis results in a *Strategy Development Map* to provide a visual representation of the data.

- Chapter 5 examines four key findings to answer the research question and sub question, implications for classroom instruction, and suggestions for future research.

## Chapter 2: Literature Review

In schools, multiplication is traditionally taught as a set of facts to be memorized and practiced by rote. This strategy alone does not foster a deep understanding of number relationships that are essential to dealing with complex multiplicative mathematical concepts efficiently and flexibly. The relationships in multiplicative reasoning are foundational to understanding fractions, ratios and proportional reasoning (Siemon et al., 2005, p. 282; Vergnaud, 1988, p. 141; Wilkins & Ulrich, 2017, p. 8). When children are not able to reason multiplicatively, it hinders the development of foundational mathematical concepts essential in their later education, which results in limited access to complex mathematical number relationships and understanding (Sowder et al., 1998; Zwanch & Wilkins, 2021, p. 166). For example, when children only memorize facts such as 6 x 7 = 42, they do not have the opportunity to use what they already know to respond such as decomposing 7 into 5 and 2 and multiplying each part by 6 (i.e., 6 x 5 + 6 x 2) to solve 6 x 7. Learning such skills will provide them with a foundation for working with the distributive property in more complex mathematics.

Children's development of multiplicative reasoning is, however, complex (Clark & Kamii, 1994; Vergnaud, 1997). Researchers in this field have, at times, held contradictory theories and viewpoints about what would best support children's learning. This literature review will explore these contrasting—at times contradictory—theories, in order to propose an effective pathway for children to develop multiplicative reasoning. This examination of the literature will include:

- an overview of the foundational skills students' need to support their development of multiplicative reasoning;
- the work of three critical theorists: Vergnaud, Schwartz, and Steffe;

- implications from these theories for student learning; and
- the theoretical framework for this research.

I turn first to examine the foundation from which this multiplicative reasoning springs.

## The Multiplicative Reasoning Foundation: Determining How Many

The mathematical strategies children use for determining *how many*, whether counting or subitizing, play different foundational roles in a child's construction of early mathematics. Understanding these differences offer insights into how each of these foundational structures impact or hinder the development of multiplicative reasoning. I begin by examining counting to create a clearer picture of how an individual counts and how a child develops counting, which is quite different from subitizing.

### *Counting: The Precursor Skill to Operations*

Understanding how to count offers children the opportunity to explain their lived experiences (Nunes & Bryant, 2022, p. 22). While seemingly an everyday event, children need to be able to coordinate a number of ideas to be successful counters. There are three components that they need to do and understand which are: (a) knowing the *counting sequence*, (b) being able to perform *one-to-one tagging* or pointing so that each item is counted once and only once, and (c) *coordinating the count* so that there is one number word per touch (Lawson, 2015, p. 12). Once children have established how to count, they begin to work towards the development of the *cardinal principle*. The cardinal principle, or cardinality, is the understanding that when you are counting a set of objects, the last number said in that count refers to the entire set (Carpenter et al., 2017, p. 13; Franke et al., 2018, p. 15). When a child has cardinality, they are demonstrating a transition from counting as a procedure to understanding the size of the set that they have

counted (Simon et al. 2021, p. 3). Without cardinality, children will struggle to develop additive reasoning.

***Subitizing: 'Grouped Objects' Versus an Individual Count***

While counting is an essential piece in understanding number, it is not enough of a foundation in and of itself. Researchers as early as 1912 began to consider how seeing 'grouped objects' instead of an individual count provided a richer understanding of quantity (Carper, 1942, p. 167; Douglass, 1925, p. 488; Freeman, 1912, p. 312). Subitizing, derived from the Latin word *subitus*, meaning suddenly, is the ability to see 'grouped objects' as a unit without having to count each object individually, as one might know that the five pips on a die face  are five without counting them. Subitizing plays a different role than counting in the development of operations (MacDonald, 2018, p. 155).

Developing and supporting children's ability to subitize is key to working with numbers efficiently as counting alone does not adequately support some mathematics concepts (Carper, 1942, p. 166). In her research, Carper (pp. 168-169) identified that children were most successful in using a grouping strategy when there were geometric arrangements without any distracting contexts, such as a picture around the dots, and when the arrangements were compact and in a regular pattern as can be seen in Figure 1. Freeman (1921) proposed that children can be supported in their development of seeing grouped objects (p. 310) because it is part of their lived experiences, whether from seeing trees planted in rows or a symmetrical table place setting of four.

**Figure 1**

*Geometric Grouped Object Arrangements (Carper, 1942, p. 168)*



*Note*. Image created based on written text from "Seeing Numbers as Groups in Primary-Grade Arithmetic,'' by D. Carper, 1942, *Elementary School Journal, 43*(3), p. 168 (https://www.jstor.org/stable/997719). Copyright 1942 by the University of Chicago Press. Created image printed with permission.

According to Clements (1999), there are two types of subitizing: *perceptual* and *conceptual*. Perceptual subitizing is the ability to quantify a group of objects by seeing without having to conduct or perform any type of calculation. This can only be done with smaller groups of objects such as the arrangements seen in Figure 1. Conceptual subitizing is the ability to combine smaller groups of objects to determine the total number of objects. An example of this is seeing the total of 8 dots on a domino by seeing two subgroups of four (p. 410).

Subitizing is an essential skill young children need to develop (Baroody, 1987; Carper, 1942; Freeman, 1912). However, just being able to "see" grouped objects will not by itself support the development of complex reasoning strategies just as counting on its own is not enough to support a deep understanding of number.

There are, then, two ways to determine quantity: counting and subitizing. As students strengthen their understanding of counting and subitizing, they can begin to develop additive and multiplicative reasoning.

**From Counting and Subitizing to Additive and Multiplicative Reasoning**

Hulbert et al. (2017) define additive and multiplicative reasoning by describing the differences between *relationships* and *actions* (see Table 2). Additive reasoning uses a *one-to-one correspondence* relationship with the actions of joining, separating or comparing (p. 5). For example, young children adding 3 apples + 6 apples, may physically count out three cubes, count out another six cubes, and push or "join" them and count them all again to get a total (Carpenter et al., 2015, p. 22; Lawson, 2015, p. 28).

For Hulbert et al. (2017) multiplicative reasoning uses a *many-to-one composite unit* relationship that involves the actions of *scaling* or *iterating* (p. 5). For example, if a child is figuring out how many apples there are when there are 6 plates, with 3 apples on each, the many-to-one composite unit relationship is the 3 apples to 1 plate, which is then scaled or iterated 6 times to find out the total number of apples (p. 4). These definitions can serve as a starting place to explore how researchers and theorists conceptualize the development of these two types of reasoning.

**Table 2**

*Examples of Additive and Multiplicative Reasoning (Hulbert et al., 2017, pp. 2-3)*

| Reasoning type | Definition | Example | Reference |
|---|---|---|---|
| additive reasoning | one-to-one correspondence relationship with the actions of joining, separating or comparing |  3 + 6 = 9 | Carpenter et al., 2012, p. 5 Hulbert et al., 2017, p. 2; 5 Lawson, 2015, p. 28 |
| multiplicative reasoning | many-to-one composite unit relationship that involves the actions of *scaling* or *iterating* |  6 x 3 = 18 | Hulbert et al., 2017, p. 3; 5 |

*Note*. The images have been adapted from *A Focus on Multiplication and Division: Bringing Research into the Classroom* by E. T. Hulbert, M. M. Petit, C. B. Ebby, E. P. Cunningham, R. E. Laird, 2017, pp. 2-3. Copyright 2017 by Routledge. Image adapted with permission.

## Two Hypotheses on the Development of Multiplicative Reasoning from Park and Nunes: A Potential Framework

Park and Nunes (2001, p. 764) argue that there are two different hypotheses of how children develop an understanding of multiplication. The first hypothesis, *the implicit theory*, specifies that every mathematical operation has an intuitive model and for multiplication, this intuitive model is repeated addition. Park and Nunes (p. 764) credit Fischbein et al. (1985) for formalizing this theory.

The second hypothesis, the *schema of correspondence theory*, stipulates that there is a distinct relationship that exists in multiplication that does not exist in addition. This theory

defines multiplication as a relationship between two quantities, known as a "ratio or a rate" (Park & Nunes, 2001, p. 764). Park and Nunes cite Piaget (1965) as the first theorist to explore this way of constructing multiplication that was later elaborated on by Vergnaud (1983, 1988) and Nunes and Bryant (1996).

Park and Nunes (2001) envisioned two instructional pathways which could be used to investigate each of the hypotheses—the implicit theory and the schema of correspondence theory. These two instructional pathways could be used to explore "what progress pupils make if taught in one way or the other" (p. 765). They conducted research with 42 children, who had a mean age of 6 years and 7 months (p. 765) at two different schools in England. Children were randomly assigned to an intervention group where they either learned multiplication as repeated addition or as schema of correspondence (as rate). Park and Nunes concluded that even though both groups showed progress in their understanding of multiplication, the *schema of correspondence* group progressed significantly further ($t = 4.09$; $df = 20$; $p = .001$) in their development of multiplicative reasoning and in solving multiplicative problems (which consisted of problems involving the operations of both multiplication and division) (p. 770).

Examining the two hypotheses of the implicit theory and the schema of correspondence theory Park and Nunes (2001) lay out a potentially comprehensive framework for organizing and examining the research literature on the development of multiplication in children. There are, however, some over–simplifications in this framework that require exploration and amplification. First, Park and Nunes identify Fischbein et al. (1985) as formalizing the implicit theory that multiplication is developed through repeated addition. While this is true, in Fischbein et al.'s analysis of their own research findings, they argue that the schema of correspondence theory is likely the better pathway for developing children's multiplicative reasoning—but

without describing how students can learn this way in school. Given this omission by Park and Nunes, it is important to understand the research study and conclusions made by Fischbein et al.

***Inconsistant Interpretations of Fischbein et al.***

Fischbein et al. (1985) conducted a study in Italy in 13 schools with 623 students in Grades 5, 7, and 9. Children were given a series of problems that included all four arithmetic operations: addition, subtraction, multiplication, and division, where they had to identify the necessary operation for solution without having to solve each problem. Fischbein et al.'s data supported the researchers' initial hypothesis that the intuitive and primitive model for multiplication was repeated addition.

Moreover, they offer two reasons as to why these primitive models are constructed. One reason is that this is the model that is "initially taught in school" and the second reason is that these intuitive models are harder to change, "because they correspond to features of human mental behavior that are primary, natural and basic" (Fischbein et al.,1985, p. 15). In the case of multiplication learned as repeated addition, this is the model that is most comfortable for teachers, and it also provides an accessible entry point for children to learn a new operation. However, it is important to note that the researchers, at the end of their article, voiced a strong caution that the model that children first learn is the model that becomes most embedded and is therefore very challenging to change—when it may in fact be a very limited model for their future development of multiplicative reasoning. For this reason, Fischbein et al. leave us with more questions than answers about how to consider an approach to multiplication in a way that does not reinforce repeated addition. In varying citations of their work by other researchers, this latter caveat is not always articulated.

Why this article had so much momentum in the research field of children's development of multiplicative reasoning is unclear. What is evident, however, is that the manner in which researchers cite this pivotal article is dependent on the argument they are trying to make about the early introduction of multiplication, as seen in Table 3. Similar to Park and Nunes' (2001) interpretation, Fischbein et al. (1985) are cited for theorizing that multiplication is based on the intuitive model of repeated addition (Carrier, 2014; DeCorte & Verschaffel, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997).

Ironically, their work is also cited to explain why repeated addition as an intuitive model is not actually the first entry point when children are learning to multiply (Steffe, 1988, p. 135). It is also cited to bolster the argument that multiplication as repeated addition is not enough (Kaput, 1986; Maffia & Mariotti, 2018). Finally, Nesher (1988) argues that it supports neither theory; instead, that it is the semantics of the word problem that dictate with which model a child will intuitively begin.

**Table 3**

*Overview of Researchers Divergent Interpretation of Fischbein et al.'s Research*

| **Implicit theory** | **Schema of correspondence theory** | | |
|---|---|---|---|
| As support for the Intuitive Model of Repeated Addition | Repeated Addition – not the first entry point | Why repeated addition is not enough to understand multiplication | Research model does not consider semantics |
| Carrier, 2014; DeCorte & Verschaffel, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997; Park & Nunes, 2001 | Steffe, 1988 | Kaput, 1986; Maffia & Mariotti, 2018 | Nesher, 1988 |

### *Enlisting the Two Theories of Multiplicative Learning as a Framework to Examine the Research*

Despite their incomplete interpretation of Fishbein et al.'s influential research, Park and Nunes' (2001) framework nonetheless offers a useful way to explore this area of research. My focus at this point will be narrowed to three crucial theorists as seen in Table 4: Vergnaud, Schwartz, and Steffe, to examine how their theories inform my understanding of children's early development of multiplicative reasoning.

**Table 4**

*Examining the Theories of Vergnaud, Schwartz, and Steffe under Two General Theories of Children's Multiplicative Development (Park & Nunes, 2001)*

| Schema of Correspondence Theory | | Implicit Theory |
|---|---|---|
| Vergnaud (1983, 1988, 1994) | Schwartz (1988, 1996) | Steffe (1988, 1992, 1994) |

**Vergnaud, Schwartz, and Steffe: Why these Three Theorists?**

The theories of multiplicative reasoning development as espoused by Vergnaud (1983, 1988, 1994) and Schwartz (1988, 1996) are based on the *schema of correspondence*, which is multiplication as a functional relationship. According to Parks and Nunes (2001, p. 764), Steffe's (1988, 1992, 1994) theory on the other hand, is based on the *implicit theory*; that is, multiplication as a form of repeated addition or an additive relationship. It should be stated, however, that Park and Nunes's categorization of Steffe's theory does not fully capture Steffe's (1988) proviso that there is an immense amount of number development that occurs before children can use repeated addition with meaning (p. 135). Regardless, Steffe focuses on the development of multiplicative reasoning as evolving from counting and addition while Schwartz and Vergnaud focus on the functional relationship that exists in multiplication and division—indicating a significantly different progression. Prior to examining these three theorists, how do other researchers view their work?

***Vergnaud, Schwartz, and Steffe: Categorization from Other Researchers***

Other researchers have noted that Schwartz's and Vergnaud's theories are distinct from other theories found under the umbrella of the schema of correspondence theory. Lesh et al. (1988, p. 108), for example, in their discussion of proportional reasoning (the relationship between quantities: i.e., a co-variation or rate), indicate that Schwartz and Vergnaud differ from

others in their point of view of how rates are formed. Nesher (1988, p. 26) describes the theories of Vergnaud and Schwartz as "dimensional analysis," i.e., an explicit description of the interconnectedness of the quantities, which considers a more sophisticated scope of multiplicative relationships than some other theories. Vergnaud's theory explores the relationships within multiplication that exist primarily between two dimensions (p. 25) and Schwartz 's theory considers dimensions in terms of referents (the meaning of each number in a multiplication context). Nesher (1988) counters their theories by stating that before children can get to the dimensional analysis, there needs to be a focus on how children understand various multiplicatively structured word problems. Greer (1992) also pays attention to the classification of word problems and the implications for theories of multiplicative reasoning development.

Greer (1992, p. 128), in *"Multiplication and Division as Models of Situations,"* outlines a brief overview of various situations that use these two operations and theories with contrasting perspectives on how these models of multiplication and division are established. In his analysis, Greer groups Vergnaud and Schwartz along with Kaput, based on how these theorists think of models for multiplication and division, further echoing Lesh et al.'s (1988) analysis that Schwartz and Vergnaud are distinct in how they define quantities. Greer separates Steffe's theory from Vergnaud and Schwartz.

In summary, Vergnaud and Schwartz as compared to Steffe represent two extreme ends of theories of development of multiplicative reasoning. Steffe's theory is important to examine as his is one of the most in-depth descriptions of the stages of multiplication as developed through repeated addition. Steffe's research (1988, 1992) examines the development of multiplicative reasoning evolving from counting, through additive reasoning, and then to multiplicative reasoning.

To flesh out their theories, I contrast and compare these three theorists and highlight the small, yet important nuances that need to be considered in children's initial introduction to multiplication. In order to compare and contrast the theories of Vergnaud, Schwartz, and Steffe, each theory will be examined through these sections:

(1) the difference between additive and multiplicative reasoning,

(2) multiplicative relationships, and

(3) multiplicative situations.

This comparison will allow for an in-depth analysis of how they explain the development of multiplicative reasoning. I start with the *schema of correspondence theory* to better understand the *functional* relationship from which multiplicative reasoning develops. Vergnaud sketches the complexity of multiplicative reasoning by describing its development in his framework of conceptual fields.

**Vergnaud: Framework of Conceptual Fields**

To help readers understand the mathematical concepts that children learn in school and in real-life applications, Vergnaud (1983) constructs his framework around *conceptual fields*. He defines a conceptual field as, "a set of problems and situations for the treatment of which concepts, procedures and representations of different but narrowly interconnected types are necessary" (p. 127). Understanding the conceptual field of multiplication affords us the opportunity to explore these concepts and the strategies that children develop over time.

***The Difference between Additive and Multiplicative Reasoning***

Vergnaud (1997) identifies two main conceptual fields of ordinary arithmetic as additive structures and multiplicative structures (p. 15). Vergnaud (1983) views these two different structures, "as sets of problems involving arithmetical operations and notions of the additive type

(such as addition, subtraction, difference, interval, translation) or the multiplicative type (such as multiplication, division, fraction, ratio, similarity)" (p. 128). Examining the conceptual field of additive structures first will establish how multiplicative structures are different.

**Vergnaud's Conceptual Field: Additive Structures.** Vergnaud's (1997) conceptual field of additive structures encompasses a variety of problem situations in both addition and subtraction (p. 16). The structures of these problem situations reveal different number relationships, similar to the problem types set out by Carpenter et al (1999, p. 12) where the unknown quantity occurs at various locations within an equation as seen in Table 5.

**Table 5**

*Examples of Equations with Unknowns in Various Places Evident Within a Context of a Problem (Vergnaud, 1997, p. 18; Carpenter et al., 1999, p. 25)*

| Sample equations with unknowns | |
|---|---|
| 9 + 4 = □ | 13 – 9 = □ |
| 9 + □ = 13 | 13 - □ = 4 |
| □+ 4 = 13 | □ - 9 = 4 |

The development of additive reasoning is supported as children solve these various problem types as they need to think about the how to solve for the unknown quantity (1997, p. 17). According to Vergnaud (1997), there are different problem situations that he organized into three categories: *part-part-whole relationships*, *state-transformation-state relationships*, and *comparison relationships,* resulting in different locations of the unknown. Part-part-whole relationships involve only two possibilities which are finding the whole based on two parts (addition) or finding a part based on knowing a whole and a part (subtraction). An example of a part-part-whole problem would be, "There are 4 green marbles and 6 blue marbles in a container.

How many marbles are there altogether?" (problem adapted from Carpenter et al., 1999, p. 12). The state-transformation-state relationships include six different situations, two of which are addition and four that are subtraction and involve solving for an unknown quantity. An example of one of these state-transformation-state situations would be, "Fred has 9 marbles. He plays a game of marbles. He now has 6 marbles. What has happened?" (Vergnaud, 1997, p. 18). The comparison relationships—also have six situations and involves the comparison of quantities. For example, "Connie has 13 marbles. Juan has 5 marbles. How many more marbles does Connie have than Juan?" (Carpenter et al., 1999, p. 12). Beyond these three initial relationships, there are other relationships within this conceptual field of additive structures that explore the use of negative numbers (1997, p. 17)

The main difference between additive and multiplicative structures is the number of quantities that are included in the structure. Summarizing Vergnaud's additive structures, Askew (2018, p. 408) highlights that the majority of these relationships are *ternary*, meaning the problem situation involves a relationship amongst three quantities where two quantities are known, and the goal is finding out the third quantity that is unknown. This differs from *quaternary* relationships that are related to multiplicative structures as seen in Table 6.

**Table 6**

*Examples of Ternary and Quaternary Relationships (Vergnaud, 1997, p. 21)*

| Characteristic of relationship | Ternary relationship | Quaternary relationship |
|---|---|---|
| Sample equation | 4 + 5 = 9 | 4 x (5x1) = 20 |
| Number of quantities | 3 | 4 |
| Situation type | additive | multiplicative |

**Vergnaud's Conceptual Field: Multiplicative Structures.** Vergnaud (1997) argues that one of the components that differentiates multiplicative structures from additive structures is that many of the multiplicative relationships are *quaternary* (p. 21) (while additive structures are consistently ternary, there are special cases when multiplicative relationships can be as well). In the discussion of early multiplication development, the types of problem situations that children will engage with are quaternary. This means that there are four different quantities that are part of the relationship of a multiplicative situation. For example, when considering the following question, "There are four bags of candy and each bag has five candies. How many candies are there altogether?" (example adapted from Schwarz, 1988), the multiplication sentence would be 4 x 5 = 20. The three quantities that are obvious are 4 bags, 5 candies per bag, and 20 candies. The quantity that is not so obvious is the quantity of 1 that connects with the 5, indicating the relationship between the number of candies and the bag. These 5 candies are contained in *1 bag* so one can infer that the same equation could be written as 4 x (5 x 1) = 20. This quantity of 1 seems invisible but it has a key role in the multiplicative relationships that exist amongst these four quantities as seen in the ratio table in Figure 2. While the "1" does not seem necessary now, there are multiplicative situations where the one will be a different number such as 5 candies per 100 grams.

**Figure 2**

*Example of Quaternary Multiplicative Relationship*

| BAGS | CANDIES |
|---|---|
| 1 | 5 |
| 4 | ☐ |

*Note*. This figure has been adapted from "The Nature of Mathematical Concepts" by G. Vergnaud in T. Nunes and P. Bryant (Eds.), 1997, *Learning and Teaching Mathematics: An International Perspective,* p. 21. Copyright 1997 by Psychology Press. Reprinted with permission.

### *Multiplicative Relationships*

Classifying this quaternary relationship, Vergnaud (1994, p. 47, 1997, p. 21) identifies three relationships that exist within simple multiplicative proportion problems: (1) a binary commutative combination, (2) the vertical unary operation, and (3) the horizontal unary operation. The word *unary* is used as an adjective to describe a single element in relation to a mathematical operation (Merriam-Webster. n.d.).

A *binary commutative combination* is the understanding that the product will be the same whether the calculation is 4 x 5 or 5 x 4, referred to as the commutative property. In the example, "There are 4 bags of candy and each bag has 5 candies. How many candies are there altogether?" 5 candies can be iterated 4 times, 5 + 5 + 5 + 5 to get a result of 20 candies. While numerically correct if students stay in the problem context and commute only the numbers thinking 4 + 4 + 4 + 4 + 4 it would seem that they are adding 4 bags, 5 times so that result should be 20 bags, not

candies. Vergnaud (1997) believes that this makes it challenging for children as the context interferes with the understanding of the commutative property in multiplication (p. 22).

Vergnaud (1997, p. 22) outlines the second relationship as *vertical unary operation* that exists vertically in one column, and notes that this relationship can be either additive or multiplicative as seen in Table 7.

**Table 7**

*Horizontal and Vertical Unary Operations (Vergnaud, 1997, p. 22)*

| Type of operation | Vertical unary operation | Horizontal unary operation |
|---|---|---|
| Direction in ratio table | BAGS \| CANDIES<br>1 \| 5 — x 4 scalar ↓<br>4 \| □ | BAGS \| CANDIES<br>1 \| 5<br>4 \| □<br>→ x 5 functional |
| Type of relationship | *Scalar relationship*: iteration of the 5, 4 times by multiplying or repeatedly adding resulting in being able to repeatedly add 5 candies without having to keep track of the bags at the same time | *Functional relationship*: understanding of the implicit relationship of 5 simultaneously representing both the number of candies and a bag or 5 candies per bag |

*Note*. This table has been adapted from "The Nature of Mathematical Concepts" by G. Vergnaud in T. Nunes and P. Bryant (Eds.), 1997, *Learning and Teaching Mathematics: An International Perspective,* p. 22. Copyright 1997 by Psychology Press. Reprinted with permission.

For example, the answer to how many candies can be solved through repeated addition. The number of candies in each bag can be repeatedly added, 5 candies + 5 candies + 5 candies + 5 candies = 20 candies (Vergnaud, 1994, p. 47). This vertical unary operation can also be seen as multiplicative by multiplying the 5 candies 4 times, as seen in Table 7. In this case, the function acts as a scalar (Vergnaud, 1988, p. 45), iterating the 5 four times. Vergnaud (1997) suggests that this vertical unary operation relates addition to multiplication, making the transition between these two operations easier for children to understand (p. 22).

Vergnaud's (1994) third relationship, the *horizonal unary operation* is more complex as it explores a functional relationship between both columns through a constant coefficient property i.e., the multiplicative relationship connecting two quantities together (p. 47). In this example, the constant coefficient is candies per bag and exists between the columns of bags and candies as seen in Table 7. The four bags are being multiplied by the constant coefficient of *5 candies per bag* to result in 20 candies. This functional relationship is much more sophisticated than the scalar relationship and fosters the development of multiplicative reasoning.

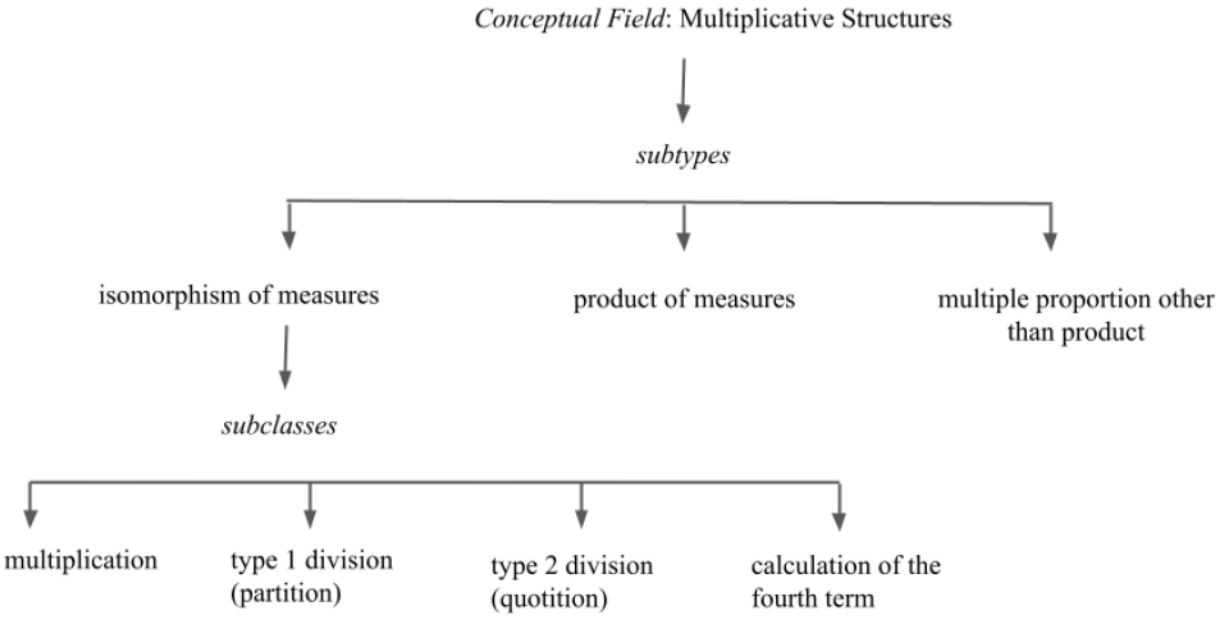
As part of describing the conceptual field, Vergnaud (1983) categorizes the various multiplicative problems and situations into subtypes. The three main subtypes of multiplicative structures that he identifies are "*isomorphism of measures, product of measures, and multiple proportion other than product*" (p. 128). I now turn to discuss the first subtype, *isomorphism of measures* as seen in Figure 3, in order to look at its subclasses:

- multiplication,
- type 1 division (partition),
- type 2 division (quotation),
- the calculation of the fourth term.

I discuss these subclasses of the first subtype as they are the most connected to the early development of multiplication and division.

**Figure 3**

*Overview of Subtype, Isomorphism of Measures within the Conceptual Field of Multiplicative Structures (Vergnaud, 1983)*




### *Multiplicative Situations: Isomorphism of Measures Subclasses*

The multiplicative structure subtype, *isomorphism of measures* "consists of a simple direct proportion between two measure spaces $M_1$ and $M_2$" (Vergnaud, 1983, p. 129) with isomorphism meaning "a one-to-one correspondence between two mathematical sets" (Merriam-Webster, n.d.). The relationship between multiplication and division within this consists of four subclasses of problems as seen in Table 8.

**Table 8**

*Vergnaud's Subclasses for the Category Isomorphism of Measures Problems (1997, p. 23)*

| Subclass | Sample problem | Known and unknown quantities |
|---|---|---|
| Multiplication<br>$M^1$ \| $M^2$<br>1 \| a<br>b \| ☐ | There are 4 bags of candy and each bag has 5 candies. How many candies are there altogether? | Multiplication<br>Bags \| Candies<br>1 \| 5<br>4 \| ☐ |
| Type 1 Division (partition)<br>$M^1$ \| $M^2$<br>1 \| ☐<br>b \| c | There are 20 candies. If the same number of candies go in each bag and there are 4 bags, how many candies go in each bag? | Type 1 Division (partition)<br>Bags \| Candies<br>1 \| ☐<br>4 \| 20 |
| Type 2 Division (quotition)<br>$M^1$ \| $M^2$<br>1 \| a<br>☐ \| c | There are 20 candies and 5 candies go in each bag. How many bags are there? | Type 2 Division (quotition)<br>Bags \| Candies<br>1 \| 5<br>☐ \| 20 |
| Calculation of the Fourth Term<br>$M^1$ \| $M^2$<br>a \| b<br>c \| ☐ | 5 candies weigh 80g. If there are 400g of candy, how many pieces of candy do I have?<br>*Note that unlike the other classes, none of quantities is the number 1 | Calculation of the Fourth Term<br>Grams (g) \| Candies<br>80 \| 5<br>400 \| ☐ |

*Note*. This table has been adapted from "The Nature of Mathematical Concepts" by G. Vergnaud in T. Nunes and P. Bryant (Eds.), 1997, *Learning and Teaching Mathematics: An International Perspective,* p. 23. Copyright 1997 by Psychology Press. Reprinted with permission.

The problems for Multiplication, and Type 1 and 2 Division are based on a problem adapted from Schwartz (1988).

Each of these subclasses involves determining the unknown fourth quantity when three other quantities are known. I will use a variation of the problem, *"There are 4 bags of candy and each bag has 5 candies. How many candies are there altogether?"* adapted from Schwartz (1988) to illustrate these subclasses.

Vergnaud (1983) defines *multiplication* as finding the total number of candies (p. 124), *type 1 division (partition)* as finding the number of candies for each bag (p. 130), and *type 2 division* as finding the number of bags (p. 133) (see Table 8). These three subclasses involve the same situations and the same numbers only the unknown varies. These subclasses are also all similar in that one of the four terms or quantities has a value of 1 as part of the problem context (p. 133). The fourth subclass, *calculation of the fourth term*, however, is different in that none of the four terms in the problem equals 1 (1983, p. 133, 1997, p. 23) as seen in Table 8. The relationship is 5 candies per 80 grams, without a quantity of 1 in the problem structure. This fourth subclass contrasts with the definition set out by Hulbert et al. (2017, p. 5) that multiplicative reasoning is based on a many-to-one relationship. Instead, this is a many-to-many relationship. The challenge of these four subclasses, as outlined by Vergnaud (1997), will continue to evolve as children explore higher numerical values, ratios, complex proportions, fractions, and decimals (p. 24). Vergnaud investigated the robustness of his theory of conceptual fields by conducting research in classrooms.

Vergnaud (1983) was part of a team of researchers who in 1978 for a span of 4-5 years conducted what he referred to as *experiments* (p. 140). One of the experiments involved 25

students in each grade from Grades 6-9. He examined how the students solved multiplication and division problems of the subclass, *calculation of the fourth term* in problems having a scalar or function ratio. By examining how students solved the problems in the various grades, Vergnaud asserts that the progression of development of multiplicative reasoning is slow, and further, that some students may still not be able to solve simple multiplicative situations by the end of early secondary school. This finding further highlights the challenge of developing multiplicative reasoning and Vergnaud's (1997) contention that multiplicative situations are different than additive ones, and each requires attention in a students' learning of mathematics. A summary of how Vergnaud differentiates between the two conceptual fields of additive structures and multiplicative structures can be seen in Table 9.

**Table 9**

*A Comparison of Vergnaud's Conceptual Fields: Additive Structures and Multiplicative Structures*

| Characteristic of conceptual field | Additive structures | Multiplicative structures |
|---|---|---|
| operations | addition and subtraction | multiplication and division |
| quantity relationship | ternary relationship | quaternary relationship |
| structures | initial relationships:<br>• part-part-whole<br>• state-transformation-state<br>• comparison | • isomorphism of measures (multiplication, division (partition), division (quotation), calculation of the fourth term)<br>• product of measures<br>• multiple proportion other than product |

Schwartz (1988, 1996), the second crucial theorist delineating the development of multiplicative reasoning agrees with Vergnaud's contention that additive and multiplicative

reasoning are quite distinct: however, Schwartz differs in a significant way from Vergnaud. Schwartz argues that the multiplicative relationship is still ternary rather than quaternary; instead, a new unit is created that defines this multiplicative relationship in one quantity rather than two separate quantities. I turn now to Schwartz's theory.

## Schwartz: Framework of Adjectival Quantities

Schwartz (1996) begins his theoretical description by arguing that if the purpose of school education is to prepare learners to understand the world around them, then it is difficult to comprehend why the focus of the learning is not on the meaning of quantity but instead working with numbers that do not have meanings attached to them (p. 6). Schwartz's theory on the development of multiplicative reasoning is based upon the stance that understanding quantity is essential to learning mathematics: he described this in *Semantic Aspects of Quantity* (1996), an unpublished paper that he had worked on for almost three decades (p. 4). It is important to note Schwartz's theory is not empirical as he did not conduct research on his theoretical ideas with students in a classroom.

### *The Difference between Additive and Multiplicative Reasoning*

Whereas Vergnaud explores the development of additive and multiplicative reasoning based on the relationships of how the quantities interact with each other, which are usually ternary for additive structures and usually quaternary for multiplicative structures, Schwartz's theory focuses on the meaning of the number (in everyday situations or word problems), defined as the *referent*, and it is this that dictates how to solve the problem and the meaning of the operations. To understand the importance and role of the referent, I turn to Schwartz's discussion of the meaning of numbers in word problems.

**The Adjectival Quantity.** A number, Schwartz (1996) states can be used as a noun or as an adjective. When it is used as a noun, such as *five*, this is called a *nominal quantity* which is a bare number without a referent. When a number is used as an adjective, describing a noun, such as *five books*, this is called an *adjectival quantity* as the quantity has a referent (Schwartz, 1988, p. 41). Schwartz (1996) emphasizes that there is a false assumption "that all the manipulations and operations that are appropriate to nominal quantity are also appropriate to adjectival quantity" (p. 7). This is particularly evident in operations. For example, a child could add the bare numbers 4 + 5 without referents to get 9; however, if the referents were 4 mm and 5 cm, this would be a very different calculation and final sum.

Schwartz (1996) identifies adjectival quantities, which are used in everyday life and in word problems, as having two distinct types, *discrete* and *continuous*. A discrete quantity uses a *count noun*. When using a count noun, the number 1 can be used with the referent. An example of this would be 1 book. The naming of a discrete quantity can be articulated as {cardinality of set, definition of set}[1] (p. 9) such as {5, books}. A continuous quantity on the other hand, uses a *mass [measurement] noun*, meaning that the number 1 cannot be used with the referent, such as 1 water (p. 8). With a mass noun, the quantity needs to be further defined with a unit of measure. The written structure for a continuous quantity would be {(magnitude, unit), attribute}. For example, an hour of driving time would be {(1, hr), driving time} (p. 9). These distinctions of quantities are important when examining operations and computations as the referents describing the number quantity impact how an answer is calculated.

Whereas Vergnaud (1997, p. 22) argues that repeated addition can act as a precursor to understanding multiplication (even though it has a functional relationship), this is not the case

[1] Schwartz uses these types of brackets to define the set, including number and meaning

with Schwartz. Schwartz (1988, 1996) identifies a definite break between the operations of addition and subtraction versus multiplication and division because the referents in these operations are distinctly different. The operations are categorized based on how the referent is dealt with either as *referent preserving* operations or *referent transforming* operations.

**Referent Preserving Operations.** Referent preserving operations describe the operations of addition and subtraction that Schwartz (1988) defines as "composing two like quantities to produce a third like quantity" (p. 41). For example, adding discrete adjectival quantities {3, apples} + {6, apples} = {9, apples} (Schwartz, 1996, p. 16), can easily be compared to adding of the nominal quantities, 3 + 6 = 9. Having said this, the comparison of adding nominal quantities is not as easily transferred when adding continuous adjectival quantities, where a unit of measure is also required. For example, to add 4 m +1 cm, a (multiplicative) conversion is required to make the referents the same; one cannot simply add 4 + 1 to get the correct answer. Therefore, with discrete and continuous quantities, the referents are important considerations even in referent preserving operations as seen in Table 10.

