SOME PROPERTIES OF EULERIAN
FAMILY OF POLYNOMIALS

A thesis submitted to

Lakehead University

in partial fulfillment of the requirements

for the degree of

Master of Arts

by

Kar-Wing Yuen

1980

ACKNOWLEDGEMENT

I wish to thank my supervisor, Professor William R. Allaway, for his advice, encouragement and patience during the preparation of this thesis.

CONTENTS

Chapter I

PRELIMINARIES

(1.1) Introduction. A large number of problems in classical analysis can be stated in the following form. Given a sequence of functions $\{f_n(x)\}$ and a function $f(x)$, find a sequence of constants $\{a_n\}$ such that, in some sense of equality,

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x) .$$

For example, if $\{f_n(x)\} = \{\cos(nx)\} \cup \{\sin(nx)\}$, then we have the classical Fourier analysis problem. From which we know that if $f(x) = x^2$; $0 < x < 2\pi$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with $a_0 = \frac{8\pi^2}{3}$, $a_n = \frac{4}{n^2}$ and $b_n = \frac{-4\pi}{n}$. The a_n's and the b_n's are easily calculated in this example because of the orthogonality of $\{\cos(nx)\} \cup \{\sin(nx)\}$. Some of the classical work has dealt with the case when $\{f_n(x)\}$ forms an orthogonal sequence of functions with respect to some inner product. Other cases might not require such an orthogonal property. For example, if we let

$$f_n(x) = (x - a)^n \text{ and } f(x) \in C^{\infty}[(a - \varepsilon, a + \varepsilon)] ,$$

then the Taylor series expansion of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - a)^n [D^n f(x)]_{x=a}}{n!}$$

and we have uniform convergence in $[a - \varepsilon, a + \varepsilon]$ for some $\varepsilon > 0$.

We give one more example showing where this problem of writing one function as a linear combination of other functions would arise. Let $f(x)$ be a function given in tabular form by

$$f(i) = y_i ;\ i = 0, 1, 2, \ldots, n ,$$

where the y_i's are given. Using this information, we want to find the approximate area under $f(x)$ for $0 \le x \le n$. The classical method of solving this problem is to use one of the difference formulas to obtain the collation polynomial

$$p(x) = a_0 + a_1 x + a_2 x(x - 1) + \ldots + a_n x(x - 1) \ldots (x - n + 1) ,$$

and then use the integral of this polynomial as an approximation to the integral of the function. One of the ways to integrate $p(x)$ is to write it in the form

$$p(x) = \sum_{k=0}^{n} b_k x^k$$

which again involves finding the coefficients c_{nk} such that

$$x(x - 1) \ldots (x - n + 1) = \sum_{k=0}^{n} c_{nk} x^k .$$

It is well known that c_{nk}'s are the Stirling numbers of the first kind.

(1.2) In this thesis, we address ourselves to the same type of problem, that is, expressing one function as a linear combination of a sequence of functions. We reduce to a minimum the analytic apparatus of analysis on the line by restricting our attention to the special case when

(i) $f(x)$ is a polynomial over the reals $\mathbb{R}$,

and (ii) $\{f_n(x)\}$ is a sequence of polynomials, or more briefly polynomial sequence, with $f_n(x)$ being exactly of degree n.

The central problem is to find an efficient way of calculating c_{nk}; $n = 0, 1, 2, \ldots$; $0 \le k \le n$ such that

$$p_n(x) = \sum_{k=0}^{n} c_{nk} q_k(x)$$

where $\{p_n(x)\}$ and $\{q_n(x)\}$ are polynomial sequences. We will call this the connecting coefficient problem. Examples of such type of coefficients are:

(i) $s(n, k)$, the Stirling numbers of the first kind mentioned in (1.1).

(ii) $S(n, k)$, the Stirling numbers of the second kind, in

$$x^n = \sum_{k=0}^{n} S(n, k) x^{(k)}$$

where $x^{(k)} = x(x - 1) \ldots (x - k + 1)$.

and (iii) (signless) Lah numbers in

$$\langle x\rangle_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} x^{(k)}$$

where $\langle x\rangle_n = x(x+1)\ \dots\ (x+n-1)$.

(1.3) Mullin and Rota in [11] point out that sequences $\{x^n\}$, $\{\langle x\rangle_n\}$, $\{x^{(n)}\}$ and many more have a common property: that of being binomial type.

(1.3.1) <u>Definition</u>: $\{p_n(x)\}$ *is called a polynomial sequence of binomial type iff* $\forall n \geq 0$,

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \quad \forall x, y .$$

This notion of sequences of binomial type goes back at least to E. T. Bell [5]. There has been a number of systematic studies of polynomials of binomial type. The first was due to Mullin and Rota [11] in which they exploited the duality between x and $\frac{d}{dx}$. Their main technique was to develop a rigorous version of the so called "Umbral Calculus" which has been widely used in the past century. Later on, Rota, Kahaner and Odlyzko [15] extended the theory to polynomial sets other than binomial type. In chapter II, we will review these authors' work and show their solution to the connecting coefficient problem for polynomials of binomial type.

Roman and Rota [14] by using the Umbral Calculus and functional analysis obtained many of the results in the earlier papers, [11] and [15]. Roman and Rota in [14] and Sweedler in [16] have attempted to unify the theory of polynomial of binomial type by using different types of algebras. Although these authors' works are very interesting, due to time and space limitation we will not consider their work in this thesis.

(1.4) Andrews in [3] introduced a q-analogue of definition (1.3.1):

(1.4.1) Definition: *A sequence of polynomials* $\{p_n(x)\}$ *is an Eulerian family of polynomials iff* $\forall n \geq 0$,

$$p_n(xy) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x) p_{n-k}(y) y^k$$

where $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_{n-k}(q)_k} \quad \text{and}$$

$$(q)_n = \begin{cases} 1 & n = 0 \\ \prod_{i=1}^{n} (1 - q^i) & n > 0 \; ; \; q \in \mathbb{R} \; . \end{cases}$$

Andrews' theory closely parallels that of Mullin and Rota [11]. He, however, was not able to obtain all the results for Eulerian family of polynomials that are analogous to those in [11], [15] and [14]. For example, Andrews did not solve the connecting coefficient problem for Eulerian family of polynomials. Adopting Andrews' theory and making full use of the simple sequence - characterization of Eulerian shift invariant operators (see prop. (3.3.1)) we are able to extend his work by obtaining the algebra isomorphism theorem for Eulerian shift invariant operators that is analogous to the results of Rota (et al) [11], [15] and [14] for shift invariant operators. We think this is an important first step in solving completely the connecting coefficient problem for the Eulerian family of polynomials.

To make our analogue more complete, we also include a chapter on Eulerian Sheffer polynomials and indicate where and how the theories diverge.

We should also mention that Edwin C. Ihrig and Mourad E. H. Ismail in their paper entitled "A q-Umbral Calculus" [9] have devised formulas for expressing an Eulerian family of polynomials in terms of monomials and vice versa. They use a more abstract approach. The idea is briefly outlined in the appendix.

Chapter II

THEORY OF POLYNOMIALS OF BINOMIAL TYPE

(2.1) Introduction. This chapter is completely devoted to the theory of polynomials of binomial type proposed by Rota (et al) ([11], [15] and [14]). We quote some of the important results as an introduction as well as references so that readers can draw the analogy when reading the later chapters. As for the proofs, they can either be found in the original publications ([11], [15] and [14]) or in Garsia's Exposé ([6]) where they are rederived.

(2.2) Fundamentals. A set of polynomials $\{p_n(x)\}$ is called *a sequence of polynomials*, briefly *a polynomial sequence*, if $p_n(x)$ is of degree precisely n in x. It is clear that $\{p_n(x)\}$ forms a basis so that any polynomial can be expressed as a linear combination of the elements of such a polynomial sequence.

(2.2.1) Definition: *A polynomial sequence* $\{p_n(x)\}$ *is said to be of binomial type iff* $\forall n \geq 0$

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y) \qquad \forall x, y .$$

The theory revolves around the interplay between the algebra of polynomials $\mathbb{P}$ and the algebra of shift invariant operators Σ. All operators considered in this thesis are assumed to be linear and are defined on $\mathbb{P}$, the linear space of all polynomials over the reals, $\mathbb{R}$.

(2.2.2) **Definition**: *An operator* T *is called a shift invariant operator iff* $\forall a \in \mathbb{R}$,

$$TE_a = E_aT$$

$$\text{where } E_a p(x) = p(x+a) \qquad \forall p(x) \in \mathbb{P} .$$

(2.2.3) **Definition**: *A delta operator* Q *is a shift invariant operator such that* Qx *is a non-zero constant.*

As the following definition will show, a delta operator is associated in a natural way with a particular sequence of polynomials.

(2.2.4) **Definition**: *A polynomial sequence* $\{p_n(x)\}$ *is called a sequence of basic polynomials, briefly a basic polynomial sequence, for the delta operator* Q *if*

(i) $p_0(x) = 1$

(ii) $p_n(0) = 0 \qquad \forall n > 0$;

and (iii) $Qp_n(x) = np_{n-1}(x) \qquad \forall n \geq 0$.

Note that the combined effort of the three requirements in the above definition is so strong that it guarantees:

(2.2.5) **Theorem**: *Every delta operator has a unique basic polynomial sequence.*

For example, $\{x^n\}$ is the basic polynomial sequence for the delta operator D , the ordinary differential operator. In addition,

$\{x^n\}$ is not only basic but also of binomial type. This turns out to be true for every basic polynomial sequence. That is:

(2.2.6) Theorem: (Mullin and Rota [11], Theorem 1)

(a) *If* $\{p_n(x)\}$ *is a basic sequence for some delta operator* Q, *then it is a sequence of polynomials of binomial type.*

(b) *If* $\{p_n(x)\}$ *is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator.*

Using the above fundamental result, they proved the following important First Expansion Theorem, a generalization of the Taylor Expansion Theorem.

(2.2.7) Theorem: (Mullin and Rota [11], Theorem 2)

Let T *be a shift invariant operator, and let* Q *be a delta operator with basic set* $\{p_n(x)\}$. *Then*

$$T = \sum_{k=0}^{\infty} \frac{a_k}{k!} Q^k$$

where

$$a_k = [Tp_k(x)]_{x=0} .$$

This very powerful result ensures that every shift invariant operator can be expressed in terms of any delta operator and its powers. The similarity between the expanded form and the formal power series suggests an isomorphism — an idea "intuited by Pincherle, and has been tacitly - and often unrigorously - used by several authors". (See Rota, Kahaner and Odlyzko [15] ch. 14.)

(2.2.8) Theorem: (Mullin and Rota [11], Theorem 3)

Let Q *be a delta operator, and let* F *be the ring of formal power series in the variable* t *, over the same field. Then there exists an isomorphism from* F *onto the ring* $\sum$ *of shift invariant operators, which carries*

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \quad \text{into} \quad \sum_{k=0}^{\infty} \frac{a_k}{k!} Q^k .$$

We use $f(Q)$ to denote $\sum_{k=0}^{\infty} \frac{a_k}{k!} Q^k$.

By using this isomorphism theorem, many of the properties in the abstract operator theory can be formulated in the more thoroughly studied theory of formal power series.

(2.3) The Umbral Calculus. In the past century, invariant theorists regarded the umbral notation, or symbolic notation, as an informal algorithmic device which allows one raising the index n to a power, and then treating the sequence $\{a_n\}$ as a sequence of powers $\{a^n\}$, while reserving the right to lower the index at the appropriate time. Computationally, the technique turned out to be very effective. However, the calculus could not be set on a rigorous foundation because no proper rules for lowering of indices were stated. Rota et al were the first to notice that the proper method is to consider a sequence $\{a_n\}$ as defined by a linear functional L on the space of polynomials: $a_n = L(x^n)$. The description of the sequence is then condensed into the properties of the linear functional. If $\{a_n(x)\}$ is a polynomial sequence, then there is a unique linear operator L on $\mathbb{P}$ such that $L(x^n) = a_n(x)$.

(2.3.1) <u>Definition</u>: *An umbral operator* T *is a linear operator which maps some basic polynomial sequence* $\{p_n(x)\}$ *into another basic polynomial sequence* $\{q_n(x)\}$, *that is,*

$$Tp_n(x) = q_n(x) .$$

Using the following facts, (Mullin and Rota [11], theorem 5)

(2.3.2) *if* T *is an umbral operator, then* T^{-1} *exists;*

(2.3.3) *if* S *is shift invariant, then* TST^{-1} *is also shift invariant*; and

(2.3.4) *if* Q *is a delta operator, then* TQT^{-1} *is also a delta operator,* they showed that

(2.3.5) T *maps every basic sequence into another basic sequence.*

The umbral composition of two polynomial sequences $\{a_n(x)\}$ and $\{b_n(x)\}$, where

$$a_n(x) = \sum_{k=0}^{n} a_{nk}x^k \quad \text{and} \quad b_n(x) = \sum_{k=0}^{n} b_{nk}x^k ,$$

is another sequence of polynomials $\{c_n(x)\}$ defined as

$$c_n(x) = \sum_{k=0}^{n} a_{nk}b_k(x) .$$

Symbolically, $c_n(x) = a_n(\underline{b}(x))$. There is a simple relation existing between umbral operators and the umbral composition of basic sequences:

(2.3.6) <u>Lemma</u>: *Let* $\{a_n(x)\}$ *and* $\{q_n(x)\}$ *be two basic polynomial sequences. If* T *is an umbral operator such that*

$$Tx^n = q_n(x) ,$$

then
$$a_n(\underline{q}(x)) = Ta_n(x) .$$

The powerful tool that Mullin and Rota use to solve the connecting coefficient problem is the following Umbral Composition Theorem.

