NON-STANDARD ANALYSIS

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ABSTRACT

This thesis is a study of several theories of Non-standard Analysis. Particular attention is paid to the theories presented by A. Robinson and E. Zakon.

Chapter I contains background information from Mathematical Logic and leads to the definition of a Non-standard Model of Analysis.

In Chapter II, we develop the direct product, the ultraproduct and the reduced ultraproduct of a set of similar structures and "construct" a non-standard model of analysis in the form of a reduced ultrapower of the set of real numbers. This model contains genuine "infinite" and "infinitesimal" elements which behave like those which we informally think of in classical analysis.

Chapter III contains the theory of Professor Abraham Robinson for first order structures and languages. The Finiteness Principle is applied in the proof of the existence of Non-standard Models of Analysis.

Chapter IV contains the theory of Non-standard Analysis presented by Professor Elias Zakon. This is the main chapter in the paper. His set-theoretical approach is based on the notion of a superstructure which contains all of the set-theoretical "objects" which exist on a set of individuals. A monomorphism is a one-to-one mapping from one superstructure into another superstructure which preserves the validity of sentences. The existence of monomorphisms is proven using ultrapowers. A Non-standard Model of Analysis is defined in terms of a monomorphism. This definition parallels the one given in Chapter I.

In Chapter V we define and prove the existence of an Extra-standard Model of Analysis, a concept which is similar to that of a Non-standard Model of Analysis. We also present Professor Robinson's theory for higher order structures and languages. We compare the theories presented by Professors Robinson and Zakon along with that of Professor M. Shimrat.

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CHAPTER I

FIRST ORDER STRUCTURE AND LANGUAGE

MODEL

This chapter contains definitions which lead to the definition of a non-standard model.

Definition

A first order structure consists of a set of individuals and, for each $n \geq 1$, a set P_n of n-ary relations such that if R' is an n-ary relation and $(a_1, \ldots, a_n)$ is an n-tuple of individuals, then either $R'(a_1, \ldots, a_n)$ holds (is true) in the structure or does not hold in the structure.

In order to discuss a structure, we need a language.

Definition

A first order language consists of

A. Atomic Symbols.

(i) Individual object symbols or constants usually denoted by the letter i with subscripts which are lower or upper case letters of the alphabet. e.g. i_a, i_N. The set of constants is arbitrary, but fixed.

(ii) Individual variables, denoted by lower case letters from the end of the alphabet. e.g. u, v, w, x. The set of variables is supposed to be infinite, but countable.

(iii) Relation symbols of order n for each $n \geq 1$, where n is the number of empty places in the symbol. e.g. R( ) and

S(, ,) are of order 1 and 3 respectively. Each set of n-ary relation symbols is of arbitrary, but specified cardinal. A first order language always contains the binary relation symbol, = (equality).

(iv) The connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\Rightarrow$ (implies), $\Leftrightarrow$ (if and only if).

(v) The universal quantifier $\forall$ and the existential quantifier $\exists$.

(vi) The brackets [and].

Functional symbols are not considered to be in the language. Functions are represented in a structure by relations. For example, a function $f(x) = y$ will be represented by a relation $S(x, y)$ defined by $S(a, b)$ if and only if $f(a) = b$.

B. Atomic Formulae are obtained by filling the empty places in relation symbols with individual constants or variables. e.g. $R(x)$, $S(i_a, i_b, y)$.

C. Well-formed Formulae are defined inductively as follows:

If X is an atomic formula, then [X] is a well-formed formula.

If X is a well-formed formula, then $[\neg X]$ is a well-formed formula.

If X and Y are well-formed formulae, then $[X \vee Y]$, $[X \wedge Y]$, $[X \Rightarrow Y]$ and $[X \Leftrightarrow Y]$ are well-formed formulae.

If X is a well-formed formula, then $[(\forall y)X]$ and $[(\exists y)X]$

are well-formed formulae, provided X does not contain either $(\forall y)$ or $(\exists y)$.

Note that the symbols X and Y are not in the language. Rather, they each represent a collection of symbols which is in the language.

Definition

An occurrence of a variable is called free if it is not in the scope of any quantifier over the same variable. Otherwise, it is a bound occurrence of the variable.

In $[(\exists x)Z]$, Z is the scope of the quantifier.

Definition

A well-formed formula which does not contain any free occurrences of variables is called a sentence. Otherwise, the formula is called a predicate.

For example, $[(\exists x)\ [\neg\ [x = i_a]]]$ is a sentence and $[(\forall x)\ [\neg\ [x = i_a] \Rightarrow [x = y]]]$ is a predicate.

Instead of considering a language and then studying structures for it, we will assume that, for a structure M, we have a language L which is "large enough" to contain a distinct name for each element of M. That is, we have a one-to-one mapping from the structure to the language so that each individual a in M has associated with it an individual constant i_a in L and each n-ary relation R' in M has associated with it an n-ary relation symbol R in L. We say that i_a denotes a and R denotes R'.

Definition

Let X be a sentence in the language L. X is said to be defined in M if each constant and n-ary relation symbol occurring in X denotes an individual or an n-ary relation respectively in M.

A sentence in L which is defined in M may or may not be true in M according to the rollowing rules:

(i) Let Y be an atomic formula. X = [Y] is a sentence where Y is of the form $R(i_{a_1}, \ldots, i_{a_n})$. R is an n-ary relation symbol and $i_{a_1}, \ldots, i_{a_n}$ are all constants.

Since X is defined in M, R denotes an n-ary relation R' in M and $i_{a_1}, \ldots, i_{a_n}$ denote individuals $a_1, \ldots, a_n$ in M respectively. X holds in M if and only if $R'(a_1, \ldots, a_n)$ holds in M. Either $R'(a_1, \ldots, a_n)$ holds in M or does not hold in M. If X does not hold in M, we say that X is false in M.

(ii) Let $X = [\neg Y]$ be defined in M. Then Y is also defined in M and X holds in M if and only if Y does not hold in M.

(iii) If $X = [Y \vee Z]$ is defined in M, then X holds in M if and only if at least one of Y and Z holds in M.

(iv) If $X = [Y \wedge Z]$ is defined in M, then X holds in M if and only if both Y and Z hold in M.

(v) If $X = [Y \Rightarrow Z]$, then X holds in M, if and only if Z holds in M or, whenever Z does not hold in M, then also Y does not hold in M.

(vi) If $X = [Y \Leftrightarrow Z]$, then X holds in M if and only if both Y and Z hold in M or both Y and Z do not hold in M.

(vii) If $X = [(\exists y)\, Z(y)]$ is defined in M, then X holds in M if and only if there exists an individual constant i_a such that $Z(i_a)$ holds in M. The constant i_a denotes an individual a of M and $Z(i_a)$ is the sentence obtained by replacing each occurrence of y in Z by i_a. If y does not occur in Z, then X holds in M if and only if Z holds in M.

(viii) If $X = [(\forall y)\ Z(y)]$ is defined in M, then X holds in M if and only if $Z(i_a)$ holds in M for every constant i_a in L which denotes an individual a of M.

Definition

Suppose X is a sentence in L which is defined in the structure M. If X is true in M, then M is a model of X. Similarly, if K is a set of sentences and if each sentence of K is true in a structure M, then M is a model of K.

Definition

Suppose that M and M' are structures. M' is called an elementary extension of M if for every sentence X defined in M, X is also defined in M' and X is true in M if and only if X is true in M'.

Let R be the set of real numbers. Consider the first order structure consisting of all real numbers and all n-ary relations of real numbers. By convention, we also use the letter R to denote this structure. Suppose that L is a first order language containing a name for each real number and a name for each n-ary relation of real numbers. Let K be the set of sentences in L which are defined in R and let K' be the set of sentences of K which are true in R. Then certainly R is a model of K'.

Any model M of K' is an elementary extension of R. Indeed, suppose that X is a sentence which is defined in R. If X is true in R, and if M is a model of K', then by the definition of model, X is true in M. Now, suppose that X is true in M. Either X is true in R or X is false in R. If X is false in R, then the sentence $[\neg X]$ is true in R and is therefore in K'. Hence, the sentence $[\neg X]$ is true in M, but this is a contradiction. Therefore, X is true in R.

Definition

Suppose that M and M' are structures such that M' is an elementary extension of M. The set K of sentences which are defined in a first order language and are true in M will contain a distinct name to denote each individual of M. Since M' is a model of K, M' will contain an individual to correspond to each individual of M. If this copy in M' of the set of individuals of M is a proper subset of the set of individuals of M', then M' is a proper elementary extension of M.

Definition

A proper elementary extension of a structure M is called a non-standard model of M. In particular, a proper elementary extension of the structure consisting of the set of all real numbers and the sets of n-ary relations of real numbers is called a non-standard model of analysis.

Definition

If all of the propositional connectives, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$ in a well-formed formula are in the scope of each of the quantifiers, then the formula is in prenex normal form.

In general, a well-formed formula is not in prenex normal form.

That is, all of the quantifiers do not occur at the beginning of the formula. But, for every sentence X, there exists an equivalent sentence X' which is in prenex normal form where two sentences X and X' are equivalent if they contain the same individual constants and relation symbols and if $[X \Leftrightarrow X']$ holds in any structure in which X is defined. The procedure for obtaining the prenex normal form of a well-formed sentence involves "factoring" the quantifiers "out of" the sentence using such well-known equivalences as $[\neg [(\exists y)\ Z]]$ to $[(\forall y)\ [\neg Z]]$, $[\neg [(\forall y)\ Z]]$ to $[(\exists y)\ [\neg Z]]$, $[[(\forall x)\ Z] \Rightarrow [X]]$ to $[(\exists x)\ [Z \Rightarrow X]]$, etc. The steps of this process are often called prenex "reductions". In carrying them out, one has to observe simple cautions to avoid "collisions" of bound variables. For example, to "factor out" the inner $\forall x$ quantifier in $[(\forall x)\ [[(\forall x)\ Z] \Rightarrow [X(x)]]]$, change first to the equivalent $[(\forall y)\ [[(\forall x)\ Z] \Rightarrow [X(y)]]]$, and then factor to get $[(\forall y)\ [(\exists x)\ [[Z(x)] \Rightarrow [X(y)]]]]$.

CHAPTER II

A NON-STANDARD MODEL OF ANALYSIS

Suppose $\{M_\lambda\}_{\lambda \varepsilon I}$ is a set of similar structures. That is, the same relations and functions are defined in each structure. The index set I is non-empty and may be finite or infinite.

Definition

The direct product of the structures M_λ, denoted $\prod_{\lambda \varepsilon I} M_\lambda$, is the set of all functions f with domain I such that $f(\lambda) \varepsilon M_\lambda$ for each $\lambda \varepsilon I$.

If a is an individual contained in each of the structures, then the constant function $f(\lambda) = a$ for all $\lambda \varepsilon I$ is identified with a.

Suppose $R(x_1, \ldots, x_n)$ is an n-ary relation in each of the structures M_λ. (Variables are placed in relations and functions for easier reading). Let $f_1, \ldots, f_n$ be elements of $\prod_{\lambda \varepsilon I} M_\lambda$. Then, we define that $R(f_1, \ldots, f_n)$ holds in $\prod_{\lambda \varepsilon I} M_\lambda$ if and only if $R(f_1(\lambda), \ldots, f_n(\lambda))$ holds in M_λ for each $\lambda \varepsilon I$.

As we commented earlier, there is no need to consider that functions, as distinct from relations are defined in our structures. However, if the definition of relations on the direct product is interpreted in the case of functions, the result is the following:

For any function $\Phi(x_1, \ldots, x_n)$ which is interpretable in each of the structures M_λ, $\Phi(f_1, \ldots, f_n)$ is interpreted in $\prod_{\lambda \varepsilon I} M_\lambda$ by the function from I into the union $\bigcup_{\lambda \varepsilon I} M_\lambda$ which has the value $\Phi(f_1(\lambda), \ldots , f_n(\lambda))$ for each $\lambda \varepsilon I$.

An idea closely related to the direct product is that of a "reduced" direct product, for whose definition we require the following new concepts.

Definition

A filter F on a non-empty set J is a non-empty family of subsets of J with the following properties:

(i) $\phi \notin F$.

(ii) If $A \varepsilon F$ and $A \subseteq B \subseteq J$, then $B \varepsilon F$.

(iii) If $A, B \varepsilon F$, then $A \cap B \varepsilon F$.

Definition

An ultrafilter on a set J is a filter F with the additional property: for each $A \subseteq J$, $A \varepsilon F$ if and only if $J - A \notin F$.

Theorem

Every filter on a non-empty set J can be extended to an ultrafilter.

Proof

Let G be the class of all filters on a non-empty set J. G is non-empty since the set consisting of J alone is a filter on J. Define the relation $\leq$ on G by $F_1 \leq F_2$ if and only if $F_1 \subseteq F_2$. Clearly $\leq$ is a partial ordering of G.