**Table 10**

*Comparison of Referent Preserving Operations and Referent Transforming Operations (Schwartz, 1988, 1996)*

| Characteristic of referent operations | Referent preserving operations | Referent transforming operations |
|---|---|---|
| Operations | addition and subtraction | multiplication and division |
| Types of quantities | extensive quantities | extensive and intensive quantities |
| Outcome of referent | referent stays the same after calculation | referent changes after the calculation |

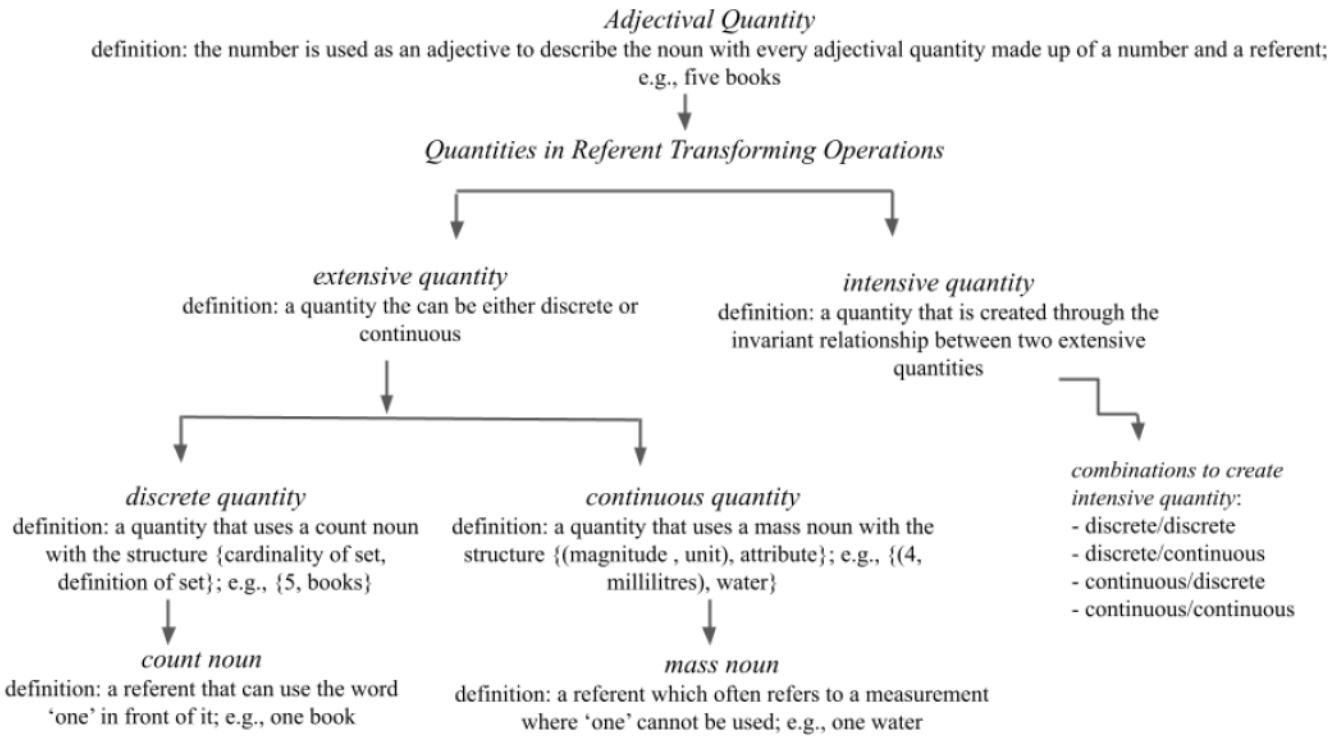
**Referent Transforming Operations.** Multiplication and division are referent transforming operations and are defined by Schwartz (1988) as, "composing two, either like or unlike, quantities to produce a third quantity that is, in general, like neither of the two original quantities" (p. 41) because it is instead transformed. For example, with 5 candies per bag x 4 bags = 20 candies, there are three different referents: candies per bag, bags, and candies. Furthermore, quantities can be described as *extensive* and *intensive* quantities (Schwartz, 1996, p. 22).

Schwartz (1988, 1996) explains that an extensive quantity can be either discrete or continuous, as described earlier when discussing adjectival quantities. An intensive quantity, on the other hand, is a quantity that is created through the invariant relationship between two extensive quantities (which are either discrete, continuous, or both) (Schwartz, 1988, 1996; Nunes & Bryant, 1996). An example of an intensive quantity would be 'candies per bag' (Schwartz, 1988, p. 46). The extensive quantity of a number of candies and the extensive quantity of a number of bags, combine together (are transformed) to form this new relationship of some number of candies per bag. See Figure 4 for an overview of the definitions for the

adjectival quantities in referent transforming operations. For Schwartz, the intensive quantity is key to the multiplicative relationship.

**Figure 4**

*Definitions Map for Adjectival Quantities in Referent Transforming Operations (Schwartz, 1988, 1996)*




### *Multiplicative Relationships – The Intensive Quantity*

The intensive quantity creates the distinction between additive relationships and multiplicative relationships, and is what distinguishes referent transforming from referent preserving operations. It is a significant part of why word problems with multiplicative structures are much more difficult for children to solve. Schwartz (1988, p. 42; 1996, p. 22) indicates that the intensive quantity cannot be counted or measured because it is a relationship. For this reason,

Schwartz states that this quantity is challenging to represent (1988, p. 43). The use of the word 'per' indicates the intensive quantity as a connection of the two extensive quantities, such as there are five candies per bag, although 'per' does not always need to be used (1988, p. 42). For example, there are five candies in each bag.

An earlier reference to the meaning of the intensive quantity comes from Piaget (1965) who defines an intensive quantity as, "the name given to any magnitude which is not susceptible of actual addition, as for example temperature. Two quantities of water at 15° and 25° respectively do not produce a mixture at 40°" (p. 244). This example illustrates that because of the referent, these two numbers cannot simply be added together. Schwartz expands this definition, by explaining that the intensive quantity is formed by various combinations of discrete and continuous quantities as seen in Table 11.

**Table 11**

*Intensive Quantities Formed in Relationship with Two Extensive Quantities (Schwartz, 1988, p. 45)*

| Extensive quantity relationships | Abbreviation | Examples |
|---|---|---|
| two sets of discrete extensive quantities | D/D | children/family<br>candies/bag |
| a set of discrete and a set of continuous extensive quantities | C/D<br>D/C | gallons/bucket<br>persons/year |
| two sets of continuous extensive quantities | C/C | miles/hour<br>grams/cubic centimeter |

The intensive quantity that Schwartz describes parallels Vergnaud's constant coefficient in that both connect two variables and can be identified by the word 'per.' There are differences, however, in how these quantities are created and their placement in a multiplication equation.

Vergnaud (1994, p. 47) explains the constant coefficient as the connection between two variables, which is found in the horizontal unary operation between two columns as seen in Table 12. The formation of the intensive quantity on the other hand is more explicit in Schwartz's theory (1988), who describes the different ways the intensive quantity can be created through the combination of two extensive quantities (see Table 11). He places the new relationship of the intensive quantity at the start of the multiplication equation and labels it 5 candies per bag.

**Table 12**

*A Comparison Chart of the Constant Coefficient and the Intensive Quantity*

| Characteristic of functional relationship | Theorists: Vergnaud (1983, 1988, 1994, 1997) | Theorists: Schwartz (1988, 1996) |
|---|---|---|
| Name of quantity | constant coefficient | intensive quantity |
| Definition | combination of two variables e.g., candies per bag | combination of two extensive quantities e.g., candies per bag |
| Formation of quantity | BAGS \| CANDIES<br>1 \| 5<br>4 \| □<br>x 5 | 5 candies (ext. quantity) 1 bag (ext. quantity)<br>5 candies per bag (intensive quantity) |
| Placement of quantity in multiplication sentence | 4 x 5 = 20<br>constant coefficient | 5 x 4 = 20<br>intensive quantity |
| Functional relationship of quantity | relationship is implied and "hidden" between columns | relationship is explicit in combination of two extensive quantities |

*Note*. The above examples are based on the problem, "There are 4 bags of candy and each bag has 5 candies. How many candies are there altogether?"(problem adapted from Schwartz, 1988)

Vergnaud (1983, p. 128) describes multiplicative structures as having a quaternary relationship. Schwartz (1988, 1996) on the other hand describes multiplicative situations as part of the multiplicative structure of the *semantic triad* combining the extensive and intensive quantities as discussed next.

### *Multiplicative Situations: The Semantic Triad*

Schwartz (1988, 1996) defines multiplicative situations within the multiplicative structure of a semantic triad. He views most multiplicative situations as grouped under one of three different semantic triads:

- intensive quantity (I), extensive quantity (E), and a second extensive quantity ($E^1$) (I E $E^1$),
- three different intensive quantities, (I $I^1$ $I^{II}$), or
- three different extensive quantities, (E $E^1$ $E^{II}$).

I examine each in turn with particular attention to the first triad, I E $E^1$ as this triad covers the most multiplicative situations that children will encounter at the primary school level.

**The Semantic Triad I E $E^1$.** In the first semantic triad, as Schwartz (1988, 1996) conceives of it, the multiplication operation expressed as I x E = $E^1$ incorporates the commutative property (although Schwartz does not discuss how children might begin to understand the commutative relationship) and the corresponding division operations are expressed as $E^1/E = I$ and $E^1/I = E$ (1988, p. 48). Continuing with the example adapted from Schwartz (1988), "There are 4 bags of candy and each bag has 5 candies. How many candies are there altogether?", the three different referents are candies per bag (I), number of bags (E) and number of candies ($E^1$) as seen in Table 13.

**Table 13**

*Examples of Meaning of Adjectival Quantities in the Semantic Triad* I E $E^1$ *(Schwartz, 1988, 1996)*

| Type of quantity | Abbreviation | Example |
|---|---|---|
| Intensive | I | candies/bag |
| Extensive | E | bags |
| Extensive | $E^1$ | candies |

The multiplication equation for this problem would be 5 candies per bag x 4 bags = 20 candies. The inverse relationships would exist for the division operations. The semantic triads for the three operations can be seen in Table 14. The semantic triad and referents demonstrate the connection between multiplication and division.

**Table 14**

*Semantic Triad I E $E^1$ for the Three Operations (Schwartz, 1988, 1996)*

| Semantic triad | Operation | Examples with referents | Unknown to be found |
|---|---|---|---|
| I x E = $E^1$ | multiplication | 5 candies/bag x 4 bags = 20 candies | How many candies ? |
| $E^1$/E = I | division (partitive) | 20 candies ÷ 4 bags = 5 candies/bag | How many candies per bag? |
| $E^1$/I = E | division (quotative) | 20 candies ÷ 5candies/bag = 4 bags | How many bags? |

It is important to note that there are two exceptions in this semantic triad I E $E^1$ that vary from the referent structure described above (Schwartz, 1988, p. 49). This first exception occurs when converting units (e.g., 2 meters to 200 centimeters) as the referent does not change, for example, the length of the bookshelf, but how the measure (unit) is described numerically does

change. The second exception is similar in that the referent does not change but the magnitude of that measure does change. Schwartz (1988) defers to Vergnaud (1983) to describe this situation through the use of a *scalar* with the assumption that the scalar does not have a referent. For example, "when one says that the weight of a second grader is {52.5, lb, body weight} and that the weight of an eighth greater is twice as much," (Schwartz, 1988, p. 49), the weight of the second grader can be multiplied by 2. Schwartz (1988, p. 50) argues that that it would be complicated to consider a referent for the 2 as being 2 pounds per pound but that there are scalar situations when not having a referent makes it less clear as to what the calculation should be. An example of this is asking someone to make a copy of a picture that would be twice as big. In this situation, Schwartz (1996) wonders if this refers to the length or the area (p. 34) and therefore wonders if there even needs to be a distinct multiplicative situation that uses a scalar, without a referent. In addition to the semantic triad, I E $E^1$ and its special cases, there are two other semantic triads.

**Semantic Triads I $I^1$ $I^{II}$ and E $E^1$ $E^{II}$.** The second semantic triad Schwartz identifies consists of three different intensive quantities, I $I^1$ $I^{II}$. Schwartz provides an example of how the intensive quantities relate in a multiplication situation, I x $I^1$ = $I^{II}$ as seen in Figure 5. In this case there are three different intensive quantities miles/gallon (I) x gallons/hour ($I^1$) = miles/hour ($I^{II}$).

**Figure 5**

*Example of I x $I^1$ = $I^{II}$ Triad (Schwartz, 1988, p. 51; 1996, p. 35)*

{(33.3 mi/gal), fuel efficiency of car on trip} x {(1.5, gal/hr), car's rate of burning fuel on trip} = {(49.95, mi/hr), average speed on trip}

The third semantic triad consists of three different extensive quantities. The structure involves Cartesian products such as determining area or the number of outfit combinations. Schwartz

(1988) provides an example of a multiplication situation for E x $E^1$ = $E^{II}$ as seen in Figure 6. As can be seen from the examples in Figure 6, there are two different extensive quantities that result in a third extensive quantity that is different from the other two. Both of these triads have related division structures. Word problems involving the two multiplicative structures of the semantic triads I $I^1$ $I^{II}$ and E $E^1$ $E^{II}$ happen later in a student's development of multiplicative reasoning.

**Figure 6**

*Examples of Semantic Triad E, $E^1$, $E^{II}$ (Schwartz, 1988, p. 51)*

(3, blouses) x (5, skirts) = (15, outfits)

(3.2, cm, width of rectangle) x (6.4, cm, length of rectangle) = (20.48, $cm^2$, area of rectangle)

In summary, Schwartz's conceptualization of multiplication and division as semantic triads reflects the complexities of multiplicative structures when one identifies the referents. Of the three theorists examined, the theories of Vergnaud and Schwartz are based on how the functional relationship is established in multiplicative situations. The third theorist, Steffe, who has also had significant influence in the field, would agree that there is a shift in children's thinking that supports the transition from additive to multiplicative reasoning: however, in his theory, how children develop this understanding is very different than outlined by the other two theorists.

Steffe's theory of development of multiplicative reasoning focuses instead on the evolution of the increasing sophistication of children's understanding of number. His theory does not consider the referents in a quantity as they appear in word problems. It is nonetheless essential to understand his work in order to have a full scope of the research and breadth of how multiplicative reasoning develops in children.

**Steffe: Framework of Number Sequence Trajectory and Coordination of Units**

Steffe's theory contrasts with the theories of Vergnaud and Schwartz. For Steffe (1988), the development of multiplicative reasoning is rooted in counting. Each increasing level of counting and unit coordination a child constructs supports their transition and shift from counting to additive reasoning, then to multiplicative reasoning (p. 122) versus the development of multiplicative reasoning emanating from a functional relationship. As a result, Steffe's theory has been characterized as a theory of multiplication developing from repeated addition (Carrier, 2014, p. 88) although the development is not isolated to repeated addition alone (Park & Nunes, 2001, p. 764).

There is a theoretical tension here however, as many researchers state that the operations of multiplication and division do not simply evolve from the operations of addition and subtraction (Hulbert et al., 2007, p. 2; Nunes & Bryant, 1996, p. 142; Schwartz, 1988, p. 46). Confrey (1994, p. 291), for example, argues that when learning multiplication, there are too many approaches that "rely too exclusively on a model of multiplication as repeated addition with an underlying basis in counting" and cites Steffe (1988), among other researchers, as part of this group. Steffe (1994) confronts this claim by arguing that he agrees with Confrey that multiplication should not *only* be based on repeated addition but states nonetheless that it is important to understand how children coordinate units from counting (p. 22). Before repeated addition can be used with meaning, *the composite unit*, as an iterable unit must be understood (Steffe, 1988, p. 135). Steffe claims that multiplicative reasoning evolves from additive reasoning, which emerges from counting based on the coordination of units within the *number sequence trajectory*.

### *The Difference Between Additive and Multiplicative Reasoning: The Number Sequence Trajectory and the Coordination of Units*

Steffe's framework is based on the evolution of learning number that children go through, called the "number sequence" (Steffe, 1988, p. 122). A trajectory of how children construct number and early counting schemes, including number sequences, as outlined in Table 15, were formulated from the *Interdisciplinary Research on Number Project* by Steffe et al. (1983). This trajectory evolves to describe the formation of units from a *perceptual counting scheme* towards a *generalized number sequence* (Olive, 2001, p. 4; Steffe, 1988, p. 122-123; Wilkins & Ulrich, 2017, p. 10) delineating how the shift occurs from additive to multiplicative reasoning. Within these stages of the number sequence, there are increasingly sophisticated modifications of the unit that children are constructing. A *unit* can be defined as an individual count (Ulrich, 2015, p. 3; Steffe, 1994, p. 14). The evolution of the unit is part of how children transition from one number sequence to another and Ulrich (2015, 2016) further labels this as levels of *coordination of units*. See Table 15 for an overview of the number sequences, the development of the composite unit, and the coordination of the unit.

**Table 15**

*Overview of Steffe's Number Sequences, the Composite Unit, and the Stages of Unit Coordination*

| Number sequence | Initial Number Sequence (INS) | Tacitly Nested Number Sequence (TNS) | Explicitly Nested Number Sequence (ENS) | Generalized Number Sequence (GNS) |
|---|---|---|---|---|
| composite unit | numerical composite | composite unit | abstract composite unit | iterable composite unit |
| coordination of units (Ulrich 2015, 2016) | one level of coordination | coordinating two levels of units | coordinating three levels of units | assimilatory coordinating, three levels of units and beyond |
| stages of reasoning (Ulrich, 2016, p. 38) | counting | early additive; early precursor to multiplicative | additive; precursor to multiplicative | multiplicative |

This focus on the unit (which are nominal quantities) is in stark contrast to Schwartz (1996) who disagrees with the idea that children's mathematics is built on working with nominal quantities (numbers that do not have a referent). This is because in Schwartz's view the operation that is required in a situation is *dependent* on the referent. Steffe's theory, however, offers a different take on the construction of the composite unit. He views this as a necessary component needed for the understanding of multiplication and division with a deeper focus on the nuances of this development and transition from counting that Schwartz does not offer. Steffe's focus and framework help to elucidate the subtle changes that occur in children's early development of multiplicative reasoning. Each stage in the number sequence provides useful detail on where children are at in their construction of the unit, with a unit being a child's understanding of what they are counting as 1 item (Ulrich, 2015, p. 3).

Olive (2001), Ulrich (2015, 2016), and Wilkins and Ulrich (2017) have conducted research to further amplify and clarify the number sequence and composition of units that are reflected in the following discussion.

### *The Evolution of Counting: Details of the Number Sequence*

Steffe's number sequence begins with early counting stages, progressing to additive and multiplicative reasoning. Each stage will be described in turn.

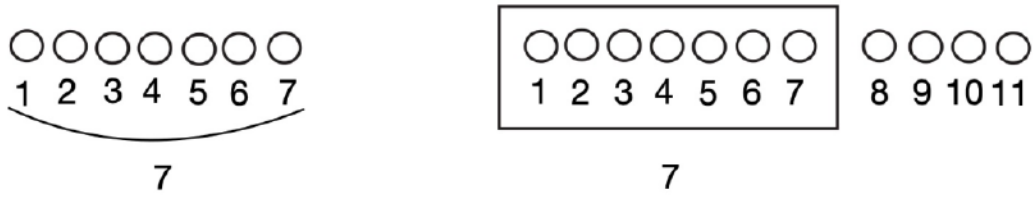
**The Number Sequence: Counting Schemes.** Counting is the beginning stage of the number sequence and Steffe (1988) breaks down the acts of counting into even smaller stages than what was outlined earlier. Before getting to the initial number sequence (INS), children will first go through the *perpetual counting scheme*, and then the *figurative counting scheme* (p. 122). The perceptual counting scheme is the stage when children can count a visible set of objects accurately. A child transitions to the figurative counting scheme when they can recreate the count of a set of objects with their fingers or with a drawing when the set of objects is covered and not visible. If more objects were added to the set however, the child would need to recount the objects from the beginning (Olive, 2001, p. 5). After children have constructed these initial counting schemes, they can navigate the next stages in the number sequence which include the Initial Number Sequence (INS), the Tacitly Nested Number Sequence (TNS), the Explicitly Nested Number Sequence (ENS), and the Generalized Number Sequence (GNS) (Steffe, 1992). These will be described in turn based on the construction of the composite unit and the coordination of the units.

**The Initial Number Sequence (INS).** When children can internalize the count and use a numeral to represent what the count would be, they are entering the Initial Number Sequence (INS) (Steffe, 1994, p. 4). Steffe uses the example with marbles hidden in cups, "There are seven

marbles in this cup (rattling marbles in the cup). Here are four marbles (placing four marbles in another cup). How many marbles are there in all?" (p. 14). A child in INS would be able to use the 7, as a *numerical composite* (Olive, 2001, p. 6; Steffe, 1988, p. 125; Wilkins & Ulrich, 2017, p. 10) resulting from the single-counting act of the "counting sequence" up to 7 (Olive, 2001). This means that the student may still need to imagine and count out the hidden "7" in order to be able *count on* 7: 8, 9, 10, 11 (Olive, 2001) as seen in Figure 7.

**Figure 7**

*Initial Number Sequence: Using 7 as a Numerical Composite and Counting On (Olive, 2001; Steffe, 1988; Ulrich, 2015)*




Another example that is indicative of a child in the INS stage is that they can count by twos but would not be able to keep track of how many twos as the child is still in a counting act (Olive, 2001, p. 6.). The numerical composite, in both examples, is based on a counting act, with no coordination with any other units. Ulrich (2015) refers to this as *one level of coordination* (p. 4).

**The Tacitly Nested Number Sequence (TNS).** In the Tacitly Nested Number Sequence (TNS), a child has constructed a *composite unit* i.e., a set of units that are grouped together, and this newly formed group is considered a unit, or as Steffe (1994) defines it, "a unit that itself is composed of units" (p. 15). Lamon (1994) refers to this as *unitizing* (p. 92). When a child understands a composite unit, they can count on by using a double count. Continuing with the

example of marbles, 7 marbles are now a composite unit of 7, and a child can keep track of 4 more marbles in conjunction with the use of the number sequence where 8 is 1, 9 is 2, 10 is 3, and 11 is 4 as seen in Figure 8 (Ulrich, 2015, p. 6). This action further develops additive reasoning but is still one level of coordination of units (Ulrich, 2015, p. 6).

**Figure 8**

*Example of Double Counting within Tacitly Nested Number Sequence (Ulrich, 2015, p. 5)*




*Note*. This figure had been adapted from "Stages in Constructing and Coordinating Units Additively and Multiplicatively (Part 1)" by C. Ulrich, November 2015, *For the Learning of Mathematics, 35*(3), p. 5. Copyright 2015 by the FLM Publishing Association. Adapted image reprinted with permission.

Children progress from one level of coordination of units to coordinating two levels of units when they can keep track of how many times a composite unit is repeated, which can be done in the form of skip counting. For example, the action of keeping track of how many twos within a set of 12 hidden objects can be seen in Figure 9 (Olive, 2001, p. 6). A child can keep track of each set of two numbers.

**Figure 9**

*Tracking of a Composite Unit within Tacitly Nested Number Sequence (Olive, 2001, p. 6)*

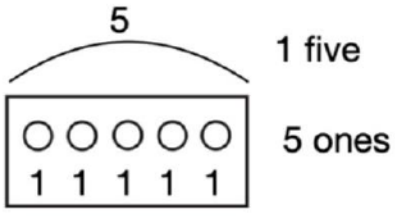
1, 2; 3, 4; 5, 6; 7, 8; 9, 10; 11, 12
1 2 3 4 5 6

Children at this stage enter an early understanding of multiplicative reasoning as they can keep track of a composite unit which they cannot do when in the initial number sequence (Olive, 2001, p. 7). It is nonetheless still a counting act as the composite unit is not yet abstract as it starts to be in the Explicitly Nested Number Sequence.

**The Explicitly Nested Number Sequence (ENS).** The Explicitly Nested Number Sequence (ENS) further builds toward the development of multiplicative reasoning as children exit out of the counting stage and can work with an *abstract composite unit* (Steffe, 1988). In other words, children understand that the composite unit is an iteration of single units as seen in Figure 10.

**Figure 10**

*Visual Representation of an Abstract Composite Unit: 5 Individual Ones Make 1 Five*



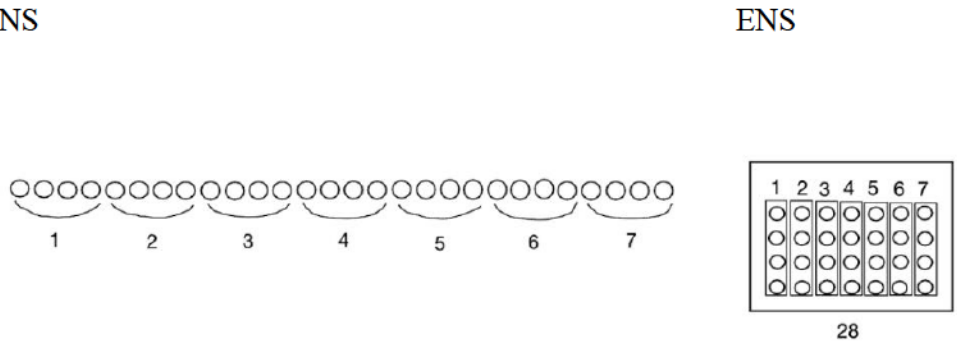

The abstract composite unit is different because there is a simultaneity children must exercise to understand that there are individual units that make the composite unit and the composite unit

contains these individual units. For example, 5 ones can be thought of as 1 five. At this point, the child has *interiorized* these two levels of units (Hackenberg & Tillema, 2009, p. 3), allowing them to work with the composite unit with reasoning strategies versus just a counting act.

In ENS, a child can begin to use *coordination of three levels of units* (Ulrich, 2016). For example, considering the expression 7 x 4 as seen in Figure 11, a child in the previous TNS stage would be able to count by ones and track or skip count the composite unit of 4 seven times using two levels of coordination  When a child has achieved the ENS stage however, they can think of the whole as 28 single units, and simultaneously as 7 composite units with 4 inside each of these units using three levels of coordination (Ulrich, 2016, p. 34).

**Figure 11**

*Representation of 7 x 4 at the TNS and ENS Stages (Ulrich, 2016, p.  34)*




*Note*. The TNS stage shows how the composite unit is kept track of. The ENS stage demonstrates 28 individual units, 7 composite units, with 4 individual units inside each of those composites.

While children can use this coordination of three levels in a situation through action they have not yet interiorized this coordination (Hackenberg & Tillema, 2009) as they are not able to

visualize the commutative property, reworking the units as composite units i.e., 4 composite units with 7 individual units inside each, or 4 x 7. This requires one more stage in the number sequence.

**The Generalized Number Sequence (GNS).** Children have achieved the Generalized Number Sequence (GNS) stage when they can move flexibly within the coordination of the three units and have, therefore, constructed an *iterable composite unit* (Olive, 2001, p. 8). This is a greater level of abstraction as the units are assimilatory, meaning that there is flexibility in being able to navigate amongst all three levels of units (Ulrich, 2016, p. 35) which is necessary for proportional reasoning (Ulrich, 2016, p. 38).

In summary, the number sequence establishes a framework for a developmental trajectory of how children construct the composite unit that evolves from one stage to the next. It is through the changes in the understanding of the composite unit that children progress to multiplicative reasoning (Steffe, 1992, p. 305).

***Multiplicative Relationships: The Evolution of the Composite Unit***

In Steffe's (1988, 1992, 1994) theory, multiplicative relationships are developed through the evolution of the construction and coordination of the composite unit leading to multiplicative reasoning at the GNS stage of the number sequence. To get to this stage, children's understanding of the composite unit increasingly advances based on the level of abstraction and interiorization of the composite unit as seen in Table 16. Hackenberg (a doctoral student of Steffe) and Tillema (2009), describe this as *multiplicative concepts*. These multiplicative concepts describe the difference between a child's coordination *in activity*, meaning that they still need to think and sort through how to use the composite unit to solve a problem versus when they have internalized it, and can work more flexibly mentally.

**Table 16**

*Levels of Whole Number Multiplicative Concepts in Connection to the Number Sequence (Hackenberg & Tillema, 2009, p. 3; Ulrich, 2016)*

| Number sequence | Multiplicative concept | Abbreviation (Ulrich, 2016) | Description (Hackenberg & Tillema, 2009, p. 3) |
|---|---|---|---|
| Initial Number Sequence (INS) | none | none | no multiplicative concepts being developed yet |
| Tacitly Nested Number Sequence (TNS) | first multiplicative concept | MC1 | coordination of two levels in activity |
| Explicitly Nested Number Sequence (ENS) | second multiplicative concept | MC2 | interiorization of coordinating two levels of units; coordination of three levels of units in activity |
| Generalized Number Sequence (GNS) | third multiplicative concept | MC3 | interiorization of coordinating of three levels of units |

*Note*. This table had been adapted from "Stages in Constructing and Coordinating Units Additively and Multiplicatively (Part 2)" by C. Ulrich, March 2016, *For the Learning of Mathematics, 36*(1), p. 38. Copyright 2016 by the FLM Publishing Association. Table adapted and reprinted with permission.

To build his theory and further solidify the stages of the number sequence trajectory, Steffe (1992) conducted research through individual *sessions* with students. During these sessions, he designed multiplicative situations to determine each student's stage of development within the trajectory.

***Multiplicative Situations: Developed During Individual Sessions with Students***

Steffe examined a small sample of children over an extended period of time to explore how children construct the composite unit in its various stages through the lens of the number sequence trajectory. He conducted a three-year study from 1985-1988 with six eight-year-olds, with two children in each stage of INS, TNS, and ENS. There were 50 teaching sessions over the three years with each child individually for approximately 30 minutes (Steffe, 1992, p. 262). Each session was designed for that child, giving them multiplicative situations to solve based on their thinking as viewed through the lens of "schemes of action and operations involving composite units" (Steffe, 1992, p. 259). A scheme, as defined by von Glasersfeld (1980, p. 82) consists of three parts: 1) an experience that invites the child into the situation, 2) an action or activity the child applies to the situation, 3) result of the action.

The focus of the study was to investigate how young children coordinated and segmented the unit (Steffe, 1992, p. 264). Steffe carefully observed children when they were solving situations involving blocks so he could better understand how children make changes in their understanding of the unit. While Steffe designed individual situations that he deemed appropriate for each's child stage of development, his situations focused on units, not referents. For example, Steffe (1992) asked a child to consider how they would figure out, "how many times he would count if he counted a pile of twelve blocks by three" (p. 267), without actually having blocks available to count. Without focusing on the referents in these situations, Steffe limits his theory of development of children's thinking to individual units or unit of units.

Steffe's (1992) multiplicative situations, including both multiplication and division, are a result of the construction of the composite unit, without a focus on referents (p. 259) which limits the depth and scope of multiplicative situations that are explored in his theory. The inherent

functional relationship is a missing piece in Steffe's theory. Steffe's intentional focus on how children coordinate units in the early stages is nonetheless, a useful part of the puzzle to understand how children begin to reason multiplicatively. All three theorists offer a perspective to consider when supporting children's development.

## Vergnaud, Schwartz and Steffe: Where They Converge and Diverge

The common goal and motivation for each of these three theorists is to provide a better mathematical experience for children in their education at school. For Vergnaud (1988), understanding the conceptual fields support us in understanding how children develop multiplicative reasoning over an extended period of time (p. 142). For Schwartz (1996), in order for children to connect to mathematics as part of their world, quantities need to have a referent as opposed to learning operations without referents and therefore devoid of meaning (p. 6). Finally, for Steffe (1988), it was important to observe how children authentically develop changes in counting, building on children's thinking, not knowledge imposed on children by adults (p. 122). At the heart of the motivation for these researchers of mathematics are children.

Each of their theories offers insights into children's development of multiplicative reasoning along with questions to continue to ponder. There are two main themes that offer both similarities and differences amongst these three theorists: (1) an identified multiplicative quantity, and (2) the role of simultaneity (the ability to keep track of two quantities or two types of units at the same time) in the construction and use of these multiplicative quantities.

### *Multiplicative Relationships from Vergnaud, Schwartz, and Steffe*

Vergnaud, Schwartz, and Steffe each identify and define a quantity or a unit that contributes to the shift from additive to multiplicative reasoning as seen in Table 17. For Steffe, the *composite unit* is what supports evolution of multiplicative relationship from counting,

whereas Vergnaud's *constant coefficient* and Schwartz's *intensive quantity* are based on a functional relationship; hence, the reason for the two broad hypotheses, schema of correspondence theory and the implicit theory as outlined by Park and Nunes (2001) earlier.

**Table 17**

*Comparison Chart for Multiplicative Relationships from Vergnaud, Schwartz, and Steffe*

| | Hypotheses (Park & Nunes, 2001) | | |
|---|---|---|---|
| Characteristics of multiplicative relationship | schema of correspondence | schema of correspondence | implicit theory |
| Name of theorist | Vergnaud (1983, 1988, 1994, 1997) | Schwartz (1988, 1996) | Steffe (1988, 1992, 1994) |
| Name of quantity or unit | constant coefficient | intensive quantity | composite unit |
| Definition | combination of two variables<br>e.g., candies per bag | combination of two extensive quantities<br>e.g., candies per bag | composite unit created by individual units |
| Formation of quantity or unit | BAGS \| CANDIES<br>1 \| 5<br>4 \| □<br>x 5 | 5 candies (extensive) 1 bag (extensive)<br>5 candies per bag (intensive quantity) | 5 individual units making one 5 unit<br> |
| Placement of quantity or unit in multiplication equation | 4 x 5 = 20<br>constant coefficient | 5 x 4 = 20<br>intensive quantity | 4 x 5 = 20<br>one level of composite unit inside another level of composite unit |
| Multiplicative relationship of quantity or unit | relationship is implied and "hidden" between columns | relationship is explicit in combination of two extensive quantities | relationship is based on the coordination of two levels of units |

*Note*. The examples above are based on the question, "There are 4 bags of candy and each bag has 5 candies. How many candies are there altogether?"(adapted from Schwartz, 1988). It is

important to note that Steffe's composite unit does not need the context as the composite unit is composed of individual units without an explicit focus on the referent.

Each quantity or unit requires a degree of *simultaneous* coordination by the student, i.e., the ability to keep track of two quantities or two types of units at the same time. How the quantity or unit is constructed is different among each of these three theorists and therefore the degree of *simultaneity* that is involved. For Steffe, simultaneity is framed around the interconnectedness of various units that do not have a referent, for Vergnaud it is part of this relationship between two quantities that is implied as seen in Table 17, and for Schwartz it is the creation of a new single quantity.

### *The Role of Simultaneity in the Development of Multiplicative Reasoning for Each Theorist*

Simultaneity is a marker of the development of multiplicative reasoning, "when children are able to think *simultaneously* about units of one and units of more than one" (Clark & Kamii, 1996, p. 43). Consideration of the subtle nuances in how simultaneity is interpreted by all three theorists is useful when evaluating the comprehensiveness of their theories. One difference is how the referent impacts simultaneity. The functional relationship for both Vergnaud and Schwartz is the relationship between two distinct quantities such as candies and bags. For Steffe, the composite unit is created without the intentional use of referents. In order for children to develop multiplicative reasoning, all three theorists agree that there needs to be a degree of simultaneity to support the shift from additive to multiplicative reasoning.

## Two Hypotheses, Three Theorists, and Implications for Student Learning

We have examined three critical theorists using the framework of Park and Nunes' (2001) two hypotheses of how students develop an understanding of multiplication: the implicit

theory (Steffe) and the schema of correspondence theory (Vergnaud and Schwartz). These two theoretical learning pathways have implications for how students develop multiplicative reasoning. Within the framework of Park and Nunes (2001) hypotheses, we now examine the research and literature on: (1) the role simultaneity has in establishing multiplicative relationships, and (2) the placement of the multiplicative quantity or unit in the writing of an equation, as both have implications for classroom instruction and student development.

***Research Studies within the Implicit Theory: The Role of Simultaneity***

Researchers have explored how the development of the composite unit with increasing level of abstraction and coordination, i.e., simultaneity, effects students' development of multiplicative reasoning. Building on Steffe's framework, Wilkins et al. (2020) conducted a three-year longitudinal study from 2012-2015 of 5747 children, beginning in grade 2 in a school in Colorado. They found that students who were at an Initial Number Sequence (counting-on) or Tacitly Nested Number Sequence (working with the composite unit) stage in grade 2, had a better chance of developing multiplicative reasoning by grade 4 than those students in grade 2 who had "high rote computational skills" (p. 529) (i.e., able to identify numbers and put together different addition pairs of numbers to make 10) but had not yet established an INS. Similarly, Tzur et al. (2010, p. 5) in their research with 12 students (9 in grade 5 and 3 in grade 4), in the Midwestern US, found that multiplicative reasoning could not be developed until children had constructed the abstract composite unit—i.e., the composite unit is an iteration of single units. This is the 'conceptual spring' that supports the shift from additive to multiplicative reasoning (p. 9).

Zwanch and Wilkins (2021) wanted to further investigate Tzur et al.'s (2010) identification of the 'conceptual spring' in their research with 326 students in grades 6 to 9, in

the rural Southeastern US. Their research indicated that if children cannot get to the stage of an advanced Tacitly Nested Number Sequence ( aTNS), by the end of grade 6, it was unlikely that multiplicative reasoning would develop in the middle grades at all (p. 166). When students are at an aTNS stage, they can mentally use the composite unit; for example, the student can see "21 as three *groups of* seven" [emphasis added] (p. 157). While the aTNS is still an additive stage, having this level of coordination or *simultaneity* makes it possible for them to develop multiplicative reasoning at the ENS stage. This echoes Steffe's (1992) conclusions that "constructing the ENS" (p. 305) (Explicitly Nested Number Sequence) is essential if children are to begin to reason multiplicatively. These studies build on Steffe's framework and further emphasize the importance of the formation of the composite unit, notably, the ability to coordinate units with units *simultaneously* to reason multiplicatively.

Lawson, in longitudinal research in Ontario, Canada from 2006 – 2012 (A. Lawson, personal communication, August 2023) identified that when children are able to keep track of the composite unit and the number of iterations at the same time, that this indicates a *transition to simultaneity*. With this transition students begin to reason multiplicatively without counting (Lawson, 2021). The role of simultaneity development is also evident in research conducted within the schema of correspondence theory.