(2.3.7) <u>Theorem</u>: *If* $P = p(D)$ *and* $Q = q(D)$ *are delta operators with respectively* $\{p_n(x)\}$ *and* $\{q_n(x)\}$ *as their corresponding basic sequences, then the umbral composition*

$$r_n(x) = p_n(\underline{q}(x))$$

is the sequence of basic polynomials for the delta operator

$$p(q(D)) .$$

(2.4) <u>Solution to the Connecting Coefficient Problem</u>. The connecting coefficients c_{nk} in

$$a_n(x) = \sum_{k=0}^{n} c_{nk} b_k(x) ,$$

where $\{a_n(x)\}$ and $\{b_n(x)\}$ are basic sequences for the delta operator $a(D)$ and $b(D)$ respectively, can now be determined alternatively by considering the polynomials

$$r_n(x) = \sum_{k=0}^{n} c_{nk} x^k ,$$

and the umbral operator T defined by

$$Tx^n = b_n(x) .$$

Clearly, $a_n(x) = Tr_n(x) = r_n(\underline{b}(x))$, (see (2.3.6)) so that $\{r_n(x)\}$ is basic with respect to the delta operator $C = c(D) = a(b^{-1}(D))$, where the last equality is obtained from

$$a(D) = c(b(D))$$

by Theorem (2.3.7).

Chapter III

THEORY OF EULERIAN FAMILY OF POLYNOMIALS

(3.1) Eulerian Family of Polynomials. We are interested in developing a theory about polynomial sequences $\{p_n(x)\}$ that have the properties: $\forall n \geq 0$

(i) degree of $p_n(x) = n$

and (ii) $p_n(xy) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) y^k \qquad \forall x, y$.

Such a polynomial set $\{p_n(x)\}$ is said to be an *Eulerian family of polynomials*. The sequence $\{(x - 1)^n\}$ will serve as an example illustrating the properties. Since

$$\begin{aligned} p_n(xy) &= (xy - 1)^n \\ &= (\overline{xy - y} + \overline{y - 1})^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (x - 1)^k y^k (y - 1)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) y^k , \end{aligned}$$

$\{(x - 1)^n\}$ is then the model polynomial sequence of the Eulerian family of polynomials. It will play the same role as $\{x^n\}$ in the set of polynomials of binomial type mentioned in chapter II.

(3.2) Eulerian Shift Invariant Operators. Let $p(x)$ be a polynomial. Multiply each term of $p(x)$ by x to obtain a new polynomial $xp(x)$. Call this the *multiplication operator* and denote it by x . Thus,

$$x \colon p(x) \to xp(x) .$$

(3.2.1) <u>Definition</u>: *Let* $a \in \mathbb{R}$. *The Eulerian shift operator* A_a *is a linear operator defined on the linear space of all polynomials by*

$$A_a p(x) = p(ax) \qquad \forall p(x) \in \mathbb{P} .$$

Using this definition we now define Eulerian shift invariant operator.

(3.2.2) <u>Definition</u>: *A linear operator* T *on the linear space of all polynomials is an Eulerian shift invariant operator if* $\forall a \in \mathbb{R}$,

$$TA_a = A_a T .$$

An example of Eulerian shift invariant operator is xD where D is the ordinary differential operator.

$$\begin{aligned} A_a (xD)x^n &= A_a n x^n \\ &= n(ax)^n . \end{aligned}$$

Also

$$\begin{aligned} (xD)A_a x^n &= xD(ax)^n \\ &= x a^n n x^{n-1} \\ &= n(ax)^n . \end{aligned}$$

(3.3) <u>Characterization of Eulerian Shift Invariant Operators</u>. Given a linear operator, how do we know that it is Eulerian shift invariant aside from the direct verification of the conditions in the definition? A simple answer is given by the following proposition which provides a distinct characterization of Eulerian shift invariance. This characterization was proven by Andrews in his paper [3].

(3.3.1) Proposition: T *is Eulerian shift invariant iff there exists a sequence* $\{t_n\}$ *of constants such that*

(3.3.1a) $$Tx^n = t_n x^n \ .$$

Proof: Assume T is Eulerian shift invariant. Then $\forall a \in \mathbb{R}$ we have $TA_a x^n = A_a T x^n$. Letting $Tx^n = \sum_{k=0}^{\infty} c_{nk} x^k$, where for each n, $c_{nk} = 0$ for all but finitely many k, we get

$$a^n \sum_{k=0}^{\infty} c_{nk} x^k = \sum_{k=0}^{\infty} c_{nk} a^k x^k \qquad \forall x$$

$$\therefore \quad \forall a \ , \ c_{nk}(a^n - a^k) = 0 \ .$$

If $c_{nk} \neq 0$, then $a^n - a^k = 0$. This implies $n = k$ since the equation is true for all $a \in \mathbb{R}$.

$$\therefore \quad Tx^n = c_{nn} x^n \ .$$

Let $t_n = c_{nn}$. Then $Tx^n = t_n x^n$.

Conversely, if $Tx^n = t_n x^n$, then

$$\begin{aligned} A_a T x^n &= A_a t_n x^n \\ &= t_n (ax)^n \\ &= a^n t_n x^n \\ &= a^n \ Tx^n \\ &= T(ax)^n \\ &= TA_a x^n \ . \end{aligned}$$

Thus, T is Eulerian shift invariant. Q.E.D.

As has been done by various authors (see [10] and [17]) we will call $\{t_n\}$ the *Fundamental sequence of the operator* T if (3.3.1a) is satisfied. It is easy to see that $\{n\}$ is the fundamental sequence for the Eulerian shift invariant operator xD.

Besides the above simple characterization for Eulerian shift invariant operators, we provide another method, in proposition (3.3.2), which may be used for the same goal. However, the proposition is intended for another purpose as we shall see later on. First, we introduce an operator which has the similar characteristic as a partial differential operator. By T_x, we mean that the operator T, when acting on a polynomial in x and y, will operate on x only and treat y as a constant.

(3.3.2) <u>Proposition</u>: T *is Eulerian shift invariant iff* $\forall p(x) \in \mathbb{P}$,

$$T_x p(xy) = T_y p(xy) .$$

<u>Proof</u>: Suppose T is Eulerian shift invariant and $\{t_n\}$ is its fundamental sequence such that

$$Tx^n = t_n x^n .$$

Let $p(x) = \sum_{k=0}^{n} a_k x^k$.

Then,
$$T_x p(xy) = T_x \sum_{k=0}^{n} a_k (xy)^k$$
$$= \sum_{k=0}^{n} a_k y^k T_x x^k$$
$$= \sum_{k=0}^{n} a_k y^k t_k x^k$$
$$= \sum_{k=0}^{n} a_k x^k t_k y^k$$
$$= \sum_{k=0}^{n} a_k x^k T_y y^k$$
$$= T_y \sum_{k=0}^{n} a_k (xy)^k$$
$$= T_y p(xy) .$$

Conversely, since $T_x p(xy) = T_y p(xy)$ is true for all $p(x) \in \mathbb{P}$, in particular we have

$$T_x (xy)^n = T_y (xy)^n .$$

Now suppose $Tx^n = \sum_{k=0}^{\infty} b_{nk} x^k$; where for each n, $b_{nk} = 0$ for all but finitely many k . Then

$$T_x (xy)^n = y^n T_x x^n$$
$$= y^n \sum_{k=0}^{\infty} b_{nk} x^k .$$

Similarly,
$$T_y (xy)^n = x^n \sum_{k=0}^{\infty} b_{nk} y^k .$$

Equating the two, we have

$$b_{nk} = 0 \quad \text{for} \quad k \neq n ,$$

and therefore $Tx^n = T_x x^n = b_{nn} x^n$. Hence T is Eulerian shift invariant. Q.E.D.

(3.4) The Algebra of Eulerian Shift Invariant Operators. Proposition (3.3.1) has some far-reaching implications. It not only provides a direct method in simplifying the process of determining whether an operator is Eulerian shift invariant or not, but also shows that every Eulerian shift invariant operator is associated with a sequence of reals in a natural way. $\mathbb{E}$, the set of all Eulerian shift invariant operators, is then expected to have the similar properties that sequences possess. One of which is that $\mathbb{E}$ forms a vector space with respect to some appropriately defined operations. They are:-

addition "+", the addition of two operators;

multiplication "$\circ$", the composition of two operators; and

scalar multiplication "λ", the scalar multiplication of an operator by a real number λ.

We use "$\hat{0}$" to denote the null Eulerian shift invariant operator, i.e.

$$\hat{0} + T = T = T + \hat{0} \quad \text{where} \quad T \in \mathbb{E} .$$

$-T$ represents the additive inverse of T.

(3.4.1) Theorem: $(\mathbb{E}, +, -, \hat{0}, \circ, \lambda)$ *is an algebra.*

Proof: From the fact that Eulerian shift invariant operators are linear operators and the set of all linear operators with respect to "+", "∘" and "λ" forms a vector space, we only have to check $\mathbb{E}$ is closed under these operations.

Let $T, S \in \mathbb{E}$ with $\{t_n\}$ and $\{s_n\}$ as their fundamental sequences respectively. Now,

$$\begin{aligned} T \circ S &= Ts_n x^n \\ &= s_n T x^n \\ &= s_n t_n x^n . \end{aligned}$$

We have thus found the fundamental sequence, namely $\{s_n t_n\}$ for $T \circ S$. Therefore $T \circ S \in \mathbb{E}$. Furthermore, $s_n t_n = t_n s_n$ implies that multiplication is commutative. "+" and "λ" can be done similarly. Q.E.D.

Re-examining the above proof, one can easily deduce:

(3.4.2) Corollary: *An Eulerian shift invariant operator* T *is invertible iff the fundamental sequence* $\{t_n\}$ *for* T *is a sequence of nonzero numbers.*

Of course, the fundamental sequence for the inverse of T is $\{\frac{1}{t_n}\}$.

Chapter IV

EULERIAN DIFFERENTIAL OPERATOR AND EULERIAN BASIC POLYNOMIAL SEQUENCES

(4.1) Eulerian Differential Operators.

(4.1.1) Definition: *An Eulerian differential operator* Q *is an operator such that* $xQ \in \mathbb{E}$ *and the fundamental sequence of constants for* xQ *is* $\{0, g_1, g_2, \ldots\}$ *with* $g_n \neq 0 \ \forall n > 0$.

Obviously, D is the simplest example of an Eulerian differential operator since $xD \in \mathbb{E}$ (see (3.2)) with $\{n\}$ as its fundamental sequence. The next example of Eulerian differential operator that we are going to discuss is the q-derivative, written as D_q , which is defined as

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x} \qquad \forall f(x) \in \mathbb{P};\ \forall q \in \mathbb{R} .$$

Since

$$xD_q x^n = x \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^n$$

xDq is therefore a Eulerian shift invariant operator whose fundamental sequence is $\{0, 1, \frac{q^2 - 1}{q - 1}, \ldots, \frac{q^n - 1}{q - 1}, \ldots\}$.

Note: $\frac{q^n - 1}{q - 1} = \sum_{k=0}^{n-1} q^k = n$ in the case $q = 1$.

Remark: At this point, we should mention that the idea of studying xQ is kindly suggested by Rota, Kahaner and Odlyzko ([15] chapter 14, problem 12 and 1). They propose that "one should begin by developing the theory of xD" ... " along a similar line but with a different invariance property than shift invariance". The invariance property that we are using is generally known as SCALE INVARIANCE.

Immediately from definition (4.1.1), we can characterize an Eulerian differential operator in a similar fashion as in proposition (3.3.1) for Eulerian shift invariant operators.

(4.1.2) Proposition: (Andrews [3], Theorem 4) Q *is an Eulerian differential operator if and only if there exists a sequence of constants* $\{g_n\}$ *with*

$$g_0 = 0 \quad \textit{and} \quad g_n \neq 0 \ \forall n > 0 \quad \textit{such that}$$

(4.1.2a)
$$Qx^n = \begin{cases} 0 & n = 0 \\ g_n x^{n-1} & n > 0 \, . \end{cases}$$

For this reason, we also call the sequence $\{g_n\}$ the fundamental sequence for Q. Eulerian differential operators work very much the same as the ordinary differential operator. The following corollary, extracted from the preceding proposition, illustrates some of the resemblances.

(4.1.3) Corollary: *Let* $p_n(x), q_n(x) \in \mathbb{P}$. *If* Q *is an Eulerian differential operator, then*

(i) $Qc = 0$ *where* c *is a constant;*

(ii) $Qp_n(x) = Qq_n(x) \Leftrightarrow p_n(x) - q_n(x) =$ *a constant;*

(iii) $Qp_{n+1}(x)$ *is a polynomial of degree* n ;

(iv) Q-*integration rule*:

$$\int_Q \sum a_i x^i = c + \sum \frac{a_i}{g_{i+1}} x^{i+1}$$ *where* c *is a constant and* $\{0, g_1, g_2, \ldots\}$ *is the fundamental sequence for* Q ;

and (v) *a polynomial* $p(x)$ *is uniquely determined by* Q, $Qp(x)$ *and* $p(a)$ *for any given* $a \in \mathbb{R}$.

Proof: We only briefly outline the proof.

(i) and (iii) are direct consequences of (4.1.2a).

(ii) can be deduced from (i).

(iv) follows from (ii) and (iii).

(v) is obtained from (iv). Q.E.D.

The relationship between Q and D is actually much deeper than merely some coincidences of characteristic resemblances mentioned above. The hidden "factor" is revealed from the following proposition.