Let $\{F_\alpha\}_{\alpha \varepsilon H}$ be a chain of elements of G where H is some index set. Consider $F = \bigcup_{\alpha \varepsilon H} F_\alpha$. F is a filter on J since (i)

$\phi \notin F$. If $\phi \in F$, then $\phi \in F_\alpha$ for some α, but F_α is a filter for each α. (ii) Suppose that $A \in F$ and $A \subseteq B \subseteq J$. $A \in F \Rightarrow A \in F_\alpha$ for some α. $A \subseteq B \subseteq J \Rightarrow B \in F_\alpha$ and therefore $B \in F$. (iii) Suppose that $A, B \in F$. Since $\{F_\alpha\}_{\alpha \in H}$ is a chain, there exists an $\alpha \in H$, say α_0, such that $A \in F_{\alpha_0}$ and $B \in F_{\alpha_0}$. $A \cap B \in F_{\alpha_0}$ since F_{α_0} is a filter. Therefore, $A \cap B \in F$ and we conclude that F is a filter on J.

Since $F = \bigcup_{\alpha \in H} F_\alpha$, $F_\alpha \leq F$ for all $\alpha \in H$. Thus, the chain $\{F_\alpha\}_{\alpha \in H}$ has F as an upper bound. Since $\{F_\alpha\}_{\alpha \in H}$ is an arbitrary chain of elements of G, we have that each chain has an upper bound and, by Zorn's Lemma, G has a maximal element. Let U be a maximal element of G. Therefore, U is a filter on J. We want to show that U also satisfies the condition: $\forall A \subseteq J$, $A \in U \Leftrightarrow J - A \notin U$. Now, A and $J - A$ cannot both belong to U since their intersection is empty. Suppose that for some $A \subseteq J$, $A \notin U$ and $J - A \notin U$. Let $V = \{B \mid B \subseteq J \text{ and } A \cup B \in U\}$. V is non-empty since $A \cup (J - A) = J \in U$. V is a filter on J since (i) $A \notin U \Rightarrow \phi \notin V$. (ii) If $B_1 \in V$ and $B_1 \subseteq B_2 \subseteq J$, then $A \cup B_1 \in U$ and $A \cup B_1 \subseteq A \cup B_2$. U is a filter implies $A \cup B_2 \in U$. Therefore $B_2 \in V$. (iii) If $B_1, B_2 \in V$, then $A \cup B_1$ and $A \cup B_2$ are elements of U. Therefore, $(A \cup B_1) \cap (A \cup B_2) = A \cup (B_1 \cap B_2) \in U$. Thus, $B_1 \cap B_2 \in V$.

But, U is a proper subset of V. Indeed, let $B \in U$. $B \subseteq A \cup B \Rightarrow A \cup B \in U \Rightarrow B \in V$. As we have seen above, $J - A \in V$, but $J - A \notin U$ by assumption. U being a proper subset of V contradicts the maximal property of U.

We conclude that for every $A \subseteq J$, $A \in U$ if and only if $J - A \notin U$. Thus, U is an ultrafilter. If F is a filter on J, then either F is an ultrafilter or F is contained in some ultrafilter on J.

Suppose that F is a filter on the index set I and that U is an ultrafilter containing F. Using U, we modify the direct product of the structures M_λ.

Definition

Suppose $R(x_1, \ldots, x_n)$ is an n-ary relation in each of the structures M_λ. Let $f_1, \ldots, f_n$ be elements of $\prod_{\lambda \varepsilon I} M_\lambda$. We now define $R(f_1, \ldots, f_n)$ holds in $\prod_{\lambda \varepsilon I} M_\lambda$ if and only if $\{\lambda \varepsilon I \mid R(f_1(\lambda), \ldots, f_n(\lambda))$ holds in $M_\lambda\} \varepsilon U$. Functions are interpreted as in the direct product. The structure obtained in this way is called an ultraproduct.

In particular, since each relation defined in the M_λ's is defined in the ultraproduct, and since each M_λ has the identity relation defined in it, we have a corresponding equivalence between any two elements f and g of the ultraproduct. f '=' g if and only if $\{\lambda \varepsilon I \mid f(\lambda) = g(\lambda)\} \varepsilon U$. This is an equivalence relation for we see that if f '=' g and g '=' h, then f '=' h since $\{\lambda \varepsilon I \mid f(\lambda) = h(\lambda)\} \supseteq \{\lambda \varepsilon I \mid f(\lambda) = g(\lambda)\} \cap \{\lambda \varepsilon I \mid g(\lambda) = h(\lambda)\}$.

Having distinguished the equivalence relation '=' from the symbol for logical identity, =, used in it's definition, we will now abandon the distinguished notation in favor of =, which is in fact to be interpreted by logical identity of equivalence classes, anyway, as follows:

Definition

The reduced ultraproduct of the structures M_λ, denoted by $M' = (\prod_{\lambda \varepsilon I} M_\lambda)_U$ is the set of equivalence classes under the equivalence relation $=$, (or '=').

If a is an individual contained in each structure M_λ, then the equivalence class of M' containing the constant function $f(\lambda) = a$ for all $\lambda \varepsilon I$ is identified with a. The reader should note how the properties of the ultrafilter are involved in what follows next.

Now, suppose $R(x_1, \ldots, x_n)$ is an n-ary relation in each of the structures M_λ. Let $\bar{f}_1, \ldots, \bar{f}_n$ be elements of M'. Then, we define $R(\bar{f}_1, \ldots, \bar{f}_n)$ holds in M' if and only if $\{\lambda \varepsilon I \mid R(f_1(\lambda), \ldots, f_n(\lambda))$ holds in $M_\lambda\} \varepsilon U$, where $f_1, \ldots, f_n$ are representatives of the equivalence classes $\bar{f}_1, \ldots, \bar{f}_n$ respectively.

This is well-defined since if $f_1', \ldots, f_n'$ are representatives of $\bar{f}_1, \ldots, \bar{f}_n$ respectively, then $\{\lambda \varepsilon I \mid R(f_1'(\lambda), \ldots, f_n'(\lambda))$ holds in $M_\lambda\} \supseteq \{\lambda \varepsilon I \mid R(f_1(\lambda), \ldots, f_n(\lambda))$ holds in $M_\lambda\} \cap$ $\{\lambda \varepsilon I \mid f_1(\lambda) = f_1'(\lambda)\} \cap \ldots \cap \{\lambda \varepsilon I \mid f_n(\lambda) = f_n'(\lambda)\}$.

For any function $\Phi(x_1, \ldots, x_n)$ which is interpretable in each of the structures M_λ, we define $\Phi(x_1, \ldots, x_n)$ in M' by $\Phi(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\Phi(f_1(\lambda), \ldots, f_n(\lambda))}$, $\lambda \varepsilon I$.

That is, the value of Φ in M' is the equivalence class of the function defined by values on the right, where $f_1, \ldots, f_n$ are representatives of the equivalence classes $\bar{f}_1, \ldots, \bar{f}_n$ respectively.

To show this is well-defined, suppose that

$f_1 = f_1', \ldots, f_n = f_n'$ and that $\Phi(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\Phi(f_1(\lambda), \ldots, f_n(\lambda))}$, $\lambda \in I$.

Now, $\{\lambda \in I \mid f_1(\lambda) = f_1'(\lambda)\} \cap \ldots \cap \{\lambda \in I \mid f_n(\lambda) = f_n'(\lambda)\} \subseteq \{\lambda \in I \mid \Phi(f_1(\lambda), \ldots, f_n(\lambda)) = \Phi(f_1'(\lambda), \ldots, f_n'(\lambda))\}$. This set is an element of U since each member of the intersection (which is finite) is an element of U. Therefore, $\overline{\Phi(f_1'(\lambda), \ldots, f_n'(\lambda))} = \overline{\Phi(f_1(\lambda), \ldots, f_n(\lambda))}$. Thus, $\Phi(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\Phi(f_1'(\lambda), \ldots, f_n'(\lambda))}$, $\lambda \in I$.

Let us consider the following example of a reduced ultraproduct. Let I be the set of all prime numbers. For each $p \in I$, let Z_p be the finite field with elements $0, \ldots, p-1$ where addition, $\oplus$, and multiplication, $\odot$, are defined by

for any $a, b \in Z_p$, $a \oplus b$ = remainder when p divides $a + b$.

$a \odot b$ = remainder when p divides $a \cdot b$.

The set $\{Z_p\}_{p \in I}$ is a set of similar structures. The index set I is infinite. Let F be the Fréchet filter on I. That is, for every $A \subseteq I$, $A \in F$ if and only if $I - A$ is finite. We can extend F to an ultrafilter U on I.

As in the general case, we obtain the reduced ultraproduct $Z' = (\prod_{p \in I} Z_p)_U$. The elements of Z' are equivalence classes under the equivalence relation $\equiv$. That is, if f and g are functions such that $f(p) \in Z_p$ and $g(p) \in Z_p$ for each $p \in I$, then $f \equiv g$

if and only if $\{p \in I \mid f(p) = g(p)\} \in U$.

Since the structures are finite fields, each has additive identity 0 and multiplicative identity 1. We identify with 0, the equivalence class of Z' containing the constant function $f(p) = 0$ for all $p \in I$. Let us denote this equivalence class by $\bar{0}$. Similarly, we obtain $\bar{1}$.

Each structure has addition and multiplication defined in it. Suppose that $\bar{f}_1$ and $\bar{f}_2$ are elements of Z'. Then, following the general case we have $\bar{f}_1 + \bar{f}_2 = \overline{f_1(p) \oplus f_2(p)}$, $p \in I$. That is, for each $p \in I$, $f_1(p)$ and $f_2(p)$ are in Z_p and $f_1(p) \oplus f_2(p)$ is defined in Z_p and takes a value in Z_p. These values, for each $p \in I$, define a function with domain I. The value of $\bar{f}_1 + \bar{f}_2$ is the equivalence class containing this function. Similarly we have that $\bar{f}_1 \cdot \bar{f}_2 = \overline{f_1(p) \odot f_2(p)}$, $p \in I$. This ends the example.

Theorem

Let X be a sentence which is defined and holds in each structure of a set $\{M_\lambda\}_{\lambda \in I}$ of similar structures and let U be an ultrafilter on the index set I. Then X holds in the reduced ultraproduct $M' = (\prod_{\lambda \in I} M_\lambda)_U$.

The proof is very direct. The properties of the ultrafilter, and the definitions above ensure the result for atomic sentences, those without quantifiers or connectives. The result for arbitrary sentences is established by induction. The proof is omitted.

Let us consider an application of this theorem. The following sentences characterize a field:

$$(\forall x)\ (\forall y)\ (\forall z)\ [(x + y) + z = x + (y + z)].$$

$$(\exists y)\ [y = 0 \wedge (\forall x)\ [x + 0 = x]].$$

$$(\forall x)\ [(\exists y)\ [y = -x \wedge [x + (-x) = 0]]].$$

$$(\forall x)\ (\forall y)\ [x + y = y + x].$$

$$(\forall x)\ (\forall y)\ (\forall z)\ [(x \cdot y) \cdot z = x \cdot (y \cdot z)].$$

$$(\exists y)\ [y = 1 \wedge (\forall x)\ [x \cdot 1 = x]].$$

$$(\forall x)\ [x \neq 0 \Rightarrow (\exists y)\ [x \cdot y = 1]].$$

$$(\forall x)\ (\forall y)\ [x \cdot y = y \cdot x].$$

$$(\forall x)\ (\forall y)\ (\forall z)\ [x \cdot (y + z) = (x \cdot y) + (x \cdot z)].$$

$$0 \neq 1.$$

Each of these sentences is true in each finite field Z_p of our previous example. Thus, by applying the above theorem, we obtain that each of these sentences is also true in the reduced ultraproduct Z'. Therefore, Z' is a field. The additive identity of the field Z' is $\bar{0}$ and the multiplicative identity is $\bar{1}$. For example, if $\bar{f}$ is any element of Z', then, by definition,

$$\bar{f} + \bar{0} = \overline{f(p) \oplus 0},\ p \in I$$

$$= \bar{f}$$

Definition

If the elements of the set $\{M_\lambda\}_{\lambda \in I}$ are all the same structure M, then M' is called a reduced ultrapower of M.

Certainly the above theorem still holds. That is, if X is a sentence which is defined and holds in a structure M, then X holds in a reduced ultrapower of M. Note also that a reduced ultrapower of a structure M is an elementary extension of M.

Now, we will develop a reduced ultrapower non-standard model of analysis. Our structure is the set of real numbers R and the index set is the set of natural numbers $N = \{0, 1, 2, \ldots\}$. Let F be the Fréchet filter on N. That is, for every $A \subseteq N$, $A \varepsilon F$ if and only if $N - A$ is finite. Let U be an ultrafilter on N containing F. Let ${}^*R = (\prod_{n\varepsilon N} R_n)_U$ be the reduced ultrapower of R where $R_n = R$ for all $n \varepsilon N$. Therefore, ${}^*R$ is the set of equivalence classes of functions $f\colon N \rightarrow R$ under the equivalence relation $=$. That is, for any functions $f, g\colon N \rightarrow R$, $f = g$ if and only if $\{n \varepsilon N \mid f(n) = g(n)\} \varepsilon U$.

