THE THEORY OF INNER MEASURES: AN AXIOMATIC APPROACH

by

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A B S T R A C T

This thesis presents an axiomatic approach to the theory of inner measures.

In Chapter I, we recognize that an axiomatic approach analogous to that for the theory of outer measures is not appropriate: such an approach involves only a finiteness concept, and consequently, as soon as countable collections are involved, it fails. It is then natural for us to speculate that our model should permit the change of limits. This idea leads us to the definition of an inner measure.

In Chapter II, we consider a space of finite measure and characterize inner measurability. We also prove that the definitions of inner measurability given by Young and Caratheodory are equivalent, and that Lebesgue's definition of measurability is equivalent to those given by Young and Caratheodory. Then, the assumption that the space has finite measure is dropped, and we study the inner measures induced by a measure.

In Chapter III, inner measures are contracted so as to guarantee that they are countably additive over their classes of inner measurable sets, and so that they always generate an outer measure. The last part of this chapter deals with conditions under which the set function μ^0 generated by an inner measure μ_c is a regular outer measure.

Finally, in Chapter IV, some relation between a sequence of contracted inner measures and the associated measure spaces is established.

NOTATION AND TERMINOLOGY

Throughout, X is given space. P(X) is the power set of X. $R = [-\infty,\infty]$. A set function is a function from a subclass of P(X) to R. If there can be no ambiguity, the intersection sign is omitted. A' is the complement of A.

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YOUNG'S DEFINITION OF INNER MEASURABILITY: p. 2.

OUTER MEASURE: Definition, p. 2.

SUPERADDITIVE SET FUNCTION: Definition, p. 4.

ALGEBRA: A subclass of P(X) which is closed under finite unions and complementation, p. 4.

σ-ALGEBRA: An algebra closed under countable unions, p. 4.

$\underline{\mu}$: p. 4.

Z^+: A set of all positive integers, p. 4.

$|A|$: Number of elements in the set A, p. 4.

MONOTONE INCREASING: A set function μ is monotone increasing (or simply called 'monotone') if whenever $A \supset B$, $\mu(A) \geq \mu(B)$, p. 4.

COUNTABLY SUPERADDITIVE: A set function μ is countably superadditive over P(X) if, whenever $\{B_n\}$ is a sequence of pairwise disjoint sets from P(X), $\mu(\bigcup_{n=1}^{\infty} B_n) \geq \sum_{n=1}^{\infty} \mu(B_n)$, p. 5.

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$M(\eta)$: p. 6.

ADDITIVE OVER A CLASS OF SETS C: Definition, p. 7.

FINITELY ADDITIVE: A set function μ is finitely additive over a subclass of $P(X)$ if, whenever $A_1, \ldots, A_n$ are pairwise disjoint sets from the subclass whose union is also in the subclass $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$, p. 9.

COUNTABLY ADDITIVE: A set function μ is countably additiver over a subclass of $P(X)$ if, whenever $\{B_n\}$ is a sequence of pairwise disjoint sets from the subclass whose union is also in the subclass, $(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$, p. 9.

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A DECREASING SEQUENCE OF SETS: $\{B_n\}$ is a decreasing sequence of sets if $B_1 \supset B_2 \supset \ldots$, p. 11.

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MONOTONE AND BOUNDED: A real sequence $\{x_n\}$ is monotone if $x_{n+1} \geq x_n$ for all n (called monotone increasing), or $x_{n+1} \leq x_n$ for all n (called monotone decreasing). $\{x_n\}$ is bounded if there exists a number K such that $|x_n| < K$ for all n, p. 11.

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μ^*-MEASURABLE: p. 12.

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μ_*-MEASURABLE: p. 13.

$\overline{\lim\limits_{n\to\infty}}\ \mu^*(K_n)$: The limit superior of a sequence of real numbers defined as $\overline{\lim\limits_{n\to\infty}}\ \mu^*(K_n) = \inf\limits_{k\to\infty} \sup\limits_{n\geq k} \mu^*(K_n)$, p. 13.

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μ^x: Definition, p. 26.

μ_0: Definition, p. 26.

μ: p. 26.

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RING: A subclass of sets of $P(X)$ closed under union and difference, p. 27.

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$\mathcal{M}(\mu_0)$: p. 27.

$\mu^x|\mathcal{M}(\mu^x)$: μ^x restricted to $\mathcal{M}(\mu^x)$ (used systematically throughout), p. 27.

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$\{C_i\}$: A sequence of μ_Δ-measurable sets such that $\mu_\Delta(\bigcup_{i=1}^{\infty} C_i)$ is finite, p. 39.

$\{\mu_{C_i}\}$: A sequence of inner measures associated with $\{C_i\}$, p. 39.

$(\mathcal{M}(\mu_{C_i}),\mu_{C_i})$: A measure space (a σ-algebra $\mathcal{M}(\mu_{C_i})$ together with a measure μ_{C_i} on it), p. 39.

$M(\mu_{\cup C_i})$: (Here, $\cup C_i$ is the shorthand for $\bigcup_{i=1}^{\infty} C_i$) p. 39.

$M(\mu_{\overline{C}_i})$: p. 39.

$\overline{C}_i$: p. 39.

μ_C-NULL: p. 42.

INTRODUCTION: SOME HISTORICAL REMARKS

In 1898, Emile Borel [1] gave a descriptive definition of a measure as follows:

(1) It is a function from a class $\mathbb{D}$ *of subsets of the real line to* $[0,\infty)$*;* $\mathbb{D}$ *is closed under differences and finite unions;*

(2) the measure of the union of a finite number of pairwise disjoint sets from $\mathbb{D}$ *equals the sum of their measures;*

(3) the measure of the difference of two sets from $\mathbb{D}$ *(a set and a subset) is equal to the difference of their measures;*

(4) every set whose measure is not zero is uncountable.

The existence of such a descriptive measure can be seen by taking $\mathbb{D}$ to be the class of all finite unions of intervals from the real line, taking the length of each interval to be its measure and extending this in the obvious way. However, the description did not include the idea of countable additivity. Based upon Borel's ideas, H. Lebesgue in 1901 [5] defined a measure for <u>any</u> set in the interval $[a,b]$ to be a non-negative number satisfying the following conditions:

(1) Two congruent sets have the same measure;

(2) the measure of the union of a finite or countable number of pairwise disjoint sets is the sum of the measures of the summands;

(3) the measure of the set $[0,1]$ *is* 1.

The importance of countable additivity for a measure was realized; however, the three conditions are incompatible as there do exist non-measurable sets in $[0,1]$. (Lebesgue did not know this at the time: the first non-measurable set was constructed by Van Vleck in 1908 [15].) Taking the length of a segment $[a,b]$ or of the interval (a,b) to be its measure m, Lebesgue gave a constructive definition of a measurable set. He first defined the outer measure $m_e(A)$ of each set $A \subset [a,b]$ by

$$m_e(A) = \inf\{\sum m((a_i,b_i)): A \subset \cup(a_i,b_i)\}.$$

He then defined the inner measure $m_i(A)$ of A to be $b-a-m_e([a,b]-A)$ and defined A to be measurable if $m_e(A) = m_i(A)$.

Lebesgue's definitions for outer and inner measures were also given by G. Vitali in 1904 [14], and by W. H. Young, also in 1904 [17]. However, Young defined measurable solely in terms of an outer measure m_e:

a set A is said to be outer-measurable if and only if, for all D such that $A \cap D = \emptyset$, $m_e(A)+m_e(D) = m_e(A \cup D)$.

Thus, a class of outer-measurable sets, a class of inner-measurable sets, and their common part (called the additive class) were constructed. It turns out that in each class, the corresponding set function is countably additive.