(4.1.4) Proposition: Q *is an Eulerian differential operator iff there exists an invertible Eulerian shift invariant operator* P *such that* $Q = PD$.

Proof: If P exists and is invertible, then its fundamental sequence $\{\pi_n\}$ has the property that $\pi_n \neq 0\ \forall n \geq 0$. (see Cor. (3.4.2))

Now, $$PDx^n = \begin{cases} 0 & n = 0 \\ n\pi_{n-1}x^{n-1} & n > 0 . \end{cases}$$

Since $n\pi_{n-1} \neq 0\ \forall n > 0$, the sequence

$$\{0, \pi_0, 2\pi_1, \ldots, n\pi_{n-1}, \ldots\}$$

is the fundamental sequence for the Eulerian differential operator Q .

Conversely, if Q is an Eulerian differential operator with fundamental sequence $\{0, g_1, g_2, \ldots\}$ such that $g_n \neq 0\ \forall n > 0$, then construct the sequence $\{\pi_n\} = \dfrac{g_{n+1}}{n+1}$. Let P be the operator defined by the property that

$$Px^n = \pi_n x^n \qquad \forall n \geq 0 .$$

Clearly, P is Eulerian shift invariant and is invertible since $\pi_n = \dfrac{g_{n+1}}{n+1} \neq 0 \quad \forall n \geq 0$. Furthermore

when $n = 0$, $\quad PDx^n = 0$

when $n > 0$, $$\begin{aligned} PDx^n &= Pnx^{n-1} \\ &= n\cdot\pi_{n-1}x^{n-1} \\ &= g_n x^{n-1} \\ &= Qx^n . \end{aligned}$$

Thus $Q = PD$. Q.E.D.

(4.2) Eulerian Basic Polynomial Sequence. The type of polynomial sequence $\{p_n(x)\}$ which satisfies the property that

$$Qp_n(x) = k_n p_{n-1}(x)$$

where k_n is independent on x, for some appropriate differential operator Q is of special interest. Extensive work has been done by Rota (et al) in the case $k_n = n$ (see [11], [15], [14] and also chapter II). Andrews in [3], has found a close analog to Rota's work by setting $k_n = 1 - q^n$. The case where $k_n = 1$ is studied by Allaway and the author in [2]. In this thesis, we use $k_n = n$ as Rota did but with a different type of invariance property.

(4.2.1) Definition: *A polynomial sequence* $\{p_n(x)\}$ *is an Eulerian basic polynomial sequence relative to the Eulerian differential operator* Q *if it satisfies the following properties:*

(i) $p_0(x) = 1$

(ii) $p_n(1) = 0 \quad \forall n > 0$

and (iii) $Qp_n(x) = np_{n-1}(x) \quad \forall n > 0$.

Examples:

(i) Direct verification will show that $\{(x - 1)^n\}$ is a sequence for the Eulerian differential operator D.

(ii) Define $$p_n(x) = \begin{cases} 1 & n = 0 \\ n!(x^n - x^{n-1}) & n > 0 , \end{cases}$$

and $$Qx^n = \begin{cases} 0 & n = 0 \\ x^{n-1} & n > 0 . \end{cases}$$

Q is then an Eulerian differential operator with fundamental sequence $\{0, 1, 1, 1, \ldots\}$. Also,

$$\begin{aligned} Qp_n(x) &= Qn!(x^n - x^{n-1}) \\ &= n!(x^{n-1} - x^{n-2}) \\ &= n((n-1)!(x^{n-1} - x^{n-2})) \\ &= np_{n-1}(x) \end{aligned}$$

and $p_n(1) = 0 \quad \forall n > 0$.

Thus $\{p_n(x)\}$ is an Eulerian basic polynomial sequence for Q .

Example (ii) shows that $\{(x - 1)^n\}$ which we have used repeatedly is not the only basic polynomial sequence in $\mathbb{P}$. However, the following proposition ensures that an Eulerian differential operator is associated with a unique Eulerian basic polynomial sequence.

(4.2.2) <u>Proposition</u>: (Andrews [3]; Conseq. of lemma 2) *Every Eulerian differential operator has a unique Eulerian basic sequence.*

<u>Proof</u>: Let Q be an Eulerian differential operator with fundamental sequence $\{g_n\}$. We want to construct a unique Eulerian basic polynomial sequence $\{q_n(x)\}$ for Q . By definition (3.1), we can

define $\quad p_0(x) = 1$,

and let $\quad p_1(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Now $\quad Qp_1(x) = aQx + Qb$

$= ag_1 \quad$ (Prop. (4.1.2) and Cor. (4.1.3(i))) .

Also $\quad Qp_1(x) = 1p_0(x) \quad$ (definition (4.2.1(iii)))

$= 1 . \quad$ (definition (4.2.1(i)))

Equating the two, we get $a = 1/g_1$.

From (4.2.1(ii)), we have

$$\left[\frac{1}{g_1} x + b\right]_{x=1} = 0 .$$

i.e. $\quad b = -1/g_1$.

Therefore, $p_0(x)$ and $p_1(x)$ are uniquely defined. Suppose $p_0(x), p_1(x), \ldots p_m(x)$ are uniquely defined. That is, for $j = 0, 1, 2, \ldots m$; $p_j(x) = \sum_{k=0}^{j} a_{jk}x^k$ with the a_{jk}'s are uniquely defined. We shall show, by induction, that $p_{m+1}(x)$ is also uniquely defined. Suppose $p_{m+1}(x) = \sum_{k=0}^{m+1} b_{m+1,k}x^k$.

$$Qp_{m+1}(x) = \sum_{k=1}^{m+1} b_{m+1,k}g_k x^{k-1}$$

$$= \sum_{k=0}^{m} b_{m+1,k+1}g_{k+1}x^k .$$

Also $\quad Qp_{m+1}(x) = (m + 1)p_m(x) = \sum_{k=0}^{m} (m + 1)a_{m,k}x^k$

$$\Rightarrow \quad b_{m+1,k+1} = \frac{(m + 1)a_{m,k}}{g_{k+1}} .$$

Thus $b_{m+1,k+1}$ is uniquely determined for $k = 0, 1, 2, \ldots m$;
and $b_{m+1,0} = -\sum_{k=1}^{m+1} b_{m+1,k}$ is given by $p_{m+1}(1) = 0$. Q.E.D.

Now an Eulerian differential operator Q is associated with two sequences, namely its Eulerian basic polynomial sequence $\{q_n(x)\}$ and its fundamental sequence of constants $\{g_n\}$. We use the term *Eulerian Triple*, which is $(Q, g_n, q_n(x))$, to emphasize their distinct relationship and to avoid the trouble of describing them, in words, repeatedly in later sections.

(4.3) <u>The Calculus of Eulerian Differential Operators</u>. For simplicity in further development, we introduce the following notations similar to the factorial sign. From the Eulerian triple $(Q, g_n, q_n(x))$, we define

$$[n; Q] = \begin{cases} g_n g_{n-1} \cdots g_1 & \text{if } n > 0 \\ 1 & \text{if } n = 0 . \end{cases}$$

Since $g_n \neq 0$ whenever $n > 0$, division is possible. Thus the n choose k notation, i.e. $\binom{n}{k}$, in factorial terminology can be extended to $\begin{bmatrix} n \\ k \end{bmatrix}; Q$, known as the generalized binomial coefficient, such that

$$\begin{bmatrix} n \\ k \end{bmatrix}; Q = \frac{[n; Q]}{[k; Q][n-k; Q]}$$

$$= \begin{cases} \dfrac{q_n q_{n-1} \cdots q_{n-k+1}}{q_k q_{k-1} \cdots q} & \text{if } 0 < k < n \\ 1 & \text{if } k = 0 \text{ or } n \\ 0 & \text{if } k < 0 \text{ or } k > n . \end{cases}$$

We isolate the following facts because they will occur quite frequently in some of the proofs in later sections.

(4.3.1) Lemma: *For a fixed Eulerian triple* $(Q, g_n, q_n(x))$, $x^kQ^k \in \mathbb{E}; \forall k \geq 0$. *Moreover,*

$$x^kQ^kx^n = \begin{cases} 0 & \text{if } k > n \\ \dfrac{[n;\ Q]}{[n-k;\ Q]}\, x^n & \text{if } k \leq n\ . \end{cases}$$

Proof: It is obvious that

$$x^kQ^kx^n = 0 \quad \text{when} \quad k > n\ .$$

When $k \leq n$,

$$\begin{aligned} x^kQ^kx^n &= x^kQ^{k-1}g_nx^{n-1} \\ &= x^kQ^{k-2}g_ng_{n-1}x^{n-2} \\ &\ \ \vdots \\ &= g_ng_{n-1}\cdots g_{n-k+1}x^n \\ &= \frac{[n;\ Q]}{[n-k;\ Q]}\, x^n\ . \end{aligned}$$

Hence, the fundamental sequence for x^kQ^k exists and is

$$\{\underbrace{0,\ 0,\ \ldots,\ 0}_{k \text{ zeroes}},\ \frac{[k;\ Q]}{[0;\ Q]}\ ,\ \frac{[k+1;\ Q]}{[1;\ Q]}\ ,\ \ldots\}$$

showing that $x^kQ^k \in \mathbb{E}$. Q.E.D.

(4.3.2) Lemma: *For a fixed Eulerian triple* $(Q, g_n, q_n(x))$,

$$x^k Q^k q_n(x) = \begin{cases} 0 & k > n \\ x^k n^{(k)} q_{n-k}(x) & k \le n , \end{cases}$$

where $n^{(k)} = n(n-1) \ldots (n-k+1)$.

Moreover, $$[x^k Q^k q_n(x)]_{x=1} = \begin{cases} n! & k = n \\ 0 & k \ne n . \end{cases}$$

Proof: The first result is obtained by iterating property (iii) of definition (4.2.1). Then,

$$\begin{aligned} [x^k Q^k q_n(x)]_{x=1} &= [x^k n^{(k)} q_{n-k}(x)]_{x=1} \\ &= \begin{cases} 0 & \text{if } k \ne n \\ n! & \text{if } k = n \end{cases} \qquad \text{see (4.2.1(i) and (ii))} \end{aligned}$$

Q.E.D.

Here we shall show another resemblance of Eulerian differential operators to the ordinary differential operator. An Eulerian differential operator Q (together with its Eulerian basic polynomial sequence) has a property similar to the Chain Rule of differentiation. In differential calculus, we have

$$\frac{d}{dx} f(\theta x) = \frac{d}{d(\theta x)} f(\theta x) \frac{d}{dx} (\theta x) .$$

In Eulerian differentiation, we have:

(4.3.3) Proposition: *Let* $(Q, g_n, q_n(x))$ *be given and* θ *a constant. Then*

$$Qq_n(\theta x) = nq_{n-1}(\theta x) \cdot \theta.$$

Proof:

$$\begin{aligned}(xQ)q_n(\theta x) &= (xQ)A_\theta q_n(x) \\ &= A_\theta (xQ) q_n(x) \\ &= A_\theta xnq_{n-1}(x) \\ &= \theta xnq_{n-1}(\theta x) \\ &= xnq_{n-1}(\theta x) \cdot \theta .\end{aligned}$$

Cancelling x, we get

$$Qq_n(\theta x) = nq_{n-1}(\theta x) \cdot \theta .$$ Q.E.D.

By induction on k, one can further deduce:

(4.3.4) Corollary: $Q^k q_n(\theta x) = n^{(k)}\theta^k q_{n-k}(\theta x) \quad \forall k \le n$.

In particular, if $k = n$, then

$$Q^n q_n(\theta x) = n!\theta^n .$$

(4.4) Characterization of Eulerian Basic Polynomial Sequence. We noticed that $\{(x - 1)^n\}$ is not only Eulerian basic (see (4.2.1)) but also an Eulerian family of polynomials (see (3.1)). This turns out to be true for every Eulerian basic polynomial sequence, and is a fundamental result in characterizing an Eulerian basic polynomial sequence.

(4.4.1) **Theorem:** (Andrews [3], Theorem 1)

(a) *If* $\{q_n(x)\}$ *is an Eulerian basic polynomial sequence for some Eulerian differential operator* Q *, then it is an Eulerian family of polynomials.*

(b) *If* $\{q_n(x)\}$ *is an Eulerian family of polynomials, then it is an Eulerian basic polynomial sequence for some Eulerian differential operator.*

Proof: (a) Since $[x^kQ^kq_n(x)]_{x=1} = \begin{cases} n! & n = k \\ 0 & n \neq k \end{cases}$, (lemma 4.3.2)

we may trivially express $q_n(x)$ in the form

$$q_n(x) = \sum_{k=0}^{n} \frac{q_k(x)}{k!} [x^kQ^kq_n(x)]_{x=1} .$$

By linearity, any polynomial $p(x)$ of degree n can be written as

$$p(x) = \sum_{k=0}^{n} \frac{q_k(x)}{k!} [x^kQ^k\ p(x)]_{x=1} .$$

Now suppose $p(x)$ is the polynomial $q_n(ax)$. Then

$$\begin{aligned} q_n(ax) &= \sum_{k=0}^{n} \frac{q_k(x)}{k!} [x^kQ^kq_n(ax)]_{x=1} \\ &= \sum_{k=0}^{n} \frac{q_k(x)}{k!} [x^kQ^kA_aq_n(x)]_{x=1} \\ &= \sum_{k=0}^{n} \frac{q_k(x)}{k!} [A_ax^kQ^kq_n(x)]_{x=1} && \text{(lemma 4.3.1)} \\ &= \sum_{k=0}^{n} \frac{q_k(x)}{k!} [A_ax^kn^{(k)}q_{n-k}(x)]_{x=1} && \text{(lemma 4.3.2)} \\ &= \sum_{k=0}^{n} \frac{q_k(x)}{k!} a^k \frac{n!}{(n-k)!} q_{n-k}(a) \\ &= \sum_{k=0}^{n} \binom{n}{k} q_k(x)\ q_{n-k}(a)a^k . \end{aligned}$$

This means that the Eulerian basic polynomial sequence $\{q_n(x)\}$ for Q is also an Eulerian family.