Since ${}^*R$ is a reduced ultrapower of R, ${}^*R$ is an elementary extension of R. For ${}^*R$ to be a non-standard model of analysis, we require that ${}^*R$ be a proper elementary extension of R. We do have this property. Indeed, consider the function $f\colon N \rightarrow R$ defined by $f(n) = n$ for all $n \varepsilon N$. This function belongs to an element, say $\bar{f}$, of ${}^*R$. Recall, that identified with each $k \varepsilon N$ is the equivalence class $\bar{k}$ containing the constant function $h(n) = k$ for all $n \varepsilon N$. In this way, N is embedded in ${}^*R$. We will show now that $\bar{f} \notin N$. That is, $\bar{f}$ is not in the copy of N in ${}^*R$. For every $k \varepsilon N$, $\{n \varepsilon N \mid n \leq k\}$ is finite. Therefore, its complement $\{n \varepsilon N \mid n > k\}$ is a member of the ultrafilter U. This implies that $\bar{f} > \bar{k}$ in ${}^*R$. ${}^*N$ is the set of natural numbers in ${}^*R$ where $\bar{g} \varepsilon {}^*N$ if an only if $\{n \varepsilon N \mid g(n) \varepsilon N\} \varepsilon U$ for $g \varepsilon \bar{g}$. Then certainly $\bar{f} \varepsilon {}^*N$ since $\{n \varepsilon N \mid f(n) \varepsilon N\} = N \varepsilon U$. Therefore,

$\bar{f} \in {}^*N - N$.

We have an element of ${}^*R$ which is not in R. Indeed, the only way that $\bar{f}$ could be an element of the copy of R in ${}^*R$ would be for $\bar{f}$ not being a natural number. We know that $\bar{f}$ is a natural number in ${}^*R$. ${}^*R$ is a proper elementary extension of R and therefore ${}^*R$ is a non-standard model of analysis.

Let us examine ${}^*R$ more closely. Note that $\bar{f}$ is larger in ${}^*R$ than every ordinary real number. We say that $\bar{f}$ is an "infinite" number. Since R is a field and ${}^*R$ is an elementary extension of R, ${}^*R$ is also a field. That is, the field axioms also hold in ${}^*R$. The additive identity of ${}^*R$ is $\bar{0}$ and since $\bar{f} \neq \bar{0}$, $\bar{f}$ has multiplicative inverse $\frac{1}{\bar{f}}$. The element $\frac{1}{\bar{f}}$ is called "infinitesimal" since $\frac{1}{\bar{f}} < \frac{1}{\bar{n}}$ for every positive integer n.

Since $\bar{f}$ is "infinite", $\bar{f} + \bar{r}$ and $\bar{f} - \bar{r}$ are "infinite" for every $r \in R$. Also, $\bar{n} \cdot \bar{f}$ is infinite for every positive $n \in N$. Thus, in ${}^*R$ we have uncountably many "infinite" numbers and "infinitesimals".

Note that ${}^*R$ is non-Archimedean since $\bar{f} > \bar{k}$ for every ordinary natural number k. This does not contradict ${}^*R$ being an elementary extension of R since the fact that R is Archimedean cannot be written as a sentence in a first order language. That is, the Archimedean principle states that if $0 < a < b$, then there exists a natural number n such that $n \cdot a > b$ and in a first order language we cannot mention anything except individuals. Since the individuals in the structure R are real numbers, we cannot mention sets of real numbers. Therefore, we cannot assert the existence of such an element in N, the set of natural numbers.

In classical analysis we speak of "infinite" and "infinitesimal" numbers although they do not exist. An "infinite" number is used in the sense of being "arbitrarily large" and an "infinitesimal" as being "arbitrarily close to zero". The "infinite" and "infinitesimal" numbers which exist in non-standard models of analysis actually "behave" like the ones we informally thought of in classical analysis. This leads to the possibility of doing calculus in $^*R$ using infinitesimals.

For example, in classical analysis, a function f which is defined on the interval (a, b) is continuous at x_0 in (a, b) if and only if for every positive number ε there exists a number δ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Now, in $^*R$, our non-standard model of analysis, let the extensions of the function f and the absolute value function be denoted by the same symbols as in R. Suppose that a, b and x_0 are in the copy of R in $^*R$ and that x_0 is in the interval (a, b). Then, f is continuous at x_0 if and only if $|f(x) - f(x_0)|$ is infinitesimal whenever $|x - x_0|$ is infinitesimal. With definitions like this, corresponding to the definitions in classical analysis, we can do calculus in $^*R$.

Now let us examine the number of non-standard models of analysis that we can obtain using an ultrafilter on the set of natural numbers N. First, how many ultrafilters exist on the set of natural numbers? The following theorem is the work of M. Shimrat.

<u>Theorem</u>

The number of ultrafilters on $N = \{0, 1, 2, \ldots\}$ is $\geq 2^{\aleph_1}$.

Proof

To every infinite ordinal $\alpha < \omega_1$ assign a particular rearrangement of $W_\alpha = \{\beta \mid \beta < \alpha\}$ into a ω-sequence. That is, a sequence $(\gamma_n)_{n<\omega}$ containing each $\beta < \alpha$ exactly once.

Consider any mapping Φ: $W_{\omega_1} \to \{0, 1\}$. That is, an ω_1-sequence of 0's and 1's. The set of all such Φ is of cardinality $2^{\aleph_1}$. We will assign to each Φ a filter base on $\mathbb{N}$ given by a certain ω_1-sequence $(F_\alpha)_{\alpha<\omega_1}$ of subsets of $\mathbb{N}$. The sets F_α are defined by transfinite induction as follows:

Suppose Φ: $W_{\omega_1} \to \{0, 1\}$.

$$\text{Define } F_0 = \begin{cases} \{0, 2, 4, \ldots\} & \text{if } \Phi(0) = 0 \\ \{1, 3, 5, \ldots\} & \text{if } \Phi(0) = 1 \end{cases}$$

Now, assume for any positive $\alpha < \omega_1$, the folloiwng condition, denoted C_α, is satisfied:

F_ξ has been defined for all $\xi < \alpha$ and

$\xi_1 < \xi_2 < \alpha \Rightarrow F_{\xi_2} - F_{\xi_1}$ is finite.

There are two cases to consider. First, suppose that α is a successor ordinal, $\alpha = \beta + 1$. Then F_β has been defined. Let $F_\beta = \{k_0, k_1, k_2, \ldots\}$ where the elements $k_0, k_1, \ldots$ are in increasing order.

$$\text{Define } F_\alpha = \begin{cases} \{k_0, k_2, k_4, \ldots\} & \text{if } \Phi(\alpha) = 0 \\ \{k_1, k_3, k_5, \ldots\} & \text{if } \Phi(\alpha) = 1 \end{cases}$$

Therefore, F_ξ has been defined for all $\xi < \alpha + 1$ and $\xi_1 < \xi_2 < \alpha + 1 \Rightarrow F_{\xi_2} - F_{\xi_1}$ if finite since $F_\beta - F_\alpha$ is void. Hence, condition $C_{\alpha+1}$ is satisfied.

Now, suppose that α is a limit ordinal. Rearrange the F_ξ into an ω-sequence (G_n) where $G_n = F_{\gamma_n}$ (with the γ_n as above). Define $H_n = \bigcap_{i=0}^{n} G_i$. By the condition C_α, in constructing H_{n+1} for any $n \geq 0$, we have removed only a finite number of elements from H_n and since $H_0 = F_0$ is infinite, each H_n is infinite. Clearly $H_n \supseteq H_{n+1}$. Let $H_n = \{k_{n0}, k_{n1}, k_{n2}, \ldots\}$ where the elements $k_{n0}, k_{n1}, k_{n2}, \ldots$ are increasing in order.

Now, define $F_\alpha = \begin{cases} \{k_{00}, k_{22}, k_{44}, \ldots\} & \text{if } \Phi(\alpha) = 0 \\ \{k_{11}, k_{33}, k_{55}, \ldots\} & \text{if } \Phi(\alpha) = 1 \end{cases}$

F_ξ has now been defined for all $\xi < \alpha + 1$. If $\beta < \alpha$ then $F_\alpha - F_\beta$ is finite. Indeed, since there is an n for which $H_\ell \subseteq F_\beta$ for all $\ell \geq n$, and by the definition of F_α, $F_\alpha - F_\beta$ contains a finite number of elements. Therefore, if $\xi_1 < \xi_2 < \alpha + 1$, then $F_{\xi_2} - F_{\xi_1}$ is finite and condition $C_{\alpha+1}$ is satisfied.

Hence, for each Φ we have defined an ω_1-sequence of subsets of N and each such sequence of subsets of N is a filter base. Now, consider the $2^{\aleph_1}$ filters generated by these filter bases. If Φ and Φ' are distinct functions with corresponding filters $\mathcal{F}$ and $\mathcal{F}'$ respectively, then $\mathcal{F}$ and $\mathcal{F}'$ are incompatible. That is, $\mathcal{F}$ and $\mathcal{F}'$ have elements K and K' respectively such that $K \cap K' = \phi$. Choose for each filter $\mathcal{F}$ an ultrafilter containing $\mathcal{F}$. We obtain a set of distinct ultrafilters of cardinality $2^{\aleph_1}$ which proves the theorem.

Now, how many distinct reduced ultrapowers of R can be obtained by considering distinct ultrafilters on the index set N? W. A. J. Luxemburg in [3] remarked that under the continuum hypothesis, all reduced ultrapowers of R obtained by using ultrafilters on N can be shown to be isomorphic. Therefore, there are at least $2^{\aleph_1}$ isomorphic reduced ultrapowers of R using the set of natural numbers as the index set.

CHAPTER III

FINITENESS PRINCIPLE

In Chapter II, we developed a non-standard model of analysis in the form of a reduced ultrapower. In this chapter, we will prove the existence of non-standard models of analysis without "constructing" one as we did in Chapter II.

An important theorem which will be applied in this chapter is the Finiteness Principle for first order languages and structures. This theorem is also called the Compactness Theorem. A proof is available in [6].

Theorem (Finiteness Principle)

Let K be a set of sentences. If every finite subset of K has a model, then K has a model.

Here, K is a set of sentences in a first order language and the models are first order structures.

Let R be the set of real numbers. Let K be the set of sentences in a first order language L which contains a distinct individual constant to denote each real number and a distinct n-ary relation symbol to denote each n-ary relation of real numbers. Again, we assume that we have a one-to-one mapping from our structure to the language. Let K' be the subset of K consisting of all sentences of K which are true in R.

Choose an individual constant b of L which does not denote any real number in R. That is, $b \neq i_r$ for every real number r. We assume that our language L contains "enough" constants to allow this, where "enough" means that, if necessary, we could embed our language L in a language L' so that the set of atomic symbols of L is a subset of the set of atomic symbols of L' and L' has the individual constant we want. Since we could then consider the language L', we may as well assume that L has such a constant. The set R of real numbers has the binary relation of logical identity defined on it. This relation is denoted in L by the symbol =. Therefore, the sentence $[i_c = i_d]$ in L holds in R if and only if $c = d$ where i_c denotes $c \in R$ and i_d denotes $d \in R$.

Let H be the set of sentences $\{[\neg [b = i_r]] \mid r \in R\}$. Here we intend that r shall range over all real numbers and i_r denotes r in the language L. Consider the set of sentences $K' \cup H$. We want to show that $K' \cup H$ has a model, say M. This model M will be a non-standard model of analysis since every sentence in K' will be true in M and M will contain an element corresponding to b.

To apply the Finiteness Principle, we need to prove that every finite subset of $K' \cup H$ has a model. Consider $K' \cup H'$ where H' is any finite subset of H. If $K' \cup H'$ has a model, then it can be shown that every finite subset of $K' \cup H$ has a model. If H' is an empty set of sentences, then $K' \cup H'$ has R as a model. Therefore, let us suppose that H' consists of the sentences $[\neg [b = i_{r_1}]], \ldots ,$ $[\neg [b = i_{r_n}]]$ where $n \geq 1$.

For any finite set of real numbers $r_1, \ldots, r_n$ we can find a real number s such that $s \neq r_k$ for all $k = 1, \ldots, n$. Therefore, R is a model of H'. Since R is certainly a model of K', R is a model of $K' \cup H'$.

Now, by the Finiteness Principle, $K' \cup H$ has a model, say $^*R$. This model $^*R$ contains an individual, say a, which is unequal (in $^*R$) to each individual of $^*R$ corresponding to an individual of R. Therefore, $^*R$ is a non-standard model of analysis.

Since R is a field and $^*R$ is a model of R, the field axioms still hold in $^*R$. Therefore, from this element a we obtain the multiplicative inverse of a, $\frac{1}{a}$ and the additive inverse of a, $-a$. Recall that in Chapter II, we obtained a non-standard model of analysis containing the "infinite" element $\bar{f}$. This element $\bar{f}$ is an example of such an element a.

The relation "unequal to" between real numbers is a particular kind of relation called "concurrent" since b is unequal to r_k concurrently for $k = 1, \ldots, n$. For the purpose of the above theorem, it was sufficient to consider this one "concurrent" binary relation. We could consider any number of concurrent relations simultaneously. We will define concurrent relation in Chapter IV and again in Chapter V and will indicate in Chapter V a proof in which more than one concurrent relation is considered simultaneously.