In 1914, C. Caratheodory [2] introduced his axiomatic definition of an outer measure:

an outer measure μ^* is a set function from $P(X)$ to $\mathbb{R}$ satisfying the following conditions:

(a) $\mu^*(\phi) = 0$;

(b) *if* $A \subset B$, *then* $\mu^*(A) \leq \mu^*(B)$;

(c) *if* $\{B_n\}$ *is any sequence of subsets of* X, *then*

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n).$$

He called a set A <u>μ^*-measurable</u> if, for any $P \subset X$, the equality

$$\mu^*(P) = \mu^*(PA) + \mu^*(PA')$$

is satisfied. He then defined an inner measure μ_* for any $A \subset X$ by

$$\mu_*(A) = sup\{\mu^*(M): \ M \subset A, \ M \ \textit{is} \ \mu^*\textit{-measurable}\}.$$

This inner measure possesses all the usual properties of the Lebesgue inner measure and, for a set A, whose outer measure is finite, to be μ^*-measurable, it is necessary and sufficient that <u>$\mu^*(A) = \mu_*(A)$</u>. Subsequently, the concept of 'inner measure' was relegated to a minor role in Measure Theory. As John von Neumann [16] pointed out, an axiomatic treatment using inner measure defined analogously to the axiomatic definition of an outer measure 'is not appropriate': the intrinsic properties of inner measures so defined are just not strong enough to generate an equivalent theory. Recently, however, interest in inner measure has been revived [12] [13], and the main object of this thesis is to present an axiomatic treatment which includes Caratheodory's inner measure as a special case.

CHAPTER I

DEFINITION OF INNER MEASURES

In this chapter, we first introduce the definition of a superadditive set function and obtain an algebra of measurable sets. We then define inner measures and extend the algebra to a σ-algebra.

Definition 1.1. *A superadditive set function on a space* X *is a function* $\underline{\mu}$ *from* $P(X)$ *to* $\mathbb{R}$ *satisfying the following conditions:*

(a) $\underline{\mu}(\phi) = 0$;

(b) *if* $A \cap B = \phi$, $\underline{\mu}(A \cup B) \geq \underline{\mu}(A) + \underline{\mu}(B)$.

Obviously, from (a) and (b), $\underline{\mu}(A) \geq 0$ for all $A \subset X$.

Throughout this thesis, $\underline{\mu}$ will denote a superadditive set function on a space X.

Example 1.2. Let $X = Z^+$. Define $\underline{\mu}$ for all $A \subset Z^+$ by

$$\underline{\mu}(A) = \begin{cases} |A|-1, & \text{if } 0 < |A| < \infty; \\ 0, & \text{if } A = \phi; \\ \infty, & \text{if } A \text{ is infinite.} \end{cases}$$

$\underline{\mu}$ so defined is a superadditive set function.

Proposition 1.3. $\underline{\mu}$ *is monotone increasing.*

Proof. Let $A \subset B$. Then,

$$\begin{aligned} \underline{\mu}(B) &= \underline{\mu}(A \cup (B-A)) \\ &\geq \underline{\mu}(A) + \underline{\mu}(B-A) \\ &\geq \underline{\mu}(A). \end{aligned}$$

Proposition 1.4. $\underline{\mu}$ *is countably superadditive.*

Proof. Let $\{B_n\}$ be a sequence of pairwise disjoint subsets of X. Then, by induction on superadditivity, we have

$$\underline{\mu}\left(\bigcup_{n=1}^{i} B_n\right) \geq \sum_{n=1}^{i} \underline{\mu}(B_n), \quad \text{for all } i, \text{ and}$$

so, by Proposition 1.3,

$$\underline{\mu}\left(\bigcup_{n=1}^{\infty} B_n\right) \geq \sum_{n=1}^{i} \underline{\mu}(B_n).$$

This implies the required result.

Remark 1.5. It is clear from the previous proposition that if we are to get some reasonable results concerning 'countable' collections of sets, as distinct from finite collections, we must impose more conditions on the superadditive set function. This will be done when we define 'inner measures'.

Definition 1.6. *Let* η *be a set function from* $P(X)$ *to* $\mathbb{R}$. *A set* $A \subset X$ *is said to be* η*-measurable if, for any* $P \subset X$ *such that* $\eta(P)$ *is finite,*

$$\eta(P) = \eta(PA) + \eta(PA').$$

This is Caratheodory's method of defining measurability. Unless otherwise specified, this method is used systematically throughout this thesis. Also, from now on, η will always denote a set function from $P(X)$ to $\mathbb{R}$ with the property that $\eta(\phi) = 0$.

We shall let <u>$M(\eta)$</u> denote the collection of all η-measurable sets, and again, this notation is used systematically throughout.

<u>Proposition 1.7</u>. *Let $A, B \in M(\eta)$. Then, $A \cup B \in M(\eta)$.*

<u>Proof</u>. Let $P \subset X$ be such that $\eta(P)$ is finite. Then, since $A \in M(\eta)$,

$$\eta(PB) = \eta(PBA) + \eta(PBA'), \qquad (1) \quad \text{and}$$

$$\eta(PB') = \eta(PB'A) + \eta(PB'A'). \qquad (2)$$

Adding equations (1) and (2), and using the measurability of B, we have

$$\eta(P) = \eta(PBA) + \eta(PBA') + \eta(PB'A) + \eta(PB'A'). \qquad (3)$$

Replacing P in (3) by $P(A \cup B)$, we have

$$\eta(P(A \cup B)) = \eta(PBA) + \eta(PBA') + \eta(PB'A) + \eta(\phi). \qquad (4)$$

Subtracting (4) from (3), we have the required result.

<u>Proposition 1.8</u>. *If $A_1, \ldots, A_n \in M(\eta)$, then $\bigcup_{i=1}^{n} A_i \in M(\eta)$.*

<u>Proof</u>. By induction, using the previous proposition.

<u>Proposition 1.9</u>. *$M(\eta)$ is closed under complementation.*

Proof. Obvious.

Proposition 1.10. *If* $A, B \in M(\eta)$, *then* $A - B \in M(\eta)$.

Proof. $A-B = (A' \cup B)' \in M(\eta)$.

Corollary 1.11. $M(\eta)$ *is an algebra.*

Proof. This follows from Propositions 1.7 and 1.9.

Proposition 1.12. *Let* A *be* η*-measurable. Let* E_1 *and* E_2 *be any subsets of* A *and* A' *respectively such that* $\eta(E_1 \cup E_2)$ *is finite. Then,*

$$\eta(E_1 \cup E_2) = \eta(E_1) + \eta(E_2).$$

Proof. Since A is η-measurable and $\eta(E_1 \cup E_2)$ is finite, we have

$$\eta(E_1 \cup E_2) = \eta((E_1 \cup E_2)A) + \eta((E_1 \cup E_2)A')$$

$$= \eta(E_1) + \eta(E_2).$$

Definition 1.13. η *is said to be additive over a class of sets* $\mathcal{C} \subset P(X)$, *if, for* $A, B \in \mathcal{C}$, *whose union is also in* $\mathcal{C}$, *and* $A \quad B = \phi$,

$$\eta(A \cup B) = \eta(A) + \eta(B).$$

Example 1.14. A superadditive set function $\underline{\mu}$ need not be additive over $M(\underline{\mu})$. For, let $X = \{1,2\}$, and define $\underline{\mu}(\{1\}) = \underline{\mu}(\{2\}) = \underline{\mu}(\phi) = 0$, and $\underline{\mu}(X) = \infty$. Then, $M(\underline{\mu}) = P(X)$, but

$$\underline{\mu}(\{1\} \cup \{2\}) \neq \underline{\mu}(\{1\}) + \underline{\mu}(\{2\})$$

Proposition 1.15. *Let* $A_1, \ldots, A_n \in M(\eta)$ *and be pairwise disjoint. Then, for any* $P \subset X$ *such that* $\eta(P)$ *is finite,*

$$\eta\left(\bigcup_{i=1}^{n} PA_i\right) = \sum_{i=1}^{n} \eta(PA_i).$$

Proof. We use induction. Since $A_2 \in M(\eta)$, we have

$$\eta(P) = \eta(PA_2) + \eta(PA_2').$$

Replacing P by $P(A_1 \cup A_2)$, we have

$$\eta\left(\bigcup_{i=1}^{2} PA_i\right) = \sum_{i=1}^{2} \eta(PA_i)$$

Hence, the proposition holds true for $n = 2$. Suppose that it holds true for $n = k$, i.e.,

$$\eta\left(\bigcup_{i=1}^{k} PA_i\right) = \sum_{i=1}^{k} \eta(PA_i)$$

To prove that it is true for $n = k+1$, we observe that, since $A_{k+1} \in M(\eta)$, we have

$$\eta(P) = \eta(PA_{k+1}) + \eta(PA'_{k+1}).$$

Replacing P by $\bigcup_{i=1}^{k+1} PA_i$, we have

$$\eta\left(\bigcup_{i=1}^{k+1} PA_i\right) = \eta(PA_{k+1}) + \eta\left(\bigcup_{i=1}^{k} PA_i\right)$$

$$= \eta(PA_{k+1}) + \sum_{i=1}^{k} \eta(PA_i)$$

$$= \sum_{i=1}^{k+1} \eta(PA_i).$$

Hence, it is true for all n.