***Research Studies within the Schema of Correspondence Theory: The Role of Simultaneity***

Research has been conducted on the focus of the multiplicative relationship amongst quantities (numbers with referents), i.e., the simultaneous interaction between quantities, including the construction of the intensive quantity. Askew et al. (2019) conducted a study with nine classes, three classes in each grade from grades 1-3, in a suburban area of Johannesburg in South Africa, to explore the development of multiplicative reasoning using Vergnaud's theory as

the framework. The mathematics intervention to support the development of multiplicative reasoning occurred over four weeks with one lesson per week. The intervention was co-designed by the researchers and the teachers and included problem solving with partners, whole class discussions, individual problem solving and the use of array images and images of the array to support multiplication and division facts. The contexts, for example: "cooking pots and [pot] legs, people and feet, boxes and eggs" (Askew et al., 2019, p. 4), were chosen to make the multiplicative relationship of the ratio, i.e., the *simultaneous* relationship, explicit. The "pre- and repeated post- and delayed post-tests included all three types of multiplicative reasoning simple proportion problems: multiplication, grouping and sharing" (Askew et al., 2019, p. 6). The delayed post-assessment found that children sustained growth made from the pre- and post-assessment, however, there was not as much evidence of children using more sophisticated strategies; instead, children were still using counting strategies in grade 3.

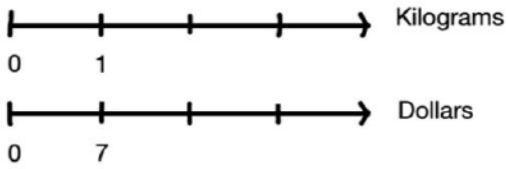
Simon and Placa (2012) wanted to think about how the intensive quantity (a *simultaneous* relationship between two quantities), which is not emphasized in the school curriculum in the United States, could be developed for whole number multiplication and division to better support children in developing multiplicative reasoning. They conducted a "theoretical exploration" (p.35) of how to develop the intensive quantity with children in grade 3 by using measurement quantities, focusing on the intensive quantities as being derived from two continuous extensive quantities. They wanted to move away from 'per one unit' situations to see if children could develop more of a multiplicative relationship and not rely on repeated addition strategies, for example by solving 3 x 4 as 3 + 3 + 3 + 3.

To support this development, they considered using the double number line as a model (p. 39). Simon and Placa (2012) gathered that even with specific attention to the construction of

the intensive quantity, children would still be able to solve problems without using a functional relationship. For example, the researchers thought they could ask, "if 1 kg of chicken costs $7, what does 6 kg (or 2.5 kg) cost?" (p. 39). The hope was that children would be able to use the double number line as a model, as seen in Figure 12, to find out the cost of 6 kg or 2.5 kg of chicken (p. 39). They thought, however, that children would still be able to reduce this situation to a referent preserving operation by adding $7 to represent each number of kilograms that were needed. For example, to figure out 6 kg or chicken, the child could add $7 for each kilogram, $7 + $7 + $7+ $7 + $7 + $7, thereby avoiding the continuous functional relationship, i.e., the *simultaneous* relationship, that exists between the two quantities represented on the number lines.

**Figure 12**

*Double Number Line Model to Support the Functional Relationship Between Two Quantities*




*Note.* The image is from "Reasoning about Intensive Quantities in Whole-Number Multiplication? A Possible Basis for Ratio Understanding," by M. A. Simon & N. Placa, 2012, *For the Learning of Mathematics, 32*(2), p. 39. Copyright 2012 by the FLM Publishing Association. Reprinted with permission.

Simon and Placa (2012) concluded that children would still be able to solve these measurement problems without using multiplicative reasoning, because "it is unlikely that a

model can be created that necessitates reasoning about intensive quantities for whole number multiplication and division problems" (p. 40).

In Japan on the other hand, the development of intensive quantities and the understanding of quantities as outlined by Schwartz (1988) is part of their curriculum. Yamanoshita and Matsushita (1996) highlight that in Japan, the Association of Mathematical Instruction indicates that repeated addition "is a way to calculate multiplication, not a meaning of it "(p. 291) because they view multiplication as a functional relationship. In the earlier grades, the development of the intensive quantity begins with creating the relationship between two discrete extensive quantities—i.e., the intensive quantity. The example given is a picture of rabbits without ears playing in a field. The children have to figure out how many ears are needed as there are "2 ears per rabbit." Children are also supported to think about intensive quantities that occur naturally such as "6 legs per beetle" (p. 292). Thus, students learn to work with the simultaneous coordination of quantities.

The importance of simultaneity and its role in the development of multiplicative reasoning is evident in both the implicit and the schema of correspondence theories. The research findings outlined here indicate how the ability to work simultaneously with more than one quantity or unit influences the development of multiplicative reasoning. How students work with the quantities or units together to solve multiplicative situations is communicated through an equation. How does the structure of an equation foster the understanding of the simultaneous interaction between the quantities or units to solve a multiplicative situation?

**Multiplicative Relationships as Reflected in the Writing of Equations**

There are differences in how the multiplicative relationship of the unit or quantity is written in the equation for each theorist.

### *The Implicit Theory: The Writing of Equations*

Steffe's (1988, 1992, 1994) theory that multiplicative reasoning in an extension of counting and addition is developed through an increasingly complex composite unit. Ulrich and Wilkens (2017) conducted research to analyze how students in grade 6 solved the problem, "You have baked 39 cupcakes and you will put them into boxes of three. How many boxes will you fill? " (p. 3). They gave the example that students at an ENS stage could solve this problem and "would be able to operate on their 13 *groups of 3s* to add or subtract boxes" [emphasis added] (Ulrich & Wilkens, 2017, p. 4). This example demonstrates how Steffe's theory of using units within units is similar to a 'groups of' model.

A 'groups of' instructional model, which is what is typically used in North America, emphasizes first the group and second what is inside the group. This is reinforced within the structure and communication of the equation for a multiplication situation with the number of groups written first, and what is inside the group second (Carpenter et al., 2015, p. 50; Hulbert et al., 2017, p. 3) as seen in Table 18. Van de Walle et al. (2022, p. 161) suggest that a suitable time to introduce the multiplication symbol is when children are communicating multiplication using repeated addition. The equation is read as 4 *groups of* 5 equals 20. How does this differ from Vergnaud and Schwartz, particularly with their focus on the functional relationship of the quantities?

**Table 18**

*The Written Order of Multiplication Equations*

| Model | 'Groups of' model | Intensive quantity model |
|---|---|---|
| Equation | 4 x 5 = 20 | 5 x 4 = 20 |
| General description | (groups) x (what's inside the group) = total of what's inside | (intensive quantity) x extensive quantity (1) = extensive quantity (2) |
| Description based on the problem: *"There are four bags of candy and each bag has five candies. How many candies are there altogether?"* (problem adapted from Schwartz, 1988) | 4 groups of 5 | 5 candies per bag x 4 bags (4 times) |
| References | Carpenter et al., 2015, p. 50; Hulbert et al., 2017, p. 3; Van de Walle et al., 2022, p. 161 | Schwartz, 1988, p. 46 |

### *The Schema of Correspondence Theory: The Writing of Equations*

Vergnaud (1983) gives us some indication of how to establish the functional relationship with his focus on multiplicative structures as a quaternary relationship. With Vergnaud's theory, the functional relationship lies within the ratio table, between two measurement spaces, and is not the construction of a new unit. Vergnaud's writing of the equation is similar to Steffe in that the quantity or unit that has the multiplicative relationship i.e., the constant coefficient or the composite unit is written second.

Schwartz, however, is the most explicit in the development of the functional relationship with 1) the creation of the intensive quantity, and 2) the placement of the intensive quantity written first in the multiplication equation as seen in Table 18. This varies from Vergnaud and

Steffe. Recording the intensive quantity first in the equation focuses in on the functional relationship (e.g., 5 candies per bag) at the outset.

Furthermore, writing the intensive quantity first builds on how children naturally solve early multiplication problems as evidenced in Lawson's (2015) research of 60 Grade 1 and 2 students. In a review of the interview videos of children's solutions when solving multiplication problems prior to instruction, she found that children consistently thought of the composite unit (intensive quantity) first and then the iterations second (A. Lawson, personal communication, July 22, 2022). Therefore, writing it in that order, builds on how children actually think about the situation.

In my review of the research and the literature, the role of simultaneity and the emphasis on the multiplicative quantity or unit when writing of equations provide insights into how to support students in developing multiplicative reasoning. Schwartz, however, offers the clearest direction on how to establish the multiplicative relationship of the intensive quantity and hence provides ideas for instruction.

**What Next and Why Schwartz?**

In summary, the two hypotheses of implicit theory and schema of correspondence theory offered a lens through which to view the three theorists: Vergnaud, Schwartz, and Steffe. From the review of the literature and the research and in consideration of the construction of the multiplicative relationship of a specified quantity or unit, one can intuit that the ability to simultaneously coordinate (simultaneity) impacts the development of multiplicative reasoning. While repeated addition is a strategy that can be used to solve multiplication problems, it is not sufficiently robust to develop and delineate the functional relationships needed for multiplicative reasoning. Under the schema of correspondence theory, Schwartz most clearly lays out the path

of multiplicative reasoning through the explicit creation of the intensive quantity, which is a functional relationship in and of itself, a new quantity that cannot be separated. This development is supported by distinguishing the operations of addition and subtraction as being referent preserving operations from multiplication and division as being referent transforming operations—not operations that merely extend from addition and subtraction. The implications for children learning through the lens of Schwartz's theory have not been extensively researched, which is the purpose of my dissertation research. The research is framed around the developmental nature of multiplicative reasoning.

**Theoretical Framework: Learning as Developmental**

When conducting research about how children construct multiplicative reasoning the key theory or lens that underpins my methodology and my analysis is that children's learning and construction of multiplicative reasoning is developmental. The development of mathematical concepts can be communicated using continuums or learning trajectories. There are learning trajectories that have been developed by researchers to illustrate the growth of multiplicative reasoning over time, as well as delineating instruction to support the development. While there are a range of definitions of learning trajectories, Clements and Sarama (2004) define learning trajectories as:

> Descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in the mathematics domain. (p. 83)

Simon (1995) refers to this as a *hypothetical learning trajectory* as it is based on what a teacher anticipates that they might see (p. 135) and there is "consideration to the learning goal, the learning activities, and the thinking and learning with which students might engage" (p. 133).

Gravemeijer (1999) describes this trajectory as *local instruction theory* which is a process of research to determine an *emergent model* that supports the development of mathematical concepts. It begins with an initial set of instructional lessons that are then explored and refined in response to what is happening in the classroom to determine "an optimal instructional sequence" (p. 156).

What is consistent with each of these definitions and descriptions of learning trajectories is the iterative process of anticipating what students will do and implementing lessons to continuously refine a pathway that is supportive of a child's mathematical development. In the area of multiplicative reasoning in particular, Steffe's (1992) number sequence trajectory discussed earlier is a first example of the iterative process of instruction, assessment of learning, and refinement of tasks. Wilkins and Ulrich (2017, p.7) have built on this trajectory to examine specific aspects such as the development of the composite unit through to skip-counting. The progression begins with a focus on manipulatives: first individual cubes, to inter-locking cubes, pre-grouped rods where you can still see the individual partitions, to solid rods that do not have any partitions. This instructional scaffolding supports the development of the composite unit from seeing individual counts to working with the composite unit more holistically and flexibly. The number sequence trajectory (Steffe, 1992) was also used in recent research by Zwanch and Wilkins (2021) to measure children's growth in multiplicative reasoning. The researchers investigated children's location on the number sequence trajectory to explore what may support or hinder a child's shift to multiplicative reasoning in the middle years (p. 151).

Continuums and trajectories offer other layers of support to educators including a deeper understanding of the mathematics that underpins the concepts as well as pedagogical strategies. Lawson's (2015, 2021) student continuum of numeracy development of multiplication and

division articulate strategies, key ideas, and overall phases that children may navigate as they develop their thinking. Her most recent work (2026) is a second example of a full trajectory of lessons designed to support students' multiplicative progress along a developmental path.

The OGAP Multiplicative Framework is a third example of a trajectory that involves both progressions and problem contexts and structures that need to be considered as children are constructing multiplicative reasoning (Hulbert et al., 2017, p. 17). Supovitz et al. (2021) explored how the development of multiplicative reasoning was supported when educators used a learning trajectory, established by the Ongoing Assessment Project (OGAP) (Hulbert et al., 2017). Children in Supovitz et al.'s study were given various multiplication and division problems and tasks to solve. The measures that were used to assess children's thinking were two rubrics, one for levels of accuracy and one for levels of sophistication. The use of a rubric based on levels of sophistication, which considered "both strategy level and the presence of conceptual errors" (p. 455), allowed them to note growth or lack of growth between strategies moving toward multiplicative reasoning. The strategies were scored using "a 0-5 scale of 0: inappropriate or nondiscernable strategy, 1: early additive strategy, 2: additive strategy, 3: early transitional strategy, 4: transitional strategy, and 5: multiplicative strategy" (pp. 454-455).

These examples of trajectories and continuums from the field highlight children's construction of multiplicative reasoning as developmental. Children are provided with greater equitable access to mathematics when they are given the opportunities to construct their thinking and the learning is built upon where they are in their development (Carpenter et al., 2017, p. 147) as the mathematics is based on their learning needs. It is through this view and philosophy of learning as developmental that this dissertation was conducted, in both the methodology and the analysis.

# Chapter 3: Methodology

## Design of the Study

This study was part of an ongoing larger multi-year project entitled, *The Multiplicative Pathway for Junior/Intermediate Students*, led by principal investigator Alex Lawson, project staff, research assistant Heather Wark, and myself. My specific contributions to the broader project are illustrated in Figure 13. My literature review expanded the project's collective understanding of how researchers and theorists understand and conceptualize the development of multiplicative reasoning in children. Furthermore, by leading the mathematics interventions in four of the seven research classrooms, I provided consistent feedback on the lesson development. Finally, the coding scheme I created for my own study provided information to inform the coding of student thinking for the project at large.

**Figure 13**

*Overview of Research Project, The Multiplicative Pathway for Junior/Intermediate Students*



My study was designed to answer the following question and sub question:

*(1) How does an instructional focus on the meaning of the referents and the development of the intensive quantity in multiplicative situations affect children's early development of multiplicative reasoning?*

*(1a) What is the role of simultaneity in the development of multiplicative reasoning in grade 3 students?*

To answer my research question and sub question, I employed a qualitative case study. Creswell and Poth (2018) define this method as:

> A qualitative approach in which the investigator explores a real-life, contemporary bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving *multiple sources of information* (e.g., observations, interviews, audiovisual material, and documents and reports) and reports a *case description* and *case themes*. (p. 96, italics in original)

My study focuses on a single case, a third-grade classroom where I implemented a four-week mathematics intervention. Specifically, I sought to investigate how children develop multiplicative reasoning when instruction is rooted in Schwartz's (1988, 1996) theory. My dual role as both educator and researcher is a common practice in mathematics education, supported by established scholars such as Askew (2018), Simon (1995), and Steffe (1992).

This research design is what Stake (1995) defines as an *intrinsic case study* because it involves "a research question, a puzzlement, a need for general understanding, and feel[ing] that we may get insight into the question by studying a particular case" (p. 3). Similarly, this research design aligns with Yin's (2018) conception of a *descriptive case study* "whose purpose is to describe a phenomenon (the "case") in its real-world context" (p. 286). In my study, the focus was on the development of multiplicative reasoning in a group of grade 3 students, providing an in-depth examination of their thinking within a real-world school context. Both of these

definitions capture the methodology needed to effectively answer my research question and sub question, offering a nuanced understanding of children's cognitive processes in an authentic educational environment.

**Participants**

The participants were students in a grade 3 class at a K-6 elementary school in a separate schoolboard in Northwestern Ontario. This grade was chosen because multiplication and division are formalized within the *Ontario Mathematics Curriculum* (2020). This purposeful sample (Creswell & Poth, 2018, p. 100) consisted of a class that had students with a wide range of academic abilities, selected in consultation with the superintendent of education responsible for mathematics.

There were 23 students in the class (12 male and 11 female). Of these, 22 students had permission to participate in the study, including 19 who had consent for video recording and 3 who had permission for data collection without video recording. After accounting for one student on a modified school day and not present during the mathematics classes and one student who was absent for the duration of the project, the final sample size consisted of 21 participants who were eight-or-nine years old.

The students in this class had a diverse range of backgrounds, including some with identified special education needs. Identifiable demographic data was not used as part of the methodology to prioritize the individual learning pathways of each child. However, one student was identified as an English Language Learner, as it was challenging to evaluate whether it was the language that impacted the response to assessment items versus the development of the mathematics concepts themselves.

**Measurement Instruments for Pre- and Post-Assessments**

Measurement instruments for the research study consisted of the set of pre- and post-assessment problems that have been utilized as part of the larger research project, *The Multiplicative Pathway for Junior/Intermediate Students*. An overview of the questions can be found in Appendix A. All students completed the initial part of the assessment. Subsequent items were presented only if the student solved the initial problems without relying on counting strategies. Students were asked a minimum of eight basic multiplication facts and continued until they consecutively answered three or four questions incorrectly or needed to use extensive calculations.

Posing problem solving tasks to children is consistent with established research in elementary mathematics (Carpenter et al., 2015; Clark & Kamii, 1996; Lawson, 2015; Steffe, 1992). Problem interviews allow children to respond in a variety of ways such as the use of counting, additive reasoning, or multiplicative reasoning, which provide researchers with insight into their mathematical thinking. The problems focused mainly on three primary multiplicative situations: multiplication, quotative division, and partitive division (Carpenter et al., 2015, p. 55). To allow for a direct comparison of students' strategy development over time, the pre- and post-assessments utilized identical questions, consistent with methodologies used in similar studies (Askew et al., 2019, p. 7; Torbeyns et al, 2006, p. 450).

**Data Collection Procedures**

The University Research Ethics Board approved the research project in March 2022 (Appendix B). An amendment was subsequently approved to include the next research classroom in the dissertation research (Appendix C). Research ethics forms were distributed by the principal, and the classroom teacher monitored their return. The teacher followed up with

students who had not yet returned their forms to gauge their interest in participating; if necessary, the teacher contacted parents or guardians by phone to address any questions or concerns. The participation letters and consent forms for students, parents/guardians, the principal, and the teacher can be found in Appendices D through G. Table 19 provides an overview of the research timeline.

**Table 19**

*Overview of the Research Study Timeline*

| Date | Research Study Action |
|---|---|
| January, 2024 | Consulted with superintendent to determine the research classroom<br>Initiated contact with the school principal |
| February, 2024 | Extended a formal invitation to the classroom teacher<br>Conducted a follow-up discussion with the principal |
| April 15, 2024 | Finalized research ethics forms<br>Coordinated with the principal and the classroom teacher regarding the delivery and distribution of ethics forms to students |
| April 16, 2024 | Delivered research ethics forms to the school<br>The principal distributed parent and student research ethic forms to students in the class |
| April 24–25, 2024 | Conducted pre-assessment interviews |
| May 6, 7, 8, 2024 | Continued pre-assessment interviews |
| May 6–31, 2024 | Implemented mathematics intervention: Classroom learning |
| June 3–4, 2024 | Facilitated consolidation and practice sessions |
| June 10–12, 2024 | Conducted post-assessment interviews |

Following the case study research design, it was imperative to collect "multiple sources of evidence" (Yin, 2018, p. 126) to analyze patterns and themes in children's early development of multiplicative reasoning, specifically the influence of simultaneity. The data collected are

organized into three categories: (1) pre- and post-assessment interviews; (2) mathematics intervention; and (3) field notes. A summary of data collection is provided in Table 20.

**Table 20**

*Summary of Data Collection*

| Data Collection Type | Description |
|---|---|
| Assessment Interviews (Pre/Post) | Interviews were video recorded to allow for detailed analysis. For students without video consent, I performed a simulation interview (and verified by the teacher when present). Jot notes were taken during all sessions to allow for review of the interview. |
| Mathematics Intervention (Class Lessons and Student Artifacts) | Classroom lessons were video recorded to identify patterns and themes. Student artifacts, including written work and whiteboard images, were collected to provide evidence of individual developmental pathways. |
| Field Notes | Consistent notes were recorded before, during, and after each intervention to capture observations and evolving questions. A separate reflection was written after each lesson specifically documenting the students' development of simultaneity. (See Appendix H for sample lesson field notes and Appendix I for sample simultaneity field notes.) |

### *Pre- and Post-Assessment Interviews*

Pre- and post-assessments were conducted through individual interviews. Interviewing participants is a key component of case study research (Yin, 2018, p. 118), offering a deeper understanding of children's cognitive processes than traditional pencil-and-paper assessments. These interviews were video recorded, because as indicated by Ginsburg (1997, p. 127), the dynamic nature of researcher-participant interaction makes it difficult to accurately remember everything. Additionally, video recording ensures consistency during coding and analysis phases.

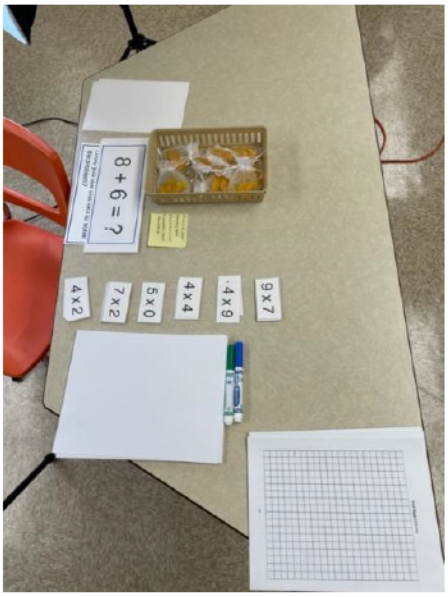
Pre-assessment interviews took place in a dedicated room at the school, with the classroom teacher present for the majority of the interviews to ensure a sense of familiarity for

the students. Following Lawson's research design (see Figure 14), I used two cameras: one to record the child, and another to record their hands and written work to clearly capture their calculation processes.

**Figure 14**

*Set-Up of Interview Space*






In preparation for the interviews, I reviewed notes from interviews I had conducted earlier in the research project based on my debriefs with Lawson. These notes consisted of specific interview probes (Creswell & Creswell, 2018, p. 191) that I could use to further elicit children's thinking (Ginsburg, 1997, p. 138). Before beginning the pre-assessment interviews,

the classroom teacher and I visited the class to discuss with the students what the space looked like so that they would be better prepared for what they would see when they entered the room. Once the student entered the room, I invited them to look through both cameras so they could see what the views looked like.

Each interview lasted approximately 20-30 minutes. During the interviews, jot notes were made using the assessment pages as found in Appendix A. The purpose of the jot notes was to provide a quick overview of which questions children solved and how they answered them. The teacher, when present during the interviews, was invited to record their own observations, which helped deepen their understanding of individual students. The pre-assessment notes supported the initial set-up of students in mathematics partners (similar academic ability and social compatibility) to foster purposeful learning conversations during instruction and to plan for differentiation.

The children were invited to solve the problems mentally or with the use of pencil and paper when they requested or through invitation by me as the interviewer. An overview of the number of interviews conducted each day can be seen in Appendix J. Lawson was present for the first pre-assessment interview to provide feedback on questioning and the process before the other interviews were conducted. For the word problems, I read the question out loud to the students and repeated the problem in a paraphrased version. A printed copy of the problem was placed in front of the student where they could refer to it if needed. This same process was followed when students were asked their multiplication facts.

As Ginsberg (1997, p. 130) articulates, it is important not to make assumptions about what the child is thinking. When asking the student questions, I would often repeat their answers in a neutral way and invite them to explain how they got their answer. During the interview, I did

not specify if the answer was correct or incorrect so that they had to provide a reason for their response to the question. If the student's response to the question was automatic, I did not always ask a follow-up question. Due to the length and cognitive demand of the of the interview, I had to be cognizant of when and how to ask questions to the students to ensure that they had the energy needed to think through each assessment item.

For those students who did not have permission to be video recorded, jot notes were made during the interview, both my myself and when present the classroom teacher. Following the interview, I made a simulation video where I talked through the student's responses to questions providing further documentation that I could later reference. There was one pre-assessment that I did not do a simulation video for as I established this process after that interview.

The interview process was consistent for all student participants except for one student. One student who had permission to be video recorded was overwhelmed with the camera set up. For this reason, I conducted the first interview in another room without the cameras. Due to this interruption, it was decided to repeat the interview, with myself present and the educator asking the questions to put the student at ease. This second interview was the one used for the analysis. For the post-assessment, the interview was conducted in a similar fashion, this time with me administering the interview without video recording, followed by a simulation video.

### *Mathematics Intervention and Data Collection*

I collected data during the mathematics intervention to further solidify my understanding of how students were developing multiplicative reasoning between the pre- and post-assessments. The 20-lesson intervention with grade 3 students, focused on the development of the intensive quantity and multiplication and division as *referent transforming* operations

(Schwartz, 1988, 1996). Given the lack of control group, it was important to try to identify learning that was not necessarily the result of a student's natural learning progression. Therefore, data identifying how students constructed the intensive quantity, their understanding of the referents, and how they solved multiplicative situations, was integral to the study.

This intervention, otherwise referred to as a *hypothetical learning trajectory*, the iterative process of designing, implementing, and refining a set of instructional tasks to support the development of a particular mathematics concept (Clements & Sarama, 1994; Simon, 1995, Gravemeijer, 1999), was embodied in my instruction of the draft lessons by Lawson and Wark (A. Lawson, personal communication, March 2022). My understanding of the existing literature on children's multiplicative development and effective instruction allowed me to translate theory into practice thereby exemplifying what Simon (1995) describes as the unique benefit of being both researcher and teacher (p. 121). As this was my fourth iteration of implementing the hypothetical learning trajectory, I was able to refine my approach and incorporate updates made to the original lessons since my previous implementation in May and June in 2023.

Instruction of the lessons began with students organizing a collection of objects (Franke et al., 2028) to physically create the intensive quantity in order to solve a multiplicative inquiry i.e., to determine some number of objects per container over some number of containers to find the total number of objects. Discussion about naming the intensive quantity and the referents was supported by playing games describing images, for example, 3 eggs per nest (intensive quantity), 4 nests (iterations), and 12 eggs (total). Furthermore, the meaning of numbers was emphasized when students solved multiplication problems and learned how to write an equation, labelling the intensive quantity and the referents. The shift to quotative division in Lesson 11 was done with an intentional connection to multiplication; rather than writing the equation as division, instead

students identified the iterations unknown in a multiplication sentence (5 x □= 15). Students also designed their own "thinking cards" to practice multiplication facts once meaning of the numbers in multiplicative situations was constructed. The pedagogy for the intervention focused on students solving problems with an intentionally matched partner (similar mathematical ability and social compatibility) and discussion centred around their thinking with some strategic direct instruction. An overview of these lessons (Lawson, 2026) can be seen in Appendix K.

I was the main instructor for the lessons. In my role as a Math Resource Teacher for many years, I had an existing professional relationship with the classroom teacher having worked in her classroom with her students over the last ten years. Additionally, I had worked with many of the students in their classroom the previous year, so they knew who I was and connected me with learning mathematics. For the majority of the lessons, the classroom teacher and a student support person were also in the room. The lessons were typically taught during the second instructional block from approximately 11:30 a.m.–1:00 p.m. There were times when the lesson would be first thing in the morning from 9:20 a.m.–10:00 a.m.

Lawson joined through Zoom for the majority of the lessons, even if only for part of the time, to witness and engage in how the students were thinking and learning. She also had the opportunity to speak with students who were willing to share their thinking or to do some practice with her. When possible, we would debrief the lessons reviewing what we noticed and discuss the pedagogical and instructional choices that were impacting children's mathematical development. We then collaborated on how to continue and modify the lesson with the students for the next day.

**Video Recordings of Classroom Lessons.** For each classroom lesson, a camera was set up at the back of the classroom. The video recording included a view of the front of the room

with the television that projected the slide-deck and the use of the document camera, the whiteboard beside the television, and backs of the students if they were sitting on the carpet. The purpose of the video recording was to capture the lesson, my pedagogical moves including questioning and scribing of student thinking, and students' responses in order to garner insights to how students were constructing their understanding of the intensive quantity and the referents in multiplicative situations.

**Student Artifacts.** Student artifacts were collected throughout the intervention. The purpose of collecting documentation of student thinking was to understand where to take the learning the next day. The gathering of student work documented each student's thinking overtime, providing another source of evidence to understand students' development of multiplicative reasoning. The student work that included pencil and paperwork tasks (e.g., independent problems, solving of a word problem, or equation writing when playing a game), were scanned and filed both electronically and as a hard copy along with images of their whiteboard work. I also made field notes throughout the intervention.

**Field Notes.** The field notes that I made during the intervention were twofold. The first set of notes I made were from the lessons and included an overview of the lesson, noticings and wonderings about student thinking, notes from debrief conversations with Lawson, and thoughts of where to take the learning the next day (see Appendix H). The second set of notes I made were my thoughts on what I was witnessing of how students were developing simultaneity. As a student researcher, and as part of the larger project, I created a draft hypothetical progression of the development of simultaneity with the intensive quantity in early multiplicative reasoning as seen in Table 21. This progression was based on the analysis of 90 videos consisting of pre- and

post-interviews of two assessment items from two of the research classrooms where I led the instruction prior to the present case study.

**Table 21**

*Draft Hypothetical Progression of the Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning*

| Direct Model → Mental Mathematics | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Strategy | no connection to the intensive quantity | subitizing the intensive quantity on a direct model, iterating and counting | drawing the intensive quantity first and then iterating | recording the intensive quantity as a number, then iterating | repeated addition with counting support | skip-counting, modelling intensive quantity with one hand and tracking with the other hand | skip-counting and keeping track of the intensive quantity | automatic |
| Further explanation | e.g., adding two numbers together<br>4 + 3 instead of 4 x 3 | | e.g., 4 candies per bag  | e.g., 4 candies per bag   | e.g., for 4 x 3 adding 4 + 4 = 8 and counting on the last 4 | using both hands – one to model the intensive quantity and the second to model the iterations | tracking the skip count on one hand | |
| Degree of simultaneity | No simultaneity → Simultaneity of intensive quantity → Simultaneity between intensive quantity and iterations | | | | | | | |

Using this progression as a template, I created a recording page (see Appendix I) to make notes on simultaneity throughout the mathematics intervention. The draft progression was modified and changed based on the analysis and coding of the data.

**Data Coding and Analysis**

The data coding and analysis were conducted in three different stages. The first stage, outlined in Table 22, established students' placement into three categories of multiplicative reasoning: Pre-Multiplicative, Beginning, and Developing. To determine this, I reviewed the number of pre-assessment items students answered along with the number of bare facts they were able to calculate. Because the number of assessment items varied based on individual

student success, I also examined students' responses to the additive question (8 + 6) to code for specific strategies: counting-all, counting-on, using up over 10, or automatic (see Table 22). This provided an initial snapshot of each child's mathematical development, allowing me to identify trends and themes that emerged specific to their entry level.

**Table 22**

*Stage 1: Determining Three Categories of Multiplicative Reasoning*

| **Stage 1:** Determining Three Categories of Multiplicative Reasoning | | | |
|---|---|---|---|
| Overview of assessment items: Interview questions and bare fact questions to determine students' multiplicative reasoning category | | | |
| Codes for 8 + 6 (Pre-) | Pre-Multiplicative: Counting-all | Beginning Multiplicative: Counting-on | Developing Multiplicative: Up over 10, Automatic |
| Students organized into multiplicative reasoning categories | | | |

Once children were organized into these three developmental categories of multiplicative reasoning, a second stage of data coding began with seven assessment items from the pre- and post-assessment interviews (see Table 23). The purpose of exploring these items was to examine strategy development, students' construction of the intensive quantity, their ability to identify referents, and their development of simultaneity.

**Table 23**

*Stage 2: Pre- and Post-Assessment Interview Analysis: Items and Coding*

| **Stage 2:** Pre- and Post-Assessment Interview Analysis: Items and Coding | |
|---|---|
| Structure of Pre- and Post-Assessment Items | Mathematics Concept |
| Seven assessment items: Pre- and post-assessment for all the multiplicative reasoning categories | additive item<br>subitizing and visual iterations (3 dots per plate x 4 plates)<br>subitizing and visual iterations (8 dots per plate x 2 plates)<br>multiplication problem with the visual support of a direct model<br>multiplication problem<br>quotative division problem<br>partitive division problem |
| Post-assessment analysis for referent identification and equation writing | multiplication problem<br>quotative division problem<br>partitive division problem |
| Additional items for students in Developing Multiplicative Reasoning Category | multiplicative comparison problem<br>double-digit multiplication problem |

I used the qualitative data analysis program ATLAS.ti to code the items. ATLAS.ti is a computerized card sorter that allows for tagging and infinite sorting based on codes that are designed by the user. I made segments of each assessment item from the student interviews which allowed me to code and analyze one question at time. The program allowed me to view videos of students with the same code to verify and examine the consistency of my analysis. For those students who were not video recorded, I put the simulation videos into ATLAS.ti so that I could code them as I did with the other videos and referred to the jot notes that were made during the interview.

Prior to my data analysis, I had set up categories for codes (Stake, 1995, p. 29) which were correctness (four levels), intensive quantity (three types), and strategy (12 types). I used an initial list of codes, some of which were based on strategies from Lawson's *Student Continuums*

*of Numeracy Development* (2015, 2021). During my coding I built upon this initial code list and created specific codes for each assessment item. The list of the initial codes can be seen in Table 24. A full list of codes that were used for the analysis can be seen in Appendix L.

**Table 24**

*Initial Codes for Early Development of Multiplicative Reasoning and Simultaneity*

| correctness | intensive quantity | strategy |
|---|---|---|
| correct | cannot create | counting by ones |
| correct with help | first | doubling |
| incorrect | second | direct model intensive quantity |
| incorrect added two numbers | | using fingers to track intensive quantity |
| | | repeated subtraction written |
| | | using fingers to support count |
| | | rhythmic counting |
| | | skip-counting |
| | | repeated subtraction mental |
| | | repeated addition |
| | | direct model subitizing intensive quantity |
| | | automatic |

To increase coding accuracy, Lawson and I viewed videos together to see if we would have the same outcome. For those videos about which I was uncertain, I conferred with Lawson, as an expert in coding student thinking to ensure that I was accurately coding the data.

In the third stage of the data coding and analysis, I built on what was initially gleaned from the pre- and post-assessments to investigate potential patterns and themes (Stake, 1995, p.

78) in children's early development of multiplicative reasoning and simultaneity. This was done by triangulating the data, which included reviewing and exploring intervention data including field notes, student artifacts, and the examination of parts of lessons from the classroom videos as seen in Table 25. For example, if I noticed in my field notes that students in the Pre-Multiplicative Reasoning stage were starting to clearly identify the referent in a problem context, I reviewed the video footage from the classroom lesson and examined the student artifacts for further clues of this development. This helped me notice if there was a pattern as "usually important meanings will come from reappearance over and over" (Stake, 1995, p. 78).

Finally, the data analysis included *pattern matching* which Yin (2018) defines as, "analyzing case study data by comparing or matching the pattern based on the collected data with a pattern defined prior to data collection (p. 287). I compared the patterns and themes that I noticed from the data against my draft hypothetical progression of *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning* (Table 21) and made changes to the progression based on this analysis.

**Table 25**

*Stage 3: Analysis of Classroom Data to Triangulate Word Problem Findings*

| **Stage 3:** Analysis of Classroom Data to Triangulate Word Problem Findings | |
|---|---|
| Triangulation of data (Stake, 1995, p. 107; Yin, 2018, p. 127) | to further triangulate and validate emerging patterns and themes on the development of multiplicative reasoning as gleaned from the assessments, intervention data including lesson videos, student artifacts, and field notes were intentionally reviewed and explored |
| Pattern matching (Yin, 2018, p. 175) | revision to my hypothetical progression of *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning* |

The data coding and analysis plan allowed me to focus specifically on how the students in the three identified multiplicative reasoning categories developed multiplicative reasoning from pre- and post-intervention.

# Chapter 4: Results and Analysis

As discussed in the last chapter, I analyzed data in three stages:

- Stage 1: Determining Three Categories of Multiplicative Reasoning;
- Stage 2: Pre- and Post-Interview Analysis: Items and Coding; and
- Stage 3: Analysis of Classroom Data to Triangulate Word Problem Findings

In Stage 1, I organized students into three categories of multiplicative reasoning. These categories were then applied in Stage 2 to analyze students' pre- and post-assessment responses. In Stage 3, I examined classroom data to further substantiate the findings identified in the assessment across the three multiplicative reasoning categories. This chapter presents the results and analysis for each stage.

## Stage 1: Determining Three Categories of Multiplicative Reasoning

The three multiplicative reasoning categories I organized students into were: *Pre-Multiplicative Reasoning*, *Beginning Multiplicative Reasoning*, and *Developing Multiplicative Reasoning*. The criteria I used to place students in these categories was: (1) the number of assessment items that the student answered, and (2) the strategy that the student used to answer the additive item, *8+6*. Results of the assessment items and strategies students used can be seen in Table 26.

**Table 26**

*Multiplicative Reasoning Categories: Initial Analysis*

| Category | Pre-Multiplicative Reasoning (purple) | | Beginning Multiplicative Reasoning (blue) | | Developing Multiplicative Reasoning (green) | |
|---|---|---|---|---|---|---|
| Strategy | incorrect | counting-all | counting-on | near doubles | up over 10 | Automatic |
| Number of students | 2 | 6* | 6** | 1 | 1 | 5 |
| Number of assessment items | 19 | | 20 | | 20 + (includes additional basic fact questions) | |

*Note*. For the following symbols:

* one student was asked additional basic fact items questions

** one student is an English Language Learner and answered all the basic facts items

The number of assessment items a child completed provided an initial indication of their level of understanding, particularly the number of basic facts questions they were able to answer. The interview ended if the student struggled with the initial eight facts that they were asked.

To further solidify my placement of students within the three multiplicative reasoning categories, I analyzed the strategy the student used to answer 8+6, based on the *Student Continuum of Numeracy Development: Addition and Subtraction* (Lawson, 2015, p. 4). The code for *automatic* was used when students were able to answer the question in three seconds or less in alignment with other mathematics research studies (Cowan et al., 2011, p. 789; Jordan et al., 2003, p. 838). The codes that I used for the analysis are described in Table 27.