CHAPTER IV

SUPERSTRUCTURES AND MONOMORPHISMS

Definition

Let $A = A_0$ be a set of individuals. An individual has no elements, but is not the empty set. In [8], E. Zakon defined the superstructure $\hat{A}$ on A_0 as $\hat{A} = \bigcup_{n=0}^{\infty} A_n$ where A_{n+1} is the set of all subsets of $\bigcup_{k=0}^{n} A_k$. Elements of A_0 are said to be of type 0 and elements of $A_{n+1} - A_n$ are of type $n + 1$.

Since $A_n \subset A_{n+1}$ for $n \geq 1$, we have $\bigcup_{k=0}^{n} A_k = A_0 \cup A_n$ and since $A_n \in A_{n+1} \subseteq \hat{A}$ for $n \geq 0$, $A_n \in \hat{A}$ for all n. Since we may express an ordered pair using the Tarski-Kuratowski definition, $(a, b) = \{\{a, b\}, \{b\}\}$ and hence an ordered n-tuple by $(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n)$, $\hat{A}$ contains all ordered n-tuples of elements of $\hat{A}$. $\hat{A}$ contains n-ary relations (set of n-tuples) where the n-tuples of the n-ary relation all have corresponding places filled by elements from the same set in $\hat{A}$.

For example, let the set R of real numbers be taken as a set of individuals. Then, the superstructure $\hat{R}$ on R contains all real numbers, all sets of real numbers and so on. $\hat{R}$ also contains, for example, sets which have as elements both real numbers and sets of real numbers. The relation of membership of a real number in a set of real numbers exists in $\hat{R}$ as a set of ordered pairs where the first place is filled by real numbers and the second place is filled by sets of real numbers.

In order to discuss a superstructure we need a language. Let L be a first order language with variables, connectives, constants, quantifiers and brackets as defined in Chapter I so that there is a one-to-one correspondence between all constants of L and all elements of $\hat{A}$. Thus, if $\hat{R}$ is the superstructure, there is a constant in L to denote each real number, each set of real numbers and so on. For any element a of a general superstructure $\hat{A}$ let its name in L be i_a. The superstructure has the relations of logical identity and membership. These relations are interpreted in L by the relation symbols = and ε respectively.

Atomic formulae are of the form $x \varepsilon y$ and $x = y$ where x and y are variables or constants (names for elements of $\hat{A}$). Well-formed formulae are constructed from atomic formulae using brackets, connectives and quantifiers. The only quantification allowed is of the form $(\exists x \varepsilon i_C)$ or $(\forall x \varepsilon i_C)$ where i_C is the name of some $C \varepsilon \hat{A}$. A reason for this restriction will be provided later in this chapter. Note that, for example, $[(\exists x \varepsilon i_C)\ x \varepsilon i_D]$ is a simplification of $[(\exists x)\ [x \varepsilon i_C \wedge x \varepsilon i_D]]$.

To illustrate a sentence, the statement that 0 is the least member of the set of natural numbers $N \varepsilon \hat{R}$ is written

$$[[i_0 \varepsilon i_N] \wedge [(\forall x \varepsilon i_N)\ i_0 \leq x]].$$

To simplify matters, we will follow the time-honored mathematical practice of using objects as their own names as in $[(\forall x \in C)\ x \in D]$ rather than $[(\forall x \in i_C)\ x \in i_D]$, although formally, sentences must contain names for the elements and not the elements themselves. This device will be used when there is no chance of confusion, for it increases readability.

Note that in the language L, the assertion $(a_1, \dots, a_m) = b$ is expressible by a well-formed sentence. Indeed, we always have $a_1, \dots, a_m \in A_n \cup A_0$ for a large n so that the assertion $\{a_1, \dots, a_m\} = b$ can be written as $(\forall x \in A_n \cup A_0)\ [x \in b \Leftrightarrow x = a_1 \vee \dots \vee x = a_m]$. Therefore, $\{\{a_1, a_2\}, \{a_2\}\} = b$, i.e., $(a_1, a_2) = b$ can be written as a well-formed sentence. Similarly for $(a_1, \dots, a_m) = b$ by induction.

Now, suppose that A and B are two sets of individuals with superstructures $\hat{A}$ and $\hat{B}$ respectively and let $\Phi\colon \hat{A} \to \hat{B}$ be a mapping of $\hat{A}$ into $\hat{B}$. We define $\Phi(a) = {}^*a$ and $\Phi[a] = \{{}^*x \mid x \in a\}$ for all $a \in \hat{A}$. Note the difference between the definitions of $\Phi(a)$ and $\Phi[a]$.

Since $A_n \in \hat{A}$ for all n, $\Phi(A_n) = {}^*A_n$. We set ${}^*\hat{A} = \bigcup_{n=0}^{\infty} {}^*A_n$.

If α is a well-formed formula, then ${}^*\alpha$ is defined to be the formula obtained from α by replacing each constant a occurring in α by ${}^*a$. Nothing is changed except the elements of $\hat{A}$. ${}^*\alpha$ is called the <u>Φ-transform</u> of α. For example, if $\alpha = [(\forall x \in C)\ x \in D]$, then

$${}^*\alpha = [(\forall x \in {}^*C)\ x \in {}^*D].$$

Again this is a simplification of the following:

If L_A and L_B are first order languages for $\hat{A}$ and $\hat{B}$ respectively and if α is a well-formed formula of L_A, then $^*\alpha$ is the formula of L_B obtained from α by replacing each occurrence in α of the name i_a for each $a \varepsilon \hat{A}$ by the name i_{*a} (in L_B) for $^*a$. Nothing is changed except the names of elements in $\hat{A}$. If $\alpha = [(\forall x \varepsilon i_C)\ x \varepsilon i_D]$, then $^*\alpha = [(\forall x \varepsilon i_{*C})\ x \varepsilon i_{*D}]$.

There is one example which should be noted.

Suppose $\alpha = [(\forall x \varepsilon i_{A_n \cup A_0})[x \varepsilon i_{A_n} \vee x \varepsilon i_{A_0}]]$, then $^*\alpha = [(\forall x \varepsilon i_{*(A_n \cup A_0)})\ [x \varepsilon i_{*A_n} \vee x \varepsilon i_{*A_0}]]$. We have $i_{*(A_n \cup A_0)}$ instead of $i_{*A_n \cup *A_0}$. The simplified case is in agreement. If $\alpha = [(\forall x \varepsilon A_n \cup A_0)\ [x \varepsilon A_n \vee x \varepsilon A_0]]$, then $^*\alpha = [(\forall x \varepsilon {}^*(A_n \cup A_0))$ $[x \varepsilon {}^*A_n \vee x \varepsilon {}^*A_0]]$.

Note that we are just now defining the Φ-transform of a sentence and we are not asserting anything about the truth of a sentence or of its Φ-transform.

Definition

If b is a member of $\hat{B}$ and $b = {}^*a$ for some $a \varepsilon \hat{A}$, then b is called a Φ-standard member of $\hat{B}$.

For example, since $A_n \varepsilon \hat{A}$ for all n, $\Phi(A_n) = {}^*A_n$ is a Φ-standard member of $\hat{B}$ for all n.

Definition

If b is a member of $\hat{B}$ and $b \varepsilon {}^*a$ for some $a \varepsilon \hat{A}$, then b is called a Φ-internal member of $\hat{B}$. That is, if b is an element of a Φ-standard member of $\hat{B}$, then b is a Φ-internal member of $\hat{B}$.

Any member of $\hat{B}$ which is not Φ-internal is called Φ-external.

For example, all elements of $^*\hat{A}$ are Φ-internal, since if $b \in {}^*\hat{A}$, then $b \in {}^*A_n$ for some n and $^*A_n$ is a Φ-standard member of $\hat{B}$.

The terms Φ-standard, Φ-internal and Φ-external are shortened to standard, internal and external respectively.

We will consider a particular kind of mapping between superstructures.

Definition

A mapping $\Phi: \hat{A} \to \hat{B}$ is called a monomorphism if and only if

(i) $\Phi(\phi) = {}^*\phi = \phi$ where ϕ is the empty set.

(ii) $\hat{A} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$ for every well-formed sentence α. That is, if the well-formed sentence α is true in $\hat{A}$, then $^*\alpha$ is true in $\hat{B}$.

We shall always identify $^*a$ with a if $a \in A_0$. Therefore, $^*a = a$ for $a \in A_0$ and $X \subseteq {}^*X$ whenever $X \subseteq (A_0)^n$ for $n \geq 1$.

Theorem Let $\Phi: \hat{A} \to \hat{B}$ be a monomorphism.

If $a, b, a_1, \ldots, a_m \in \hat{A}$, then

(i) $^*(a - b) = {}^*a - {}^*b$

(ii) $^*(a \cap b) = {}^*a \cap {}^*b$

(iii) $^*(a \cup b) = {}^*a \cup {}^*b$

(iv) $^*\{a\} = \{^*a\}$

(v) $^*\{a_1, \ldots, a_m\} = \{^*a_1, \ldots, {}^*a_m\}$

(vi) $^*(a_1, \ldots, a_m) = (^*a_1, \ldots, {}^*a_m)$.

Regarding (i), (ii) and (iii), if a and b are individuals, then $a \cup b = a \cap b = a - b = \phi$ by definition.

Proof

We will prove (iii) and (iv). The other proofs are similar, and illustrate what is perhaps the basic proof technique of non-standard analysis, the construction of appropriate sentences and their transformation under the monomorphism.

(iii) If $a, b \in \hat{A}$, then $a \cup b \in \hat{A}$. Let $\alpha = [(\forall x \in (a \cup b)) [x \in a \vee x \in b]]$. Then $^*\alpha = [(\forall x \in {}^*(a \cup b)) [x \in {}^*a \vee x \in {}^*b]]$. $\hat{A} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$ and we have $^*(a \cup b) \subseteq {}^*a \cup {}^*b$. Now, let $\beta = [(\forall x \in a) (\forall y \in b) [x \in a \cup b \wedge y \in a \cup b]]$. $^*\beta = [(\forall x \in {}^*a) (\forall y \in {}^*b) [x \in {}^*(a \cup b) \wedge y \in {}^*(a \cup b)]]$. $\hat{A} \models \beta \Rightarrow \hat{B} \models {}^*\beta$ and we have $^*a \cup {}^*b \subseteq {}^*(a \cup b)$. Therefore $^*(a \cup b) = {}^*a \cup {}^*b$.

(iv) $a \in \hat{A} \Rightarrow \{a\} \in \hat{A}$. Let $\alpha = [(\forall x \in \{a\}) [x = a]]$. Then $^*\alpha = [(\forall x \in {}^*\{a\}) [x = {}^*a]]$. $\hat{A} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$. Therefore $^*\{a\} = \{^*a\}$ and the proof of (iv) is complete. We can use (iii) and (iv) to prove (v).

$$\{a_1, \ldots, a_m\} = \bigcup_{k=1}^{m} \{a_k\}$$

Therefore $^*\{a_1, \ldots, a_m\} = \bigcup_{k=1}^{m} {}^*\{a_k\} = \bigcup_{k=1}^{m} \{^*a_k\} = \{^*a_1, \ldots, {}^*a_m\}$.

In the previous example, the Φ-transform of the sentence $[(\forall x \in C) x \in D]$ is $[(\forall x \in {}^*C) x \in {}^*D]$. If Φ is a monomorphism, then we have by (ii) of the definition of a monomorphism, if C is a subset of D (in $\hat{A}$), then $^*C$ is a subset of $^*D$ (in $\hat{B}$).

Note that if α is the sentence $[(\forall x \in A_n \cup A_0) [x \in A_n \vee x \in A_0]]$ and Φ is a monomorphism, then by (iii) of the above theorem we can now write

$$^*\alpha = [(\forall x \in {}^*A_n \cup {}^*A_0) [x \in {}^*A_n \vee x \in {}^*A_0]].$$

Earlier, we noted the difference between the definitions of $\Phi(a)$ and $\Phi[a]$ for $a \in \hat{A}$. If Φ is a monomorphism, then by (v) of the above theorem, if C is a <u>finite</u> set in $\hat{A}$, then $\Phi(C) = \Phi[C]$.

<u>Theorem</u>

If $\Phi: \hat{A} \to \hat{B}$ is a monomorphism from $\hat{A}$ into $\hat{B}$, then each standard member of $\hat{B}$ is also an internal member of $\hat{B}$.

<u>Proof</u>

Suppose b is a standard member of $\hat{B}$. That is, $b = {}^*a$ for some $a \in \hat{A}$. Therefore, $\hat{A} \models [a \in A_n]$ for some n and since Φ is a monomorphism, $\hat{B} \models [{}^*a \in {}^*A_n]$. ${}^*a \in {}^*A_n \Rightarrow b \in {}^*A_n \Rightarrow b \in {}^*\hat{A}$. Since members of ${}^*\hat{A}$ are internal, b is an internal member of $\hat{B}$.

Note that a monomorphism is a one-to-one mapping since by (ii) of the definition of a monomorphism, $\hat{A} \models [\neg [a = b]] \Rightarrow \hat{B} \models [\neg [{}^*a = {}^*b]]$, for $a, b \in \hat{A}$.

