INTERVAL-BASED UNCERTAIN REASONING

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Abstract

This thesis examines three interval based uncertain reasoning approaches: reasoning under interval constraints, reasoning using necessity and possibility functions, and reasoning with rough set theory. In all these approaches, intervals are used to characterize the uncertainty involved in a reasoning process when the available information is insufficient for single-valued truth evaluation functions. Approaches using interval constraints can be applied to both interval fuzzy sets and interval probabilities. The notion of interval triangular norms, or interval t-norms for short, is introduced and studied in both numeric and non-numeric settings. Algorithms for computing interval t-norms are proposed. Basic issues on the use of t-norms for approximate reasoning with interval fuzzy sets are studied. Inference rules for reasoning under interval constraints are investigated. In the second approach, a pair of necessity and possibility functions is used to bound the fuzzy truth values of propositions. Inference in this case is to narrow the gap between the pair of the functions. Inference rules are derived from the properties of necessity and possibility functions. The theory of rough sets is used to approximate truth values of propositions and to explore modal structures in many-valued logic. It offers an uncertain reasoning method complementary to the other two.

Chapter 1

INTRODUCTION

Uncertain reasoning seems to be a characteristic of human thinking and is therefore an essential subject of artificial intelligence. In many situations, uncertainty is inevitable due to a lack of knowledge, or incompleteness and unreliability of the available information. There are two fundamental issues involved in uncertain reasoning, representation of uncertain information and inference with such information.

Traditionally, single-valued measures, such as probability functions, are used to handle uncertain information where the uncertainty of a proposition is expressed with a single number. These approaches have a number of practical problems [32]. It may be unrealistic to expect an expert to provide precise and reliable probability functions. The maintenance of consistency using single-valued measures may be a difficult task. To resolve these problems, various proposals have been suggested using the notion of numeric and non-numeric intervals [2, 3, 8, 23, 30, 46, 44, 47]. The results of these studies have resulted in many interval-based tools for uncertain reasoning, such as incidence calculus [5, 44], belief and plausibility functions [35], necessity and possibility functions [14], interval fuzzy sets [31], interval probabilities [32, 44], and rough sets [29].

In interval-based approaches, the uncertainty of a proposition is not represented by a single number but by an interval. Instead of providing the exact value, one gives a range within which lies the actual value. Based on the interpretations of such an interval representation of uncertainty, the main objective of this thesis is to analyze three different interval-based uncertain reasoning approaches. They are uncertain reasoning using interval constraints, necessity and possibility functions, and rough sets, respectively.

In reasoning under interval constraints, an interval is interpreted as confining a family of uncertainty measures. That is, the degree of uncertainty, or the truth, of a proposition is bounded by intervals. An inference process is formulated as constraint propagation [30]. The main tasks are to derives, from the given interval constraints on certain propositions, the intervals for propositions whose truth values are not provided, and to tighten the initial intervals. An advantage of this approach is that no *ad hac* assumption is introduced. This view of interval-based inference provides a unified framework for reasoning with several types of uncertainties, such as interval fuzzy sets and interval probabilities.

A fuzzy set is defined in terms of a function from a universe to the unit interval $[0,1]$. That is, the membership of each element belonging to a fuzzy set is a single value between 0 and 1. The intersection and union of fuzzy sets are defined in terms of max-min system, probabilistic-like system, and more generally triangular norms and conorms (t-norms and t-conorms for short). Such a single-value-based system is commonly known as the type-1 fuzzy set system. In practical applications, there is also a need to represent the membership of an element by using a fuzzy set in $[0,1]$, instead of a single value. This system is known as the type-2 fuzzy set systems. Operations on type-2 fuzzy sets are defined by extending the operations on the type-1 fuzzy sets. Studies on operations of type-2 fuzzy sets have been concentrated mainly on the max-min system [9, 25]. In addition, inference with type-2 fuzzy sets is computationally expensive. In order to overcome this difficulty, special cases of type-2 fuzzy sets have been considered [14, 31]. For example, membership function of fuzzy sets may be restricted to fuzzy intervals of $[0, 1]$. If ordinary subintervals of $[0, 1]$ are used to represent membership, one obtains the interval fuzzy sets commonly known as the Φ-fuzzy sets [31].

Although it is important to study type-2 fuzzy sets in general case based on the calculus of fuzzy quantities [14], it is equally important to study some special cases. The particular characteristics of each special case may offer more efficient algorithms and more insights that may not be obtainable in the general case. For example, one can derive closed-form solutions of fuzzy set operations for fuzzy intervals [4, 14]. Kenevan and Neapolitan studied interval fuzzy sets based on the usual max-min system [20]. Dubois and Prade [14], and Goodman *et al.* [18] studied the same problem of extending max-min system to interval fuzzy sets, in connection to Lukasiewicz many-valued logic and interval analysis. Turksen discussed the notion of interval-valued fuzzy sets constructed from the disjunctive and conjunctive normal forms, DNF and CNF, in which certain types of t-norms can be used [38]. Operations on interval-valued fuzzy sets are defined by considering all possible combinations of DNF and CNF [39]. On the other hand, in Φ-fuzzy sets, an interval is merely regarded as the range within which lies the true membership [20, 31]. The computation of fuzzy set operations may be simplified. Dubois and Prade introduced the notion of twofold fuzzy sets, which is a special kind of Φ-fuzzy sets such that the lower bound of a twofold fuzzy set is included in the core of the upper bound [13]. More specifically, the lower and upper bounds are interpreted as bounds of necessity and possibility. Consequently, the max-min system is used to define operations on twofold fuzzy sets. Bonissone proposed an approximate reasoning model with interval representation of uncertainty, in which four operations are defined using t-norms [2, 3, 4].

In order to perform interval fuzzy set operation, the notion of t-norms is extended to interval t-norms. Interval t-norms are defined as two-place functions on the closed subintervals of $[0, 1]$ by drawing results from interval computation. Using interval t-norms, operations on interval fuzzy sets can be efficiently computed, i.e., by computing only values of two extreme points of intervals. Inference with interval fuzzy set is based on the computation of interval-valued membership function. Intersection and union of two interval fuzzy sets are obtained by interval t-norms and t-conorms of their membership intervals. A set of inference rules is presented based on interval t-norms. Inference using interval L-fuzzy sets is also considered.

The use of interval probabilities is another example of reasoning under interval constraints. In practice, the exact probability of a proposition may not be available, and an interval may be adopted. Quinlan proposed a set of inference axioms to tighten the bound of the probabilities of propositions [32]. The inference axioms used by Quinlan do not necessarily produce the tightest probability bounds. By refining these axioms, it is possible to infer tighter probability bounds [44].

In the approach with necessity and possibility functions, the truth value of a proposition is bounded by a pair of necessity and possibility functions. The necessity function provides sure threshold for the truth value while the possibility gives the maximal possible point. A practical problem with this approach is that an expert may have difficulties in supplying precise and consistent necessity and possibility functions. That is, the bound provided by experts may not necessarily be necessity and possibility functions. In this case, inference may be formulated by updating the existing bounds so that they will be as close as possible to a pair of necessity and possibility functions.

The theory of rough sets offers another interval-based method. The interval approximations stem from a lack of sufficient information or incomplete information. A set is assumed to be precisely defined. However, the available information, given in terms of equivalence classes, does not allow us to describe the set exactly. In other words, it may be impossible to describe a precisely defined set with equivalence classes. In this case, a pair of lower and upper approximations is obtained. The lower approximation contains all elements necessarily belonging to the set, while the upper approximation contains all elements possibly belonging to the set. The theory of rough sets is closely related to modal logic. Inference with rough sets can be done in a similar manner as in modal logic [45]. More specifically, the theory of rough sets is extended in this thesis to investigate the approximation of one many-valued logic by another many-valued logic with fewer truth values.

The rest of the thesis is organized as follows. In Chapter 2, basic notions, such as lattice, fuzzy sets, rough sets, uncertainty measures, and logics, are introduced. In Chapter 3, a framework of interval computations is presented. It is applied to the study of interval t-norms. In Chapter 4, three interval-based uncertain reasoning approaches are examined. Finally, a summary of the thesis and future work are presented in Chapter 5.

Chapter 2

BASIC NOTIONS

In this chapter, some elementary concepts involved in uncertain reasoning are briefly reviewed. A summary of various uncertainty measures is presented.

2.1 Lattice

A binary relation $\mathcal{R}$ in a set L is called a *partial order relation* on L if $\mathcal{R}$ is reflexive, antisymmetric, and transitive. The ordering is denoted by $\leq$ and the pair $(L, \leq)$ is called a *partially ordered set* or *poset.*

Let $(L, \leq)$ be a poset and let $A \subseteq L$. An element $x \in L$ is an upper bound for A if for all $a \in A$, $a \leq x$. An element $x \in L$ is a least upper bound for A if x is an upper bound for A and $x \leq y$ for any upper bound y for A. Similarly, any element $x \in L$ is a lower bound for A if for all $a \in A$, $x \leq a$. An element $x \in L$ is a greatest lower bound for A if x is a lower bound for A and $y \leq x$ for any lower bound y for A. In a poset $(L, \leq)$, for $x, y \in L$, if their *greatest lower bound* exists, it is called the *meet* of x and y and denoted by $x \otimes y$; if their *least upper bound* exists, it is called the *join* of x and y and denoted by $x \oplus y$.

Definition 2.1 *If $(L, \leq)$ is a partially ordered set such that $a, b \in L$ implies there exists $a \otimes b$ in L, $(L, \leq)$ is a called a meet semi-lattice (lower semi-lattice). If there* *exists $a \oplus b$ for every pair $a, b \in L$, L is called a join semi-lattice (upper semi-lattice). If there exist both $a \otimes b$ and $a \oplus b \in L$, L is called a lattice.*

In the following discussion, it is assumed that there exist a universal least element and a universal greatest element, represented by 0 and 1, respectively. In this case, a lattice may also be denoted by $(L, \oplus, \otimes, 0, 1)$. Any lattice possesses the following properties: for any $a, b, c \in L$,

L1. Commutative laws

$$a \otimes b = b \otimes a, \qquad a \oplus b = b \oplus a;$$

L2. Associative laws

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c, \qquad a \oplus (b \oplus c) = (a \oplus b) \oplus c;$$

L3. Idempotent laws

$$a \otimes a = a \oplus a = a;$$

L4. Absorption laws

$$a \oplus (a \otimes b) = a \otimes (a \oplus b).$$

For an element $x \in L$, if there exists an element x' satisfying $x \oplus x' = 1$ and $x \otimes x' = 0$, x' is called a *complement* (denoted by $\ominus$) of x. A lattice is called *complemented* if every $x \in L$ has a complement. A lattice L is called *distributive* if, for $a, b, c \in L$, it satisfies the distributive law:

L5. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c), \qquad a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c).$

A complemented and distributive lattice is called a Boolean algebra, denoted by $(L, \oplus, \otimes, \ominus, 0, 1)$.

2.2 Fuzzy Sets

The notion of fuzzy sets was introduced by Zadeh [48]. It is a natural extension of classical set when membership is no longer all-or-nothing. Let U be a classical set called the *universe*. A fuzzy set A on U is defined by a membership function, which is a mapping from U to the unit interval, namely,

$$\mu_A : U \longrightarrow [0,1]. \tag{2.1}$$

Fuzzy set inclusion ($\subseteq$) is defined point-wise as:

$$A \subseteq B \;\text{ iff }\quad \text{for any } x \in U, \mu_A(x) \leq \mu_B(x). \tag{2.2}$$

Fuzzy set union ($\cup$), intersection ($\cap$), and complement ($\sim$) may be defined by the max-min system proposed by Zadeh:

$$\begin{aligned}
\mu_{A\cup B}(x) &= \max(\mu_A(x), \mu_B(x)), \\
\mu_{A\cap B}(x) &= \min(\mu_A(x), \mu_B(x)), \\
\mu_{\sim A}(x) &= 1 - \mu_A(x).
\end{aligned}$$

Many other proposals have been made for fuzzy set intersection and union. An important class of such operations can be formulated by t-norms and t-conorms, including the max-min and product operations.

