Compact Sets in Banach Spaces

A Thesis Submitted to Lakehead University

In Partial Fulfillment of the Requirements

for the Degree of Master of Science

by

Ling Yu ©

1990.

Thesis Adviser: Dr. John H. M. Whitfield.

ISBN 0-315-63185-6

Compact Sets in Banach Spaces

Ling Yu

Master of Science

1990.

Abstract

Connections between several compact spaces are studied in this thesis. Proofs are given when one implies another, and so are counterexamples when one does not. The spaces discussed in this thesis are: uniform Eberlein - compact (UEC) space, Eberlein - compact (EC) space, Talagrand -compact (TC) space, Gul'ko - compact (GC) space, Corson - compact (CC) space, Radon-Nikody´m - compact (RN) space, Rosenthal - compact (RC), Valdivia - compact (VC) space. The main results are:

UEC $\Rightarrow$ EC $\Rightarrow$ TC $\Rightarrow$ GC $\Rightarrow$ CC $\Rightarrow$ VC;

UEC $\Leftarrow\neq$ EC $\Leftarrow\neq$ TC $\Leftarrow\neq$ GC $\Leftarrow\neq$ CC $\Leftarrow\neq$ VC;

EC $\Rightarrow$ RN, RN $\neq\Rightarrow$ EC; RN $\neq\Rightarrow$ CC, and CC $\neq\Rightarrow$ RN;

EC + $w(K) \leq \omega \Rightarrow$ RC; EC $\Leftrightarrow$ RN + CC, VC $\Leftrightarrow$ CC or VC $\supset [0, \omega_1]$;

Also, we show: RN $\neq\Rightarrow$ VC, RN $\Leftarrow\neq$ VC, RC $\neq\Rightarrow$ EC; RN $\neq\Rightarrow$ TC, RN $\neq\Rightarrow$ GC; and the equivalency of definitions of κ - analytic and K - analytic, κ - countably determined and countably determined.

ACKNOWLEDGEMENT

The author wishes to express her gratitude to her supervisor, Dr. John H. M. Whitfield whose grants, academic and professional advice, suggestions and encouragements made this thesis possible. Special thanks are due to the Mathematical Sciences Department of Lakehead University for its hospitality and many stimulating ideas. Also, appreciation is expressed to Dr. Marian Fabian for helpful conversations concerning many topics in the thesis. Finally, she is grateful to her husband, Xing, for his love and support during her Master's program.

List of Special Symbols

Symbols which will be used in this thesis are list below:

$\|X\|$ or card(X)	the cardinality of set X.
ω	the cardinality of a countable set.
ω_1	the cardinality of the first uncountable ordinal.
ω_2	the cardinality of the second uncountable ordinal.
Φ	empty set.
N	set of natural numbers.
R	set of real numbers.
R^n	n - dimensional vector spaces.
A^B	product space.
(X, τ)	a topological space X with topology τ .
$\subseteq$	set containment.
$\cap$	set intersection.

$\cup$	set union.
i.e.	that is.
iff	if and only if.
$\Rightarrow$	imply.
$\Leftarrow$	only if
$\neq\Rightarrow$	do not imply.
$\Leftrightarrow$	if and only if.
$(X, \|\|.\|\|)$	normed space X with the norm $\|\|.\|\|$.
$B(X)$	closed unit ball of a normed space X.
$\rho(x, \varepsilon)$	$\{y \in X: \rho(x,y) \leq \varepsilon$, where ρ is a metric on $X\}$.
$f\|A$	function f restricted to the set A.

Contents

Chapter 1

Introduction

1.1 Historical Notes and Main Results

Compactness has long been of interest to topologists and played a significant role in analysis. In this thesis we study several notions of compactness that have been of particular interest to functional analysts in the past couple of decades. We focus our attention on the connections between these compact spaces.

The compact spaces we are going to study are uniform - Eberlein compact, Eberlein - compact, Talagrand - compact, Gul'ko - compact, Corson - compact, Valdivia - compact, Rosenthal - compact, and Radon - Nikody'm compact spaces. The term Eberlein - compact was coined by Lindenstrauss [23] in 1972 in his survey paper. The Talagrand - compactness was introduced by Talagrand in 1979. Talagrand - compact spaces, however, are based on κ - analytic spaces which were first introduced by Choquet [9] in 1953. Gul'ko [19] in 1979 introduced the notion of Gul'ko - compactness. Corson - compact spaces were first studied by Corson [11] in 1959 . Valdivia - compact spaces were first introduced by Argyros [30] in 1988. Rosenthal - compact spaces were first studied by Rosenthal [35] in 1977. And, Radon - Nikody'm compact spaces were introduced by Namioka [27] in 1987.

Through our research, we find that

$$UEC \Rightarrow EC \Rightarrow TC \Rightarrow GC \Rightarrow CC \Rightarrow VC;$$

but, $UEC \Leftarrow\neq EC \Leftarrow\neq TC \Leftarrow\neq GC \Leftarrow\neq CC \Leftarrow\neq VC;$

$EC \Rightarrow RN$, but $RN \neq\Rightarrow EC$;

$RN \neq\Rightarrow CC$, $CC \neq\Rightarrow RN$;

$RN \neq\Rightarrow RC$, $RC \neq\Rightarrow RN$;

Furthermore, we have

$EC \Leftrightarrow RN + CC$, $VC \Leftrightarrow CC$ or $[0, \omega_1] \subseteq VC$;

$EC + w(K) \leq \omega \Rightarrow RC$;

We show

$VC \neq\Rightarrow RN$, $RN \neq\Rightarrow VC$, $RN \neq\Rightarrow TC$, $RN \neq\Rightarrow GC$;

$RC \neq\Rightarrow EC$, and equivent definitions of κ - analytic and K - analytic and κ - counbtably determined and countable determined.

1.2 Basic Definitions and Theorems

Before starting to discuss these different compact spaces, we briefly recall some definitions and theorems which are required in the thesis.

Definition 1.2.1: Let (X, τ) be a topological space.

(a) The weight of X is $w(X) = \inf\{|\beta|: \beta \subseteq \tau, \beta \text{ is a basis}\}$.

(b) The density character of X is $d(X) = \inf\{|Y|: Y \subseteq X, Y \text{ is dense in } X\}$

(c) X is said to be scattered iff for any $C \subseteq X$, C has at least one isolated point.

(d) X is called a Polish space iff it is a complete separable metrizable space.

Definition 1.2.2: A compact Hausdorff space (K, τ) is called angelic iff for any subset $C \subseteq K$ and any $x \in \tau\text{-}cl(C)$, there is a sequence $(x_n)_{n=1}^{\infty} \subseteq C$ such that $(x_n)_{n=1}^{\infty}$ converges to x.

Definition 1.2.3: Let (X, τ) be a topological space.

(a) $C(X)$ is defined by $C(X) = \{g: X \to R: g \text{ is continuous}\}$

(b) The pointwise topology on C(X) means that any net $(g_\alpha)_{\alpha \in A}$ of C(X) converges in the pointwise topology to g of C(X) $\Leftrightarrow$ $g_\alpha(x)$ converges to $g(x)$ for all $x \in X$.

Definition 1.2.4: Let (X, τ) be a Polish space. A function $g: X \to R$ is called a Baire - 1 function iff there is a sequence $(g_n)_{n=1}^{\infty}$ of $C(X)$ such that $(g_n(x))_{n=1}^{\infty}$ converges to $g(x)$ for all $x \in X$. The set of all Baire - 1 functions on X together with the pointwise topology is denoted as $\beta_1(X)$.

Definition 1.2.5: Let (X, τ) be a topological space, ρ be any metric on X. For any non - empty set $C \subset X$, $\rho(C) = \sup\{\rho(x, y): x, y \in C\}$. X is fragmented by ρ iff for each $\varepsilon > 0$ and any non - empty $C \subset X$, there is an open set O such that $O \cap A \neq \Phi$ and $\rho(O \cap A) \leq \varepsilon$.

Definition 1.2.6: A topological space (X, τ) is called norm - fragmented iff X is a subset of a normed linear space and is fragmented by the metric of the norm.

Theorem 1.2.1: (Namioka [27]) A topological space (X, τ) is ρ - fragmented iff for any $\varepsilon > 0$ and for any $C \subset A$ which is τ – closed, there is τ - open subset O of X such that $C \cap O \neq \Phi$ and $\rho(C \cap O) \leq \varepsilon$.

Proof: $\Rightarrow$ It is obvious.

$\Leftarrow$ For any $\varepsilon > 0$ and $C \subset X$, there is a τ – open subset O of X such that

τ - cl(C) $\cap$ O $\neq \Phi$ and $\rho(\tau$ - cl(C) $\cap$ O) $\leq \varepsilon$. Clearly, $\rho(C \cap O) \leq \varepsilon$. To see that C $\cap$ O $\neq \Phi$, suppose that C $\cap$ O = Φ. Since τ - cl(C) $\cap$ O $\neq \Phi$, there exists $x \in \tau$ - cl(C) $\cap$ O. Hence, there is a net $(x_\alpha)_{\alpha \in A}$ of C such that $\lim_{\alpha \in A} x_\alpha = x$. Therefore, for the neigbourhood O of x, there is λ_0 in the directed set A such that for any $\alpha \in A, \alpha > \lambda_0, x_\alpha \in O$. Since $(x_\alpha)_{\alpha \in A} \subset C$, this is a contradiction.

<u>Theorem 1.2.2:</u> (Namioka [27]) Let (X, τ) be a topological space and ρ be a metric on X. Then the following conditions are equivalent.

(1) The space (X, τ) is ρ - fragmented.

(2) For any τ - closed C $\subset$ X, the set

A(C) = {x: $x \in C$ such that i: $(C, \tau) \to (C, \rho)$ is continuous at x} $\neq \Phi$, where i is the identity map.

Proof. (1) $\Rightarrow$ (2): Suppose that (X, τ) is ρ - fragmented. Let C be a τ - closed subset of X. By Theorem 1.2.1, there is a τ - open subset O of X such that C $\cap$ O $\neq \Phi$ and $\rho(C \cap O) \leq \varepsilon$, which means exactly that A(C) $\neq \Phi$.

(2) $\Rightarrow$ (1): Suppose that for any τ - closed subset C of X, A(C) $\neq \Phi$. Then there exists $x \in C$ such that for any $\varepsilon > 0$, there is a τ - open subset O of X such

that for any $y \in C \cap O$, $y \in \rho(x, \varepsilon)$. By the theorem 1.2.1, (X, τ) is ρ - fragmented.

Theorem 1.2.3: Let (X, τ) be a topological space which is fragmented by a metric ρ, then X is fragmented by any metric whose topology is identical with that of ρ.

Proof. It follows from Theorem 1.2.2 easily.

In the case that Y is a normed linear space over R, we define the dual of Y, Y^*, as follows: $Y^* = \{h : Y \to R: h \text{ is continuous and linear on } Y\}$. The weak topology on Y is defined by $x_\alpha \to x$ in the weak topology of Y iff $g(x_\alpha) \to g(x)$ for any $g \in Y^*$. This is denoted by (Y, weak). Also, the weak* topology on Y^* is defined by $f_\alpha \to f$ in the weak* topology of Y^* iff $f_\alpha(x) \to f(x)$ for any $x \in Y$. This weak* topological space is denoted by (Y^*, weak*). When (K, τ) is a compact topological space, we define the sup - norm on C(K) as follows: $||f||_\infty = \sup_{x \in K}|f(x)|$ for any $f \in C(K)$. it is a classical result that C(K) is a Banach space with this norm. Hence, we are able to define the weak topological space (C(K), weak) and the weak* - topological space ($C^*(K)$, weak*).

Definition 1.2.7: A Banach space X is called an Asplund space iff every separable subspace of X has a separable dual.

Definition 1.2.8: A Banach space X is called weakly compactly generated (W.C.G.) iff there exists a weakly compact set $K \subseteq X$ such that the linear span of K is dense in X.

Chapter 2

Compact Spaces

2.1 Introduction

In this chapter, we introduce each of the notions of compactness that is considered in this thesis. Also, some relationships between types of compacta are given in this chapter.

2.2 Definitions of Compact Spaces

In the following definitions of compact spaces, we always let (K, τ) be a compact topological space.

Definition 2.2.1: K is said to be uniform Eberlein - compact (UEC) iff K is homeomorphic to a weakly compact subset of a Hilbert space X.

Definition 2.2.2: K is said to be Eberlein - compact (EC) iff K is homeomorphic to a weakly compact subset of a Banach space X.

<u>Comments:</u> From Definition 2.21 and 2.2.2, it is easy to see that

UEC $\Rightarrow$ EC, since every Hilbert space is a Banach space.

<u>Definition 2.2.3:</u> K is said to be <u>Radon - Nikody'm compact</u> (RN) iff K is homeomorphic to weak* - compact subset of X *, where X is an Asplund space.

<u>Definition 2.2.4:</u> K is said to be <u>Rosenthal - compact</u> (RC) iff K is homeomorphic to a subset of $\beta_1(X)$ (from τ on K to the pointwise topology on $\beta_1(X)$, where X is a Polish space).

<u>Definition 2.2.5:</u> A topological space (X, τ) is said to be <u>κ - analytic</u> iff X is continuous image of a $K_{\sigma\delta}$ set, i.e. $X = f(\cap_{n=1}^{\infty}\cup_{m=1}^{\infty}K_{m,\,n})$, where $f\colon (Y, \tau_1) \rightarrow (X, \tau)$, f is contiuous, $K_{m,n}$ is compact in (Y, τ_1) for any $m,n \in N$.

<u>Definition 2.2.6:</u> K is said to be <u>Talagrand - compact</u> (TC) iff C(K) is a κ - analytic set in its weak topology.

<u>Definition 2.2.7:</u> A topological space (X, τ) is said to be <u>κ - countably determined</u> iff there is a compact topological space (Y, τ_1) such that $X \subseteq Y$ and there is a

sequence of τ_1 - closed subsets, $(K_n)_{n=1}^{\infty}$, of Y such that for every $x \in X$, there is $\Phi \neq N(x) \subseteq N$ such that $x \in \cap_{n \in N(x)} K_n$.

Definition 2.2.8: K is said to be Gul'ko - compact (GC) iff C(K) is a κ - countably determined set in its weak topology.

Definition 2.2.9: K is said to be Corson - compact (CC) iff there is a set C such that K is homeomorphic to a subset of $\Sigma(R^C) = \{ x \in R^C: \text{supp}(x) \text{ is countable}\}$, where $\text{supp}(x) = \{c \in C: x_c \neq 0\}$, (from τ on K to the product topology on $\Sigma(R^C)$) .

Definition 2.2.10: K is said to be Valdivia - compact (VC) iff $K \subseteq [0, 1]^C$ for some set C such that $K \cap \Sigma([0, 1]^C)$ is dense in K.

Comments: Following Definition 2.2.9 and Definition 2.2.10, it is easy to see that CC $\Rightarrow$ VC.

Theorem 2.2.1 (Corson [11], Proposition 1) Any metrizable space is a Corson - compact space.

Proof. Let X be a metric space. Bing [7] showed that there is a sequence $(A_i)_{i=1}^{\infty}$, where each A_i is a family of open sets in the metric topology τ, such that $\cup_{i=1}^{\infty} A_i$ is a base of τ and each $x \in X$ is contained in an open set which meets at most one member of $(A_i)_{i=1}^{\infty}$, i.e. there is $O \in \tau$ and at most one $j \in N$ such that $x \in O \cap O(A_j) \neq \Phi$, where $O(A_j) \in A_j$. For any open set $O(A_i) \in A_i$, there is a countable familty of open sets $B_j(O(A_i)) \subseteq O(A_i)$, such that $O(A_i) = \cup_{j=1}^{\infty} B_j(O(A_i))$.

Let $U = \{B\colon$ B is a kind of $B_j(O(A_i))$ for $i, j \in N\}$.

For each $B \in U$, there is $i \in N$, and $j \in N$ such that $B = B_j(O(A_i))$. By [Wilansky [46], Theorem 4.3.3] which says that every semimetric space is normal, then X is normal, since X is metric space. By Urysohn's lemma, we can define a continuous function $f_B : X \to [0, 1]$ such that

$$f_B(x) = 1 \quad x \in B,$$

$$f_B(x) = 0 \quad x \notin A_i.$$

By [Kelley [22], Lemma 4.5] the evaulation function $f : X \to \Pi\{f_B(x)\colon B \in U\}$, where $(f(x))_B = f_B(x)$, is a continuous function. Since there are countably many B in U, then $f(X) \subseteq \Sigma([0, 1]^U)$. Hence, X is Corson - compact.

2.3 Eberlein - compact Implies Talagrand - compact

The main theorem of this section is that every Eberlein - compact space is a Talagrand - compact space. But, before we show it, we need some theorems. One of these is the following one which is well - known and connects a W.C.G. space with Eberlein- compact space (see Diestel [16], p.152):

Theorem 2.3.1 Let (K, τ) be a compact Hausdorff topological space. Then the following conditions are equivalent:

(1) K is Eberlein - compact;

(2) C(K) is a W.C.G. Banach space;

(3) The closed unit ball of $C^*(K)$ is Eberlein compact in its weak*- topology.

The key theorem for the proof of the main theorem of this section is Theorem 2.3.2 which was first proved by Talagrand [40]. The proof given here follows that of Rogers and Jayne [32].

Theorem 2.3.2 (Talagrand [40]) If X is a W.C. G. Banach space, then X is κ -

analytic in its weak topology.