(b) Suppose $\{q_n(x)\}$ is an Eulerian family of polynomials and $q_0(x) = c$, a non zero constant. Then, $\forall n \geq 0$, $\forall a \in \mathbb{R}$,

$$q_n(ax) = \sum_{k=0}^{n} \binom{n}{k} q_k(x) q_{n-k}(a) a^k . \tag{4.4.1b}$$

In particular, when $n = 0$ and $x = a$, we have

$$q_0(a^2) = q_0(a) q_0(a)$$

i.e. $$c = c^2$$

$$\Rightarrow \quad q_0(x) = c = 1 .$$

Next, we are going to show by induction that $q_n(1) = 0 \ \forall n > 0$.
Putting $n = 1$ and $a = 1$ in (4.4.1.b), we have

$$q_1(x) = q_0(x) q_1(1) + q_1(x) q_0(1) = q_1(1) + q_1(x)$$

$$\Rightarrow \quad q_1(1) = 0 .$$

Assume $q_n(1) = 0$ is true for $n = 1, 2, 3, \ldots m$. When $n = m + 1$ amd $a = 1$, (4.4.1b) becomes

$$\begin{aligned} q_{m+1}(x) &= q_0(x) q_{m+1}(1) + 0 + \ldots + 0 + q_{m+1}(x) q_0(1) \\ &= q_{m+1}(1) + q_{m+1}(x) \end{aligned}$$

$$\Rightarrow \quad q_{m+1}(1) = 0 .$$

Hence, by induction, $q_n(1) = 0 \ \forall n > 0$.

Now, we define the operator Q by:

$$Qq_n(x) = \begin{cases} nq_{n-1}(x) & \forall n > 0 \\ 0 & n = 0 \ . \end{cases}$$

All we need to show is that xQ is Eulerian shift invariant.

$$\begin{aligned}
(xQ)A_a q_n(x) &= xQq_n(ax) \\
&= xQ \sum_{k=0}^{n} \binom{n}{k} q_k(x)q_{n-k}(a)a^k \\
&= x \sum_{k=1}^{n} \binom{n}{k} kq_{k-1}(x)q_{n-k}(a)a^k \\
&= x \sum_{k=0}^{n-1} \binom{n}{k+1} (k+1)\, q_k(x)q_{n-k-1}(a)a^{k+1} \\
&= nax \sum_{k=0}^{n-1} \binom{n-1}{k} q_k(x)q_{n-1-k}(a)a^k \\
&= nax\, q_{n-1}(ax) \\
&= A_a(xnq_{n-1}(x)) \\
&= A_a(xQ)q_n(x)
\end{aligned}$$

$$\Rightarrow \quad (xQ)A_a = A_a(xQ)$$

Hence, xQ is Eulerian shift invariant. Q.E.D.

Chapter V

THE VECTOR SPACE ISOMORPHISM DETERMINED BY AN EULERIAN DIFFERENTIAL OPERATOR

(5.1) Representation of Eulerian Shift Invariant Operators. In (4.3.1), we have already seen that for an Eulerian differential operator Q, x^kQ^k is Eulerian shift invariant for all $k \geq 0$. The fact that $(\mathbb{E}, +, -, \hat{0}, \circ, \lambda)$ is an algebra (see theorem (3.4.1)) enables us to generalize the idea so that any linear combination of x^kQ^k ; $k = 0, 1, 2, \ldots$ is also Eulerian shift invariant. More precisely,

(5.1.1) Lemma: *Given an Eulerian differential operator* Q, *for any sequence* $(a_n) \in \mathbb{R}^{\mathbb{N}_0}$,

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} x^kQ^k \in \mathbb{E} .$$

Note: The presence of the $k!$ is needed in accordance to our later development.

It is the converse (of the above lemma) that interests us the most. As a matter of fact, it turns out to be one of our basic results. It is also regarded as a generalization of the Taylor expansion theorem for the Eulerian shift invariant operators.

(5.1.2) <u>Theorem</u>: [3; Theorem 2] (Taylor Expansion of Eulerian Shift invariant Operators). *Let* $T \in \mathbb{E}$ *and* $(Q, g_n, q_n(x))$ *an Eulerian triple. Then*

(5.1.2a)
$$T = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k Q^k$$

where $a_k = [Tq_k(x)]_{x=1}$.

<u>Proof</u>: Since $\{q_n(x)\}$ is Eulerian basic for Q , it is an Eulerian family of polynomials and therefore

$$q_n(xy) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q_k(y) x^k q_{n-k}(x) .$$

i.e.
$$q_n(xy) = \sum_{k=0}^{n} \frac{q_k(y)}{k!} x^k n^{(k)} q_{n-k}(x)$$

i.e.
$$q_n(xy) = \sum_{k=0}^{n} \frac{q_k(y)}{k!} x^k Q^k q_n(x) \qquad \text{(lemma (4.3.2))}$$

Applying T_y to both sides, we have

$$T_y q_n(xy) = \sum_{k=0}^{n} \frac{T_y q_k(y)}{k!} x^k Q^k q_n(x)$$

but $T_y q_n(xy) = T_x q_n(xy)$ (proposition (3.3.2))

$$\therefore \quad T_x q_n(xy) = \sum_{k=0}^{n} \frac{T_y q_k(y)}{k!} x^k Q^k q_n(x) .$$

Setting $y = 1$, we get

$$Tq_n(x) = [T_x q_n(xy)]_{y=1}$$

$$= \sum_{k=0}^{n} \frac{[T_y q_k(y)]_{y=1}}{k!} x^k Q^k q_n(x)$$

$$= \sum_{k=0}^{n} \frac{[Tq_k(x)]_{x=1}}{k!} x^k Q^k q_n(x) .$$

This can be extended, by linearity to any polynomial. Hence,

$$T = \sum_k \frac{a_k}{k!} x^k Q^k$$

where $a_k = [Tq_k(x)]_{x=1}$ Q.E.D.

We should also mention that the above theorem is a generalization of the Taylor expansion theorem.

With minor adjustment of the a_k's , one can rewrite (5.1.2a) as:

$$T = \sum_{k=0}^{\infty} \frac{b_k}{[k;\ Q]} x^k Q^k$$

$$\text{where } b_k = \frac{[k;\ Q]}{k!} [Tq_k(x)]_{x=1}$$

and $[k;\ Q]$, the generalized factorial defined in (4.3) comes from the Eulerian triple $(Q,\ g_n,\ q_n(x))$. The latter version turns out to be more convenient to work with in some cases.

We have just defined a mapping:

$$V_Q : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{E} \text{ such that}$$

$$V_Q((a_n)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n Q^n \ .$$

We are going to show that V_Q is a vector space isomorphism where $\mathbb{R}^{\mathbb{N}_0}$ is a vector space of sequences with respect to

(i) addition "+"; i.e. $(a_n) + (b_n) = (a_n + b_n)$

and (ii) scalar multiplication "λ" i.e. $\lambda(a_n) = (\lambda a_n)$

that we are already familiar with. We break the proof of the isomorphism up into individual theorems because we think some of the results are so important that special attention is required.

Theorem (5.1.2) proves that V_Q is *surjective*. Next, we are going to show that it is also *injective*.

(5.1.3) Theorem: (Unique Representation Theorem). *Let* $M = \sum_{k=0}^{\infty} a_k x^k Q^k$ *and* $N = \sum_{k=0}^{\infty} b_k x^k Q^k$ *be two Eulerian shift invariant operators.* $M = N$ *iff* $a_k = b_k \ \forall k \geq 0$.

Proof: To show that $a_k = b_k \ \forall k \geq 0$, it suffices to prove

$$\sum_{k=0}^{\infty} a_k x^k Q^k = \hat{0} \Rightarrow a_k = 0 \ \forall k \geq 0 .$$

$$\text{First,} \quad \sum_{k=0}^{\infty} a_k x^k Q^k x^0 = \hat{0}\, x^0 \Rightarrow a_0 = 0 .$$

Assume $a_k = 0$ for $k = 0, 1, 2, \ldots, m-1$.

$$\sum_{k=0}^{\infty} a_k x^k Q^k x^m = \hat{0}\, x^m \Rightarrow \sum_{k=0}^{m} a_k x^k Q^k x^m = 0$$

$$\text{i.e.} \quad a_m g_m g_{m-1} \cdots g_1 x^m = 0$$

This implies $a_m = 0$ since $g_n \neq 0 \ \forall n > 0$ and $x \neq 0$ in general. By induction, we conclude that $a_k = 0 \ \forall k \geq 0$.

The converse is obvious. Q.E.D.

(5.1.4) Lemma: V_Q *preserves addition.*

Proof: Let $T = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k Q^k$ and $S = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k Q^k$ be two elements of $\mathbb{E}$ where $(a_k), (b_k) \in \mathbb{R}^{\mathbb{N}_0}$ and, from Theorem (5.1.2), $a_k = [Tq_k(x)]_{x=1}$ and $b_k = [Sq_k(x)]_{x=1}$. From Theorem (3.4.1), $T + S \in \mathbb{E}_Q$. Thus by Theorem (5.1.2), $T + S = \sum_{k=0}^{\infty} \frac{c_k}{k!} x^k Q^k$ where

$$\begin{aligned} c_k &= [(T + S)q_k(x)]_{x=1} \\ &= [Tq_k(x) + Sq_k(x)]_{x=1} \\ &= [Tq_k(x)]_{x=1} + [Sq_k(x)]_{x=1} \\ &= a_k + b_k \ . \end{aligned}$$

That means,

$$\begin{aligned} V_Q((a_n) + (b_n)) &= V_Q((a_n + b_n)) \\ &= \sum_{n=0}^{\infty} \frac{a_n + b_n}{n!} x^n Q^n \ . \end{aligned}$$

When the infinite sum acts on a polynomial of fixed degree, it becomes a finite sum. Thus,

$$\begin{aligned} V_Q((a_n) + (b_n)) &= \sum_{n} \frac{a_n + b_n}{n!} x^n Q^n \\ &= \sum_{n} \frac{a_n}{n!} x^n Q^n + \sum_{n} \frac{b_n}{n!} x^n Q^n \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n Q^n + \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n Q^n \quad \text{(This is done by adding zeroes.)} \\ &= V_Q((a_n)) + V_Q((b_n)) \ . \end{aligned}$$

Hence addition is preserved under V_Q . Q.E.D.

In similar manner, we can prove:

(5.1.5) Lemma: V_Q *preserves scalar multiplication.*

Finally, we can conclude that:

(5.1.6) Theorem: *The mapping* $V_Q\colon \mathbb{R}^{\mathbb{N}_0} \to \mathbb{E}$ *is a vector space isomorphism such that*

$$V_Q((a_n)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n Q^n .$$

Since $\mathbb{E}$ is an $\mathbb{R}$-algebra, there exists for each Eulerian differential operator Q, a multiplication on $\mathbb{R}^{\mathbb{N}_0}$, $\otimes$, such that

$$V_Q((a_n) \otimes (b_n)) = V_Q((a_n)) \circ V_Q((b_n)) .$$

The ring isomorphism problem is to determine the sequence

$$(c_n) = (a_n) \otimes (b_n)$$

in terms of the sequences (a_n), (b_n) and the operator Q.

We will replace $\mathbb{E}$ by $\mathbb{E}_Q = \{ \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k Q^k \colon (a_k) \in \mathbb{R}^{\mathbb{N}_0}\}$ i.e. by its representation under V_Q when we are considering this ring isomorphism problem.

Chapter VI

THE ALGEBRA ISOMORPHISM DETERMINED BY THE ORDINARY DIFFERENTIAL OPERATOR D

(6.1) Vector Multiplication on $\mathbb{E}$. We have seen in the last chapter that given an Eulerian differential operator, Q, we can determine a vector space isomorphism $V_Q: \mathbb{R}^{\mathbb{N}_0} \to \mathbb{E}_Q$. Our next goal is to extend this to an algebra isomorphism. To do that, we must devise a vector multiplication on $\mathbb{E}_Q$. The most natural choice would be the composition of operators in $\mathbb{E}_Q$.

Now for any $n, m \geq 0$, x^nQ^n and $x^mQ^m \in \mathbb{E}_Q$. Their composition is also Eulerian shift invariant since

$$x^nQ^nx^mQ^m = \sum_{k=0}^{\infty} H_k^{n,m}(Q) \frac{x^kQ^k}{k!} \quad \text{where}$$

$$H_k^{n,m}(Q) = [x^nQ^nx^mQ^mq_k(x)]_{x=1} . \quad \text{(see theorem (5.1.2))}$$

Moreover, since the operators $\sum_{n=0}^{\infty} a_n \frac{x^nQ^n}{n!}$ and $\sum_{m=0}^{\infty} b_m \frac{x^mQ^m}{m!}$ use only finitely many summands for any particular polynomial, we have

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n \frac{x^nQ^n}{n!}\right) \circ \left(\sum_{m=0}^{\infty} b_m \frac{x^mQ^m}{m!}\right) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_n}{n!}\frac{b_m}{m!} x^nQ^nx^mQ^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_n}{n!}\frac{b_m}{m!}\left(\sum_{k=0}^{\infty} H_k^{n,m}(Q) \frac{x^kQ^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_n}{n!}\frac{b_m}{m!} H_k^{n,m}(Q)\right)\frac{x^kQ^k}{k!} .\end{aligned}$$

It is therefore of interest to calculate the coefficients $H_k^{n,m}(Q)$. We state some general properties of $H_k^{n,m}(Q)$ for any Q and calculate their values when $Q = D$.