The concept of fuzzy sets can be extended to L-fuzzy sets if the special lattice $([0,1], \max, \min, 0, 1)$ is replaced by an arbitrary lattice $(L, \oplus, \otimes, 0, 1)$. The intersection and union in L-fuzzy sets are given by:

$$\begin{aligned}
\mu_{A\cup B}(x) &= \mu_A(x) \oplus \mu_B(x), \\
\mu_{A\cap B}(x) &= \mu_A(x) \otimes \mu_B(x).
\end{aligned}$$

These operations can also be defined by lattice t-norms and t-conorms.

A concept closely related to L-fuzzy sets is m-flou set, which was first introduced by Gentilhomme [16]. An m-flou set can be denoted by an m-tuple $A = (E_1, \ldots, E_m)$ of ordinary subsets of U such that:

$$E_1 \subseteq \ldots \subseteq E_m. \tag{2.3}$$

Operations on m-flou sets are defined as follows: for $A = (E_1, \ldots, E_m)$ and $B = (F_1, \ldots, F_m)$,

Union	$A \cup B = (E_1 \cup F_1, \ldots, E_m \cup F_m)$
Intersection	$A \cap B = (E_1 \cap F_1, \ldots, E_m \cap F_m)$
Complement	$\overline{A} = (\overline{E_1}, \ldots, \overline{E_m})$
Inclusion	$A \subseteq B$ iff $E_i \subseteq F_i, \quad i = 1, \ldots, m.$

An m-flou set is a particular case of L-fuzzy set where L is a finite linearly ordered set of m elements. There is a structural isomorphism between them [10].

When the membership of a fuzzy set itself is a fuzzy set, it is called type-2 fuzzy set. Following the same argument, one can define type-m fuzzy sets recursively from type-$(m-1)$ fuzzy sets. A special type-2 fuzzy sets is Φ-fuzzy sets [34], or called interval valued fuzzy sets, whose memberships are closed subintervals of $[0, 1]$.

2.3 Rough Sets

Let U denote a finite and non-empty set called the universe, and let $\mathcal{R} \subseteq U \times U$ denote an equivalence relation on U. The pair $apr = (U, \mathcal{R})$ is called an approximation space. The equivalence relation $\mathcal{R}$ partitions the set U into disjoint equivalence classes. Such a partition of the universe is denoted by $U/\mathcal{R}$. If two elements $x, y \in U$ belong to the same equivalence class, x and y are indistinguishable. The equivalence classes of $\mathcal{R}$ and the empty set $\emptyset$ are called the elementary or atomic sets in the approximation space $apr = (U, \mathcal{R})$.

Given an arbitrary set $A \subseteq U$, it may not always be possible to describe A precisely using the equivalence classes of $\mathcal{R}$. In this case, one may characterize A by a pair of lower and upper approximations:

$$\begin{aligned} \underline{apr}(A) &= \bigcup_{[x]_{\mathcal{R}} \subseteq A} [x]_{\mathcal{R}}, \\ \overline{apr}(A) &= \bigcap_{[x]_{\mathcal{R}} \cap A \neq \emptyset} [x]_{\mathcal{R}}, \end{aligned}$$

where

$$[x]_{\mathcal{R}} = \{y \mid x\mathcal{R}y\} \tag{2.4}$$

is the equivalence class containing x. The pair $(\underline{apr}(A), \overline{apr}(A))$ is called the rough set with respect to A. The lower approximation $\underline{apr}(A)$ is the union of all the elementary sets which are subsets of A, and the upper approximation $\overline{apr}(A)$ is the union of all the elementary sets which have a non-empty intersection with A. An element in the lower approximation necessarily belongs to A, while an element in the upper approximation possibly belongs to A.

2.4 Uncertainty Measures

There are many kinds of uncertainty measures such as fuzzy measures, probability functions, belief and plausibility functions, and necessity and possibility functions. Each class is governed by a set of axioms. A brief summary of uncertainty measures, adopted from Yao, Wong, and Wang [47], is presented next.

A frame is a finite set $\Theta = \{\theta_1, \ldots, \theta_n\}$ containing all possible answers to a given *question* [35]. A proposition can be seen as a subset of Θ, and all possible propositions form the power set 2^{Θ}. The uncertainty of a proposition is defined by a measure satisfying certain axioms.

2.4.1 Fuzzy measures

A fuzzy measure f satisfies the following axioms [37]:

$$\text{(F1)} \qquad f(\emptyset) = 0,$$

$$\text{(F2)} \qquad f(\Theta) = 1,$$

$$\text{(F3)} \qquad A \supseteq B \Longrightarrow f(A) \geq f(B).$$

where A and B are subsets of Θ.

2.4.2 Belief and plausibility functions

A belief function f is defined by axioms (F1), (F2) and the superadditivity axiom:

(F4) For every positive integer n and every collection $A_1, \ldots, A_n \subseteq \Theta$,

$$f(A_1 \cup A_2 \ldots \cup A_n) \geq \sum_i f(A_i) - \sum_{i<j} f(A_i \cap A_j)$$
$$\pm \cdots + (-1)^{n+1} f(A_1 \cap A_2 \ldots \cap A_n). \tag{2.5}$$

The dual of a belief function is called plausibility function f':

$$f'(A) = 1 - f(\sim A), \tag{2.6}$$

It can be equivalently defined by axioms (F1), (F2) and the subadditivity axiom:

(F5) For every positive integer n and every collection $A_1, \ldots, A_n \subseteq \Theta$,

$$f'(A_1 \cap A_2 \ldots \cap A_n) \leq \sum_i f'(A_i) - \sum_{i<j} f'(A_i \cup A_j)$$
$$\pm \ldots + (-1)^{n+1} f'(A_1 \cup A_2 \ldots \cup A_n). \tag{2.7}$$

The interval $[f(A), f'(A)]$ indicates the range of the truth value of proposition A.

2.4.3 Probability functions

A probability function f is defined by axioms (F1), (F2) and the additivity axiom:

$$\text{(F6)} \quad A \cap B = \emptyset \Longrightarrow f(A \cup B) = f(A) + f(B).$$

Belief and plausibility functions are closely related to probability functions as explained in the following theorem proved by Dempster [8].

Theorem 2.1 *Let $\underline{f}$ be a belief function on Θ and $\overline{f}$ be the corresponding plausibility function. Then for all $A \subseteq \Theta$,*

$$\begin{aligned} \underline{f}(A) &= \inf_{f \in \mathcal{I}} f(A), \\ \overline{f}(A) &= \sup_{f \in \mathcal{I}} f(A), \end{aligned}$$

where $\mathcal{I}$ is the family of probability functions defined by $\mathcal{I} = \{P \mid \underline{f}(A) \leq P(A) \leq \overline{f}(A)\}$.

From this theorem, it is clear that if $\underline{f}(A) = \overline{f}(A)$ for all $A \subseteq \Theta$, they become probability functions. In other words, probability functions are special cases of belief and plausibility functions.

2.4.4 Necessity and possibility functions

A necessity function f is determined by axioms (F1), (F2) and

$$\text{(F7)} \quad f(A \cap B) = \min\{f(A), f(B)\}. \tag{2.8}$$

The dual f' is a possibility function defined by axioms (F1), (F2) and

$$\text{(F8)} \quad f'(A \cup B) = \max\{f'(A), f'(B)\}. \tag{2.9}$$

They are special belief and plausibility functions commonly known as consonant belief functions.

2.4.5 Relationships between uncertainty measures

In summary, necessity and possibility functions are consonant belief and plausibility functions. Probability functions are both belief and plausibility functions. Belief and plausibility functions are fuzzy measures.

2.5 Logic

The basic items of logic are propositions. Propositions communicate judgements or beliefs. Logic is the study of formal methods of manipulating propositions.

2.5.1 Classical propositional logic

Let Φ be a set of propositional variables. In propositional logic, a sentence based on Φ is derived by elements of Φ connected by logic connectives, such as negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and implication ($\longrightarrow$). Any sentence is considered to be either *true* or *false*, i.e., it is associated with a *truth value*. A propositional language formed from Φ, denoted by $L(\Phi)$, is the smallest set containing the truth values (true and false) and elements of Φ, and being closed under negation, conjunction, disjunction and implication.

An important part of logic is the study of *arguments*. It deals with conclusions that can be said to follow from given premises. In classical propositional logic, the validity of an argument can be decided by *truth table*. An assignment of truth values to propositional variables is called a *valuation*, which can be represented by an *evaluation function* v, mapping from $L(\Phi)$ to true or false. A formula that is true in every valuation is said to be *valid*. Valid formulas in propositional logic are called *tautologies*.

Propositional logic is truth-functional. The truth value of a compound sentence in propositional logic can be calculated solely from the truth values of its constituent sentences. The rules for using the connectives in truth value calculation can be conveniently expressed in the form of truth tables. A truth table can be constructed for any given sentence by listing all the ways in which truth values can be assigned to its constituent propositional variables, and using the truth tables for the connectives, calculating the value of the sentence in each case. A set of sentences G *entails* a sentence ϕ, if every valuation that makes all the sentences in G true also makes ϕ true. If no valuation makes all the sentences in G true then G is said to be *inconsistent*.

A *formal* system enables the validity of arguments in a language to be decided without reference to the notion of true and false. A formal system has alphabets and grammar of the language, as well as axioms and rules of inference. A *proof* in such a system is a sequence of sentences, each of which is either an axiom or is derived from earlier members of the sequence by the rules of inference. The final sentence of such a sequence is said to be a *theorem* of the system. The formal system for classical propositional logic is *sound* and *complete* because all its theorems are tautologies and all tautologies are among its theorems.

2.5.2 Non-classical propositional logic

Based on classical propositional logic, various non-classical logics have been proposed. Non-classical logics play a very important role in the field of artificial intelligence. Examples of non-classical propositional logics are modal logic, many-valued logic, fuzzy logic, probabilistic logic, and possibilistic logic.

- Modal logic

Modal propositional logic is an extension of classical propositional logic that deals with necessity and possibility. In addition to the standard logic connectives, negation, conjunction, disjunction, and implication, modal logic introduces two additional unary connectives, the *necessity* (denoted by $\Box$) and its dual $\neg\Box\neg$, the possibility (denoted by $\Diamond$).

A modal logic system $L(\Phi)$ contains all the theorems of classical propositional logic and all instances of the schema of distribution, i.e., for ϕ and ψ in $L(\Phi)$,

$$\Box(\phi \longrightarrow \psi) \longrightarrow (\Box\phi \longrightarrow \Box\psi). \tag{2.10}$$

It is complete for the inference rule of modus ponens and necessitation [36].

- **Many-valued and fuzzy logics**

Many-valued logic extends the valuation set from $\{false, true\}$ to a set of more than two elements. The simplest many-valued logic is the three-valued logic, where the valuation set is $\{false, uncertain, true\}$ or $\{0, u, 1\}$. Following is the truth table for a three-valued logic proposed by Lukasiewicz: for $\phi, \psi \in L(\Phi)$,

ϕ	$\neg\phi$
1	0
u	u
0	1

	$\phi \wedge \psi$			$\phi \vee \psi$			$\phi \longrightarrow \psi$		
ψ / ϕ	1	u	0	1	u	0	1	u	0
1	1	u	0	1	1	1	1	u	0
u	u	u	0	1	u	u	1	1	u
0	0	0	0	1	u	0	1	1	1

The set $\{0, u, 1\}$ equipped with an order relation $0 \leq u \leq 1$ is a lattice. The conjunction and disjunction in Lukasiewicz logic can in fact be characterized by the meet and join of lattice operators. Such a three-valued logic can be easily extended to a m-valued logic which uses the lattice $\{0, \frac{1}{m-1}, \ldots, \frac{m-2}{m-1}, 1\}$ with the usual $\leq$ relation. The conjunction and disjunction are defined by min and max. If the unit interval $[0, 1]$ is used, one obtains the fuzzy logic system, commonly known as max-min system, proposed by Zadeh [48].