Proof. Suppose that X is a W.C.G. Banach space. Then there is a weakly compact subset $C \subseteq X$ such that w - cl(span(C)) = X, where w - cl(span(C)) denotes the closure of span(C) in (X, weak). Let

$A = \{a: a = (r_1, \ldots, r_n), r_i$ is a rational number, $n \in N\}$. If we set $C(a) = r_1C + \ldots + r_nC$, then C(a) is a weakly compact set in X. Since A is countable, let $(C(n))_{n=1}^{\infty}$ be an enumeration of the sequence $\{C(a): a \in A\}$.

Then for any positive natural number $m \in N$ and considering X as a subset of X^{**},

$$X \subset \cup_{n=1}^{\infty} C(n) + (1/m)B(X) \subset \cup_{n=1}^{\infty} C(n) + (1/m)B(X^{**}),$$

so, $$X \subseteq \cap_{m=1}^{\infty} \cup_{n=1}^{\infty} (C(n) + (1/m)B(X^{**})).$$

Let $$x^{**} \in \cap_{m=1}^{\infty} \cup_{n=1}^{\infty} (C(n) + (1/m)B(X^{**})),$$

then for any $m \in N$, there is $x_m \in \cup_{n=1}^{\infty} C(n) \subseteq X$ such that

$$\|x^{**} - x_m\| \leq 1/m. \qquad (*)$$

The Cauchy sequence $(x_m)_{m=1}^{\infty}$ converges to some point in X, and because of (*), we know x^{**} is the limit of $(x_m)_{m=1}^{\infty}$. Hence,

$$X = \cap_{m=1}^{\infty} \cup_{n=1}^{\infty} (C(n) + (1/m)B(X^{**})).$$

Since $C(n) + (1/m)B(X^{**})$ is weak* - compact in the weak* - topology of X^{**}, X is a $\kappa_{\sigma\delta}$ set in (X**, weak*). Hence X is a $\kappa_{\sigma\delta}$ in (X, weak).

Theorem 2.3.3 Every Eberlein - compact space is a Talagrand - compact space.

Proof. Suppose K is Eberlein - compact, then C(K) is a W.C.G. Banach space by Theorem 2.3.1. Following Theorem 2.3.2, C(K) is κ - analytic in (C(K), weak). So, K is Talagrand - compact.

2.4 Talagrand - compact Implies Gul'ko - compact

The basic aim of section 2.4 is to show that every Talagrand - compact space is a Gul'ko - compact space. We will use Talagrand's theorem which says that every K - Souslin set is κ - countably determined.

Definition 2.4.1: Let $S = \{s: \ s = (n_1, \dots, n_i), \ n_i \in N, 1 \le i < \omega\}$ and $\Sigma = \{\sigma: \ \sigma = (n_1, \dots, n_i, \dots), n_i \in N\}$. We say $s < \sigma$ for any $s \in S$ and $\sigma \in \Sigma$ iff s and σ have the same first i terms.

Definition 2.4.2: For $S' \subseteq S$, suppose that $(B_s)_{s \in S'}$ is a class of subsets of X.

Define B_σ by: $B_\sigma = \cap_{s<\sigma} B_s$, where $s \in S'$. The set $A = \cup_{\sigma\in\Sigma} B_\sigma$ is called the <u>nucleus</u> associated with class $(B_s)_{s \in S'}$; sometimes, we say that the set A is obtained from $(B_s)_{s\in S'}$ by <u>Souslin's operation</u>.

<u>Definition 2.4.3:</u> Let (X, τ) be a Hausdorff topological space. Any $C \subseteq X$ which is obtained by Soulin's operation from a class of compact subsets of X is called a <u>X - Souslin set</u>.

<u>Theorem 2.4.1</u> (Talagrand [41], Proposition 1.1) Let (K, τ) be a compact topological space and $A \subseteq K$. If there is a family $(B_s)_{s\in S}$, where B_s is compact in (K, τ), and a $\Sigma' \subseteq \Sigma$ such that $A = \cup_{\sigma\in\Sigma'} \cap_{s<\sigma} B_s$, then there is $(K_n)_{n=1}^\infty$, where K_n compact in (K, τ), such that for any $x \in A$ there exists $N(x) \subseteq N$ such that $x \in \cap_{n\in N(x)} K_n$.

Proof. Since $A = \cup_{\sigma\in\Sigma'} \cap_{s<\sigma} B_s$, then for any $x\in A$, there is $\sigma \in \Sigma' \subseteq \Sigma$ such that $x \in \cap_{s<\sigma} B_s$. Since $|S| = \omega$, let $(K_n)_{n=1}^\infty$ be an enumeration of $(B_s)_{s\in S}$. Then for any $x \in A$, there is $N(x) \subseteq N$ such that $x \in \cap_{n\in N(x)} K_n$, where K_n is compact in (X, τ) for $n \in N$.

Theorem 2.4.2 Let (X, τ) be a Hausdorff topological space and K be κ - analytic. Then K is an X - Souslin set.

Proof. Since X is κ - analytic, without loss of generality, we can assume that $K = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} K_{m,n}$, where $K_{m,n}$ is compact in (X, τ). Since, $X = \cup_{m=1}^{\infty} \cap_{n=1}^{\infty} K_{m,n}$, then K is a X - Souslin set.

Theorem 2.4.3 Every Talagrand - compact space is a Gul'ko - compact space.

Proof: Using Theorem 2.4.1 and Theorem 2.4.2, it is easy to get this result.

2.5 Gul'ko - compact implies Corson - compact

In this section, we will show Gul'ko's theorem [19] which says that every Gul'ko - compact space is a Corson - compact. First, we prove some lemmas.

Definition 2.5.1: Let (X, τ) be a topological space. A set $C \subseteq C(X)$ is said to distinguish points of X iff for any $x, y \in K$ $x \neq y$, there is $f \in C$ such that $f(x) \neq f(y)$.

Definition 2.5.2: Let (K, τ) be a compact Hausdorff space and $C \subseteq C(K)$ distinguishes points of K. The pair (A, B), where $A \subseteq K$ and $B \subseteq C$, is called C - conjugate iff for any $x \in K$, there is a $y \in A$ such that for any $f \in B$, $f(x) = f(y)$, and for all $f \in C$, there is $g \in B$ such that for any $x \in A$, $f(x) = g(x)$ (y and g are unique, since C distinguishes points of K).

Definition 2.5.3: Let (K, τ) be a compact Hausdorff space and $C \subseteq C(K)$. The pair (A, B), where $A \subseteq K$ and $B \subseteq C$, is called C - preconjugate iff $\{x|B: x \in A\}$ is dense in $\{x|B: x \in K\}$ and $\{f|A: f \in B\}$ is dense in $\{f|A: f \in C\}$, where the topology on C is the pointwise topology.

Lemma 2.5.1 Let (K, τ) be a compact Hausdorff space, $C \subseteq C(K)$ distinguish points of K. If (A, B), where $A \subseteq K$ and $B \subseteq C$, is C - conjugate, then (A, B) is C - preconjugate.

Proof. Suppose (A, B) is C - conjugate. Then, by Definition 2.5.2, for any $x \in K$, there is a $y \in A$ such that for any $f \in B$, $f(x) = f(y)$. That implies $\{x|B: x \in A\}$ is dense in $\{x|B: x \in K\}$. By Definition 2.5.2 again, for any $f \in C$, there is $g \in B$ such that for any $x \in A$, $f(x) = g(x)$. That implies $\{f|A: f \in B\}$ is dense in $\{f|A: f \in C\}$. So, (A, B) is C - preconjugate.

<u>Lemma 2.5.2</u> Let (K, τ) be a compact Hausdorff space and $C \subseteq C(K)$ distinguishes points of K. If (A, B), where $A \subseteq K$ and $B \subseteq C$, is C - conjugate, then $\{x|B: x \in A\} = \{x|B: x \in K\}$ and $\{f|A: f \in B\} = \{f|A: f \in C\}$.

Proof. Since $A \subseteq K$, then $\{x|B: x \in A\} \subseteq \{x | B: x \in K\}$. From Definition 2.5.2, for any $x \in K$, there is $y \in A$ such that for any $f \in B$, $f(x) = f(y)$. That means there is $y \in A$ such that $x|B = y|B$. So, $\{x|B: x \in K\} \subseteq \{x|B: x \in A\}$. Hence, $\{x|B: x \in A\} = \{x|B: x \in K\}$.

Using the same method, we can prove $\{f|A: f \in B\} = \{f|A: f \in C\}$.

<u>Lemma 2.5.3</u> Let (K, τ) be a compact Hausdorff space. Then for any $C \subseteq C(K)$, $M \subseteq K$, $L \subseteq C$, α an infinite cardinal number, and the cardinal number of M and of L is not greater than α, respectively, there is a C - preconjugate pair (A, B) such that $M \subseteq A$, $L \subseteq B$ and the cardinal number of A and of B is not greater than α, respectively.

Proof. We are going to construct (A, B) which satisifes the condition of Lemma 2.5.3. Let $A_1 = M$, construct B_1, A_2 such that $A_1 \subseteq A_2$, $B_1 \subseteq C$,

$\{x|B_1: x \in A_2\}$ is dense in $\{x|B_1: x \in K\}$ and $\{f|A_1: f \in B_1\}$ is dense in $\{f|A_1: f \in C\}$. To do so,

<u>Step 1:</u> let $R(A_1) = \{g|A_1: g \in C\}$. Since $d(R(A_1)) \leq w(R(A_1)) \leq |R(A_1)| \leq \alpha$ is true for the pointwise topology on $R(A_1)$, then there is $U(A_1) \subseteq R(A_1)$ which is dense in $R(A_1)$ and $|U(A_1)| \leq \alpha$. Set $B_1 = L \cup \{g: g|A_1 \in U(A_1)\}$. Hence, $\{f|A_1: f \in B_1\}$ is dense in $\{f|A_1: f \in C\}$.

<u>Step 2:</u> let $R(B_1) = \{x|B_1 : x \in K\}$. Since we have $d(R(B_1)) \leq w(R(B_1)) \leq |R(B_1)| \leq \alpha$ in the pointwise topology, we can find $U(B_1) \subseteq R(B_1)$ which is dense in $R(B_1)$ and $|U(B_1)| \leq \alpha$. Set $A_2 = A_1 \cup \{x: x|B_1 \in U(B_1)\}$.

From above, we know $A_1 \subseteq A_2$ and $\{x|B_1: x \in A_2\}$ is dense in $\{x|B_1: x \in K\}$.

Using the same method, we can get non - decreasing sequences $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ such that

(1) $\{x|B_n: x \in A_{n+1}\}$ is dense in $\{x|B_n: x \in K\}$;

(2) $\{f|A_n: f \in B_n\}$ is dense in $\{f|A_n: f \in C\}$.

Let $A = \cup_{n=1}^{\infty} A_n$, $B = \cup_{n=1}^{\infty} B_n$, then (A, B) is C - preconjugate. To see this, let $g \in C$, $\varepsilon > 0$, and $x_1, \dots, x_n \in A$. Since $A = \cup_{n=1}^{\infty} A_n$ and $(A_n)_{n=1}^{\infty}$ is non - decreasing, there is $m \in N$ such that $x_1, \dots, x_n \in A_m$. Since $\{f|A_m: f \in B_m\}$ is dense in $\{f|A_m: f \in C\}$, then there is $f \in B_m \subseteq B$ such that $|f(x_i) - g(x_i)| < \varepsilon$, for $i = 1, \dots, n$. That means $\{f|A: f \in B\}$ is dense in $\{f|A: f \in C\}$. In the same way, one can prove $\{x|B: x \in A\}$ is dense in $\{x|B: x \in K\}$.

<u>Lemma 2.5.4</u> Let (K, τ) be a compact Hausdorff space and $C \subset C(K)$. If there are non - decreasing families $(A_\xi)_{\xi<\eta}$ and $(B_\xi)_{\xi<\eta}$, ξ and η are ordinals, such that (A_ξ, B_ξ) are C - preconjugate pairs for $\xi < \eta$, then $(\cup_{\xi<\eta} A_\xi, \cup_{\xi<\eta} B_\xi)$ is C - preconjugate.

Proof. The proof is similar to that in Lemma2.5.1. Let $A = \cup_{\xi<\eta} A_\eta$, $B = \cup_{\xi<\eta} B_\eta$. For any $g \in C$, $x_1, \dots, x_n \in A$, and any $\varepsilon > 0$, since $(A_\xi)_{\xi<\eta}$ is non - decreasing, there is $\xi_0 < \eta$ such that $x_1, \dots, x_n \in A_{\xi_0} \subseteq A$ and since (A_{ξ_0}, B_{ξ_0}) is C - preconjugate, there is $f \in B_{\xi_0} \subseteq B$ such that $|f(x_i) - g(x_i)| < \varepsilon$ for $i = 1, \dots, n$. This shows that $\{f|A : f \in B\}$ is dense in $\{f|A : f \in C\}$. In the same way, one can prove $\{x|B: \ x \in A\}$ is dense in $\{x|B : \ x \in K\}$.

Lemma 2.5.5 Let (K, τ) be a compact Hausdorff space and $C \subset C(K)$. If (A, B) is C - preconjugate, then $\{x|B_1: \ x \in A\} = \{x|B_1: \ x \in K\}$, where $A_1 = \tau$ - cl(A) and $B_1 = cl_C B$.

Proof. Since $A_1 \subseteq K$, $\{x|B_1: \ x \in A_1\} \subseteq \{x|B_1: \ x \in K\}$. (1)

For any $x_0 \in K$, we need to prove there is $z_0 \in A_1$ such that $x_0|B_1 = z_0|B_1$. To do so, since (A, B) is C - preconjugate, for any $y \in K$ and any finite subset $D \subseteq B$, there is $x_D \in A$ such that $|f(y) - f(x_D)| < 1/|D|$. (*)

Then $\{x_D: D \subseteq B \text{ and } |D| < \omega\}$ is a net in A. Since K is compact, $\{x_D: \ D \subseteq B \text{ and } |D| < \omega \}$ has a limit point z in K. From (*), we know $y|B = z|B$. When considering x and y as functions on C, then, x, y are continuous. So, $y|B_1 = z|B_1$. Hence, $\{x|B_1: \ x \in K\} \subseteq \{x|B_1: \ x \in A_1\}$. (2)

By (1) and (2), $\{x|B_1: \ x \in K\} = \{x|B_1: \ x \in A_1\}$.

Lemma 2.5.6 Let (K, τ) be a compact Hausdorff space. If there are $M \subseteq K$, $C_n \subseteq C(K)$, $n \in N$, $L \subset \cup_{n=1}^{\infty} C_n = C$ and the cardinal number of M , L is not greater than an infinite cardinal number α, then there are $A \subseteq K$, $B \subseteq C$ which satisy $M \subseteq A$, $L \subseteq B$ and the cardinal number of A, B is not greater than α, respectively, and $(A, B \cap C_n)$ is C_n - preconjugate for any $n \in N$.

Proof. Let $N = \cup_{n=1}^{\infty} N_n$, where $N_n \cap N_m = \Phi$ for any $m \neq n$, $|N_n| = \omega$, and for any $n \in N$, there is a unique N_m such that $n \in N_m$. Hence for $M \subseteq K$, $B \cap C_{n(1)} \subseteq C$, by Lemma 2.5.3, there is A_1, B_1 such that $(A_1, B_1 \cap C_{n(1)})$ is $C_{n(1)}$ - preconjugate. Using the same method, we can get non - decreasing sequences $(A_m)_{m=1}^{\infty}$ and $(B_m)_{m=1}^{\infty}$ such that $(A_m, B_m \cap C_{n(m)})$ is $C_{n(m)}$ - precongugate. Then, by the Lemma 2.5.4, for $A = \cup_{m=1}^{\infty} A_m$, $B = \cup_{m=1}^{\infty} B_m$, $(A, B \cap C_n)$ is C_n - preconjugate.

Lemma 2.5.7 Let (K, τ) be a Gul'ko - compact space. Suppose B(K) is the unit ball of C(K). Let $(C_n)_{n=1}^{\infty}$ be a sequence of compact subsets of $[-1, 1]^K$ with the product topology, closed under finite intersections, such that

(1) for every $g \in B(K)$, there is $\Phi \neq N_g \subseteq N$ with $g \in \cap_{n \in N(g)} C_n \subseteq B(K)$, where B(K) is considered a subset of $[-1, 1]^K$.

Let $A \subseteq K$, $B \subseteq \cup_{n=1}^{\infty} C_n \cap B(K)$ such that

(2) $(A, B \cap C_n \cap B(K))$ is $C_n \cap B(K)$ - preconjugate for $n \in N$.

Then, (A_1, B_1) is $B(K)$ - conjugate, where $A_1 = \tau$ - cl(A), $Q = cl_{B(K)}B$.

Proof. Following Lemma 2.5.2, condition (2) and Lemma 2.5.6, we have

$$\{x|B_1 \cap C_n: x \in A_1\} = \{x|B_1 \cap C_n: x \in K\}, \text{ for any } n \in N. \quad (*)$$

Since condition (*) is true for any $n \in N$, then

$$\{x|\cup_{n=1}^{\infty} B_1 \cap C_n: x \in A_1\} = \{x|\cup_{n=1}^{\infty} B_1 \cap C_n: x \in K\}.$$

From condition (1), we have $B_1 \subseteq \cup_{n=1}^{\infty} C_n$,

$\{x|B_1: x \in A_1\} = \{x | B_1: x \in K\}$.

We are going to show that $\{f|A_1: f \in B_1\} = \{f|A_1: f \in B(K)\}$.

Since $B_1 \subseteq B(K)$, $\{f|A_1: f \in B_1\} \subseteq \{f|A_1: f \in B(K)\}$.