Proposition 1.16. *If $\eta(X)$ is finite, then η is finitely additive over $M(\eta)$.*

Proof. This follows from Proposition 1.15 upon replacing P by X.

Proposition 1.17. *If $\eta(X)$ is finite, then, for $A,B \in M(\eta)$,*

$$\eta(A \cup B) = \eta(A) + \eta(B) + \eta(AB).$$

Proof. $\eta(A \cup B) = \eta(A \cup (B-A))$

$$= \eta(A) + \eta(B-A)$$

$$= \eta(A) + \eta(B-BA)$$

$$= \eta(A) + \eta(B) - \eta(BA).$$

Example 1.18. If $\eta(X)$ is finite, then η is finitely additive over $M(\eta)$, which is an algebra. However, $\underline{\mu}$ need not be countably additive over $M(\underline{\mu})$. The following is an example.

For all $A \subset Z^+$, define

$$\underline{\mu}(A) = \begin{cases} 0, & \text{if } A' \text{ is infinite;} \\ 1, & \text{if } A' \text{ is finite.} \end{cases}$$

Then, $M(\underline{\mu}) = \{A$: if A is finite, then A' is infinite, and if A

is infinite, then A' is finite}. η is then finitely but not countably additive over $M(\underline{\mu})$.

Definition 1.19. *A set A is said to be* η*-null if A is* η*-measurable and* $\eta(A) = 0$.

Proposition 1.20. *A subset of a* $\underline{\mu}$*-null set is* $\underline{\mu}$*-null.*

Proof. Let $A \subset B$ and B be $\underline{\mu}$-null. By monotonicity, Proposition 1.3, $\underline{\mu}(A) = 0$; we shall show that $A \varepsilon M(\underline{\mu})$. For any $P \subset X$ such that $\underline{\mu}(P)$ is finite,

$$\begin{aligned} \underline{\mu}(P) &\geq \underline{\mu}(PA) + \underline{\mu}(PA') \\ &= \underline{\mu}(PA') \\ &\geq \underline{\mu}(PB') \\ &= \underline{\mu}(PB) + \underline{\mu}(PB') \\ &= \underline{\mu}(P), \quad \text{since } B \varepsilon M(\underline{\mu}). \end{aligned}$$

And so, $A \varepsilon M(\underline{\mu})$.

Remark 1.21. $\underline{\mu}(A) = 0$ does not imply that A is $\underline{\mu}$-measurable. The following is an example.

Let $X = \{1,2,3\}$, and define

$$\underline{\mu}(A) = \begin{cases} |A|-1, & \text{if } A \neq \phi, \ A \subset X; \\ 0, & \text{if } A = \phi. \end{cases}$$

Then, $\underline{\mu}(\{1\}) = 0$, but $\{1\}$ is not $\underline{\mu}$-measurable, because $\underline{\mu}(X) \neq \underline{\mu}(\{1\}) + \underline{\mu}(\{2,3\})$.

Definition 1.22. *An inner measure on a space X is a superadditive set function, μ_Δ, from $P(X)$ to $\mathbb{R}$ such that:*

> *if $\{B_n\}$ is a decreasing sequence of subsets of X, and $\mu_\Delta(B_1)$ is finite, then*
>
> $$\lim_{n\to\infty} \mu_\Delta(B_n) = \mu_\Delta\left(\bigcap_{n=1}^{\infty} B_n\right).$$

We shall call the above condition the *'i/m condition'*. Throughout, μ_Δ will denote an inner measure on a space X.

Remark 1.23. We note that in Remark 1.21, the superadditive set function $\underline{\mu}$ is an inner measure.

Remark 1.24. The introduction of the i/m condition in Definition 1.22 permits the change of limits which is crucial in obtaining countable collections of sets (Remark 1.5). It is important that we did not specify the kind of sets in the sequence $\{B_n\}$; this is an important feature in constructing our theory of inner measures. Also, we restrict $\mu_\Delta(B_1)$ to be finite so as to guarantee the existence of finite limits for the sequence $\{\mu_\Delta(B_n)\}$, since it is monotonic and bounded. Nevertheless, the restriction is intended to keep in accordance with that of $\eta(P)$ in Definition 1.6, in defining η-measurability.

Remark 1.25. From Example 1.18, we see that a superadditive set function need not be an inner measure.

Remark 1.26. We shall call a set function μ_* a Caratheodory's inner measure if μ_* is induced by an outer measure μ^* in the form of

$$\mu_*(A) = \sup\{\mu^*(M) : M \subset A,\ M \text{ is } \mu^*\text{-measurable}\}.$$

We shall show that Caratheodory's inner measure is an inner measure. The proof that it is a superadditive set function is straight-forward and is omitted. We shall only prove that μ_* satisfies the i/m condition. Before doing that, we first state a definition and a lemma.

Definition 1.27. *Let* α, β *be set functions from* $P(X)$ *to* $\mathbb{R}$. *Let* $K \subset A \subset X$. K *is said to be an α-measurable kernel, with respect to β, for* A, *if* K *is α-measurable and β-measurable, and* $\alpha(K) = \beta(A)$.

If there can be no ambiguity, 'with respect to β' will be dropped.

Lemma 1.28. *Let* μ_* *be Caratheodory's inner measure induced by an outer measure* μ^*. *Then, there exists a* μ^**-measurable kernel, with respect to* μ_*, *for any* $A \subset X$.

Proof. If $\mu_*(A) = \infty$, the result holds. Suppose that $\mu_*(A)$ is finite. Then, given any n, there exists $M_n \subset A$, M_n being μ^*-measurable, such that

$$\mu_*(A) - \frac{1}{n} < \mu^*(M_n)$$

$$\leq \mu^*\left(\bigcup_{n=1}^{\infty} M_n\right).$$

Hence, $\mu_*(A) \leq \mu^*(\bigcup_{n=1}^{\infty} M_n)$. Equality follows from the monotonicity of μ_*. $\bigcup_{n=1}^{\infty} M_n$ is μ^*-measurable since $M(\mu^*)$ is a σ-algebra [Munroe [7], Theorem 11.2, p. 87], and also, it is fairly trivial to check that $\bigcup_{n=1}^{\infty} M_n$ is also μ_*-measurable. Hence, $\bigcup_{n=1}^{\infty} M_n$ is a μ^*-measurable kernel for A.

Proposition 1.29. *Caratheodory's inner measure,* μ_*, *is an inner measure.*

Proof. In view of Remark 1.26, we need only prove the i/m condition. Let $\{B_n\}$ be a decreasing sequence of sets such taht $\mu_*(B_1)$ is finite. Corresponding to each B_n, there exists a μ^*-measurable kernel K_n. Since $\{B_n\}$ is a decreasing sequence of sets, $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n = \bigcap_{n=1}^{\infty} B_n$. By [Munroe [7], Corollary 10.8.1, p. 84], we have

$$\mu^*\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} K_n\right) \geq \overline{\lim_{n\to\infty}}\, \mu^*(K_n).$$

Hence,

$$\mu_*\left(\bigcap_{n=1}^{\infty} B_n\right) \geq \mu_*\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} K_n\right)$$

$$= \mu^*\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} K_n\right), \quad \text{since } \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} K_n \in M(\mu^*),$$

$$\geq \overline{\lim_{n\to\infty}}\, \mu^*(K_n)$$

$$= \overline{\lim_{n\to\infty}}\ \mu_*(B_n)$$

$$= \lim_{n\to\infty} \mu_*(B_n), \quad \text{since limit exists,}$$

$$\geq \mu_*\left(\bigcap_{n=1}^{\infty} B_n\right).$$

Remark 1.30. There are two possible ways of generating an inner measure: Lebesgue's method (which we shall specify below) and Caratheodory's method (which we have stated in Remark 1.26). We shall show that an inner measure is not necessarily generated by either method. We first define a Lebesgue's inner measure.