- **Probabilistic and possibilistic logics**

When the evaluation function becomes a probability function $P : L(\Phi) \longrightarrow [0,1]$, the logic is called probabilistic logic. The logic calculation is governed by the laws of probability. The truth value of conjunction and disjunction of two propositions, being probabilities, cannot be computed from the truth values of the two propositions involved. For example, given the probabilities of two propositions, $P(\phi)$ and $P(\psi)$, in general one cannot get the probabilities $P(\phi \wedge \psi)$ and $P(\phi \vee \psi)$. Similar to probabilistic logic, possibilistic logic can be obtained when the evaluation function becomes a possibility function. Possibilistic logic is truth functional with respect to disjunction, but is not truth functional with respect to conjunction. The property of non-truth-functional of probabilistic and possibilistic logics makes them a very important class of non-classical many-valued logics.

Chapter 3

INTERVAL COMPUTATIONS AND T-NORMS

This chapter examines some basic concepts of interval computations and their applications in defining interval t-norms. Interval computations provide a mathematical basis for interval-based reasoning. The special class of interval computations based on interval t-norms plays a key role in reasoning with interval fuzzy sets.

3.1 Interval Computations

Let A and B be two subsets of universe U. A function f on U can be extended to a function F on 2^U by the following extension rule :

$$F(A, B) = \{f(x, y) \mid x \in A, y \in B\}. \tag{3.11}$$

The extension rule forms the basis of set-based computations [46]. This framework may be set in a wider context of universal algebra, in which the arguments and value of a function are not single points but subsets [7]. In the study of power structures, Brink lifted any function f over a set A to an extended function F over subsets of A by employing the same extension rule [1]. The algebraic properties of the extended function F can be studied based on the properties of f.

A subclass of set-based computations are interval computations. In this special case, one considers intervals of U that are defined by an order relation on U. More specifically, function f is extended to F such that both domain and range of F are intervals in U [28]. Interval computations are applicable to both numeric and non-numeric cases.

3.1.1 Numeric interval computations

Numbers have been widely adopted because of their richness of arithmetic properties and operations. Some of these properties and operations can be naturally extended to interval numbers. An *interval number* $[\underline{a}, \overline{a}]$ with $\underline{a} \leq \overline{a}$ is a closed set on real numbers $\Re$:

$$[\underline{a}, \overline{a}] = \{x \mid \underline{a} \leq x \leq \overline{a}\}. \tag{3.12}$$

The set of all interval numbers is denoted by $I(\Re)$. Degenerate intervals of the form $[a, a]$ are equivalent to real numbers. The notion of interval numbers provides a tool for representing a real number by specifying its lower and upper endpoints. This representation scheme is very useful in situations where a precise measurement of a physical quantity is impossible (i.e., inexact experimental measurements), or where a real number can not be stored with sufficient precision in a computer due to space limitation (i.e., insufficient representation). Arithmetic operations can be performed with interval numbers through the arithmetic operations on their members [26, 27]. Let A and B be two interval numbers, and let $*$ denote an arithmetic operation $+$, $-$, $\cdot$ or $/$ on pairs of real numbers. An arithmetic operation $*$ may be extended to pairs of interval numbers A, B:

$$A * B = \{x * y \mid x \in A, y \in B\}. \tag{3.13}$$

The result $A * B$ is again a closed and bounded interval unless $0 \in B$ and the operation $*$ is division (in which case, $A * B$ is undefined). In fact, the following formulas can be used: for $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$:

$$\begin{aligned}
A + B &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
A - B &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
A \cdot B &= [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})], \\
A \,/\, B &= [\underline{a}, \overline{a}] \cdot [1/\overline{b}, 1/\underline{b}] \text{ for } 0 \notin [\underline{b}, \overline{b}].
\end{aligned} \tag{3.14}$$

In the special case where both A and B are positive intervals, the multiplication can be simplified to:

$$A \cdot B = [\underline{a}\underline{b}, \overline{a}\overline{b}], \qquad 0 \leq \underline{a} \leq \overline{a}, \;\; 0 \leq \underline{b} \leq \overline{b}. \tag{3.15}$$

Many properties of the arithmetic operations on pairs of real numbers can be carried over to the new arithmetic operations on pairs of interval numbers. For example, the addition operation $+$ on interval numbers is also associative and commutative.

The arithmetic of interval numbers can be easily extended to any function. Let f be a function from $\Re \times \Re$ to $\Re$. Using the extension rule (3.11), the corresponding function F of interval numbers can be obtained. Operations such as addition, subtraction, multiplication, and division are only special cases. However, in general, there is no guarantee that the extended function F is an interval-valued function. The following theorem provides a sufficient condition for F to be interval-valued.

Theorem 3.1 *Suppose f is a continuous function from $\Re \times \Re$ to $\Re$. Given any pair of closed and bounded intervals A and B in $\Re$, $F(A, B)$ is a closed interval, namely,*

$$F(A, B) = [\inf_{x \in A, y \in B} f(x, y), \sup_{x \in A, y \in B} f(x, y)]. \tag{3.16}$$

Proof. By definition, for all $x \in A$ and $y \in B$, $f(x, y) \in F(A, B)$. In order to prove the sufficiency of the theorem, it is necessary to show that for any number m in $F(A, B)$, there exist two numbers, say $h_1 \in A$ and $h_2 \in B$, such that $f(h_1, h_2) = m$.

Suppose function f attains its least value in $A \times B$ at $x = a$ and $y = b$, and its greatest at $x = c$ and $y = d$. The value m must be within $f(a, b)$ and $f(c, d)$, i.e.,

$$f(a, b) \leq m \leq f(c, d).$$

Apparently, $f(c, b)$ lies between $f(a, b)$ and $f(c, d)$. Furthermore, $f(c, b)$ is either greater than or equal to m or less than m. Assume $f(c, b) \leq m$, it follows:

$$f(a, b) \leq f(c, b) \leq m \leq f(c, d).$$

Function f can be regarded as a one place function when its first variable is fixed in point c. Recall that f is a continuous function. According to Bolzano's intermediate value theorem, $f(c, b) \leq m \leq f(c, d)$ implies there must exist a point h such that $h \in [b, d]$ and $f(c, h) = m$.

The case of $f(c, b) > m$ can be similarly proven. Combining both cases, one can conclude that for an arbitrary number m in $F(A, B)$, there must exist two numbers $h_1 \in A$ and $h_2 \in B$ such that $f(h_1, h_2) = m$. □

Theorem 3.1 suggests that the extended interval-valued function corresponding to a continuous function can be easily computed. It is sufficient to find only the maximum and minimum values. If it is further assumed that the function is isotonic, the computation reduces to only endpoints of intervals as shown in the following corollary [14].

Corollary 3.1 *Suppose f is a continuous isotonic function from $\Re \times \Re$ to $\Re$, that is, for all $x, x', y, y' \in \Re$,*

$$(x \leq x', y \leq y') \Longrightarrow f(x, y) \leq f(x', y'). \tag{3.17}$$

Then,

$$F(A, B) = [f(\underline{a}, \underline{b}), f(\overline{a}, \overline{b})]. \tag{3.18}$$

This corollary trivially follows from the fact that a continuous isotonic function reaches its minimum and maximum values at ending points of an interval.

3.1.2 Non-numeric interval computations

The argument used in numeric interval computations may be applied to non-numeric cases [46]. Let L be a lattice. A closed interval $A = [\underline{a}, \overline{a}]$ of L, with $\underline{a} \leq \overline{a}$, is the set of elements bounded by $\underline{a}$ and $\overline{a}$. That is,

$$A = [\underline{a}, \overline{a}] = \{x \in L \mid \underline{a} \preceq x \leq \overline{a}\}. \tag{3.19}$$

Let $I(L)$ denote the set of all intervals formed from L. Operations $\otimes$ and $\ominus$ may be extended to elements of $I(L)$ as follows:

$$\begin{aligned} [\underline{a}, \overline{a}] \otimes [\underline{b}, \overline{b}] &= \{x \otimes y \mid x \in [\underline{a}, \overline{a}], y \in [\underline{b}, \overline{b}]\}, \\ [\underline{a}, \overline{a}] \ominus [\underline{b}, \overline{b}] &= \{x \ominus y \mid x \in [\underline{a}, \overline{a}], y \in [\underline{b}, \overline{b}]\}. \end{aligned} \tag{3.20}$$

For simplicity, the same set of symbols has been used for operations on both L and $I(L)$. In general, these operations may not be closed on $I(L)$.

Example 3.1 Interval computations in a non-distributive lattice. Consider a non-distributive lattice given in Figure 3.1. For two intervals $[a, 1]$ and $[b, 1]$,

$$[a, 1] \otimes [b, 1] = \{0, a, b, 1\},$$

which is not an interval. Similarly, for two intervals $[0, a]$ and $[0, c]$,

$$[0, a] \ominus [0, c] = \{0, a, c, 1\},$$

is also not an interval. □

Operations $\otimes$ and $\ominus$ on L have isotonicity properties similar to equation (3.17), namely,

$$\begin{aligned} (a \leq a', b \leq b') &\Longrightarrow a \otimes b \leq a' \otimes b', \\ (a \leq a', b \leq b') &\Longrightarrow a \ominus b \leq a' \ominus b'. \end{aligned} \tag{3.21}$$

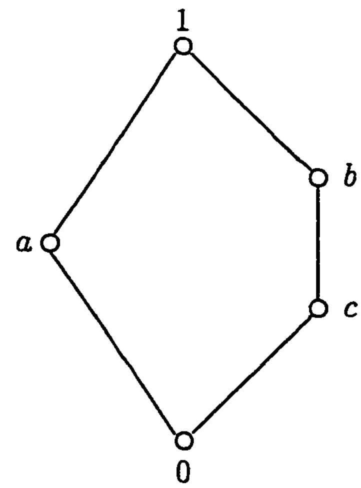

Figure 3.1: A non-distributive lattice

It is expected that a simple computation method can be used if extended operations are closed on $I(L)$. As shown by the following theorem, a sufficient condition for these operations to be closed is that the lattice L is distributive. In addition, the extended operations can be easily computed by considering only ending points of intervals.

Theorem 3.2 *Suppose L is a distributive lattice. Then,*

$$\begin{aligned} [\underline{a}, \overline{a}] \otimes [\underline{b}, \overline{b}] &= [\underline{a} \otimes \underline{b}, \overline{a} \otimes \overline{b}], \\ [\underline{a}, \overline{a}] \oplus [\underline{b}, \overline{b}] &= [\underline{a} \oplus \underline{b}, \overline{a} \oplus \overline{b}]. \end{aligned} \tag{3.22}$$

Moreover, $I(L)$, with operations $\otimes$ and $\oplus$, forms a distributive lattice.