For any $g \in B(K)$, by condition (1), we have $\Phi \neq N(g) \subseteq N$ such that

$g \in \cap_{i \in N(g)} C_i \subseteq B(K)$. Since $g \in C_i \cap B(K)$ and $\{f|A: f \in B \cap C_i \cap B(K)\}$ is dense in $\{f|A: f \in C_i \cap B(K)\}$, there is a net $(f_\alpha)_{\alpha \in A} \subset B \cap C_i \cap B(K) \subset C_i \cap B_1$ such that $\lim_{\alpha \in A} f_\alpha(x) = g(x)$ for $x \in A$. Let $f_g \in C_i \cap B_1$ be a continuous function on K such that $f_g(x) = \lim_{\alpha \in A} f_\alpha(x)$ for $x \in A$. So, $\{f|A_1: f \in B(K)\} \subseteq \{f|A_1: f \in B_1\}$.

Hence, $\{f|A_1: f \in B_1\} = \{f|A_1: f \in B(K)\}$. Therefore, (A_1, B_1) is $B(K)$ - conjugate according to Lemma 2.5.2.

Lemma 2.5.8 (Talagrand [41]) Let (K, τ) is a Gul'ko compact space. If $B(K)$ is the unit ball of $C(K)$ in the pointwise topology, then $w(K) = d(K) = d(B(K))$.

Lemma 2.5.9 (Corson [12]) Let (K, τ) be compact Hausdorff space. If $C(K)$ in its pointwise topology is Lindelof, then K is angelic.

Theorem 2.5.1 If (K, τ) is a Gul'ko - compact space, then K is Corson - compact.

Proof. We are going to use transfinite induction.

First, suppose $w(K) = \omega$. Then K has a countable base and, being compact space, K is metrizable. By Theorem 2.2.1, K is Corson - compact.

Secondly, suppose that $\alpha > \omega$ and for any cardinal $\omega \leq \beta < \alpha$, if $w(K) = \beta$ and K is Gul'ko - compact, then K is Corson - compact.

Thirdly, suppose $w(K) = \alpha$. By the Lemma 2.5.6, $w(K) = d(K) = d(B(K))$.

Let $U(K) = \{x_\eta: \omega \leq \eta < \alpha\}$ and $V(B(K)) = \{g_\eta: \omega \leq \eta < \alpha\}$ be dense subsets of K and B(K), respectively. From Lemma 2. 5.1 to Lemma 2.5.5, we can construct a family $\{(A_\eta, B_\eta): \omega \leq \eta < \alpha\}$ such that (A_η, B_η) is B(K) - conjugate for $\omega \leq \eta < \alpha$, satisfying

(i) $x_\eta \in A_{\eta+1}$, $g_\eta \in B_{\eta+1}$;

(ii) $\{A_\eta: \omega \le \eta < \alpha\}$ and $\{B_\eta: \omega \le \eta < \alpha\}$ are increasing families;

(iii) if η is a limit ordinal, then $A_\eta = cl_K(\cup_{\xi<\eta} A_\xi)$, $B_\eta = cl_{B(K)}(\cup_{\xi<\eta} B_\xi)$; and

(iv) $w(A_\eta) = d(A_\eta) = d(B(K)) \le |\eta|$ for $\omega \le \eta < \alpha$.

For $\omega \le \eta < \alpha$, by the induction hypothesis, (A_η, τ) is a Corson - compact space. That means there is a homeomorphism f_η: $A_\eta \rightarrow \Sigma(R^{C_\eta})$ for some C_η. Without loss of generality, we can assume

$\{C_\eta: \omega \le \eta < \alpha\}$ is pairwise disjoint. Let $C = N \cup \{C_{\eta+1}: \omega \le \eta < \alpha\}$. Define a homeomorphism f: $K \rightarrow \Sigma([0, 1]^C)$ by setting for each $x \in K$,

$f(x)(c) = f_\omega(y_x)(c)$, $y_x \in A_\omega$, $c \in N$

(using the fact that (A_ω, B_ω) is B(K) - conjugate);

$f(x)(c) = f_{\eta+1}(y_x)(c) - f_{\eta+1}(z_x)(c)$, $y_x \in A_{\eta+1}$, $z_x \in A_\eta$, for $c \in C_{\eta+1}$

(using the fact that families A_η, B_η are increasing).

It is shown below that for any $x \in K$, $A = \{\eta : \omega \le \eta < \alpha$ and $y_x \ne z_x\}$ is countable, wheace f: $K \rightarrow \Sigma(R^C)$ is a homeomorphism, because each f_η is a homeomorphism. To see that A is countable, suppose there is $x_0 \in K$ such that the set $A = \{\eta\colon \omega \leq \eta < \alpha, y_{x(o)} \neq z_{x(o)}\}, y_{x(o)} \in A_{\eta+1}, z_{x(o)} \in A_\eta\}$ is uncountable. Then, there is an uncountable subset $B \subset A$ such that $y_{x(o)} \neq z_{x(o)}$, $y_{x(o)} \in A_\xi, z_{x(o)} \in A_\zeta$ for all $\xi, \zeta \in B, \xi \neq \zeta$.

Let $\tau = \sup B \leq \alpha$. we choose B such that $cf(\tau) \geq \omega_1$, since if $cf(\tau) = \omega$ and $\tau > \omega_1$, then there is τ_0 with $\omega_1 \leq \tau_0 < \tau$ such that $cf(\tau_0) \geq \omega_1$ and $\tau_0 \cap B$ is uncountable. Then, choose τ_0 instead of τ.

We know $y_{x(o)} = \lim_{\xi \in B} z_{x(o)}$, where $y_{x(o)} \in A_\tau, z_{x(o)} \in A_\xi$. From Lemma 2.5.9, there is $(\xi_n)_{n=1}^{\infty} \subset B$ with $y_{x(o)} = \lim_{n \in N} z_{x(o)}$, where $z_{x(o)} \in A_{\xi(n)}$. Then, for $\zeta = \sup_{n \in N} \xi_n$, we have $z_{x(o)} = u_{x(o)} = y_{x(o)}$ for $z_{x(o)} \in A_\zeta, u_{x(o)} \in A_\xi, \zeta \leq \xi \leq \eta$, and $y_{x(o)} \in A_\tau$. This is a contradiction which completes the proof of Theorem 2.5.1.

2.6 Conclusion

Based on the results in this chapter, we have that every uniform Eberlein - compact space is a Eberlein - compact space, every Eberlein - compact space is a Talagrand - compact space, every Talagrand - compact space is a Gul'ko - compact space, every Gul'ko - compact space is a Corson - compact space, and every Corson - compact space is a Valdivia compact space, i.e.

$UEC \Rightarrow EC \Rightarrow TC \Rightarrow GC \Rightarrow CC \Rightarrow VC.$

Chapter 3

Some Counterexamples

3.1 Introduction

In chapter 2, we proved UEC $\Rightarrow$ EC $\Rightarrow$ TC $\Rightarrow$ GC $\Rightarrow$ CC $\Rightarrow$ VC. In this chapter, we are going to show that Eberlein - compact does not imply uniform Eberlein - compact, Talagrand - compact does not imply Eberlein - compact, Gul'ko - compact does not imply Talagrand - compact, Corson - compact does not imply Gul'ko - compact, and Valdivia - compact does not imply Corson - compact.

3.2 Eberlein - Compact Does Not Imply Uniform Eberlein - Compact

In this section, we are going to give a counterexample which shows there is an Eberlein - compact space which is not a uniform Eberlein - compact space. The example is due to Benyamini and Starbird [6]. First, some lemmas are required.

Lemma 3.2.1 (Juha'sz [21]) Let C be any uncountable set. If

$\{A_\alpha: A_\alpha \subseteq C$ is finite for $\alpha \in \Psi\}$ is an uncountable family of subsets of C, then

there are a $\Psi_1 \subseteq \Psi$, $|\Psi_1| \geq \omega_1$, a finite set $B \subseteq C$ and a pairwise disjoint family

$\{B_\alpha: B_\alpha \subseteq C$ is finite for $\alpha \in \Psi_1\}$ such that for any $\alpha \in \Psi_1$, $A_\alpha = B \cup B_\alpha$.

Lemma 3.2.2 Let C be any set and $K = \{B: B \subseteq C$ and B is finite$\}$. If

(1) any $B \in K$ and $D \subseteq B$ implies $D \in K$;

(2) there is no infinite increasing chain in K;

then K is a weakly compact set in $c_0(\Gamma)$, for some set Γ.

Note: this mean K is Eberlein - compact.

Proof. We identify sets in K with their characteristic functions. Then, for some $\Gamma, K \subseteq c_0(\Gamma)$.

We will show K is a weakly compact subset of $c_0(\Gamma)$. Let U be a limit point of K and $V \subset U$ be any finite set. Since the weak topology on $c_0(\Gamma)$ is the same as the pointwise topology on $c_0(\Gamma)$, there is a $B_0 \in K$ such that $V \subseteq B_0$. By (1), $V \in K$. Since there is no infinite increasing chain in K, then U is finite. Hence $U \in K$. So, K is weakly compact set in $c_0(\Gamma)$.

<u>Lemma 3.2.3</u> Let (K, τ) be a weakly compact subset of Hilbert space X. If a discrete set $C \subseteq K$ has a unique limit point c_0, then for any $c \in C$, there is τ-open subset U_c of K satisfying

(a) $c \in U_c$;

(b) there is a countable pairwise disjoint family $(C_n)_{n=1}^{\infty}$ of C such that

$C = \cup_{n=1}^{\infty} C_n$ and for any $c_1, \ldots, c_{n+1} \in C_n$, $c_1 \neq c_2 \neq \ldots \neq c_{n+1}$,

$\cap\{U_{c(i)}: 1 \leq i \leq n+1\} = \Phi$.

Proof. Since X is a Hilbert space, then there is an inner product on it. For any $x, y \in X$, we let (x, y) denote its inner product. For convenience and without loss of generality, suppose K is the closed unit ball of X with the weak topology and c_0 is the origin of X.

We will define an equivalence relation on X.

$a \cong b$ iff there is a sequence $c_1, c_2, \ldots, c_j$ such that $c_1 = a, c_2, \ldots, c_j = b$ and $(c_i, c_{i+1}) \neq 0$.

From the definition above, we get

(1) if A and B are different equivalence classes and $a \in A$ and $b \in B$, then $(a, b) = 0$;

(2) the cardinal number of the equivlence classes is at most countable.

We will construct $(C_n)_{n=1}^{\infty}$ as follows:

Step 1: From (2), $C = \cup_{n=1}^{\infty} E_m$, where in each E_m, the elements are mutually orthogonal.

Step 2: C_n's are a partition of C such that for any $c_1, c_2 \in C_n$, $c_1 \neq c_2$, $(c_1, c_2) = 0$ and $\|c\|^4 > n^{-1}$ for any $c \in C_n$.

For any $c \in C_n$, let $U_c = \{x \in K: (x, c)^2 > n^{-1}\}$. Then, $c \in U_c$ and U_c is τ - open. Let $c_1, \ldots, c_{n+1} \in C_n$, $c_1 \neq c_2 \neq \ldots \neq c_{n+1}$. Suppose $x \in \cap\{U_{c(i)}: 1 \leq i \leq n+1\}$. Then $1 \geq \|x\|^2 \geq \Sigma_1^{n+1}(x, c_i)^2 \geq (n+1)\backslash n > 1$, a contradiction. So, $\cap\{U_{c(i)}: 1 \leq i \leq n+1\} = \Phi$.

Theorem 3.2.1 (Benyamini and Starbird [6]) There is a Eberlein - compact space (E, τ) which is not uniform Eberlein - compact.

Proof. Let $C = R \times (\Pi_{i=2}^{\infty}\{1, \ldots, i\})$. So any element $c \in C$ looks like $c = (x, m_1, \ldots, m_i, \ldots)$, where $x \in R$ and $m_i \in \{1, \ldots, i\}$. We consider the product topology τ on C.

For any $n \in N$, let F_i: $C \to N^i$ be a projection defined by

$F_i(x, m_1, \ldots, m_i, \ldots) = (m_1, \ldots, m_i)$.

For each $i \in N$, we will define a family A_i of subsets of C such that for any $D \in A_i$, $|D| = i$.

<u>Step 1:</u> Let $A_1 = \{\{c\}: c \in C\}$.

<u>Step 2:</u> We define A_i, $i \geq 2$, as follows:

For $m_1, \ldots, m_{i-1}, m_k \in \{1, \ldots, k\}$, define

$A(m_1, \ldots, m_{i-1}) = \{\{c_1, \ldots c_j, \ldots, c_i\}: F_i(c_j) = (m_1, \ldots, m_{i-1}, j) \text{ for } c_j \in C, 1 \leq j \leq i\}$.

Then, let $A_i = \cup\{A(m_1, \ldots, m_{i-1}):$ for any sequence $m_1, \ldots, m_{i-1}\}$.

Now, define $E = \{B: B \subseteq C$ and $B \subseteq \cup_{i=1}^{\infty} A_i\}$. Since $\{\chi_B: B \in E\} \subseteq c_0(\Gamma)$ for some Γ, the topology on E is the relative weak topology for $c_0(\Gamma)$.

We will show E is Eberlein - compact but not uniform Eberlein - compact.

First, we show E is Eberlein - compact. From the definition of E, we know that E is a collection of finite sets and condition (1) of Lemma 3.2.2 is satisfied. We will show for any $i \neq j$ and any $U \in A_i$, $V \in A_j$, $\text{card}(U \cap V) \leq 1$, there is no infinite increasing chain in E, from which condition (2) of Lemma 3.2.2 follows.

Suppose $i > j$ and $U \in A_i$, $V \in A_j$. From the definition of A_i, we know for any $m, n \in U$, they have the same jth coordinate, but for any $m, n \in V$, they may have different jth coordinates. So, $\mathrm{card}(U \cap V) \leq 1$. From Lemma 3.2.2, E is Eberlein - compact.

Secondly, we will show E is not uniform Eberlein - compact using Lemma 3.2.3. To see this, we construct a discrete set $M \subseteq E$ which has a unique limit point m_0 such that for any $m \in M$, there is relatively open subset U_m of E satisfying

(a) for any $m \in M$, $m \in U_m$;

(b) for any countable pairwise disjoint family $(M_n)_{n=1}^{\infty}$ in M such that for any

$m_1, \ldots, m_{n+1} \in M_n$, $m_1 \neq m_2 \neq \ldots \neq m_{n+1}$, $\cap\{U_{m(i)}: 1 \leq i \leq n+1\} = \Phi$, then

$M \neq \cup_{n=1}^{\infty} M_n$.

Let $M = A_1 = \{\{c\}: c \in C\}$. M is discrete in E and has a unique limit point, the empty set Φ. Let $\{U_m: m \in M\}$ be a family of open sets such that $m \in U_m$. Also, let $(M_n)_{n=1}^{\infty}$ be a family of pairwise disjoint sets in M such that for any

$m_1, \ldots, m_{n+1} \in M_n$, $m_1 \neq m_2 \neq \ldots \neq m_{n+1}$, $\cap\{U_{m(i)}: 1 \leq i \leq n+1\} = \Phi$.

Without loss of generality, assume that U_m is a basic open set for any $m \in M$, i.e. there is a finite set $F_m \subset C$ such that $B \in U_m \Leftrightarrow B \in E$ and there is a $B_m \in U_m$ such that $B \cap F_m = B_m \cap F_m$. Since F_m is finite and $m \in U_m$, we can suppose $m \in F_m$. To complete the proof, we will show that $M \neq \cup_{n=1}^{\infty} M_n$. To do this, we will first verify the following claim.

<u>Claim:</u> if $j \in N$, $j \geq 2$, and any finite sequence $m_1, \ldots, m_{j-1}$ is fixed, then there is i_0, $1 \leq i_0 \leq j$ such that card($M_{j-1} \cap F_j^{-1}(m_1, \ldots, m_{i-1}, i_0)) \leq \omega$.To prove this claim, suppose card($M_{j-1} \cap F_j^{-1}(m_1, \ldots, m_{i-1}, i)) > \omega$, for each i, $1 \leq i \leq j$. We choose an uncountable subset $E_i \subseteq A_{j-1} \cap F_j^{-1}(m_1, \ldots, m_{i-1}, i)$ for $1 \leq i \leq j$. For every $m \in E_i$, there is a finite set F_m as above and $\{F_m: m \in E_i\}$ is an uncountable family. By Lemma 3.2.1, there are a $\Psi_i \subseteq E_i$ with $|\Psi_i| > \omega$, a finite set $B(E_i) \subseteq C$ and a pairwise disjoint family $\{B_m: B_m \subseteq C$ is finite for $m \in \Psi_i\}$ such that for any $m \in \Psi_i$, $A_m = B(E_i) \cup B_m$. Since $B(E_1) \cup \ldots \cup B(E_j)$ is finite, we can assume the E_i's are chosen such that $m \notin B(E_1) \cup \ldots \cup B(E_j)$ for any $m \in E_1 \cup \ldots \cup E_j$.

Fix $a(1) \in \Psi_1$. Since $B_{a(1)}$ is finite, we can choose $a(2) \in \Psi_2$ such that $a(2) \notin B_{m(1)}$, and since $\{B_m: m \in \Psi_2\}$ is pairwise disjoint, $a(1) \notin B_{a(2)}$. Using the same method, we can get a finite sequence $a(1), \ldots a(j)$ with $a(i) \notin B_{a(k)}$ for

$1 \leq i, k \leq j$ and $i \neq j$. Since $a(i) \notin B(E_1) \cup \ldots \cup B(E_j)$ for $1 \leq i \leq j$, we have

$a(i) \notin A_{a(k)}$ for $1 \leq i, k \leq j$ and $i \neq k$. So, for i, $1 \leq i \leq j$,

$\{a(1), \ldots, a(j)\} \cap A_{a(i)} = \{a(i)\} \cap A_{a(i)}$. Since $a(i) \in U_{a(i)}$ and $U_{a(i)}$ is a basic open set for any $1 \leq i \leq j$, $\{a(1), \ldots, a(j)\} \in \cap\{U_{a(i)}: 1 \leq i \leq j\}$. This contradicts the definition of M_n. This verifies the claim.