Definition 1.31. *Let μ^* be an outer measure and $\mu^*(X)$ be finite. A set function μ_1 generated in the form of*

$$\mu_1(A) = \mu^*(X) - \mu^*(A'), \quad \textit{for all} \quad A \subset X,$$

is called a Lebesgue's inner measure.

μ_1 so defined need not be an inner measure. For, suppose it is an inner measure. Then,

$$\mu_1(X) = \mu^*(X), \quad \text{and so,}$$

$$\mu^*(A) = \mu_1(X) - \mu_1(A').$$

Let A, B be any subsets of X:

$$\begin{aligned}\mu_1(X) - \mu_1((A \cup B)') &= \mu^*(A \cup B) \\ &\le \mu^*(A) + \mu^*(B) \\ &= \mu_1(X) - \mu_1(A') + \mu_1(X) - \mu_1(B').\end{aligned}$$

Hence, we have

$$\mu_1(A') + \mu_1(B') \le \mu_1(X) + \mu_1(A'B'). \qquad (*)$$

Now, let $X = \{1,2,3\}$, and define

$$\mu_\Delta(A) = \begin{cases} 0, & \text{if } |A| \le 1; \\ 1, & \text{if } |A| \ge 2. \end{cases}$$

μ_Δ so defined is an inner measure. Let $A = \{1\}$, and $B = \{2\}$. Applying the result of (*), we have a contradiction.

Next, we consider Caratheodory's inner measure.

Let $X = \{1,2,3\}$. Define μ_Δ as in Remark 1.21. μ_Δ so defined is an inner measure. Now, suppose μ_Δ is induced by an outer measure μ^* as in Remark 1.26. Suppose that $\{1\}$ is μ^*-measurable; then $\{2,3\}$ must be μ^*-measurable since $\mathcal{M}(\mu^*)$ is a σ-algebra. Hence, we have

$$\mu_\Delta(\{1,2,3\}) = \mu_\Delta(\{1\}) + \mu_\Delta(\{2,3\}),$$

which is not the case, and so $\{1\}$ is not μ^*-measurable, and $\{2,3\}$ is not μ^*-measurable. Similarly, $\{2\}$ and $\{1,3\}$, $\{3\}$ and $\{1,2\}$ are all non-μ^*-measurable. Thus only ϕ and X are μ^*-measurable, and then by our supposition, $\mu_\Delta(\{2,3\})$ must be 0 if μ^* induces μ_Δ -- but $\mu_\Delta(\{2,3\}) = 1$.

<u>Proposition 1.32</u>. *If* $A_1,\ldots,A_n,\ldots \in M(\mu_\Delta)$, *then* $\bigcup_{n=1}^{\infty} A_n \in M(\mu_\Delta)$.

<u>Proof</u>. Let $P \subset X$ be such that $\mu_\Delta(P)$ is finite. Let $M_i = \bigcup_{n=1}^{i} A_n$. From the increasing sequence of sets $\{PM_i\}$, we have an increasing sequence $\{\mu_\Delta(PM_i)\}$, and from $\{PM_i'\}$, we have a decreasing sequence $\{\mu_\Delta(PM_i')\}$. Both real sequences converge, and so,

$$\begin{aligned}\mu_\Delta(P) &= \lim_{i\to\infty} \mu_\Delta(PM_i') + \lim_{i\to\infty} \mu_\Delta(PM_i)\\ &= \mu_\Delta(\bigcap_{i=1}^{\infty} PM_i') + \lim_{i\to\infty} \mu_\Delta(PM_i), \text{ by the i/m condition,}\\ &\leq \mu_\Delta(\bigcap_{i=1}^{\infty} PM_i') + \mu_\Delta(\bigcup_{i=1}^{\infty} PM_i)\\ &\leq \mu_\Delta(P).\end{aligned}$$

Hence, $\bigcup_{i=1}^{\infty} M_i \in M(\mu_\Delta)$, and thus, $\bigcup_{n=1}^{\infty} A_n \in M(\mu_\Delta)$.

<u>Corollary 1.33</u>. $M(\mu_\Delta)$ *is a* σ*-algebra.*

<u>Proof</u>. This follows from Corollary 1.11 and Proposition 1.32.

<u>Proposition 1.34</u>. *Let* $A_1,\ldots,A_n,\ldots \in M(\mu_\Delta)$ *be pairwise disjoint.*

Then, for any $P \subset X$ *such that* $\mu_\Delta(P)$ *is finite,*

$$\mu_\Delta\left(\bigcup_{n=1}^{\infty} PA_n\right) = \sum_{n=1}^{\infty} \mu_\Delta(PA_n).$$

Proof. Since $\{P(\bigcup_{n=1}^{i} A_n)'\}$ is a monotonically decreasing sequence of sets, we have

$$\lim_{i\to\infty} \mu_\Delta(P(\bigcup_{n=1}^{i} A_n)') = \mu_\Delta(\bigcap_{i=1}^{\infty} P(\bigcup_{n=1}^{i} A_n)')$$

$$= \mu_\Delta(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{i} PA_n')$$

$$= \mu_\Delta(\bigcap_{n=1}^{\infty} PA_n').$$

From Proposition 1.15, $\mu_\Delta(\bigcup_{n=1}^{i} PA_n) = \sum_{n=1}^{i} \mu_\Delta(PA_n)$. Since $\bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{i} A_n \in M(\mu_\Delta)$, we have

$$\mu_\Delta(\bigcup_{n=1}^{\infty} PA_n) + \mu_\Delta(\bigcap_{n=1}^{\infty} PA_n') = \mu_\Delta(\bigcup_{n=1}^{i} PA_n) + \mu_\Delta(P(\bigcup_{n=1}^{i} A_n)')$$

$$= \sum_{n=1}^{i} \mu_\Delta(PA_n) + \mu_\Delta(P(\bigcup_{n=1}^{i} A_n)')$$

Take limits with respect to i, we have

$$\mu_\Delta(\bigcup_{n=1}^{\infty} PA_n) + \mu_\Delta(\bigcap_{n=1}^{\infty} PA_n') = \lim_{i\to\infty} \sum_{n=1}^{i} \mu_\Delta(PA_n) + \lim_{i\to\infty} \mu_\Delta(P(\bigcup_{n=1}^{i} A_n)')$$

$$= \sum_{n=1}^{\infty} \mu_\Delta(PA_n) + \mu_\Delta(\bigcap_{n=1}^{\infty} PA_n').$$

And so, $\mu_\Delta(\bigcup_{n=1}^{\infty} PA_n) = \sum_{n=1} \mu_\Delta(PA_n)$.

<u>Corollary 1.35</u>. *For any sequence* $\{A_n\}$ *of sets from* $M(\mu_\Delta)$, *and any* $P \subset X$ *such that* $\mu_\Delta(P)$ *is finite,*

$$\mu_\Delta(\bigcup_{n=1}^{\infty} PA_n) \leq \sum_{n=1}^{\infty} \mu_\Delta(PA_n).$$

<u>Proof</u>. We decompose $\bigcup_{n=1}^{\infty} A_n$ into the union of pairwise disjoint, μ_Δ-measurable, sets. That is, we have

$$\mu_\Delta(\bigcup_{n=1}^{\infty} PA_n) = \mu_\Delta(\bigcup_{i=1}^{\infty} P(A_i - \bigcup_{n=0}^{i-1} A_n)),$$

where A_0 is defined to be ϕ. By the previous proposition, we have

$$\mu_\Delta(\bigcup_{n=1}^{\infty} PA_n) = \sum_{i=1}^{\infty} \mu_\Delta(P(A_i - \bigcup_{n=0}^{i-1} A_n))$$

$$\leq \sum_{i=1} \mu_\Delta(PA_i).$$

In case if $\mu_\Delta(X)$ is finite, then we have, in general,

$$\mu_\Delta(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu_\Delta(A_n).$$

CHAPTER II

CHARACTERIZATION OF INNER MEASURABILITY AND THE INDUCTION OF INNER MEASURES

In this chapter, we assume that for each subset of X, there exists a $\underline{\mu}$-measurable kernel with respect to $\underline{\mu}$, and that $\underline{\mu}(X)$ is finite. We recall that $\underline{\mu}$ is a superadditive set function (Definition 1.1) used throughout this thesis. We then characterize $\underline{\mu}$-measurability. Basing on the characterization, and assuming that μ^* is a regular outer measure (which we shall define below) and that $\mu^*(X)$ is finite, we prove easily the equivalence of Young's (see Introduction) and Caratheodory's definitions of inner measurability. Also, we prove that Lebesgue's inner measure (Definition 1.31) and Caratheodory's inner measure (Remark 1.26) are identical, and Lebesgue's condition for measurability (i.e., A is *measurable* if $\mu_*(A) = \mu^*(A)$, see Introduction) is also equivalent to Young's and Caratheodory's.