Proof. The inclusion $[\underline{a}, \overline{a}] \otimes [\underline{b}, \overline{b}] \subseteq [\underline{a} \otimes \underline{b}, \overline{a} \otimes \overline{b}]$ follows trivially from the properties of lattice, namely, $\underline{a} \leq x \leq \overline{a}$ and $\underline{b} \leq y \leq \overline{b}$ imply $\underline{a} \otimes \underline{b} \leq x \otimes y \leq \overline{a} \otimes \overline{b}$. Now suppose $z \in [\underline{a} \otimes \underline{b}, \overline{a} \otimes \overline{b}]$. It is necessary to show there exists a pair $x \in [\underline{a}, \overline{a}]$ and $y \in [\underline{b}, \overline{b}]$ such that $x \otimes y = z$. Let $x = \underline{a} \oplus z$ and $y = \underline{b} \oplus z$. By the assumption of z

and properties of lattices, it can be verified that $x \in [\underline{a}, \overline{a}]$ and $y \in [\underline{b}, \overline{b}]$. Moreover,

$$\begin{aligned} x \otimes y &= (\underline{a} \oplus z) \otimes (\underline{b} \ominus z) \\ &= (\underline{a} \otimes \underline{b}) \ominus z \\ &= z. \end{aligned}$$

Therefore, $[\underline{a}, \overline{a}] \otimes [\underline{b}, \overline{b}] = [\underline{a} \otimes \underline{b}, \overline{a} \otimes \overline{b}]$. Similarly, it can be shown that the operation $\oplus$ is also closed. The fact that if L is a distributive lattice, $I(L)$ is a distributive lattice can be easily checked. In particular, the order relation on intervals is given by $[\underline{a}, \overline{a}] \preceq [\underline{b}, \overline{b}]$ if and only if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$. □

To differentiate it from the original lattice L, we call $I(L)$ an interval lattice. If L is a Boolean algebra, the complement operation $\ominus$ may be extended as follows:

$$\begin{aligned} \ominus[\underline{a}, \overline{a}] &= \{\ominus x \mid x \in [\underline{a}, \overline{a}]\} \\ &= [\ominus\overline{a}, \ominus\underline{a}]. \end{aligned} \tag{3.23}$$

$I(L)$ is not a Boolean lattice but a complete distributive lattice.

One may extend functions on a lattice to functions on an interval lattice in the same way that meet and join are extended. Using the extension rule (3.11), for any function f on a lattice, the corresponding extended function F can be derived from f. The extended function F may not form an interval in L. For instance, as showed in Example 3.1, the lattice operations join and meet are usually not closed when extended from L to $I(L)$. It may be difficult to find the conditions under which function F is closed in $I(L)$. For the lattice join and meet, Theorem 3.2 states a sufficient condition for extended operations to be closed.

3.2 T-norms and Interval T-norms

Using the results from interval computations, the notion of t-norms can be extended to interval t-norms and interval lattice t-norms.

3.2.1 Numeric t-norms

The concept of t-norms first appeared in the study of probability metric space, and was later adopted for the investigations of fuzzy set operations. A t-norm is a function from $[0,1] \times [0,1]$ to $[0,1]$ and satisfies the following conditions: for $a, b, c \in [0,1]$,

(i). Boundary conditions

$$t(0,0) = 0, \qquad t(1,a) = t(a,1) = a;$$

(ii). Monotonicity

$$(a \leq c, b \leq d) \Longrightarrow t(a,b) \leq t(c,d);$$

(iii). Symmetry

$$t(a,b) = t(b,a);$$

(iv). Associativity

$$t(a, t(b,c)) = t(t(a,b), c)).$$

Some commonly used t-norms are $t_{\mathrm{b}}(a,b) = \max(0, a+b-1)$, $t_{\min}(a,b) = \min(a,b)$, the product operator $t_{\mathrm{p}}(a,b) = a \cdot b$, and t_{w} defined by boundary conditions and $t_{\mathrm{w}}(a,b) = 0$, for $(a,b) \in [0,1) \times [0,1)$. These t-norms are related by inequality [11]:

$$t_{\mathrm{w}}(a,b) \leq t_{\mathrm{b}}(a,b) \leq t_{\mathrm{p}}(a,b) \leq t_{\min}(a,b). \tag{3.24}$$

Moreover, any t-norm is bounded by t_{w} and $t_{\min}$, i.e.,

$$t_{\mathrm{w}}(a,b) \leq t(a,b) \leq t_{\min}(a,b). \tag{3.25}$$

Suppose $n : [0,1] \longrightarrow [0,1]$ is an operation called negation. The dual of a t-norm is called a t-conorm, which is a function s mapping $[0,1] \times [0,1]$ to $[0,1]$ and satisfying the boundary conditions:

(i′). Boundary conditions

$$s(1,1) = 1, \qquad s(a,0) = s(0,a) = a,$$

and conditions (ii)-(iv). Suppose the negation operation is defined by $n(a) = 1 - a$. The t-conorm s corresponding to a t-norm t is given by:

$$\begin{aligned} s(a,b) &= n(t(n(a), n(b))) \\ &= 1 - t(1-a, 1-b). \end{aligned} \tag{3.26}$$

The t-conorms of $t_{\min}$, t_{p} and t_{b} are $s_{\max}(a,b) = \max(a,b)$, $s_{\mathrm{p}}(a,b) = a + b - ab$, and $s_{\mathrm{b}}(a,b) = \min(1, a+b)$, respectively.

Based on the results from interval computations, the notion of t-norms on single values in $[0,1]$ can be extended to subintervals of $[0,1]$. Let $I([0,1])$ denote the set of all closed subintervals of $[0,1]$. For a given t-norm t, an extended t-norm is defined by:

$$T(A,B) = \{t(x,y) \mid x \in A, y \in B\}. \tag{3.27}$$

Similarly, an extended t-conorm is defined by:

$$S(A,B) = \{s(x,y) \mid x \in A, y \in B\}. \tag{3.28}$$

In general, $T(A,B)$ and $S(A,B)$ may not necessarily be subintervals of $[0,1]$. According to Corollary 3.1, they are indeed intervals for the class of continuous t-norms. In this case, the results of interval t-norms can be easily computed by considering only extreme points of intervals. They are referred to as interval t-norms and t-conorms.

Theorem 3.3 *Suppose t is a continuous t-norm. For any two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, the interval t-norm produces the following interval:*

$$T(A, B) = [t(\underline{a}, \underline{b}), t(\overline{a}, \overline{b})]. \tag{3.29}$$

The interval t-conorm of a continuous t-conorm s produces the interval:

$$S(A, B) = [s(\underline{a}, \underline{b}), s(\overline{a}, \overline{b})]. \tag{3.30}$$

Proof. This theorem trivially follows from the property of t-norms and Corollary 3.1. □

For the negation operation $n(a) = 1 - a$, an extended negation on intervals of $[0, 1]$ is defined by:

$$\begin{aligned} N([\underline{a}, \overline{a}]) &= \{n(x) \mid x \in [\underline{a}, \overline{a}]\} \\ &= \{1 - x \mid x \in [\underline{a}, \overline{a}]\} \\ &= [1, 1] - [\underline{a}, \overline{a}] \\ &= [1 - \overline{a}, 1 - \underline{a}]. \end{aligned} \tag{3.31}$$

An interval t-conorm is related to an interval t-norm in terms of extended negation by:

$$\begin{aligned} S(A, B) &= N(T(N(A), N(B))) \\ &= [1, 1] - T([1, 1] - A, [1, 1] - B). \end{aligned} \tag{3.32}$$

When degenerated intervals of the form $[a, a]$ are used, interval t-norms reduce to t-norms. Interval t-norms are interval extension of t-norms [27]. Interval t-norms corresponding to $\min(a,b)$, $a \cdot b$, and $\max(0, a+b-1)$ can be computed by:

$$T_{\min}(A,B) = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})],$$

$$T_{\mathrm{p}}(A,B) = [\underline{ab}, \overline{a}\overline{b}],$$

$$T_{\mathrm{b}}(A,B) = [\max(0, \underline{a} + \underline{b} - 1), \max(0, \overline{a} + \overline{b} - 1)].$$

The corresponding interval t-conorms are:

$$S_{\max}(A,B) = [\max(\underline{a}, \underline{b}), \max(\overline{a}, \overline{b})],$$

$$S_{\mathrm{p}}(A,B) = [(\underline{a} + \underline{b} - \underline{ab}), (\overline{a} + \overline{b} - \overline{a}\overline{b})],$$

$$S_{\mathrm{b}}(A,B) = [\min(1, \underline{a} + \underline{b}), \min(1, \overline{a} + \overline{b})].$$

Properties of interval t-norms can be obtained from t-norms. Consider the following relation defined on intervals [14, 27]:

$$A \preceq B \Longleftrightarrow (\underline{a} \le \underline{b}, \overline{a} \le \overline{b}). \tag{3.33}$$

With this relation, the counterpart of equation (3.24) can be expressed as:

$$T_{\mathrm{b}}(A,B) \preceq T_{\mathrm{p}}(A,B) \preceq T_{\min}(A,B). \tag{3.34}$$

With this relation, properties of an interval t-norm can be formally stated in parallel to that of a t-norm.

Theorem 3.4 *An interval t-norm T has the following properties: for any closed intervals $A, B, C \subseteq [0,1]$,*

(I). Boundary conditions

$$T([0,0],[0,0]) = [0,0], \qquad T([1,1],A) = T(A,[1,1]) = A;$$

(II). Monotonicity

$$(A \preceq C, B \preceq D) \Longrightarrow T(A,B) \preceq T(C,D);$$

(III). Symmetry

$$T(A,B) = T(B,A);$$

(IV). Associativity

$$T(A,T(B,C)) = T(T(A,B),C).$$

Proof. From Corollary 3.1 and the definition of t-norms, properties of interval t-norm T can be verified as follows: for $A, B, C, D \subseteq [0,1]$,
Boundary conditions:

$$T([0,0],[0,0]) = [t(0,0), t(0,0)] = [0,0],$$

$$T(A,[1,1]) = [t(\underline{a},1), t(\overline{a},1)] = A.$$

Monotonicity:

$$\begin{aligned}(A \preceq C, B \preceq D) &\Longrightarrow (\underline{a} \leq \underline{c}, \overline{a} \leq \overline{c}, \underline{b} \leq \underline{d}, \overline{b} \leq \overline{d}) \\ &\Longrightarrow (t(\underline{a},\underline{b}) \leq t(\underline{c},\underline{d}), t(\overline{a},\overline{b}) \leq t(\overline{c},\overline{d})) \\ &\Longrightarrow [t(\underline{a},\underline{b}), t(\overline{a},\overline{b})] \preceq [t(\underline{c},\underline{d}), t(\overline{c},\overline{d})] \\ &\Longrightarrow T(A,B) \preceq T(C,D).\end{aligned}$$

Symmetry:

$$\begin{aligned}(t(\underline{a},\underline{b}) = t(\underline{b},\underline{a}), t(\overline{a},\overline{b}) = t(\overline{b},\overline{a})) &\Longrightarrow [t(\underline{a},\underline{b}), t(\overline{a},\overline{b})] = [t(\underline{b},\underline{a}), t(\overline{b},\overline{a})] \\ &\Longrightarrow T(A,B) = T(B,A).\end{aligned}$$

Associativity:

$$(t(\underline{a}, t(\underline{b},\underline{c})) = t(t(\underline{a},\underline{b}),\underline{c}), (t(\overline{a}, t(\overline{b},\overline{c})) = t(t(\overline{a},\overline{b}),\overline{c}))$$

$$\Longrightarrow T(A,T(B,C)) = T(T(A,B),C).$$

$\square$

An interval t-conorm S can be defined by the boundary conditions

(I′). Boundary conditions

$$S([1,1],[1,1]) = [1,1], \qquad S([0,0],A) = S(A,[0,0]) = A.$$

and properties (II)-(IV). These properties are counterparts of properties of t-norms. The intervals $[0,0]$ and $[1,1]$ play an important role in the characterization of interval t-norms.

In the above discussion, interval t-norms are defined as interval extension of t-norms. Conversely, an interval t-norm may regarded as an interval-valued function from $I([0,1]) \times I([0,1])$ to $I([0,1])$, satisfying properties (I)-(IV). Moreover, for each interval t-norm, a t-norm can be defined.

Theorem 3.5 *Let $T : I([0,1]) \times I([0,1]) \longrightarrow I([0,1])$ be an interval-valued function satisfying properties (I)-(IV). The function $t : [0,1] \times [0,1] \longrightarrow [0,1]$,*

$$t(a,b) = T([a,a],[b,b]), \tag{3.35}$$

is a t-norm.