From the claim, choose m_1 such that $\mathrm{card}(M_1 \cap F_1(m_1)) \leq \omega$. Repeating this, we can get sequence $m_1, \ldots, m_j, \ldots$ such that $\mathrm{card}(M_i \cap F_i(m_1, \ldots, m_i)) \leq \omega$, for $i \in N$. Since $(\cup_{j=1}^{\infty} M_j) \cap (\cap_{j=1}^{\infty}\{ F_j^{-1}(m_1, \ldots, m_j))$

$$\subseteq \cup_{j=1}^{\infty}(M_j \cap F_j^{-1}(m_1, \ldots, m_j)). \qquad (*)$$

Then, $\mathrm{card}(\cup_{j=1}^{\infty}M_j) \cap (\cap_{j=1}^{\infty}F_j^{-1}(m_1, \ldots, m_j)) \leq \omega$. Now,

$\mathrm{card}(\cap_{j=1}^{\infty} F_j^{-1}(m_1, \ldots, m_j)) > \omega$. By (*), $M \neq \cup_{n=1}^{\infty}M_n$, which completes the proof of Theorem 3.2.1.

3.3 Talagrand - compact Does Not Imply Eberlein - compact

The main aim of this section is to construct a Talagrand - compact space such that it is not Eberlein - compact. We begin with some definitions and theorems which are required for the proof of the main theorem.

Definition 3.3.1: Let C be any non - empty set. Suppose $\mathcal{B} = \{B: B \subseteq C\}$ is a family of C. $\mathcal{B}$ is said to be adequate iff

(1) $\mathcal{B}$ contains all singletons in C;

(2) for any $B \in \mathcal{B}$, if $A \subseteq B$, then $A \in \mathcal{B}$;

(3) for any subset A of C, if any finite set $F \subseteq A$ implies $F \in \mathcal{B}$, then $A \in \mathcal{B}$.

Theorem 3.3.1 For any set C and an adequate family $\mathcal{B}$ on C, let

$K = K(\mathcal{B}) = \{\chi_B: B \in \mathcal{B}\} \subseteq \{0, 1\}^C$. Then K is compact subset of $(\{0, 1\}^C, \tau)$, where τ is the product topology on $\{0, 1\}^C$.

Proof. By the Tychonoff Theorem, $\{0, 1\}^C$ is compact. Suppose $(\chi_{B(\alpha)})_{\alpha \in A}$ is any net in K which is convergent to χ_B. By Kelley [22], Theorem 3.4, we know , in the product topology, $\chi_{B(\alpha)} \to \chi_B$ iff $\chi_{B(\alpha)}(x) \to \chi_B(x)$ for all $x \in C$. Let F be a finite subset of B. Then, $\chi_{B(\alpha)}(f) \to \chi_B(f)$ for any $f \in F$. Hence, $\chi_{B(\alpha)}(f) \to 1$ for $f \in F$. That implies $f \in B(\alpha)$ eventually. So, there is $\alpha_0(f)$ such that for any $\alpha > \alpha_0(f)$, $f \in B(\alpha)$. Hence, $f \in \cap\{B(\alpha): \alpha > \alpha_0(f)\}$. Since F is finite, let $\alpha_0 = \max\{\alpha_0(f): f \in F\}$. Then, $F \subseteq \cap\{B(\alpha): \alpha > \alpha_0\} \in \mathcal{B}$. By Definition 3.3.1, Condition (3), $B \in \mathcal{B}$. So, K is closed and it follows that K is compact.

<u>Definition 3.3.2:</u> Let C be any set and $\mathcal{B}$ be any adequate family on it. We define $C^* = C \cup \{\infty\}$ and the topology, $\tau(\mathcal{B})$, on C^* as follows: for any $c \in C$, $\{c\} \in \tau(\mathcal{B})$. A subbase for the neighborhoods of ∞ is the family $\{\{\infty\} \cup \{C \sim B\}: B \in \mathcal{B}\}$.

M. Talagrand in [41] proved the following theorem:

<u>Theorem 3.3.2</u> Let C be any set and $\mathcal{B}$ be any adequate family on it. Then

(1) $(K(\mathcal{B}), \rho)$ is Talagrand - compact iff $(C^*, \tau(\mathcal{B}))$ is a κ - analytic space, where $\tau(\mathcal{B})$ and C^* are defined in Definition 3.3.2, ρ is the pointwise topology on $K(\mathcal{B})$, and $K(\mathcal{B})$ is defined in Theorem 3.3.1.

(2) $(K(\mathcal{B}), \tau)$ is Eberlein - compact iff $(C^*, \tau(\mathcal{B}))$ is σ - compact.

Proof. We will prove (1).

$\Rightarrow$ Suppose that $K = K(\mathcal{B}) = \{\chi_B: B \in \mathcal{B}\}$ is Talagrand - compact. Then $(C(K(\mathcal{B})), \rho)$ is κ - analytic, where $\mathcal{B}$ is an adequate family on set C and ρ is the pointwise topology. For $C^* = C \cup \{\infty\}$ and the topology $\tau(\mathcal{B})$ on C^* as defined in definition 3.3.2, define $h: C^* \to C(K(\mathcal{B}))$ by

$$h(c)(\chi_B) = \chi_B(c) \quad \text{for } c \in C;$$

$$h(c)(\chi_B) = 0 \quad \text{for } c = \infty.$$

<u>Claim 1:</u> h is continuous. To see this, suppose $c_\alpha \to c$. If $c \in C$, then $h(c_\alpha)(\chi_B) = \chi_B(c_\alpha) \to \chi_B(c)$ for any $B \in \mathcal{B}$. If $c = \infty$, for $B \in \mathcal{B}$, since $V = (C \sim B) \cup \{\infty\}$ is a neighborhood of ∞, there is α_0 such that for any $\alpha > \alpha_0$ $c_\alpha \in V$. That means that $c_\alpha \notin B$ for $\alpha > \alpha_0$. Hence, $h(c_\alpha)(\chi_B) = 0$ for $\alpha > \alpha_0$. So, $h(c_\alpha)(\chi_B) = \chi_B(c_\alpha) \to h(\infty)(\chi_B) = \chi_B(\infty) = 0$. This verifies that h is continuous.

<u>Claim 2:</u> h is one - to - one. To see this, suppose $h(c) = h(c')$ where $c \neq c'$. That means that $h(c)(\chi_B) = h(c')(\chi_B)$ for any $B \in \mathcal{B}$. Without loss of generality, suppose $c \neq \infty$. Then $h(c)(\chi_{\{c\}}) = \chi_{\{c\}}(c) = 1 \neq \chi_{\{c\}}(c') = h(c')(\chi_{\{c\}})$, a contradiction. Hence, h is one - to - one.

<u>Claim 3:</u> h^{-1} is continuous. To see this, suppose $h(c_\alpha)(\chi_B) \to h(c)(\chi_B)$ for any $B \in \mathcal{B}$. If $c \neq \infty$, let $B = \{c\}$. Since $h(c_\alpha)(\chi_{\{c\}}) = \chi_{\{c\}}(c_\alpha) \to h(c)(\chi_{\{c\}}) = \chi_{\{c\}}(c) = 1$,

there is α_0 such that for any $\alpha > \alpha_0$, $c_\alpha = c$. Hence, $c_\alpha \to c$. If $c = \infty$, since $h(c_\alpha)(\chi_B) \to 0$ for any $B \in \mathcal{B}$, there is α_0 such that $\alpha > \alpha_0$ implies $|h(c_\alpha)(\chi_B)| < 1\backslash 2$. So, $c_\alpha \notin B$. Therefore, $c_\alpha \in U = (C \sim B) \cup \{\infty\}$. Hence, $c_\alpha \to \infty$. This veries that h^{-1} is continuous.

Claim 4: $h(C^*)$ is closed in $(C(K(\mathcal{B})), \rho)$. To see this, let $f \in \rho$ - $cl(h(C^*))$, if there is a $c \in C$ such that $f(\chi_{\{c\}}) = 1$, then $f = h(c)$, since $f = \lim_\alpha h(c_\alpha)$ for some $(c_\alpha)_{\alpha \in A} \subseteq C$, $1 = f(\chi_{\{c\}}) = \lim_\alpha h(c_\alpha)(\chi_{\{c\}})$, which means that there is α_0 such that whenever $\alpha > \alpha_0$, $c_\alpha = c$. So, $f \in h(C^*)$; if, for every $c \in C$, $f(\chi_{\{c\}}) = 0$, then for every finite subset F of C, $f(\chi_F) = 0$. Since $f = \lim_\alpha h(c_\alpha)$ for some $(c_\alpha)_{\alpha \in A} \subseteq C$, $f(\chi_F) = \lim_\alpha h(c_\alpha)(\chi_F) = \lim_\alpha \chi_F(c_\alpha) = 0$. Since, for any $B \in \mathcal{B}$, $\chi_B = \lim_{F \subseteq B} \chi_F$ where F is finite and F is continuous, $f(\chi_B) = \lim_{F \subseteq B} f(\chi_F) = 0$. That means $f = 0$. Hence, $f \in h(C^*)$. Therefore, $h(C^*)$ is closed.

From Claim 1, 2, 3, 4, , we know that f is a homeomorphism from $(C^*, \tau(\mathcal{B}))$ to $(h(C^*), \rho)$. Since $(C(K(\mathcal{B})), \rho)$ is κ - analytic and $h(C^*)$ is closed, $(C^*, \tau(\mathcal{B}))$ is κ - analytic.

$\Leftarrow$ Suppose that $(C^*, \tau(\mathcal{B}))$ is κ - analytic. h is defined as above. Let $\mathcal{A}(h(C^*))$ be the smallest algebra containing $h(C^*)$ and 1. Define:

$A_1 = h(C^*) \cup 1$, $A_2 = \{a + a'\colon\ a, a' \in A_1\}$; $A_3 = \{a \times a'\colon\ x \in R, a \in A_2\}$.

Continuing this process, we can get a sequence of compact subsets, $(A_i)_{i=1}^{\infty}$, of $(C(K(\mathcal{B})), \rho)$. Now, $h(C^*) \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$, $\mathcal{A}(h(C^*)) = \cup_{i=1}^{\infty} A_i$, by Stone - Weierstrass Theorem which says that if K is compact and $A \subseteq C(K)$ is an algebra containing constant functions such that for any $k, k' \in K$, there is $f \in A$ such that $f(k) \neq f(k')$, then A is dense in C(K) in supremum norm, i.e. norm - $cl(\mathcal{A}(h(C^*))) = C(K(\mathcal{B}))$.

<u>Claim:</u> If norm - cl(Y) = C(X) and Y is κ - analytic, then $(C(X), \rho)$ is κ - analytic.

For any $f \in C(X)$, there is $g \in Y$ such that $\| f - g \| < 1$. For $f - g \in C(X)$, there is $f_1 \in B(Y)$ such that $\| f - g - f_1 \| < 1\backslash 2$. Generally, there is $f_{n+1} \in (1\backslash 2^{n+1})B(Y)$ such that $\| f - g - f_1 - \ldots - f_{n+1} \| < 1\backslash 2^n$. Hence, $f = g - f_1 - f_2 - \ldots - f_n - \ldots$. Define

$\delta\colon Y \times B(Y) \times (1\backslash 2)B(Y) \times (1\backslash 4)B(Y) \times \ldots \to C(X)$ by $\delta(g, f_1, \ldots) = g + f_1 + \ldots$.

Hence, δ is continuous and onto. Hence C(X) is κ - analytic.

By Claim above, we know that $C(K(\mathcal{B}))$ is κ - analytic.

Definition 3.3.3: Let (X, τ) be a Hausdorff topological space. X is said to be K - analytic iff it is the image of a Polish space under a compact valued upper semi - continuous map (u.s.c.), i.e. there is a Polish space Y, a compact valued function f which is u.s.c. such that $X = \cup_{\sigma\in Y} f(\sigma)$.

Definition 3.3.4: Let (X, τ) be a Hausdorff topological space. X is said to be countably determined iff it is the image of a separable metrizable space under a compact valued upper semi - continuous map.

Definition 3.3.5: Let (X, τ) be a Hausdorff topological space. For any set $C \subseteq X$, C is said to be an analytic set iff there is a continuous map $f\colon \Sigma = N^N \to X$ with $f(\Sigma) = C$.

Rogers and Jayne ([32], Corollary 2.4.3) proved:

Theorem 3.3.3: Every Polish space is a continuous image of Σ with the product topology on it, i.e. every Polish space is an analytic set.

We will show in the following that κ - countably determined (see Definition 2.2.7) is equivalent to countably determined and κ - analytic (see Definition 2.2.5) is equivalent to K - analytic.

<u>Theorem 3.3.4</u> (X, τ) is K - analytic iff X is the image of Σ under a compact valued upper semi - continuous map.

Proof. $\Rightarrow$ Suppose that (X, τ) is K - analytic. From Defintion 3.3.3 and Theorem 3.3.3, it is obvious that X is the image of Σ under a compact valued upper semi - continuous map.

$\Leftarrow$ Since it is known that Σ is a Polish space, by Definition 3 3.3, (X, τ) is K - analytic.

<u>Theorem 3.3.5</u> (X, τ) is countably determined iff X is the image of a subset Σ' of Σ under a compact valued upper semi - continuous map.

Proof. Using Definition 3.3.4 and Theorem 3.3.3, the proof is similar to Theorem 3.3.4.

Now, we are ready to show that the definitions are equivalent.

<u>Theorem 3.3.6</u> Let (X, τ) be a Hausdorff topological space. X is κ - countably determined iff X is countably determined.

Proof. $\Rightarrow$ Suppose that (X, τ) is κ - countably determined. That means that there is a compact set (Y, τ_1) such that $X \subseteq Y$, τ is the relative topology of τ_1 on

X, and there is a sequence of τ_1 - closed subsets, $(K_n)_{n=1}^{\infty}$, of Y, such that for every $x \in X$, there is $\sigma \in \Sigma$ such that $x \in \cap_{i=1}^{\infty} K_{\sigma(i)}$, where $\sigma = (\sigma(1), \ldots, \sigma(i), \ldots)$. For any $\sigma \in \Sigma$, define $f(\sigma) = \cap_{i=1}^{\infty} K_{\sigma(i)}$. Let $\Sigma' = \{\sigma \in N^N: f(\sigma) \cap X \neq \Phi\}$, and $g: \Sigma' \to X$ such that $g(\sigma) = f(\sigma) \cap X$. So, $g(\Sigma') = X$, g is compact valued in X and u. s. c. To see that g is u.s.c., recall that $\{V_s : s \in S\}$ is a base for the product topology on Σ, where $S = \{(s(1), \ldots s(i), \ldots, s(n)): n, s(i) \in N, i \leq i \leq n\}$ and $V_s = \{\sigma \in \Sigma: \sigma > s\}$. For $\sigma \in \Sigma'$ and any open set U, where $g(\sigma) \subseteq U$, by a compactness argument, there is $m \in N$ such that $\cap_{i=1}^{m} K_{\sigma(i)} \cap X \subseteq U$. So, for $s = (\sigma(1), \ldots, \sigma(m))$, $\sigma \in V_s$ and $g(V_s) \subseteq U$, since any $\eta \in V_s$ implies $\eta > s$, $g(\eta) = \cap_{i=1}^{\infty} K_{\eta(i)} \cap X \subseteq \cap_{i=1}^{m} K_{\sigma(i)} \cap X \subseteq U$. Hence, g is u.s.c. and by Theorem 3.3.5, (X, τ) is countably determined.

$\Leftarrow$ Suppose that (X, τ) is countably determined. Let (Y, τ_1) be a compact set which contains (X, τ) and τ is the relative topology from τ_1. For any $s \in S$, let $K_s = cl_Y(\cup_{\sigma > s} f(\sigma))$, where $\sigma \in \Sigma' \subseteq \Sigma$. So, K_s is compact. $\{K_s: s \in S\}$ is a countable family of compact sets. Since for any $x \in X$, there is $\sigma \in \Sigma'$ such that $x \in f(\sigma)$. We claim that $f(\sigma) = \cap_{i=1}^{\infty} K_{\{\sigma(1), \ldots, \sigma(i)\}}$. From the definition of K_s it is obvious that $f(\sigma) \subseteq \cap_{i=1}^{\infty} K_{\{\sigma(1), \ldots, \sigma(i)\}}$. To see $\cap_{i=1}^{\infty} K_{\{\sigma(1), \ldots, \sigma(i)\}} \subseteq f(\sigma)$, suppose that $x \notin f(\sigma)$. Let $U = Y \sim \{x\}$. Then U is open. There is V, open, such that $f(\sigma) \subseteq V \subseteq cl_Y(V) \subseteq U$. Since f is u.s.c., there is $i \in N$, such that $f(V_{\{\sigma(1), \ldots, \sigma(i)\}}) \subseteq V$. Hence $cl_Y(\cup_{\sigma > \{\sigma(1), \ldots, \sigma(i)\}} f(\sigma)) \subseteq cl_Y(V) \subseteq U$. Therefore, $K_{\{\sigma(1), \ldots, \sigma(i)\}} \subseteq cl_Y(V) \subseteq U$. Since $U = Y \sim \{x\}$, $x \notin K_{\{\sigma(1), \ldots, \sigma(i)\}}$. Thus, $\cap_{i=1}^{\infty} K_{\{\sigma(1), \ldots, \sigma(i)\}} \subseteq f(\sigma)$, and X is κ - countably determined.