Definition 2.1. *Let* α, β *be set functions from* $P(X)$ *to* $\mathbb{R}$. *Let* $A \subset F \subset X$. *F is said to be an* *α-measurable cover, with respect to β, for A,* *if F is* α*-measurable and* β*-measurable, and* $\alpha(F) = \beta(A)$.

If there can be no ambiguity, 'with respect to β' will be dropped.

Definition 2.2. *If every subset of X has a* μ^**-measurable cover, with respect to the outer measure* μ^*, *then* μ^* *is said to be a* *regular outer measure* [*Munroe* [7], *p. 94*].

Proposition 2.3. *A set* $A \subset X$ *is* $\underline{\mu}$*-measurable iff* $\underline{\mu}(A) + \underline{\mu}(A') = \underline{\mu}(X)$.

Proof. If A is $\underline{\mu}$-measurable, then obviously, we have $\underline{\mu}(A) + \underline{\mu}(A') = \underline{\mu}(X)$. We now prove the converse. Let K_1 and K_2 be $\underline{\mu}$-measurable kernels for A and A' respectively, and for any $P \subset X$, let K be a $\underline{\mu}$-measurable kernel for P. Since K_1 is $\underline{\mu}$-measurable, we have

$$\underline{\mu}(K_1) + \underline{\mu}(K_1') = \underline{\mu}(X).$$

Also, we have assumed that

$$\underline{\mu}(A) + \underline{\mu}(A') = \underline{\mu}(X).$$

From the two equations above, we obtain

$$\underline{\mu}(A') = \underline{\mu}(K_1').$$

From $K_2 \subset A' \subset K_1'$, we have $KK_2 \subset KA' \subset KK_1'$, and obtain

$$\begin{aligned}
\underline{\mu}(KK_1') - \underline{\mu}(KA') &\leq \underline{\mu}(KK_1') - \underline{\mu}(KK_2) \\
&= \underline{\mu}(K(K_1'-K_2)), \text{ by Prop. 1.16,} \\
&\leq \underline{\mu}(K_1'-K_2) \\
&= \underline{\mu}(K_1') - \underline{\mu}(K_2) \\
&= \underline{\mu}(A') - \underline{\mu}(A') \\
&= 0.
\end{aligned}$$

Hence, $\underline{\mu}(KA') = \underline{\mu}(KK_1')$. It follows that

$$\begin{aligned}\underline{\mu}(PA) + \underline{\mu}(PA') &\geq \underline{\mu}(KK_1) + \underline{\mu}(KA') \\ &= \underline{\mu}(KK_1) + \underline{\mu}(KK_1') \\ &= \underline{\mu}(K) \\ &= \underline{\mu}(P) \\ &\geq \underline{\mu}(PA) + \underline{\mu}(PA').\end{aligned}$$

And so, A is $\underline{\mu}$-measurable.

Proposition 2.4. *A set* $A \subset X$ *is* $\underline{\mu}$*-measurable iff for all* $D \subset X$ *such that* $A \cap D = \emptyset$,

$$\underline{\mu}(A) + \underline{\mu}(D) = \underline{\mu}(A \cup D).$$

Proof. The proof is straight-forward if we apply Proposition 2.3, and is omitted.

Proposition 2.5. *A set* $A \subset X$ *is* $\underline{\mu}$*-measurable if there exist* $\underline{\mu}$*-measurable sets* E *and* F *such that* $E \subset A \subset F$ *and* $\underline{\mu}(F-E) = 0$.

Proof. A-E is a subset of F-E which is $\underline{\mu}$-null. By Proposition 1.20, A-E is $\underline{\mu}$-null, and so $A = (A-E) \cup E \in M(\underline{\mu})$.

Proposition 2.6. *Let* A, B *be any two subsets of* X. *If there exist* $\underline{\mu}$*-measurable sets* $M_1 \supset A$, *and* $M_2 \supset B$ *such that* $\underline{\mu}(M_1 M_2) = 0$, *then* *(a)* $\underline{\mu}(A \cup B) = \underline{\mu}(A) + \underline{\mu}(B)$;

(b) the union of the $\underline{\mu}$*-measurable kernels for* A *and* B *is a* $\underline{\mu}$*-measurable kernel for* $A \cup B$.

<u>Proof</u>. Let K_1, K_2 and K be $\underline{\mu}$-measurable kernels for A, B and $A \cup B$ respectively, and M_1 and M_2 be as stated. Then,

$$K\text{-}M_2 \subset A, \quad \text{and} \quad K\text{-}M_1 \subset B.$$

Also,

$$\begin{aligned} \underline{\mu}(K\text{-}M_2) + \underline{\mu}(K\text{-}M_1) &= \underline{\mu}((K\text{-}M_2) \cup (K\text{-}M_1)) \\ &= \underline{\mu}(K\text{-}M_1M_2) \\ &= \underline{\mu}(K) - \underline{\mu}(M_1M_2) \\ &= \underline{\mu}(K). \end{aligned}$$

Hence, we have

$$\begin{aligned} \underline{\mu}(A) + \underline{\mu}(B) &\geq \underline{\mu}(K) \\ &= \underline{\mu}(A \cup B). \end{aligned}$$

Also, $\underline{\mu}(A \cup B) \geq \underline{\mu}(K_1 \cup K_2)$

$= \underline{\mu}(K_1) + \underline{\mu}(K_2)$

$= \underline{\mu}(A) + \underline{\mu}(B)$, and so, both (a) and (b) are proved.

Remark 2.7. In the previous proposition, if $B = A'$, then A is $\underline{\mu}$-measurable by Proposition 2.3. This result holds analogously for outer measures.

Lemma 2.8. [*Munroe* [7], *Theorem 12.2, p. 96*]. *If* μ^* *is a regular outer measure and* $\mu^*(X)$ *is finite, then* $A \subset X$ *is* μ^**-measurable iff*

$$\mu^*(A) + \mu^*(A') = \mu^*(X).$$

Proof. Omitted.

Proposition 2.9. *Let* μ^* *be an outer measure, and* $\mu^*(X)$ *be finite. Then,* μ^* *is regular iff the following two definitions are identical:*

(a) $\mu_1(A) = \mu^*(X) - \mu^*(A')$;

(b) $\mu_*(A) = sup\{\mu^*(M) : M \subset A, M \in \mathcal{M}(\mu^*)\}$.

Proof. Suppose μ^* is regular. Then, there exists a μ^*-measurable cover F for A', and let K be a μ^*-measurable kernel, with respect to μ_*, for A. Then, we have

$$\begin{aligned} \mu_1(A) &= \mu^*(X) - \mu^*(A') \\ &= \mu^*(X) - \mu^*(F) \\ &= \mu^*(F') \\ &\leq \mu_*(A), \quad \text{since } F' \subset A \text{ and } F \in \mathcal{M}(\mu^*). \end{aligned}$$

Also,

$$\mu_*(A) = \mu^*(K)$$

$$= \mu^*(X) - \mu^*(K')$$

$$\leq \mu^*(X) - \mu^*(A')$$

$$= \mu_1(A).$$

Hence, $\mu_1(A) = \mu_*(A)$ for all $A \subset X$.

Conversely, let K be a μ^*-measurable kernel, with respect to μ_1, for A. Then,

$$\mu^*(X) - \mu^*(A') = \mu^*(K),$$

i.e.,

$$\mu^*(A') = \mu^*(X) - \mu^*(K)$$

$$= \mu^*(K').$$

Hence, K' is a μ^*-measurable cover for A'.