Also note that although it need not be defined this way we actually have (ii) of the definition of a monomorphism as $\hat{A} \models \alpha \Leftrightarrow \hat{B} \models {}^*\alpha$. Indeed, suppose $\hat{B} \models {}^*\alpha$. Now, either $\hat{A} \models \alpha$ or $\hat{A} \not\models \alpha$. If $\hat{A} \not\models \alpha$, then $\hat{A} \models \neg[\alpha]$. Since $\neg[\alpha]$ is a well-formed sentence, $\hat{A} \models \neg[\alpha] \Rightarrow \hat{B} \models {}^*[\neg[\alpha]]$. In obtaining the Φ-transform of a sentence, we change a to ${}^*a$ for every element a of $\hat{A}$ occurring in the sentence. Nothing else is changed. Therefore, ${}^*[\neg[\alpha]] = \neg[{}^*\alpha]$. Thus, $\hat{A} \models \neg[\alpha] \Rightarrow \hat{B} \models \neg[{}^*\alpha]$. This is a contradiction of $\hat{B} \models {}^*\alpha$. Therefore $\hat{B} \models {}^*\alpha \Rightarrow \hat{A} \models \alpha$ and we have $\hat{A} \models \alpha \Leftrightarrow \hat{B} \models {}^*\alpha$.

If K is the set of sentences which are true in $\hat{A}$ and if Φ

is a monomorphism of $\hat{A}$ into $\hat{B}$, then the sentences of K are also true in $\hat{B}$. Thus, a monomorphism provides us with another model of K. For example, suppose that $A = A_0$ is a set of individuals with superstructure $\hat{A}$ and with a binary operation, $\cdot$, defined on A. Let K be the set of sentences which define $\cdot$ as being a binary operation and which assert that elements of A form a group under the operation. If Φ is a monomorphism from $\hat{A}$ into another superstructure $\hat{B}$, then the sentences of K will also be true in $\hat{B}$. These sentences assert that there is a binary operation, ${}^*\cdot$, defined on ${}^*A_0$ and that the elements of ${}^*A_0$ form a group under ${}^*\cdot$.

<u>Theorem</u>

If Φ is a monomorphism from a superstructure $\hat{A}$ into a superstructure $\hat{B}$, then the internal members of $\hat{B}$ are exactly the elements of ${}^*\hat{A} = \bigcup_{n=0}^{\infty} {}^*A_n$.

<u>Proof</u>

We know already that if $b \in {}^*\hat{A}$, then b is internal. Suppose now that b is an internal member of $\hat{B}$. Then $b \in {}^*a$ for some $a \in \hat{A}$ and so $a \in A_{n+1}$ for some $n \geq 0$. Thus
$\hat{A} \models [(\forall x \in a)\ [x \in A_n \cup A_0]] \Rightarrow \hat{B} \models [(\forall x \in {}^*a)\ [x \in {}^*A_n \cup {}^*A_0]]$ and
$b \in {}^*a \Rightarrow b \in {}^*A_n \cup {}^*A_0$. Thus, $b \in {}^*\hat{A}$.

<u>Theorem</u>

No internal element can belong to any $y \in {}^*A_0$.

<u>Proof</u>

Since A_0 consists of individuals, $\hat{A} \models [(\forall x \in A_n)\ (\forall y \in A_0)\ [x \notin y]]$

for every n. Therefore, $\hat{B} \models [(\forall x \in {}^*A_n)\ (\forall y \in {}^*A_0)\ [x \notin y]]$ for every n.

If x is internal, then $x \in {}^*\hat{A} \Rightarrow x \in {}^*A_n$ for some n. Thus, $\hat{B} \models [(\forall y \in {}^*A_0)\ [x \notin y]]$ and the proof is complete.

However, if a is a non-empty set in $\hat{A}$, then $^*a$ has internal elements in $\hat{B}$ and is therefore non-empty. Indeed, if $a \in A_{n+1}$ for some $n \geq 0$ and a is non-empty, $\hat{A} \models [(\exists x \in A_n \cup A_0)\ x \in a]$. Thus, $\hat{B} \models [(\exists x \in {}^*A_n \cup {}^*A_0)\ x \in {}^*a]$ and $^*a$ has at least one internal element.

Note that for any $a \in \hat{A}$, $^*a$ may have external members. If $^*a \in {}^*A_0$ has external members, then $^*a$ is not a genuine individual.

We define a strict monomorphism which excludes these possibilities.

Definition

A monomorphism Φ from $\hat{A}$ into $\hat{B}$ is strict if and only if all members of $^*\hat{A}$ have internal elements only (if any at all).

Therefore, if Φ is strict, the members of an internal set of sets are internal sets themselves. Note that although all members of internal sets are internal, not all internal sets need be standard. Hence the fact that a is internal if and only if a is a member of a standard set characterizes internal, but does not characterize standard.

Also, if Φ is strict, elements of $^*A_0$ are genuine individuals. Indeed, any $y \in {}^*A_0$ cannot contain any internal elements and, since Φ is strict, these are the only elements that y could contain.

If $\Phi\colon \hat{A} \to \hat{B}$ is a strict monomorphism of $\hat{A}$ into $\hat{B}$ and if X is a sentence in L which is defined and true in $\hat{A}$, then let us note that X is true in $\hat{B}$ with any quantifiers in X relativized to internal sets.

To illustrate, let Φ: $\hat{A} \to \hat{B}$ be a strict monomorphism of $\hat{A}$ into $\hat{B}$. Let X be the sentence $(\forall Y \in A_{n+1} \cup A_0)(\forall Z \in A_{n+1} \cup A_0)(\exists W \in A_{n+1})$

$$(\forall x \in A_n \cup A_0)[x \in W \Leftrightarrow x \in Y \cap Z].$$

X states that for every two sets in $\hat{A}$, there is a set which is their intersection. The sentence X is true in $\hat{A}$ for each n. Consider the Φ-transform of X. ${}^*X = (\forall Y \in {}^*A_{n+1} \cup {}^*A_0)(\forall Z \in {}^*A_{n+1} \cup {}^*A_0)(\exists W \in {}^*A_{n+1})$

$$(\forall x \in {}^*A_n \cup {}^*A_0)[x \in W \Leftrightarrow x \in Y \cap Z].$$

Φ is a monomorphism implies that X is true in $\hat{B}$ for each n and since $W \in {}^*A_{n+1}$ implies that $W \in {}^*\hat{A}$, W is internal. We have that W and $Y \cap Z$ have the exact same elements in ${}^*A_n \cup {}^*A_0$. It remains to prove that all elements of W and $Y \cap Z$ are in ${}^*A_n \cup {}^*A_0$.

The following sentence is true in $\hat{A}$ for each m:

$(\forall u \in A_{n+1})(\forall v \in A_m)[v \in u \Rightarrow v \in A_n \cup A_0]$. Hence, $(\forall u \in {}^*A_{n+1})(\forall v \in {}^*A_m)[v \in u \Rightarrow v \in {}^*A_n \cup {}^*A_0]$ is true in $\hat{B}$ for each m. Now, if $v \in u \in {}^*A_{n+1}$, then u is internal since ${}^*A_{n+1}$ is standard and, by the definition of a strict monomorphism, v is also internal. Hence, $v \in {}^*A_m$ for some m and, by the above, $v \in u \Rightarrow v \in {}^*A_n \cup {}^*A_0$.

Therefore, all elements of W and $Y \cap Z$ are in ${}^*A_n \cup {}^*A_0$. Each of Y and Z is a member of a standard set ${}^*A_{n+1} \cup {}^*A_0 = {}^*(A_{n+1} \cup A_0)$ and therefore internal. Therefore, we have that for every two <u>internal</u> sets in $\hat{B}$, there is an <u>internal</u> set in $\hat{B}$ which is their intersection.

<u>Theorem</u>

Every monomorphism Φ: $\hat{A} \to \hat{B}$ can be transformed into a strict one.

<u>Proof</u>

For any $y \in {}^*\hat{A}$, we want y to have internal elements only (if any). $y \in {}^*\hat{A} \Rightarrow y \in {}^*A_n$ for some $n \geq 0$. Suppose that $y \in {}^*A_0$. We replace y by an individual (possibly outside $\hat{B}$) so that distinct elements of ${}^*A_0$ are replaced by distinct individuals. We assume that $\hat{A}$ and $\hat{B}$ are in some "universe" which has "enough" such individuals. Therefore, y has no elements at all. Any $y \in {}^*A_0$ cannot contain any internal elements, so we would not be changing any internal elements which are not in ${}^*A_0$.

If $y \in {}^*A_n$ for $n \geq 1$, we replace y by $y \cap {}^*\hat{A}$. This removes any external elements from y. We carry out this process in steps for $n = 1, 2, \ldots$.

Therefore, for any element $y \in {}^*\hat{A}$, y has internal elements only (if any) and Φ is a strict monomorphism. It can be argued, by the usual inductive process on the length of α, that if α is a sentence, and Φ is a non-strict monomorphism replaced by a strict monomorphism Φ' according to the above scheme, we still have $\hat{A} \models \alpha \Leftrightarrow \hat{B} \models {}^*\alpha$. The proof formalizes the intuitive truth that since ${}^*\alpha$ makes no assertion about external entities, their existence is irrelevant to the truth of ${}^*\alpha$.

Up to now, our results have been based on the definition of a monomorphism of one superstructure into another superstructure. Now, we will prove the existence of monomorphisms by constructing one.

Given a superstructure $\hat{A}$ we will construct a monomorphism of $\hat{A}$ into another superstructure $\hat{B}$. This monomorphism will be developed from an ultrapower of $\hat{A}$.

Recall the following definitions.

A <u>filter</u> F on a non-empty set J is a non-empty family of subsets of J satisfying

(i) $\phi \notin F$.

(ii) If $A \in F$ and $A \subseteq B \subseteq J$, then $B \in F$.

(iii) If $A, B \in F$, then $A \cap B \in F$.

An <u>ultrafilter</u> on a set J is a filter F on J with the additional property

$$\forall A \subseteq J, A \in F \Leftrightarrow J - A \notin F.$$

Applying a theorem proved in Chapter II, we can extend a filter F on J to an ultrafilter U.

Let J be a non-empty set and U an ultrafilter on J. Let $\hat{A}$ be a superstructure on a set of individuals $A = A_0$. Let M be the set of all maps of the form $f\colon J \to D$, $D \in \hat{A}$. Binary relations $\dot{\varepsilon}$ and $\doteq$ are defined on M as follows:

$$\forall f, g \in M,\ f \mathrel{\dot{\varepsilon}} g \text{ if and only if } \{i \in J \mid f(i) \in g(i)\} \in U$$

$$\text{and } f \doteq g \text{ if and only if } \{i \in J \mid f(i) = g(i)\} \in U.$$

M is called the U-ultrapower of $\hat{A}$ (over J). For each $c \in \hat{A}$, let $\bar{c}$ denote the constant function on J with value c. That is, $\bar{c}(i) = c$, $\forall i \in J$. Therefore, $\bar{c} \in M$. In particular, $\bar{\phi} \in M$ and $\bar{A}_n \in M$ for $n = 0, 1, \ldots$.

<u>Theorem</u>

For any $f, g \in M$

(i) If $g \mathrel{\dot{\varepsilon}} f \mathrel{\dot{\varepsilon}} \bar{A}_{n+1}$, then $g \mathrel{\dot{\varepsilon}} \bar{A}_n$ or $g \mathrel{\dot{\varepsilon}} \bar{A}_0$.

(ii) $(\forall f \in M)(\exists n)\ f \mathrel{\dot{\varepsilon}} \bar{A}_n$.

Proof

(i) $f \dot{\varepsilon} \bar{A}_{n+1} \Rightarrow \{i \varepsilon J \mid f(i) \varepsilon A_{n+1}\} \varepsilon U$ and

$g \dot{\varepsilon} f \Rightarrow \{i \varepsilon J \mid g(i) \varepsilon f(i)\} \varepsilon U$.

$\{i \varepsilon J \mid g(i) \varepsilon A_n \cup A_0\} \supseteq \{i \varepsilon J \mid g(i) \varepsilon f(i)\} \cap \{i \varepsilon J \mid f(i) \varepsilon A_{n+1}\}$. Thus $\{i \varepsilon J \mid g(i) \varepsilon A_n \cup A_0\} \varepsilon U$. Since $A_n \cap A_0 = \phi$ for $n \geq 1$, either $\{i \varepsilon J \mid g(i) \varepsilon A_n\} \varepsilon U$ or $\{i \varepsilon J \mid g(i) \varepsilon A_0\} \varepsilon U$. Therefore, $g \dot{\varepsilon} \bar{A}_n$ or $g \dot{\varepsilon} \bar{A}_0$.