Choquet [10] proved:

<u>Theorem 3.3.7</u> In a metric space, the κ - analytic sets coincide with analytic sets.

<u>Theorem 3.3.8</u> Let (X, τ) be a Hausdorff topological space. X is κ - analytic iff X is K - analytic.

Proof. $\Rightarrow$ Suppose that (X, τ) is κ - analytic. Without losing of generality, we assume that $X = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} K_{n,m}$, where $K_{n,m}$ is compact. Set

$K'_{n, m} = K_{n, m} \cup \{x_0\}$, for some fixed $x_0 \in X$. For any $\sigma = (\sigma(1), \dots \sigma(n), \dots) \in \Sigma$, let $f(\sigma) = \cap_{n=1}^{\infty} K'_{n, \sigma(n)}$. Hence, for any $\sigma \in \Sigma$, $f(\sigma)$ is non - empty and compact. Using the same argument as in Theorem 3.3.6, we can get f is u.s.c.. Furthermore, $X = \cup_{\sigma \in \Sigma} f(\sigma)$. To see this, for any $x \in X$, and for any $n \in N$, there is $m(n) \in N$ such that $x \in K'_{n, m(n)}$, since $X = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} K'_{n, m}$. Let $s = \{m(1), \dots m(n), \dots\}$. Then, $x \in \cap_{n=1}^{\infty} K'_{n, m(n)} = f(\sigma)$. Therefore, X is K - analytic.

$\Leftarrow$ Suppose that (X, τ) is K - analytic. Define graph(f) = $\{(\sigma, x) : x \in f(\sigma)\}$. Talagrand [41] proved that graph(f) $\subseteq N^N \times K$ for some compact $K \supset X$ and graph(f) is closed. Since $\Sigma = N^N$ is a metric space, by Theorem 3.3.7, Σ is a $K_{\sigma\delta}$ set. Further, it is easy to see that $\Sigma \times K$ is also a $K_{\sigma\delta}$ set. Since graph(f) is a closed set of a $K_{\sigma\delta}$ set, then graph(f) is a $K_{\sigma\delta}$ set. Let p_1 be the projection from $\Sigma \times K$ to the first coordinate. Since p_1 is continuous, then $X = p_1(\text{graph}(f))$ is κ - analytic.

<u>Theorem 3.3.9</u> (Talagrand [41]) There is a compact Hausdorff space (K, τ) which is Talagrand - compact, but not Eberlein - compact.

Proof. We are going to construct a topological space $(K, \tau) \subseteq \{0, 1\}^{\Sigma}$ such that K is Talagrand - compact, but not Eberlein - compact by using Theorem 3.3.2.

Let $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$., where

$$\mathcal{B}_n = \{B \subseteq \Sigma: \text{ if } \sigma \neq \rho \in B, \text{ then } \sigma|n = \rho|n, \sigma{+}1|n \neq \rho{+}1|n\}.$$

We will show that for any $n \in N$, $\mathcal{B}_n$ is an adequate family on Σ.

First, it is obvious that singletons are in $\mathcal{B}_n$.

Secondly, for any $B \in \mathcal{B}_n$, suppose $A \subseteq B$ and any $\sigma \neq \rho \in A$. Then, $\sigma \neq \rho \in B$. By the definition of $\mathcal{B}_n$,, $\sigma|n = \rho|n$, $\sigma{+}1|n \neq \rho{+}1|n$. Hence $A \in \mathcal{B}_n$.

Thirdly, suppose that D is any subset of Σ such that for any finite subset F of A, $F \in \mathcal{B}_n$. Then, for any $\sigma \neq \rho \in D$, $\{\sigma, \rho\} \in \mathcal{B}_n$. Hence, $\sigma|n = \rho|n$, $\sigma{+}1|n \neq \rho{+}1|n$. Therefore, $D \in \mathcal{B}_n$.

From Definition 3.3.1, $\mathcal{B}_n$ is an adequate family.

Furthermore, we are going to show that $\mathcal{B}$ is an adequate family on Σ.

First, it is obvious that singletons are in $\mathcal{B}$.

Secondly, suppose that $A \in \mathcal{B}$. For any $D \subseteq A$, we would like to show that $D \in \mathcal{B}$. To see this, since $A \in \mathcal{B}_n$ for some $n \in N$ and $\mathcal{B}_n$ is an adequate family, then $D \in \mathcal{B}_n$. Hence, $D \in \mathcal{B}$.

Thirdly, for A $\subseteq \Sigma$, suppose that any finite subset of A belongs to Σ. Let F, H be finite subsets of A and F $\neq$ H. From definitions of $\mathcal{B}$ and $\mathcal{B}_n$ for n $\in$ N, it is easy to see that there is m $\in$ N such that F, H $\in \mathcal{B}_m$. Hence, if G is any finite subset A, then G $\in \mathcal{B}_m$. Since$\mathcal{B}_m$ is an adequate family, then A $\in \mathcal{B}_m$. So, A $\in \mathcal{B}$.

From Definition 3.3.1,$\mathcal{B}$ is an adequate family.

Let $\Sigma^* = \Sigma \cup \{\infty\}$, $\tau(\mathcal{B})$ as defined in Definition 3.3.2.

Claim: (Σ^*, $\tau(\mathcal{B})$) is K - analytic. To see this, define f: $\Sigma \to (\Sigma, \tau(\mathcal{B}))$ by $f(\sigma) = \{\sigma,\infty\}$. Hence, f($\sigma$) is compact in (Σ^*, $\tau(\mathcal{B})$), and $\Sigma^* = \cup_{\sigma\in\Sigma} f(\sigma)$. We are going to show that f is u.s.c.. Let U is any open set such that $f(\sigma) \subseteq U$. Hence, there is B $\in \mathcal{B}$ such that $U = (\Sigma \backslash B) \cup \{\infty\} \cup \{\sigma\}$. If we can find an open set V $\subseteq \Sigma$ such that $f(V) = V \cup \{\infty\} \subseteq U = (\Sigma \backslash B) \cup \{\infty\} \cup \{\sigma\}$, then f is u.s.c.. To see this, since B $\in \mathcal{B}$, by the definitions of $\mathcal{B}$ and $\mathcal{B}_n$'s , there is a m $\in$ N such that B $\in \mathcal{B}_m$., which means that there is a finite sequence $s = (s_1, \ldots s_m) \in S$ such that for any $\sigma, \rho \in$ B, we have $\sigma|m = s = \rho|m$, and $\sigma+1|m \neq \rho+1|m$. Choose t, t $\neq$ s and

$|t| = m$, where $|t|$ is the length of the finite sequence t, and $V = \{ \sigma\in\Sigma: t < \sigma \}$. So,

$V \cap B = \Phi$. Hence, $f(V) = V \cup \{\infty\} \subseteq U = (\Sigma \backslash B) \cup \{\infty\}$. Therefore, $(\Sigma^*, \tau(\mathcal{B}))$ is K - analytic. By Theorem 3.3.8, $(\Sigma^*, \tau(\mathcal{B}))$ is κ - analytic.

Set $K = K(\mathcal{B}) = \{\chi_B\colon B \in \mathcal{B}\} \subseteq \{0, 1\}^{\Sigma}$. By Theorem 3.3.2, K with the product topology is Talagrand - compact. We are going to show K is not Eberlein - compact. Using Theorem 3.3.2, it is enough to show that $(\Sigma^*, \tau(\mathcal{B}))$ is not σ - compact.

To show $(\Sigma^*, \tau(\mathcal{B}))$ is not σ - compact, suppose

$\Sigma = \cup\{\Sigma_n : n \in N,\ \Sigma_n \cup \{\infty\}$ is compact in $(\Sigma^*, \tau(\mathcal{B}))\}$ with usual topology, τ_1.

Let $\Sigma = \{\sigma = (n_1, \ldots, n_m, \ldots)\colon n_m \in N,\ m \in N\}$,

$N^m = \{a = (n_1, \ldots, n_m)\colon n_i \in N, 1 \le i \le m\}$.

Define: $F_m\colon \Sigma \to N^m$ by $F_m(n_1, \ldots, n_m, \ldots) = (n_1, \ldots, n_m)$ for any $m \in N$. By the Baire category therem, there is $n(0) \in N$ such that $\tau_1 - \mathrm{Int}(\tau_1 - \mathrm{cl}(\Sigma_{n(0)})) \neq \Phi$. We can choose $(n_1, \ldots, n_m)$ such that

$U = F_m^{-1}(n_1, \ldots, n_m) \subseteq \tau_1 - \mathrm{Int}(\tau_1 - \mathrm{cl}(\Sigma_{n(0)})) \neq \Phi$. For any $k \in N$, let

$U_k = F_{m+1}{}^{-1}(n_1, \ldots, n_m, k)$. So, $U_k \subseteq U$ for any $k \in N$. Since $\Sigma_{n(0)} \cap U_k \neq \Phi$, for any $k \in N$, choose $a_k \in \Sigma_{n(0)} \cap U_k$. So, $\{a_k\colon k \in N\} \in \mathcal{B}_m \subseteq \mathcal{B}$, by the definitions of $\{a_k: k \in N\}$. Since $\{a_k: k \in N\} \subseteq \Sigma_{n(0)} \cup \{\infty\}$ is compact, then there is at least a limit point x in Σ. From the definition of $\{a_k: k \in N\}$, we know that $x \notin \Sigma$.

Claim: $\{\infty\}$ is not a limit point of $\{a_k: k \in N\}$. To see this, let $V = \{\infty\} \cup (\Sigma \sim \{a_k: k \in N\})$. Hence $V \cap \{a_k: k \in N\} = \Phi$. So, $x \neq \{\infty\}$. This is a contradiction. Hence, Σ^* is not σ - compact. Therefore, K is not Eberlein - compact.

3.4 Gul'ko - compact Does Not imply Talagrand - compact

We will use a counterexample developed by Talagrand in [42] to show that there is a Gul'ko - compact space that is not Talagrand - compact.

Theorem 3.4.1 There is a countably determined space which is not K - analytic.

Proof: Let $P = \{a: a = (a(1), \ldots, a(n)) \in S \text{ and } a(i) < a(i+1)\}$. Define a partial ordering, $\leq$, on P by $a = (a(1), \ldots, a(n)), b = (b(1), \ldots, b(m)) \in P$, $a \leq b$ iff $n \leq m$ and $a(i) = b(i)$ for $i \leq n$. So, $(P, \leq)$ is a partially ordered set.

$M \subseteq P$ is a tree iff for any $a \in M$, if $b \leq a$, then $b \in M$.

Let T_0 = {M: M is a tree in P}. By identifying a tree M with its characteristic function, we can consider that $T_0 \subseteq \{0, 1\}^P$.

Claim: T_0 is closed in $\{0, 1\}^P$ with the product topology. To see this, let $\chi_{M(\alpha)} \to \chi_M$, where $M(\alpha)$ is a tree in P. Suppose $a \in M$ and $b \leq a$. Then, $\chi_{M(\alpha)}(a) \to 1$. So, there is a α_0 such that for any $\alpha > \alpha_0$, $\chi_{M(\alpha)}(a) = 1$, this means $a \in M(\alpha)$ for $\alpha > \alpha_0$. Since $M(\alpha)$ is a tree, then $b \in M(\alpha)$ for $\alpha > \alpha_0$. Hence, $\chi_{M(\alpha)}(b) = 1$. This implies $b \in M$. So, M is a tree. Since P is countable, $\{0, 1\}^P$ is compact and metrizable. So, T_0 is a compact metrizable space with topology denoted by τ.

$M \in T_0$ has an infinite branch if M contains an increasing sequence $(a^n)_{n=1}^{\infty}$ with the length of a^n going to infinity.

Let T_1 = {M: $M \in T_0$ and M has an infinite branch}.

For any $n \in N$, set $P_n = \{a \in P: 1 \leq a_i \leq n\}$. Suppose $M \in T_0$. The basic neighbourhoods of M are $U_n(M) = \{Y: Y \in T_0 \text{ and } Y \cap S_n = M \cap S_n\}$ for $n \in N$.

Let $\mathcal{A}_0 = \{\mathcal{C} : \mathcal{C} \subseteq T_0, \mathcal{C}$ is finite such that if $\mathcal{C} = \{M_1, \ldots, M_n\}$, then there is a

$Y \in T_0$ and $a = (a(1), \ldots, a(n)) \in Y$ such that $M_i \in U_{a(i)}(Y)\}$.

Let $A_1 = \cap\{B : A_0 \subseteq B$ and B is an adequate family of $T_0\}$.

Claim: Suppose that $C \in A_1$ and M is any limit point of C. Then, $M \in T_1$. (*)

To see this, suppose $(M_n)_{n=1}^{\infty} \subseteq C$ such that $M_n \to M$ with $M_n \neq M$ for $n \in N$.

Since for each n, there is $f(n) \in N$ such that $M_n \in U_{f(n)}(M) \sim U_{f(n)+1}(M)$.

Without loss of generality, we can assume $f(n) \geq n$ for each $n \in N$, and

$(f(n))_{n=1}^{\infty}$ is increasing.

Fix n, then there is a $C \in A_0$ such that $M_1, \ldots, M_n \in C$. So, by the definition

of A_0, $C = \{M'_1, \ldots, M'_k\}$ and there is a $Y \in T_0$, $a = (a^n(1), \ldots, a^n(k)) \in Y$ such

that $M'_i \in U_{a^n(i)}(Y)$. Clearly $k \geq n$. Now, $M_i = M'_{g(i)}$, for $1 \leq i \leq n$, where $g(i) \leq k$.

For

$i \neq j$, $g(i) \neq g(j)$. Since $f(i) \geq i$ and $a^n(i) \geq i$, there is $1 \leq k \leq n$ such that both f(k) and

$a^n(g(k)) \geq n(0)\backslash 2$. Let $m = \min\{f(k), a^n(g(k))\}$. Hence, $M_k \in U_m(M)$ implies

$M_k \in U_m(Y)$. From the definitions of $U_m(M)$ and $U_m(Y)$, we get $U_m(M) = U_m(Y)$.

Let $q(n) = \max\{i: f(i) < n\backslash\backslash 2\}$ and $i \leq q(n)$. Then,

$M'_{g(i)} = M_i \notin U_{f(i)+1}(M) = U_{f(i)+1}(Y)$. Since $f(i)+1 \leq m$, then $i \leq a^n(g(i)) \leq f(i)$. So,

there are at least i elements in $(a^n(1), \ldots, a^n(k))$ which are less than or equal to

f(i). Moreover, $(a^n(1), \ldots, a^n(q(n))) \in Y$, since Y is a tree and

$(a^n(1), \ldots, a^n(q(n))) \leq (a^n(1), \ldots, a^n(k)) \in Y$. Now $i \leq q(n)$, $a^n(i) \leq f(i) \leq (n\backslash 2) - 1$.

So, $(a^n(1), \ldots, a^n(q(n))) \in M$. Hence, for any $n \in N$, there are $q(n)$ and

$(a^n{}_1, \ldots, a^n{}_{q(n)})$ such that $(a^n{}_1, \ldots, a^n{}_{q(n)}) \in M$.

Fix i. Since $a^n(i) \leq f(i)$ for any $n \in N$, then, there is a $\{n_k\}_{k=1}^{\infty}$ such that $a^{n_{k(i)}}$ eventually equals to a(i). In fact, for n sufficiently large, we get

$(a^n(1), \ldots, a^n(q(n))) = (a(1), \ldots, a(n)) \in M$ and $|(a(1), \ldots, a(n))| \to \infty$ when $n \to \infty$.

Hence, $M \in T_1$. This verifies the claim.

Let $T = T_0 \sim T_1$, $A = \{B \subseteq T: B \in A_1\}$.

Claim: A is an adquate family on T, and any $B \in A$ implies that B is closed. To see this,

(1) suppose $B \in A$. Then, B is closed, since B has no limit point;

(2) if $B \in A$, then $B \in T \subseteq T_0$, $\{B\} \in A_1$, since A_1 is an adequate family on T_0.

So, $\{B\} \in A$;

(3) suppose $C \in A$ and $B \subseteq C$. Then, $B \in A_1$, since A_1 is an adequate family

on T_0. By the definition of A, $B \in A$.

(4) suppose that $B \subseteq T$ such that for any finite $F \subseteq B$, $F \in A$. So, $F \in A_1$. Then,

$B \in \mathcal{A}_1$, since $\mathcal{A}_1$ is an adequate family on T_0. Therefore, $B \in \mathcal{A}$.

Hence, we have verified the claim.

Let $T^* = T \cup \{\infty\}$. Define a topology, $\tau(\mathcal{A})$, on T^* as following: for any $M \in T$, $\{M\}$ is open. The basic neighbourhoods of $\{\infty\}$ are $T^* \sim D$, where D is the finite union of elements of $\mathcal{A}$.

Define h: $(T, \tau) \to (T^*, \tau(\mathcal{A}))$ by $h(t) = \{t, \infty\}$ for any $t \in T$. Hence, h is compact valued and u.s.c.. To see h is u.s.c., let $D = \cup_{i=1}^{\infty} B_i$, where $B_i \in \mathcal{A}$, such that $\{\infty, t\} \subseteq T^* \sim D$. That implies $t \notin D$. Since D is closed in T, there is an open set U with $t \in U$ such that $h(U) \subseteq T^* \sim D$. Hence, f is u.s.c.. Since (T, τ) can be embedded in a Polish space, then (T, τ) is countably determined.