Remark 2.10. From this proposition, it follows that μ_1 is, in this case, an inner measure because μ_* is, by Remark 1.26, and Lebesgue's is identical with Caratheodory's.

Proposition 2.11. *Let* μ^* *be a regular outer measure and* $\mu^*(X)$ *is finite.* μ_* *is Caratheodory's inner measure induced by* μ^*. *The following are equivalent:*

(a) $A \varepsilon M(\mu_*)$;

(b) $A \varepsilon M(\mu^*)$;

(c) $\mu_*(A) = \mu^*(A)$.

<u>Proof</u>. Since $X \varepsilon M(\mu^*)$, we have $\mu^*(X) = \mu_*(X)$. (a) implies (b): Let $A \varepsilon M(\mu_*)$, and let K_1 and K_2 be μ^*-measurable kernels for A and A' respectively. Then, we have

$$\begin{aligned} \mu^*(X) &= \mu_*(X) \\ &= \mu_*(A) + \mu_*(A') \\ &= \mu^*(K_1) + \mu^*(K_2) \\ &= \mu^*(X) - \mu^*(K_1') + \mu^*(X) - \mu^*(K_2'). \end{aligned}$$

Hence,

$$\begin{aligned} \mu^*(X) &= \mu^*(K') + \mu^*(K') \\ &\geq \mu^*(A') + \mu^*(A) \\ &\geq \mu^*(X). \end{aligned}$$

By Lemma 2.8, $A \varepsilon M(\mu^*)$.

(b) implies (c): This is obvious.

(c) implies (a): Let $\mu_*(A) = \mu^*(A)$. Then, we have

$$\mu_*(A) + \mu_*(A') = \mu^*(A) + \mu_*(A')$$

$= \mu^*(A) + \mu^*(X) - \mu^*(A)$, by Prop. 2.9,

$= \mu^*(X)$

$= \mu_*(X)$.

Remark 2.12. In this case, we have proved farily easily that Lebesgue's condition for measurability is equivalent to Caratheodory's, and hence, also to Young's.

Definition 2.13. *Define* $\mu^x(A)$ *and* $\mu_0(A)$ *by*

(a) $\mu^x(A) = inf\{\mu(F) : F \supset A, F \in R\}$, *if such a cover exists;*

$= \infty$, *otherwise;*

(b) $\mu_0(A) = sup\{\mu(E) : E \subset A, E \in R\}$.

Here, μ is a measure on a ring R which has the property that $\mu(A)$ is finite for all $A \in R$, and also, that whenever $\{A_n\}$ is a sequence of sets from R such that $\sum_{n=1}^{\infty} \mu(A_n)$ is finite, then $\bigcup_{n=1}^{\infty} A_n \in R$.

μ^x is known to be an outer measure [12], and the proof that μ_0 is an inner measure is merely a duplicate of that given in Proposition 1.29. T. P. Srinivasan [12] proves that $M(\mu^x) = M(\mu_0)$. Also, he shows that it is not necessarily true that $\mu^x|M(\mu^x) = \mu_0|M(\mu_0)$. In our case, we shall study the relation between $M(\mu_0)$ and $M(\mu_*)$, and also μ_0 and μ_*, where μ_* is a

Caratheodory's inner measure induced by μ^X (Remark 1.26).

Proposition 2.14. $M(\mu_0) = M(\mu_*)$.

Proof. Direct proving is not easy. Instead, we apply a proof similar to that given in [12] and get $M(\mu^X) = M(\mu_*)$. Since we have pointed out already that $M(\mu^X) = M(\mu_0)$, we have $M(\mu_0) = M(\mu_*)$.

Remark 2.15. Since μ^X is a measure on $M(\mu^X)$, μ_* is then a measure on $M(\mu_*)$ because, obviously, $\mu_*|M(\mu_*) = \mu^X|M(\mu^X)$. We have also mentioned that $\mu^X|M(\mu^X)$ is not necessarily equal to $\mu_0|M(\mu_0)$, we conclude that the two inner measures so induced are not identical even on $M(\mu_*)$. More specifically, $\mu_*(A) \geq \mu_0(A)$, for all $A \subset X$. However, if we define $\mu_X(A)$ by

$$\mu_X(A) = sup\{\mu_0(M) : M \subset A, M \in M(\mu_0)\},$$

then it is routine to check that for any $A \subset X$, there exists a μ_0-measurable kernel, with respect to μ_X, for A. If $\mu_X(A) = \infty$, then $\mu_X(A) = \mu_0(A)$. We consider those sets A such that $\mu_0(A)$ is finite and let K be a μ_0-measurable kernel, with respect to μ_X, and E be a μ-measurable kernel, with respect to μ_0, for A. Then,

$$\begin{aligned} \mu_X(A) &= \mu_0(K) \\ &\leq \mu_0(A) \\ &= \mu(E) \end{aligned}$$

$$\leq \mu_x(A).$$

And so, in this case, $\mu_x(A) = \mu_0(A)$.

CHAPTER III

CONTRACTION OF INNER MEASURES AND INDUCTION OF OUTER MEASURES

The definition of an inner measure μ_Δ allows us to obtain a σ-algebra of μ_Δ-measurable sets. In the case of Caratheodory's inner measure μ_*, μ_* is an inner measure which is a measure on $M(\mu_*)$. It is not, however, generally true that an inner measure μ_Δ is always a measure on $M(\mu_\Delta)$. In Example 1.14, the superadditive set function $\underline{\mu}$ is an inner measure and $M(\underline{\mu})$ is a σ-algebra. However, $\underline{\mu}$ is not a measure on $M(\underline{\mu})$.

In this chapter, we shall 'contract' (which we shall define below) the inner measure μ_Δ, so that it is always a measure on $M(\mu_\Delta)$, and that it always generates an outer measure.

Definition 3.1. *For any* $A \subset X$, *define* $\underline{\mu_C(A)}$ *by*

$$\mu_C(A) = \mu_\Delta(CA),$$

where C *is* μ_Δ*-measurable and* $\mu_\Delta(C)$ *is finite.*

μ_C is called a *'contraction'* of μ_Δ.

Definition 3.2. *A set* $A \subset X$ *is* $\underline{\mu_C\text{-measurable}}$ *iff for any* $P \subset X$ *such that* $\mu_\Delta(P)$ *is finite,*

$$\mu_C(P) = \mu_C(PA) + \mu_C(PA').$$

[The restriction that $\mu_\Delta(P)$ be finite is immaterial in this chapter. It becomes important in Chapter IV.]

The following proposition is an obvious consequence of results of Chapter I, and the fact that $\mu_C(X)$ is finite. The proof is omitted.

Proposition 3.3. *If* μ_Δ *is an inner measure on* X, $C \in M(\mu_\Delta)$ *and* $\mu_\Delta(C)$ *is finite, then the contraction* μ_C *of* μ_Δ *has the following properties:*

(a) it is an inner measure on X;

(b) $M(\mu_C)$ *is a* σ*-algebra;*

(c) let $A_1, A_2, \ldots, A_n, \ldots \in M(\mu_\Delta)$ *be pairwise disjoint. Then, for any* $P \subset X$,

$$\mu_C\left(\bigcup_{n=1}^{\infty} PA_n\right) = \sum_{n=1}^{\infty} \mu_C(PA_n);$$

(d) it is countabl additive over $M(\mu_C)$.

Proposition 3.4. $M(\mu_\Delta) \subset M(\mu_C)$.

Proof. Let $B \in M(\mu_\Delta)$. For any $P \subset X$, we have

$$\begin{aligned} \mu_C(P) &\geq \mu_C(PB) + \mu_C(PB') \\ &= \mu_\Delta(CPB) + \mu_\Delta(CPB') \\ &= \mu_\Delta(CP), \quad \text{since } B \in M(\mu_\Delta), \\ &= \mu_C(P), \quad \text{and so, } B \in M(\mu_C). \end{aligned}$$

Example 3.5. It follows from the previous proposition that μ_C is also a measure on $M(\mu_\Delta)$. However, the reverse inclusion need not hold:

Let $A \subset (X-C)$ be such that A is not μ_Δ-measurable. Then, for any $P \subset X$, we have

$$\begin{aligned}\mu_C(P) &\geq \mu_C(PA) + \mu_C(PA')\\ &= \mu_\Delta(CPA) + \mu_\Delta(CPA')\\ &= \mu_\Delta(CP)\\ &= \mu_C(P).\end{aligned}$$

It follows that A is μ_C-measurable.