**Table 27**

*Strategies for Additive Item 8 + 6*

| Strategy | Description of strategy |
|---|---|
| Counting-all | The student counted both the 8 and the 6, often drawing it out, and counted everything from 1 again to get the total amount. |
| Counting-on | The student counted on 6 more from 8, keeping track of the count, often on their fingers. |
| Near doubles | The student used the near double fact of 8 + 8 to help them solve 8 + 6. |
| Mental calculation | The student needed time to think and calculate their answer but could not explain their reasoning. |
| Up over 10 | The student decomposed the 6 into 2 + 4, to add 8 + 2 = 10, adding the remaining 4, 10 + 4 = 14. |
| Automatic | The student answered the question in 3 seconds or less. |

The analysis of an additive 8 + 6 item to determine student readiness for multiplicative reasoning, is consistent with recent research conducted by Tzur et al. (2021, p. 456). In their study of 31 fourth-grade students Tzur et al. categorized participants based on their strategy for solving 8 + 7: "pre-multiplicative" for those counting-on, "emerging" for those using doubling, and "developed" for those using Break Apart Make Ten (BAMT), also known as "up over 10" (Lawson, 2015). Unlike Tzur et al., who excluded students relying on more foundational counting strategies, my study includes all students, regardless of their level of counting proficiency.

Learning multiplication and division is part of the Ontario Elementary Mathematics Curriculum (Ontario Ministry of Education, 2020) for grade 3 and, therefore, it was important for me to examine all students' development of multiplicative reasoning to fully reflect a typical

classroom's range of developmental stages. My multiplicative reasoning categories therefore vary from Tzur's et al.'s (2021) study; in the current study, students in the counting-all stage were considered *Pre-Multiplicative Reasoning* (8 students); *Beginning Multiplicative Reasoning* (7 students) if they were counting-on or using near doubles; and *Developing Multiplicative Reasoning* (6 students) if students were able to use up over 10 or answer automatically.

To examine changes in students' multiplicative reasoning, I organized them into three categories based on their performance on the pre-assessment. This classification created the conditions for a more focused analysis on the subtle nuances of multiplicative reasoning development within each of these categories. Moreover, it enabled me to determine whether students in all categories benefited from the mathematics intervention. With these categories determined, I analyzed specific assessment items to identify patterns of development both holistically and within each category where appropriate.

**Stage 2: Pre- and Post-Interview Analysis: Items and Coding**

I examined seven items, as outlined in Table 28, from the pre- and post-assessment for children's early development of multiplicative reasoning and the role of simultaneity within this development—as outlined in the research question and sub question.

**Table 28**

*Seven Assessment Items for Analysis*

| Mathematics concept | Assessment item |
|---|---|
| Additive item | 8 + 6 |
| Subitizing and visual iterations (3 dots per plate x 4 plates) |  How many dots do you see? How many would there be if there were two plates? How many would there be if there were four plates? |
| Subitizing and visual iterations (8 dots per plate x 2 plates) |  How many dots do you see? How many dots would there be if I had two plates? |
| Multiplication problem with the visual support of a direct model | I have some candies in bags. There are 2 candies per bag and there are 8 bags. I'm wondering, how many candies are there altogether? |
| Multiplication problem | Joshua has packages of gum with 4 pieces per package. He has 3 packages. How many pieces of gum does he have? |
| Quotative division problem | You have 15 cookies. You are going to put them in bags with 3 cookies per bag. How many bags can you fill? |
| Partitive division problem | The teacher has 24 pencils that she wants to put equally into 6 containers. How many pencils should she put in each container? |

During the interviews, students were asked to solve each assessment item mentally. A marker and paper were provided only if requested or if the student needed support to get started. The total sample size item was $n = 21$ students. Data analysis for each assessment item is presented as follows:

1. Identification of the specific assessment item
2. The purpose of the assessment item
3. Description of the codes used to categorize student strategies
4. A map of students' pre- and post-assessment strategy development

5. An overall report of correct versus incorrect responses to both pre- and post-assessment
6. Analysis of strategy development within each multiplicative reasoning category (Pre-, Beginning, and Developing)
7. Additional item analysis where applicable

For word problems, further analysis was conducted on the post-assessment data regarding students' writing of equations and the understanding of referents. The presentation of the data analysis of equation writing and referent identification will be as follows:

8. Description of codes used for referents and equation correctness
9. Analysis of equation writing and referent identification within each multiplicative reasoning category (Pre-, Beginning, and Developing)
10. A summary of student equation-writing

The seven assessment items were analyzed for both correctness and strategy usage. Correctness was categorized using three codes: *correct, correct with help,* and *incorrect*. The *correct with help* code was used when the student's initial response was not correct but they arrived at the correct answer independently after receiving minimal probes (Creswell & Creswell, 2018, p. 191), such as repeating their answer or inviting them to check out their response again. The *incorrect* code was applied if further teacher intervention or extensive conversation was required to arrive at the correct answer.

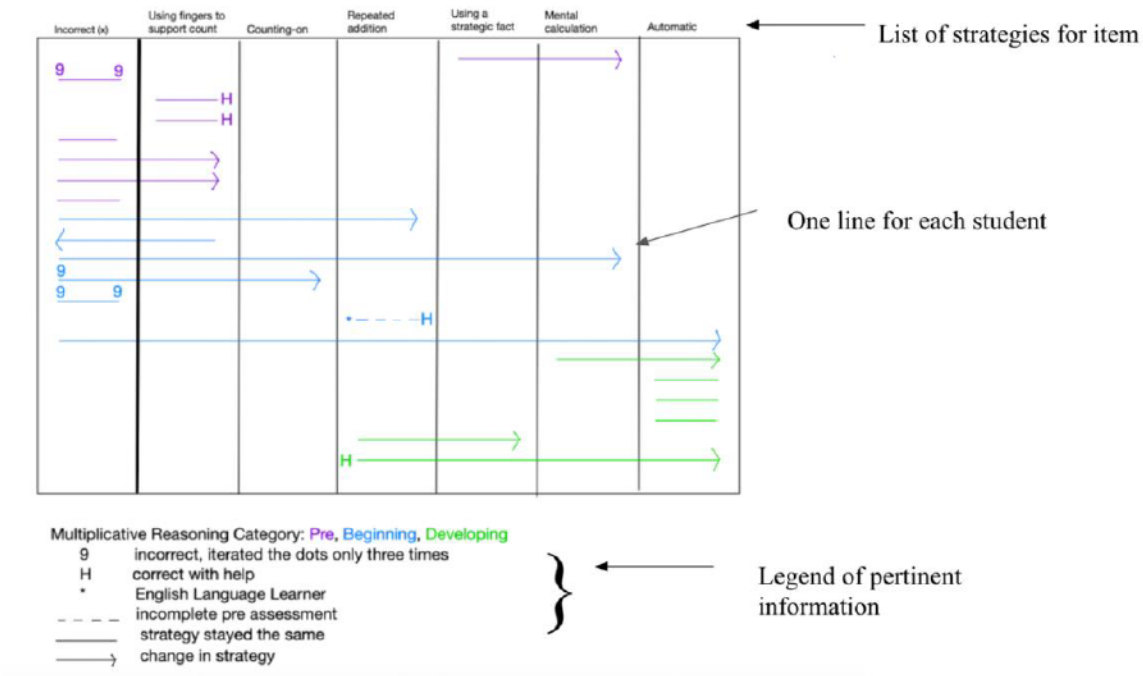
Codes were also used to indicate the students' strategies and the use of the intensive quantity. Some of the codes that I used across multiple items are consistent with the names of strategies as outlined in Lawson's *Student Continuums of Numeracy Development* (2015, 2021). In addition, I created item-specific codes for strategies that emerged only in particular

assessments. It is important to note that the list of strategies presented for each item is not exhaustive; it reflects only those strategies that this group of students used. I describe the strategy codes associated with each assessment item.

The data for each item are illustrated on a *Strategy Development Map* in order to examine development within each of the three-colour coded multiplicative reasoning categories. Each line reflects one student's development from pre- to post-assessment, appearing in the same order on each map as seen in Figure 15. The legend indicates the symbols used to mark additional pertinent information, including items where an inconsistent error is identified. I summarize the analysis for each item by highlighting key observations within the three multiplicative reasoning categories, providing overall observations on the development of multiplicative reasoning, based on where students were in their additive thinking and their early attempt at solving multiplicative situations.

**Figure 15**

*Overview Details of Example Strategy Development Map*




As discussed in Chapter 3, each of these items was part of a 20 to 30-minute interview with every student. As an interviewer, I had to make careful decisions about when and how I wanted students to expend their energy when explaining their thinking. The students worked very hard and were eager to share their ideas. As they got to know me as an educator during the mathematics intervention, they appeared more comfortable during the post-assessment. This section analyzes how students approached these seven items both before and after the intervention.

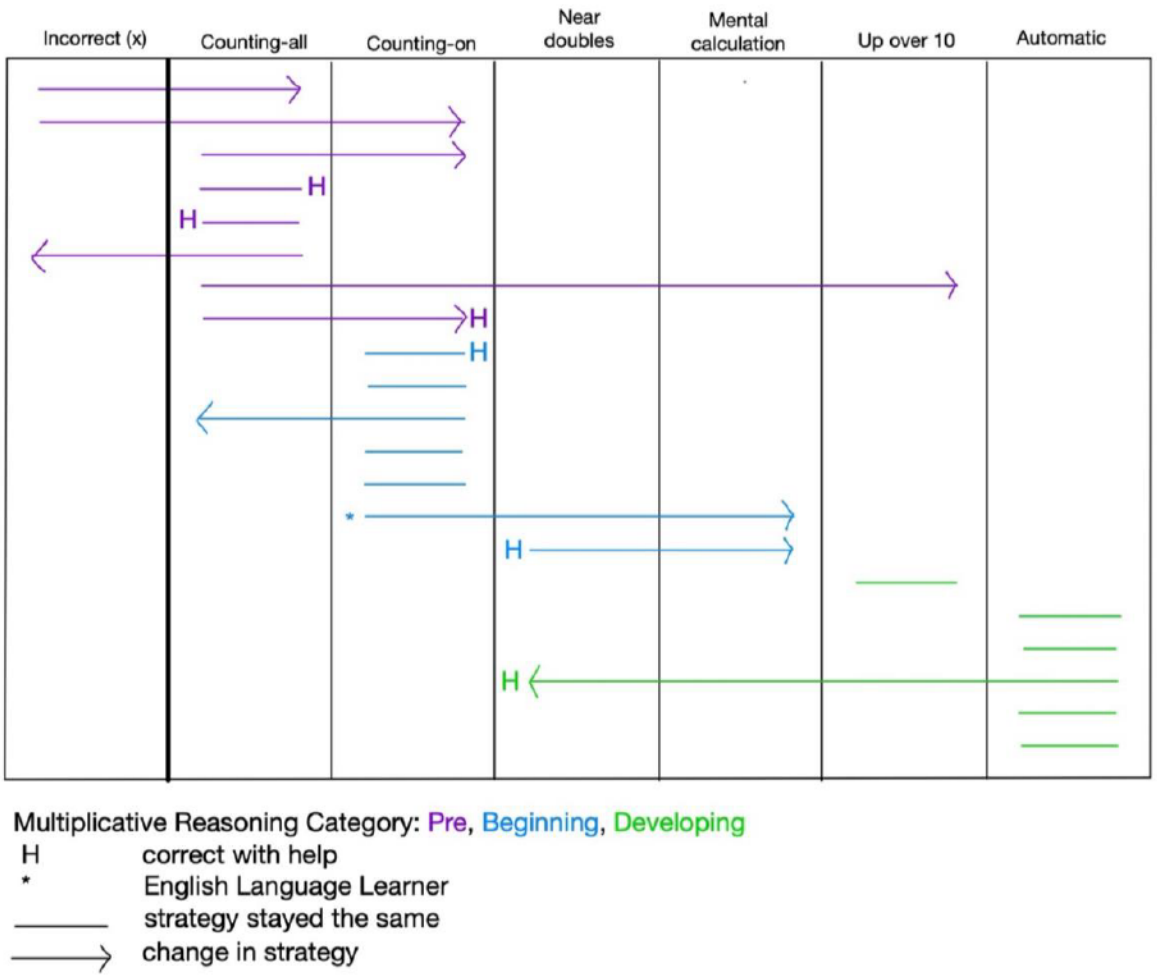
***Additive Item: 8+6***

The first item is a further analysis of 8 + 6, for the purpose of examining changes in students' additive thinking from the pre- and post-assessment—even though it was not a predominant focus of instruction. The purpose of including this item was to understand the

strategies students use to solve an additive problem and how this impacts their ability to develop multiplicative reasoning. The analysis of an additive question aligns with Steffe's (1994) number sequence trajectory, where students' ability to count-on, at an *Initial Number Sequence*, marks progress toward constructing multiplicative reasoning. The codes for this item were outlined earlier in Table 27. The strategies and progress students made from pre- to post-assessment are outlined in Figure 16.

**Figure 16**

*Strategy Development Map: Pre- and Post-Addition of 8 + 6*




*Note*. Each line in the figure represents one student. Descriptions of the strategy codes can be found in Table 27.

Overall, there were 17 students (81%, n=21) who solved this item correctly in the pre-assessment, while 2 (10%) solved it correctly with help, and 2 (10%) provided incorrect answers. In the post-assessment, 15 students (71%, n=21) solved it correctly, 5 (24%) correctly with help, while 1 (5%) continued to be incorrect.

In the Pre-Multiplicative Reasoning Category (purple), there were 6 of the 8 students (75%) who solved the pre-assessment question correctly, predominantly using a "counting-all" strategy (with one student requiring help). The two students who answered incorrectly attempted a "counting-on" strategy. Nonetheless, I observed subtle changes between the pre- and post-assessments that indicate a shift in their additive thinking. By the post-assessment, 7 of the 8 students (88%) solved this question correctly (one with help), with five students demonstrating a shift in their strategy to greater sophistication by the post-assessment.

In the Beginning Multiplicative Reasoning Category (blue), all 7 students (100%) were able to arrive at a correct answer to 8 + 6, with 2 of the 7 students in the pre-assessment and 1 of the 7 students in the post-assessment solving it correctly with help. The students in this category, however, demonstrated little to no shift in the sophistication of strategies they used, with their predominant strategy being counting-on in both the pre- and post-assessment. The challenge of making a shift from counting-on algins with Steffe's (1992) research. He gives the example of Zachary, who was able to count-on and in the pre-multiplicative stage, *Initial Number Sequence* (INS), and remained at this stage, even with the support of learning multiplication in his grade 3 class and through the intervention interviews (p. 304).

The results of students in the Developing Multiplicative Reasoning Category (green), were in stark contrast to the Pre- and Beginning Multiplicative Reasoning Categories. All 6 students (100%) were able to solve this question correctly in both the pre- and post-assessments with 1 of the 6 students solving it correctly with help in the post-assessment. Most of the students in this category solved 8 + 6 automatically in both the pre- and post-assessments.

In summary, by the post-assessment, 7 students (33%, n=21) were counting-on, and 9 (43%) were using a strategy beyond counting-on, with four of those students solving it

automatically. The fine-grained findings of the additive item, 8 + 6, reaffirmed the benefit of placing students in multiplicative reasoning categories to evaluate their readiness for multiplicative reasoning (Tzur et al., 2021) and their change or lack of change in strategies within each of the multiplicative reasoning categories. I now move to the analysis of items that were supported in the mathematics intervention in the development of multiplicative reasoning. The analysis of students' ability to subitize and iterate, focuses on the development of simultaneity.

***Subitizing with Visual Iterations Items: 3 dots per plate x 4 plates***

The first subitizing and visual iterations' item involved a dot plate with three dots in blue. With this dot plate, further questions were asked where students had to visualize various iterations of this plate. The questions for the item can be found in Table 29.

**Table 29**

*Subitizing and Visual Iterations Assessment Item: 3 dots per plate x 4 plates*

| Dot Plate | Item | Question |
|---|---|---|
|  | 01a | How many dots do you see? How many would there be if there were two plates? |
| | 01b | How many would there be if there were four plates? |

For this assessment item, the card with 3 dots per plate was held up for a few seconds for students to view and was not shown again when the students were asked to imagine iterating the plate. The purpose of this task was to understand whether students can subitize (as being able to see 'grouped objects' can provide a different understanding of quantity than single counting alone {Carper, 1942, p. 167; Douglass, 1925, p. 488; Freeman, 1912, p. 312}) and also, mentally iterate a set of 'grouped objects'. A further purpose of this item was to examine student's development of *simultaneity*—as they are attempting to both visualize the group of dots and keep

track of the iterations. The codes used to identify the strategies that students used when iterating the visual can be found in Table 30.

**Table 30**

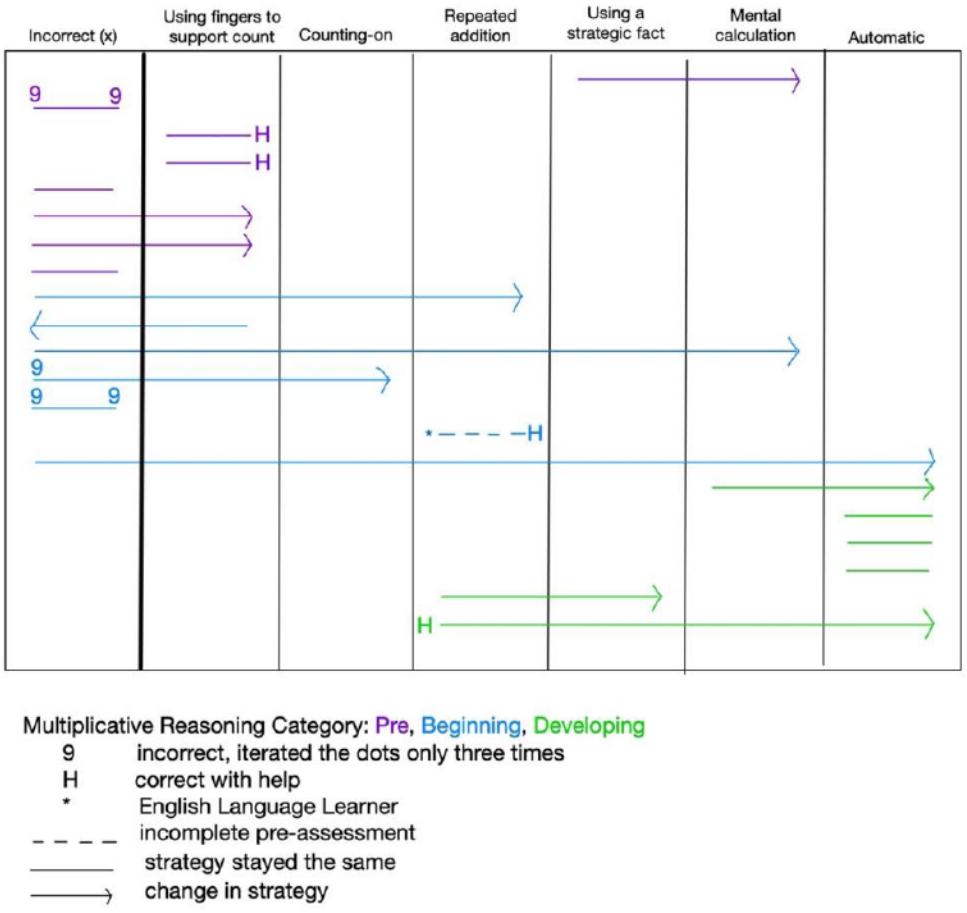
*Strategies for Subitizing and Visualizing Iterations*

| Strategy | Description of strategy |
|---|---|
| Using fingers to support count | The student used their fingers to be able to count the total number of dots. |
| Counting-on | The student was able to arrive at 9 dots and used their fingers to count and track 3 more. |
| Repeated addition | The student repeatedly added the number of dots together. |
| Using a strategic fact | The student used 3 x 3 = 9 and added 3 more dots. |
| Mental calculation | The student needed time to think and calculate their answer. |
| Automatic | The student answered the question in three seconds or less. |

As outlined in Table 29, there were a series of questions with 3 dots per plate. Students first had to identify 3 dots per plate, double the plates, and then answer how many dots there would be if there were 4 plates. Students sometimes only added on one more iteration. Therefore, I indicated when students only iterated the plates 3 times instead of 4 resulting in an answer of 9. After they came up with an answer of 9, I asked them how many plates that would be and most often they would come up with 3 plates. When I asked them how many dots there would be if there were four plates, they were able to arrive at the answer of 12. I still marked this as incorrect due to the amount of discussion that was required. I created a strategy development map for 3 dots per plate x 4 plates as found in Figure 17 to examine changes between the pre- and post-assessment.

**Figure 17**

*Strategy Development Map: Pre- and Post-Subitizing and Visual Iterations: 3 dots per plate x 4 plates*




*Note.* Each line in the figure represents one student. Descriptions of the strategy codes can be found in Table 30.

Overall, 9 students (45%, n=20) were able to solve it correctly in the pre-assessment increasing to 13 students (62%, n=21) who were able to solve it correctly in the post- assessment. One student, who was an English Language Learner, was marked as incomplete in the pre-

assessment as it was unclear if the language interfered with their understanding of the task. There were differences in students' strategies and progress from pre- and post-assessments in each multiplicative reasoning category.

In the Pre-Multiplicative Reasoning Category (purple), 3 of the 8 students (38%) solved the question incorrectly in both the pre- and post-assessment with one student who said '9' as an answer both times. Two of the eight students (25%) solved the question incorrectly in the pre-assessment but were able to solve it correctly in the post-assessment. Two students who were able to accurately use their fingers to calculate the solution in the pre-assessment, needed help to solve it correctly in the post-assessment—demonstrating the fragility of the use of a counting strategy. While there were some improvements in students' ability to solve the problem correctly, the results point to the inherent challenge of being able to simultaneously hold onto a mental image of a group of dots and successfully iterate them.

In the Beginning Multiplicative Reasoning Category (blue), only 1 of the 7 students (14%) was able to solve 3 dots x 4 correctly in the pre-assessment increasing to 5 of the 7 students (71%) who solved it in the post-assessment. There were changes in strategy development, with 4 of the 5 students (80%), in the post-assessment, who solved it correctly using a strategy other than counting, therefore demonstrating a shift in simultaneity.

For students in the Developing Multiplicative Reasoning Category (green), all 6 students (100%) answered this item correctly in both the pre- and post-assessments with 3 out of 6 students (50%) answering this question automatically in the pre-assessment. This result signified a very different starting point for students in the Developing Multiplicative Reasoning Category than what the Pre- and Beginning Multiplicative Reasoning Categories demonstrated. By the post-assessment, 5 of the 6 students (83%) were able to answer this question automatically.

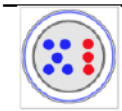
Nonetheless, students in the Pre- and Beginning Multiplicative Reasoning Categories progressed in their ability to accurately subitize and iterate a dot arrangement. My analysis continued with a second subitizing and visual iterations assessment item.

***Subitizing and Visual Iterations: 8 dots per plate (5 red + 3 blue) x 2 plates***

The second subitizing and visual iterations' item involved a dot plate with two colours of five red dots and three blue dots. For this assessment item, the card with five blue dots and three red dots, was held for a few seconds and sometimes held up again for a second time. It was not shown to the students again when they were asked to double the image. The questions that students were asked can be seen in Table 31. The purpose of this assessment item was to determine if the students were able to subitize two different coloured amounts, combine them for a total of 8 dots (defined by Clements (1999, p. 404) as *conceptual subitizing*), and mentally iterate the image.

**Table 31**

*Subitizing and Visual Iterations Assessment Item: 8 dots per plate (5 red + 3 blue) x 2*

| Dot Plate | Item | Question |
|---|---|---|
|  | 02i | How many dots do you see? |
| | 02ii | How many dots would there be if I had two plates? |

The codes used to identify the strategies that students used when subitizing and iterating the visual can be found in Table 32.

**Table 32**

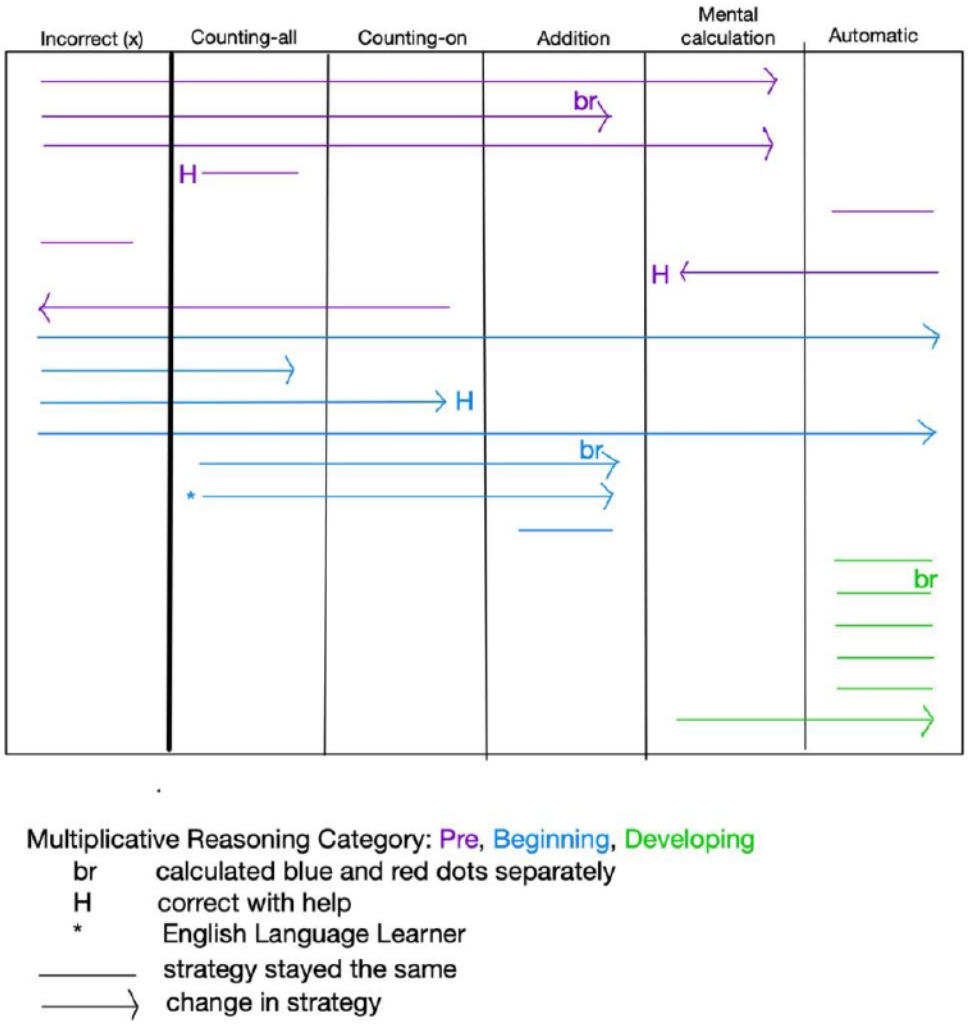
*Strategies for Subitizing and Visual Iterations: 8 dots per plate (5 red + 3 blue) x 2 plates*

| Strategy | Description of strategy |
|---|---|
| Counting-all | The student counted out 8 on their fingers, starting at 1, then repeated that count of 8 on their fingers, this time starting at 9, to arrive at 16. |
| Counting-on | The student counted-on from 8 using their fingers to count and track 8 more. |
| Addition | The student added the number of dots together either as two sets of 8 dots or as two sets of 5 red dots and two sets of 3 blue dots. |
| Mental calculation | The student required time to think to arrive at the answer. |
| Automatic | The student answered the question in three seconds or less. |

I created a strategy development map for 8 dots per plate (5 blue dots + 3 red dots) x 2 as found in Figure 18 to examine changes between the pre- and post-assessment. For the visual iteration of (5 blue dots + 3 red dots) x 2, part of the analysis was identifying when students incorporated the use of the two colours as part of their strategy. If students added the two blue dots together (5 + 5), added the two red dots together (3+3) and then added the two subtotals together, (10+6), I coded this strategy as addition.

**Figure 18**

*Strategy Development Map: Pre- and Post-Subitizing and Visual Iterations 8 dots per plate (5 red + 3 blue) x 2*




*Note.* Each line in the figure represents one student. Descriptions of the strategy codes can be found in Table 32.

Overall, 13 students (62%, n=21) were able to solve this item correctly in the pre-assessment and 19 students (90%, n=21) were able to solve this item correctly in the post-

assessment. There continued to be differences within each of the three multiplicative reasoning categories.

The Pre-Multiplicative Reasoning Category (purple) fared better in this subitizing and visual iteration assessment item, when they were doubling 8 dots per plate, than the previous subitizing and visual iterations' item. There were 4 of the 8 students (50%) who solved it correctly in the pre-assessment, (including one correctly with help), to 6 of the 8 students (75%) who solved it correctly in the post-assessment, (including one correctly with help). Of these six students that solved this item correctly in the post-assessment, only one of them used a counting strategy, meaning five demonstrated the use of more efficient strategies.

In the Beginning Multiplicative Reasoning Category (blue), 3 of the 7 students (43%) solved this item correctly in the pre-assessment, increasing to all 7 students (100%) solving it correctly in the post-assessment, with 1 of these students solving it correctly with help. There was a shift in strategy development, from only 1 of the 7 students (14%) who was able to solve it correctly with a strategy other than counting in the pre-assessment, to 5 of the 7 students (71%), in the post-assessment.

For students in the Developing Multiplicative Reasoning Category (green), in both the pre- and post-assessments, all 6 students (100%) answered this question correctly. In the pre-assessment, 5 of the 6 students (83%) were able to answer this question automatically moving to all 6 students (100%) solving this automatically by the post-assessment.

Overall, there were greater changes for students in terms of correctness and strategy development with 8 dots per plate x 2 plates than 3 dots per plate x 4 plates, particularly in the Pre- and Beginning Multiplicative Reasoning Category, as there were less iterations of which students needed to keep track. My analysis continued examining student responses to a

multiplication problem with support of a direct model, offering further indications of students' development towards multiplicative reasoning.

***Multiplication Problem (2 candies per bag x 8 bags) with the Visual Support of a Direct Model***

There were two components to this assessment item and resulting analysis. In the first part of the assessment, students were asked to solve a real-world problem with the support of a direct model aligned with a context as seen in Figure 19. The second part of the item was an assessment of students' ability to orally recite their skip counting sequence by 2s.

**Figure 19**

*Multiplication Problem with the Visual Support of a Direct Model*

Question: *I have some candies in bags. There are 2 candies per bag and there are 8 bags. I'm wondering, how many candies are there altogether?*



After the question was asked, the physical model of 2 candies per bag, 8 bags, was placed in front of the student.

The purpose this assessment was to note students' skip counting fluency. Wilkins and Ulrich (2017) in their article, *The Role of Skip Counting in Children's Reasoning* argue that skip counting is an important stage in a child's development of multiplicative reasoning worth analyzing.

The focus of the data analysis was on strategy development and fluency of the skip counting sequence. After coding the pre- and post-assessment, I noticed that I did not

consistently ask for the referent in the interview for this question. Therefore, the responses without a referent were still coded as correct. There was one student in the pre-assessment and three students in the post-assessment who made a referent error, answering 16 candies per bag instead of saying 16 candies which was easily resolved through conversation.

The solutions were coded with one of four strategies: counting-all, coordinating to track the intensive quantity, coordinating with fluency, and automatic. The explanation for the distinctions amongst these strategies is found in Table 33.

**Table 33**

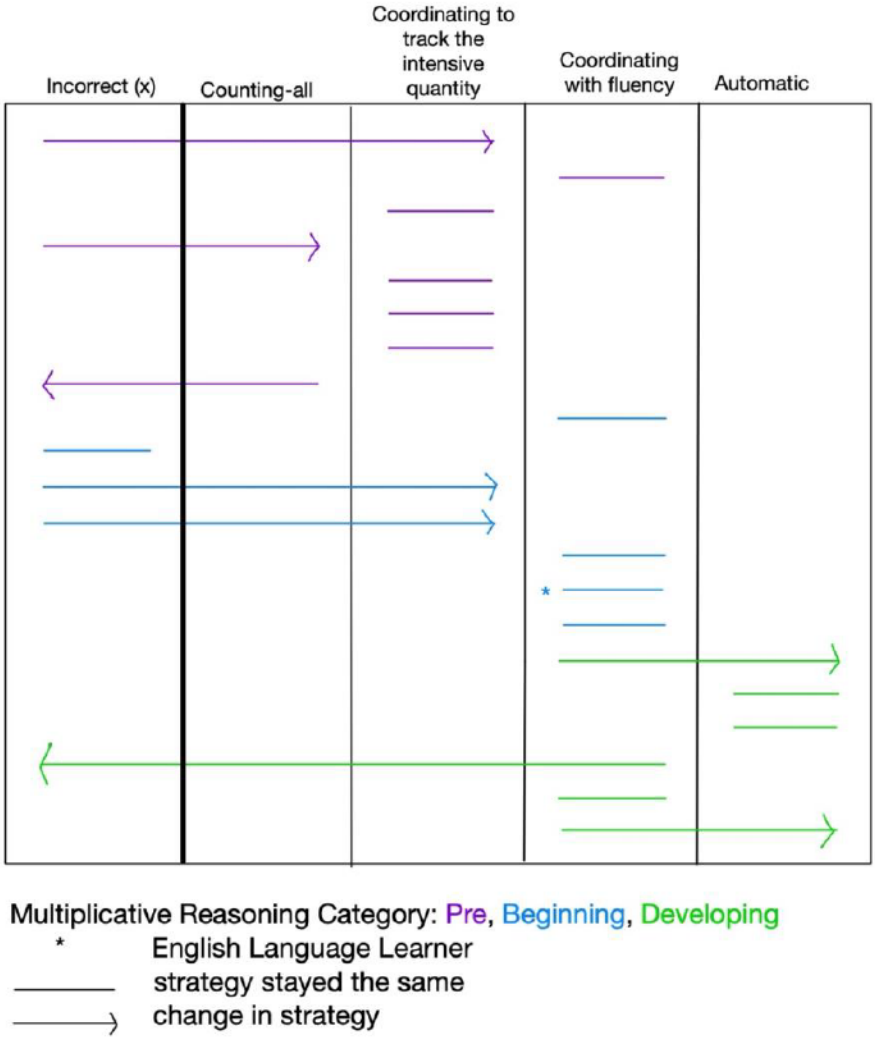
*Codes of Strategies for Multiplication Problem with the Visual Support of a Direct Model*

| Strategy | Description of strategy |
|---|---|
| Counting-all | The student individually counted most or all the objects. |
| Coordinating to track the intensive quantity | The student touched an object, (e.g., a bag with candies inside) as they are skip counting, however, there were pauses within the skip counting sequence. |
| Coordinating with fluency | The student touched each of the bags, skip counting by twos with little or no pauses within their skip counting sequence. |
| Automatic | The student answered the question in three seconds or less without touching the objects. |

When coding the strategies, I made the decision to code the differences between coordinating *with or without fluency* to determine if there was a change in simultaneity when there was increased fluency in the skip count. The strategies that students used with the support of a direct model are highlighted in Figure 20.

**Figure 20**

*Strategy Development Map: Pre- and Post-Multiplication Strategies with the Visual Support of a Direct Model*




*Note.* Each line in the figure represents one student. Descriptions of the strategy codes can be found in Table 33.

Overall, 16 students (76%, n=21) were able to solve this question correctly in the pre-assessment increasing to 18 students (86%, n=21) able to solve this correctly in the post-

assessment. The analysis of the strategy development uncovered changes in students' thinking from pre- and post-assessment within and amongst the three multiplicative reasoning categories.

Students in the Pre-Multiplicative Reasoning Category (purple), were not fluent in their skip counting sequence in connection with the direct model (2 candies per bag) in either the pre- or the post-assessment. By the post-assessment, there were still students who did not know their skip counting sequence by 2's fluently to 20, as seen in 5 of the 8 students (63%) who were still coordinating to track the intensive quantity, meaning that there were thinking pauses when they were touching each of the bags, needing to either count by ones or think of the next number within the skip counting sequence.

In the Beginning Multiplicative Reasoning Category (blue), there were more students who had a fluent skip counting sequence, with 4 of the 7 students (57%) able to answer this question correctly in the pre-assessment. They were able to coordinate with fluency by connecting the quantity with the direct model, being less dependent on counting each candy one at a time. In the post-assessment, 6 of the 7 students (86%) solved this question correctly. There were two students in this category who solved it incorrectly the first time but by the post-assessment were able to coordinate to track the intensive quantity.

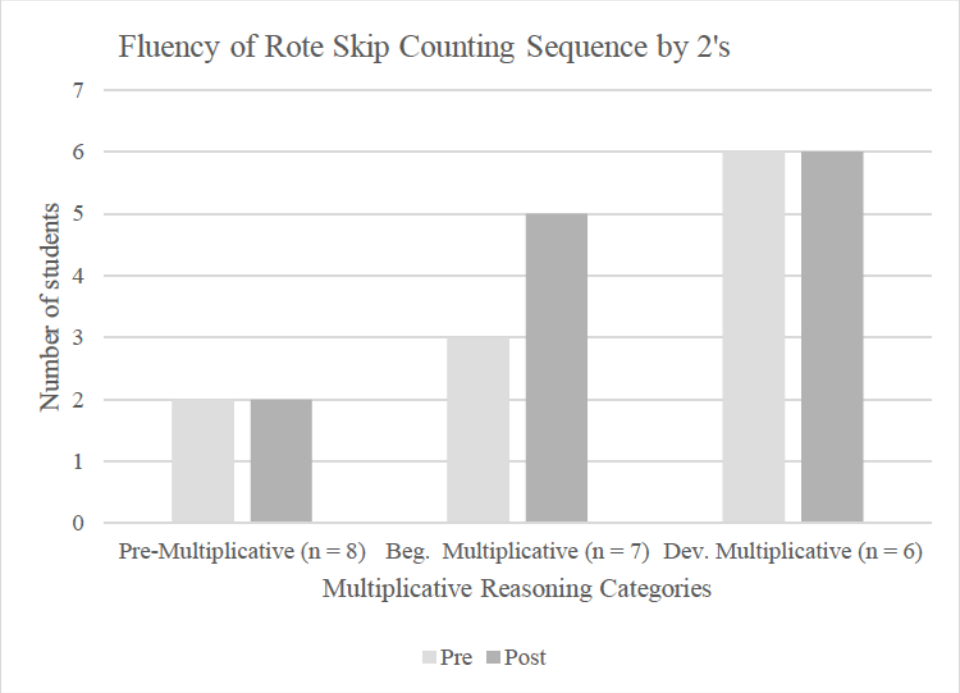
This was very different than students in the Developing Multiplicative Reasoning Category (green) where in the pre-assessment, 4 of the 6 students (67%) were coordinating with fluency and 2 of the 6 students (33%) solved the problem automatically understanding that 2 x 8 = 16. By the post-assessment, 4 of the 6 students (67%) solved this problem automatically.