We are going to show that $(T^*, \tau(\mathcal{A}))$ is not K - analytic. To prove this, we need some definitions.

For $M \in T_0$, we define M^1 by

$M^1 = \{a \in M$: there is a $b \in P$ such that $a \leq b$ and $b \in M\}$.

Using transfinite induction, for any $\alpha < \omega_1$, we can define M^α by:

1) if $\alpha = \beta + 1$, then $M^{\alpha+1} = (M^\alpha)^1$;

2) if α is a limit ordinal and $\alpha = \sup_n \alpha(n)$, then $M^\alpha = \cap_{n=1}^{\infty}\{M^{\alpha(n)}\}$.

If for some $\alpha < \omega_1$ $M^\alpha = \Phi$, we denote o(M) by $o(M) = \min\{\alpha: M^\alpha = \Phi\}$. If not, $o(M) = \omega_1$.

For any $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_n) \in P$, if $a_m < b_1$, we define a+ b by $a + b = (a_1, \ldots, a_m, b_1, \ldots, b_n)$. So, $a + b \in P$.

Let $a \in P$ and $M \in T_0$, define a|M by $a|M = \{b: a + b \in M\}$.

Now, we are ready to prove that $(T^*, \tau(A))$ is not K - analytic.

Suppose that $(T^*, \tau(A))$ is K - analytic. By Theorem 3.3.4, there is a compact valued u.s.c. mapping p from Σ onto T^*. For any $s \in N^n$, let $E_s = \cup_{s<\sigma} p(\sigma)$, where $\sigma \in \Sigma$.

Claim: There are sequences, $a^n = (a(1), a(2), \ldots, a(n)) \in P$, $b^n = (b(1), b(2), \ldots, b(n)) \in N^n$ and a sequence of trees, $M_1, \ldots, M_n$, for $n \in N$, such that

(1) $U_{a(i)}(M_i) = U_{a(i)}(M_n)$ for any $1 \le i \le n$;

(2) $(a(1), a(2), \ldots, a(n - 1)) \in M_n$;

(3) $\{o(s_n|M): M \in U_{a(n)}(M_n) \cap E_{b^n}\}$ is unbounded. To see this,

step 1: Since $T^* = \cup_{n=1}^{\infty} E_n$ and for any $\alpha < \omega_1$ there is a $M \in T_0$ such that

$o(M) = \alpha$, then, there is a $b(1)$ such that o is not bounded on $E_{b(1)}$. From the definition of $n|M$, we know $o(M) \leq \sup_n o(n|M)+1$. So, there is $a(1)$ such that $\{o(a(1)|M): M \in E_{b(1)}\}$ is unbounded. Since there are only finitely many sets of the type $U_{a(1)}(M)$, choose M_1 such that $\{o(a(1)|M: M \in U_{a(1)}(M_1) \cap E_{b(1)}\}$ is unbounded.

<u>step 2:</u> Assume $a^n = (a(1), \ldots, a(n)) \in P$, $b^n = (b(1), \ldots b(n)) \in N^n$, and trees M_1, $M_2, \ldots, M_n$ are chosen such that they satisfy conditions of the claim.

<u>step 3:</u> Now, $E_{b^n} = \cup_m E_{b^n+m}$. There exists $b(n+1)$ such that $\{o(a^n|M: M \in U_{a(n)}(M_n) \cap E_{b^n+1})$ is unbounded. Since $o(a^n|M) \leq \sup_m o(a^n+m|M) +1$, for any $m > a(n)$, then we can choose $a(n+1) = m$ such that $\{o(a^{n+1}|M): M \in U_{a(n)}(M_n) \cap E_{b^n+1}\}$ is unbounded. Since there are finitely many sets of type $U_{a(n+1)}(M)$, choose M_{n+1} such that $\{o(a^{n+1}|M: M \in U_{a(n+1)}(M_{n+1}) \cap E_{b^n+1}\}$ is unbounded. Notice that $a^n \in M_n$, otherewise $o(a^n|M) = \Phi$, for $M \in U_{a(n)}(M_n)$. Moreover, there is $M \in T_0$ such that $U_{a(n)}(M) = U_{a(n)}(M_n)$. So, the claim is proved.

For each n, let $X_n \in E_{b^n} \cap U_{a(n)}(M) \cap T$. For each k, $X_1, \ldots, X_k \in \mathcal{A}_0$, so,

$B = (X_n)_{n=1}^{\infty} \in \mathcal{A}_1$ and $B \subseteq T$. Then B is closed and discrete. Let

$b = (b(n))_{n=1}^{\infty}$. Since p(b) is compact, then $p(b) \cap B$ is a finite set. Let

$C = B \sim p(b)$. Then C is closed and $C \cap p(b) = \Phi$. Hence, there is $s \in N^n$, $s < b$,

such that $\rho(\sigma) \cap C = \Phi$ for any $\sigma > s$, $\sigma \in \Sigma$. Hence, $E_s \cap C = \Phi$. Now,

$s = (b(1), ..., b(n))$. Then $X_k \in C \sim E_s$ for any $k \geq n$, except possibly finitely

many. This is a contradiction. So, $(T^*, \tau(\mathcal{A}))$ is not K - analytic.

<u>Theorem 3.4.2</u> There is a Gul'ko - compact topological space which is not a Talagrand - compact topological space.

Proof. It follows from Theorem 3.3.6, Theorem 3.3.8, and Theorem 3.4.1 directly.

3.5 Corson - compact Does Not Imply Gul'ko - compact

Alster and Pol [1] first constructed a counterexample which says there is a Corson - compact topological space which is not Talagrand. This space turns out to be a Gul'ko - compact (see Argyros, Mercourakis and Negrepontis [3]).

<u>Theorem 3.5.1</u> (Pol [31]) If a topological space (X, τ) is κ - analytic then there exists subsets of X, $A_{i(1), \ldots, i(k)}$, where $(i(1), \ldots, i(k)) \in S$, such that

(1) $X = \cup_{i=1}^{\infty} A_i$, $A_{i(1), \ldots, i(k)} = \cup_{i=1}^{\infty} A_{i(1), \ldots, i(k), i}$.

(2) if $(i(k))_{k=1}^{\infty} \in \Sigma$ and $a_k \in A_{i(1), \ldots, i(k)}$, then the sequence $(a_k)_{k=1}^{\infty}$ has a limit point in X.

<u>Theorem 3.5.2</u> (Alster and Pol [1]) There is a Corson - compact topological space which is not Gul'ko - compact.

Proof. Let $C = [0, 1]$, $<$ be the usual ordering on C, $\lessdot$ be a well - ordering of C, and $\mathcal{B} = \{B \subseteq C:\ <$ and $\lessdot$ coincide on B$\}$. It is easy to check that $\mathcal{B}$ is an adequate family. Define $K = K(\mathcal{B}) = \{\chi_B\colon B \in \mathcal{B}\}$. Then, K is a compact subset of $\{0, 1\}^C$. Now, since $B \in \mathcal{B}$, $|B| \leq \omega$. Hence, for any $x \in K$, supp{x} is countable, i.e. $x \in \Sigma(\{0, 1\}^C)$. Therefore, K is Corson - compact. Let $C^* = C \cup \{\infty\}$ and the topology $\tau(\mathcal{B})$ on C^* as defined in definition 3.3.2.

<u>Claim:</u> K is not Talagrand - compact. By Theorem 3.3.2, it is enough to show that C^* is not κ - analytic. To see this, suppose that C^* is κ - analytic. By Theorem 3.5.1, there are $\{A_{i(1), \ldots, i(k)}\}$, where $i(1), \ldots, i(k)$ is finite sequence of natural numbers such that $C^* = \cup_{i=1}^{\infty} A_i$, and

$A_{i(1), \ldots, i(k)} = \cup_{i=1}^{\infty} A_{i(1), \ldots, i(k), i}$.

Below we will choose $(i(1), \ldots, i(k)) \in N^k$ and distinct points $a_k \in A_{i(1), \ldots, i(k)}$ such that $\{a_1, \ldots, a_k, \ldots\}$ is a discrete set in C^*, this will contradict Theorem 3.5.2.

<u>Step 1:</u> Since $C^* = \cup_{i=1}^{\infty} A_i$, there is $i \in N$ such that A_i is uncountable. Choose $i(1)$ such that $A_{i(1)}$ is uncountable. Choose $a_1 \in A_{a(1)}$ such that $B_1 = \{a \in A_{i(1)}: a_1 < a\}$ is uncountable. To see this, let $s = \inf A_{i(1)}$. If $s \in A_{a(1)}$, let $a_1 = s$. If $s \notin A_{a(1)}$, since there is a sequence $\{b_n\}_{n=1}^{\infty} \subseteq A_{i(1)}$ such that $b_n \to s$, there is $n(0)$ such that $\{a \in A_{i(1)}: a_{n(0)} < a\}$ is uncountable. Choose $a_1 = a_{n(0)}$.

<u>Step 2:</u> Since $A_{i(1)} = \cup_{i=1}^{\infty} A_{i(1),i}$, there is $i \in N$ such that $B_1 \cap A_{i(1),i}$ is uncountable. Choose $i(2)$ such that $B_1 \cap A_{i(1),i(2)}$ is uncountable. Choose $a_2 \in B_1 \cap A_{i(1),i}$ such that $a_1 \ll a_2$ and $B_2 = \{a \in B_1 \cap A_{i(1),i(2)} : a_2 < a\}$ is uncountable and $a_1 < a_2$. To see this, let $D = \{a: a \ll a_1\}$. Then D is countable. Hence, $([a_1, 1] \cap B_1 \cap A_{i(1),i(2)}) \sim D$ is uncountable. Choose $a_2 \in ([a_1, 1] \cap B_1 \cap A_{i(1),i(2)}) \sim D$, $a_2 \neq a_1$ such that $M_2 = \{a \in ([a_1, 1] \cap B_1 \cap A_{i(1),i(2)}) \sim D: a_2 < a\}$ is uncountable. Let

$B_2 = \{ a \in (B_1 \cap A_{i(1),i(2)}) \sim D\colon\ a_2 < a\}$. Then, $M_2 \subseteq B_2$. Hence, B_2 is uncountable. Hence, we get (i(1), i(2)) and (a_1, a_2) such that $a_{i(k)} \in A_{i(k)}$ for k =1, 2, and $a_1 < a_2$, $a_1 \lessdot a_2$.

Step 3: Continue this process to obtain $(i(k))_{k=1}^{\infty} \subseteq N$ and $(a_k)_{k=1}^{\infty}$ such that $a_{i(k)} \in A_{i(k)}$ and $a_k < a_{k+1}$, $a_k \lessdot a_{k+1}$.

Claim: $B = (a_k)_{k=1}^{\infty}$ has no limit point. To see this, first, note that $B \in \mathcal{B}$. Secondly, since a_k's are distinct and on C one has the discrete topology, the only possible limit point of B is ∞. However, $C^* \sim B$ is a neigborhood of ∞. Hence, ∞ is not a limit point of B. Therefore, B has no limit points. This contradicts to Theorem 3.5.2. So, C^* is not κ - analytic. Hence, K is not Talagrand - compact.

3.6 Valdivia - compact Does Not Imply Corson - compact

Deville and Godefroy [14] construct a counterexample which says that Valdivia - compact does not generally imply Corson - compact.

Lemma 3.6.1 Suppose that α is any ordinal such that cf(α) is uncountable, where cf(α) is the cofinality of α, defined by $cf(\alpha) = \inf\{\beta\colon \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_\beta$,

$\alpha_\eta < \alpha$ for $\eta < \beta\}$. If there is a function $f: [0, \alpha] \to K \subseteq [0, 1]^C$ where f is continuous from the order topology to the product topology on $[0, 1]^C$, where K is closed and C is some uncountable set such that $K \cap \Sigma[0, 1]^C$ is dense in K, then $f(\alpha) \notin \Sigma([0, 1]^C)$.

Proof. Suppose that $f(\alpha) \in \Sigma([0, 1]^C)$. By the definition of $\Sigma([0, 1]^C)$, we know that $|D| = |\{d: d \in C \text{ and } f(\alpha)(d) \neq 0\}| \leq \omega$. Let $d \in D$. Since f is continuous, for any n, there is β_n such that if $\eta > \beta_n$, then $| f(\eta)(d) - f(\alpha)(d)| < 1\backslash n$. Let $\xi_d = \sup_n \beta_n$. Then, $\xi_d < \alpha$. So, if $\eta > \xi_d$, then $f(\eta)(d) = f(\alpha)(d)$. Let $\zeta = \sup_{d \in D} \xi_d$. Then, $\zeta < \alpha$. Therefore, if $\eta > \zeta$, then $f(\eta)(d) = f(\alpha)(d)$ for any $d \in D$.

We build an increasing sequence $(\zeta_n)_{n=1}^{\infty}$ in $[\zeta, \alpha)$ such that $f(\zeta_{n+1})(d) = 0$, for every $d \in \text{supp } f(\zeta_i) \sim D$ for any some $i \leq n$. To see this,

<u>Step 1:</u> Let $\zeta_0 = \zeta$. Choose $\zeta_1 > \zeta_0$. Hence, for $d \in \text{supp } f(\zeta_0) \sim D$, $f(\zeta_1)(d) = 0$.

<u>Step 2:</u> Suppose that for $n \in N$, the finite increasing sequence $(\zeta_m)_{m<n}$ is chosen such that if $i \neq j$ and $i, j < n$, then $(\text{supp } f(\zeta_i) \sim D) \cap (\text{supp } f(\zeta_j) \sim D) = \Phi$.

<u>Step 3:</u> Choose $\zeta_n > \zeta_{n-1}$ such that $(\text{supp } f(\zeta_i) \sim D) \cap (\text{supp } f(\zeta_j) \sim D) = \Phi$ for any $i \neq j$ and $i, j \leq n$, since α has uncountable cofinality. This completes the

construction.

Let $\eta = \lim_{n\to\infty}\zeta_n$. So, if $c \in D$, $f(\eta)(c) = \lim_{n\to\infty} f(\zeta_n)(c) = f(\alpha)(c)$; if $c \notin D$, then there is at most one n such that $f(\zeta_n)(c) \neq 0$, since for any $i \neq j$, $(\text{supp } f(\zeta_i) \sim D) \cap (\text{supp } f(\zeta_j) \sim D) = \Phi$. Hence, $f(\eta)(c) = 0$ for $c \notin D$. So, for $c \in D$, $f(\eta)(c) = f(\alpha)(c)$. Since f is one to one, $\eta = \alpha$. This contradicts that α has uncountable cofinality.

Theorem 3.6.1 (Deville and Godefroy [14]) There is a topological space which is Valdivia - compact, but not Corson - compact.

Proof: We are going to show that $[0, \omega_1]$ is Valdivia - compact, but not Corson - compact.

From Lemma 3.6.1, it can be shown that $[0, \omega_1]$ is not Corson - compact. To see this, suppose that $[0, \omega_1]$ is Corson - compact, then there is a homeomorphism, g, from $[0, \omega_1]$ onto a closed subset K of $\Sigma([0, 1]^C)$ for some set C, by the definition of Corson - compact. Following Lemma 3.6.1, $g(\omega_1) \notin \Sigma([0, 1]^C)$. This is a contradiction.

We will show that $[0, \omega_1]$ is Valdivia - compact. Define

$h: [0, \omega_1] \to E \subseteq [0, 1]^{[0, \omega_1]}$ by $h(\alpha) = \chi_{[0, \alpha)}$. We know that

$E = \{\chi_{[0, \alpha)}: \alpha < \omega_1\}$ is compact in $[0, 1]^{[0, \omega_1]}$, by Theorem 3.3.1. It is easy to see that h is continuous. Hence, h is a homeomorphism from $[0, \omega_1]$ to E. Since $E \cap \Sigma([0, 1]^{[0, \omega_1]}) = E \sim \{f(\omega_1)\}$, then, $E \cap \Sigma([0, 1]^{[0, \omega_1]})$ is dense in E. Hence, $[0, \omega_1]$ is Valdivia - compact.

3.7 Conclusion

In this chapter, we have given examples to show that none of the implications below can be reversed: UEC $\Rightarrow$ EC $\Rightarrow$ TC $\Rightarrow$ GC $\Rightarrow$ CC $\Rightarrow$ VC.

Chapter 4

Radon Nikody'm Compactness

4.1 Introduction

We have already shown in Chapter 1 to Chapter 3 that

$$UEC \Rightarrow EC \Rightarrow TC \Rightarrow GC \Rightarrow CC \Rightarrow VC, \text{ and}$$

$$UEC \not\Leftarrow EC \not\Leftarrow TC \not\Leftarrow GC \not\Leftarrow CC \not\Leftarrow VC.$$

In Chapter 4, we are interested in exploring the connections between Radon - Nikody'm compactness and those notions of compactness previously considered. The notion of Radon Nikody'm compactness was introduced by Namioka [27], most of the results in this chapter are due to him.

4.2 Radon Nikody'm, Uniform Eberlein and Eberlein Compactness

First, we will prove that Eberlein - compact implies Radon - Nikody'm compact.

Lemma 4.2.1 (Davis et al. [13]) Suppose that $(X, ||.||)$ is a Banach space and

$S \subseteq X$ is convex, symmetric and bounded. Then, the gauge $||.||_n$ of the set $U_n = 2^n S + 2^{-n} B(X)$ is a norm equivalent to $||.||$. For any $x \in X$, define $|||x||| = (\Sigma_{n=1}^{\infty} ||x||_n^2)^{1/2}$, and let $C = \{x \in X: |||x||| < \infty\}$, $U = B(C) = \{x \in X: |||x||| \leq 1\}$ and i denote the identity embedding from C into X. Then,

(1) $S \subseteq U$

(2) $(C, |||.|||)$ is a Banach space and i is continuous.