The t-norm t may be considered as the projection of the interval t-norm T on $[0,1]$, which is the set of all degenerated intervals of the form $[a,a]$, $a \in [0,1]$.

3.2.2 Non-numeric t-norms

Let L be a lattice. A lattice t-norm is a function $\mathrm{t} : L \times L \longrightarrow L$, satisfying the same conditions (i)-(iv) as ordinary t-norms, except that elements a, b and c are now in L. Lattice t-conorm is a function $\mathrm{s} : L \times L \longrightarrow L$ and satisfies the same conditions (ii)-(iv) and (i′) as ordinary t-conorm except that the domain and range are $L \times L$

and L, respectively [41]. The monotonicity conditions are stated with respect to the order relation of a lattice.

A negation operation on a lattice L is a function $\mathbf{n} : L \longrightarrow L$ such that, for all $a \in L$ [41],

1. $\mathbf{n}(0) = 1, \qquad \mathbf{n}(1) = 0;$
2. $\mathbf{n}$ is strict decreasing;
3. $\mathbf{n}(\mathbf{n}(a)) = a.$

With respect to a negation operation, a lattice t-norm uniquly defines a t-conorm, and vice versa.

Theorem 3.6 *Suppose L is a lattice with a negation $\mathbf{n}$. For each t-norm $\mathtt{t}$ on L, there is a dual t-conorm with respect to $\mathbf{n}$: for any $a, b \in L$,*

$$\mathtt{s}(a, b) = \mathbf{n}(\mathtt{t}(\mathbf{n}(a), \mathbf{n}(b))),$$

and for each t-conorm $\mathtt{s}$ on L, there is a dual t-norm with respect to $\mathbf{n}$: for any $a, b \in L$,

$$\mathtt{t}(a, b) = \mathbf{n}(\mathtt{s}(\mathbf{n}(a), \mathbf{n}(b))).$$

Proof. It is easy to verify that the above expression satisfies the conditions of symmetry and associativity. Boundary and monotonicity conditions are verified as follows:
Boundary conditions:

$$\begin{aligned} \mathbf{n}(\mathtt{t}(\mathbf{n}(1), \mathbf{n}(1))) &= \mathbf{n}(\mathtt{t}(0, 0)) \\ &= \mathbf{n}(0) \\ &= 1, \end{aligned}$$

$$
\begin{aligned}
\mathbf{n}(\mathbf{t}(\mathbf{n}(0), \mathbf{n}(a))) &= \mathbf{n}(\mathbf{t}(1, \mathbf{n}(a))) \\
&= \mathbf{n}(\mathbf{n}(a)) \\
&= a.
\end{aligned}
$$

Monotonicity:

$$
\begin{aligned}
(a \leq c, b \leq d) &\Longrightarrow (\mathbf{n}(c) \leq \mathbf{n}(a), \mathbf{n}(d) \leq \mathbf{n}(b)) \\
&\Longrightarrow (\mathbf{t}(\mathbf{n}(c), \mathbf{n}(d)) \leq \mathbf{t}(\mathbf{n}(a), \mathbf{n}(b))) \\
&\Longrightarrow (\mathbf{n}(\mathbf{t}(\mathbf{n}(a), \mathbf{n}(b))) \leq \mathbf{n}(\mathbf{t}(\mathbf{n}(c), \mathbf{n}(d)))).
\end{aligned}
$$

Therefore, $\mathbf{n}(\mathbf{t}(\mathbf{n}(a), \mathbf{n}(b)))$ is a t-conorm. It can aslo be verified that $\mathbf{n}(\mathbf{s}(\mathbf{n}(a), \mathbf{n}(b)))$ is a t-norm. □

As shown in the following theorem, lattice t-norms and t-conorms may be considered as a generalization of the standard lattice operators of meet and join.

Theorem 3.7 *Suppose $(L, \oplus, \otimes, 0, 1)$ is a lattice. The meet operation $\otimes$ is a lattice t-norm, and the join operation $\oplus$ is a lattice t-conorm.*

Proof. By properties L1 and L2 of a lattice, $\otimes$ satisfies the conditions of symmetry and associativity. It also satisfies the boundary and monotonicity conditions, for $a, b, c, d \in L$,

$$
\text{Boundary conditions :} \quad 0 \otimes 0 = 0, \qquad a \otimes 1 = 1 \otimes a = a;
$$

$$
\text{Monotonicity :} \quad (\, a \leq c, b \leq d \,) \Longrightarrow a \otimes b \leq c \otimes b \leq c \otimes d.
$$

Similarly, $\oplus$ satisfies lattice t-conorm conditions. □

The lattice counterpart of t-norm t_w is defined by: for any $a, b \in L$, $\mathbf{t}_w(a, b)$ are the boundary conditions if one of a and b is 1; otherwise, $\mathbf{t}_w(a, b) = 0$. The lattice counterpart of t-conorm s_w is defined by: for any $a, b \in L$, $s_w(a, b)$ are the boundary conditions if one of a and b is 0; otherwise, $s_w(a, b) = 1$. Any lattice t-norm is bounded by t_w and the lattice meet operation, and a t-conorm is bounded by s_w and join.

Theorem 3.8 *A lattice t-norm* $t(a, b)$ *is bounded by*

$$t_w(a, b) \leq t(a, b) \leq a \otimes b, \tag{3.36}$$

and a lattice t-conorm $s(a,b)$ *is bounded by*

$$a \oplus b \leq s(a, b) \leq s_w(a, b). \tag{3.37}$$

Proof. According to boundary conditions and monotonicity conditions,

$$(t(a, b) \leq a, t(a, b) \leq b) \Longrightarrow t(a, b) \leq a \otimes b.$$

Thus, the right inequality holds. It can also be proven that the left inequality holds. If one of $a, b \in L$ is 1, $t(a, b) = t_w(a, b)$. Otherwise, the following holds,

$$t_w(a, b) = 0 \Longrightarrow t_w(a, b) \leq t(a, b).$$

Therefore inequality (3.36) holds for t-norms. Similarly, inequality (3.37) holds for t-conorms. □

Lattice t-norms in a three element lattice and a four element lattice are presented below. They are connected to many-valued logic.

Example 3.2 Lattice t-norms in a three element lattice. Consider a three element lattice $L = \{0, u, 1\}$, in which the order relation is given by:

$$0 \leq u \leq 1.$$

The following two tables define lattice t-norms in the three element lattice.

lattice t-norm t_1:

$y \backslash x$	1	u	0
1	1	u	0
u	u	u	0
0	0	0	0

lattice t-norm t_2:

$y \backslash x$	1	u	0
1	1	u	0
u	u	0	0
0	0	0	0

The elements in the lattice may be interpreted as representing three truth values. The above two t-norms may be regarded as the truth tables of logical conjunction in three-valued logic proposed by Lukasiewicz, Kleene, Heyting and Rrechenbach [33]. The first t-norm is the min function. The second t-norm which only gives the other value for (u, u) is in agreement with Lukasiewicz's interpretation of three-valued logic. □

The three element lattice in the above example is a linear lattice. In general, based on the notion of directed algebra, Mayor and Torrens has studied all possible t-norms on a finite linear lattice [24].

Example 3.3 Lattice t-norms in a four element lattice. Consider a four element lattice $L = \{0, a, b, 1\}$ characterized by the order relation:

$$0 \leq \begin{matrix} a \\ b \end{matrix} \leq 1,$$

where two elements a and b are not comparable. All acceptable lattice t-norms are given below:

lattice t-norm t_1:

$y \backslash x$	1	a	b	0
1	1	a	b	0
a	a	a	0	0
b	b	0	b	0
0	0	0	0	0

lattice t-norm t_2:

$y \backslash x$	1	a	b	0
1	1	a	b	0
a	a	0	0	0
b	b	0	b	0
0	0	0	0	0

lattice t-norm t_3:

$y \backslash x$	1	a	b	0
1	1	a	b	0
a	a	a	0	0
b	b	0	0	0
0	0	0	0	0

lattice t-norm t_4:

$y \backslash x$	1	a	b	0
1	1	a	b	0
a	a	0	0	0
b	b	0	0	0
0	0	0	0	0

The four elements can be interpreted as four truth values in a four-valued logic. The above four lattice t-norms may be regarded as truth tables for conjunction in four-valued logics. The incomparability of a and b enables the logics to handle more general situations where two propositions are not comparable. The first lattice t-norm is the min function, and the last $\mathrm{t_w}$. □

Using the extension rule (3.11), interval lattice t-norms or t-conorms can be defined based on lattice t-norms or t-conorms. Let L be a lattice and $I(L)$ the corresponding interval lattice. For a given t-norm t on L, it can be extended by: for any $A, B \in I(L)$,

$$\mathcal{T}(A, B) = \{\mathrm{t}(x, y) \mid x \in A, y \in B\}. \tag{3.38}$$

Similarly, an extended t-conorm is defined by:

$$\mathcal{S}(A, B) = \{\mathrm{s}(x, y) \mid x \in A, y \in B\}. \tag{3.39}$$

In general, $\mathcal{T}(A, B)$ and $\mathcal{S}(A, B)$ may not necessarily be intervals of L. If $\mathcal{T}$ and $\mathcal{S}$ are closed in $I(L)$, they are called interval lattice t-norms and t-conorms. Although it is difficult to state the conditions for such functions in general cases, one can deduce from Theorem 3.2 that the extended meet and join functions on interval lattice are interval lattice t-norm and t-conorm if the lattice is distributive. Interval lattice t-norm has the same properties (I)-(IV) as its numeric counterpart. Similarly, interval lattice t-conorm has the same properties as numeric interval t-conorms.

Chapter 4

INTERVAL-BASED UNCERTAIN REASONING

This chapter analyzes three different interval-based uncertain reasoning methods. One is based on the interpretation that intervals are constraints on a family of truth evaluation functions. Any member of the family can be the actual, but perhaps unknown, evaluation function. Examples of this class are reasoning with interval fuzzy sets and interval probabilities. Alternatively, when it is difficult to specify a particular class of truth evaluation functions, one can specify the properties that must be satisfied by the lower and upper bounds of intervals. For example, the lower bound is a necessity function, while the upper bound is a possibility function. Finally, when inferencing with two sets of truth values, it may be necessary to represent the truth values in one set by the truth values in the other. The theory of rough sets provides a systematic tool for solving this problem.

4.1 Uncertain Reasoning under Interval Constraints

Given a propositional language $L(\Phi)$, a truth evaluation function can be defined as a mapping $v : L(\Phi) \longrightarrow V$, where V is the set of truth values, such as the unit interval $[0, 1]$ or a lattice. This is an extension of classical propositional logic, in which the truth value of a proposition is not only true or false, but can be any value in V. In an inference system, one may choose different classes of evaluation functions satisfying certain axioms, such as different uncertainty measures. For example, if the evaluation is a probability function, a probabilistic inference system is obtained. If evaluation function satisfies the max-min rule of fuzzy sets, one obtains the max-min fuzzy logic system. In general, each pair of t-norms and t-conorms defines a fuzzy logic system. In practice, it may be difficult to specify precisely and consistently the actual evaluation function. One may only be able to provide a pair of lower and upper bounds which state the range of the actual evaluation function. In other words, intervals are provided instead of single points. In the absence of any information, the trivial interval [0,1] may be used.