Definition 3.6. *For any* $A \subset X$, *define* $\mu^C(A)$ *by*

$$\mu^C(A) = \inf\{\mu_C(F) : F \supset A,\ F \in M(\mu_C)\},$$

where μ_C *is a contraction of* μ_Δ *defined in Definition 3.1.*

Remark 3.7. Since $X \in M(\mu_C)$, F always exists. It is obvious from the definition that $\mu^C(A)$ is finite for all $A \subset X$, and $\mu^C(\phi) = 0$, and also, μ^C is monotone.

Proposition 3.8. *For any* $A \subset X$, *there exists a* μ_C*-measurable set* $F \supset A$ *such that* $\mu_C(F) = \mu^C(A)$, *where* μ^C *is defined in Definition 3.6.*

<u>Proof</u>. Let A be any subset of X. Given $\frac{1}{n}$, there exists $F_n \in M(\mu_C)$ and $F_n \supset A$ such that

$$\mu_C(F_n) < \mu^C(A) + \frac{1}{n}.$$

This holds true for all n. Hence, we have

$$\mu_C\left(\bigcap_{n=1}^{\infty} F_n\right) < \mu^C(A) + \frac{1}{n}.$$

And so,

$$\mu_C\left(\bigcap_{n=1}^{\infty} F_n\right) \leq \mu^C(A).$$

Equality follows immediately from the monotonicity of μ^C. Since $M(\mu_C)$ is a σ-algebra, by Proposition 3.4(b), $\bigcap_{n=1}^{\infty} F_n \in M(\mu_C)$, and $\bigcap_{n=1}^{\infty} F_n$ is our F.

<u>Corollary 3.9</u>. *μ^C defined in Definition 3.6 is subadditive.*

<u>Proof</u>. Let $A \cap B = \phi$. Let $F_1, F_2 \in M(\mu_C)$ such that $F_1 \supset A$, $F_2 \supset B$, and $\mu_C(F_1) = \mu^C(A)$ and $\mu_C(F_2) = \mu^C(B)$. Then,

$$\begin{aligned}
\mu^C(A) + \mu^C(B) &= \mu_C(F_1) + \mu_C(F_2) \\
&= \mu_C(F_1 \cup F_2) + \mu_C(F_1F_2) \\
&= \mu^C(F_1 \cup F_2) + \mu_C(F_1F_2) \\
&\geq \mu^C(A \cup B).
\end{aligned}$$

Corollary 3.10. *In Proposition 3.9, F is a μ_C-measurable cover, with respect to μ^C, for A.*

Proof. We need only prove that $F \in M(\mu^C)$.

Let P be any subset of X, and $F_0 \in M(\mu_C)$ be such that $F_0 \supset P$ and $\mu_C(F\ ) = \mu^C(P)$. Then,

$$\begin{aligned} \mu^C(P) &= \mu_C(F_0) \\ &= \mu_C(F_0F) + \mu_C(F_0F'), \quad \text{since } F \in M(\mu_C), \\ &= \mu^C(F_0F) + \mu^C(F_0F') \\ &\geq \mu^C(PF) + \mu^C(PF'), \quad \text{since } \mu^C \text{ is monotone,} \\ &\geq \mu^C(P), \quad \text{by Corollary 3.9.} \end{aligned}$$

And so, $F \in M(\mu^C)$.

Corollary 3.11. *μ^C defined in Definition 3.6 is an outer measure.*

Proof. In view of Remark 3.7, we need only prove that μ^C is countably subadditive.

Let $\{B_n\}$ be any sequence of subsets of X, and F_n be the corresponding μ_C-measurable cover for B_n, for all n. Then, we have

$$\mu^C(\bigcup_{n=1}^{\infty} B_n) \leq \mu^C(\bigcup_{n=1}^{\infty} F_n)$$

$$= \mu_C\left(\bigcup_{n=1}^{\infty} F_n\right)$$

$$\leq \sum_{n=1}^{\infty} \mu_C(F_n), \quad \text{by Corollary 1.35,}$$

$$= \sum_{n=1}^{\infty} \mu^C(B_n).$$

<u>Remark 3.12</u>. Our model, in addition to being a generalization of inner measures as considered by Caratheodory, Lebesgue and Srinivasan, can also be used as a starting point in the development of Measure Theory. The concept of inner measures is also applied to topological spaces [Topsøe [13]].

In the following, we ask the question: Given the inner measure μ_C defined in Definition 3.1, we obtain a measure $\mu_C|M(\mu_C)$ on $M(\mu_C)$ which is a σ-algebra. We then induce an outer measure μ^C by Definition 3.7, and from μ^C, we induce Caratheodory's inner measure μ_C' (Remark 1.26). What is the relation between μ_C and μ_C', and between $M(\mu_C)$ and $M(\mu_C')$?

Obviously, $M(\mu_C')$ and $M(\mu^C)$ are both σ-algebras, and μ^C is a regular outer measure. Since $\mu^C(X)$ is finite, the outer-measurability criterion of Lemma 2.9 can be applied. Likewise, a μ^C-measurable kernel exists, with respect to μ_C', for each subset of X, by Lemma 1.28. Also, since $\mu_C'(X)$ is finite, the μ_C'-measurability criterion (Proposition 2.3) is applicable. It follows from Proposition 2.12 that $M(\mu^C) = M(\mu_C')$, and $\mu^C|M(\mu^C)$ and $\mu_C'|M(\mu_C')$ are identical. We can show easily that $M(\mu_C) \subset M(\mu^C)$:

Let $A \varepsilon M(\mu_C)$. Then, we have

$$\mu^C(A) + \mu^C(A') = \mu_C(A) + \mu_C(A')$$

$$= \mu_C(X)$$

$$= \mu^C(X), \quad \text{as desired.}$$

And so, we have the relation:

$$M(\mu_C) \subset M(\mu^C) = M(\mu_C').$$

It is unlikely that we can then relate μ_C and μ_C'. However, if we assume that a μ_C-measurable kernel exists, with respect to μ_C, for any subset of X, then we have

$$\mu_C(A) \leq \mu_C'(A), \quad \text{for all} \quad A \subset X.$$

We next consider a set function μ^0 as follows:

Definition 3.13. *For any $A \subset X$, define $\underline{\mu^0(A)}$ by*

$$\mu^0(A) = \mu_C(X) - \mu_C(A'),$$

where μ_C is the contracted inner measure defined in Definition 3.1.

Proposition 3.14. *Let μ^0 be the set function defined in Definition 3.13, and μ^C be the regular outer measure defined in Definition 3.6. Then, for each $A \subset X$, there exists a μ_C-measurable kernel, with respect to μ_C, for A, iff μ^0 and μ^C are identical.*

Proof. By assuming that for each $A \subset X$, there exists a μ_C-measurable kernel, with respect to μ_C, for A, we first prove that for each A, there exists a μ_C-measurable set $E \supset A$ such that $\mu_C(E) = \mu^0(A)$.

Let K be a μ_C-measurable kernel, with respect to μ_C, for A'. Then,

$$\begin{aligned} \mu^0(A) &= \mu_C(X) - \mu_C(A') \\ &= \mu_C(X) - \mu_C(K) \\ &= \mu_C(K'), \text{ and } K' \text{ is the desired} \end{aligned}$$

set E.

It follows that, for all $A \subset X$,

$$\mu^0(A) \geq \mu^C(A).$$

Now, let F be a μ_C-measurable cover, with respect to μ^C, for A. Then,

$$\begin{aligned} \mu^C(A) &= \mu_C(F) \\ &= \mu_C(X) - \mu_C(F') \\ &\geq \mu_C(X) - \mu_C(A') \\ &= \mu^0(A). \end{aligned}$$

Hence, we have $\mu^0(A) = \mu^C(A)$, for all $A \subset X$.