(3) $i^{**} : C^{**} \rightarrow X^{**}$ is one to one and $(i^{**})^{-1}(X) = C$.

(4) C is reflexive iff S is weakly relatively compact.

Proof. (1) If $x \in S$, then $||x||_n \leq 2^{-n}$, for any $n \in N$. So, $|||x||| \leq 1$. That means $x \in U$.

(2) Let $X_n = (X, ||.||_n)$ and $Y = \Sigma_{n=1}^{\infty} X_n$ with l_1- norm. We define a function $f: C \rightarrow Y$ by $f(x) = (i(x), i(x), i(x), ...)$. Then, f is a linear isometric embedding, and $f(C) = \{y = (y_n): y \in Y \text{ and } y_n = y_1 \text{ for } n \in N\}$ is a closed subspace of Y. Therefore, $(C, |||.|||)$ is a Banach space. We consider i as the composition of f and the projection of U onto the first coordinate. So, i is continuous.

(3) Since $f^{**}(x^{**}) = (i^{**}(x^{**}), i^{**}(x^{**}), i^{**}(x^{**}), ...)$ for any $x^{**} \in C$, and since f is an isometry, $(f^{**})^{-1}(0) = \{0\}$, hence f^{**} is one to one, $(f^{**})^{-1}(f(C)) = C$.

(4) We claim that weak - cl(U) = i^{**} ($B(C^{**})$) in (X^{**}, weak). To see this, first, we know that $B(C^{**})$ is weak* - compact in (C^{**}, weak*) (see the Alaoglu Theorem [18]), U = B(C) is weak* dense in $B(C^{**})$ (see the Goldstine Theorem [18]), and i^{**} is weak* continuous. Hence, $i^{**}(B(C^{**}))$ is closed in (X^{**}, weak*), since $i^{**}(B(C^{**}))$ is compact in (X^{**}, weak*). And i^{**} (U) = U is dense in $i^{**}(B(C^{**}))$.

Suppose that S is weakly relatively compact, i.e. weak - cl(S) is compact in (X, weak). Then, $U \subseteq 2^n$ weak - cl(S) + $2^{-n} B(X^{**})$, for any $n \in N$. Furthermore, 2^n weak - cl(S) + $2^{-n} B(X^{**})$ are weak* closed in (X^{**}, weak*), for $n \in N$, hence they contain i^{**} ($B(C^{**})$). Since

$\cap_{n=1}^{\infty}(2^n$ weak - cl(S) + $2^{-n} B(X^{**})) \subseteq \cap_{n=1}^{\infty}(X + 2^{-n}B(X^{**})) = X$, then,

i^{**} ($B(C^{**})) \subseteq X$. So, by (3), $C^{**} \subseteq C$. That means that C is reflexive.

It is easy to get the other part of the implication directly from (1).

Theorem 4.2.1 (Davis et al. [13]) Every weakly compact subset K of a Banach space is weak to weak affinely homeomorphic to a subset of a reflexive Banach space.

Proof. In Lemma 4.2.1, let S= cl - conv(K $\cup$ (- K)). By the Krein - Smulian Theorem which says that the closed convex hull of a weakly compact subset of a Banach space is itself weakly compact (see [18]), S is weakly relatively compact. By (4) of Lemma 4.2.1, C is reflexive. Therefore,

$A = i^{-1}(K)$ is weakly compact. Hence, $i|A$ is the homeomorphism we need.

The following theorem is proven in Diestel and Uhl [17].

Theorem 4.2.2 Suppose that X is a reflexive Banach space. Then $X^{**} = X$ has the Radon - Nikody'm property.

Theorem 4.2.3 (Namioka [27]) Suppose that (K, τ) is a compact Hausdorff topological space which is an Eberlein - compact or scattered compact. Then (K, τ) is Radon - Nikody'm - compact.

Proof. Supppose that (K, τ) is an Eberlein - compact space. Then K is homeomorphic to a weakly compact subset, F, of a Banach space X. By Theorem 4.2.1, F is weak to weak homeomorphic to a subset of a reflexive Banach space X_1. Hence, K is homeomorphic to a weakly compact subset of a reflexive space X_1. Following Theorem 4.2.2, X_1 has the Radon - Nikody'm property. So, (K, τ) is RN compact.

Suppose that (K, τ) is scattered - compact. By Rudin in [37], K^{**} is isomorphic to $l_1(K)$, where we identify K with F. Since $l_1(K)$ has the Radon - Nikody'm property by [17], then K is homeomorphic to a weak* compact subset of K^{**}. Therefore, K is RN compact, which completes the proof.

So, EC $\Rightarrow$ RNC. The next main result is RNC $\not\Longrightarrow$ EC. First, we note the following characterization for Eberlein - compact spaces due to Rosenthal [33].

<u>Theorem 4.2.4</u> A compact Hausdorff space K is Eberlein - compact iff there is a sequence $\mathcal{A} = (\alpha_n)_{n=1}^{\infty}$ such that

(1) If $n \in N$ and any $A \in \alpha_n$, then A is an open F_σ - set of K.

(2) If $x \neq y$, then there is $n \in N$ and $A \in \alpha_n$ such that either $x \in A$ and $y \notin A$; or $x \notin A$ and $y \in A$.

(3) If $x \in K$ and $n \in N$, then x belongs to a finite number of sets in α_n, i.e. each α_n is point - finite.

The sequence $\mathcal{A}$ is called an EC - structure.

<u>Theorem 4.2.5</u> (Bennett et al. [5]) $[0, \omega_1]$ is not Eberlein - compact.

Proof. Suppose that $[0, \omega_1]$ is Eberlein - compact. By Theorem 4.2.4, there is an EC - structure $\mathcal{A}$. Let $\mathcal{A}' = \{A \in \mathcal{A} : \omega_1 \in A\}$. Then, $\mathcal{A}'$ is countable by Theorem 4.2.4, Condition (3). Hence, there is an ordinal $\beta < \omega_1$ such that $(\beta, \omega_1) \subseteq \cap\{A: A \in \mathcal{A}'\}$. Then for any $\xi, \eta \in (\beta, \omega_1)$ and $\xi \neq \eta$, if $A \in \mathcal{A}'$ which separates ξ, η, then $\omega_1 \notin A$. Such a set A must be bounded away from ω_1, so that the family*A* ~ *A* ' is a point - countable cover of (β, ω_1) by bounded open sets. Using the Pressing Down Lemma, we know that such a cover cannot exist.

Theorem 4.2.6 (Namioka [27]) There is a topological space (K, τ) which is Radon - Nikody'm compact but not Eberlein - compact.

Proof. $[0, \omega_1]$ is such a space, since it is Hausdorff and scattered in the order topology and by Theorem 4.2.3, $[0, \omega_1]$ is Radon - Nikody'm compact. However, by Theorem 4.2.5, we know that $[0, \omega_1]$ is not an Eberlein - compact.

The following two corollaries are immediate since UEC $\Rightarrow$ EC.

Corollary 4.2.7 If a topological space (K, τ) is uniform Eberlein - compact, then (K, τ) is Radon - Nikody'm compact.

Corollary 4.2.8 Radon - Nikody'm compact does not imply uniform Eberlein - compact.

4.3 Radon - Nikody'm and Corson Compactness

We will show RNC $\not\Longrightarrow$ CC. The following theorem is due to Namioka [27], but, the proof is different from the original one.

Theorem 4.3.1 Radon - Nikody'm compact does not imply Corson - compact.

Proof. $[0, \omega_1]$ with the order topology is Radon - Nikody'm compact as shown in Theorem 4.2.6. By Theorem 3.6.1, $[0, \omega_1]$ is not Corson - compact.

Also, we give a counterexample which will show that CC $\not\Longrightarrow$ RNC.

Theorem 4.3.2 (Stegall [38]) Let X be a Banach space. X* has the Randon - Nikody'm property, i.e. X is an Asplund space, iff every weak* - compact subset of X* is norm - fragmented.

Theorem 4.3.3 (Namioka [27]) Suppose that (K, τ) is Baire and ρ is a metric on K. If (K, τ) is ρ - fragmented, then the set

$C = \{x \in X:\ i:\ (K, \tau) \to (K, \rho)$ at x is continuous$\}$ is a dense G_δ set of (K, τ).

Proof. Let $U_\varepsilon = \cup\{ V:\ V$ is τ - open and ρ - diam$(V) \leq \varepsilon\}$. Since (K, τ) is ρ -

fragmented, then it is easy to see that U_ε is τ - open and τ - dense in K.

Let C = {x∈ X: i : (K, τ) → (K, ρ) at x is continuous}. Then

$C = \cap_{n=1}^{\infty} U_{1/n}$. Since (K, τ) is a Baire space, then C is a dense G_δ subset of (K, τ).

<u>Theorem 4.3.4</u> (Namioka [27]) Suppose that (K, τ) is a Radon - Nikody'm compact space. Then, there is G_δ subset C of K such that C is dense in K and it is metrizable in the relative topology. Hence, if x ∈ C, then x is a G_δ point in K.

Proof. By Theorem 4.3.2, we can suppose that K is weak* - compact subset of the dual X* of a Banach space X and K is norm - fragmented. Let C = {x∈ X: i : (K, weak) → (K, norm) at x is continuous}. Following Theorem 4.3.3 directly, we know that C is a dense G_δ subset of (K , weak*). Since i: (C, weak*) → (C, norm) is a homeomorphism and (C, norm) is metrizable, then, (C, weak*) is metrizable. For any x ∈ C, by the definition of C, it is obvious that {x} is a G_δ subset in (C, weak*). Hence, {x} is a G_δ set in (K, weak*).

The following theorem essentially comes from Namioka.

Theorem 4.3.5 Corson - compact does not imply Radon - Nikody'm compact.

Poof. Let (K, τ) be Corson - compact. Suppose K is Radon - Nikody'm compact. By Theorem 4.3.4, for (K, τ), there is a dense G_δ subset of $C \subseteq K$ such that C is metrizable in the relative topology. But, Todorcevi´c, (see Chapter 6, Theorem 9.11 [28]), built a Corson - compact space which does not have any metrizable subspace.

4.4 Radon Nikody'm, Talagrand and Gul'ko Compactness

Since TC $\Rightarrow$ GC $\Rightarrow$ CC, we have the following immediate corollaries to Theorem 4.3.1.

Corollary 4.4.1 Radon - Nikody'm compact does not generally imply Talagrand - compact.

Corollary 4.4.2 Radon - Nikody'm compact does not generally imply Gul'ko - compact.

4.5 Radon Nikody'm and Valdivia compactness

The main results of this section are VC $\not\Longrightarrow$ RN which is Theorem 4.5.1, and RN $\not\Longrightarrow$ VC which is Theorem 4.5.2.

Theorem 4.5.1 Valdivia - compact does not imply Radon - Nikody'm compact.

Proof. Suppose that Valdivia - compact does imply Radon - Nikody'm compact. Since any Corson - compact is Valdivia - compact, then Corson - compact implies Ra*don - Nikody'm compact. This contradicts Theorem 4.3.6.

The proof that $[0, \omega_2]$ is not Valdivia is given by Yabouri (see [14]).

Theorem 4.5.2 There is a Radon - Nikody'm compact which is not Valdivia - compact.

Proof. We will show that $[0, \omega_2]$ with the order topology is Radon - Nikody'm compact, but not Valdivia - compact.

Since $[0, \omega_2]$is a scattered - compact in the order topology, by Theorem 4.2.3, $[0, \omega_2]$ is Radon - Nikody'm compact.

We are going to show that $[0, \omega_2]$ is not Valdivia - compact. Suppose that $[0, \omega_2]$ is Valdivia - compact. By the definition of Valdivia - compact, $[0, \omega_2]$ is homeomorphic to a subset $F \subseteq [0, 1]^C$ for some set C such that $F \cap \Sigma([0, 1]^C)$ is dense in F. We will identify F and $[0, \omega_2]$. By Lemma 3.6.1, $\omega_2 \notin [0, 1]^C$. Hence, there is a $A \subseteq C$ such that $\omega_2(x) \neq 0$ for any $x \in A$ and $|A| = \omega_1$. Since $|A| < \omega_2$, there is η such that $\beta(x) = \omega_2(x)$ for any $x \in A$. Therefore, $[\eta, \omega_2] \cap \Sigma([0, 1]^C) = \Phi$. This contradicts that $[0, \omega_2] \cap \Sigma([0, 1]^C)$ is dense in $[0, \omega_2]$.

4.6 Conclusion

The results of this chapter can be summarized as follows:

UEC $\Rightarrow$ RN, UEC $\Leftarrow\neq$ RN;

EC $\Rightarrow$ RN, EC $\Leftarrow\neq$ RN;

RN $\neq\Rightarrow$ TC, RN $\neq\Rightarrow$ GC;

RN $\neq\Rightarrow$ CC, CC $\neq\Rightarrow$ RN;

RN $\neq\Rightarrow$ VC, VC $\neq\Rightarrow$ RN;

Chapter 5

Rosenthal Compactness

5.1 Introduction

Up to this point, we have not discussed the connections between Rosenthal - compact and the other types of compact spaces. Unfortunately, not many results have been obtained. Some of the known results are shown in this chapter.

5.2 Rosenthal and Radon Nikody'm Compactness

We are going to show RN $\not\Longrightarrow$ RC.

Theorem 5.2.1 (Rosenthal [35]) Suppose that (K, τ) is a Rosenthal - compact space. Then K is angelic.

The following theorem is due to Namioka, but the proof is different from the original one.

Theorem 5.2.2 (Namioka [27]) There is a topological space (K, τ) which is Radon

- Nikody'm compact but not Rosenthal - compact.

Proof. From Theorem 4.2.6, we know that $[0, \omega_1]$ is Radon - Nikody'm compact.

Since $[0, \omega_1]$ is not angelic, then $[0, \omega_1]$ is not Rosenthal - compact by Theorem 5.2.1.

We are going to give a counterexample which will show that RC $\not\Longrightarrow$ RN.

<u>Theorem 5.2.3</u> (Namioka [27]) A topological space (K, τ) which is Radon - Nikody'm compact is herditarily Lindelof iff it is metrizable.

<u>Theorem 5.2.4</u> (Namioka [27]) There is a topological space (K, τ) which is Rosenthal - compact, but not Radon - Nikody'm compact.

Proof. Let $K = \{(x, y): 0 \le x \le 1, y = 0, 1\} \sim \{(0, 0), (1, 1)\}$ and define an order on K by $(x, y) \le (x_1, y_1)$ iff $x < x_1$ or $x = x_1$ and $y < y_1$. K with the order topology τ is compact. It can be shown that (K, τ) is hereditarily Lindelöf, but not metrizable. Therefore (K, τ) is not Radon - Nikody'm compact by Theorem 5.2.3.

We will show that (K, τ) is Rosenthal - compact. Let

$F = \{g: g: [0, 1] \to \{0, 1\}$ and if $t \le t_1$ then $g(t) \le g(t_1)$ and $g(0) = 0, g(1) = 1\}$.

If we induce the pointwise topology, τ_1, on it, then (F, τ_1) is Rosenthal - compact.

(K, τ) is homeomorphic to (F, τ_1). So, (K, τ) is Rosenthal - compact.

5.3 Rosenthal and Eberlein Compactness

Argyros, Mercourakis and Negrepontis showed [3] that every Eberlein - compact of weight at most 2^{ω} is Rosenthal - compact. However, since

EC $\Rightarrow$ RN, RC $\neq \Rightarrow$ RN, then RC $\neq \Rightarrow$ EC and RC $\neq \Rightarrow$ UEC.

5.4 Conclusion

Evidently there are no known connections between RC and the remaining notions of compactness, namely, TC, GC, CC and VC.

Chapter 6

Some Recent Discoveries

6.1 Introduction

In 1989, Deville and Godefroy [14] showed that a topological space (K, τ) is Valdivia - compact if and only if it is Corson or $[0, \omega_1] \subseteq K$. This theorem supplies a clear picture of the relationship between Corson - compact and Valdivia - compact. In the same year, Orihuela, Schachermayer and Valdivia [30] proved that a Radon - Nikody'm and Corson - compact space is Eberlein compact which answered an open question proposed by Namioka [27] in 1987. In this chapter, we will give the proofs of these two theorems.

6.2 Conditions for a Valdivia - compact to be Corson - Compact

We are going to give conditions for a Valdivia - compact to be Corson - compact.

Lemma 6.2.1 Suppose that (X, τ) is a Hausdorff topological space and a net of functions, $\{f_\alpha: 0 \le \alpha \le \beta\}$, which is from X to itself satisfies the following conditions

(1) $f_\alpha f_\xi = f_\xi f_\alpha = f_\alpha$ for $0 \le \alpha \le \xi \le \beta$;

(2) for $x \in X$, the function $g_x: \alpha \to f_\alpha(x)$ is continuous from $[0, \beta]$ to (X, τ),

then $g_x([0, \beta])$ is homeomorphic to a well ordered space for every $x \in X$.

Proof. Fix $x \in X$. Let $C = g_x([0, \beta]) \subseteq X$. We say $x_1 \le x_2$ iff $\inf\{g_x^{-1}(x_1)\} \le \inf\{g_x^{-1}(x_2)\}$. C is well ordered by the relation, $\le$.

Claim: $g_x^{-1}(z)$ is an interval in $[0, \beta]$. To see this, let $\alpha = \inf\{g_x^{-1}(z)\}$ and $\beta = \sup\{g_x^{-1}(z)\}$. For any $\alpha < \gamma < \beta$, by condition (1), $f_\gamma f_\alpha(x) = f_\gamma(z) = f_\alpha(x) = z$. Since there is a $\zeta \in g_x^{-1}(z)$ such that $\gamma < \zeta \le \beta$, $f_\gamma(x) = f_\gamma(f_\zeta(x)) = f_\gamma(z) = z$. This verifies the claim.