For the second part of the data analysis for this question, I analyzed students' fluency of their rote skip counting sequence, which I asked them to say aloud after they solved the problem. Students were coded as being fluent or not in their sequence, with fluency being defined as

having little or no pause when they were skip-counting as outlined in Figure 21. The data analysis included all 21 students; however, this graph indicates the number of students who were fluent in each of the three multiplicative reasoning categories.

**Figure 21**

*Fluency of Rote Skip Counting Sequence by 2's in Pre- and Post-Assessments*




In the Pre-Multiplicative Reasoning Category, while there were some changes by the post- assessment of how far students were able to recite the sequence without having to pause; overall, students in this group were still working on the fluency of their rote sequence. This contrasts with the Beginning Multiplicative Reasoning Category, where there were changes for two students in their fluency. There were no changes in the Developing Multiplicative Reasoning Category because all students exhibited fluency in their skip-counting sequence in the pre-assessment.

***Multiplication Word Problem for 4 pieces of gum per package x 3 packages (4 x 3 = □)***

In this multiplication assessment item, students were asked the following word problem: *Joshua has packages of gum with 4 pieces per package. He has 3 packages. How many pieces of gum does he have?* The purpose of this assessment item was to determine students' understanding of a multiplicative situation where the total is unknown.

The strategies that students used provided more detailed information about their development towards multiplicative reasoning. The meaning of the codes for this item can be found in Table 34. The analysis for correctness differed slightly from other items. I used a new code to indicate when students added the two number together in this problem instead of multiplying them to provide further insight into how students thought of multiplication before instruction.

**Table 34**

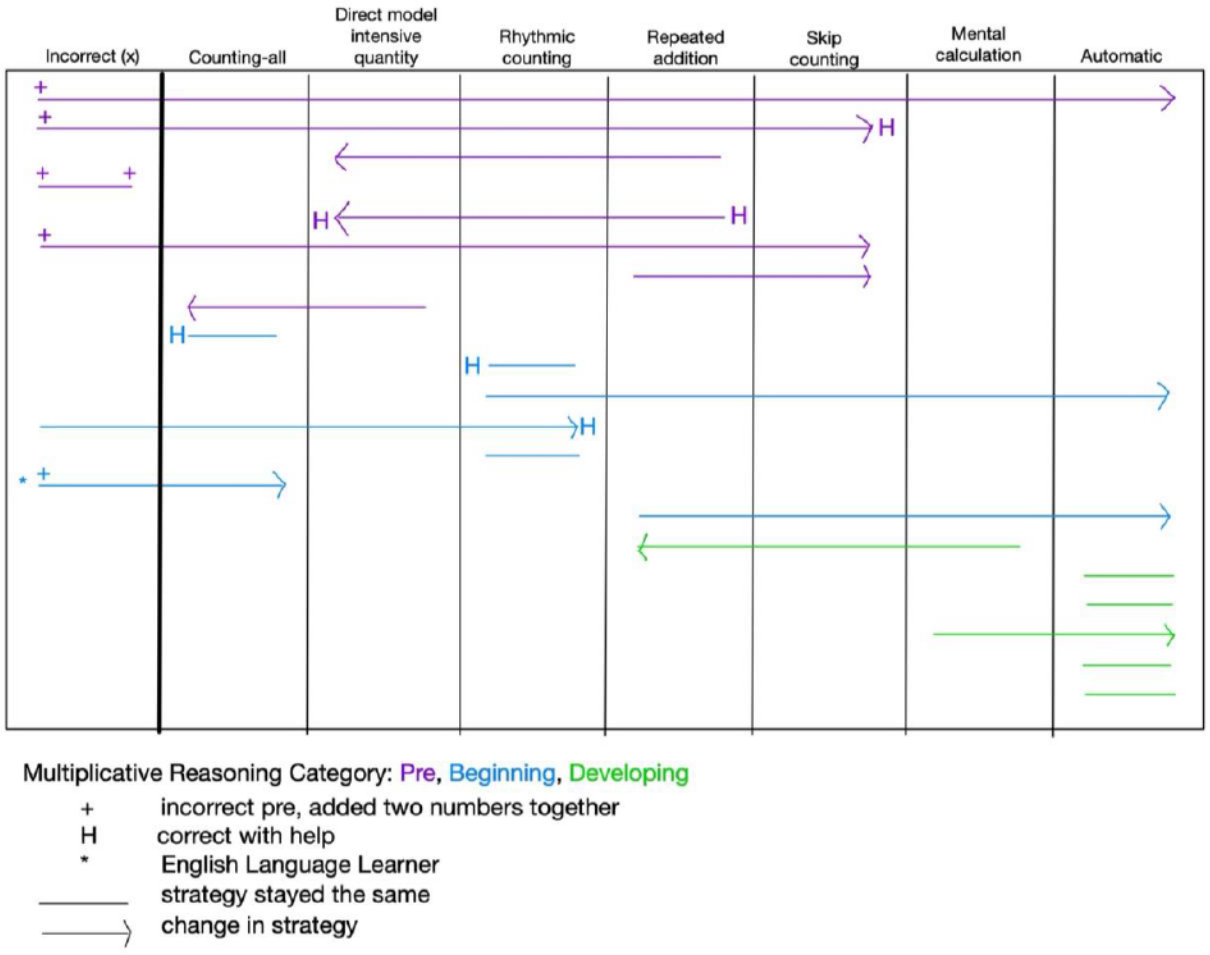
*Codes of Strategies for Multiplication Problem*

| Strategy | Description of strategy |
|---|---|
| Counting-all | The student drew three packages separately, then filled in each piece of gum inside the package, counting them individually: no simultaneity evident. |
| Direct modelling of the intensive quantity | The student drew a representation of the intensive quantity. They drew out a package and filled in the 4 pieces of gum or drew 4 pieces of gum and then circled the pieces. This entire process was repeated for a total of three iterations. |
| Rhythmic counting | The student counted aloud, emphasizing every multiple of 4 (e.g., 1, 2, 3, **4**; 5, 6, 7, **8**; 9, 10, 11, **12**). Occasionally, the student used 4 fingers to track these iterations, sometimes beginning the process after the number 4. |
| Repeated addition | The students added 4, three times. |
| Skip counting | The students orally skip counted by 4s (e.g., 4, 8, 12). |
| Mental calculation | The student required time to think and calculate the answer and did not have a clear explanation of how they solved it. |
| Automatic | The student answered the question in three seconds or less. |

I analyzed the strategies that students used and examined if there were any changes or shifts in each students' development as can be seen in Figure 22.

**Figure 22**

*Strategy Development Map: Pre- and Post-Multiplication (4 pieces of gum per package x 3 packages) (4 x 3 = □)*




*Note.* Each line in the figure represents one student. Descriptions of the strategy codes can be found in Table 34.

Overall, the number of students who incorrectly added the numbers together dropped from 6 students (29%, n=21) in the pre-assessment (including one English Language Learner) to only 1 student (5%, n=21) in the post-assessment. Additionally, the number of students who

answered the question correctly without hesitation (automatically) increased from 4 (19%) to 8 (38%).

In the Pre-Multiplicative Reasoning Category (purple), 4 of the 8 students (50%) in the pre-assessment, added the two numbers together, 4 + 3 = 7, and by the end of the intervention, only 1 student (13%) added the numbers together. Within this category, three students reverted to a less sophisticated strategy in the post-assessment. Despite this, for the two students who went from repeated addition back to modelling the intensive quantity in the post-assessment, this may nonetheless represent a positive shift in simultaneity. In the pre-assessment, the repeated addition may have only represented the "4 pieces of gum" rather than the intensive quantity of "4 pieces of gum per package". The students' drawings, in the post-assessment, however, demonstrated their understanding of the intensive quantity (drawing a representation of a pack and then the gum inside and then iterating).

In the Beginning Multiplicative Reasoning Category (blue), 5 of the 7 students (71%) were able to answer this question correctly in the pre-assessment (one correct with help). There was one student who answered this question incorrectly by adding the two numbers together, but it is uncertain if this was a result of them being an English Language Learner, and therefore not fully understanding the context of the problem. By the post-assessment, all 7 students (100%) were able to answer this question correctly (one correct with help). There were still two students, in the post-assessment, who used counting-all and did not create the intensive quantity, given that they drew out the packages first and then filled in the pieces of gum. The students' strategies in this group remain at early developmental stages of multiplicative reasoning.

In the Developing Multiplicative Reasoning Category (green), 4 of the 6 students (67%) in the pre-assessment, and 5 of the 6 students (83%) in the post-assessment answered this

question automatically, demonstrating a different level of development of multiplicative reasoning than in the Pre-Multiplicative Reasoning Category. The one student who could not solve it automatically had a strategy code change from a mental calculation to repeated addition. This does not necessarily indicate a change in strategy development as the student was not able to clearly identify their strategy in the pre-assessment; therefore, the code mental calculation was used. However, in the post-assessment, they clearly articulated using the strategy of repeated addition to solve the problem.

**Equation and Referent Analysis for the Multiplication Problem.** I undertook another level of analysis for this multiplication problem where in the post-assessment, the students were asked to write the equation and explain the meaning of the numbers by identifying the referent. All referents were analyzed and coded, even if the student incorrectly solved the problem. The codes for the equations and the referents are indicated in Table 35.

**Table 35**

*Codes for Equation Correctness and Referent Identification for Multiplication Problem (Post-Assessment)*

| Equation correctness and referents | Description |
|---|---|
| Equation correct | The equation was coded as correct if all components of the equation were written correctly, including symbol and number notation, for example 4 x 3 = 12 and 3 x 4 =12 were both correct notations for the multiplication problem. |
| Equation incorrect | The equation was coded as incorrect if any component of the equation was written incorrectly, including a symbol or number error, for example: 4 + 3 = 12. |
| Intensive quantity (per) | The student identified the intensive quantity as *pieces of gum per package*. Students were also coded as having the full referent if they were able to identify a relationship between two quantities such as 'how many were in each package.' |
| Intensive quantity (no per) | The student only described one extensive quantity within the intensive quantity, i.e. indicating 'pieces' without connecting it to the packages. |
| Iterations | The student identified the iterations as *packages*. An alternate description of a container, such as bags, was also accepted. |
| Total (with referent) | The student identified the total as *pieces of gum*. It was accepted if the student said only 'pieces' or 'gum.' |
| Total (no referent) | The student identified the total without using a referent. |

How the students recorded the equation and identified the referents are summarized in Table 36. Within each of the three multiplicative categories, I started with the incorrect writing of equations first to help create an overview of the progression.

**Table 36**

*Analysis of Equations and Referents for Multiplication Problem (Post-Assessment)*

| Reasoning category | Pre-Multiplicative Reasoning | | Beginning Multiplicative Reasoning | | Developing Multiplicative Reasoning | |
|---|---|---|---|---|---|---|
| | Equation | Referent | Equation | Referent | Equation | Referent |
| Equation and referent analysis | x (4 + 3 = 7) | IQ, IT, T | x 4 + 3 = 12 | iq, IT, T | ✓ 4 x 3 = 12 | IQ, IT, t |
| | x 4 = 3 x 12 | IT, it, t | ✓ 4 x 3 = 12 | iq, IT, t | ✓ 4 x 3 = 12 | IQ, IT, T |
| | x 4 + 3 = 12 H | IQ, IT, T | ✓ 4 x 3 = 12 | iq, T, T | ✓ 3 x 4 = 12 | iq, IT, T |
| | x 4 + 3 = 12 | IQ, IT, T | ✓ 3 x 4 = 12 H | iq, ~~it~~, t | ✓ 4 x 3 = 12 | iq, IT, T |
| | ✓ 4 x 3 = 12 | IQ, IT, T | ✓ 4 x 3 = 12 | IQ, IT, T | ✓ 4 x 3 = 12 | IQ, IT, T |
| | ✓ 3 x 4 = 12 H | IQ, IT, t | ✓ 4 x 3 = 12 | IQ, IT, t | ✓ 4 x 3 = 12 | IQ, IT, T |
| | ✓ 4 x 3 = 12 | IQ, IT, t | ✓ 4 x 3 = 12 | IQ, IT, T | | |
| | ✓ 4 x 3 = 12 | IQ, IT, t | | | | |

*Note*. Each entry is a different student.

✓ indicates a correct equation and x indicates an incorrect equation.

( ) around an equation indicates that the student did not solve the problem correctly; 'H' indicates that the student solved it correctly with help; no bracket indicates that the student solved it correctly.

Initials used to indicated full referents: intensive quantity (IQ), iterations (IT), and total (T). The lower-case version of these referents "iq," "it," and "t" indicates a partial referent. A strike through indicates no referent indicated for the number.

After the students wrote the equation, they were asked to identify the referent for each number. I will comment on this analysis holistically as there were no striking differences in referent identification amongst the three multiplicative reasoning categories. Overall, 14 students (67%, n=21) were able to identify the intensive quantity, and 20 students described the

iterations fully (95%, n=21). For the total, 13 students (62%, n=21) identified the total with a referent either after they wrote the equation or when they solved the problem. I considered coding 'total' as a complete referent, which would be all 21 students because this part of the instruction was supporting students to understand that when they multiplied the two numbers together, they were finding the total. The iterations appear to be the easiest referent for students. The intensive quantity and the total are slightly more complex, as they both involve pieces of gum but one is the total number and the other the number per package.

Further analyzing the equation writing I categorized the correctness of equations into sub-categories as seen Table 37 to determine if there were consistent errors and where students placed the intensive quantity in the equation.

**Table 37**

*Analysis of Equation Writing for Multiplication Problem (Post-Assessment)*

| Incorrect: Correct for student solution | Incorrect: Symbol error; correct solution | Correct equation |
|---|---|---|
| 4 + 3 = 7 | 4 = 3 x 12 | 3 x 4 = 12<br>3 students |
| | 4 + 3 = 12<br>3 students | 4 x 3 = 12<br>13 students |

Overall, for the students who wrote their equation correctly, 13 of the 16 students (81%) wrote the equation as 4 x 3 = 12, beginning with the intensive quantity as emphasized during instruction. Apart from the student who added the two numbers together, the other four students who wrote the equation incorrectly arrived at the correct answer to the problem. When I was debriefing with a student after the post-assessment interview and I asked about the '+' sign, they immediately realized that they should have written the 'x' symbol. This finding aligns with

Kamii (2000, p. 78) where she discusses this very idea when young children are learning addition—she argues that it is the reasoning of the operation that is important as the symbol for addition is a convention that may come later. Kato et al. (2002), in their research study of 60 students aged 3 to 7, expand on this by suggesting that children have reached a higher level of abstraction when they are able to record a number of objects in a set with both a number and the meaning of that number, such as 6 beans (p.39). Therefore, in the current study, the analysis of the equation writing and the referents indicates that students understand the meaning of the numbers in a multiplication context, which requires a further level of abstraction.

To further understand how students develop multiplicative reasoning, it was important to examine their understanding in different types of multiplicative situations. Therefore, my analysis continued by examining students' responses to two types of division word problems. I start with quotative division.

***Quotative Division Word Problem: 15 cookies ÷ 3 cookies per bag (3 x □= 15)***

The next item in the mathematics assessment was a quotative division problem, where the students know the total and the intensive quantity, while the number of iterations is unknown. Students were asked to solve the following problem: *You have 15 cookies. You are going to put them in bags with 3 cookies per bag. How many bags can you fill?* In this problem, students know the number of cookies per bag and need to determine the number of iterations to reach the total. While this problem is more challenging than multiplication, it is nonetheless closer to multiplication than it is to partitive division. In partitive division, the unknown is the intensive quantity, which is more challenging to solve for because you do not know the amount that needs to be iterated. The purpose of this assessment, therefore, was to understand how students solved a multiplicative situation with a different unknown, in this case the number of iterations. An

explanation for the strategy codes that were used for this quotative division problem can be found in Table 38.

**Table 38**

*Codes of Strategies for Quotative Division Problem*

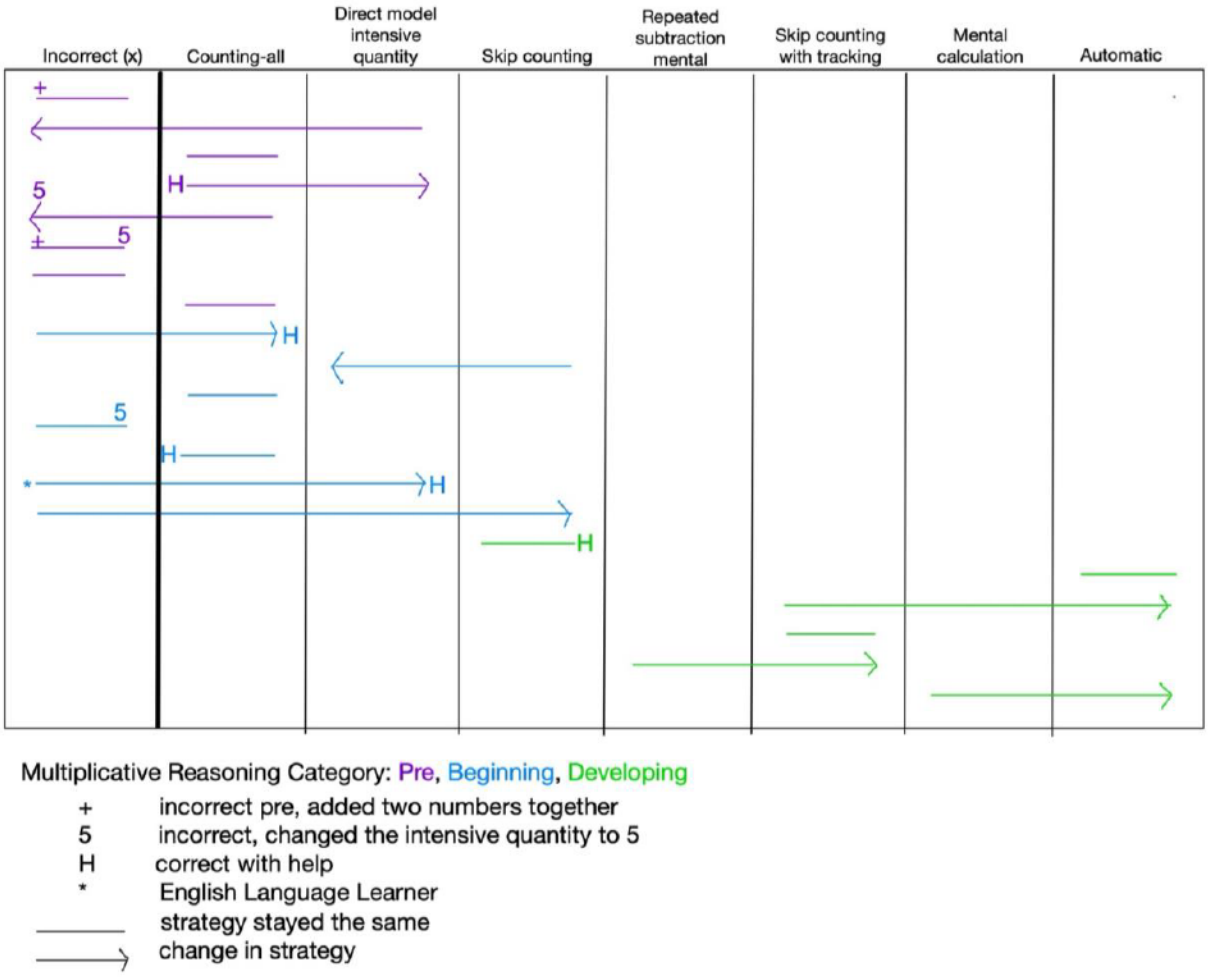
| Strategy | Description of Strategy |
|---|---|
| Counting-all | The student drew a representation of three cookies separately, went back to circle each group of three, and counted the number of circles to find out the number of bags; no simultaneity evident. |
| Direct modelling of the intensive quantity | The student drew a representation of the intensive quantity. They drew a bag and filled in three cookies or drew the cookies and circled them. This entire process was repeated until the total of 15 cookies was reached. |
| Skip-counting | The student skip counted by three and used a piece of paper to record and track the iterations (e.g., 3, 6, 9, 12, 15). |
| Repeated subtraction written | The student repeatedly subtracted 3 from 15, keeping track of their calculations on paper. |
| Repeated subtraction mental | The student repeatedly subtracted 3 from 15, keeping track of the iterations on their fingers. |
| Skip-counting with tracking | The student counted by 3s while simultaneously keeping track of the iterations on their fingers. |
| Mental calculation | The student required think time to arrive at the answer. |
| Automatic | The student answered the question in three seconds or less. |

The development of the student strategies from the pre- to post-assessment can be seen in Figure 23. Incorrect answers were further coded to indicate if the students added the two numbers together. I also indicated if students answered 5 and when asked, said it was ‘5 cookies per bag,’ thereby commuting numerically but not making sense of the referent.

**Figure 23**

*Strategy Development Map: Pre- and Post-Quotative Division (15 cookies ÷ 3 cookies per bag)*

*(*3 x □= 15)




*Note.* Each line in the figure represents one student. Descriptions of these codes can be found in Table 39.

Overall, 12 students (57%, n=21) solved it correctly in the pre-assessment and 13 students (62%, n=21) solved it correctly in the post-assessment. In the pre-assessment, there

were two students who solved the problem by incorrectly adding the two numbers together, whereas, by the post-assessment, there were no students who made this error.

In the Pre-Multiplicative Reasoning Category (purple), strategies for these students remained in the counting phases. Two students mentally changed the intensive quantity to '5 cookies per bag" and said '3' for the answer. When I reiterated the question to them as '3 cookies per bag,' they could not adjust their answer. I suggest that this indicates two things: 1) the students were able to connect that 5 x 3 would equal 15 as they used 5 as a starting place, and 2) the referents, the meaning of the numbers play a more challenging role in the development of multiplicative reasoning as the students were not able to successfully solve the problem again. Even though the answer is '5,' there is a substantial difference between the answer meaning '5 cookies per bag,' the intensive quantity, versus '5 bags,' the iterations. What is interesting to note is that while these two students solved the question incorrectly, there was nonetheless a shift in their strategy development. The students went from adding the two numbers in the problem together or counting out everything to being able to try to use a known fact, demonstrating a shift in understanding.

In the Beginning Multiplicative Reasoning Category (blue), 4 of the 7 students (57%) solved this problem incorrectly in the pre-assessment, with one being an English Language Learner. By the post-assessment, only 1 of the 7 students (14%) answered this question incorrectly—using 5 cookies per bag. While more students solved this question correctly, they remained largely in the early counting stages which demonstrates little evidence of simultaneity.

In the Developing Multiplicative Reasoning Category (green), all 6 students (100%) answered the question correctly in the pre-assessment with strategies that demonstrated a simultaneous connection between the intensive quantity and the iterations. There was a change

from only 1 out of 6 students (17%) being able to solve the question automatically in the pre-assessment to 3 out of 6 students (50%) in the post-assessment.

**Equation and Referent Analysis for Quotative Division Problem.** As part of the post-assessment, students were asked to write the equations and identify the referents, in a quotative division problem. The codes for the analysis are described in Table 39. Similar to the multiplication problem, all referents were analyzed and coded, even if the student incorrectly solved the problem.

**Table 39**

*Codes for Equation Correctness and Referents for Quotative Division Problem (Post-Assessment)*

| Equation Correctness and Referents | Description |
|---|---|
| Equation correct | The equation was coded as correct if all components of the equation were written correctly, including symbol and number notation, for example 3 x 5 = 15 and 15 ÷ 3 = 5. |
| Equation incorrect | The equation was coded as incorrect if any component of the equation was written incorrectly, including a symbol or number error, for example, 15 x 3 = 5. |
| Intensive quantity (per) | The student identified the intensive quantity as *cookies per bag*. |
| Intensive quantity (no per) | The student only described one extensive quantity within the intensive quantity, such as indicating 'cookies' without connecting it to the bags. |
| Iterations | The student identified the iterations as *bags*. |
| Total (with referent) | The student identified the total as *cookies*. |
| Total (no referent) | The student identified the total without using a referent. |
| Incomplete data for referent | There was insufficient information to determine the student's ability to independently identify the referent. Therefore, a referent code was not applied. |

A detailed analysis of equations and referents for the quotative problem summarized in Table 40, where I started with the incorrect writing of equations first to help create an overview of the progression.

**Table 40**

*Analysis of Equations and Referents for Quotative Division Problem (Post-Assessment)*

| Reasoning category | Pre-Multiplicative Reasoning | | Beginning Multiplicative Reasoning | | Developing Multiplicative Reasoning | |
|---|---|---|---|---|---|---|
| | Equation | Referent | Equation | Referent | Equation | Referent |
| Equation and referent analysis | x (3 ≠ 10 15) | IQ, IT, T | x (3 x 15 = 3 + 15 =) | incomplete data | ✓ 3 x 5 = 15 H | iq, IT, t |
| | x (3 x ? = 15 3 x 15 = 15) | incomplete data | x 15 x 3 = 5 | IQ, IT, T | ✓ 3 x 5 = 15 15 ÷ 3 = 5 | IQ, IT, T |
| | x (15 – 3 = 12, 10, 7 ,4, 1) | ~~iq~~, IT, T | x 15 + 3 = 5 | IQ, IT, T | ✓ 15 ÷ 3 = 5 | IQ, IT, T |
| | x (15 x 3 = 19) | ~~iq~~, ~~it~~, T | x 15 x 3 = 5 5 x 3 = 15 | IQ, IT, T | ✓ 5 x 3 = 15 | IQ, IT, T |
| | x (3 x 15 15 – 3) | incomplete data | ✓ 3 x 5 = 15 H | IQ, IT, t | ✓ 3 x 5 = 15 | IQ, IT, T |
| | x no equation; wrote 5 bags | ~~iq~~, ~~it~~, T | ✓ 5 x 3 = 15 | IQ, IT, T | ✓ 15 ÷ 3 = 5 | IQ, IT, T |
| | x 15 x 3 = 5 | IQ, IT, T | ✓ 3 x 5 = 15 H | iq, IT, T | | |
| | x 15 x 3 = 5 15 x 5 = 3 | ~~iq~~, IT, T | | | | |

*Note.* Each equation or groups of equations represent a different student.

✓ indicates a correct equation and x indicates an incorrect equation

( ) around an equation indicates that the student did not solve the problem correctly; 'H' indicates that the student solved it correctly with help

Initials used to indicated full referents: intensive quantity (IQ), iterations (IT), and total (T)

The lower-case version of these referents "iq," "it," and "t" indicates a partial referent. A strike through indicates no referent indicated for the number.

There were differences amongst the three multiplicative reasoning categories in students' ability to fully identify the referents and write equations in the quotative division solutions. In the Pre-Multiplicative Reasoning Category (purple), 1 of the 8 students (13%) was able to identify all three referents correctly and solve the problem correctly. There were three students where I was unable to decipher if they were able to independently identify the referents when reviewing our conversations during the mathematics interview. In this category while there were three students who could identify the referents no one was able to write the equation correctly. One of these three students first wrote 3 x ? = 15 but could not follow through with their thinking.

The Beginning Multiplicative Reasoning Category (blue) had 4 of the 7 (57%), who were able to identify all three referents and solve the problem correctly. There were 3 of the 7 students (43%) able to correctly write an equation and one student who wrote it correctly with help. While one student was able to write a correct equation with their second attempt, this was done through discussion; therefore, I coded their equation as incorrect based on their first attempt. Finally, in the Developing Multiplicative Reasoning Category (green), 5 of the 6 students (83%) were able to identify all three referents, correctly solve the problem, and accurately write an equation.

I conducted further analysis of the equation writing, in particular to examine symbol errors. The subcategories of equation analysis can be found in Table 41.

**Table 41**

*Quotative Division Equation Analysis*

| Incorrect | Incorrect: Symbol error; solution correct | Equation fully correct |
|---|---|---|
| 3 ≠ 10 15 | 15 x 3 = 5/ 15 x 5 = 3 | 3 x 5 = 15 |
| 3 x ? = 15/3 x 15 = 15 | 15 + 5 = 3 | 2 students |
| 15 x 3 = 19 | | |
| 3 x 15/ 15-3 | 15 x 3 = 5 | 5 x 3 = 15 |
| 15 – 3 = 12 (repeated subtraction) | 2 students | 2 students |
| 3 x 15/3 + 15 | 15 x 3 = 5/ 5 x 3 = 15 (with help) | 15 ÷ 3 = 5 |
| | | 2 students |
| | | 3 x 5 = 15/15 ÷ 3 = 5 |
| | | 1 student (both operations) |

When examining the students' equation writing holistically, it is notable that the five students who made symbol errors still solved the problem correctly. This supports Kamii's (2000, p. 78) contention that understanding mathematical conventions follows conceptual understanding. Some students experimented with the order of numbers, writing 15 or 3 first, often omitting the division symbol entirely. Among the nine students who wrote the equation correctly, six (67%) expressed it as a multiplication problem, which is how it was introduced during instruction. This suggests that quotative division is more easily connected to multiplication because it utilizes the same numerical relationship, with only the number of iterations being unknown. In contrast, partitive division involves the unknown intensive quantity, which proves more abstract.

***Partitive Division Word Problem: 24 pencils ÷ 6 containers ( □ x 6 = 24)***

The partitive division word problem, where the intensive quantity is unknown, was the third problem asked of the students. The question was: *The teacher has 24 pencils that she wants to put equally into 6 containers. How many pencils should she put in each container?* The

purpose of this item was to understand a student's ability to solve a multiplicative situation where the intensive quantity is unknown. The codes for the strategies are outlined in Table 42. This type of problem proved to be the most challenging as there were eight students who answered this incorrectly in both the pre- and post-assessment. It is important to note that there was a very limited focus on partitive division during the mathematics intervention.

**Table 42**

*Codes of Strategies for Partitive Division Problem*

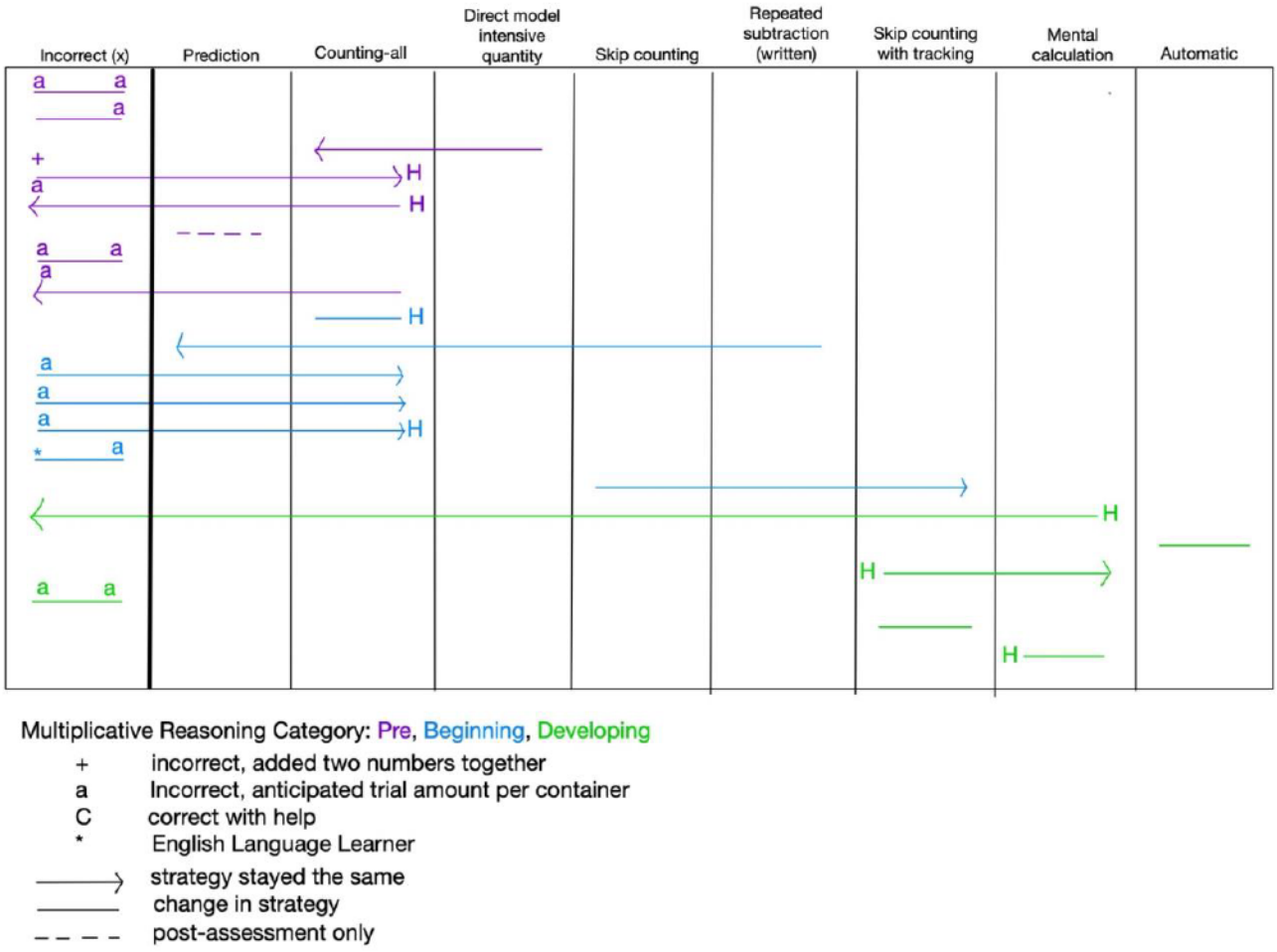
| Strategy | Description of Strategy |
|---|---|
| Counting-all | The student drew a representation of all 6 containers and then filled in each individual pencil, usually with an anticipated amount, counting each tick as they went. |
| Prediction | The student made a prediction of "4." While the answer was correct, they couldn't explain or prove this answer. |
| Direct modelling of the intensive quantity | The student anticipated a trial amount and represented the pencils using tick marks, leaving a space in-between each set to represent the containers. |
| Skip-counting | The student skip counted by a trial amount of 4, using paper to support record and keep track of the count (e.g., 4, 8, 12, 16, 20, 24). |
| Repeated subtraction written | The student repeatedly subtracted an amount from 24, keeping track of their calculations on paper. |
| Skip-counting with tracking | The student skip counted by a trial amount and kept track of the iterations on their fingers. |
| Mental calculation | The student required time to think of the answer and may have used a strategy that demonstrates flexible thinking (e.g., 4, 8 ,12, and then double that to get 24). |
| Automatic | The student arrived at the answer in three seconds or less. |

The analysis for the strategy development can be seen in Figure 24. There was one student that I did not pose this question to in the pre-assessment because they solved the previous

two multiplicative situations by adding both numbers together. Their data is indicated by a dotted line. Overall, 7 students (35%, n=20) answered the question correctly in the pre-assessment and in the post-assessment, 9 students (43%, n=21) answered this question correctly.

**Figure 24**

*Strategy Development Map: Pre- and Post-Partitive Division (24 pencils ÷ 6 containers)* ( □ x 6 = 24)




*Note.* Each line in the figure represents one student. Descriptions of the strategy codes can be found in Table 43.

In the Pre-Multiplicative Reasoning Category (purple), only 1 of the 8 students (13%) was able to answer this question correctly in both the pre- and post-assessment. The students'

attempted strategies for this question continued to rely heavily on counting in both the pre- and post-assessment. In the post-assessment, 1 student answered it correctly, 5 students who answered it incorrectly anticipated a certain number of pencils that would go into each container, and 2 students anticipated a trial amount of 4 but they were not able to accurately iterate 4, six times to get to 24.

The students in the Beginning Multiplicative Reasoning Category (blue) fared slightly better in the both the pre- and post-assessments for this question. In the pre-assessment, 3 of the 7 students (43%) answered this question correctly. By the post-assessment, 4 of the 7 students (57%) were able to answer this question correctly and 2 of the 7 students (29%) were able to answer it correctly with help, for a total of 6 of the 7 students (85%). It should be noted that some of their reasoning strategies were still fragile. For example, one student thought that the answer would be 4. They represented 6 containers in their drawing but only filled in three of the containers, not following through with the full solution to be sure that it would work.

This question also proved to be challenging for students in the Developing Multiplicative Reasoning Category (green) with only 1 of the 6 students (17%) solving this question automatically in the pre- and post-assessment. The answers from students in the post-assessment revealed how students were understanding this multiplicative situation. In the post-assessment, one student immediately answered 4, but when I asked them, "4 what?" their response was '4 containers.' When I reminded them that there were 6 containers, they needed to re-work the problem, resulting in an incorrect answer in their post-assessment of 3 pencils in each container and 8 containers (3 x 8). Another student, in the post-assessment, who counted by 3s up to 24, did not recognize that he did 8 iterations not 6, and recorded his equation as 3 x 6 = 24. On the other hand, one student in the post-assessment was able to commute the problem, iterating 6,

four times, and understood that the four iterations would mean '4 pencils per container." The challenges students had in all the three reasoning categories, highlights the complexity of solving a problem where the intensive quantity is unknown, as they needed to figure out how many pencils go inside each container, creating the simultaneous relationship of pencils per container.