(ii) $f \varepsilon M \Rightarrow f\colon J \to D$, $D \varepsilon \hat{A}$ and $D \varepsilon \hat{A} \Rightarrow D \varepsilon A_{n+1}$ for some $n \geq 0$. Thus, $\{i \varepsilon J \mid f(i) \varepsilon A_n \cup A_0\} = J \varepsilon U$ and since $A_n \cap A_0 = \phi$ for $n \geq 1$, we have that either $\{i \varepsilon J \mid f(i) \varepsilon A_n\} \varepsilon U$ or $\{i \varepsilon J \mid f(i) \varepsilon A_0\} \varepsilon U$. In either case, we have $f \dot{\varepsilon} \bar{A}_n$ for some $n \geq 0$.

Note that M has elements $\bar{A}_n$ for each n and has the relation $\dot{\varepsilon}$ while ${}^*\hat{A}$ has elements ${}^*A_n$ for each n and has the relation ε. We modify the ultrapower M so that $\dot{\varepsilon}$ is replaced by ε. Also the relation $\doteq$ is replaced by $=$. This modification is carried out in steps and requires the axiom of choice.

Let ${}^*A_0$ be the reduced ultrapower of A_0 over J. That is, for any functions f and g from J into A_0, $f = g$ if and only if $\{i \varepsilon J \mid f(i) = g(i)\} \varepsilon U$ and ${}^*A_0$ is the set of equivalence classes under the equivalence relation $=$.

Let B_0 be a set of individuals resulting from replacing distinct equivalence classes of ${}^*A_0$ by distinct individuals. Let $\hat{B}$ be the superstructure on B_0.

We replaced the equivalance classes by individuals since we wanted to obtain a superstructure and equivalence classes are not individuals according to the definition of an individual in this chapter. In other chapters, for example Chapter V, an individual is not necessarily defined as it is here.

Suppose that $f \in M$ and $f \mathrel{\dot{\varepsilon}} \bar{A}_0$. Therefore, $E = \{i \in J \mid f(i) \in A_0\} \in U$. Let $g\colon J \to A_0$ be a function from J into A_0 such that g has the same respective values on E as f does. It is possible to find such a function g. Let $\bar{g}$ be the equivalence class of ${}^*A_0$ to which g belongs. We choose f', called the <u>fiber</u> of f, to be the individual in B_0 replacing $\bar{g}$. Then, $f' \in \hat{B}$ since $f' \in B_0$. The same index set J and ultrafilter U are used in M to define $\doteq$ and $\dot{\varepsilon}$ and in ${}^*A_0$ to define the equivalence classes. Therefore, if $f_1, f_2 \mathrel{\dot{\varepsilon}} \bar{A}_0$, then $f_1 \doteq f_2 \Leftrightarrow f_1' = f_2'$. If $f \mathrel{\dot{\varepsilon}} \bar{A}_0$ has a particular individual value, say a, on a member of U, and if $\bar{a}$ is the equivalence class of ${}^*A_0$ containing the constant function from J into A_0 with that particular value, then f' will be the individual in B_0 replacing $\bar{a}$.

If $f \mathrel{\dot{\varepsilon}} \bar{A}_1$ and $g \mathrel{\dot{\varepsilon}} f$, then $g \mathrel{\dot{\varepsilon}} \bar{A}_0$ by (i) of the preceding theorem. Therefore, g' has already been chosen. Choose $f' = \{g' \mid g \mathrel{\dot{\varepsilon}} f\}$. Then, $f' \in \hat{B}$ since $g' \in \hat{B}$ for every $g \mathrel{\dot{\varepsilon}} f$.

Suppose that $f' \in \hat{B}$ has been chosen for $f \mathrel{\dot{\varepsilon}} \bar{A}_k$ for $k = 0, \ldots, n$. If $f \mathrel{\dot{\varepsilon}} \bar{A}_{n+1}$, and $g \mathrel{\dot{\varepsilon}} f$, then $g \mathrel{\dot{\varepsilon}} \bar{A}_n$ or $g \mathrel{\dot{\varepsilon}} \bar{A}_0$ by (i) of the preceding theorem. In either case, g' has already been chosen. Choose $f' = \{g' \mid g \mathrel{\dot{\varepsilon}} f\}$. Again $f' \in \hat{B}$ since $g' \in \hat{B}$ for every $g \mathrel{\dot{\varepsilon}} f$.

For each $f \in M$, $f \mathrel{\dot{\varepsilon}} \bar{A}_n$ for some $n = 0, 1, \ldots$ by (ii) of the preceding theorem. Therefore, f' has been chosen. Let

$M' = \{f' \mid f \in M\}$. The $\dot{\varepsilon}$ of M has been replaced by ε since $g \mathrel{\dot{\varepsilon}} f \Leftrightarrow g' \in f'$. Note also that the $\doteq$ of M has been replaced by $=$. If $f \mathrel{\dot{\varepsilon}} \bar{A}_n$ for some n and $g \doteq f$, then $g \mathrel{\dot{\varepsilon}} \bar{A}_n$ for the same n since $\{i \in J \mid g(i) \in A_n\} \supseteq \{i \in J \mid g(i) = f(i)\} \cap \{i \in J \mid f(i) \in A_n\}$. If $f, g \in M$ such that $f \doteq g$ and $f, g \mathrel{\dot{\varepsilon}} \bar{A}_0$, then we have seen that $f \doteq g \Leftrightarrow f' = g'$. Suppose that $f, g \mathrel{\dot{\varepsilon}} \bar{A}_n$ for some n.

$$f \doteq g \Leftrightarrow \{i \in J \mid f(i) = g(i)\} \in U$$

$$\Leftrightarrow \{i \in J \mid (\forall h \in M)\ (h(i) \in f(i) \Leftrightarrow h(i) \in g(i))\} \in U$$

$$\Leftrightarrow (\forall h \in M)\ [h \mathrel{\dot{\varepsilon}} f \Leftrightarrow h \mathrel{\dot{\varepsilon}} g]$$

$$\Leftrightarrow (\forall h' \in M')\ [h' \in f' \Leftrightarrow h' \in g']$$

$$\Leftrightarrow f' = g'.$$

Thus, for any $f, g \in M$, $f \doteq g \Leftrightarrow f' = g'$.

$M' \subseteq \hat{B}$ since for every $f \in M$, $f' \in \hat{B}$. M' is called the modified U-ultrapower of $\hat{A}$ (over J). Even though the elements of M are mappings and not equivalence classes and the elements of B_0 are individuals and not equivalence classes, this modification of M to M' actually involves mapping equivalence classes of functions which are individuals on a member of U into equivalence classes of functions which are individuals for every $i \in J$, mapping equivalence classes of functions which are sets of individuals on a member of U into sets of equivalence classes of functions which are individuals for every $i \in J$ and so on.

$\hat{B}$ also contains for example, sets which have as elements both individuals replacing equivalence classes and sets of such individuals.

We define a map $\Phi: \hat{A} \to \hat{B}$ by $\Phi(a) = \bar{a}' = {}^*a$ for every $a \in \hat{A}$. $\bar{a}'$ is the fiber of the constant function $\bar{a} \in M$. In particular, $\Phi(\phi) = \phi$ and $\Phi(A_n) = (\bar{A}_n)'$ for every n.

One can now prove, by induction on the complexity of sentences as will be done in the Monomorphism Theorem, the

<u>Ultrapower Theorem</u>

Let $\alpha = \alpha(x_1, \ldots, x_m)$ be a well-formed formula in L with $x_1, \ldots, x_m$ its only free variables and let $f_1', \ldots, f_m' \in M'$. Then the sentence ${}^*\alpha(f_1', \ldots, f_m')$ holds in M' if and only if $\{i \in J \mid \alpha(f_1(i), \ldots, f_m(i))$ holds in $\hat{A}\} \in U$. $\alpha(f_1(i), \ldots, f_m(i))$ is defined in $\hat{A}$ since for any $i \in J$ and any $k = 1, \ldots, m$, $f_k(i) \in \hat{A}$.

Our aim is to show that Φ as defined is a monomorphism. In fact, we have the

<u>Monomorphism Theorem</u>

The mapping $\Phi: \hat{A} \to \hat{B}$ defined by $\Phi(a) = \bar{a}' = {}^*a$ for every $a \in \hat{A}$, is a strict monomorphism of $\hat{A}$ into $\hat{B}$. Moreover, M' is exactly the set ${}^*\hat{A}$ of all internal elements of $\hat{B}$.

<u>Proof</u> To prove that Φ is a monomorphism, we must prove that $\hat{A} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$ for every well-formed sentence α. The proof is by induction on the number of logical symbols $\exists, \neg, \vee$ in α. We consider only these three logical symbols since all of the other

logical symbols $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\forall$ can be expressed in terms of $\exists$, $\neg$, $\vee$. Indeed, if X, Y and Z are well-formed formulae, then $[X \wedge Y]$ is equivalent to $\neg[[\neg X] \vee [\neg Y]]$, $[X \Rightarrow Y]$ is equivalent to $[[\neg X] \vee Y]$, $[X \Leftrightarrow Y]$ is equivalent to $[[X \Rightarrow Y] \wedge [Y \Rightarrow X]]$, and $[(\forall y)\ Z(y)]$ is equivalent to $[\neg[(\exists y)\ [\neg Z(y)]]]$.

First, suppose that α does not contain any of $\exists$, $\neg$, $\vee$. Thus, α is an atomic sentence and $\alpha = [a \in b]$ or $\alpha = [a = b]$ where $a, b \in \hat{A}$. We have that $a \in b \Rightarrow \bar{a} \ddot{\in} \bar{b}$ since $\{i \in J \mid \bar{a}(i) \in \bar{b}(i)\} = J \in U$. Now, $\bar{a} \dot{\in} \bar{b} \Rightarrow \bar{a}' \in \bar{b}'$ and since $\bar{a}' = {}^*a$ and $\bar{b}' = {}^*b$, we obtain $a \in b \Rightarrow {}^*a \in {}^*b$. Similarly, $a = b \Rightarrow \bar{a} \doteq \bar{b} \Rightarrow \bar{a}' = \bar{b}'$ and we obtain $a = b \Rightarrow {}^*a = {}^*b$. Therefore, if α is an atomic sentence, then $\hat{A} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$.

Suppose that $\alpha(x_1, \ldots, x_m)$ and $\beta(y_1, \ldots, y_k)$ are well-formed formulae in L where $x_1, \ldots, x_m$ are the only free variables in α and $y_1, \ldots, y_k$ are the only free variables in β. Suppose that for any $a_1, \ldots, a_m, b_1, \ldots, b_k \in \hat{A}$, $\hat{A} \models \alpha(a_1, \ldots, a_m) \Rightarrow \hat{B} \models {}^*\alpha({}^*a_1, \ldots, {}^*a_m)$ and $\hat{A} \models \beta(b_1, \ldots, b_k) \Rightarrow \hat{B} \models {}^*\beta({}^*b_1, \ldots, {}^*b_k)$. We must prove that $\hat{A} \models \neg\alpha(a_1, \ldots, a_m) \Rightarrow \hat{B} \models {}^*[\neg\alpha(a_1, \ldots, a_m)]$ and $\hat{A} \models [\alpha(a_1, \ldots, a_m) \vee \beta(b_1, \ldots, b_k)] \Rightarrow \hat{B} \models {}^*[\alpha(a_1, \ldots, a_m) \vee \beta(b_1, \ldots, b_k)]$. For any $a_1, \ldots, a_m, b_1, \ldots, b_k \in \hat{A}$, ${}^*a_1, \ldots, {}^*a_m, {}^*b_1, \ldots, {}^*b_k \in M'$. Since U is an ultrafilter and by the Ultrapower Theorem, we have $\hat{B} \models \neg[{}^*\alpha({}^*a_1, \ldots, {}^*a_m)]$

$\Leftrightarrow \{i \in J \mid \alpha(\bar{a}_1(i), \ldots, \bar{a}_m(i))$ holds in $\hat{A}\} \notin U$

$\Leftrightarrow \{i \in J \mid \neg\alpha(\bar{a}_1(i), \ldots, \bar{a}_m(i)$ holds in $\hat{A}\} \in U$

$\Leftrightarrow \hat{B} \models {}^*[\neg\alpha(a_1, \ldots, a_m)]$.

Similarly, we obtain

$\hat{B} \models {}^*[\alpha(a_1, \ldots, a_m) \vee \beta(b_1, \ldots, b_k)]$

$\Leftrightarrow \hat{B} \models [{}^*\alpha({}^*a_1, \ldots, {}^*a_m) \vee {}^*\beta({}^*b_1, \ldots, {}^*b_k)]$.

We have, by the Induction Hypothesis, $\hat{A} \models \neg\alpha(a_1, \ldots, a_m) \Rightarrow \hat{B} \models \neg[{}^*\alpha({}^*a_1, \ldots, {}^*a_m)]$. That is, we obtain a contradiction of the Induction Hypothesis with $\hat{A} \models \neg\alpha(a_1, \ldots, a_m) \Rightarrow \hat{B} \models {}^*\alpha({}^*a_1, \ldots, {}^*a_m)$. Similarly, we have by the Induction Hypothesis, $\hat{A} \models \alpha(a_1, \ldots, a_m) \vee \beta(b_1, \ldots, b_k) \Rightarrow \hat{B} \models {}^*\alpha({}^*a_1, \ldots, {}^*a_m) \vee {}^*\beta({}^*b_1, \ldots, {}^*b_k)$.