The assignment of intervals can be formally described by two mappings $v_* : L(\Phi) \longrightarrow V$ and $v^* : L(\Phi) \longrightarrow V$. They specify an interval $[v_*(\phi), v^*(\phi)]$ within which lies the actual truth value of the proposition ϕ [42]. A set of lower and upper bounds is said to be *consistent* if there exists an evaluation function v such that for all $\phi \in L(\Phi)$,

$$v_*(\phi) \leq v(\phi) \leq v^*(\phi). \tag{4.40}$$

On the other hand, if an evaluation function v satisfies condition (4.40), it is is said to be bounded by (v_*, v^*) [40]. A consistent set of lower and upper bounds (v_*, v^*) can be interpreted as constraints on the evaluation function v. In fact, they determine the following maximal family of evaluation functions:

$$\mathcal{V} = \{v \mid v_*(\phi) \leq v(\phi) \leq v^*(\phi) \ \ for \ \ every \ \ \phi \in L(\Phi)\}. \tag{4.41}$$

For the set $L(\Phi)$, a pair of bounds $v_{0*} : L(\Phi) \longrightarrow V$ and $v_0^* : L(\Phi) \longrightarrow V$ is called the tightest bounds if every $v \in \mathcal{V}$ is bounded by (v_{0*}, v_0^*) and there does not exist another pair of bounds inside (v_{0*}, v_0^*) having this property. If a pair of lower and upper bounds is consistent, the tightest bounds are unique [44].

When the available information is insufficient, unavailable, or inconsistent, the mappings (v_*, v^*) may not be the tightest bounds. In order to carry out inference, it is necessary to tighten the given bounds, to infer the information of the propositions which is initially not available, and to resolve inconsistency [44]. The interval-based inference may be formulated as a process of constraint of propagation. Two examples are given in the rest of this section to illustrate the usefulness of such a framework.

4.1.1 Inference with interval fuzzy sets

In their book, Klir and Yuan [22] briefly discussed the problem of approximate reasoning using interval fuzzy sets. They suggested that t-norms can be used to carry out this task by directly applying them to the lower and upper bounds of interval fuzzy sets and relations. In the light of interval t-norms, a systematic analysis of basic issues is provided, such as interpretations of set-theoretic operations on interval fuzzy sets.

An important feature of this formulation is that the interpretation, "an interval fuzzy set is a set of fuzzy sets", is used as a basic notion. This is similar to the study of conditional events by Goodman [17]. Under certain conditions, operations on interval fuzzy sets are derived automatically using techniques of interval computation. In contrast, many studies use interval fuzzy set operations as basic notions, which are typically defined by the component-wise application of fuzzy set operations [13, 22, 31]. Although both approaches produce the same mathematical results, the reformulation may enhance the understanding of interval fuzzy reasoning by providing a concrete interpretation of interval fuzzy sets.

A Φ-fuzzy or an interval fuzzy set F can be described by a membership function:

$$\mu_{\mathcal{A}} : U \longrightarrow I([0,1]), \tag{4.42}$$

where U is called a universe [13, 22]. The membership interval $\mu_{\mathcal{A}}(u) = [\underline{\mu}_{\mathcal{A}}(u), \overline{\mu}_{\mathcal{A}}(u)]$ of element u may be interpreted as the range of the true membership. Any value in the interval may actually be the true membership. An interval fuzzy set can be described equivalently by a set of fuzzy sets bounded by two fuzzy sets $\underline{A}$ and $\overline{A}$, namely,

$$\begin{aligned} \mathcal{A} &= [\underline{A}, \overline{A}] \\ &= \{X \mid \underline{A} \subseteq X \subseteq \overline{A}\} \\ &= \{X \mid \forall u \in U(\mu_{\underline{A}}(u) \leq \mu_X(u) \leq \mu_{\overline{A}}(u))\}. \end{aligned} \tag{4.43}$$

Any fuzzy set inside $[\underline{A}, \overline{A}]$ may be the true fuzzy set. This offers a slightly new interpretation of interval fuzzy sets as compared to the traditional views that focus mainly on memberships. Interval fuzzy sets can be considered as a generalization of crisp interval sets [43].

By interpreting an interval fuzzy set as a family of fuzzy sets bounded by two fuzzy sets, one may immediately apply the technique from interval computation to define interval fuzzy set operations: for $\mathcal{A} = [\underline{A}, \overline{A}]$ and $\mathcal{B} = [\underline{B}, \overline{B}]$,

$$\begin{aligned} \sim \mathcal{A} &= \{\sim X | X \in [\underline{A}, \overline{A}]\}, \\ \mathcal{A} \cap \mathcal{B} &= \{X \cap Y | X \in [\underline{A}, \overline{A}], Y \in [\underline{B}, \overline{B}]\}, \\ \mathcal{A} \cup \mathcal{B} &= \{X \cup Y | X \in [\underline{A}, \overline{A}], Y \in [\underline{B}, \overline{B}]\}. \end{aligned} \tag{4.44}$$

Typically, t-norms and t-conorms are used to define fuzzy set intersection and union [4, 11, 22]. Suppose t and s are a pair of continuous t-norm and t-conorm. According to Theorem 3.3, operations on interval fuzzy sets defined by equation (4.44) can be expressed component-wise as:

$$
\begin{aligned}
\mu_{\sim\mathcal{A}}(u) &= \{1 - x \mid x \in \mu_{\mathcal{A}}(u)\} \\
&= [1,1] - \mu_{\mathcal{A}}(u), \\
\mu_{\mathcal{A}\cap\mathcal{B}}(u) &= \{t(x,y) \mid x \in \mu_{\mathcal{A}}(u), y \in \mu_{\mathcal{B}}(u)\} \\
&= T(\mu_{\mathcal{A}}(u), \mu_{\mathcal{B}}(u)), \\
\mu_{\mathcal{A}\cup\mathcal{B}}(u) &= \{s(x,y) \mid x \in \mu_{\mathcal{A}}(u), y \in \mu_{\mathcal{B}}(u)\} \\
&= S(\mu_{\mathcal{A}}(u), \mu_{\mathcal{B}}(u)).
\end{aligned} \tag{4.45}
$$

That is, the definition of interval fuzzy set operations by interval t-norms and t-conorms is a natural consequence of the interpretation given by equation (4.43) and the use of continuous t-norms and t-conorms for fuzzy set operations.

In parallel to the study of the degree of membership in fuzzy sets, one may consider the degree of truth in fuzzy logic. Given a proposition ϕ, let an interval $[\underline{a}, \overline{a}]$ denote the range of its truth value, written ϕ: $[\underline{a}, \overline{a}]$. Inference with interval truth value involves the derivation of truth values and tightening of the derived intervals. Suppose t and s are a pair of continuous t-norm and t-conorm, and T and S are the corresponding interval t-norm and t-conorm. One can use the following set of inference rules:

(R1) ϕ: $[\underline{a}, \overline{a}] \Longrightarrow \neg\phi$: $[1 - \overline{a}, 1 - \underline{a}]$;

(R2) $(\phi$: $[\underline{a}, \overline{a}], \psi$: $[\underline{b}, \overline{b}]) \Longrightarrow \phi \wedge \psi$: $T([\underline{a}, \overline{a}], [\underline{b}, \overline{b}])$;

(R3) $(\phi$: $[\underline{a}, \overline{a}], \psi$: $[\underline{b}, \overline{b}]) \Longrightarrow \phi \vee \psi$: $S([\underline{a}, \overline{a}], [\underline{b}, \overline{b}])$;

(R4) $(\phi$: $[\underline{a}, \overline{a}], \phi$: $[\underline{b}, \overline{b}]) \Longrightarrow \phi$: $[\max(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})]$;

(R5) $(\phi$: $[\underline{a}, \overline{a}], \phi \wedge \psi$: $[\underline{b}, \overline{b}]) \Longrightarrow \phi$: $[\max(\underline{a}, \underline{b}), \overline{a}]$;

(R6) $(\phi\colon [\underline{a}, \overline{a}], \phi \vee \psi\colon [\underline{b}, \overline{b}]) \Longrightarrow \phi\colon [\underline{a}, \min(\overline{a}, \overline{b})]$.

Inference rules (R1)-(R3) can be used to derive bounds of truth values for composite propositions, while rules (R4)-(R6) can be used to tighten the bounds. Similar rules have been used in a number of studies [5, 20, 44].

Let L be a distributive lattice and $I(L)$ can be the induced interval lattice. An interval L-fuzzy set $\mathcal{A}$ can be described by a membership function:

$$\mu_{\mathcal{A}} : U \longrightarrow I(L). \tag{4.46}$$

Corresponding to interval L-fuzzy sets, one may develop interval-valued fuzzy logic in which the truth value of a proposition is an interval of a lattice. In this case, the following rules may be used:

(R2′) $(\phi\colon [\underline{a}, \overline{a}], \psi\colon [\underline{b}, \overline{b}]) \Longrightarrow \phi \wedge \psi\colon [\underline{a}, \overline{a}] \otimes [\underline{b}, \overline{b}]$;

(R3′) $(\phi\colon [\underline{a}, \overline{a}], \psi\colon [\underline{b}, \overline{b}]) \Longrightarrow \phi \vee \psi\colon [\underline{a}, \overline{a}] \ominus [\underline{b}, \overline{b}]$;

(R4′) $(\phi\colon [\underline{a}, \overline{a}], \phi\colon [\underline{b}, \overline{b}]) \Longrightarrow \phi\colon [\underline{a} \oplus \underline{b}, \overline{a} \otimes \overline{b}]$;

(R5′) $(\phi\colon [\underline{a}, \overline{a}], \phi \wedge \psi\colon [\underline{b}, \overline{b}]) \Longrightarrow \phi\colon [\underline{a} \oplus \underline{b}, \overline{a}]$;

(R6′) $(\phi\colon [\underline{a}, \overline{a}], \phi \vee \psi\colon [\underline{b}, \overline{b}]) \Longrightarrow \phi\colon [\underline{a}, \overline{a} \otimes \overline{b}]$.

They may be considered as counterparts of rules (R2)-(R3). The definition of negation depends on the choice of a negation operation in a lattice.

Example 4.1 Kleene's three-valued logic. In this example, we show that Kleene's three-valued logic can be easily interpreted as an interval generalization of two-valued logic, based merely on the semantics of two-valued logic and interval computation. Consider the standard two-valued logic with the Boolean lattice $L = \{T, F\}$. In this case, $I(L) = \{[F, F], [F, T], [T, T]\}$. The interval $[F, F]$ indicates that the proposition is *false*, while the interval $[T, T]$ indicates that the proposition is *true*. On the other hand, the interval $[F, T]$ indicates, although the proposition must in fact be either true or false, the available information is insufficient to determine what its specific truth status may be. Similar interpretation of three-valued logic has been explored by Goodman *et al.* in the study of conditional events [18]. According to Theorem 3.2, such an interval-valued logic is characterized by the following truth tables:

ϕ	$\neg\phi$
$[T,T]$	$[F,F]$
$[F,T]$	$[F,T]$
$[F,F]$	$[T,T]$

	$\phi \wedge \psi$			$\phi \vee \psi$		
ϕ \ ψ	$[T,T]$	$[F,T]$	$[F,F]$	$[T,T]$	$[F,T]$	$[F,F]$
$[T,T]$	$[T,T]$	$[F,T]$	$[F,F]$	$[T,T]$	$[T,T]$	$[T,T]$
$[F,T]$	$[F,T]$	$[F,T]$	$[F,F]$	$[T,T]$	$[F,T]$	$[F,T]$
$[F,F]$	$[F,F]$	$[F,F]$	$[F,F]$	$[T,T]$	$[F,T]$	$[F,F]$

	$\phi \rightarrow \psi$			$\phi \leftrightarrow \psi$		
ϕ \ ψ	$[T,T]$	$[F,T]$	$[F,F]$	$[T,T]$	$[F,T]$	$[F,F]$
$[T,T]$	$[T,T]$	$[F,T]$	$[F,F]$	$[T,T]$	$[F,T]$	$[F,F]$
$[F,T]$	$[T,T]$	$[F,T]$	$[F,T]$	$[F,T]$	$[F,T]$	$[F,T]$
$[F,F]$	$[T,T]$	$[T,T]$	$[T,T]$	$[F,F]$	$[F,T]$	$[T,T]$

Each entry in the above tables is computed based on equation (3.22). For example, $[F,T] \wedge [T,T] = [F \wedge F, T \wedge T] = [F,T]$. These truth tables coincide with that of Kleene's three-valued logic [21, 33]. The interval-valued logic therefore provides an interpretation of three-valued logic in terms of standard two-valued logic. □

4.1.2 Inference with interval probabilities

In probabilistic logic, the evaluation function $v : L(\Phi) \longrightarrow V$ is chosen to be a probability function $P : L(\Phi) \longrightarrow [0,1]$. When it is difficult to provide a single probability function, a pair of bounds (P_*, P^*) are associated with propositions. They are consistent if there exists a probability function P bounded by P_* and P^*. A pair of consistent bounds define a maximal family of the probability functions bounded by the bounds.