(i) $H_k^{n,m}(Q) = 0$ if $k < m$ or $k < n$

This follows directly from the differential properties of Q .

(ii) $H_k^{n,m}(Q) = H_k^{m,n}(Q)$

This is because $x^n Q^n x^m Q^m = x^m Q^m x^n Q^n$.

(iii) $$H_k^{0,m}(Q) = \begin{cases} k! & k = m \\ 0 & k \neq m \end{cases}$$

<u>Proof</u>: $$H_k^{0,m}(Q) = [x^0 Q^0 x^m Q^m q_k(x)]_{x=1}$$

$$= [x^m Q^m q_k(x)]_{x=1}$$

$$= \begin{cases} k! & k = m \\ 0 & k \neq m \end{cases} \qquad \text{(see lemma (4.3.2))}$$

and (iv) $$H_k^{m,k}(Q) = \begin{cases} \dfrac{k![k;\ Q]}{[k-m;\ Q]} & \text{if } m \leq k \\ 0 & \text{if } m > k \end{cases}$$

<u>Proof</u>: $$H_k^{m,k}(Q) = [x^m Q^m x^k Q^k q_k(x)]_{x=1}$$

$$= [x^m Q^m x^k k!\, q_0(x)]_{x=1}$$

$$= \begin{cases} \left[x^m k!\, q_0(x) \cdot \dfrac{[k;\ Q]}{[k-m;\ Q]}\right]_{x=1} & \text{if } m \leq k \\ 0 & \text{if } m > k \end{cases}$$

$$= \begin{cases} \dfrac{k![k;\ Q]}{[k-m;\ Q]} & \text{if } m \leq k \\ 0 & \text{if } m > k\ . \end{cases}$$

(6.1.1) <u>Lemma</u>: $H_k^{n,m}(D) = \begin{cases} \frac{n!}{(k-n)!}\frac{m!}{(k-m)!}\frac{k!}{(m+n-k)!} & ; m, n \le k \le m+n \\ 0 & ; \text{otherwise.} \end{cases}$

<u>Proof</u>: When $m > k$ or $n > k$, $H_k^{n,m}(D) = 0$ from (i) above.

When $k > m + n$, $H_k^{n,m}(D) = 0$ because the factor $(x - 1)$ still exists after the action of $x^n D^n x^m D^m$ and $H_k^{n,m}(D)$ will then be zero when x is replaced by 1.

We are left with the case $m, n \le k \le m + n$. Without loss of generality, we can assume $m \ge n$, and let $m = n + t$ and $k = m + s$.

$$H_k^{n,m}(D) = [x^n D^n x^m D^m (x-1)^{m+s}]_{x=1}$$

$$= [x^n D^n x^m \frac{(m+s)!}{s!}(x-1)^s]_{x=1}$$

$$= \frac{(m+s)!}{s!}[x^n D^n x^{n+t}(x-1)^s]_{x=1}$$

$$= \frac{(m+s)!}{s!}\left[x^n \sum_{j=0}^{n}\binom{n}{j}(D^{n-j}x^{n+t})(D^j(x-1)^s)\right]_{x=1}$$

$$= \frac{(m+s)!}{s!}\left[x^n \sum_{j=0}^{n}\binom{n}{j}\frac{(n+t)!}{(t+j)!}x^{t+j}\frac{s!}{(s-j)!}(x-1)^{s-j}\right]_{x=1}$$

When we substitute $x = 1$ in, the only non zero term is the one with $j = s$.

$$\therefore \quad H_k^{n,m}(D) = \frac{(m+s)!}{s!}\binom{n}{s}\frac{(n+t)!}{(t+s)!}s!$$

$$= k!\frac{n!}{(n-(k-m))!(k-m)!}\frac{m!}{((m-n)+(k-m))!}$$

$$= \frac{n!}{(k-n)!}\frac{m!}{(k-m)!}\frac{k!}{(n+m-k)!} \qquad \text{Q.E.D.}$$

Thus,

$$\left(\sum_{n=0}^{\infty} a_n \frac{x^n D^n}{n!}\right) \circ \left(\sum_{m=0}^{\infty} b_m \frac{x^m D^m}{m!}\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_n}{n!}\frac{b_m}{m!}\frac{n!}{(k-n)!}\frac{m!}{(k-m)!}\frac{k!}{(n+m-k)!}\right)\frac{x^k D^k}{k!}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\sum_{m=k-n}^{k} \frac{a_n}{(k-n)!}\frac{b_m}{(k-m)!}\frac{k!}{(m+n-k)!}\right)\frac{x^k D^k}{k!}$$

(This is because (i) $\frac{1}{(k-n)!} = 0$ whenever $n > k$,

(ii) $\frac{1}{(k-m)!} = 0$ whenever $m > k$,

and (iii) $\frac{1}{(m+n-k)!} = 0$ whenever $m < k - n$.)

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\sum_{m=0}^{n} \frac{a_n}{(k-n)!}\frac{b_{m-n+k}}{(k-(m-n+k))!}\frac{k!}{((m-n+k)+n-k)!}\right)\frac{x^k D^k}{k!}$$

(replacing m by $m - n + k$)

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\sum_{m=0}^{n} \frac{a_n}{(k-n)!}\frac{b_{m-n+k}}{(n-m)!}\frac{k!}{m!}\cdot\frac{n!}{n!}\right)\frac{x^k D^k}{k!}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\sum_{m=0}^{n} a_n b_{m-n+k}\binom{k}{n}\binom{n}{m}\right)\frac{x^k D^k}{k!}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\binom{k}{n}a_n \sum_{m=0}^{n} b_{m-n+k}\binom{n}{m}\right)\frac{x^k D^k}{k!}$$

It should be understood from the above proof that $H_k^{n,m}(D)$ can be determined explicitly only because the Eulerian basic polynomials for D are known to be $\{(x-1)^n\}$ and the Leibnitz rule enables us to calculate the n^{th} derivative of the product of two functions. Complication arises in simplifying the coefficients $H_k^{n,m}(Q)$ in the general case. However, as we shall see later on, our algebra isomorphism does not require the knowledge of the simplified form of $H_k^{n,m}(Q)$.

Before we go any further, we want to make a side-trip to study a special type of series known as *Formal Newton Series.*

(6.2) <u>A Motivation to the Study of Formal Newton Series</u>. Mullin and Rota in [11] proved that every shift invariant operator T , when expanded with respect to a delta operator P i.e. $T = \sum_{k=0}^{\infty} a_k \frac{P^k}{k!}$, corresponds in a natural way to the formal power series $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ (also see chapter II). Such an isomorphism permits one to investigate the properties of the ring of shift invariant operators via the well explored formal power series. We are going to develop the similar type of isomorphism.

Since an Eulerian shift invariant operator is expanded neither in powers of Eulerian differential operator Q , nor in powers of xQ , but in a rather peculiar form:

$$x^k Q^k \; ; \; k = 0, 1, 2, \ldots \; ,$$

there was no reason to expect that it would have any relation with the formal power series again, and indeed it did not. The following lemma which appears in Riordan [13], explains why we choose the Formal Newton Series.

(6.2.1.) <u>Lemma</u>: $x^n D^n = xD(xD - 1) \ldots (xD - n + 1) \quad \forall n \geq 1.$

The structural resemblance between

$$x^n D^n = xD(xD - 1) \ldots (xD - n + 1) \qquad \text{and}$$

$$s^{(n)} = s(s-1) \ldots (s - n + 1)$$

strongly suggests that the isomorphic image of $\sum a_k \frac{x^k D^k}{k!}$, if any, should be of the form $\sum a_k \frac{s^{(k)}}{k!}$. Such a series is known as a Formal Newton Series.

(6.3) Formal Newton Series: This type of series can be justified by interpreting these series as functions from $\mathbb{N}_0$ to $\mathbb{R}$. The justification is given by the following results ((6.3.1) to (6.3.5)).

(6.3.1) Definition: *A formal Newton series function is a function* f *from* $\mathbb{N}_0$ *to* $\mathbb{R}$ *such that*

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}$$

where (a_k) *is a sequence of reals and*

$$s^{(k)} = \begin{cases} s(s-1)\ldots(s-k+1) & k > 0 \\ 1 & k = 0 \end{cases}.$$

Remark: (i) For our purpose, it is necessary for us to restrict the domain of f to $\mathbb{N}_0$ so that f remains as a finite sum. i.e. the sum only goes up to $k = s$ because the rest of the terms are zero.

(ii) From here on, the term Formal Newton Series is used while its property as function is understood.

(iii) For more detailed description on Formal Newton Series, see Allaway's "Extension of Sheffer Polynomial sets", [1].

(6.3.2) Lemma: *Two formal Newton series* $\sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}$ *and* $\sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}$ *are equal if and only if* $a_k = b_k \quad \forall k \geq 0$.

Proof: If $\sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} = \sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}$, then for all integer $n \geq 0$

$\sum_{k=0}^{n} a_k \frac{n^{(k)}}{k!} = \sum_{k=0}^{n} b_k \frac{n^{(k)}}{k!}$. When $n = 0$, we have $a_0 = b_0$.

When $n = 1$, we have $a_0 + a_1 \cdot 1 = b_0 + b_1 \cdot 1$

$$\text{i.e. } a_1 = b_1 .$$

Now assume $a_i = b_i$ for $i = 0, 1, 2, \ldots m$.

Then
$$\sum_{k=0}^{\infty} \frac{a_k (m+1)^{(k)}}{k!} = \sum_{k=0}^{\infty} b_k \frac{(m+1)^{(k)}}{k!}$$

implies
$$\sum_{k=0}^{m} a_k \frac{(m+1)^{(k)}}{k!} + a_{m+1} \frac{(m+1)^{(m+1)}}{(m+1)!} = \sum_{k=0}^{m} b_k \frac{(m+1)^{(k)}}{k!} + b_{m+1} \frac{(m+1)^{(m+1)}}{(m+1)!}$$

$$\text{i.e. } a_{m+1} = b_{m+1}$$

Hence, by induction, $a_k = b_k \quad \forall k \geq 0$.

The converse is obvious. Q.E.D.

(6.3.3) Lemma: *Two formal Newton series* $\sum_{m=0}^{\infty} a_m \frac{s^{(m)}}{m!}$ *and* $\sum_{m=0}^{\infty} b_m \frac{s^{(m)}}{m!}$ *are equal if and only if* $\forall n \geq 0$

(6.3.2a)
$$\left[\Delta^n \sum_{m=0}^{\infty} a_m \frac{s^{(m)}}{m!}\right]_{s=0} = \left[\Delta^n \sum_{m=0}^{\infty} b_m \frac{s^{(m)}}{m!}\right]_{s=0}$$

where Δ is the advancing difference operator defined as

$$\Delta f(s) = f(s+1) - f(s) .$$

<u>Proof</u>: (6.3.3a) can be rewritten as

$$\left[\sum_{m=0}^{\infty} a_m \Delta^n \frac{s^{(m)}}{m!}\right]_{s=0} = \left[\sum_{m=0}^{\infty} b_m \Delta^n \frac{s^{(m)}}{m!}\right]_{s=0} . \tag{6.3.3b}$$

Noting that

$$\Delta^n \frac{s^{(m)}}{m!} = \begin{cases} \dfrac{s^{(m-n)}}{(m-n)!} & n < m \\ 1 & n = m \\ 0 & n > m \end{cases}$$

we have, from (6.3.3b),

$$\left[\sum_{m=0}^{\infty} a_m \frac{s^{(m-n)}}{(m-n)!}\right]_{s=0} = \left[\sum_{m=0}^{\infty} b_m \frac{s^{(m-n)}}{(m-n)!}\right]_{s=0}$$

i.e.

$$\left[\sum_{m=0}^{\infty} a_{m+n} \frac{s^{(m)}}{m!}\right]_{s=0} = \left[\sum_{m=0}^{\infty} b_{m+n} \frac{s^{(m)}}{m!}\right]_{s=0} .$$

When the sums are evaluated at $s = 0$, the only non-zero terms are when $m = 0$. Thus $a_n = b_n$; $n = 0, 1, 2, \ldots,$ showing that the two Newton series are equal. The converse is obvious. Q.E.D.

We shall use $\mathfrak{S}$ to denote the set of all Formal Newton series. Binary operations can be defined on $\mathfrak{S}$. Let $f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}$ and $g(s) = \sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}$ be two typical elements of $\mathfrak{S}$. Addition, $+$, is defined as:

$$f(s) + g(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} + \sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}$$

$$= \sum_{k=0}^{\infty} (a_k + b_k) \frac{s^{(k)}}{k!} .$$

Multiplication, $\times$, is defined as the product of the two series, i.e.

$$f(s) \times g(s) = \left(\sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}\right)\left(\sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}\right) .$$

To see that $\mathfrak{S}$ is closed under $\times$, we need the following lemma.

(6.3.4) Lemma: $s^{(n)}s^{(m)} = \sum_{k=0}^{n+m} \frac{k!}{(n+m-k)!} \frac{n!}{(k-n)!} \frac{m!}{(k-m)!} \frac{s^{(k)}}{k!}$, $\forall n, m \geq 0$.