Therefore, $\hat{A} \models \neg\alpha(a_1, \ldots, a_m) \Rightarrow B \models {}^*[\neg\alpha(a_1, \ldots, a_m)]$ and $\hat{A} \models [\alpha(a_1, \ldots, a_m) \vee \beta(b_1, \ldots, b_k)] \Rightarrow \hat{B} \models {}^*[\alpha(a_1, \ldots, a_m) \vee \beta(b_1, \ldots, b_k)]$.

Now, suppose that $\beta(x_1, \ldots, x_m, y)$ is a well-formed formula with $x_1, \ldots, x_m, y$ its only free variables and suppose that for any $a_1, \ldots, a_m, b \in \hat{A}$,

$$\hat{A} \models \beta(a_1, \ldots, a_m, b) \Rightarrow \hat{B} \models {}^*\beta({}^*a_1, \ldots, {}^*a_m, {}^*b).$$

We need to show that for any $a_1, \ldots, a_m, C \in \hat{A}$, $\hat{A} \models (\exists y \in C)\, \beta(a_1, \ldots, a_m, y) \Rightarrow \hat{B} \models (\exists y \in {}^*C)\, {}^*\beta({}^*a_1, \ldots, {}^*a_m, y)$. Suppose that $\hat{A} \models (\exists y \in C)\, \beta(a_1, \ldots, a_m, y)$. We can fix $d \in C$ such that $\hat{A} \models \beta(a_1, \ldots, a_m, d)$. Therefore, by the Induction Hypothesis, $\hat{B} \models {}^*\beta({}^*a_1, \ldots, {}^*a_m, {}^*d)$. Since ${}^*d \in {}^*C$,

$\hat{B} \models (\exists y \in {}^*C)\ {}^*\beta({}^*a_1, \ldots, {}^*a_m, y)$.

Every well-formed formula in L is constructed from atomic formulae using connectives and quantifiers. Thus, the above steps show that if α is a well-formed sentence in L, then $\hat{A} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$. Therefore, Φ is a monomorphism.

The proof of $M' = {}^*\hat{A} = \bigcup_{n=0}^{\infty} {}^*A_n$ is as follows:

If $f' \in M'$, then f' is the fiber of a map $f \in M$. $f \in M \Rightarrow f \dot{\in} \bar{A}_n$ for some $n \Rightarrow f' \in (\bar{A}_n)' = {}^*A_n \Rightarrow f' \in {}^*\hat{A}$.

Now, if $x \in {}^*\hat{A}$, then $x \in {}^*A_n$ for some n. Thus, $x \in (\bar{A}_n)'$ and $x = f'$ for some $f' \in M'$. Therefore, $x \in M'$ and the proof is complete.

To prove that Φ is strict, we need to prove that if $y \in {}^*\hat{A}$ and $x \in y$, then $x \in {}^*\hat{A}$. Now, if f' is any element of $M' = {}^*\hat{A}$ and $x \in f'$, then $x = g'$ for some $g' \in M'$ by construction since $f' = \{g' \mid g \dot{\in} f\}$. Therefore $x \in {}^*\hat{A}$ and Φ is strict.

Consider the following example of the construction of a monomorphism:

Let $R = R_0$ be the set of real numbers and let $\hat{R}$ be the super-structure on R_0. Let N be the set of natural numbers and let F be the Fréchet filter on the index set N. We can extend F to an ultrafilter U on N. Let B_0 be a set of distinct individuals replacing the equivalence classes of ${}^*R_0$, the reduced U-ultrapower of R_0 over N. We obtain the strict monomorphism $\Phi: \hat{R} \to \hat{B}$ as in the general case. Let us examine this monomorphism.

$$R \in \hat{R} \quad \text{and} \quad \Phi(R) = \bar{R}' = {}^*R = \{f' \mid f \dot{\in} \bar{R}\}.$$

Let $r \in R$. Then $\bar{r} \in M$ and $\bar{r}' = {}^*r \in \hat{B}$ is the individual replacing the equivalence class containing the constant function on N with value r. Certainly for $\bar{r} \in M$, $\bar{r} \dot{\in} \bar{R}$, and so ${}^*r \in {}^*R$. We identify ${}^*r$ with r and we can embed R into ${}^*R$.

In the same way, ${}^*N$ contains a copy of N. This copy of N is a proper subset of ${}^*N$. Indeed, consider the function $f \in M$ defined by $f(n) = n$, $\forall n \in N$. Recall that this function occurred in Chapter II in the example of a non-standard model of analysis. Certainly $f \dot{\in} \bar{N}$ and therefore $f' \in {}^*N$. But f' is not in the copy of N. Indeed, suppose $f' = \bar{k}'$ for some $k \in N$. $\bar{k} \in M$ is the constant function $\bar{k}(n) = k$ for all $n \in N$. $f' = \bar{k}' \Rightarrow f \doteq \bar{k} \Rightarrow \{n \in N \mid f(n) = k\} \in U$. This means that f takes the value k on an infinite subset of the index set N and this is a contradiction of the definition of the function f. Therefore, the copy of N in ${}^*N$ is a proper subset of ${}^*N$.

<u>Definition</u>

A binary relation b is said to be <u>concurrent</u> on a set $D \subseteq$ domain of b if and only if for any finite number of elements $a_1, \ldots, a_m$ of D, there exists some y in the range of b such that (a_k, y) satisfies b for $k = 1, \ldots, m$.

For example, the relation $<$ on the natural numbers is a concurrent binary relation since given any finite set $\{a_1, \ldots, a_m\}$ of natural numbers we can find a natural number which is larger than each of $a_1, \ldots, a_m$.

<u>Definition</u>

A monomorphism $\Phi\colon \hat{A} \to \hat{B}$ is said to be <u>enlarging</u> (and ${}^*\hat{A}$ is called an enlargement of $\hat{A}$) if and only if for each concurrent binary relation $b \in \hat{A}$, there is some $y \in {}^*\hat{A}$ such that $({}^*a, y)$ satisfies ${}^*b$ for all a in the domain of b simultaneously. The enlargement is called strict if Φ is strict.

In the construction of a monomorphism one can make special choices of the index set and the ultrafilter to make Φ enlarging. This is involved in the proof of the

Enlargement Theorem

For every superstructure $\hat{A}$, there is a superstructure $\hat{B}$ and a monomorphism $\Phi: \hat{A} \to \hat{B}$ which is strict and enlarging.

Now, suppose that A_0 is an infinite set of individuals, Then, by the axiom of choice, A_0 contains a countable subset A_N which can be identified with the set of natural numbers N. Thus $\hat{A}$ contains a binary relation $<$ on the elements of A_N corresponding to the binary relation $<$ on the natural numbers. This relation extends, under a monomorphism Φ, to a total ordering ${}^*<$ of ${}^*A_N$. We have that $A_N \subseteq {}^*A_N$ and $< \subseteq {}^*<$ so that ${}^*<$ coincides with $<$ when restricted to A_N. Since $<$ is a concurrent binary relation, if Φ is enlarging, there exists some $y \in {}^*A_N$ such that ${}^*a < y$ for every $a \in A_N$. Such an element y is called infinite.

$${}^*A_N - A_N = \{n \in {}^*A_N \mid n \text{ is infinite}\}.$$

Even if Φ is not enlarging we can have infinite elements. In the preceding example, where R_0 is the set of real numbers, we have that the copy of N in ${}^*N$ is a proper subset of ${}^*N$ so that

${}^*N - N \neq \phi$.

Definition

Given a monomorphism $\Phi\colon \hat{A} \to \hat{B}$ where A_0 contains a copy A_N of the set of natural numbers N, we call ${}^*\hat{A}$ a non-standard model of $\hat{A}$ if and only if ${}^*A_N - A_N \neq \phi$. (Actually ${}^*\hat{A}$ is a model of the set of sentences of the language which are defined and true in $\hat{A}$).

As we have seen above, if A_0 is infinite and $\Phi\colon \hat{A} \to \hat{B}$ is enlarging, then ${}^*\hat{A}$ is a non-standard model of $\hat{A}$, since ${}^*A_N - A_N \neq \phi$.

Theorem

Let $\Phi\colon \hat{R} \to \hat{B}$ be the strict monomorphism of our previous example. Then, there is no least infinite natural number. That is, ${}^*N - N$ has no least member.

Proof

Suppose that a is any element of ${}^*N - N$. Certainly $a \neq {}^*0$ since ${}^*0 = \Phi(0)$ for $0 \in N$. It is true in the set of ordinary natural numbers that each $n \neq 0$ is the successor of another natural number. Therefore, it is also true in ${}^*N$ and we have $a = b \; {}^*\!+ \; {}^*1$ for an element b of ${}^*N$. The binary relation ${}^*+$ is the extension of the relation $+$ of N. Now, b is also an element of ${}^*N - N$ since, if b is in the copy of N, then so is a, but $a \in {}^*N - N$. Therefore, $b \in {}^*N - N$ and $b \; {}^*\!< a$. This proves that there is no least infinite natural number.

Now, suppose that in our first order language described in the beginning of this chapter, we had not required that all quantification be of the form $(\exists x \in C)$ or $(\forall x \in C)$ for $C \in \hat{A}$.

Without this restriction we could write

$$\alpha = [(\forall y)\ [(\forall w)\ [w \in y \Rightarrow w \in N]] \wedge [(\exists v)\ [v \in y]] \Rightarrow [(\exists x)\ [x \in y \wedge [(\forall z)\ z \in y \Rightarrow x \leq z]]]]$$

which states that every non-empty subset of N has a least member. We would obtain, $\hat{R} \models \alpha \Rightarrow \hat{B} \models {}^*\alpha$. Therefore,

$$\hat{B} \models [(\forall y)\ [(\forall w)\ [w \in y \Rightarrow w \in {}^*N]] \wedge [(\exists v)\ [v \in y]] \Rightarrow [(\exists x)\ [x \in y \wedge [(\forall z)\ z \in y \Rightarrow x \leq z]]]].$$

Now, ${}^*N - N$ is a subset of ${}^*N$ and ${}^*N - N \neq \phi$. Therefore, we would conclude that ${}^*N - N$ has a least member. This contradicts the preceding theorem.

The restriction on quantification does not lead us to a contradiction. Recall the relativization of quantifiers to internal sets. With the restriction on quantification, we have the sentence

$$X = [(\forall y \in R_1)\ [(\forall w \in R)\ [w \in y \Rightarrow w \in N]] \wedge [(\exists v \in R)\ [v \in y]] \Rightarrow [(\exists x \in N)\ [x \in y \wedge [(\forall z \in N)\ z \in y \Rightarrow x \leq z]]]].$$

We determined that the Φ transform of X would read "every internal non-empty subset of the set ${}^*N$ has a least member". Since ${}^*N - N$ has no least member, we must conclude that ${}^*N - N$ is not internal. That is, ${}^*N - N$ is external in ${}^*N$ and we have no contradiction at all.

Note that the above example of the relativization of quantifiers to internal sets is, in fact, true for all monomorphisms and not just for strict monomorphisms. Since ${}^*R_1$ is standard, $y \in {}^*R_1$ implies that y is internal. Therefore, "$\forall y \in {}^*R_1$" reads "for all internal sets in ${}^*R_1$". The least member x is internal since it is an element of the standard set ${}^*N$.

CHAPTER V

EXTRA-STANDARD MODEL OF ANALYSIS HIGHER ORDER STRUCTURE AND LANGUAGE

In what we have done up to this point, "analysis" has meant the set of all sentences which are true of the set of ordinary real numbers, the usual model of analysis. We will return to this pattern later in this chapter. This set K of sentences is vast beyond comprehension, and there is no means of constructing more than a small part of K. Analysis usually means the facts that can be deduced from a very restricted subset of K called "axioms". For such restricted sets of axioms, the incompleteness phenomena of Gödel are present. We shall prove below that we can obtain models of the set of axioms which are not elementary extensions of R and hence certainly not reduced ultrapowers of R.

Since the term "non-standard model" has been precisely defined, we introduce a temporary designation to describe the kind of model we want to discuss, an extra-standard model. An extra-standard model of analysis is a model, say M, of some set of axioms for analysis such that M contains more individuals than there are ordinary real numbers. We proceed to outline the proof of the following theorem.

Theorem

There are extra-standard models of analysis which are not reduced ultrapowers of the set of real numbers.

Proof

The proof is divided into two parts.

(i) There are models, N', of the set of natural numbers 0, 1, 2, ... which are not reduced ultrapowers of the usual model, N.

(ii) Using N', as in (i), we can construct an extra-standard model, R', of analysis. We then show that R' cannot be a reduced ultrapower of the set of real numbers R.

Proof of (i)

If M is a model of Peano Arithmetic, and M is a reduced ultrapower of the model M, then M is an elementary extension of M (by a theorem stated in Chapter II).

Gödels Completeness Theorem states that a sentence A is provable from the set of axioms Γ if and only if A is true in each model of Γ.