**Equation and Referent Analysis for Partitive Division Problem.** There were sizeable differences amongst the three multiplicative reasoning categories in students' equation writing and understanding of the referents. All referents were analyzed and coded, even if the student incorrectly solved the problem. The codes for the analysis are described in Table 43.

**Table 43**

*Codes for Equation Correctness and Referents for Partitive Division Problem (Post-Assessment)*

| Equation Correctness and Referents | Description |
|---|---|
| Equation correct | The equation was coded as correct if all components of the equation were written correctly, including symbol and number notation, for example 4 x 6 = 24 and 24 ÷ 4 = 6. |
| Equation incorrect | The equation was coded as incorrect if any components of the equation were written incorrectly, including a symbol or number error, for example, 24 + 6 = 4. |
| Intensive quantity (per) | The student identified the intensive quantity as *pencils per container*. |
| Intensive quantity (no per) | The student only described one extensive quantity within the intensive quantity, such as indicating 'pencils' without connecting it to the containers. |
| Iterations | The student identified iterations as *containers*. |
| Total (with referent) | The student identified total as *pencils*. |
| Total (no referent) | The student identified the total without using a referent. |
| Incomplete data for referent | There was insufficient information to determine the student's ability to independently identify the referent. Therefore, a referent code was not applied. |

The detailed analysis of equations and referents for the partitive word problem can be found in Table 44, with the incorrect writing of equations listed first within each of the three multiplicative reasoning categories.

**Table 44**

*Analysis of Equations and Referents for Partitive Division Problem (Post-Assessment)*

| Reasoning category | Pre-Multiplicative Reasoning | | Beginning Multiplicative Reasoning | | Developing Multiplicative Reasoning | |
|---|---|---|---|---|---|---|
| | Equation | Referent | Equation | Referent | Equation | Referent |
| Equation and referent analysis | x (24 = 6 - 3) | IQ, IT, T | x 24 x 6 = 4 | IQ, IT, T | x (3 x 8 = 24) | Iq, IT, t |
| | x (6 x 6 = 24<br>4 x 6 = 24) | IQ, IT, t | x 4 x 6 = 16 | iq, IT, T | x (3 x 6 = 24) | IQ, IT, T |
| | x (24 x 6 = 3) | ~~iq~~, IT, T | x 24 + 6 = 4 | IQ, IT, T | ✓ 24 ÷ 6 = 4 | IQ, IT, T |
| | x (24 + 6 = 30) | ~~iq~~, IT, T | ✓ (4 x 6 = 24) | iq, IT, t | ✓ 6 x 4 = 24 | IQ, IT, T |
| | x 24 - 6 | incomplete data | ✓ 4 x 6 = 24 | incomplete data for IQ only, IT, T | ✓ 4 x 6 = 24<br>24 ÷ 6 = 4 | IQ, IT, T |
| | x no equation H | iq, IT, ~~t~~ | ✓ 6 x 4 = 24 | iq, IT, t | ✓ 24 ÷ 6 = 4<br>6 x 4 = 24<br>24 ÷ 4 = 6 | IQ, IT, T ** |
| | x 24 + 6 = 4 | IQ, IT, T | ✓ 4 x 6= 24 H | iq, IT, t | 4 x 6 = 24 | |
| | x 24 x 6 = 4 | IQ, IT, T | | | | |

*Note*. Each equation or groups of equations represent a different student.

✓ indicates a correct equation and x indicates an incorrect equation

( ) around an equation indicates that the student did not solve the problem correctly; 'H' indicates that the student solved it correctly with help; no bracket indicates that the student solved it correctly.

** The student was able to correct the referents with help and then proceeded to change the 4 and 6 in the equation.

Initials used to indicated full referents: intensive quantity (IQ), iterations (IT), and total (T)

The lower-case version of these referents "iq," "it," and "t" indicates a partial referent. A strike through indicates no referent indicated for the number.

In the Pre-Multiplicative Reasoning Category (purple), 2 of the 8 students (25%) were able to solve the problem correctly and identify all three referents. However, none of the students were able to write the equation correctly, including one student who could not write an equation at all. One student changed their original equation to a correct equation; however, it did not connect with how they solved the problem. Therefore, I coded the equation as incorrect.

In the Beginning Multiplicative Reasoning Category (blue), 2 of the 7 students (29%) identified all three referents and correctly solved the problem although they could not write a correct equation. There was one student who correctly wrote the equation as 4 x 6 = 24, but their answer to the problem was 6 pencils per container and 4 containers. This student is an English Language Learner so it is inconclusive as to why this error may have occurred.

For the Developing Multiplicative Reasoning Category (green), 3 of the 6 students (50%) were able to identify the referents, solve the problem correctly and correctly write an equation. One student was able to describe the referents correctly with help and changed their writing of the equation once they were able to self-correct the referent from 6 pencils per container to 6 containers.

I conducted further analysis by dividing the written equations into subcategories as seen in Table 45. I wanted to find if there were consistent errors and if the students used a multiplication or division equation for this problem.

**Table 45**

*Partitive Division Equation Analysis (Post-Assessment)*

| Incorrect | Incorrect: Symbol error; correct solution | Equation correct for student solution | Equation fully correct |
|---|---|---|---|
| Incorrect solution<br>24 = 6 + 3<br>6 x 6 = 24<br>24 + 6 = 30<br>24 – 6<br>24 x 6 = 3<br>3 x 6 = 24<br><br>Correct solution<br>4 x 6 = 16 | 24 x 6 = 4<br>2 students<br><br>24 + 6 = 4<br>2 students | 3 x 8 = 24<br><br>4 x 6 = 24<br>(referent error) | 6 x 4 = 24<br>2 students<br><br>4 x 6 = 24<br>2 students<br><br>24 ÷ 6 = 4<br><br>4 x 6 = 24/24 ÷ 6 = 4<br><br>24 ÷ 6 = 4/6 x 4 = 24<br>24 ÷ 4 = 6/4 x 6 = 24<br><br>2 students (both operations) |

Consistent with the other items, students who made a symbol error when writing the equation, were able to solve the problem correctly and most students recorded a multiplication sentence. For those students who wrote an incorrect equation, most, 8 of the 11 (73%), began with the number 24. For those students who answered this problem correctly, 7 of the 12 (68%) were able to correctly write a corresponding equation. Of these 7 students, 4 wrote a multiplication equation and 2 students wrote both multiplication and division equation.

***Summary of Seven Assessment Items Analysis***

The purpose of the analysis of these seven assessment items was to examine students' progress in their development of multiplicative reasoning within the three multiplicative reasoning categories. These items required a range of mathematics concepts starting with the early stages of number development, including subitizing, through to how they solved various

multiplicative situations. As part of my analysis, I also wanted to see if there were any changes in multiplicative reasoning of the six students in the Developing Multiplicative Reasoning Category. They were therefore asked one or two more challenging additional items in both the pre- and post-assessments. As seen in the pre-assessment, students in this reasoning category demonstrated a stronger foundation of readiness for learning multiplication and division at the outset of the mathematics intervention.

***Multiplicative Word Problem: Analysis for Students in Developing Multiplicative Category on Extra Items (n=6)***

Each student in the Developing Multiplicative Reasoning Category (green) was asked the following problem: *'Three candies cost 8 cents. How much do 12 candies cost?'* The purpose of this question was to examine students' understanding of multiplicative reasoning at a higher level of complexity in a multiplicative situation that involved a ratio without a unit rate. Furthermore, I wondered, did students in this category develop greater multiplicative reasoning skills during the mathematics intervention even though this particular problem type was not addressed?

The codes used for strategies for this question are found in Table 46. These strategies are not an exhaustive list; instead, they are the strategies that were seen in this cohort.

**Table 46**

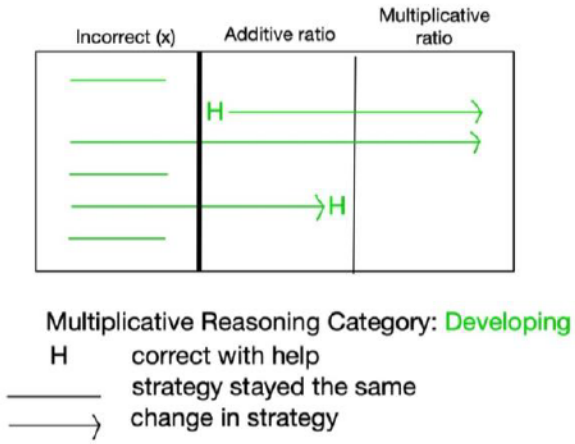
*Codes of Strategies for Multiplicative Word Problem*

| Strategy | Description of Strategy |
|---|---|
| Additive ratio | The student kept track of both the number of candies and the cents, either mentally or in writing. For example: 3:8; 6:16; 9:24; 12:32. |
| Multiplicative ratio | The student demonstrated an understanding of proportionally: since 3 candies cost 8 cents, they multiplied both the candies and the cents by 4 to determine that 12 candies cost 32 cents. |

The strategy development for this question can be seen in Figure 25.

**Figure 25**

*Strategy Development Map: Pre- and Post-Multiplicative Reasoning Problem (n = 6)*




*Note.* Each line in the figure represents one student in the Developing Multiplicative Reasoning Category. Descriptions of the strategy codes can be found in Table 46.

Initially, only one of the six students (17%) correctly solved the problem: "Three candies cost 8 cents. How much do 12 candies cost?" and even then, they required assistance. The

remaining students incorrectly attempted to add 8 twelve times. In the post-assessment, however, the success rate rose to 50% (3 of the 6 students). One student successfully shifted from an additive to a multiplicative ratio, while two students moved from incorrect responses to an additive ratio strategy (one of whom achieved the correct answer with help). Although, these types of problems were not part of the instruction, some students in the Developing Multiplicative Reasoning Category were further challenged with a double-digit multiplication problem as discussed next.

***Multiplication Word Problem: 12 rows x 18 plums per row***

Some of the students in the Developing Multiplicative Reasoning Category were asked to solve the following problem: '*Plums in a grocery store are arranged in 12 rows. There are 18 plums per row. Do you know how many plums there are?'* The purpose of this item was to investigate students' understanding of concepts that go beyond the grade 3 curriculum in Ontario, before they may have learned a formal procedure (algorithm) of how to multiply two, two-digit numbers as well as their ability to visualize an array. The codes that were used for this item can be seen in Table 47.

**Table 47**

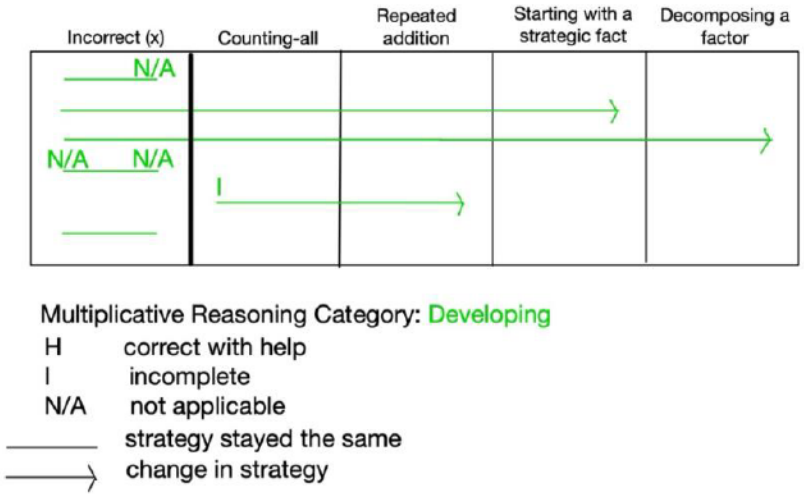
*Codes of Strategies for Double-Digit Multiplication Problem*

| Strategy | Description of Strategy |
|---|---|
| Counting-all | The student attempted to represent 18 plums in a row by drawing tick marks and iterating twelve times, with the intent to go back and count them all once they had been drawn. |
| Repeated addition | The student repeatedly added one factor to get the answer. |
| Starting with a strategic fact | The student used a fact that they knew to build upon to get the answer. For example, 12 x 12 = 144. |
| Decomposing a factor | The student used the distributive property (e.g., 12 x 18 = (12 x 8) + (12 x 10). |
| Not applicable | The student was not asked this question during the interview. |

Students were only asked this question if they used efficient strategies when solving the partitive word problem and they had an entry point when solving the previous multiplicative word problem. Therefore, the code *not applicable* was used when students were not asked this question during either the pre- or post-assessment. There were 5 of the 6 students (83%) in the Development Multiplicative Reasoning Category who were asked to solve this question in the pre-assessment, with 4 out of the 6 students (67%) who were asked this question in the post-assessment. Out of these five students in the pre-assessment, four of them solved this problem incorrectly. One student had a strategy to draw out 18 tick marks, 12 times and count them all to find the answer. Since I knew the strategy they were going to use, I told them they did not have to draw them all out. From the pre- and post-assessment, there was evidence in a shift in strategies as seen in Figure 26.

**Figure 26**

*Strategy Development Map: Pre- and Post-Double-digit Multiplication Problem (12 rows x 18 plums per row) (n=6)*




*Note.* Each line in the figure represents one student in the Developing Multiplicative Reasoning Category. Descriptions of the strategy codes can be found in Table 47.

In the post-assessment, one student attempted an algorithm and ended up with an answer of 96. When he looked at the answer he self-corrected by using the known fact of 12 x 12 = 144, a fact that he was practicing during the mathematics intervention and added 12 six more times to get the answer. Since he was able to self-correct in the moment, I coded this solution as correct. The student who was going to concretely model by drawing out 18 tick marks, 12 times and counting them all, changed to repeatedly adding 12, 18 times, keeping track of the iterations on a row of squares they drew. The analysis of these two items highlights growth for some students in the Developing Multiplicative Reasoning Category.

**Summary of Item Analysis for Subitizing, Additive Reasoning, and Skip Counting**

In order to develop a general description of students' strategies and progress within each category, I identified the prevalent methods students used during pre- and post-interviews as summarized in Table 48. There were identifiable trends and patterns in both students' readiness to develop multiplicative reasoning and the progression or growth that occurred within each of the three reasoning categories.

**Table 48**

*Summary of Student Thinking with the Three Multiplicative Reasoning Categories: Strategies Supportive of the Development of Multiplicative Reasoning*

| Mathematics Development | Multiplicative Reasoning Category | | |
|---|---|---|---|
| | Pre-Multiplicative Reasoning | Beginning Multiplicative Reasoning | Developing Multiplicative Reasoning |
| | Subitizing and Visual Iterations | | |
| Pre | subitize and double a small amount | subitize and double a small amount and iterate a visual using a counting strategy | subitize and iterate with an efficient strategy or automatically |
| Post | can accurately subitize and iterate the visual with support from fingers<br><br>can mentally hold the image | can accurately subitize and iterate the visual using additive strategies<br><br>can mentally work with the image | subitize and iterate with automaticity |
| | Additive Reasoning | | |
| Pre | counting-all | counting-on | automatic |
| Post | counting-on | counting-on | automatic |
| | Skip counting by 2s | | |
| Pre | skip counting sequence not fluent with pauses after the count of 8<br><br>able to track and coordinate the iterations<br><br>reliant on the model | skip counting sequence has errors with pauses after the count of 8<br><br>coordinating the iterations somewhat fluently<br><br>reliant on the model | skip counting sequence is fluent<br><br>coordinating iterations with fluency<br><br>model supports tracking the iterations |
| Post | more fluent skip counting sequence with pauses after the count of 10<br><br>coordinating the iterations somewhat fluently<br><br>reliant on the model | more fluent skip counting sequence<br><br>coordinating iterations with fluency<br><br>model needed to support tracking the iterations | skip counting sequence fluent<br><br>answer to question is automatic<br><br>model not required |

The skip counting section includes the analysis of the multiplication problem with a direct model and students' ability to recite the skip counting sequence by 2s.

Overall, students initially in the Pre-Multiplicative Reasoning Category (purple) were able to subitize and double a small amount, calculate a simple addition question by counting-all, and skip count by 2s up to 8 fluently (coordinating with a direct model). By the post-assessment, they were able to accurately visualize and iterate an image using their fingers to support their count, solve an addition question by counting-on, and fluently count by 2s past 10. There were subtle shifts in their efficiency of iterating a visual image and in their counting strategies.

At the outset of the mathematics intervention students in the Beginning Multiplicative Reasoning Category (blue) were also able to subitize and double a small amount as well as mentally iterate an image using a counting strategy. They used counting-on to solve a simple addition question and were only fluent in their skip count by 2s to 8, and somewhat fluent with their coordination of the sequence with a direct model. By the post-assessment they were able to accurately subitize and iterate a visual model using additive strategies; however, they remained at the counting-on stage when solving an addition question. There was improved fluency in their skip counting sequence by 2s and they were able use the direct model fluently to keep track of their iterations. The shifts indicate a change from early counting stages towards additive reasoning.

The students in the Developing Multiplicative Reasoning Category (green) already had the ability to subitize and iterate as well as add simple addition questions automatically, without needing time to calculate. The shift for these students was evident in their skip counting sequence moving from a fluent sequence with tracking using a direct model to no longer needing the model to support their skip count by 2s by the post-assessment.

To further solidify my summary description of the student strategies in the word problems and their understanding of the referents in connection to the writing of equations, I returned to my classroom data. I used multiple sources of data, including classroom lesson videos, student artifacts, and daily field notes, to better understand and qualitatively corroborate my interview findings as described in Stage 3 of my analysis.

**Stage 3: Analysis of Classroom Data to Triangulate Word Problem Findings**

In my analysis of the word problems in the pre- and post-assessment, there were themes that I identified amongst and within the three multiplicative reasoning categories that I wanted to further investigate in support of answering my research question and sub research question. I looked at four areas to gather further evidence, investigating how these areas developed over time in the classroom. Each of the areas will be discussed in turn:

- How students constructed their understanding of the intensive quantity and the referents.
- How they developed simultaneity.
- How they learned to write equations with meaning: both symbols and referents.
- How students developed key ideas of multiplicative reasoning: associative property and the relationship between multiplication and division

I highlight these areas using specific examples from the classroom data.

***Construction of the Intensive Quantity and the Referents***

For students to be able to recognize that the numbers in a multiplicative situation are different than in addition they need to understand the meaning of the numbers. Schwartz (1988, 1996) defines multiplication and division as operations that are '*referent transforming*' as the referents change because of the operation. Understanding the meaning of the numbers was a challenge for students in the Pre-Multiplicative Reasoning Category as evidenced in the student

artifacts in *Lesson 7* (May 15, 2024). Students were given three problems to solve on their own; all problems used the numbers 8 and 10: two multiplication problems on the front page and one addition problem on the back page (Lawson, 2026). In the Pre-Multiplicative Reasoning Category, 5 of the 8 students (63%) multiplied the two numbers together in the addition problem for an answer of 80 (10 x 8 = 80) instead of adding the two numbers together (10 + 8 = 18). They were still working on recognizing the difference between an addition and multiplication situation and the role of the numbers within each of those operations.

It is important to note that evaluating student's knowledge from their artifacts is challenging as there are many variables to consider, such as conversations students are having with other students or educators. Nonetheless, the number of students in this early group that multiplied instead of added signals the continued time some students need to understand the difference between an additive versus a multiplicative word problem.

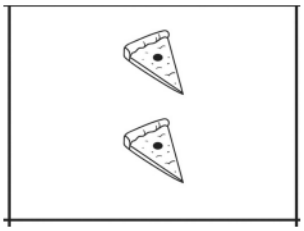
To further differentiate multiplicative and additive situations, Schwartz (1988, 1996) points to the importance and role of the referents, particularly the complexity of the intensive quantity, a unique relationship between two other quantities that only exists in multiplicative situations. This was evident in the data analysis for partitive division, where '4' is only the correct answer if the corresponding referent of 'pencils per container' is attached. The process of developing an understanding of the correct referent with a number was also apparent during a warm-up activity in *Lesson 9* (May 17, 2024).

Students were using a bingo card that was designed with images to support identifying the various referents and writing a corresponding equation based on a clue that I read (Lawson, 2026). In this example, the clue I read was the intensive quantity of '2 pepperoni per slice.' One

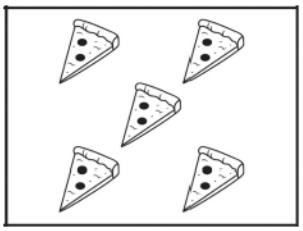
student from the Beginning Multiplicative Reasoning Category identified an image that had a total of '2 pepperoni' (1 per slice or 2 slices) as seen in Figure 27.

**Figure 27**

*Images from Bingo Card (Lawson, 2026)*



1 pepperoni per slice, 2 slices



2 pepperoni per slice, 5 slices

*Note.* These images are like the images that the students used for the research project in 2024. The images have been updated to include the publication images from "What to look for 2: Understanding and developing student thinking in multiplicative reasoning" by A. Lawson, 2026. Copyright 2026 by Pearson. Reprinted with permission.

This brought up the important discussion of the difference between both quantities of '2' where in one example, the '2' was the total number of pepperoni, and the other example was the intensive quantity, '2 pepperoni per slice.'

***Strategy to Support the Development of Simultaneity***

During the intervention, I worked on supporting students on the strategy of skip counting and tracking the iterations at the same time. Wilkins and Ulrich (2017, p.13) identify students' ability to skip count and track as an important transition strategy for students' thinking toward simultaneity (although this strategy does not necessarily mean a student is thinking multiplicatively). In *Lesson 10* (May 21, 2024) as viewed in the video recording of the classroom lesson, I asked the students how many times we would we need to count by 5s to get to 100.

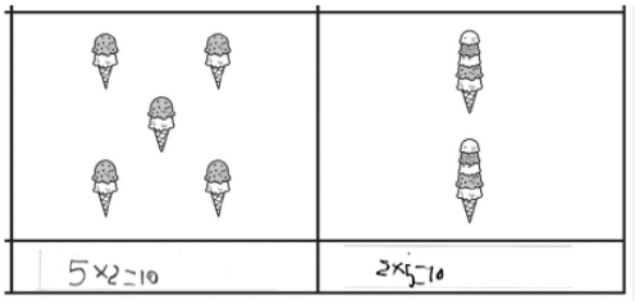
The responses within the three multiplicative reasoning categories provide examples of how students are developing the ability to skip count and track the iterations. In the Pre-Multiplicative Reasoning Category, one student said 10 and changed their mind to 3. Another student said it would be two or three. In the Beginning Multiplicative Reasoning Category, a student said 10, and in the Developing Multiplicative Reasoning Category three students said the correct answer of 20. These examples demonstrate that while students need to know their skip counting sequence, they need to be able to effectively skip count and track the iterations, developing the simultaneity needed to solve multiplicative problems.

### *Writing of Equations with Meaning*

While writing equations, students practiced connecting mathematical notation and the meaning of the numbers. During *Lesson 8* (May 16, 2024), students practiced writing equations based on verbal prompts. For example, after I announced a total, such as *10 scoops*, students identified the corresponding image on their bingo cards and recorded a matching equation, as seen in Figure 28.

**Figure 28**

*Bingo Images and Student Equations for '10 scoops'*

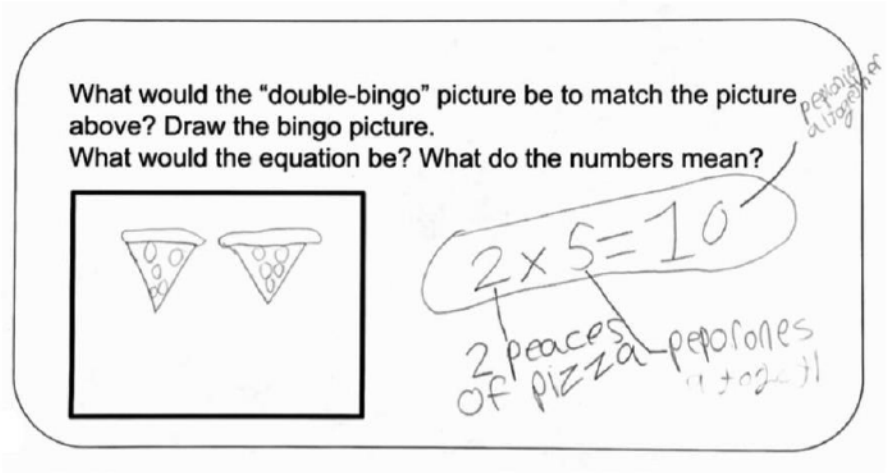



*Note.* These images are like the images that the students used for the research project in 2024. The images have been updated to include the publication images from "What to look for 2: Understanding and developing student thinking in multiplicative reasoning" by A. Lawson, 2026. Copyright 2026 by Pearson. Reprinted with permission.

Students were grappling with writing meaningful equations differently across the three multiplicative reasoning categories. Within the Pre-Multiplicative Reasoning Category, two students recorded only the referents that I read aloud underneath the image, while the third was not sure how to begin. In the Beginning Multiplicative Reasoning Category, one student practicing with an image on the bingo card, identified referents but missed using 'per' to distinguish the intensive quantity. As shown in Figure 29, the student wrote *2 pieces of pizza x 5 pepperoni = 10 pepperoni altogether* (student spelling errors corrected).

**Figure 29**

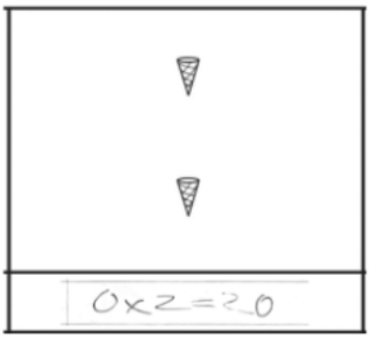
*Student Work for '10 pepperoni altogether'*




For the Developing Multiplicative Reasoning Category, explaining the zero in the meaning of an equation proved challenging. With the image of *0 scoops per cone, 2 cones*, one student wrote the equation as 1 + 1 = 2 instead of 0 x 2 = 0. Another student initially wrote the equation as 0 x 2 = 2, and then after discussion was able to correct it to 0 x 2 = 0, as seen in Figure 30.

**Figure 30**

*Student Work for '0 scoops per cone, 2 cones'*




*Note.* This image is like the image that the students used for the research project in 2024. The image has been updated to include the publication image from "What to look for 2: Understanding and developing student thinking in multiplicative reasoning" by A. Lawson, 2026. Copyright 2026 by Pearson. Reprinted with permission.

These examples highlight the thinking that is required to write a mathematical equation with understanding.

***Development of Multiplicative Reasoning***

Within the Developing Multiplicative Reasoning Category, three students demonstrated further growth in their strategy development, as evidenced by their responses to additional questions in the pre- and post-assessments. During the intervention, one student successfully articulated the relationship between 5 and 10 (noting that 5 is half of 10), when connecting 5s and 10s multiplication facts, drawing on the associative property. Another student consistently recorded both multiplication and division equations for the division word problems in the post-assessment, consistently making connections between these operations during mathematics

instruction, and demonstrating a strong grasp of the inverse relationships between these two operations. These examples highlight the growth in multiplicative reasoning achieved during the intervention.

**Summary of Strategy Development in the Three Multiplicative Reasoning Categories**

The data from the classroom fleshed out and qualitatively corroborated much of the data from the interviews. To create a full synopsis of the students in the three multiplicative categories areas according to their solving of the word problems (multiplication, partitive and quotative division) and the class data, I summarize the progress of their strategy development in Table 49.

**Table 49**

*Summary Strategy Development within Word Problems According to the Multiplicative Category*

| Mathematics development timing | Multiplicative reasoning category | | |
|---|---|---|---|
| | Pre-Multiplicative Reasoning | Beginning Multiplicative Reasoning | Developing Multiplicative Reasoning |
| | Word problem strategies for multiplication | | |
| Pre | limited understanding of multiplication context; added the numbers together | able to solve the problem with help within counting stages | using efficient strategies or solving the problem automatically |
| Post | shift from adding the two numbers to understanding how to create the intensive quantity<br><br>shift in strategy development to the strategy of skip counting | shifted to more refined strategies particularly rhythmic counting | automatic |
| | Word problem strategies for quotative division | | |
| Pre | cannot solve the problem correctly | the problem is solved either incorrectly or correctly using counting strategies | strategies involving simultaneity e.g., skip counting and tracking |
| Post | strategies shift to greater simultaneity, e.g., creating the intensive quantity either mentally or with a model though still not able to solve the problem correctly | can solve the problem using counting strategies | shift in efficiency of strategies moving towards automaticity |
| | Word problem strategies for partitive division | | |
| Pre | cannot solve the problem correctly | the problem is solved either incorrectly or correctly with the use counting strategies | can solve the problem with trial-and-error amounts using strategies involving simultaneity, e.g., skip counting |
| | some students able anticipating a trial amount | can anticipate a trial amount | counting and tracking |
| Post | more students able to use trial and error amounts but with limited success | can solve the problem with counting strategies using trial and error amounts | shift in efficiency of strategies moving towards automaticity |
| | Communicating mathematical thinking through equation writing | | |
| Overall | support is still needed to accurately use symbols and identify the referents when writing equations | shifting towards accuracy in equation writing and referent identification | accurately able to communicate mathematical thinking with an equation and an understanding of the referents |

Students in the Pre-Multiplicative Reasoning Category (purple), had limited understanding of how to solve the three multiplicative situations in the pre-assessment. For the multiplication problem, they added the numbers together and they could not correctly solve the quotative and partitive division problems. There were noticeable shifts by the post-assessment where the majority of these students were able to understand the multiplication problem was not addition, leading to a change in strategy development of skip counting. In quotative division, by the post-assessment, they were able to use strategies that required simultaneity including the creation of the intensive quantity mentally or with a direct model, however, they still struggled to answer the question correctly. In partitive division, students could anticipate a trial amount but had challenges testing these numbers out successfully in the context of the problem. By the end of the intervention, these students still required further support to accurately communicate writing an equation and identifying the referents.

In the Beginning Multiplicative Reasoning Category (blue), students initially were able to solve the multiplication problem with help using counting strategies. For quotative and partitive division, students were either unable to accurately solve the problem or correctly solve the problem using counting strategies. By the post-assessment, these students shifted to using more refined strategies, such as rhythmic counting to solve the multiplication problem and solved both types of division problems correctly using counting strategies. Their equations indicate that they are heading towards accuracy in their symbol communication and referent identification.

In the Developing Multiplicative Reasoning Category (green), all students were able to solve the multiplication and quotative division situations correctly prior to the intervention. In multiplication, the majority of students solved this problem automatically in the pre- and post-assessment. In both division multiplicative situations in the pre-assessment, the students were

able to solve the problems using strategies that demonstrated simultaneity (e.g., skip counting and tracking). Students demonstrated a strong understanding of the referents and were able to accurately use symbols to write an equation.

The results of the data analysis, within and amongst the three categories of multiplicative reasoning, highlight both the process and pace of student development. In the discussion, I will synthesize these findings with current literature to provide further insight into the developmental trajectory of multiplicative reasoning in children.

# Chapter 5: Discussion and Conclusions

The complexity of how children develop multiplicative reasoning has been on the mind of many researchers and theorists for decades (see e.g., Hulbert et al., 2017; Greer, 1992; Zwanch & Wilkins, 2020). Given the range of theories and viewpoints, how should educators take on this complex task of supporting students in their development of multiplicative reasoning at the outset of their formal introduction to multiplication and division? To explore this quandary, I examined the viewpoints of three influential theorists: Vergnaud, Schwartz, and Steffe with contrasting theories of multiplicative reasoning development. Park and Nunes (2001) argue that there are two hypotheses of multiplication development: Vergnaud and Schwartz fall under the *schema of correspondence theory,* where multiplication is based on the relationship between two quantities (i.e., ratio or rate) and Steffe falls under *implicit theory*, with the understanding of multiplication as evolving from repeated addition.

While grouped with Vergnaud, Schwartz (1988, 1996) was unique in explicitly identifying and defining the difference in quantities that exist in multiplication and division—in particular the intensive quantity. In Schwartz's (1996) unpublished article *Semantic Aspects of Quantity*, he argues that most numbers we use are connected to a noun and that the number acts as an adjective (e.g., ten cars), defining this as an *adjectival quantity*. He goes on to further and explicitly argue:

> if it is true that the fundamental reason for the presence of mathematics in the school curriculum is to equip people with a set of analytic tools for modeling their world, then it is at least plausible to explore the question of why it is that we do not teach mathematics of such quantity in schools but rather the mathematics number devoid of referents. (p. 6)

This quote captures the purpose of my research, and therefore I set out to answer the following research question and sub question:

*1) How does an instructional focus on the meaning of the referents and the development of the intensive quantity in multiplicative situations affect children's early development of multiplicative reasoning?*

*(1a) What is the role of simultaneity in the development of multiplicative reasoning in grade 3 students?*

I examined video recordings of pre- and post-assessment mathematics interviews with 21 third-grade students. Based on their pre-assessments, I organized them into three multiplicative reasoning categories: *Pre-*, *Beginning*, and *Developing*. By identifying themes and patterns across these categories, I arrived at four key findings:

1. Students' initial stage of additive reasoning (i.e., Pre- and Beginning Multiplicative) significantly influenced their development of multiplicative reasoning—particularly those relying on single-counting strategies.
2. All students, even those using single-counting, successfully constructed the intensive quantity and identified some or all referents, marking the emergence of early, albeit for some fragile, multiplicative reasoning.
3. Rather than remaining solely on an additive reasoning pathway, all students are now positioned on or are approaching a developmental pathway toward multiplicative reasoning.
4. To fully capture student thinking, I needed to include more detailed indicators in my draft hypothetical progression, *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning*.

For each of these key findings, I will discuss how they answer my research question and sub question, with connections to the existing literature and research. I will then outline my final

conclusions, share implications for classroom instruction, and offer considerations for future research.

We begin the discussion with the first key finding to help us understand how students' responses to a simple addition question is an illuminating measure of their progression toward multiplicative reasoning.

**Key Finding #1: Progress of the 14 Students Using Single-Counting**

My first key finding is that a students' stage of additive reasoning prior to the intervention significantly influenced their subsequent development of multiplicative reasoning, in particular for those students relying on single-counting strategies in the Pre-Multiplicative and Beginning Multiplicative Reasoning Category. To investigate this relationship, I used students responses on the addition question "8 + 6" prior to the intervention as a key indicator to place them in one of the three multiplicative reasoning categories: Pre-, Beginning, and Developing. This classification allowed me to examine how students' initial readiness correlated with their progress.

In the class of 21 students, 14 (67%) were using a single-counting strategy to solve 8 + 6 prior to the start of the intervention. Of these 14 students, 8 were in the Pre-Multiplicative Reasoning Category and 6 were in the Beginning Multiplicative Reasoning Category. In the pre-assessment two students solved the question incorrectly, six used a "counting-all" strategy, and six used a "counting-on" strategy. There are mathematical consequences for students who continue to be dependent on counting strategies in later grades.

Hopkins et al. (2020) clearly captures this issue in their article entitled, "*Is counting hindering learning? An investigation into children's proficiency with simple addition and their flexibility with mental computation strategies."* In their research with 166 grade 3 and 4 students,

approximately 30% of them were at the stage where they were counting-on from a larger number to add. These students could not accurately retrieve basic addition facts and were less flexible than their peers with mental computation strategies (p.52). Hopkins et al. noted that previous discussions about the student struggles with counting tended to be focused on students with learning disabilities —however, this is no longer the case as it applied to a much wider range of student abilities (p. 65).

This raises a critical question: Can students who remain reliant on single-counting successfully develop multiplicative reasoning? Or conversely, is there a viable entry point into developing multiplicative reasoning when foundational additive reasoning has not yet been solidified?

***Readiness for Multiplicative Reasoning of Students Using Single-Counting Strategies***

Drawing on Steffe's (1988, 1992, 1994) number sequence trajectory, Wilkins et al. (2020) conducted a three-year study of 5747 children. They found that students who were at an *Initial Number Sequence* (counting-on) or Tacitly Nested Sequence (working with the composite unit) stage in grade 2, had a better chance of developing multiplicative reasoning by grade 4 than peers who had 'high rote computational skills' (p. 529) (i.e., the ability to identify numbers and pair two numbers to make 10) but lacked the ability to count-on. In my own research, eight students had not yet reached the counting-on stage at the outset of the intervention. It reveals that students who are still using counting strategies have further to go to be able to reason multiplicatively. When students are in the earliest counting stage, it makes it harder for them to develop multiplicative reasoning and the shift to support students beyond this counting stage is not easy to facilitate. Steffe (1992) captures the issue, stating: "knowing *what* has to be done by

children to construct the TNS [Tacitly Nested Number Sequence] helps us to form goals to help them do it, but it does not necessarily make achieving these goals easy" (p. 305).

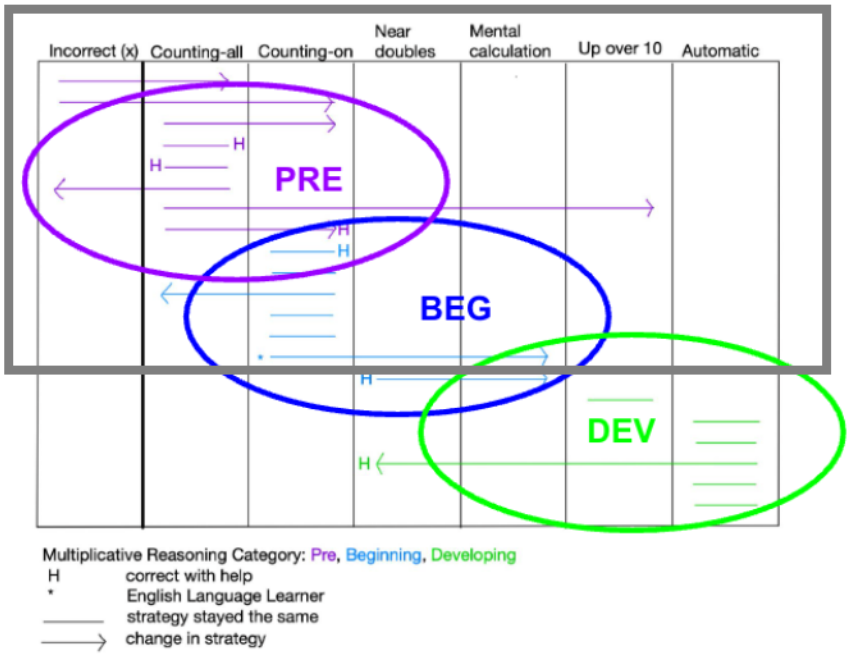
Developing multiplicative reasoning is hard for students still in the counting stage. In Steffe's (1988, 1992, 1994) number sequence trajectory, his pathway describes the development of multiplicative reasoning as an evolution from counting to repeated addition but does so without a focus on the meaning of the numbers. Schwartz's (1988, 1996) theory, on the other hand, offers an alternative departure point by interrupting the traditional counting-based pathway. Therefore, my analysis sought to determine whether students in the counting stage could nonetheless approach multiplicative reasoning, *if* instruction (rooted in Schwartz's theory) prioritized the construction of the intensive quantity.