Proof: $\frac{s^{(n)}}{n!} \frac{s^{(m)}}{m!} = \binom{s}{n}\binom{s}{m}$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n+m-k}{m-k}\binom{s}{m+n-k} \quad \text{(Riordan [13]; pp. 15)}$$

$$= \sum_{k=0}^{n} \binom{m}{k}\binom{n+m-k}{m}\binom{s}{m+n-k} \quad \text{(binomial identity)}$$

$$= \sum_{k=m+n}^{k=m} \binom{m}{n+m-k}\binom{k}{m}\binom{s}{k} \quad \text{(replacing } k \text{ by } n+m-k\text{)}$$

$$= \sum_{k=m}^{m+n} \binom{m}{n+m-k}\binom{k}{m}\binom{s}{k} \quad \text{(reversing the order of summation)}$$

$$= \sum_{k=m}^{m+n} \frac{m!}{(n+m-k)!(k-n)!} \frac{k!}{(k-m)!m!} \frac{s!}{(s-k)!k!}$$

$$= \sum_{k=m}^{m+n} \frac{k!}{(n+m-k)!(k-m)!(k-n)!} \frac{s^{(k)}}{k!}$$

$$= \sum_{k=0}^{n+m} \frac{k!}{(n+m-k)!(k-m)!(k-n)!} \frac{s^{(k)}}{k!}$$

This is done by adding m zeroes as the first m terms and this agrees with the general term since

$$\frac{1}{(k-m)!} = 0 \quad \text{whenever} \quad k < m .$$

Hence, $$s^{(n)} s^{(m)} = \sum_{k=0}^{n+m} \frac{k!}{(n+m-k)!} \frac{m!}{(k-m)!} \frac{n!}{(k-n)!} \frac{s^{(k)}}{k!}$$ Q.E.D.

<u>Remark</u>: Lemma (6.3.4) is merely another disguised formulation of the Pfaff-Saalschutz summation. That is, (6.3.4) is a special case of

$$\sum_{k\geq 0} \binom{m-t}{k} \binom{n+t}{t+k} \binom{s+t+k}{m+n} = \binom{s+t}{m} \binom{s}{n}$$

with $t = 0$. For more details, see Andrews [4; pp. 98] and Riordan [13; pp. 15].

(6.3.5) <u>Lemma</u>: *The product of two formal Newton series is another formal Newton series. More precisely*

$$f(x) \times g(s) \equiv \left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!} \right) \times \left(\sum_{m=0}^{\infty} b_m \frac{s^{(m)}}{m!} \right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} b_{m-n+k} \right) \frac{s^{(k)}}{k!}$$

<u>Proof</u>: $$f(s) \times g(s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_n}{n!} \frac{b_m}{m!} s^{(n)} s^{(m)}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_n}{n!} \frac{b_m}{m!} \sum_{k=0}^{m+n} \frac{k!}{(m+n-k)!} \frac{n!}{(k-n)!} \frac{m!}{(k-m)!} \frac{s^{(k)}}{k!}$$

(lemma (6.3.4))

Noting that $\frac{1}{(m+n-k)!} = 0$ whenever $k > m + n$, we can write

$$f(s) \times g(s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_n}{(k-n)!} \frac{b_m}{(k-m)!} \frac{k!}{(m+n-k)!} \frac{s^{(k)}}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_n}{(k-n)!} \frac{b_m}{(k-m)!} \frac{k!}{(m+n-k)!} \frac{s^{(k)}}{k!}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} b_{m-n+k} \right) \frac{s^{(k)}}{k!}$$

by the same argument as in (6.1.1). Q.E.D.

(6.3.6) The distributive law also holds since

$$\left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \times \left(\sum_{m=0}^{\infty} b_m \frac{s^{(m)}}{m!} + \sum_{m=0}^{\infty} c_m \frac{s^{(m)}}{m!}\right)$$

$$= \left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \times \left(\sum_{m=0}^{\infty} (b_m + c_m) \frac{s^{(m)}}{m!}\right) \quad \text{(definition of addition)}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m}(b_{m-n+k} + c_{m-n+k})\right)\frac{s^{(k)}}{k!} \quad \text{(lemma (6.3.5))}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} b_{m-n+k} + \sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} c_{m-n+k}\right)\frac{s^{(k)}}{k!}$$

$$= \sum_{k=0}^{s}\left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} b_{m-n+k}\right)\frac{s^{(k)}}{k!} + \sum_{k=0}^{s}\left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} c_{m-n+k}\right)\frac{s^{(k)}}{k!} \quad \text{(see the remark on (6.3.1))}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} b_{m-n+k}\right)\frac{s^{(k)}}{k!} + \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} \binom{n}{m} c_{m-n+k}\right)\frac{s^{(k)}}{k!}$$

(This is done by adding infinitely many zeroes to the Newton series.)

$$= \left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \times \left(\sum_{m=0}^{\infty} b_m \frac{s^{(m)}}{m!}\right) + \left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \times \left(\sum_{m=0}^{\infty} c_m \frac{s^{(m)}}{m!}\right)$$

(lemma (6.3.5))

Finally, we conclude that $(\mathbb{S}, +, -, \hat{0}, \times, \lambda)$ is an algebra since the definitions of the operations were transported via the interpretation as functions, and these are subalgebras (of the algebra of all functions from $\mathbb{N}_0$ to $\mathbb{R}$).

(6.4) The Algebra Isomorphism Theorem. With all the preliminary results done, we can now state the isomorphic relationship between $\mathbb{E}_D$ and $\mathbb{S}$.

(6.4.1) Theorem: *There exists an algebra isomorphism* ϕ_D *from* $(\mathbb{E}_D, +, -, \hat{0}, \circ, \lambda)$ *to* $(\mathbb{S}, +, -, \hat{0}, \times, \lambda)$ *such that*

$$\phi_D : \sum_{k=0}^{\infty} a_k \frac{x^k D^k}{k!} \to \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} .$$

Proof: In view of theorem (5.1.6), we only have to show that ϕ_D preserves multiplication.

Let $U = \sum_{k=0}^{\infty} a_k \frac{x^k D^k}{k!}$ and $V = \sum_{k=0}^{\infty} b_k \frac{x^k D^k}{k!}$.

Then,

$$U \circ V = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_n}{n!} \frac{b_m}{m!} \frac{n!}{(k-n)!} \frac{m!}{(k-m)!} \frac{k!}{(n+m-k)!} \right) \frac{x^k D^k}{k!}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} b_{m-n+k} \binom{n}{m} \right) \frac{x^k D^k}{k!} \qquad \text{(see (6.1.1))}$$

Under the mapping,

$$\phi_D(U \circ V) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \binom{k}{n} a_n \sum_{m=0}^{n} b_{m-n+k} \binom{n}{m} \right) \frac{s^{(k)}}{k!}$$

$$= \left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!} \right) \times \left(\sum_{m=0}^{\infty} b_m \frac{s^{(m)}}{m!} \right) \qquad \text{(lemma (6.3.5))}$$

$$= \phi_D(U) \times \phi_D(V) .$$ Q.E.D.

In the above theorem, we have restricted ourselves in expanding a given Eulerian shift invariant operator in terms of the differential operator D only. Our next goal, of course, is to generalize the above version so that it is applicable when an Eulerian shift invariant operator is expanded in terms of ANY given Eulerian differential operator.

(6.4.2) <u>Corollary</u>: *There exists an algebra isomorphism* ϕ_Q *from* $(\mathbb{E}_Q, +, -, \hat{0}, \circ, \lambda)$ *to* $(\mathfrak{S}, +, -, 0, \times, \lambda)$ *such that*

$$\phi_Q: \sum_{k=0}^{\infty} a_k \frac{x^k Q^k}{k!} \to \sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}$$

where
$$b_k = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} (-1)^{n-j} \frac{[j;\ Q]}{[j-k;\ Q]} \frac{a_k}{k!}$$

<u>Proof</u>: From lemma (4.3.1) and theorem (5.1.2), we have

$$\sum_{k=0}^{\infty} a_k \frac{x^k Q^k}{k!} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \sum_{n=0}^{\infty} [x^k Q^k (x - 1)^n]_{x=1} \frac{x^n D^n}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} [x^k Q^k \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} x^j]_{x=1} \frac{a_k}{k!} \frac{x^n D^n}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=k}^{n} \binom{n}{j} (-1)^{n-j} \frac{[j;\ Q]}{[j-k;\ Q]} \frac{a_k}{k!} \frac{x^n D^n}{n!} \quad \text{(lemma (4.3.1))}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=k}^{n} \binom{n}{j} (-1)^{n-j} \frac{[j;\ Q]}{[j-k;\ Q]} \frac{a_k}{k!} \frac{x^n D^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} (-1)^{n-j} \frac{[j;\ Q]}{[j-k;\ Q]} \frac{a_k}{k!} \frac{x^n D^n}{n!} \quad \text{(see (4.3))}$$

The required result follows from Theorem (6.4.1). Q.E.D.

For simplicity in the discussion of our later works, by isomorphism theorem, we are referring to the one described in Theorem (6.4.1) rather than the more complicated version stated in (6.4.2). It should, however, be understood that the theory is equally applicable to the generalized version with minor adjustments.

We discuss some of the consequences of the algebra isomorphism theorem.

(6.4.3) Corollary: Q *is an Eulerian differential operator if and only if* xQ *corresponds, under the isomorphism, to a formal Newton series*

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} \quad \textit{such that}$$

$$f(0) = 0 \quad \textit{and} \quad f(n) \neq 0 \quad \forall n \geq 1 .$$

Proof: If Q is an Eulerian differential operator, then xQ is, by definition, Eulerian shift invariant with the expansion $\sum_{k=0}^{\infty} a_k \frac{x^k D^k}{k!}$. Under the isomorphism, the corresponding formal Newton series for xQ is

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} .$$

$$\sum_{k=0}^{\infty} a_k \frac{x^k D^k}{k!} 1 = xQ1 = 0 \Rightarrow a_0 = 0$$

This means $f(0) = 0$.

Thus
$$Q = \sum_{k=1}^{\infty} a_k \frac{x^{k-1} D^k}{k!}$$

and
$$\begin{aligned} Qx^n &= \left(\sum_{k=1}^{\infty} a_k \frac{x^{k-1} D^k}{k!} \right) x^n \\ &= \sum_{k=1}^{n} a_k \frac{x^{k-1}}{k!} n^{(k)} x^{n-k} \\ &= \left(\sum_{k=1}^{n} a_k \frac{n^{(k)}}{k!} \right) x^{n-1} \\ &= f(n) x^{n-1} . \end{aligned}$$

Also $Qx^n = g_n x^{n-1}$ where $\{g_n\}$ is the fundamental sequence for Q. Therefore $g_n = f(n)$.

Since, by definition, $g_n \neq 0 \;\; \forall n \geq 1$, we have $\forall n \geq 1$

$$f(n) \neq 0 .$$

Conversely, if xQ corresponds, under the isomorphism, to a formal Newton series

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} \quad \text{with} \quad f(0) = 0 \quad \text{and} \quad f(n) \neq 0 \quad \forall n \geq 1 ,$$

then xQ is an Eulerian shift invariant operator such that

$$xQ = \sum_{k=1}^{\infty} a_k \frac{x^k D^k}{k!} \; ; \quad \text{or equivalently} \quad Q = \sum_{k=1}^{\infty} a_k \frac{x^{k-1} D^k}{k!} .$$

Since $\quad Qx^0 = \sum_{k=1}^{\infty} a_k \frac{x^{k-1}D^k}{k!} \; 1 = 0$, and

for $n \geq 1$,

$$Qx^n = \left(\sum_{k=1}^{\infty} a_k \frac{x^{k-1}D^k}{k!} \right) x^n$$

$$= \sum_{k=1}^{\infty} a_k \frac{x^{k-1}}{k!} n^{(k)} x^{n-k}$$

$$= \left(\sum_{k=1}^{\infty} a_k \frac{n^{(k)}}{k!} \right) x^{n-1}$$

$$= f(n) x^{n-1} ,$$

there exists a sequence of constants $\{0, f(1), f(2), \ldots f(n) \ldots\}$ for Q. In addition, $f(n) \neq 0 \;\; \forall n \geq 1$, Q is therefore an Eulerian differential operator. Q.E.D.

<u>Remark</u>: With the notations in the preceding corollary, one can easily see that the fundamental sequence for Q is $\{f(n)\}_{n=0}^{\infty}$.

Chapter VII

EULERIAN SHEFFER POLYNOMIALS

(7.1) Introduction. In [15], Rota, Kahaner and Odlyzko extend the theory of polynomials of binomial type to Sheffer polynomials. A Sheffer set with respect to the delta operator Q is a sequence of polynomials $\{s_n(x)\}$ such that

(i) $Qs_n(x) = ns_{n-1}(x)$

and (ii) $s_0(x) = c \neq 0$.

Comparing with the definition of basic polynomial sequence given in (2.2.4), one can easily see that a Sheffer set has less restriction than a basic polynomial sequence. A Sheffer set is therefore not necessarily unique with respect to a delta operator. However, a Sheffer set still possesses the similar type of characterization that a basic set has (see Theorem (2.2.6)). That is:

(7.1.1) Theorem: (Rota, Kahaner and Odlyzko [15], section 5, prop. 6). *A sequence* $\{s_n(x)\}$ *is a Sheffer set with respect to a basic set* $\{q_n(x)\}$ *if and only if*

$$s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)q_{n-k}(y) \ .$$

Proof: Refer to original paper [15].

We shall develop the Eulerian analog of some of these facts.

(7.2) Eulerian Sheffer Sets.