Quinlan proposed a method of interval-based probabilistic inference [32]. A subset of the inference rules related to the primitive connectives negation ($\neg$) and conjunction ($\wedge$) is summarized below:

(P1) $\quad P_*(\phi) \longleftarrow \max\{P_*(\phi), 1 - \phi^*(\neg\phi)\};$

(P2) $\quad P^*(\phi) \longleftarrow \min\{P^*(\phi), 1 - \phi_*(\neg\phi)\};$

(P3) $\quad P_*(\phi \wedge \psi) \longleftarrow \max\{P_*(\phi \wedge \psi), P_*(\phi) + P_*(\psi) - 1\};$

(P4) $\quad P^*(\phi \wedge \psi) \longleftarrow \min\{P^*(\phi \wedge \psi), P^*(\phi) + P^*(\psi) - 1\};$

(P5) $\quad P_*(\phi) \longleftarrow \max\{P_*(\phi), P_*(\phi \wedge \psi)\};$

(P6) $\quad P^*(\phi) \longleftarrow \min\{P^*(\phi), P^*(\phi \wedge \psi) + (1 - P_*(\psi))\}.$

The symbol $\longleftarrow$ represents assignment operation. Quinlan's inference rules can increase the lower bound and decrease the upper bound, therefore tighten the bounds. However, they may not necessarily produce the tightest bounds, because they are based on the following inequality:

$$P(\phi) + P(\psi) - 1 \leq P(\phi \wedge \psi) \leq \min\{P(\phi), P(\psi)\},$$

which is derived from the following probability axiom by replacing $P(\phi \vee \psi)$ with the trivial bound $[0, 1]$:

$$P(\phi \wedge \psi) = P(\phi) + P(\psi) - P(\phi \vee \psi). \tag{4.47}$$

If information is available for probability of $\phi \vee \psi$ and $\phi \wedge \psi$, Yao [44] showed that Quinlan's rules can be improved. That is, $P(\phi \vee \psi)$ and $P(\phi \wedge \psi)$ can be assigned tighter bounds. Using interval computation, the right hand of equality (4.47) in terms of intervals becomes

$$[P_*(\phi), P^*(\phi)] + [P_*(\phi), P^*(\phi)] - [P_*(\phi \vee \psi), P^*(\phi \vee \psi)]$$
$$= [P_*(\phi) + P_*(\psi) - P^*(\phi \vee \psi), P^*(\phi) + P^*(\psi) - P_*(\phi \vee \psi)]$$

This may tighten the bound of $P(\phi \wedge \psi)$ further than Quinlan's method. More specifically, the following improved inference rules are obtained [44]:

(P3′) $P_*(\phi \wedge \psi) \longleftarrow \max\{P_*(\phi \wedge \psi), P_*(\phi) + P_*(\psi) - P^*(\phi \vee \psi)\}$;

(P4′) $P^*(\phi \wedge \psi) \longleftarrow \min\{P^*(\phi \wedge \psi), P^*(\phi), P^*(\psi), P^*(\phi) + P^*(\psi) - P_*(\phi \vee \psi)\}$;

(P5′) $P_*(\phi) \longleftarrow \max\{P_*(\phi), P_*(\phi \wedge \psi), P_*(\phi \wedge \psi) + P_*(\phi \vee \psi) - P^*(\psi)\}$;

(P6′) $P^*(\phi) \longleftarrow \min\{P^*(\phi), P^*(\phi \wedge \psi) + P^*(\phi \vee \psi) - P_*(\psi)\}$.

However, it should be pointed out that they still do not necessarily produce the tightest bounds.

Example 4.2 An inference network with four propositions. Consider an inference network consisting of propositions $\phi, \psi, \phi \wedge \psi$ and $\phi \vee \psi$. The probabilities of ϕ, ψ and $\phi \wedge \psi$ are $[0.5, 0.6], [0.6, 0.7]$ and $[0.4, 0.45]$. According to Quinlan's rules, the probability of $\phi \vee \psi$ lies in $[0.6, 1]$, while Yao's rules result in an interval $[0.65, 0.9]$, which is tighter than Quinlan's result. □

4.2 Inference with Necessity and Possibility Functions

In their semantic approach to the computation of truth, Dubois and Prade put forth that the fuzzy truth value of a proposition can be constrained by a pair of necessity and possibility functions [12, 36]. Given a proposition ϕ and available knowledge

$\mathcal{B}$ referring to the same universe U, the truth value of ϕ depends on the information $\mathcal{B}$, and can be represented by the valuation set $t(\phi|\mathcal{B})$. ϕ and $\mathcal{B}$ can be translated into $M(\phi)$ and $M(\mathcal{B})$ by means of meaning computation such as PRUF proposed by Zadeh [49], where $M(\phi)$ and $M(\mathcal{B})$ are subsets of U. The valuation set $t(\phi|\mathcal{B})$ is determined by the matching process between $M(\phi)$ and $M(\mathcal{B})$. When proposition ϕ and information $\mathcal{B}$ are vague and fuzzy, both $M(\phi)$ and $M(\mathcal{B})$ are fuzzy sets. In this case, the valuation set $t(\phi|\mathcal{B})$ is a fuzzy set of $[0,1]$ which can be interpreted as a fuzzy truth value with membership function:

$$\mu_{t(\phi|\mathcal{B})}(v) = \begin{cases} 0 & \text{if } \mu^{-1}_{M(\phi)}(v) = \emptyset, \\ \sup_u\{\pi(u) \mid \mu_{M(\phi)}(u) = v\} & \text{otherwise,} \end{cases} \tag{4.48}$$

where $\pi(u)$ is the possibility distributed in $u \in U$ and $\mu_{t(\phi|\mathcal{B})}(v)$ can be regarded as the grade of possibility that the truth value of ϕ is v. The fuzzy truth value can be constrained by the possibility function $\Pi(\phi)$ and necessity function $N(\phi)$ defined by

$$\begin{aligned} \Pi(\phi) &= \sup_u \min(\mu_{M(\phi)}(u), \pi(u)) \\ N(\phi) &= 1 - \Pi(\neg\phi) = \inf_u \max(\mu_{M(\phi)}(u), 1 - \pi(u)). \end{aligned}$$

It can be proved that the truth value v with degree of possibility 1 is in between $N(\phi)$ and $\Pi(\phi)$ [12].

Dubois and Prade's approach provides an interval representation of the truth values of propositions through a pair of necessity and possibility functions. The narrower the bound, the more precise the truth value. In practice, the initial bounds assigned to propositions may not necessarily be necessity and possibility functions. In this case, the inference is to infer lower and upper bounds that are as close as possible to necessity and possibility functions.

A pair of necessity and possibility functions, N and Π, have the following properties:

(PN1) $\quad N(\phi) = 1 - \Pi(\neg\phi)$;

$$\text{(PN2)} \quad \Pi(\phi) = 1 - N(\neg\phi);$$

$$\text{(PN3)} \quad \Pi(\phi) \geq N(\phi);$$

$$\text{(PN4)} \quad N(\phi) \geq N(\phi \wedge \psi);$$

$$\text{(PN5)} \quad N(\phi \wedge \psi) = \min(N(\phi), N(\psi));$$

$$\text{(PN6)} \quad N(\phi \vee \psi) \geq \max(N(\phi), N(\psi));$$

$$\text{(PN7)} \quad \Pi(\phi \wedge \psi) \leq \min(\Pi(\phi), \Pi(\psi));$$

$$\text{(PN8)} \quad \Pi(\phi \vee \psi) = \max(\Pi(\phi), \Pi(\psi)).$$

From these properties, one can see that $N(\phi \wedge \psi)$ and $N(\phi \vee \psi)$ are always less than $\Pi(\phi \wedge \psi)$ and $\Pi(\phi \vee \psi)$. Both N and Π are monotonic with respect to set inclusion. Therefore, they satisfy the additional properties:

$$\text{(PN9)} \quad N(\phi \wedge \psi) \leq \Pi(\phi \wedge \psi) \leq \min(\phi \wedge \psi) \leq \Pi(\phi);$$

$$\text{(PN10)} \quad N(\phi \wedge \psi) \leq N(\phi) \leq \max(N(\phi)N(\psi)) \leq N(\phi \vee \psi) \leq \Pi(\phi \wedge \psi).$$

Let N and Π be initial bounds that may not be necessity and possibility functions. The following inference rules can be used to update these bounds:

$$\text{(I1)} \quad \Pi(\phi) \longleftarrow \inf\{\Pi(\phi); 1 - N(\neg\phi), \Pi(\phi \vee \psi)\};$$

$$\text{(I2)} \quad N(\phi) \longleftarrow \sup\{N(\phi), 1 - \Pi(\neg\phi), N(\phi \wedge \psi)\};$$

$$\text{(I3)} \quad \Pi(\phi \wedge \psi) \longleftarrow \inf\{\Pi(\phi \wedge \psi)), \min(\Pi(\phi), \Pi(\psi))\};$$

$$\text{(I4)} \quad N(\phi \wedge \psi) \longleftarrow \sup\{N(\phi \wedge \psi)), \min(N(\phi), N(\psi))\};$$

$$\text{(I5)} \quad \Pi(\phi \vee \psi) \longleftarrow \inf\{\Pi(\phi \vee \psi)), \max(\Pi(\phi), \Pi(\psi))\};$$

$$\text{(I6)} \quad N(\phi \vee \psi) \longleftarrow \sup\{N(\phi \vee \psi)), \max(N(\phi), N(\psi))\}.$$

Using the above rules, one can increase the lower bound and decrease the upper bound. The soundness of these inference rules can be seen from properties (PN1)-(PN10). For example, (PN1) and (PN10) gives rule (I1).

The necessity and possibility functions are a special class of belief and plausibility functions. The argument may be applied to inference with belief and plausibility functions in general.

4.3 Inference with Rough Sets

Let $(B, \oplus, \otimes, \ominus, 0, 1)$ be a finite Boolean algebra, and $(B_0, \oplus, \otimes, \ominus, 0, 1)$ be a sub-Boolean algebra of B. That is, B_0 contains both elements 0 and 1, and is closed under $\ominus$, $\otimes$, and $\oplus$. Assume that the available information is only sufficient for us to consider elements of B_0. If an element not in B_0 is encountered, one must represent it in terms of elements of B_0. The theory of rough sets provides a systematic method to perform this task.