Conversely, we assume that μ^0 and μ^C are identical. By the way μ^C is defined, there exists a μ_C-measurable cover F, with respect to μ^C, for each $A \subset X$. Also, since we have assumed that $\mu^0(A) = \mu^C(A)$, we have

$$\mu_C(X) - \mu_C(A') = \mu_C(F).$$

This implies that

$$\begin{aligned} \mu_C(A') &= \mu_C(X) - \mu_C(F) \\ &= \mu_C(F'), \quad \text{and} \quad F' \end{aligned}$$

is a μ_C-measurable kernel, with respect to μ_C, for A'. That is, for each subset of X, there exists a μ_C-measurable kernel, with respect to μ_C.

<u>Corollary 3.15</u>. *Let μ^0 be the set function defined in Definition 3.13, and μ^C be the regular outer measure defined in Definition 3.6. Then, for each $A \subset X$, there exists a μ_C-measurable set $E \supset A$ such that $\mu_C(E) = \mu^0(A)$ iff μ^0 and μ^C are identical.*

<u>Proof</u>. By assuming the existence of such a μ_C-measurable set E in the corollary, we have $\mu^0(A) = \mu^C(A)$ for all $A \subset X$, by following the same proof given in Proposition 3.14. Conversely, if μ^0 and μ^C are identical, then, by the same proposition, there exists a μ_C-measurable kernel, with respect to μ_C, for each subset of X. This implies the existence of such a μ_C-measurable set E as stated in the corollary.

Remark 3.16. Outer measures are frequently obtained by inducing them from measures. We have shown that an outer measure can always be generated from an inner measure (Corollary 3.11), and in terms of an inner measure as defined in Definition 3.13.

CHAPTER IV

SEQUENCES OF CONTRACTIONS OF INNER MEASURES

In this chapter, we consider an increasing sequence $\{C_i\}$ of μ_Δ-measurable sets such that $\mu_\Delta(\bigcup_{i=1}^{\infty} C_i)$ is finite. $\{\mu_{C_i}\}$ is the associated sequence of inner measures contracted from μ_Δ, and obviously, $(M(\mu_{C_i}),\mu_{C_i})$ are measure spaces for all i.

Proposition 4.1. *(a) For any* $A \subset X$ *such that* $\mu_\Delta(A)$ *is finite,* $\mu_{\cup C_i}(A) = \lim_{i\to\infty} \mu_{C_i}(A)$;

(b) $M(\mu_{\cup C_i}) = \bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i})$; *where* $\overline{C}_i$ *is defined to be* $C_i - C_{i-1}$, *for* $i = 1,2,\ldots,$ *and* $C_0 = \emptyset$.

Proof. Part (a): Let $\mu_\Delta(A)$ be finite. Then,

$$\begin{aligned}
\mu_{\cup C_i}(A) &= \mu_\Delta(\bigcup_{i=1}^{\infty} C_iA) \\
&= \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_iA) \\
&= \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_iA), \text{ by Proposition 1.34,} \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_iA) \\
&= \lim_{n\to\infty} \mu_\Delta(\bigcap_{i=1}^{n} \overline{C}_iA) \\
&= \lim_{n\to\infty} \mu_\Delta(\bigcap_{i=1}^{n} (C_i - C_{i-1})A) \\
&= \lim_{n\to\infty} \mu_\Delta(C_nA)
\end{aligned}$$

$$= \lim_{n\to\infty} \mu_{C_n}(A), \quad \text{as desired.}$$

Part (b): We shall first show that $M(\mu_{\cup C_i}) \subset \bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i})$.
Let $B \in M(\mu_{\cup C_i})$. Then, for any $P \subset X$ such that $\mu_\Delta(P)$ is finite, we have

$$\mu_\Delta(\bigcup_{i=1}^{\infty} C_iP) = \mu_\Delta(\bigcup_{i=1}^{\infty} C_iPB) + \mu_\Delta(\bigcup_{i=1}^{\infty} C_iPB').$$

Also, since $\mu_\Delta(\bigcup_{i=1}^{\infty} C_iP) = \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_iP)$, we have

$$\mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_iP) = \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_iPB) + \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_iPB').$$

Applying Proposition 1.34, we have

$$\sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_iP) = \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_iPB) + \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_iPB')$$

$$= \sum_{i=1}^{\infty} (\mu_\Delta(\overline{C}_iPB) + \mu_\Delta(\overline{C}_iPB')),$$

which gives

$$\sum_{i=1}^{\infty} (\mu_\Delta(\overline{C}_iP) - (\mu_\Delta(\overline{C}_iPB) + \mu_\Delta(\overline{C}_iPB'))) = 0.$$

But, $\mu_\Delta(\overline{C}_iP) \geq \mu_\Delta(\overline{C}_iPB) + \mu_\Delta(\overline{C}_iPB')$, for all i, hence, we have

$$\mu_\Delta(\overline{C}_iP) = \mu_\Delta(\overline{C}_iPB) + \mu_\Delta(\overline{C}_iPB'), \quad \text{for all } i,$$

and so, $B \in \bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i})$.

Next, we shall prove that $\bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i}) \subset M(\mu_{\cup C_i})$. Let $B \in \bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i})$. Then, for any $P \subset X$ such that $\mu_\Delta(P)$ is finite, we have

$$\begin{aligned}
\mu_{\cup C_i}(P) &\geq \mu_{\cup C_i}(PB) + \mu_{\cup C_i}(PB') \\
&= \mu_\Delta(\bigcup_{i=1}^{\infty} C_i PB) + \mu_\Delta(\bigcup_{i=1}^{\infty} C_i PB') \\
&= \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_i PB) + \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_i PB') \\
&= \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_i PB) + \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_i PB'), \quad \text{by Proposition 1.34,} \\
&= \sum_{i=1}^{\infty} (\mu_\Delta(\overline{C}_i PB) + \mu_\Delta(\overline{C}_i PB')) \\
&= \sum_{i=1}^{\infty} (\mu_{\overline{C}_i}(PB) + \mu_{\overline{C}_i}(PB')) \\
&= \sum_{i=1}^{\infty} \mu_{\overline{C}_i}(P), \quad \text{since } B \in \bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i}), \\
&= \sum_{i=1}^{\infty} \mu_\Delta(\overline{C}_i P) \\
&= \mu_\Delta(\bigcup_{i=1}^{\infty} \overline{C}_i P), \quad \text{by Proposition 1.34,} \\
&= \mu_\Delta(\bigcup_{i=1}^{\infty} C_i P) \\
&= \mu_{\cup C_i}(P).
\end{aligned}$$

Hence, $B \in M(\mu_{\cup C_i})$.

<u>Proposition 4.2</u>. $M(\mu_{C_1}) \supset M(\mu_{C_2}) \supset M(\mu_{C_3}) \supset \dots$.

<u>Proof</u>. From the previous proposition, we have $M(\mu_{\cup C_i}) = \bigcap_{i=1}^{\infty} M(\mu_{\overline{C}_i})$. This implies that

$$M(\mu_{C_{n+1}}) = \bigcap_{i=1}^{n+1} M(\mu_{\overline{C}_i}), \quad \text{for } n = 1,2,3,\dots .$$

It follows that

$$\begin{aligned} M(\mu_{C_{n+1}}) &= \bigcap_{i=1}^{n+1} M(\mu_{C_i - C_{i-1}}) \\ &= \left(\bigcap_{i=1}^{n} M(\mu_{C_i - C_{i-1}})\right) \cap M(\mu_{C_{n+1} - C_n}) \\ &= M(\mu_{C_n}) \cap M(\mu_{C_{n+1} - C_n}). \end{aligned}$$

Hence, we have $M(\mu_{C_{n+1}}) \subset M(\mu_{C_n})$, for $n = 1,2,3,\dots$.

Note that by Example 3.6, the set inclusion can be proper.

<u>Remark 4.3</u>. There is another advantage of contracting an inner measure μ_Δ: we can obtain more μ_C-null sets, and more μ_C-measurable sets of finite measures. Here, μ_C is the contracted inner measure. We have shown in Example 3.6 that there may exist sets which are μ_C-measurable but not μ_Δ-measurable. In the example, A is actually a μ_C-null set. On the other hand, if we pick any set $A \supset C$ such that A is not μ_Δ-measurable, then clearly, A is μ_C-measurable and $\mu_C(A) = \mu_\Delta(C)$. In this case, $\mu_C(A)$ is finite if $\mu_\Delta(C)$ is.

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