My data suggest that students who were still in counting stages at the start of the intervention nonetheless benefited and developed some understanding of the mathematics concepts that they need to reason multiplicatively—as captured in the strategy development maps. The maps provide a visual representation of how students within and amongst the three multiplicative reasoning categories, including those students still in the counting stages, were able to demonstrate changes in their mathematical understanding on their pathway towards the development of multiplicative reasoning. The colour coding of the three multiplicative reasoning categories and the lines representing each student on the *strategy development maps* provide a visual indicator of students' thinking from the pre- and post-assessment. The strategy development maps are designed with increased sophistication of strategies and levels of simultaneity progressing from left to right. One can readily see the distinctions amongst the three categories in Figure 31. As an example, I use the strategy development map of pre- to post-solution strategies of the additive item "8 + 6" to highlight these distinctions.

Overall, students in the Pre-Multiplicative Reasoning Category start near the left-hand side, in the Beginning Multiplicative Reasoning Category students hover around the middle, and in the Developing Multiplicative Reasoning Category the students are at the right-hand side. Amongst all three categories, from the pre- and post-assessments, there is a general shift towards the right with very few examples of dramatic movement from one side of the strategy development map to the other.

**Figure 31**

*Student Placements in the Multiplicative Reasoning Categories on the Strategy Development Maps (n = 21) With a Focus on 14 Students Using Single-Counting*

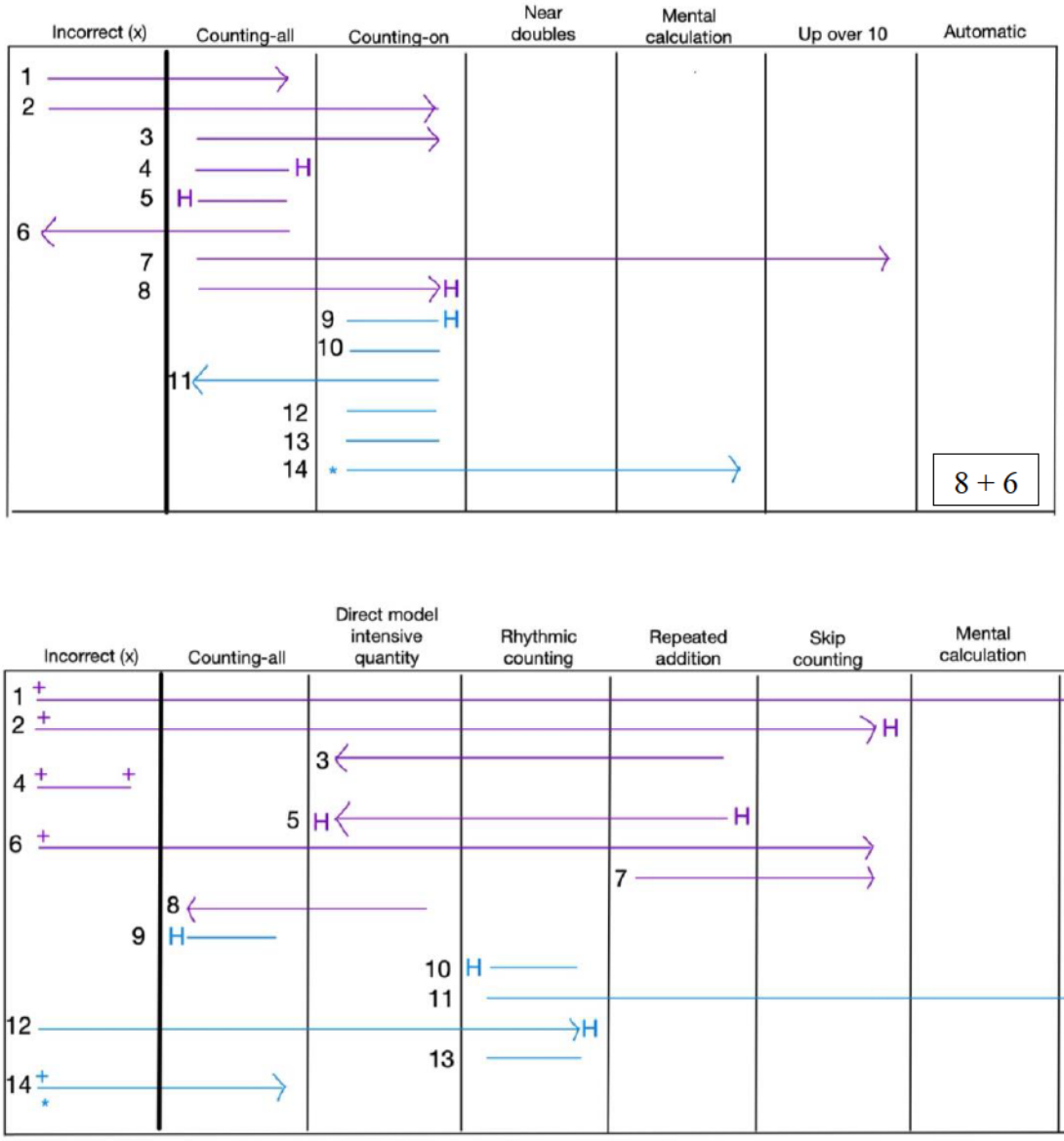



Categorizing students in this way allowed me to analyze the subtle nuances that occur in their early development of multiplicative reasoning, details that might otherwise have been missed, in particular for students who relied on counting strategies. Specifically, how did these students navigate the transition toward multiplicative reasoning, or what prevented them from

doing so? To understand how the 14 students who initially used single-counting strategies progressed during the intervention, we can compare the strategy development maps for the addition problem (8 + 6) and the multiplication problem (4 x 3 = total unknown), as seen in Figure 32. Numbers have been added for each student in the maps, to allow for a comparison between their additive and multiplicative reasoning development. As the maps demonstrate, there were some students in both the Pre-Multiplicative (purple) and the Beginning Multiplicative (blue) categories showed progress in both the additive and multiplicative reasoning from the pre- to post-assessment.

**Figure 32**

*Strategy Development Maps, for Single Counting Students Pre- and Post-Assessment for 8 + 6 and Multiplication Problem (4 x 3 = □) (n=14)*




*Note.* Full legends for both of these *Strategy Development Maps* can be found in Chapter 4 on p. 100 and p. 117.

For the 8 students placed in the Pre-Multiplicative Reasoning Category, in their pre-assessment, 6 of the 8 students were counting-all, with 2 of the 8 students incorrect. By the post-assessment, 7 of the 8 students were able to answer 8 + 6 correctly, with 4 of the 8 students solving this question using a counting-on strategy or higher. In the multiplication problem, in the pre-assessment, 4 of the 8 students tried to solve the problem by adding the two numbers together, with only 1of the 8 students who added the numbers together by the post-assessment. When we compare the strategy development between the additive and multiplication problem, 3 of the 8 students (students 1, 2, 7) demonstrated strategy changes in both assessment items (one student correct with help).

Monitoring shifts in additive thinking within the Pre-Multiplicative Reasoning Category is important, as the transition from "counting-all" to "counting-on" is complex. Students often oscillate between these strategies, applying counting-on in some contexts while regressing to counting-all in others. Tzur and Lambert (2011), in their study of 37 grade 1 students, concluded that this transition, moving toward a higher level of numerical abstraction, is complex and *non-linear* (p. 447). Notably, there were some students within this category who were able to make progress both in their additive strategies and multiplicative strategies.

These results are particularly interesting in comparison to the six single-counting students in the Beginning Multiplicative Reasoning Category. Within this group, there was minimal development regarding the additive item; 4 of 6 students continued to rely on "counting-on", in both the pre- and post-assessment (with one student correct after help). However, 5 of the 6 students demonstrated progress in either correctness or strategy development on the multiplication problem. Specifically, 2 students (students 11 and 14) improved their strategy, 2

(students 9 and 10) transitioned from correct with help to correct in the strategy they were using, and 1 (student 12) moved from an incorrect response to solving the problem with help.

These examples highlight how some students were able to develop strategies to accurately solve a multiplicative situation, despite still being in a counting stage, with some students concurrently improving their additive strategy development. Another shift in thinking for some students in the Pre-Multiplicative and Beginning Multiplicative categories was in their ability to subitize and iterate a visual image.

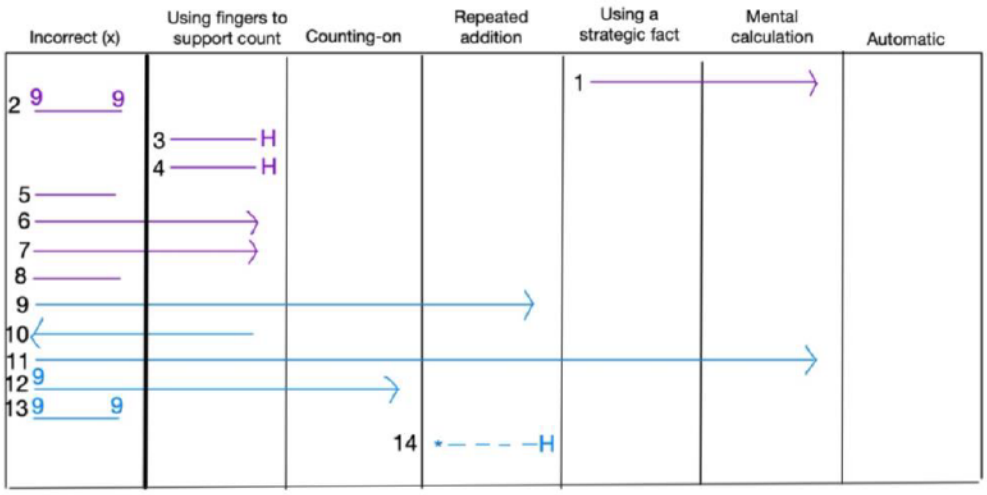
***Subitizing and Simultaneity: A Path to Developing Multiplicative Reasoning?***

Subitizing, the ability to see 'grouped objects' as a unit without having to count each object individually, I argue, fosters students' development towards multiplicative reasoning because it interrupts a counting pathway. Subitizing is an important skill to develop as evidenced in research by Starkey and McCandliss (2014). In their research with 378 students from K to 3, they found that students' ability to '*groupitize*,' or put together subsets of grouped arrangements correlated with students' fluency in operations (p. 134).

If we focus on the 14 students who were in counting stages prior to the start of the mathematics intervention (all in the Pre- and Beginning Multiplicative Reasoning Categories), the majority could not successfully subitize and iterate a visual image in either assessment item. Nonetheless, some of these students made notable changes from the pre- and post-assessment as seen in Figure 33. There were 6 of the 14 students (students 1, 6, 7, 9, 11, and 12) who demonstrated growth in the 3 dots per plate x 4 plates item and 9 of the 14 students (students 1-3 and 9-14) who demonstrated growth in the 8 dots per plate x 2 plates item.

**Figure 33**

*Changes from Pre- and Post-Assessment in Subitizing Items with Students Using Single Counting Strategies at the Outset of the Mathematics Intervention (n=14)*




*Note.* Full legends for both of these *Strategy Development Maps* can be found in Chapter 4 on p. 104 and p. 108.

This change in subitizing strategies reveals students' shift in simultaneity as they were able to hold onto a mental image and accurately iterate it, even while possibly still using a counting strategy. Being able to hold an image and work with it, in this case a set of dots as a

quantity to be iterated, is a step towards the pathway of developing multiplicative reasoning. These changes in strategy development occurred even though these students were still solving an addition calculation by counting-all or counting-on.

If students in my study who were in counting stages were nonetheless able to approach the pathway to develop multiplicative reasoning, how did this happen? This leads us to consider the second key finding. What role did learning about the intensive quantity and the respective referents, play in students' development of early multiplicative reasoning?

**Key Finding #2: Construction of the Intensive Quantity and Identification of Referents**

My second key finding is that the construction of the intensive quantity and identification of some or all the referents was possible for all students, even students using single-counting, marking the emergence of early, albeit for some fragile, multiplicative reasoning. The intervention for all 21 students provided opportunities for them to construct and develop the meaning of the intensive quantity and referents in multiplicative situations. Students actively worked on constructing the intensive quantity through social interactions with their peers (Santrock et al., 2010) in order for them to grapple with these initial multiplicative relationships.

***The Construction of the Intensive Quantity***

The intervention focused on students learning about the meaning of multiplication and division with an emphasis on the construction of the intensive quantity. Students worked on constructing the meaning of the intensive quantity by building it with materials (e.g., 10 beans per tray), through quick images (e.g., an image of 3 eggs per nest), and practicing using the term 'per' to connect two quantities (e.g., 4 pieces of gum per package) (Lawson, 2026). During the intervention and the analysis, I observed and examined students construction of the intensive quantity in various learning opportunities (e.g., games, solving problems, equation writing) to

ensure that their understanding of the intensive quantity went beyond simply using the word 'per.'

Schwartz (1988) described the various ways that the intensive quantity can be created. With students learning about the intensive quantity for the first time, the instruction focused on the creation of the intensive quantity by combining two discrete extensive quantities. An example of this is *10 beans per tray*, the two quantities being 10 *beans* and one *tray* in a relationship to create the unique intensive quantity, 'beans per tray.' According to Yamanoshita and Matsushita (1996), this is how children learn about the intensive quantity in Japan beginning in Grade 2.

Initiating the development of the intensive quantity with discrete quantities differs however from how Simon and Placa (2012) conceive of this development. They conducted a *theoretical exploration* on how students in grade 3 might construct the intensive quantity, expanding on the definition set out by Schwartz. This theoretical exploration involved thinking about students' use of the double-number line measurement model for solving multiplication and division problems to develop the connection between two continuous quantities by creating the intensive quantity. They concluded that this is not a model that supports children thinking multiplicatively. However, their interpretation of the construction of intensive quantity with continuous quantities was more challenging. This is why, counter to Simon and Placa, I suggest that students can develop an understanding of the intensive quantity if they begin with two discrete quantities.

I do agree, however, with Simon and Placa's (2012) conclusion that the intensive quantity is challenging for students to construct in grade 3. I saw this evidence in my analysis of

students in all three multiplicative reasoning categories, where they were grappling with the meaning of the numbers in each quantity across all situations as can be seen in Table 50.

**Table 50**

*Identification of Intensive Quantity and Referents in Three Multiplicative Situations (n = 21)*

| Multiplicative situation | Intensive quantity | Iterations | Total |
|---|---|---|---|
| Multiplication | 14/21 (67%) | 20/21 (95%) | 13/21 (62%) |
| Quotative division | 12/21 (57%) | 16/21 (76%) | 16/21 (76%) |
| Partitive division | 11/21 (52%) | 20/21 (95%) | 14/21 (67%) |

In each of these problem types, there were variations on how successful students were in identifying the referents. For the division problems, just over half of the students were able to fully label the intensive quantity, a relationship that connects two extensive quantities, whereas identifying the iterations and total were easier. This challenge—identifying the referents in all situations, gives us an insight into why multiplicative situations are different and tougher to understand than additive ones. While not universal, many students could construct the meaning of these numbers.

Let us consider, for example, student responses to the partitive division word problem, *The teacher has 24 pencils that she wants to put equally into 6 containers. How many pencils should she put in each container?* When a student incorrectly arrived at an answer of '4 containers' and I reminded them that there were 6 containers and I wanted to know how many pencils would be in each container— most needed to start the problem all over again. Had the students not been asked to identify the referents, they could have answered '4,' without correctly identifying the meaning of the number in the context of the problem.

This is why I suggest that the construction of the intensive quantity and the understanding of the referents herald the beginning of the separate pathway towards developing multiplicative reasoning. Being able to construct the intensive quantity, I propose, is an early indicator of establishing simultaneity. I define *simultaneity within the intensive quantity* as the ability to understand and work with the relationship in the construction between two extensive quantities that exists in the intensive quantity (Clark & Kamii, 1996; Schwartz, 1988, 1996).

### *The Pathway to Multiplicative Reasoning: Understanding the Meaning of the Intensive Quantity and the Referents*

Approaching the pathway to the development of multiplicative reasoning started with students learning to construct and understand the intensive quantity. As Hulbert et al. (2017) articulated from conducting research over 14 years as part of the *Ongoing Assessment Project (OGAP)* in the United States, "multiplication and division are not simply extensions of addition and subtraction" (p. 2). Instead, Schwartz's (1988, 1996) description of the referents in the operations provides a framework that interrupts the singular progression of the operations, creating a clear distinction between learning multiplication and division, as the referent *transforming* operations versus addition and subtraction, which are referent *preserving* operations. I conceptualize and highlight this disruption in Figure 34 (Lawson et al., in press), using the two hypotheses of the development of multiplication established by Parks and Nunes (2001).

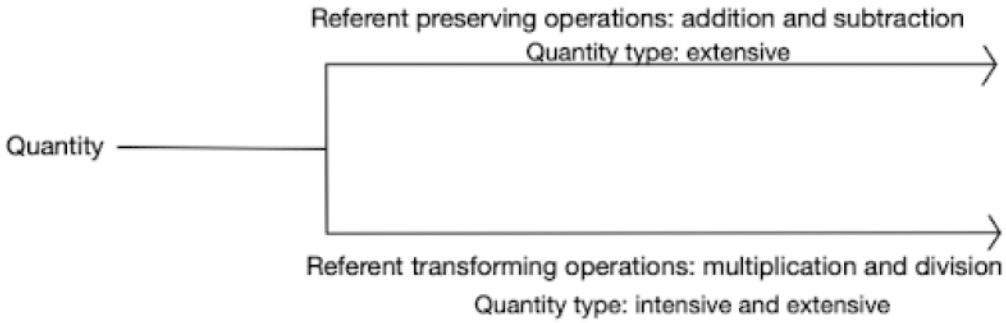
**Figure 34**

*Hypotheses of Pathways for Learning Multiplication (Park and Nunes, 2001)*

*Extension of Addition: The Implicit Theory*




*Interruption of Operation Pathway: The Schema of Correspondence Theory Referencing Schwartz (1988)*




*Note.* This figure is from the chapter, 'Developing Multiplicative Reasoning with 8- to 10-Year-Olds Through a Focus on the Intensive Quantity by Alex Lawson, Melanie Biesenthal, and Heather Wark in J. Holm, S. Ferreyro-Mazieres, A. Kajander, & E. Chernoff (Eds.,), *Teaching and Learning Elementary School Mathematics: Canadian Perspectives in an International Context.* Springer (in press). Reprinted with Permission.

One can see that these two separate pathways, are different, the implicit theory (i.e., repeated addition or 'groups of' instruction) versus the schema of correspondence theory (i.e., intensive

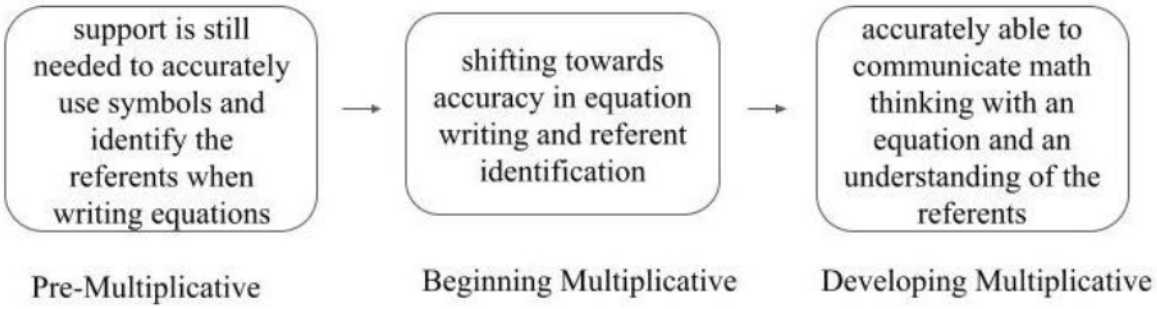
quantity instruction) both in the meaning of the numbers and the relationship between the numbers. My understanding of how students thought about the referents in multiplicative situations was brought to the forefront in their equation writing (an instructional focus in the intervention).

***Evidence of the Construction of the Intensive Quantity in Equation Writing***

As illustrated in Figure 35, the sophistication of students' equations varied across the three multiplicative reasoning categories. Students in the Pre-Multiplicative Reasoning Category needed support to convey mathematical symbols and referents accurately while students in the Developing Multiplicative Reasoning Category were able to independently write equations with meaning. Noticing these differences is important because, if the purpose of equations is to communicate mathematical ideas, is it not essential that students understand exactly what it is that they are communicating?

**Figure 35**

*Equation Writing within the Three Student Multiplicative Reasoning Categories*




These descriptions give an indication of the stages students may go through when they learn to write equations and aligns with Hiebert's (1988) article, *A Theory of Developing Competence*

*with Written Mathematical Symbols*. His theory suggests that students need to first be able to connect the meaning of the referents to symbols (p. 336) and that these connections are best supported using students' day-to-day experiences (p. 338). The analysis allowed me to see that there were students who understood the problem and answered it correctly, yet were unable to correctly write an equation, as indicated in Table 51.

**Table 51**

*Comparison of Incorrect Equation Writing with Correct Solutions*

| Multiplicative situation | Number of students who wrote an incorrect equation (n=21) | Number and percentage of these same students who solved the problem correctly |
|---|---|---|
| Multiplication (4 x 3) | 5 | 4/5 (80%) |
| Quotative division (15 ÷ 3) | 12 | 6/12 (50%) |
| Partitive division (24 ÷ 6) | 13 | 7/13 (54%) |

For example, in the multiplication word problem, of the five students who wrote the equation incorrectly, only one student solved it incorrectly while the other four students solved the problem correctly. These examples highlight that students may first understand the meaning of the problem context while the symbolism and communication of the mathematical situation come second.

It is important, therefore, for us to consider how students develop meaning of the intensive quantity and the referents as well as the multiplicative relationships they use when solving various multiplicative situations. It is in the third key finding, where I consider how students' strategy development in various situations provide us with further clues into how they develop multiplicative reasoning.

**Key Finding #3: The Pathway to Develop Multiplicative Reasoning**

My third key finding is that all students are now positioned on or are approaching a developmental pathway toward multiplicative reasoning rather than just being solely on an additive reasoning pathway. Evidence for this pathway development is found in the following areas:

- the qualitative codes developed to track the early development of multiplicative reasoning,
- students' limited use of repeated addition to correctly solve word problems,
- observable changes in students' skip counting strategies, and
- the continued progress of students when challenged with more complex multiplicative situations.

To examine this key finding, I will begin by discussing the specific codes used to analyze these developmental shifts.

***Unique Codes to Identify the Development of Multiplicative Reasoning***

The codes I developed to analyze the construction of the intensive quantity revealed students' progress in their development of multiplicative reasoning—a trajectory I would have missed had I been focused exclusively on counting and repeated addition. Unlike Steffe (1988), whose framework did not emphasize the intensive quantity or the referents, my codes highlight the beginning stages of multiplicative relationships as students start to construct and understand intensive quantities through instruction and word problems. Examples of these codes can be seen in Table 52.

**Table 52**

*Examples of Strategy Codes Specific to the Development of the Intensive Quantity*

| Strategy code | Description of code |
|---|---|
| Directly modelling of the intensive quantity | The student drew a representation of the intensive quantity. For example, if the intensive quantity was 4 pieces of gum per package, they drew out a package and filled in the pieces of gum or drew 4 pieces of gum and circled the pieces. This entire process was repeated for the number of iterations. |
| Coordinating to track the intensive quantity | The student touched an object (e.g., a bag with candies inside) as they were skip counting, however, there were pauses within the skip counting sequence. |
| Skip counting with tracking | The student skip counted by a number while simultaneously keeping track of the iterations on their fingers. |

*Note*. All codes and descriptions that were used in the analysis can be found in Tables 27, 30, 32-35, 38-39, 42-43, 46-47 in Chapter 4.

When students can draw a representation of the intensive quantity and iterate it, they are coordinating (Ulrich, 2016) the *simultaneous* relationship between two quantities (simultaneity), the same as when they can skip count and track (with or without a direct model).

Further to this, I also developed unique codes for incorrect answers when there were multiple students who incorrectly solved the problem in the same way. Examples of these codes can be seen in Table 53. The incorrect answers, particularly in the division questions, reveal how students were thinking about and using strategies that involved simultaneity. I used a '5' in the quotative problem (3 x □ = 15), to indicate when students commuted the problem but did not understand that the '5' was not the intensive quantity and answered incorrectly, '3 bags.' In the partitive division question (□ x 6 = 24), I indicated if the students were able to anticipate an

amount, meaning they anticipated the intensive quantity of how many pencils needed to go in each container that was either incorrect or correct. There were some students who anticipated the correct amount of '4,' but were unable to prove it was 'pencils per container' and therefore could not confirm their answer to the problem. The specific codes I developed provide insight into how the students were grappling with multiplicative reasoning as different from additive reasoning.

**Table 53**

*Examples of Codes for Incorrect Responses in Multiplicative Situation Problems (n=21)*

| Multiplicative situation | Incorrect error code | Code description | # of students pre-assessment | # of students post-assessment |
|---|---|---|---|---|
| Multiplication: Joshua has packages of gum with 4 pieces per package. He has 3 packages. How many pieces of gum does he have? | + | added the two numbers together | 4 | 1 |
| Quotative division: You have 15 cookies. You are going to put them in bags with 3 cookies per bag. How many bags can you fill? | + | added the two numbers together | 2 | 0 |
| | 5 | changed the intensive quantity to 5 bags (answering 3 bags) | 0 | 3 |
| Partitive division: The teacher has 24 pencils that she wants to put equally into 6 containers. How many pencils should she put in each container? | + | added the two numbers together | 1 | 0 |
| | a | anticipated an incorrect amount per container or correct with no follow through | 6 | 7 |

### *The Limited Evidence of Repeated Addition*

A second indicator that students are on the pathway to develop multiplicative reasoning rather than additive reasoning is the limited evidence of repeated addition (or subtraction)

strategies used by students. By the post-assessment as can be seen in Table 54, there was only one student who used repeated addition when correctly solving the three multiplicative situation word problems.

**Table 54**

*Number of Correct Responses Using the Strategy of Repeated Addition or Subtraction (n=21)*

| Multiplicative situation | Number of students using repeated addition or subtraction in correct responses: pre-assessment | Number of students using repeated addition or subtraction in correct responses: post-assessment |
|---|---|---|
| Multiplication (4 x 3) | 4 | 1 |
| Quotative division (15 ÷ 3) | 1 | 0 |
| Partitive division (24 ÷ 6) | 1 | 0 |

When we consider pathways of multiplicative development as either evolving from counting through repeated addition to multiplication (Steffe, 1988) or multiplication and division as a different pathway from addition and subtraction (Schwartz, 1988, 1996), we find these students do not use repeated addition. Instead, they create the intensive quantity by drawing a direct model, rhythmically counting, skip counting, or knowing it automatically.

***Changes in Students' Skip Counting Strategies***

Increased sophistication in skip counting using a direct model is the third indicator that students are approaching or are on the pathway of developing multiplicative reasoning, distinct from additive reasoning. At this stage, students simultaneously connect the intensive quantity, the number being skip counted by, with the number of iterations (simultaneity). My examination of pre- and post- assessment revealed a clear developmental trajectory. In the interviews, students were asked a multiplication question supported by a direct model as seen in Figure 36.

**Figure 36**

*Assessment Item: Multiplication Problem with the Visual Support of a Direct Model*

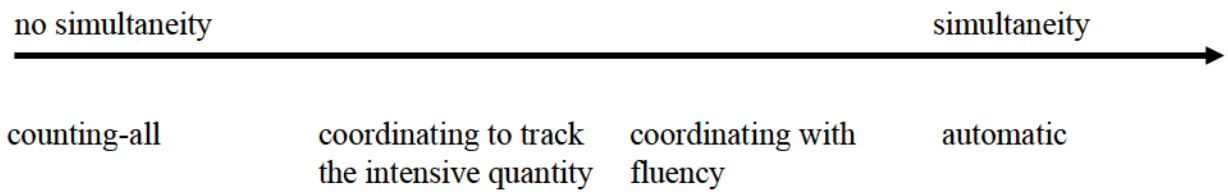
Question: *I have some candies in bags. There are 2 candies per bag and there are 8 bags. I'm wondering, how many candies are there altogether?*



The codes I created for the analysis were: counting-all, coordinating to track the intensive quantity, coordinating with fluency, and automatic. A student who was *counting-all*, was counting each individual candy. When students were actively coordinating to track the intensive quantity, they had pauses in their skip count sequence and would methodically touch the bags as they were counting. By the time they reached counting with fluency, the bags were touched very quickly to keep track of the iterations and when the response was automatic there was no contact with the direct model. As a result of coding this item, I propose an evolution of skip counting in conjunction with the direct model as can be seen in Figure 37. As students became more fluent with their skip counting, they relied less on the model, becoming more *simultaneous* in coordinating their counting and the iteration processes (simultaneity).

**Figure 37**

*Development of Skip Counting in Connection with a Context and a Direct Model*




This transition in their skip counting with a model differs from Wilkins and Ulrich (2017) where they suggest instead making "subtle changes in the use of manipulatives" (p. 14) to foster students' development of the composite unit. Their model is an evolution from counting as it starts with individual cubes to representing the composite unit, then interlocking cubes, to a rod with marked partitions on the bottom, and finally to solid rods without partitions. Their focus is an additive reasoning pathway, connecting individual cubes to create the composite unit rather than the construction of the intensive quantity (i.e., the relationship of two quantities interconnected with a referent such as 'candies per bag').

The progress in skip counting observed in my study did not stem from changes to the model itself, but instead from the students' skip counting strategies becoming increasingly multiplicative as they began to coordinate their skip count (intensive quantity) with the number of iterations. During the intervention, the students were challenged to use their skip counting sequence in a multiplicative situation. In the data analysis, I gave examples of the varied response students gave when anticipating how many times they would need, for example, to count by 5s to reach 100. When we were practicing skip counting, we verified mental calculations by skip counting and tracking iterations with their fingers. Other times, we incorporated physical movement (e.g., jumping) to kinaesthetically represent the skip counting process.

Both methods—correct skip counting using a direct model or skip counting while tracking on their fingers, are evidence that students are approaching or already on the pathway of multiplicative reasoning, particularly for those in the Pre- and Beginning categories. In contrast, students in the Developing Multiplicative Reasoning Category were already either coordinating with fluency or knew the answers automatically without needing a direct model. The differences in reasoning among students in this latter reasoning category became more apparent in my examination of the two additional problems they solved.

***Growth on the Pathway of Multiplicative Reasoning When Solving Complex Multiplicative Situations***

For students in the Developing Multiplicative Reasoning Category, the earlier and easier word problems did not fully expose their strategy development. This fourth indicator provides evidence of continued growth for students who were able to solve multiplicative situations of greater complexity. In the pre-assessment, five of six students incorrectly answered the question, "*Three candies cost 8 cents. How much do 12 candies cost?"* by attempting to repeatedly add 8, 12 times. By the post-assessment, three students solved it correctly (one with help), by identifying the multiplicative relationship between 3 candies and their cost. The remaining three students continued to rely on unsuccessful repeated addition, unable to grasp the multiplicative relationship.

The results were similar for the problem, "*Plums in a grocery store are arranged in 12 rows. There are 18 plums per row. Do you know how many plums there are?"* By the post-assessment, the three students who had solved the previous question correctly also solved this one; two used multiplicative strategies while one used repeated addition. Overall, 50% of the six students in this category made greater progress along the multiplicative reasoning pathway than

their peers. These three students consistently demonstrated elements of multiplicative reasoning throughout the mathematics intervention.

Regarding students who showed less progress, two required educator support to complete all pencil and paper tasks in class. Their limited engagement with the practice component may have contributed to their lack of development. A third student had acquired mathematics facts through rote memorization. Even though this student knew many of their facts, they could not solve complex problems. Even though there were various levels within the Developing Multiplicative Reasoning category, at the outset of the intervention, the six students in this category demonstrated multiplicative reasoning in their pre-assessment interview.

In summary, all 21 students are approaching or are on a pathway of developing multiplicative reasoning rather than on additive reasoning alone. This is evidenced by the unique coding patterns observed, a limited reliance on repeated addition, refinements in skip counting strategies, and the increased capacity for some students to solve problems involving complex multiplicative relationships. Each of these progressions reflects a student's growing ability to work with a quantity that has a functional relationship (i.e., the intensive quantity) to navigate various multiplicative situations (Schwartz, 1988, 1994).

For students to go beyond counting and adding strategies into multiplicative reasoning, they first need to construct the functional relationship (intensive quantity), by *simultaneously* thinking of two quantities (e.g., candies per bag) and use this relationship to solve problems. There are elements of simultaneity, between and amongst the quantities in a multiplicative situation that students need to attend to as part of the pathway to develop multiplicative reasoning. We turn to the fourth key finding where I discuss what this development of simultaneity looks like to on the pathway of multiplicative reasoning.

**Key Finding #4: Revision of My Hypothetical Progression on the Development of Simultaneity**

My fourth key finding is that my draft hypothetical progression *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning* (which can be found in Chapter 3), needed more detailed indicators to fully capture students' thinking. To investigate the role of simultaneity in students' multiplicative reasoning, I examined: (1) simultaneity within the intensive quantity, and (2) simultaneity in early multiplicative situations between the intensive quantity and the iterations. This builds on descriptions of simultaneity in the existing literature using Schwartz's theory (1988, 1996) of the intensive quantity and the referents which can be seen in Table 55.

**Table 55**

*Definitions of Simultaneity*

| Term | Definition | References |
|---|---|---|
| Simultaneity within the intensive quantity | the ability to understand and work with the relationship in the construction between two extensive quantities that exists in the intensive quantity (e.g., candies per bag) | Clark & Kamii, 1996<br>Schwartz, 1988, 1996 |
| Simultaneity in early multiplicative situations | the simultaneous interactions of the intensive quantity and the iterations (i.e., the ability to keep track of the intensive quantity and the iterations at the same time) when solving multiplicative situations including multiplication, quotative and partitive division, involving whole numbers and discrete quantities (e.g., keeping track of 2 candies per bag and the number of bags at the same time) | Ulrich, 2015, 2016<br>Lawson, 2015, 2016<br>Schwartz, 1988, 1996 |

To revise my initial draft learning progression as outlined in my methodology, I referred to the summary charts of the seven assessment items outlined in the analysis. I examined the post-

assessment summaries and labelled students' use of simultaneity, within the three multiplicative reasoning categories as *early*, *developing*, or *established* as seen in Table 56. *Early simultaneity* is evidence that simultaneity is starting to be constructed (i.e., drawing the intensive quantity as opposed to drawing empty groups then placing items in the group), *developing simultaneity* is evidence that simultaneity is occurring (i.e., strategies that go beyond single-counting) and *established simultaneity* indicates an automatic response to the assessment item, correctly identifying the referent when applicable, in which case simultaneity is assumed.

**Table 56**

*Summary of Simultaneity Development in Post-Assessment Items for Three Categories of Students*

| Mathematics concept | Level of simultaneity development | | |
|---|---|---|---|
| | Pre-Multiplicative | Beginning Multiplicative | Developing Multiplicative |
| Subitizing and visual iterations (both items) | early | developing | established |
| Additive reasoning | early | early | established |
| Skip counting by 2s with direct model | early | developing | established |
| Multiplication word problem | developing | developing | established |
| Quotative division word problem | early | developing | established |
| Partitive division word problem | early | early | developing |

Based on my data analysis, I created and revised *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning* as outlined in Table 57. As this is an *early* progression, the focus is on the creation of the intensive quantity as the connection of two discrete quantities, such as 4 candies and 1 bag to create the intensive quantity of 4 candies per bag. The progression may look different as students move towards working with intensive quantities that have continuous referents such as kilometres per hour. I use *simultaneity indicators* to specify how the evidence of simultaneity correlates with students' development of multiplicative reasoning.

**Table 57**

*The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning*

| Simultaneity indicator | Development of simultaneity → | | | |
|---|---|---|---|---|
| | No simultaneity | Early | Developing | Established |
| Construction of the intensive quantity as a model | cannot independently create the intensive quantity in a drawing or with physical objects | can create the intensive quantity in a drawing or with physical objects and is dependent on the model to calculate | can create the intensive quantity in a drawing or with physical objects and the model is there to support keeping track of the iterations | a model is no longer needed |
| Use of the intensive quantity in multiplicative situations | inability to create the intensive quantity and therefore cannot use it in multiplicative situations | effectively uses the intensive quantity to solve multiplication problems | effectively uses the intensive quantity to solve multiplication and quotative division problems | effectively uses the intensive quantity to solve multiplication, quotative, and partitive division problems |
| Referent identification and equation writing in multiplicative situations | may be able to identify some of the referents but not able to use the information to solve a problem (e.g., adding numbers together in a multiplication problem) or write an equation | can partially identify the referents though not always able to connect the two quantities in the intensive quantity (e.g., 2 pepperoni or 2 per slice versus 2 pepperoni per slice) and needs support to write an equation | can identify the intensive quantity orally and in a written equation, connecting the quantities using the word per in multiplication contexts and sometimes in division contexts | can clearly identify all three referents in multiplication and division contexts and accurately write an equation |
| Strategy development | strategies do not show evidence of simultaneity (e.g., counting-all by separately drawing quantities) | strategies show some evidence of simultaneity; support is needed (e.g., pencil and paper; direct model) to keep track of the intensive quantity (which is produced through individual counts) and the iterations (e.g., counting-all by drawing both quantities of the intensive quantity consecutively) | strategies demonstrate simultaneity where consideration is given to the intensive quantity and the iterations at the same time using mostly mental strategies (e.g., skip counting and tracking) | strategies are almost or close to being automatic and involve mental calculations |

These four simultaneity indicators: construction of the intensive quantity as a model, use of the intensive quantity in multiplicative situations, referent identification and equation writing in multiplicative situations, and strategy development, provide specific details on where students are in their development of simultaneity. In summary, these four key findings provide an understanding of the impact on one class of grade 3 students' early development of multiplicative reasoning when there is an instructional focus on the development of the intensive quantity.