Consider an element $a \in B$. One can associate two elements of B_0 with a as follows:

$$\underline{apr}(a) = \bigvee\{b \mid b \in B_0, b \leq a\},$$
$$\overline{apr}(a) = \bigwedge\{b \mid b \in B_0, a \leq b\}. \tag{4.49}$$

The pair $\underline{apr}(a)$ and $\overline{apr}(a)$ is referred to as the lower and upper approximations of a. By definition, they are the best approximations of a in the sense that $\underline{apr}(a)$ is the largest element in B_0 satisfying $b \leq a$, while $\overline{apr}(a)$ is the smallest element in B_0 satisfying $a \leq b$. Such a formulation is taken from Gehrke and Walker [15], in which they use completely distributive lattice by generalizing Pawlak's original proposal [29]. Since the Boolean algebra B is finite, B_0 is an atomic Boolean algebra. Let $At(B_0)$ denote the set of atoms of B_0, the lower and upper approximations can be equivalently defined by:

$$\begin{aligned}\underline{apr}(a) &= \bigvee\{b \mid b \in At(B_0), b \leq a\},\\ \overline{apr}(a) &= \bigvee\{b \mid b \in At(B_0), a \wedge b \neq 0\}.\end{aligned} \tag{4.50}$$

This definition is originally used by Pawlak, in which the Boolean algebra is the power set of the universe, and the atoms of the sub-Boolean algebra are the equivalence classes [29].

It can be easily verified that the following properties hold: for $a, b \in B$,

(L1) $\underline{apr}(a) = \ominus\overline{apr}(\ominus a)$;

(L2) $\underline{apr}(1) = 1$;

(L3) $\underline{apr}(a \otimes b) = \underline{apr}(a) \otimes \underline{apr}(b)$;

(L4) $\underline{apr}(a) \oplus \underline{apr}(b) \leq \underline{apr}(a \oplus b)$;

(L5) $a \leq b \Longrightarrow \underline{apr}(a) \leq \underline{apr}(b)$;

(L6) $\underline{apr}(0) = 0$;

(L7) $\underline{apr}(a) \leq a$;

(L8) $a \leq \underline{apr}(\overline{apr}(a))$;

(L9) $\underline{apr}(a) \leq \underline{apr}(\underline{apr}(a))$;

(L10) $\overline{apr}(a) \leq \underline{apr}(\overline{apr}(a))$;

(U1) $\overline{apr}(a) = \ominus\underline{apr}(\ominus a)$;

(U2) $\overline{apr}(0) = 0$;

(U3) $\overline{apr}(a \oplus b) = \overline{apr}(a) \oplus \overline{apr}(b)$;

(U4) $\overline{apr}(a \otimes b) \leq \overline{apr}(a) \otimes \overline{apr}(b)$;

$$(U5) \quad a \leq b \Longrightarrow \overline{apr}(a) \leq \overline{apr}(b);$$

$$(U6) \quad \overline{apr}(1) = 1;$$

$$(U7) \quad a \leq \overline{apr}(a);$$

$$(U8) \quad \overline{apr}(\underline{apr}(a)) \leq a;$$

$$(U9) \quad \overline{apr}(\overline{apr}(a)) \leq \overline{apr}(a);$$

$$(U10) \quad \overline{apr}(\underline{apr}(a)) \leq \underline{apr}(a);$$

$$(K) \quad \underline{apr}(\ominus a \oplus b) \leq \ominus \underline{apr}(a) \oplus \underline{apr}(b);$$

$$(LU) \quad \underline{apr}(a) \leq \overline{apr}(a).$$

Properties (L1) and (U1) state that two approximations are dual to each other. Properties with the same number may therefore be regarded as dual properties.

Inference with rough sets deals with the lower and upper approximations of truth values in different systems or with respect to different experts. Suppose a Boolean algebra B is used by one system, say S_1, and any proposition in this system has an exact truth value taken from B. On the other hand, another system, say S_2, may only use a sub-Boolean algebra B_0 to represent its truth values. When statements from S_1 are considered in system S_2, it may not always be possible to specify their truth exactly. One has to consider approximations of the truth values in B by truth values in B_0.

Given a proposition ϕ, let a denote its truth value in B, which is represented by a pair of lower and upper approximations $(\underline{apr}(a), \overline{apr}(a))$ in B_0. Based on the property of rough sets, we can obtain the following inference rules:

$$(R1'') \quad \phi\colon (\underline{apr}(a), \overline{apr}(a)) \Longrightarrow \neg\phi\colon (\ominus\overline{apr}(a), \ominus\underline{apr}(a));$$

$$(R2'') \quad (\phi\colon (\underline{apr}(a), \overline{apr}(a)), \psi\colon (\underline{apr}(b), \overline{apr}(b))) \Longrightarrow$$

$$\phi \wedge \psi\colon (\underline{apr}(a) \otimes \underline{apr}(b), \overline{apr}(a \otimes b));$$

$$\text{(R3}'')\quad (\phi\colon (\underline{apr}(a), \overline{apr}(a)), \psi\colon (\underline{apr}(b), \overline{apr}(b))) \Longrightarrow$$

$$\phi \vee \psi\colon (\underline{apr}(a \oplus b), \overline{apr}(a) \oplus \overline{apr}(b)).$$

These rules are much weaker than their counterparts in interval fuzzy sets. In both rules (R2″) and (R3″), only one of the lower and upper approximations may be derived from the lower and upper approximations of the two propositions involved. Since rough sets provide the best lower and upper approximations, other rules are no longer needed.

Based on the approximation of truth values, we may introduce modal structure in many-valued logic [33]. More specifically, a necessity operator is defined in terms lower approximations and a possibility operator defined in terms of upper approximations. That is, for any elements $a \in B$, $\Box a = \underline{apr}(a)$ and $\Diamond a = \overline{apr}(a)$. Consequently, the truth value of modal propositions are defined by:

$$\begin{aligned} \Box\phi\colon \underline{apr}(a) &\iff \phi\colon a, \\ \Diamond\phi\colon \overline{apr}(a) &\iff \phi\colon a. \end{aligned} \tag{4.51}$$

If the maximum element 1 is chosen to be the designated truth value, following modal expressions are tautologies:

(i) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;

(ii) $\Box\phi \to \Diamond\phi$;

(iii) $\Box\phi \to \phi$;

(iv) $\phi \to \Box\Diamond\phi$;

(v) $\Box\phi \to \Box\Box\phi$;

(vi) $\Diamond\phi \to \Box\Diamond\phi$,

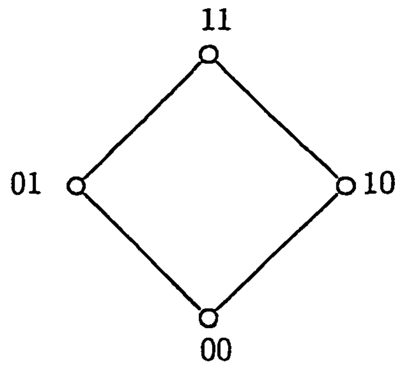

Figure 4.1: A four elements Boolean algebra

where $\phi \rightarrow \psi$ is defined by $\neg\phi \vee \psi$ as in standard two-valued logic. In other words, each of the above formulas takes the designated truth value 1 for every assignment of values to the variables in it. The modal logic system S_5 also obeys these axioms [6, 19]. One may say that this many-valued logic captures the theorems of modal logic system S_5 in the sense that every theorem of S_5 is a tautology of the many-valued system [33].

Example 4.3 A four-valued modal logic. Consider a four-valued logic system in which the truth values are drawn from a Boolean algebra given in Figure 4.1. It can be interpreted as the product of two classical two-valued logic systems, namely, the system $C_2 \times C_2$ as referred to by Rescher [33]. The truth value 11 can be interpreted as complete truth, 00 as complete falsity. They are complements of each other. Both 01 and 10, complement to each other, are regarded as partial truth or falsity.

If only complete truth or falsity can be used, we may consider the approximations of the partial truth. In other words, we want to approximate elements of Boolean algebra $B = \{00, 01, 10, 11\}$ by elements of the sub-Boolean algebra $B_0 = \{00, 11\}$. In this case, we have:

$$\underline{apr}(00) = \underline{apr}(01) = \underline{apr}(10) = 00, \qquad \underline{apr}(11) = 11,$$

$$\overline{apr}(01) = \overline{apr}(10) = \overline{apr}(11) = 11, \qquad \overline{apr}(00) = 00. \tag{4.52}$$

Although a proposition may take a partial truth as its value, the lower and upper approximations take either complete truth or complete falsity. □

4.4 Summary

In this chapter, three interval-based inference approaches have been developed. They can be applied to different situations depending on the interpretation of the intervals involved. In the inference with interval constraints, the properties of the truth evaluation function are known and therefore may be used to narrow the bounds. Inference with interval fuzzy sets and interval probabilities are carried out in this framework. In the approach with necessity and possibility, the properties of the evaluation function are not available, but the properties of bounds are known. That is, the evaluation function is bounded by a pair of necessity and possibility functions. If the initial bounds are not such functions, one can update these bounds based on properties that must be satisfied. Rough sets offer a different approach for interval-based uncertain reasoning. The truth value of a proposition is known but cannot be exactly described by truth values in another related logic with fewer truth values. It is therefore approximated by an interval. Inference with rough set theory is similar to inference using modal logic.

Chapter 5

CONCLUSION

In this thesis, we have studied some fundamental issues related to interval-based uncertain reasoning. A framework of interval computations is presented, which is a special case of set-based computations. Algorithms for interval computations are examined. Their applications in uncertain reasoning are explored.

One of the fundamental assumption of interval-based uncertain reasoning is that the truth values of propositions are intervals. They reflect the uncertainty that is inevitable in human reasoning, and hence in any intelligent systems attempting to model such a reasoning process. There are at least three interpretations of intervals involved. They lead to different uncertain reasoning methods.

In reasoning under interval constraints, intervals are interpreted as constraints that define a family of truth evaluation functions. Any member the family may in fact be the actual truth evaluation function. In other words, the available information is insufficient for us to give one evaluation function. Intervals are used to define the family of all evaluation functions compatible with the given information. The inference is formulated as a process of constraint propagation. Reasoning using interval fuzzy sets and interval probabilities are example of this class. Uncertain reasoning with interval fuzzy sets can be understood as an extension of single-value-based many-valued logic to interval-value-based many-valued logic. A basic concept used is the notion of interval t-norms. They can either be derived from continuous t-norms or be defined by using a set of axioms similar to that of t-norms. The standard L-fuzzy sets can also be extended to interval L-fuzzy sets. Interval t-norms can be computed by simply applying the corresponding t-norms on both lower and upper bounds of interval fuzzy sets. Inference with both numeric and lattice based interval fuzzy sets has been examined. Inference with interval probabilities can be regarded as the constraint of upper and lower bounds of the actual probabilities of various propositions. Quinlan's inference rules have been refined to get tighter bounds based on the results of interval computations.

In some situations, it may be difficult to identify the properties of the evaluation function. However, one may, instead, use intervals and define properties of the bounds. The interval representation reflects our inability to define precisely an evaluation function. For this class, we have discussed reasoning with necessity and possibility functions. It is assumed that the fuzzy truth values of propositions are bounded by a pair of necessity and possibility functions. The necessity function indicates the sure threshold of the truth value while the possibility function gives the maximal possible truth value. One of the main tasks of inference in this class is to derive the tightest bounds that obey the required properties. Inference rules can be obtained based on the properties of necessity and possibility functions.

Finally, there may also be cases where a precisely known truth value must be approximated. This may happen in the combination of opinions from many experts, where different experts use different many-valued logic systems. The theory of rough sets provides a solution to this problem. In this case, intervals are interpreted as approximations. Inference with rough sets explores modal structures in many-valued logic. Lower and upper approximations can be used to introduce necessity and possibility operations. In other words, inference based on rough sets can be understood in terms of many-valued modal logic. More specifically, interval-based inference using rough sets is related to modal logic system S_5.

In this thesis, we have only considered the extension of t-norms to interval t-norms in a numeric framework and the extension of standard lattice operations. It is useful to study the notion of t-norms and their interval extensions using other mathematical structures. Some initial results along this line have been reported by Mayor and Torrens using totally ordered sets [24], and by Wu [41] using complete lattices. The formulation of rough sets using Boolean algebras corresponds to the original proposal of Pawlak. It would be interesting to examine other generalized rough set models discussed by Yao and Lin [45].

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IMAGE EVALUATION
TEST TARGET (QA-3)

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