Hence, g_x is continuous from $[0, \beta]$ with its order topology to $C = g_x([0, \beta])$ with the topology τ_1 induced by the relation $\le$. If we can prove that $\tau = \tau_1$ on $C = g_x([0, \beta])$, then C is homeomorphic to the well ordered set C with regard to the relation $\le$.

Suppose that A is τ_1- closed. Then, $g_x^{-1}(A)$ is closed in $[0, \beta]$. Since

$g_x(g_x^{-1}(A)) = A$ and $g_x^{-1}(A)$ is compact, then A is compact in τ. Hence A is τ - closed following the fact that (C, τ) is Hausdorff. That means $\tau_1 \subseteq \tau$.

Suppose that A is τ - closed, but not τ_1 - closed. There is $x_0 \in A$, a net $(x_\alpha)_{\alpha\in\Psi} \subseteq A$ such that $(x_\alpha)_{\alpha\in\Psi} \to x_0$ in (C, τ), but $(x_\alpha)_{\alpha\in\Psi}$ is not convergent to x_0 in (C, τ_1). Hence, there is an interval (y_1, y_2), $x_0 \in (y_1, y_2)$, such that for any $0 \le \eta < \beta$, there is $\alpha(\eta) > \eta$ such that $x_{\alpha(\eta)} \notin (y_1, y_2)$. So, we get a subnet $(x_{\alpha(\eta)})_{\eta\in\Psi}$ which is not a subset of (y_1, y_2). Since $(x_{\alpha(\eta)})_{\eta\in\Psi} \to x_0$, then $x_0 \notin (y_1, y_2)$. Contradiction. Hence, $\tau \subseteq \tau_1$.

Therefore, $\tau = \tau_1$. This completes the Lemma.

<u>Lemma 6.2.2</u> (Argyros [14]) Suppose that X is a compact subset of $([0, 1]^C, \tau_1)$, where τ_1 is the product topology on $[0, 1]^C$ for some set C such that $X \cap \Sigma([0, 1]^C)$ is dense in X. If $A \subset C$ and $|A| \ge \omega$, then there is a $B \subset C$ such that $A \subset B$, $|A| = |B|$ and $r_B(X) \subseteq X$, where r_B: $[0, 1]^C \to [0, 1]^C$ is defined by:

$$r_B(x)(k) = x(k) \quad \text{if} \quad k \in B;$$

$$r_B(x)(k) = 0 \qquad \text{if} \quad k \notin B.$$

Theorem 6.2.1 (Deville and Godefroy [14]) If (K, τ) is a Valdivia - compact space, then K is Corson - compact iff there is no closed subset of K which is homeomorphic to $[0, \omega_1]$.

Proof. $\Rightarrow$ Let (K, τ) be Valdivia - compact. Suppose that K is Corson - compact and B is any closed subset of K. Since B is Corson -compact, by Theorem 3.6.1, B is not homeomorphic to $[0, \omega_1]$.

$\Leftarrow$ Suppose that (K, τ) is Valdivia - compact, but not Corson - compact. Hence, there is a set C such that $K \subseteq [0, 1]^C$ and $D \cap \Sigma([0, 1]^C)$ is dense in K, where we identify K with its image in $[0, 1]^C$, and there is a $x_0 \in K$ such that $x_0 \notin \Sigma([0, 1]^C)$. Let $A \subseteq C$ such that $|A| = \omega_1$ and $x_0(a) \neq 0$ for every $a \in A$. We rewrite $A = \{a_\alpha: 0 \leq \alpha \leq \omega_1\}$.

Claim: There is a net, $\{A_\alpha: 0 \leq \alpha \leq \omega_1\}$, of subsets of C such that

(1) $A_\alpha \subseteq A_\beta$ for $\alpha < \beta$;

(2) $|A_\alpha| = \omega$ for $\alpha \leq \omega_1$;

(3) $a_\alpha \in A_{\alpha+1}$;

(4) $A_\alpha = \cup_{\beta < \alpha} A_\beta$ for any limit ordinal α.

(5) $r_{A\alpha}(K) \subseteq K$.

To see this, since $x_0 \in V_\alpha = \{x \in K: x(a_\alpha) \neq 0\}$ which is an open subset of K and $K \cap \Sigma([0, 1]^C)$ is dense in K, we can choose $x_\alpha \in V_\alpha \cap \Sigma([0, 1]^C)$. In Theorem 3.6.1, let $A = \{c \in C: x_0(c) \neq 0\}$, for $\alpha = 0$;

$A = A_\alpha \cup \{c \in C: x_\alpha(c) \neq 0\}$ for $\alpha \neq 0$.

Using Lemma 6.2.2, build $A_{\alpha+1}$ from A_α such that $r_{A\alpha+1}(K) \subseteq K$. If α is a limit ordinal, let $A_\alpha = \cup_{\beta < \alpha} A_\beta$. By this construction, one can see that $\{A_\alpha: 0 \leq \alpha \leq \omega_1\}$ satisfies condition (1) to (5).

Let $f_\alpha = r_\alpha$ and K = X in Lemma 6.2.1, we can get $g_x([0, \omega_1])$ which is homeomorphic to $[0, \xi]$ for some ordinal ξ. From condition (2) and (3) of Lemma 6.2.2, it is easy to see that $g_x([0, \omega_1])$ is uncountable. Hence, K includes a subset homeomorphic to $[0, \omega_1]$.

Corollary 6.2.2 Let (K, τ) be a topological space. Then, K is Valdivia - compact iff K is Corson - compact, or $[0, \omega_1] \subseteq K$.

6.3 Conditions for Radon - Nikody'm to be Eberlein - compact

Based on a result of Orihuela, Schachermayer and Valdivia in [29], we show that every Radon - Nikody'm and Corson compact space is Eberlein - compact. First, we give some definitions and prove some lemmas.

Definition 6.3.1: A class Ψ of compact Hausdorff space is called a perfect class iff Ψ is stable by taking continuous images, countable products, and closed subspaces.

Definition 6.3.2: Let X be a linear space. A set S in X is called absolutely convex iff for any $y_1, y_2 \in S$, $\lambda y_1 + \mu y_2 \in S$ whenever $|\lambda| + |\mu| \leq 1$.

Definition 6.3.3: Suppose that X is a Banach space and C is an absolutely convex and weak* compact subset of X^*. Denote by $|.|$ the seminorm on X dual to C, i.e. $|x| = \sup\{\langle x, y\rangle : y \in C\}$. Let $(Y, |\,.\,|)$ be the Banach space obtained by completing the equivalence classes modulo $|.|$ of X and $i: X \to Y$ be the canonical map.

Let $\|.\|_n$ be the norm on Y whose unit ball is given by

$$B(Y, \|.\|_n) = cl(2^n i(B(X, \|.\|) + 2^{-n}B(Y, |.|)).$$

For $1 \leq p < +\infty$, define $\|x\|_p = (\Sigma_{n=1}^{\infty}\|x\|_n^p)^{1\backslash p}$ and $\|x\|_0 = \max\{\|x\|_n\}$. Let $F_p = \{x \in Y: \|x\|_p < +\infty\}$ and $F_0 = \{x \in Y: \|x\|_0 < +\infty$ and $\lim_n\|x\|_n = 0\}$.

<u>Lemma 6.3.1</u> Suppose that X is a Banach space and C is an absolutely convex and weak* compact subset of X* that belongs to a perfect class Ψ. Then both $(B(F_p^*), \text{weak}^*), (B(F_0^*), \text{weak}^*) \in \Psi$.

Proof. For $1 \leq p < +\infty$, denote

$\Sigma_p = \Sigma_p(Y, \|\cdot\|_n)$

$= \{z = (x_n): x_n \in Y$ and $\|z\| = (\Sigma_{n=1}^{\infty}\|x_n\|_n^p)^{1\backslash p} < +\infty\}$, and

$\Sigma_0 = \Sigma_0(Y, \|\cdot\|_n)$

$= \{z = (x_n): x_n \in Y$ and $\|z\| = \sup\{\|x_n\|_n\} < +\infty$ and $\lim_n\|x\|_n = 0\}$.

F_p (respectively, F_0) is isometrically isomorphic to the diagonal of

Σ_p (respectively, Σ_0), i.e. for $z = (x_n) \in \Sigma_p$ such that $x_1 = x_2 = \ldots$ whence $B(F_p^*)$ (respectively, $B(F_0^*)$) is a continuous image of $B(\Sigma_p^*)$ (respectively, $B(\Sigma_0^*)$) with respect to the weak* - topology. Hence, it is sufficient to show that $(B(\Sigma_p^*)$, weak*), $(B(\Sigma_0^*)$, weak*) $\in \Psi$.

First, we note that the identity on Y induces a continuous injection of norm 1 and with dense range from Σ_p into Σ_p, for $1 \le p < +\infty$, and from Σ_0 into Σ_0. Whence $B(\Sigma_p^*)$, $B(\Sigma_0^*)$ are weak* - homeomorphic to a subset of $B(\Sigma_1^*)$, it is sufficient to show that $B(\Sigma_1^*)$ is in Ψ.

Secondly, we note that $B((Y, ||.||)^*)$ may naturally be identified with a subset of $2^n C$. Hence, $B((Y, ||.||)^*) \in \Psi$. Since $(B(\Sigma_p^*)$, weak*) is homeomorphic to $\Pi_{n=1}^{\infty}(B((Y, ||.||_n)^*)$, weak*), then $(B(\Sigma_1^*)$, weak*) $\in \Psi$.

<u>Lemma 6.3.2</u> Suppose that K is a weak* - compact subset of the dual X* of a Banach space X where K is norm - fragmented such that its weak* closed absolutely convex hull C belongs to a perfect class Ψ of compact spaces. Then, space F_p and F_0 are Asplund and $B(F_p^*)$, $B(F_0^*) \in \Psi$ for any

$1 \leq p < +\infty$.

Proof. By the Stegall Theorem [39], F_p and F_0 are Asplund spaces. By Lemma 3.5.1, $(B(F_p^*), \text{weak}^*)$, $(B(F_0^*), \text{weak}^*) \in \Psi$.

Theorem 6.3.1 Let (K, τ) be a topological space. Then, K is Eberlein -compact iff K is Radon - Nikody'm and Corson.

Proof. $\Rightarrow$ Suppose that K is Eberlein - compact. Since EC $\Rightarrow$ TC $\Rightarrow$ GC $\Rightarrow$ CC, K is Corson - compact. Following Theorem 4.2.3, K is Radon - Nikody'm compact. Hence, if K is Eberlein - compact, then K is Radon - Nikody'm compact and Corson - compact.

$\Leftarrow$ Suppose that K is Radon - Nikody'm and Corson - compact. So, K is homeomorphic to a weak* - compact subset of the dual X* of a Banach space X such that the dual norm fragments K. Jayne, Namioka and Rogers [20] proved that every regular Borel probability measure on K has separable support, and Argyros, Mercourakis and Negrepontis [3] proved that if K is a Corson - compact space with this property, then B(C*(K)) is Corson - compact. Hence, the weak* closed absolutely convex hull C of K is a Corson - compact, since it is the continuous image of B(C*(K)) in (C*(K), weak*). We apply Lemma 6.3.2 to the perfect class of Corson compact spaces and find an Asplund space F with weak* Corson compact dual unit ball such that K is homeomorphic to a weak* compact subset of the dual F*. Orihuela, Schachermayer, and Valdivia [30] showed that F is weakly compactly generated and thus K is Eberlein compact.

Reference

[1] K. Alster and R. Pol. On function spaces of compact subspaces of Σ - products of the real line. Fund. Math. 107 (1980), 135 - 143.

[2] D. Amir and J. Lindenstrauss. The structure of weakly compact sets in Banach spaces. Ann. of Math. (2) 88 (1968), 35 - 46.

[3] S. Argyros, S. Mercourakis and S. Negrepontis. Functional - analytic properties of Corson - compact spaces. Studia Math. 89 (1988), 197 - 229.

[4] S. Argyros. Corson - compact spaces of bounded order - type. To appear.

[5] H. R. Bennett, D. J. Lutzer and J. M. Van. Wouwe. Linearly ordered Eberlein compact spaces. Topology Appl. 12 (1981), 11 - 18.

[6] Y. Benyamini and T. Starbird. Embedding weakly compact sets into Hilbert space. Israel. J. Math. 23 (1976), 137 - 141.

[7] R. H. Bing. Metrization of topological spaces. Canad. J. Math. 3 (1951), 175 - 186.

[8] Nicolas Bourbaki. General Topology. Addison - Wesley Publishing Company, (1966).

[9] G. Choquet. Theory of capacities. Ann. Inst. Fourier (Grenoble) 5 (1953), 131 - 297.

[10] G. Choquet. Lectures on Analysis. W. A. Benjamin, Inc. 1969.

[11] H. H. Corson. Normality in subsets of product spaces. Amer. J. Math. (1959), 785 - 796.

[12] H. H. Corson. The weak topology of a Banach space. Trans. Amer. Math. Soc. 101 (1961), 1 - 15.

[13] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski. Factoring weakly compact operators. J. Funct. Anal. 17 (1974), 311 - 327.

[14] Robert. Deville and Gilles Godefroy. Some applications of projective resolutions of identity. To appear.

[15] G. Godefroy. Compacts de Rosenthal. Pacific. J. M. 91 (1980), 293 - 306.

[16] J. Diestel. Geometry of Banach Spaces - Selected Topics, Lecture Notes in Mathematics 485, Springer - Verlag (1975).

[17] J. Diestel and J. J. Uhl. Jr. Vector measures, Math. Surveys 15. Amer. Math. Soc. , Providence, (1977).

[18] N. Dunford and J. Schwartz. Linear Operators, Part 1. Interscience Publishers, Inc., (1957).

[19] S. P. Gul'ko. On the structure of spaces of continuous functions and their complete paracompactness. Russian Math. Surveys 34 (1979), 36 - 44.

[20] J. E. Jayne, I. Namioka and C. A. Rogers. Norm fragmented weak* - compact sets, preprint.

[21] I. Juhász. Cardinal functions in topology. Math. Centrum Tract. 34, Amsterdam, (1971).

[22] John. L. Kelly. General Topology. Springer - Verlag, (1975).

[23] J. Lindenstrauss. Weakly compact sets - their topological properties and the Banach spaces they generate. Ann. of Math. (2) 9 (1972), 235 - 273.

[24] S. Mercourakis. On weakly countably determined Banach spaces. Trans. Amer. Math. Soc. 300 (1987), 307 - 327.

[25] E. Michael and I. Namioka. Barely continuous function. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 24 (1976), 889 - 892.

[26] I. Namioka. Separate continuity and joint continuity. Pacific J. Math. 51, (1974), 515 - 531.

[27] I. Namioka. Radon - Nikody'm compact spaces and fragmentability. Mathematika 34 (1987), 258 - 281.

[28] S. Negrepontis. Banach spaces and topology, Handbook of Set - Theoretic Topology, ed. K. Dunen and J.E. Vaughan. Elsevier Science Publishers, (1984), 1045 - 1142.

[29] E. Odell and H. P. Rosenthal. A double - dual characterization of separable Banach spaces containing l_1. Israel J. Math. 20 (1975), 375 - 384.

[30] J. Orihuela, W. Schachermayer, and M. Valdivia. Every Radon - Nikody'm

Corson compact space is Eberlein compact. To appear.

[31] R. Pol. A theorem on the weak topology of C(X) for compact scattered X. Fund. Math. 106 (1980), 135 - 140.

[32] C. A. Rogers and J. E. Jayne. Analytic Sets. Academic Press, London, (1980).

[33] H.P. Rosenthal. The heredity problem for weakly compactly generated Banach spaces. Compositio. Math. 28 (1974), 83 - 111.

[34] H. P. Rosenthal. A characterization of Banach spaces containing l_1. Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411- 2413.

[35] H. P. Rosenthal. Point - wise compact subsets of the first Baire class. Amer. J. Math. 99 (1977), 362 - 378.

[36] H. P. Rosenthal. Some recent discoveries in the isomorphic theory of Banach spaces. Bull. Amer. Math. Soc. 84 (1978), 803 - 831.

[37] W. Rudin. Continuous functions on compact spaces without perfect

subsets. Proc. Amer. Math. Soc. 8 (1957), 39 - 42.

[38] C. Stegall. The duality between Alplund spaces and spaces with the Radon - Nikody'm property. Israel J. Math. 29 (1978), 408 - 412.

[39] C. Stegall. The Radon - Nikody'm property in conjugate Banach spaces II. Trans. Amer. Math. Soc. 264 -2 (1981), 507 - 519.

[40] M. Talagrand. Sur une conjecture de H. H. Corson. Bull. Sci. Math. 99 (1975), 211 - 212.

[41] M. Talagrand. Espaces de Banach faiblement κ - analytiques. Ann. of Math. (2) 110 (1979), 407 - 438.

[42] M. Talagrand. A new countably determined Banach space. Israel J. Math. 47 (1984), 75 - 80.

[43] Angus E. Taylor and David C. Lay. Introduction to Functional Analysis, 2nd Edition. John Wiley and Sons, (1980).

[44] M. Valdivia. Some properties of weakly countably determined Banach

spaces. Studia Math. 93 (1989), 137 - 144.

[45] L. Vasak. On one generalization of weakly compactly generated Banach spaces. Studia Math. 70 (1981), 11-19.

[46] Albert Wilansky. Topology for Analysis. Ginn, (1970).