Gödels Second Incompleteness Theorem states that if Peano Arithmetic is consistent then the sentence CON is not provable in Peano Arithmetic. CON is the sentence $\neg \exists x \, Prf(x, 0 = 1)$ where $Prf(x, 0 = 1)$ is a formula of Peano Arithmetic which expresses the statement that "x is the Gödel number of a proof of 0 = 1". Therefore, there must be a model M of Peano Arithmetic in which CON is false. CON is true in the set of natural numbers N which is the usual model of Peano Arithmetic. CON is true in any model isomorphic to N. Therefore, M cannot be an elementary extension of any model isomorphic to N. Hence, M cannot be a reduced ultrapower of any model isomorphic to N. (There will be an infinite number in M which acts like the number of a proof of 0 = 1). Therefore, there are models, N', of the set of natural numbers which are not reduced ultrapowers of N.

Proof of (ii)

Just as we construct the set of real numbers R from the set of natural numbers N, we can construct R' from the model N' of (i). That is, we start with the elements of N' and construct the set I' of "integers", the elements of N' along with their "negatives". Next, we construct the set Q' of "rationals", the set of ordered pairs in which the first position is filled by elements of I' and the second position is filled by "non-zero" elements of I'. Next, we consider the set of all Cauchy sequences of elements of Q' and define an equivalence relation on the set of sequences. Two sequences are equivalent if they converge to the same limit. We denote the set of equivalence classes by R'. Note that R' contains a copy of the set of real numbers R. Indeed, N' contains a copy of the set of natural numbers N, and in the construction of R', we obtain a copy of the set of real numbers R from this copy of N.

Now, R' is not a reduced ultrapower of R. Indeed, if this were so, then R' would be an elementary extension of R, but the statement

$$[(\exists x)\ [x \in i_N \wedge Prf(x,\ 0 = 1)]]$$

is true in R' where i_N is interpreted by N', but not true in R, where i_N is interpreted by N. Therefore, R' cannot be an elementary extension of R and hence R' cannot be a reduced ultrapower of R. R' is an extra-standard model of analysis.

We now return to a discussion of theories in which "analysis" means the set of all sentences which are true of the set of ordinary real numbers, the usual model of analysis.

Suppose that we want to be able to reference more than one "type" of entity, for example, real numbers and sets of real numbers. We can proceed in two ways.

The first way, derived from the standard method used to develop mathematics within set theory, is to form the superstructure $\hat{R}$ on the set of real numbers as in Chapter IV. The superstructure is a "first order" structure since it formally refers to only one "type" of entity, a set. We adopt a first order language for $\hat{R}$ and use its variables uniformly to reference all "types" of sets which we intuitively think of as being different. For example, $\hat{R}$ contains $R = R_0$, the set of real numbers and R_1, the set of all sets of real numbers, but, set-theoretically, each is just a "set" and, hence, of the same formal "type".

Instead of this, we could consider a structure in which these intuitively different "types" of entities are formally distinguished. Such a structure is called a <u>higher order structure</u>. The class T of types is defined inductively as follows:

(i) 0 is a type (natural number zero)

(ii) If n is a positive integer and $\tau_1, \ldots, \tau_n$ are types, then $(\tau_1, \ldots, \tau_n)$ is a type.

Individuals are of lowest type (ie. 0) and sets of individuals and relations between individuals are of higher type than individuals.

Consider the following example of a higher order structure to be denoted by M. M contains all real numbers. Each real number is assigned type 0. M contains all sets of real numbers and each set of real numbers has type (0). For example, the set of natural numbers N and the set of real numbers *R* each has type (0). M contains the relation of logical identity between real numbers. This relation is assigned type (0, 0) since the relation is a set of ordered pairs in which real numbers (entities of type 0) fill both the first and second places. M contains the relation of membership of real numbers in a set of real numbers and this relation has type (0, (0)). M also contains the relations of addition and multiplication of real numbers and these relations both have type (0, 0, 0), since they are each a set of ordered triples of real numbers. This example indicates the method for forming types.

In order to discuss a higher order structure we need a higher order language. A <u>higher order language</u> consists of constants to denote each entity of each type in the structure, and connectives, quantifiers and brackets just as in a first order language. Regarding variables, we could have a distinct set consisting of an infinite number of variables for each type τ, so that for any variable in the language we know the type of the entity that it represents. For each type τ we would have a set of relation symbols of the form $R_\tau(\, , \ldots , \,)$ where the numbers of empty places and the types of entities that fill the empty places depend on τ. For example, let the addition and multiplication of real numbers be denoted by $S_{(0,\,0,\,0)}(\, , \, , \,)$ and $P_{(0,\,0,\,0)}(\, , \, , \,)$ respectively. We write

$S_{(0,\ 0,\ 0)}(i_a, i_b, i_c)$ and $P_{(0,\ 0,\ 0)}(i_a, i_b, i_c)$ for $a + b = c$ and $a \cdot b = c$ respectively where each of a, b and c is a real number (entity of type 0) denoted by i_a, i_b and i_c in the language respectively.

Customarily in a formal language we denote relation symbols in the form $R(\ ,\ )$ where $R(i_a, i_b)$ might denote, for example, $a < b$. There is no reason why the same could not be denoted by a symbol such as $\Phi_\tau(\ , \ldots ,\ )$ with only one symbol $\Phi_\tau(\ , \ldots ,\ )$ for each type τ. The number of empty places and the entities that fill the places depend on τ. The first argument position of $\Phi_\tau(\ , \ldots ,\ )$ is filled by a symbol which denotes the particular relation of type τ being described. The reasons for preferring this will appear. In this case, we need only one set of variables since the position which a variable fills in a relation symbol determines the type of entity which it represents.

To illustrate an application of this relation symbol, suppose that the relation of membership of a real number in a set of real numbers is denoted in the language by the symbol ε. Then, ε has type (0, (0)). The fact that the set of natural numbers, denoted in the language by the constant i_N, is non-empty, is written

$$(\exists x)\ \Phi_{(0,\ (0))}(\varepsilon, x, i_N).$$

Note that the position which a symbol fills indicates the type of entity which it represents. Thus, ε has type (0, (0)), x has type 0 and i_N has type (0).

If the symbols S and P denote addition and multiplication of real numbers respectively, and if i_a, i_b, i_c denote a, b, and c respectively, then $a + b = c$ is written $\Phi_{(0, 0, 0)}(S, i_a, i_b, i_c)$ and $a \cdot b = c$ is written $\Phi_{(0, 0, 0)}(P, i_a, i_b, i_c)$.

Note that there is now no reason to consider that the symbols ε, P, S, which denote these relations are of a different kind than the other constants i_N, i_a, i_b, i_c used in the formulae.

We must make sure that whenever we write a formula, each variable or constant in the formula always fills positions of the same type. Such formulae are called "stratified" and, by restricting attention to stratified formulae we observe the spirit of a higher order type theory by not allowing such formulae as $\Phi_{(0, (0))}(\varepsilon, x, x)$ which expresses $x \varepsilon x$, which lead to paradoxes of set theory.

We assume that a higher order language for a higher order structure contains "enough" constants to be put into a one-to-one correspondence with the entities of the structure of each type. For example, if M is the higher order structure described early in this chapter with the set of real numbers as the set of individuals, and if L is a higher order language for M, then L contains a constant i_r to denote each real number r, a constant i_B to denote each set of real numbers B, the symbol $=$ to denote the relation of logical identity of real numbers, the symbol ε to denote the membership relation of a real number in a set of real numbers and symbols S and P to denote addition and multiplication of real numbers respectively. These are the constants denoting all of the entities of M. M certainly does not contain all entities of all types that can enjoy a set-theoretic existence based on the set of real numbers. If M did contain all possible set-theoretic entities of all types then M would be called a <u>full structure</u>.

Let K be the set of sentences formulated in the language L which are defined in M and let K' be the set of these sentences which are true in M. A higher order structure $^*M$ is called a <u>higher order model</u> of K' if all the sentences of K' are true in $^*M$. It can be shown, as for first order models, that a sentence which is defined in M is true in M if and only if it is true in $^*M$.

M can be embedded in $^*M$. Indeed, if a is an entity of M of type τ, then this fact will be included in sentences of K' containing i_a. The sentences are true in $^*M$, so there will be an entity of type τ in $^*M$, say $^*a$, which corresponds to a. We identify a and $^*a$ for every entity a of M and this provides us with the embedding. The mapping $a \to {}^*a$ is one-to-one since different entities of M are denoted by different constants in L, which in turn, denote different entities of $^*M$.

Even if M is a full structure, $^*M$ need not be full. The entities which are present in $^*M$ are called internal entities. If $^*M$ is not full, then there are entities which do not exist in $^*M$. These entities are called external. If an entity of $^*M$ corresponds to an entity of M under the embedding of M into $^*M$, then this entity of $^*M$ is called standard.

Of course, any higher order structure could be thought of as a substructure (in an appropriately defined sense) of the full structure over its set of individuals. One can consider that our structures simply ignore some the the entities of the full structure.

If the copy in $^*M$ of the set of individuals of M is a proper subset of the set of individuals of $^*M$, then $^*M$ is a proper extension of M and $^*M$ is called a higher order non-standard model of M. (Actually $^*M$ is a model of K', the set of sentences which are defined and true in M).

The Finiteness Principle, which is stated in Chapter III for first order languages and structures is also true for higher order languages and structures. Therefore, if K is a set of sentences in a higher order language such that every finite subset of K has a higher order model, then K has a higher order model. The proof of this involves the following:

We add to our language L a one-place relation symbol $\Theta_\tau(\ )$, for each type τ, which allow us to state the type of an entity. Our new language is denoted by L'. For example, if r is a real number (entity of type 0) we write $[\Theta_0(i_r)]$. We also transform each sentence of L to a sentence of L'. For example, $X = [(\exists x)\ \Phi_{(0,\ (0))}(\varepsilon,\ x,\ i_N)]$ becomes $X_\lambda = [(\exists x)\ [\Theta_0(x) \wedge \Phi_{(0,\ (0))}(\varepsilon,\ x,\ i_N)]]$.

For a sentence X of L, the new sentence X_λ of L' is called the type transform of X.

We associate a first order structure M_λ with M. The individuals of M_λ are the individuals and relations of M. The set of relations of M_λ consists of relations to interpret the relation symbols $\Theta_\tau(\ )$ and $\Phi_\tau(\ ,\ \ldots\ ,\ )$. Then, we prove that a sentence X in L is defined and true in M if and only if it's type transform X_λ is defined and true in M_λ. At this point, we apply the Finiteness Principle for first order languages and structures to prove the same principle for higher order languages and structures.

We can see from this outline of the proof that the proof involves putting all entities of all types in the higher order structure M into one "type" of entity, the individuals of the first order structure M_λ. This one "type" parallels the superstructure of Chapter IV. Now, each individual of the first order structure is denoted in the language L' by a constant, and each relation of M_λ is denoted by a relation symbol in L'.

Recall the following definitions.

Definition

A binary relation b of a structure M is called concurrent if for every finite set $\{a_1, \ldots, a_n\}$ of elements in the domain of b, there exists an element y in the range of b such that $b(a_i, y)$ holds in M for $i = 1, \ldots, n$.

Definition

A model $^*M$ of a structure M is called an enlargement of M if, for every concurrent binary relation b of M, there exists an element y in $^*M$ such that $^*b(^*x, y)$ holds in $^*M$ for every x in the domain of b where $^*b$ and $^*x$ in $^*M$ correspond to b and x in M respectively. We say that $^*M$ bounds each concurrent relation b and that b has y as a bound.

As we proved in Chapter IV, if the set of individuals of a structure M is infinite, then any enlargement $^*M$ of M is a non-standard model of M. Indeed, the binary relation $\neq$ is concurrent if the set of individuals is infinite, and since $^*M$ is an enlargement of M, $^*M$ bounds this binary relation. We obtain an individual y of $^*M$ such that $y \neq {^*a}$ for every individual a of M. Therefore, $^*M$ is a proper extension of M and $^*M$ is a non-standard model of M.

Theorem

Each structure M has an enlargement.

The proof of this theorem involves considering more than one concurrent binary relation simultaneously as we noted in Chapter III. Let K be the set of all sentences which are true in M and let H be the set of sentences consisting of one sentence for each concurrent binary relation b, stating that b has a bound. We prove that $K \cup H$ has a model, say $^*M$. Since $^*M$ is a model of K and $^*M$ bounds each concurrent binary relation of M simultaneously, $^*M$ is an enlargement of M.

If M is a full higher order structure with set A of individuals, then M resembles the superstructure $\hat{A}$ on A since each of M and $\hat{A}$ contains all the set-theoretic entities that exist based on the set of individuals A. There is one difference. The theory of types does not allow a set containing different types of elements. A set has type (τ) for some type τ, so that each element of the set has type τ. This excludes, for example, a set containing as elements both individuals and sets of individuals. Such a set exists in a superstructure since, for each $n \geq 0$, A_{n+1} is the set of all subsets of $A_0 \cup A_n$.

Shimrat in [7] developed ultrapowers and mappings similar to those of Zakon in [8]. The main difference in these two approaches is that Zakon defined monomorphisms and then constructed ultrapowers to prove the existence of monomorphisms, while Shimrat constructed ultrapowers and then used them to define monomorphism.

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