(7.2.1) **Definition**: *A polynomial sequence* $\{s_n(x)\}$ *is an Eulerian Sheffer set for the Eulerian differential operator* Q *if*

(i) $s_0(x) = c \neq 0$

and (ii) $Qs_n(x) = ns_{n-1}(x)$.

An example of an Eulerian Sheffer set is $s_n(x) = (x + 1)^n$ for D because

$$s_0(x) = 1 \neq 0 ,$$

and $Ds_n(x) = D(x + 1)^n = n(x + 1)^{n-1} = ns_{n-1}(x)$.

From the above definition, it is not difficult to see that an Eulerian Sheffer set $\{s_n(x)\}$ should somehow be related to the Eulerian basic sequence $\{q_n(x)\}$ for Q . One simple relation between the two is given in the following lemma.

(7.2.2) **Lemma**: (Andrews [3]; Theorem 12) *Let* $\{s_n(x)\}$ *be an Eulerian Sheffer set with respect to the Eulerian differential operator* Q *whose basic polynomial sequence is* $\{q_n(x)\}$. *Then*

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(1) q_k(x) . \qquad (7.2.2a)$$

Proof: Since $s_0(x) = c = s_0(1) = s_0(1)q_0(x)$, (7.2.2a) is true for $n = 0$. Assume (7.2.2a) is true for $n = 0, 1, \ldots, m - 1$.

Then
$$\begin{aligned} Qs_m(x) &= ms_{m-1}(x) \\ &= m \sum_{k=0}^{m-1} \binom{m-1}{k} s_{m-1-k}(1) q_k(x) \\ &= m \sum_{k=1}^{m} \binom{m-1}{k-1} s_{m-k}(1) q_{k-1}(x) \\ &= \sum_{k=1}^{m} \binom{m}{k} s_{m-k}(1) k q_{k-1}(x) \end{aligned}$$

$$= \sum_{k=0}^{m} \binom{m}{k} s_{m-k}(1) Q q_k(x) \qquad (Qq_0(x) = 0)$$

$$= Q\left(\sum_{k=0}^{m} \binom{m}{k} s_{m-k}(1) q_k(x)\right)$$

$$\Rightarrow s_m(x) = \sum_{k=0}^{m} \binom{m}{k} s_{m-k}(1) q_k(x) + c \quad \text{where } c \text{ is a constant.}$$

When $m = 0$, $s_0(x) = s_0(1) + c \Rightarrow c = 0$.

$$\therefore s_m(x) = \sum_{k=0}^{m} \binom{m}{k} s_{m-k}(1) q_k(x) \ .$$

Hence, by induction, (7.2.2a) is true $\forall n \geq 0$. Q.E.D.

The following theorem provides an important characterization for the Eulerian Sheffer set. It was first quoted by Andrews [3]. Our proof is different than his.

(7.2.3) Theorem: *$\{s_n(x)\}$ is an Eulerian Sheffer set with respect to Q if and only if $s_n(x)$, with $s_0(1) = c \neq 0$, satisfies*

(7.2.3a)
$$s_n(xy) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(x) x^k q_k(y)$$

where $\{q_n(x)\}$ is the Eulerian basic set for Q .

Proof: If $s_n(x)$ satisfies (7.2.3a), we have in particular

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(1) q_k(x) \ .$$

Then,
$$Qs_n(x) = \sum_{k=1}^{n} \binom{n}{k} s_{n-k}(1) k q_{k-1}(x)$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} s_{n-k}(1) q_{k-1}(x)$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} s_{n-k-1}(1) \cdot 1^k \cdot q_k(x)$$

$$= n s_{n-1}(x) \ .$$

Hence, $\{s_n(x)\}$ is an Eulerian Sheffer set. Conversely, if $\{s_n(x)\}$ is an Eulerian Sheffer set, then $s_0(1) = c \neq 0$, and

$$s_m(xy) = \sum_{n=0}^{m} \binom{m}{n} s_{m-n}(1) q_n(xy) \qquad \text{(lemma (7.2.2))}$$

$$= \sum_{n=0}^{m} \binom{m}{n} s_{m-n}(1) \sum_{k=0}^{n} \binom{n}{k} q_{n-k}(x) x^k q_k(y)$$

$$= \sum_{n=0}^{m} \sum_{k=0}^{n} \binom{m}{n}\binom{n}{k} s_{m-n}(1) q_{n-k}(x) x^k q_k(y)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{m}{n}\binom{n}{k} s_{m-n}(1) q_{n-k}(x) x^k q_k(y) \qquad (s_j(x) = 0 \quad \forall j < 0)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{m}{n+k}\binom{n+k}{k} s_{m-n-k}(1) q_n(x) x^k q_k(y) \qquad \text{(conversion formula)}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{m}{n+k}\binom{n+k}{k} s_{m-n-k}(1) q_n(x) x^k q_k(y)$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{m}{k}\binom{m-k}{n} s_{m-n-k}(1) q_n(x) x^k q_k(y) \qquad \text{(binomial identity)}$$

$$= \sum_{k=0}^{\infty} \binom{m}{k} x^k q_k(y) \sum_{n=0}^{\infty} \binom{m-k}{n} s_{m-n-k}(1) q_n(x)$$

$$= \sum_{k=0}^{\infty} \binom{m}{k} x^k q_k(y) \sum_{n=0}^{m-k} \binom{m-k}{n} s_{m-n-k}(1) q_n(x) \qquad (s_j(x) = 0 \quad \forall j < 0)$$

$$= \sum_{k=0}^{\infty} \binom{m}{k} x^k q_k(y) s_{m-k}(x) \qquad \text{(lemma (7.2.2))}$$

$$= \sum_{k=0}^{m} \binom{m}{k} s_{m-k}(x) x^k q_k(y) \qquad (s_j(x) = 0 \quad \forall j < 0)$$

which is the required result. Q.E.D.

(7.3) Properties of Eulerian Sheffer Set. Since An Eulerian Sheffer set has less restriction than the Eulerian basic polynomial sequence, one can easily see that some of the properties that a basic set has are equally applicable to an Eulerian Sheffer set. One example is

$$Qs_n(\theta x) = ns_{n-1}(\theta x) \cdot \theta \qquad \forall \theta \in \mathbb{R} .$$

However, it is the restriction

$$q_n(1) = 0 \quad \forall n \geq 1$$

that makes an Eulerian basic set "unique". That is to say, given an Eulerian Sheffer set $\{s_n(x)\}$ with respect to Q, we can always find another, infintiely many in fact, Eulerian Sheffer set $\{\overline{s}_n(x)\}$ for the same Q. This is described in the following proposition.

(7.3.1) Proposition: *If* $\{s_n(x)\}$ *is an Eulerian Sheffer set with respect to* Q *whose Eulerian basic set is* $\{q_n(x)\}$, *then for a fixed real value of* θ ,

(7.3.1a) $$\overline{s}_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(\theta) q_k(x)$$

is again an Eulerian Sheffer set with respect to Q .

Proof: Clearly, $\overline{s}_0(x) = s_0(\theta)$

$= c \neq 0$ by definition.

$$\text{Also, } Q\bar{s}_n(x) = Q \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(\theta) q_k(x)$$

$$= \sum_{k=1}^{n} \binom{n}{k} s_{n-k}(\theta) k q_{k-1}(x)$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} s_{n-k}(\theta) q_{k-1}(x)$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} s_{n-1-k}(\theta) q_k(x)$$

$$= n\bar{s}_{n-1}(x)$$

Thus, $\bar{s}_n(x)$ is also an Eulerian Sheffer set for Q . Q.E.D.

It is obvious that $\bar{s}_n(x) \equiv s_n(x)$ when $\theta = 1$. As a simple illustration, we use $p_n(x) = (x - 1)^n$ and $s_n(x) = x^n$ as respectively the Eulerian basic polynomial sequence and Eulerian Sheffer set for D . If we put $\theta = 1 + \phi$ in (7.3.1a), we have

$$\bar{s}_n(x) = \sum_{k=0}^{n} \binom{n}{k} (1 + \phi)^{n-k} (x - 1)^k$$

$$= (x + \phi)^n$$

which, as ϕ changes, generates a series of Eulerian Sheffer sets with respect to the same operator D . However, (7.3.1a) is not "powerful" enough to generate all the Eulerian Sheffer sets. For instance, $\hat{s}_n(x) = (x - 1)^n + x^n$ is another Eulerian Sheffer set for D but it cannot be generated from $(x + \phi)^n$; $\forall\phi \in \mathbb{R}$.

So far our theory is a close parallel to the result of Rota, Kahanar and Odlyzko ([15]). The following result exhibits the beginning of the divergence of the theories. In section 5, proposition 1, of [15], Rota (et al) stated that there is an invertible shift invariant operator T relating the basic sequence $\{q_n(x)\}$ of the delta operator Q to a Sheffer set $\{s_n(x)\}$ with respect to the same delta operator by

$$T^{-1}s_n(x) = q_n(x) .$$

In our theory, we found that such an operator, if it exists, only generates Eulerian Sheffer sets which are constant multiples of the Eulerian basic set.

(7.3.2) <u>Proposition</u>: *Let* $\{q_n(x)\}$ *and* $\{s_n(x)\}$ *be, respectively, the Eulerian basic polynomial sequence and Eulerian Sheffer set for* Q . *Define* S *as*

$$S: q_n(x) \to s_n(x) .$$

If S *is Eulerian shift invariant, then* $S = cI$ *where* $c = s_0(x)$ *and* I *is the identity map.*

<u>Proof</u>: Let $\{c_n\}$ and $\{g_n\}$ be respectively the fundamental sequences for S and Q . We first show that S commutes with Q , i.e. $SQ = QS$.

$$QSq_n(x) = Qs_n(x) = ns_{n-1}(x) = nSq_{n-1}(x) = SQq_n(x) .$$

Next, we show that the c_n's are all equal. Since $g_{n+1} \neq 0 \ \forall n \geq 0$, $\frac{1}{g_{n+1}}$ is well defined, we can write

$$
\begin{aligned}
Sx^n &= SQ \frac{1}{g_{n+1}} x^{n+1} \\
&= \frac{1}{g_{n+1}} QSx^{n+1} \\
&= \frac{1}{g_{n+1}} Qc_{n+1} x^{n+1} \\
&= \frac{c_{n+1}}{g_{n+1}} g_{n+1} x^n \\
&= c_{n+1} x^n .
\end{aligned}
$$

But $Sx^n = c_n x^n$ by definition.

Thus $c_0 = c_1 = c_2 \ldots\ldots = c_n = \ldots$.

In particular, $Sq_0(x) = s_0(x)$

i.e. $S1 = c$.

This implies $Sx^n = cx^n$.

Hence, $S = cI$. Q.E.D.

Remark: The above result is immediate from the fact that $QS = SQ$ with S being Eulerian shift invariant while Q is not.

The subsequent development of Rota's (et al) work on Sheffer polynomials depends very much on the existence of such an invertible shift invariant operator. In our case, however, it is not of too much interest to study an operator that "creates" Eulerian Sheffer sets which merely are constant multiples of the original Eulerian basic sequence. In fact, it is the diversion of our theory that makes drawing further analogs Rota's (et al) theory difficult.

Chapter VIII

APPLICATIONS

(8.1) In this chapter, we provide some simple applications using the expansion theorem and the algebra isomorphism theorem.

(8.1.1) According to the isomorphism theorem (Theorem (6.4.1)), the image of $s^{(n)}$ is $x^n D^n$; $n = 0, 1, 2, \ldots$. Thus from the following well known relationship

(8.1.1a)
$$s^{(n)} = \sum_{k=0}^{n} s(n, k) s^k ,$$

where $s(n, k)$ is the Stirling numbers of the first kind, we obtain

$$x^n D^n = \sum_{k=0}^{n} s(n, k)(xD)^k .$$

Then
$$x^n D^n x^m D^m = \left(\sum_{k=0}^{n} s(n, k)(xD)^k\right)\left(\sum_{j=0}^{m} s(m, j)(xD)^j\right)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{m} s(n, k) s(m, j)(xD)^{k+j}$$

$$= \sum_{k=0}^{n} \sum_{j=k}^{m+k} s(n, k) s(m, j - k)(xD)^j .$$

It provides an alternate rule in determining $H_k^{n,m}(D)$ in (6.1.1).

(8.2.1) Our second application comes from the inverse formula of (8.1.1a), i.e.