**Conclusions**

Multiplicative reasoning is challenging for students to develop because of how the quantities interact and relate to each other in multiplicative situations; they are significantly different from additive situations. Schwartz's (1988, 1996) theory highlights the difference between the operations of addition and subtraction and multiplication and division. He emphasizes the complexity of learning multiplication and division through his identification of the intensive quantity and how the referents change in multiplicative situations. This differs from the theories of Vergnaud (1982, 1988, 1994) who focuses on a quaternary relationship in multiplication and division, and Steffe (1988, 1992, 1994) whose theory builds on the increased sophistication of the use of the composite unit, evolving from counting and adding.

My research question and sub question sought to examine how children develop multiplicative reasoning when the focus of instruction is based on Schwartz's (1988, 1996) theory through the development of the intensive quantity and the meaning of the referents. The key findings provide evidence that students, even those at a single-counting stage, are able to approach or access a separate pathway to develop multiplicative reasoning, different from simply

an extension of addition. This is demonstrated when they construct the *simultaneous relationship* of two quantities to create the intensive quantity and develop calculation strategies that require increasing levels of *simultaneous interactions* amongst quantities in multiplicative situations, suggesting that simultaneity is a component in the development of multiplicative reasoning.

During the mathematics intervention, I observed students across all three multiplicative reasoning categories, Pre-, Beginning, and Developing, as they grappled with interpreting numbers in multiplicative situations. Although their progress varied, all students were approaching or on a pathway toward developing multiplicative reasoning in their ability to work on constructing the meaning of the intensive quantity, the referents, and the number relationships that do not exist in addition. They were not just getting stronger at solving multiplication and division problems, they were also, to varying degrees, working with the functional relationship strategically in their identification of meaning of their answers and communication of their thinking in equations. This is important to recognize because introductory multiplication and division problems can be solved with counting and adding strategies or with a memorized fact, without students necessarily understanding the multiplicative number relationships in the problem.

Even struggling students who remained in counting were able to manage the cognitive load of dealing with two quantities simultaneously. These students progressed but well behind their peers given their continued use of single-counting. Students should not remain stuck in single-counting as they build an understanding of multiplicative relationships. While most students understood the intensive quantity relationship, single-counting will impede their progress. Askew et al. (2019) in their study of three classrooms across grades 1-3, following a four-lesson intervention focused on multiplicative reasoning found that while students across all grades

improved their ability to recognize multiplicative situations, their computational strategies remained stagnant; even some of the third-graders continued to rely on inefficient counting strategies. Therefore, while students may begin to recognize multiplicative relationships, how they proceed on the pathway to develop multiplicative reasoning requires continued and focused attention. With this in mind, we turn to the implications for classroom practice.

**Implications for Classroom Instruction**

The study's key findings suggest several implications for classroom instruction:

- Focus on moving students beyond single-counting strategies in addition prior to the beginning of learning multiplication and division.
- Support students to construct the intensive quantity at the outset of instruction on multiplication and division so they immediately learn to understand the functional relationship of a quantity that does not exist in addition. Use subitizing and visual iterations tasks to support the development of the intensive quantity.
- Incorporate teaching strategies that foster the development of mental mathematics, e.g., skip counting and tracking strategies without the use of pencil and paper to prevent an over-reliance on physical or drawn models of the intensive quantity.
- Intentionally distinguish between the meaning of numbers in multiplicative situations (e.g., intensive quantity) versus additive situations.

Based on these implications for classroom instruction, continued research would build on my findings from this study.

**Considerations for Future Research**

To further build on the findings in this study, I offer four suggestions for consideration. The

first is to the examine the instructional impact of developing multiplicative reasoning with a focus on understanding of the intensive quantity and the referents with a classroom that has fewer students who are using single-counting strategies. A limitation of this study was that, of the 21 students, 14 were using inefficient counting strategies at the outset of instruction. Therefore, further analysis is needed with a group of students, where many of them are at least at an additive stage of reasoning. Further to this, a second suggestion would be to run this research study again with a control group of students learning multiplication and division with instruction focusing on 'groups of' (implicit theory) to compare the pathway of development of multiplicative reasoning.

A third suggestion for future research is to investigate whether my hypothetical progression, *The Development of Simultaneity with the Intensive Quantity in Early Multiplicative Reasoning,* helps educators identify a student's level of simultaneity and plan subsequent instructional steps.

A fourth and final suggestion is to examine the impact of skip counting and tracking exercises, with and without a direct contextual model, on second-grade students. This research would determine how such exercises improve readiness for multiplication and division in grade 3. Future research in these four suggested areas would provide additional insights to the complex discussion of children's complex development of early multiplicative reasoning.

**Closing Statement**

As educators, our instructional choices have significant implications for student learning. Relying on multiplication models as 'repeated addition' or 'groups of', while intended to be supportive of children's understanding of mathematics concepts, may in fact be restricting their growth. This is important for us to address because the failure to develop multiplicative

reasoning carries high stakes. It limits students' access to more complex mathematical concepts, such as fractions, ratios and proportional reasoning (Siemon et al., 2005, p. 282; Vergnaud, 1988, p. 141; Wilkins & Ulrich, 2017, p. 8) that are needed in life as well as in future career opportunities (Byun et al., 2015, p. 439). Therefore, further investigation into how instruction using the intensive quantity and identifying the referents might support students' development of multiplicative reasoning while students are *simultaneously* constructing their own meanings of the numbers on a multiplicative pathway, is an inquiry worth pursuing.

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# Appendix A: Short Instrument for Multiplication Interview Grade 3

Student Name & Gr. _____ Date _____Researcher_____ Teacher

**Early Multiplication and Division Interview Items (Short Version)**

| | | Auto | Think | CM | Comments |
|---|---|---|---|---|---|
| 3 | Dot Plate 3 | | | | |
| 6 | Two plates **Imagined** | | | | |
| 12 | Four plates **imagined** | | | | |
| 8 | Dot Plate 5 + 3 | | | | |
| 16 | Doubled | | | | |
| 14 | 8 + 6 | | | | |
| 16 | There are 2 coins per bag (show a bag and shake it) and there a 8 bags. How many coins are there? | | | | |
| | (If they don't skip count to solve) When you were younger, maybe you would have skip counted by 2s. Can you show me how you might do that? | | | | |
| 12 | Joshua has packages of gum. There are 4 pieces of gum per package. He has 3 packages. How many pieces of gum does he have? | | | | |

| | | Auto | Think | CM | Comments |
|---|---|---|---|---|---|
| 5 | You have 15 cookies. You are going to put them in bags with 3 cookies per bag. How many bags can you fill?<br>Can you write the equation for this problem? | | | | |
| 9 | Manusha has 5 cats and 4 dogs. How many pets does she have? | | | | |
| 6 | The teacher has 24 pencils that she wants you to put equally into 6 containers. How many pencils should she put into each container?"<br>Can you write the equation for this problem? | | | | |
| 32 | Three candies cost 8 cents, how much do 12 candies cost? | | | | |

Student Name & Gr. _____ Date _____Researcher_____ Teacher

| | | Auto | Think | CM | Comments |
|---|---|---|---|---|---|
| 30 | Can you draw a rectangle that is 5 squares across by 6 squares down on this paper? Can you figure out how many squares are in the rectangle without counting every square one at a time? How did you find the number of squares? | | | | |
| 216 | Plums in a grocery store display are arranged in 12 rows. There are 18 plums per row. How many plums are there? Can you write the equation for this problem? | | | | |
| 42 | Jerusha who works after school made 6 dollars a day for 7 days. How much money did she make? If you knew 6 x 5 was 30, how could you use that fact to solve the problem? | | | | |

Student Name & Gr.:_____ Date: _____ Researcher _____ VER 5

| | Fact | answer | Speed Assessment | | | Strategy Assessment | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | < 3s | >3<10 | >10s | Counting by ones | Rhythmic Counting/Skip Counting/ Repeated Addition | Working with the Relationship | Automatic/ Memorized |
| | | Write in answer given if incorrect | ✓ for correct X for incorrect | | | | Circle the number they used as the composite. If different from provided, write in the number used. | Write in the mathematics used to calculate the answer. | |
| 1 | 4x2 | 8 | | | | | 4 2 | | |
| 2 | 10x6 | 60 | | | | | 10 6 | | |
| 3 | 2x6 | 12 | | | | | 2 6 | | |
| 4 | 5x3 | 15 | | | | | 5 3 | | |
| If you were going to explain to a younger student (someone in grade 1) what 5 x 3 means how would you explain it? | | | | | | | | | |
| If you are getting all "counting by ones" or if the student doesn't yet understand multiplication, end the interview | | | | | | | | | |
| 5 | 7x2 | 14 | | | | | 7 2 | | |
| 6 | 10x4 | 40 | | | | | 10 4 | | |
| 7 | 2x8 | 16 | | | | | 2 8 | | |
| 8 | 5x5 | 25 | | | | | 5 | | |
| 9 | 9x2 | 18 | | | | | 9 2 | | |
| 10 | 4x10 | 40 | | | | | 4 10 | | |
| 10 | 10x9 | 90 | | | | | 10 9 | | |
| 11 | 2x5 | 10 | | | | | 2 5 | | |
| 12 | 5x7 | 35 | | | | | 5 7 | | |
| If you are getting all "counting by ones, skip counting/counting rhythmically, repeated addition," end the interview (if this is true except for the x10 and or the 10x facts also end the interview) | | | | | | | | | |
| 16 | 5x0 | 0 | | | | | 5 0 | | |
| 17 | 8x1 | 8 | | | | | 8 1 | | |

Student Name & Gr.:_____ Date: _____ Researcher _____ VER 5

| | Fact | answer | Speed Assessment | | | Strategy Assessment | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | < 3s | >3<10 | >10s | Counting by ones | Rhythmic Counting/Skip Counting/ Repeated Addition | Working with the Relationship | Automatic/ Memorized |
| | | Write in answer given if incorrect | ✓ for correct X for incorrect | | | | Circle the number they used as the composite. If different from provided, write in the number used. | Write in the mathematics used to calculate the answer. | |
| 18 | 0x4 | 0 | | | | | 0 4 | | |
| 19 | 1x6 | 6 | | | | | 1 6 | | |
| | | | | | | If any of the 0 and 1 facts are incorrect end the interview. | | | |
| 20 | 4x4 | 16 | | | | | 4 | | |
| 21 | 8x4 | 32 | | | | | 8 4 | | |
| 22 | 4x6 | 24 | | | | | 4 6 | | |
| 23 | 3x3 | 9 | | | | | 8 4 | | |
| 24 | 3x6 | 18 | | | | | 3 6 | | |
| 25 | 9x3 | 27 | | | | | 9 3 | | |
| | | | | | | If Gr. 3 end the interview | | | |
| 26 | 4x9 | 36 | | | | | 4 9 | | |
| 27 | 4x7 | 28 | | | | | 4 7 | | |
| 28 | 7x3 | 21 | | | | | 7 3 | | |
| | | | | | | If you are getting all "counting by ones, skip counting/counting rhythmically, repeated addition," end the interview. | | | |
| 29 | 9x7 | 63 | | | | | 9 7 | | |
| 30 | 6x6 | 36 | | | | | 6 | | |
| 31 | 9x8 | 72 | | | | | 9 8 | | |
| 32 | 7x6 | 42 | | | | | 7 6 | | |
| 33 | 9x4 | 36 | | | | | 9 4 | | |

Student Name & Gr.:_____ Date: _____ Researcher _____ VER 5

| | Fact | answer | Speed Assessment | | | Strategy Assessment | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | < 3s | >3<10 | >10s | Counting by ones | Rhythmic Counting/Skip Counting/ Repeated Addition | Working with the Relationship | Automatic/ Memorized |
| | | Write in answer given if incorrect | ✓ for correct X for incorrect | | | | Circle the number they used as the composite. If different from provided, write in the number used. | Write in the mathematics used to calculate the answer. | |
| 34 | 9x6 | 54 | | | | | 9 6 | | |
| 35 | 8x7 | 56 | | | | | 8 8 | | |
| 36 | 8x6 | 48 | | | | | 8 6 | | |
| 37 | 5x8 | 40 | | | | | 5 8 | | |
| 38 | 6x9 | 54 | | | | | 6 9 | | |
| 39 | 8x8 | 64 | | | | | 8 | | |
| 40 | 9x9 | 81 | | | | | 9 | | |

IN - Incorrect response
SC - Skip Counting by one of the factors
RA - Repeatedly added one of the factors
CR - Counted Rhythmically by one of the factors
DP - Distributive Property (e.g., n x 9 = n x 10 - n x 1)
AP - Associative Property (e.g., 4 x n = double 2 x n)
CP - Commutative Property (e.g., 6 x 7 = 7 x 6)

# Appendix B: Research Ethics Board: Approval of Research Project, Multiplicative Pathway for Junior/Intermediate Students

Lakehead UNIVERSITY

**Research Ethics Board**
t: (807) 343-8283
research@lakeheadu.ca

March 7, 2022

**Principal Investigator**: Dr. Alexandra Lawson
**Project Staff/Research Assistant**: Heather Wark
**Student Researcher:** Melanie J. Biesenthal
Faculty of Education
Lakehead University
955 Oliver Road
Thunder Bay, ON P7B 5E1

Dear Dr. Alexandra Lawson, Heather Wark, and Melanie Biesenthal:

**Re: Romeo File No: 1469119**

On behalf of the Research Ethics Board, I am pleased to grant ethical approval to your research project titled, "The Multiplicative Pathway for Junior/ Intermediate Students".

Ethics approval is valid until March 7, 2023. Please submit a Request for Renewal to the Office of Research Services via the Romeo Research Portal by February 7, 2023, if your research involving human participants will continue for longer than one year. A Final Report must be submitted promptly upon completion of the project. Access the Romeo Research Portal by logging into myInfo at:

https://erpwp.lakeheadu.ca/

During the course of the study, any modifications to the protocol or forms must not be initiated without prior written approval from the REB. You must promptly notify the REB of any adverse events that may occur.

Best wishes for a successful research project.

Sincerely,

Dr. Kristin Burnett
Chair, Research Ethics Board

/sa

# Appendix C: Approval of Amendment for Research Project, Multiplicative Pathway for Junior/Intermediate Students

**Date**: June 08, 2023

**To**: Dr. Alexandra Lawson, Primary Investigator

**From**: Dr. Claudio Pousa, Chair, Research Ethics Board

**Subject**: Approval of Amendment for REB Romeo #1469119

---

Thank you for your request for amendments for your project titled "The Multiplicative Pathway for Junior/ Intermediate Students".

Your request to include Melanie Biesenthal's use of data in your project as part of her dissertation data is acceptable to the Research Ethics Board. Thank you for providing the revised documentation.

Please continue to advise us of any future changes to your research project.

/sw

# Appendix D: Student Information Letter for Participation in Research Study

Lakehead UNIVERSITY Faculty of Education

*Is there a better way to learn multiplication? (Student Letter)*

April 2024

Dear Student,

My name is Alex Lawson. I am a researcher in mathematics education at Lakehead University. Melanie Biesenthal (K-12 Coordinator of Mathematics) and I are conducting research to determine the best way to teach multiplication. My research may help us to find better way to teach math to other children.

Many children find learning multiplication difficult. Your teacher and Mrs. Biesenthal are going to teach you multipication in a new way. I want to find out if this way is better than what we usually do. I would like to interview you before and after your teacher teaches you multiplication. I will find out what you know about multiplication before the instruction, and then after the instruction.

In the interview Mrs. Biesenthal will ask you some questions, and record your answers by video. If at anytime you no longer want to participate, you can just so say and she will end the interview.

Your work (without your name) may be included in any reports or information I share with teachers. If it turns out that this is a better way to teach and I want to share your video in a book on multiplication, I will contact you and your parent again to obtain your permission.

If you want to be part of the research please speak to your parent or guardian. Show them their letter and complete the consent form with them. Return it to your teacher.

Thank you,

Alex Lawson

Melanie Biesenthal

# Appendix E: Parent/Guardian Letter for Participation in Research Study

Lakehead UNIVERSITY

Faculty of Education

*Determining An Effective Instructional Pathway Toward Fluent And Accurate Multiplication In Junior Age Children*
(Parent's Information Letter)

Dear Parent/Guardian of Potential Participant,

I am Alex Lawson, a researcher in mathematics education at Lakehead University. Melanie Biesenthal (K-12 Coordinator of Mathematics for the (School Board Name) and PhD student), my team and I are conducting research on a more effective way to teach children multiplication and division. I am writing to see whether you would consider allowing your child to participate in the research.

The research will start in April. Your child's teacher, (Teacher Name), is interested in and willing to undertake the research in order to improve their instruction. They will follow the Ontario Math Curriculum but, use instructional ideas that have proven more successful internationally. Children who struggle with multiplication often go on to have difficulties in their intermediate and high school math classes. We hope to learn to better teach this foundational mathematics.

All children in the class will have the same instruction regardless of their participation in the research. I am writing to ask your permission to interview your child before and after their instruction and to video record that interview. We will ask them a series of mathematical questions to determine what they know before and after the unit. I am also asking your permission to collect, duplicate, and then return your child's math work as well as to video record some of the lessons in which your child might appear.

Your child's participation is entirely voluntary and will in no way affect their instruction in the classroom. If you agree to allow them to participate my team and I will analyze their data to determine whether the instruction has proven more effective than past instruction. If they participate they can withdraw at any time for any reason from the research until July 1, 2024. I foresee no harm any greater than what would normally happen in the classroom.

I will provide a summary of the findings upon completion should you wish to see it. The research will be published in Melanie Biesenthal's dissertation, and may be published in academic journals or, a professional book for teachers. If the instruction proves to be more effective, we will share this in professional development with teachers in Ontario and at the university with our teacher candidates. Your child's identity and work will be anonymized in any publication. The only exception to this is if we use a video from their interview or classroom lesson as an exemplar in professional development sessions with teachers or teacher candidates. The data will be stored for a minimum of 5 years at Lakehead University.

Please keep this information letter for your files. If agree to allow your child to participate please complete the consent form below and return it to your child's teacher. Feel free to contact me at any time at alawson@lakeheadu.ca.

This research study has been reviewed and approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca.

Alex Lawson PhD
Lakehead University
807-343-8720
alawson@lakeheadu.ca

Melanie Biesenthal MEd
Lakehead University and (School Board Name)
mjbiesen@lakeheadu.ca/(School Board Email)

Lakehead UNIVERSITY

Faculty of Education

*Determining An Effective Instructional Pathway Toward Fluent And Accurate Multiplication In Junior Age Children* (Parent/Guardian Information Letter)

**MY CONSENT:**

I agree to the following by signing below.

- I have read and understand the information contained in the Information Letter
- I agree to have my child participate
- I understand the risks and benefits to the study
- My child is a volunteer and can withdraw from the study at any time for any reason until their data is incorporated into the study (after July 1, 2024)
- That the data will be securely stored at Lakehead University for a minimum period of 5 years following completion of the research project
- I understand that the research findings will be made available to me upon request by email
- My child will remain anonymous unless their video is used as an exemplar in professional development with teachers or teachers candidates only
- I understand that Alex Lawson will contact me for permission is she wishes to use a video of my child in her publications for teachers
- All of my questions have been answered

☐ My child will remain anonymous unless their video is used as an exemplar in professional development with teachers or teacher candidates only (PLEASE INTIAL)

_____ _____ _____

Name of Parent/Guardian Signature of Name of Parent/Guardian Date

Date

_____ _____ _____

Name of Potential Participant (Child) Signature of Potential Participant (Child) Date

# Appendix F: Principal Research Participation and Consent Form

Lakehead UNIVERSITY

Faculty of Education

*Determining An Effective Instructional Pathway Toward Fluent And Accurate Multiplicative Reasoning In Junior Age Children* (Principal Letter)

Dear (Principal Name),

I am Alex Lawson, a researcher in mathematics education at Lakehead University. Melanie Biesenthal (K-12 Coordinator of Mathematics for the (School Board Name) and PhD student), my team and I are conducting research on a more effective way to teach children multiplication and division. As you know some children struggle to learn multiplication (the facts, and how to reason multiplicatively). This puts them at a significant disadvantage in many areas of later mathematics

I am writing to see whether you would consider participating in the research.

If you agree to having the research conducted in your school, (Teacher Name), will teach the same unit to all children regardless of participation. We have been meeting with (Teacher Name), to determine the needs of their students. The unit reflects the needs of their students coupled with the international research on effective instruction of multiplication and division. I am interested in gathering data about the efficacy of the unit on students' learning.

Your participation would involve allowing us to conduct research in one of your classrooms. This would include having you:

- speak to the students about the research,
- answer any questions, and
- hand out the forms
- locate an interview room or storage closet.

This would involve your participating teacher:

- collecting consent forms,
- or allowing interviews of your participating students,
- teaching the intervention unit,
- collecting written data throughout the unit,
- allowing filming of some lessons, and
- checking in with me to give feedback on the lessons as convenient.

Your participation is voluntary and should you consent to participate you can still withdraw your school from the research study at any time for any reason until July 1, 2024. The research will be published in Melanie Biesenthal's dissertation, and may be published in academic journals or, a professional book for teachers. The information may be shared in professional development sessions for teachers and teacher candidates.

The data will be stored for a minimum of 5 years at Lakehead University. I do not foresee any risk or harm as a result of participating in the research. I see potential benefits to your students and the participating teacher and wider community if the intervention improves student learning.

Lakehead UNIVERSITY

Faculty of Education

Please keep this information letter for your files. If agree to allow your teacher and students to participate please complete the consent form below and return it to me. Please feel free to contact me at any time with any questions regarding the research.

This research study has been reviewed and approved by your board of education. This research has also been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca.

Alex Lawson PhD
Lakehead University
807-343-8720
alawson@lakeheadu.ca

Melanie Biesenthal MEd
Lakehead University & (School Board Name)
mjbiesen@lakeheadu.ca/(School Board Email)

✂----------**Please tear off and return the following** ----------

*Determining An Effective Instructional Pathway Toward Fluent And Accurate Multiplicative Reasoning*

*In Junior Age Children* (Principal Consent Form)

**MY CONSENT:**

I agree to the following:

- ✓ I have read and understand the information contained in the Information Letter
- ✓ I agree to participate
- ✓ I understand the risks and benefits to the study
- ✓ I understand I may withdraw my consent at any time for any reason without penalty until July 1, 2024 when writing begins.
- ✓ I understand that the data will be securely stored at Lakehead University for a minimum period of 5 years following completion of the research project
- ✓ My board, school, and identity will be anonymized in any publication.
- ✓ I understand that the research findings will be made available to me upon request by email
- ✓ If Alex Lawson wishes to publish a professional book on the unit she will contact me before contacting the teacher and parents for their permission
- ✓ All of my questions have been answered
- ✓ The teacher and students' data will remain confidential unless their video is used only as an exemplar in professional development with teachers or teachers candidates

_____ _____

Name of Principal Date

_____

Signature of Principal

# Appendix G: Teacher Research Participation and Consent Form

Lakehead UNIVERSITY

Faculty of Education

*Determining An Effective Instructional Pathway toward*
*Fluent And Accurate Multiplicative Reasoning In Junior Age Children* (Teacher's Information Letter)

Dear (Teacher Name),

I am Alex Lawson, a researcher in mathematics education at Lakehead University. Melanie Biesenthal (K-12 Coordinator of Mathematics for the (School Board Name) and PhD student), my team and I are conducting research on a more effective way to teach children multiplication and division. As you know some children struggle to learn multiplication (the facts and how to reason multiplicatively). We hope to improve instruction in this area for students in Ontario. I am writing to see whether you would consider participating in the research. Our past experience is that teachers find participating in our research benefits their instruction in the classroom.

If you agree to participating in the research, you will teach a unit we have designed together, to all children—regardless of their participation. The unit reflects the needs of your students coupled with the international research on effective instruction of multiplication and division.

As a participant you would:

- collect the student and parent information/consent forms,
- meet for 2 hours prior to the unit to review it,
- teach the intervention unit,
- collect written data throughout the unit,
- allow me to film parts of the instruction,
- allow me to conduct pre and post interviews, and
- discuss the progress of the unit when convenient

You are a volunteer and you may withdraw your consent at any time for any reason without penalty until July 1, 2024. The research will be published in Melanie Biesenthal's dissertation, and may be published in academic journals or, a professional book for teachers. The information may be shared in professional development sessions for teachers and teacher candidates. Please note that the data will be stored for a minimum of 5 years at Lakehead University.

Please keep this information letter for your files. If you agree to participate and your principal agrees as well please complete the consent form below and return it to me. Please feel free to contact me directly with any questions.

This research study has been reviewed and approved by your board of education. It has also been reviewed and approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca. Thank for your support.

Lakehead UNIVERSITY

Faculty of Education

Alex Lawson PhD
Lakehead University
807-343-8720
alawson@lakeheadu.ca

Melanie Biesenthal MEd
Lakehead University & (School Board Name)
mjbiesen@lakeheadu.ca/(School Board Email)

*Determining An Effective Instructional Pathway Toward*

*Fluent And Accurate Multiplicative Reasoning In Junior Age Children* (Teacher's Consent Form)

**MY CONSENT:**

I agree to the following:

- ✓ I have read and understand the information contained in the Information Letter
- ✓ I agree to participate
- ✓ I understand the risks and benefits to the study
- ✓ I understand I may withdraw my consent at any time for any reason without penalty until July 1, 2024 when the analysis will be complete
- ✓ I understand that the data will be securely stored at Lakehead University for a minimum period of 5 years following completion of the research project
- ✓ I understand that the research findings will be made available to me upon request by email
- ✓ My board, school, and identity will be anonymized in any publication
- ✓ All of my questions have been answered

☐ My identity will remain anonymous unless a video with me in it is used only as an exemplar in professional development with teachers or teachers candidates. (Initial Box)

- ✓ If Alex Lawson wishes to include a video of me in a future book for teachers she will return to me to ask permission.

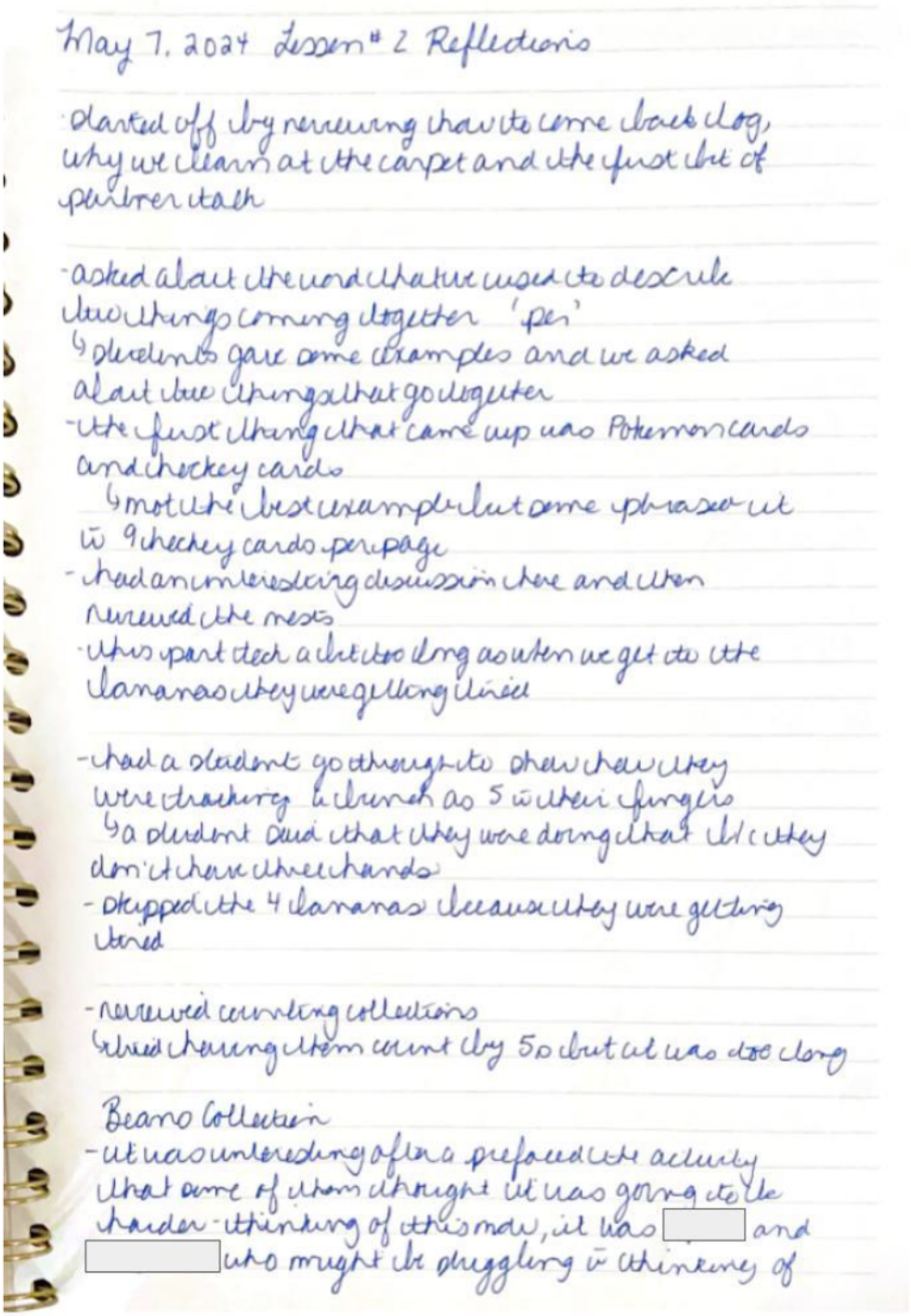
_____ _____

Name of Teacher Date

_____

Signature of Teacher

Appendix H: Example of Lesson Field Notes



May 7, 2024 Lesson # 2 Reflections

- started off by reviewing how to come back to log, why we learn at the carpet and the first bit of number talk

- asked about the word that we used to describe two things coming together 'pair'
  - students gave some examples and we asked about two things that go together
- the first thing that came up was Pokemon cards and hockey cards
  - not the best example but some phrased it to 9 hockey cards per page
- had an interesting discussion there and then reviewed the rest
- this part took a bit too long as often we get to the bananas they were getting tired

- had a student go through to show how they were tracking & drew each of 5 with their fingers
  - a student said that they were doing that b/c they don't have three hands
- skipped the 4 bananas because they were getting bored

- reviewed counting collections
  - tried having them count by 5s but it was too long

Beans Collection
- it was interesting after & prefaced the activity that some of them thought it was going to be harder - thinking of this now, it was [redacted] and [redacted] who might be struggling w thinking of

the multiplicative relationships of, for example, 5 beans per cup

- for some groups, the 60 was too much to count
- a couple of groups said that there were 55 beans which would have affected the consolidation at the end.

- all wondering about the ice cube trays
  - do they hinder or help?
  - caused some confusion of what to count - the tray or the spots?
- it does produce a double relationship of 30 per tray or 3 per cup/spot

- after the chart discussion back a bread

For Wednesday
- Minds On is Bananas
- Independent Questions
- Bunjo

Reflection
- the ones that struggled in the count or are on earlier strategies did not get the right # Simultaneity in a different way...
- this seemed harder for them today and I wonder if it is because keeping the iterations consistent and then figuring out the # of containers + not necessarily building at the same time that all required knowing both
- more abstract perhaps?

# Appendix I: Example of Simultaneity Field Notes

**Simultaneity Observation Template**

**Lesson:** 2 (Common Counting Collections) **Date:** May 7, 2024

| Direct Model | | | | | | Mental Mathematics | | |
|---|---|---|---|---|---|---|---|---|
| Strategy | No connection to the intensive quantity | subitizing the intensive quantity on a direct model, iterating and counting | drawing the intensive quantity first and then iterating | recording the intensive unit as a number, then iterating | repeated addition with counting support | skip-counting, modelling intensive quantity with one hand and tracking with the other hand | skip-counting and keeping track of the intensive quantity | automatic |
| Further Explanation | e.g., adding two numbers together<br>4 + 3 instead of 4 x 3 | | e.g., 4 candies per bag  | e.g., 4 candies per bag  | e.g., for 4 x 3 adding 4 + 4 = 8 and counting on the last 4 | using both hands - one to model the intensive quantity and the second to model the iterations | tracking the skip count on one hand | |
| Degree of Simultaneity | No simultaneity | Simultaneity of intensive unit | | | | Simultaneity between intensive quantity and iterations | | |

- big # of 60 was challenging for some groups → even to build and create the iterations accurately

Trouble in Iterations

[redacted] / [redacted]

[redacted] / [redacted] needed to practice counting

[redacted] / [redacted] → hard time going through the count # of iterations

ended up w/ 75

Capable of iterations

[redacted] / [redacted] → able to calculate well; used the ice cube trays and had a hard time determining the

Did not count 60...

[redacted] / [redacted] their iterations leaps were correct, incorrect total.

[redacted] / [redacted]

© Melanie Biesenthal May 2024

Appendix J: Overview of Pre- and Post-Assessment Interviews

| Date | Number and Type of Interviews |
|---|---|
| | Pre-Assessment Interviews |
| April 24, 2024 | 7 videorecorded interviews |
| April 25, 2024 | 6 videorecorded interviews |
| | 1 non-videorecorded interview with note to re-do, due to frustration level of the student |
| April 29, 2024 | 1 non-videorecorded interview (second attempt with classroom teacher leading the interview) |
| May 6, 2024 | 3 videorecorded interviews |
| May 7, 2024 | 1 non-videorecorded interview |
| | 1 videorecorded interview |
| May 8, 2024 | 2 non-videorecorded interviews |
| | Post-Assessment Interviews |
| June 10, 2024 | 4 videorecorded interviews |
| June 11, 2024 | 4 non-videorecorded interviews |
| | 5 videorecorded interviews |
| June 12, 2024 | 8 videorecorded interviews |

Appendix K: Overview of the Lessons Taught for the Math Intervention

| Date | Lesson Overview |
|---|---|
| May 6, 2024 | Counting collections; Quick images |
| May 7, 2024 | Quick images; Common collection (Beans) |
| May 8, 2024 | Quick images; Word problems (Independent – Beans); Bingo v1 |
| May 9, 2024 | Writing multiplication equations with quick images; Word problems (Independent) v2 |
| May 13, 2024 | Referents for addition and multiplication problems; 1xs and 10xs String; Connect 4 game |
| May 14, 2024 | Difference between addition and multiplication; 10xs Skip-counting; Quick images |
| May 15, 2024 | Introduction to commutative property (Tim Bits); Independent work |
| May 16, 2024 | Two ways to think about 2 facts; Quick images; Double fun bingo |
| May 17, 2024 | 'I Spy' to review with bingo cards; Thinking cards |
| May 21, 2024 | x2; Ms. Moneybags problems; Thinking cards practice |
| May 22, 2024 | Missing parts equation; Introduction to the division symbol |
| May 23, 2024 | Missing part equations review 'Make it True;' Practice of writing Multiplication and division word problems |
| May 24, 2024 | Writing of word problems (reviewing intensive quantities and iterations) *this session led by classroom teacher (I was not present) |
| May 27, 2024 | Introduction to the array; Magic trick |
| May 28, 2024 | Magic trick; Array (con't)<br>*Note – educators from other research iterations of the project and early math intervention teachers were present |
| May 29, 2024 | Skip-counting warm-up; Patterns of 5x's |
| May 30, 2024 | Skip-counting warm-up; Thinking cards 5x's; Game: Target 10x's and 5x's |
| May 31, 2024 | Skip-counting warm-up; Introduction to fractions; Double fun bingo |
| June 3, 2024 | Consolidation of learning: Finishing up of word problems; Thinking cards; Connect 4 (10x's or 2x's) |
| June 4, 2024 | Sharing of word problems: Student would record equation on whiteboard<br>*led by classroom teacher |

*Note*. Many of the lessons listed above have been published by A. Lawson, 2026, *What to look for 2: Understanding and developing student thinking in multiplicative reasoning,* Pearson.

Appendix L: List of Codes for Analysis

| Code Category | Specific Codes |
|---|---|
| Correctness | correct |
| | correct with help |
| | incomplete |
| | incorrect |
| Equation | equation correct |
| | equation incorrect |
| Intensive Quantity | cannot create |
| | first |
| | second |
| Referent | intensive quantity (no per) |
| | intensive quantity (per) |
| | iterations |
| | total (with referent) |
| | total (no referent) |
| Skip counting rote by 2 | fluent |
| | not fluent |
| Strategy | automatic |
| | coordinating to track intensive quantity |
| | coordinating with fluency |
| | counting-all |
| | counting-on |
| | decomposing a factor |
| | direct model intensive quantity |
| | direct model subitizing intensive quantity |
| | doubling |
| | fingers to track intensive quantity |
| | mental calculation |
| | near doubles |
| | prediction |
| | proportional reasoning |
| | ratio |
| | repeated addition |
| | repeated subtraction mental |
| | repeated subtraction written |
| | rhythmic counting |
| | skip count to 8 |
| | skip counting |
| | skip counting with tracking |
| | up over 10 |
| | using a known fact |