(8.2.1a)
$$s^n = \sum_{k=0}^{n} S(n, k) s^{(k)} ,$$

where $S(n, k)$ is the Stirling number of the second kind. Under the isomorphism, (8.2.1a) becomes

(8.2.1b) $$(xD)^n = \sum_{k=0}^{n} S(n, k)k! \frac{x^k D^k}{k!} \quad .$$

$(xD)^n$; $n = 0, 1, 2, \ldots$, is clearly Eulerian shift invariant and therefore the right hand side of (8.2.1b) is its expansion, in the sense of Theorem (5.1.2). Thus the Stirling number of the second kind can be determined systematically by the following formula

$$S(n, k) = \frac{1}{k!} [(xD)^n (x - 1)^k]_{x=1} \quad .$$

(8.3.1) We use $[x]_n$ to denote the rising factorial sequence,

i.e. $$[x]_n = x(x + 1)(x + 2) \ldots (x + n - 1) \quad .$$

Similarly, $[xD]_n = xD(xD + 1) \ldots (xD + n - 1)$ which is understood to be Eulerian invariant and therefore by the expansion theorem,

$$[xD]_n = \sum_{k=0}^{n} J_{n,k} \frac{x^k D^k}{k!} \quad \text{for some} \quad J_{n,k} \in \mathbb{R} \, .$$

Riordan in [13] proved that

$$[xD]_n = n! L_n^{(-1)}(-xD) \; ; \; (xD)^n \equiv x^n D^n \; ,$$

where $L_n^{(p)}(x)$, the generalized Laguerre polynomial, is given by

$$L_n^{(p)}(x) = \sum_{k=0}^{n} \binom{n + p}{n - k} \frac{(-x)^k}{k!} \quad .$$

Thus,

(8.3.1a) $$[xD]_n = \sum_{k=0}^{n} \binom{n - 1}{n - k} \frac{n!}{k!} x^k D^k \quad .$$

By the isomorphism theorem, we have

$$[s]_n = \sum_{k=0}^{n} \binom{n - 1}{n - k} \frac{n!}{k!} s^{(k)} \quad .$$

For reference purposes, besides (8.2.1b) and (8.3.1a), we also list the following identities, which appear in Riordan [13].

$$(Dx)^n = \sum_{k=0}^{n} k!\ S(n+1, k+1) \frac{x^k D^k}{k!}$$

$$(Dx)^{(n)} = nx^{n-1}D^{n-1} + x^n D^n$$

The isomorphic image of Dx is $s + 1$. This follows directly from the identity

$$Dx = I + xD .$$

(8.4.1) We extend a little further the discussion on the inverse of an Eulerian shift invariant operator mentioned in (3.4.2). Let F be an invertible Eulerian shift invariant operator with fundamental sequence $\{f_n\}$, and $f(s) = \sum_{k=0}^{\infty} \frac{a_k s^{(k)}}{k!}$ its corresponding formal Newton series (under the isomorphism). Clearly, we have

$$Fx^n = f_n x^n \ ;\ f_n \neq 0 \quad \forall_n \geq 0$$

and

$$F^{-1}x^n = \frac{1}{f_n} x^n \qquad \text{(Corollary (3.4.2))} .$$

F^{-1} which again is Eulerian shift invariant can therefore be expanded as

$$F^{-1} = \sum_{k=0}^{\infty} b_k \frac{x^k D^k}{k!}$$

where

$$\begin{aligned} b_k &= [F^{-1}(x-1)^k]_{x=1} \\ &= \left[F^{-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} x^j\right]_{x=1} \\ &= \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{f_j} x^j\right]_{x=1} \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{f_j} . \end{aligned}$$

But $f_j = f(j) = \sum_{n=0}^{j} \frac{a_n}{n!} j^{(n)}$ (see Corollary (6.4.3)).

Therefore, $$F^{-1} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{\sum_{n=0}^{j} \frac{a_n}{n!} j^{(n)}} \frac{x^k D^k}{k!}$$

We have thus expressed the coefficients of the expansion of the inverse operator F^{-1} , i.e. b_k's , in terms of the a_k's in the expansion of F .

(8.5) As a partial solution to the connecting coefficients problem mentioned in Chapter I, we give the following theorem.

(8.5.1) Theorem: *Given Eulerian triples* $(P, \pi_n, p_n(x))$ *and* $(Q, g_n, q_n(x))$, *then*

$$p_n(a) = \sum_{k=0}^{n} \frac{[Q^k p_n(x)]_{x=1}}{k!} q_k(a) \qquad a \in \mathbb{R} .$$

Proof: Given any real number a , the Eulerian shift operator A_a can be written, according to the expansion theorem, as

$$A_a = \sum_{k=0}^{\infty} a_k \frac{x^k Q^k}{k!}$$

where $$a_k = [A_a q_k(x)]_{x=1} = [q_k(ax)]_{x=1} = q_k(a)$$

i.e. $$A_a = \sum_{k=0}^{\infty} \frac{q_k(a)}{k!} x^k Q^k \quad \text{and equivalently,}$$

(8.5.1a) $$A_a = \sum_{n=0}^{\infty} \frac{p_n(a)}{n!} x^n P^n .$$

Since $x^k Q^k$ is also Eulerian shift invariant (Lemma (4.3.1)), it can be expanded again, in terms of P , such that

$$A_a = \sum_{k=0}^{\infty} \frac{q_k(a)}{k!} \left(\sum_{n=0}^{\infty} \frac{[x^k Q^k p_n(x)]_{x=1}}{n!} \; x^n P^n \right) , \quad \text{i.e.}$$

(8.5.1b) $$A_a = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{[Q^k p_n(x)]_{x=1}}{k!} q_k(a) \right) \frac{x^n P^n}{n!} .$$ Note that $[x^k Q_k]_{x=1} = [Q^k]_{x=1}$.

Equating (8.5.1a) and (8.5.1b), we conclude, by the unique representation theorem, that

$$p_n(a) = \sum_{k=0}^{n} \frac{[Q^k p_n(x)]_{x=1}}{k!} q_k(a) \qquad \text{Q.E.D.}$$

We shall use our model sequence $\{(x - 1)^n\}$, substituting $\{p_n(x)\}$ or/and $\{q_n(x)\}$ in the above formula, to obtain some specific results.

(8.5.2) (i) When $q_n(a) = p_n(a) = (a - 1)^n$,

then $$(a - 1)^n = \sum_{k=0}^{n} \frac{[D^k (x - 1)^n]_{x=1}}{k!} (a - 1)^k$$

$$= \sum_{k=0}^{n} \left[\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} D^k x^j \right]_{x=1} \frac{(a - 1)^k}{k!}$$

$$= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} (-1)^{n-j} \binom{j}{k} (a - 1)^k$$

from which we deduce the well known binomial identity:

$$\sum_{j=k}^{n} \binom{n}{j}\binom{j}{k} (-1)^{n-j} = \begin{cases} 1 & k = n \\ 0 & \text{otherwise.} \end{cases}$$

(ii) When $q_k(a) = (a - 1)^k$,

then $$p_n(a) = \sum_{k=0}^{n} \frac{[D^k p_n(x)]_{x=1}}{k!} (a - 1)^k .$$

Since any polynomial $p(x)$ is a linear combination of the Eulerian basic set $\{p_n(x)\}$, the above equation will also hold for all polynomials $p(x)$, that is,

$$p(x) = \sum_{k=0} \frac{[D^k p(x)]_{x=1}}{k!} (x - 1)^k$$

which is precisely the Taylor series of $p(x)$ expanded at $x = 1$.

APPENDIX

Ihrig and Ismail [9] introduced a q-umbral calculus by introducing $*$, known as the star product, on $K[x]$, the vector space of all polynomials over a field K of characteristic zero. $*$ is defined by the following:

Definition: [9; definition 2.1] $_{*}K[x]$ *will denote the algebra of polynomials equipped with the usual addition, the usual multiplication by scalars and by the star product* "*" *defined on the given set of polynomials* $\{p_n(x)\}$ *by*

$$p_n * p_m = p_{n+m} .$$

They pointed out that from a given polynomial $p(x) = \sum_j c_j x^j$, one can define the associated polynomial $p^*(x)$ via a $*$ multiplication such that

$$p^*(x) = \sum_j c_j x^{j*}$$

where q^{n*} means $\underbrace{q * q * q * \dots * q}_{n \text{ times}}$ for all $q \in \mathbf{P}$.

From this they claimed that almost any sequence of polynomials may be considered the same as any other sequence as long as one is allowed to alter the multiplication. This is more precisely stated as follows:

(9.1.1) Theorem: (The Umbral Lemma) [9: theorem 3.1] *Let* $\{p_n(x)\}$ *and* $\{b_n(x)\}$ *be two sequences of polynomials. Then* (a) *and* (b) *are equivalent.*

(a) *There is a unique multiplication* $*$ *on* $K[x]$ *such that*

(i) $p_n(x) = b_n^*(x)$; $n = 0, 1, 2, \ldots$

(ii) *There exists a mapping* $S: (K[x], \cdot) \to (K[x], *)$ *such that* S *is an isomorphism and*

$$S(x) = x \ , \ S(1) = 1$$

(b) $p_0(x) = b_0(x)$ *and* $p_1(x) = b_1(x)$.

As a result of the lemma, they illustrated that one may treat any polynomial as a monomial by proving the following theorem:

(9.1.2) Theorem: [9: theorem 3.2] *Let* $\{p_n(x)\}$ *be any polynomial set. Then there exists a product* $*$ *such that*

$$p_n(x) = (c(x - a))^{n*}$$

where

$$p_1(x) = c(x - a) \ .$$

Theorem (9.1.2) provides the foundation for the concept of formally manipulating certain families of polynomials $\{p_n(x)\}$ as if $p_n(x) = x^n$ and hence less obvious results of $\{p_n(x)\}$ can be deduced. Thus, the main object is to develop formulas for expressing an Eulerian family of polynomials (throughout this section, definition (1.4.1) is used for the Eulerian family of polynomials) in terms of monomials and vice versa. To do that, they used Tensor product to define the product of two functionals L and M by

$$\langle LM | x^n \rangle = \langle L | x^n \rangle \langle M | x^n \rangle \ .$$

It turns out that:

(9.1.3) Theorem: [9: theorem 4.1] *A polynomial set* $\{p_n(x)\}$ *is an Eulerian family iff*

$$\langle LM|p_n(x)\rangle = \sum_{0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle L|p_k(x)\rangle\langle M|x^k p_{n-k}(x)\rangle .$$

For a given Eulerian family $\{p_n(x)\}$ such that

$$p_n(x) = \sum_{m=0}^{n} c_{n,m} x^m \; ; \; n = 0, 1, 2, \ldots ,$$

then d_{nm} in

$$x^n = \sum_{m=0}^{n} d_{n,m} p_m(x) \; ; \; n = 0, 1, 2, \ldots$$

is given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q c_{n-m,n-m}/c_{nn} \qquad \text{(see [9]; Theorem 4.2, 4.3).}$$

The inverse problem, namely expressing the p's in terms of the monomials, requires the generating function techniques mentioned in Andrews' paper [3]:

(9.1.4) Theorem: ([3]; corollary of theorem 6) *If* $p_n(x)$ *is an Eulerian family of polynomials, and if* c_n *is the leading coefficient of* $p_n(x)$, *then*

(9.1.4a) $$\sum_{n\geq 0} \frac{p_n(x)t^n}{(q)_n} = \frac{f(xt)}{f(t)} ,$$

where $$f(t) = \sum_{n\geq 0} \frac{c_n t^n}{(q)_n} .$$

From (9.1.4a), it is not difficult to see that

(9.1.4b)
$$x^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \theta_m(x)$$

where

$$\theta_n(x) = \begin{cases} 1 & n = 0 \\ (x-1)(x-q)\dots(x-q^{n-1}) & n > 0 \end{cases}$$

is an Eulerian family. The inverse relation to (9.1.4b) is given by:

(9.1.5) <u>Theorem</u>: (Gauss' Binomial Theorem) [9; theorem 4.5] *The polynomials* $\theta_n(x)$ *are given explicitly by*

$$\theta_n(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^{n-m} q^{(n-m)(n-m-1)/2} x^m .$$

The object of this appendix is to provide a brief reference to Ihrig and Ismail's work on Eulerian family of polynomials. Some of their important results have not been mentioned. Interested readers should refer to the original publication.

REFERENCES

[1] Wm. R. Allaway, Extensions of Sheffer polynomial sets, SIAM, J. Math. Anal. Vol. 10, No. 1, January 1979, pp. 38-48.

[2] Wm. R. Allaway and K. W. Yuen, Ring Isomorphism for the family of Eulerian Differential Operators, J. of Math. Analysis & Apps., Vol. 77, No. 1, Sept. 80.

[3] G. E. Andrews, "On the foundations of combinatorial theory V, Eulerian Differential Operators:, Studies in Applied Mathematics, Vo. L, No. 4 December 1971, Massachusetts Institute of Technology, Cambridge, Massachusetts.

[4] G. E. Andrews, "Identities in combinatorics, I: On sorting two ordered sets", Discrete Mathematics 11 (1975), pp. 97-106.

[5] E. T. Bell, Postulational bases for the umbral calculus, Amer. J. Math. 62 (1940), pp. 717-724.

[6] A. M. Garsia, An Expose of Mullin-Rota Theory of Polynomials of Binomial Type, Linear and Multilinear Algebra, 1973, Vo. 1, pp. 47-65.

[7] J. Goldman and G.-C. Rota, "On the foundations of combinatorial theory IV, finite vector spaces and Eulerian generating functions", Studies in Applied Mathematics, 49 (1970), pp. 239-258.

[8] H. W. Gould, Combinatorial Identities, Morgantown, WV, 1922 (privately printed).

[9] E. C. Ihrig and M. E. H. Ismail, "A q-Umbral calculus", Technical report No. 49, Arizona State University.

[10] M. E. H. Ismail, Classification of Polynomial Sets, M.Sc. Thesis, University of Alberta, Edmonton, Alberta.

[11] R. Mullin and G.-C. Rota, "On the foundations of combinatorial theory, III theory of binomial enumeration", Graph Theory and Its Applications edited by B. Harris, Academic Press, New York (1970), pp. 167-213.

[12] E. D. Rainville, Special Functions, Macmillan, New York, 1960.

[13] J. Riordan, "Combinatorial Identities", John Wiley & Sons, Inc. New York, 1968.

[14] S. M. Roman and G.-C. Rota, The Umbral Calculus, Advances in Mathematics, Vol. 27, No. 2 February 1978, pp. 95-188.

[15] G.-C. Rota, D. Kahaner, and A. Odlyzko, "On the foundations of combinatorial theory VIII, finite operator calculus", Journal of Mathematical Analysis and Applications, Vol. 42, No. 2, June 1973, pp. 684-760.

[16] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[17] M. Ward, A certain class of polynomials, Ann. Math., Vol. 31 (1930), pp. 43-51.