METRIZATION

AND

SIGMA BASES

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ABSTRACT

In this thesis we use the concept of sigma base to study the metrizability of various topological spaces. Chapter I is devoted to the definition of basic terms, a proof of Urysohn's Metrization Theorem, and an introduction to sigma structures.

In Chapter II various sigma bases are defined and their inter-relationships, as well as their relationships to other topological concepts, are studied. The Nagata-Smirnov Metrization Theorem and the Bing Metrization Theorem are among the theorems studied in this chapter.

First countable spaces that have useful sigma base characterizations are studied in Chapter III. In particular, sigma base characterizations are developed for semi-metric spaces and developable spaces. We prove a theorem due to Heath, that a semi-metric space with a point countable base is developable. We also prove that a developable paracompact space is metrizable.

Sigma refinements that are used in Michael's characterizations of paracompactness are introduced in Chapter IV and then used to develop M_i spaces as defined by Ceder. We prove that M_i spaces are paracompact and perfectly normal and prove that first countable M_3 spaces are semi-metric spaces. Finally, we generalize both the Nagata-Smirnov Theorem and Bing's Theorem, by proving that a

topological space is metrizable if and only if it is T_1 and has a sigma cushioned point countable pair-base.

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L.D.H.

CONTENTS

Chapter I

Introductory Concepts

1.1 Introduction.

We assume that the reader is familiar with set theory and consequently we draw freely upon set-theoretic theorems and definitions without recourse to proof or explanation.

In 1.2 the notions of topological spaces and bases for topologies are introduced. Section 1.3 gives definitions of most of the basic terms that will be used throughout the work, although occasionally the definition of a particular concept may be deferred until needed.

Following the definition of metric spaces in 1.4, the problem of metrization of topological spaces is introduced. Urysohn's metrization theorem is studied in some detail in order to acquaint the reader with the type of proof that one encounters when studying these problems. Then we consider the metric structure of the Euclidean plane and, from a brief analysis of various bases for this structure, formulate the concept of sigma structure (σ-structure). In succeeding chapters, various σ-structures are introduced and studied in a topological setting.

1.2 Topological Spaces.

Definition 1.2.1.

A topological space (X,τ) is a set X and a family τ of subsets of X such that:

(1) $\emptyset \in \tau$ ($\emptyset$ is the empty set);

(2) $X \in \tau$;

(3) if $U,V \in \tau$ then $U \cap V \in \tau$;

(4) the union of an arbitrary collection of elements of τ is a member of τ.

The elements of X are referred to as points. τ is called a topology on X and the elements of τ are called open sets. Sometimes X is referred to as a topological space, or just as a space, the family τ being understood.

We mention, in passing, that the same set can be given different topologies by choosing different families of sets to form τ. In fact, many examples and counter examples in topology are constructed by imposing different topologies on the Euclidean plane.

Definition 1.2.2.

A base B for a topology τ on X is a collection B of open sets such that if $x \in U$ and $U \in \tau$ then there exists some $V \in B$ such that $x \in V \subset U$. Equivalently, B is a base for τ if and only if each member of τ is the

union of members of B and each member of B is in τ.

Definition 1.2.3.

A subbase S for a topology τ on X is a collection of open sets such that the collection of finite intersections of members of S forms a base for τ. If X is a nonempty set, then any family of subsets of X is a subbase for a unique topology on X.[1]

1.3 Basic Definitions.

In the following definitions (X,τ) is a topological space and the sets in question are subsets of X.

Definition 1.3.1.

The complement of a set A is denoted by A^c and is the set $X - A$.

Definition 1.3.2.

A set A is closed if $A^c \in \tau$ (that is, if A^c is open).

Definition 1.3.3.

The closure of a set A is denoted by $\overline{A}$ and is defined by $\overline{A} = \cap \{C \mid A \subset C \text{ and } C \text{ is a closed subset of } X\}$. It is an easy consequence of the definitions of topological space and closed set that $\overline{A}$ is a closed set.

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[1] For a more complete discussion of the definitions of base and subbase see Kelley [10], pages 46-48.

Definition 1.3.4.

A point x is an accumulation point of a set B if every open set containing x contains some $y \neq x$ such that $y \in B$. It is not difficult to show that $\overline{A} = A \cup A'$ (where A' denotes the set of all accumulation points of A).

Definition 1.3.5.

The interior of a set A is denoted by A° and is the union of all open subsets of A. A set A is a neighbourhood of a point x if $x \in A^\circ$.

Definition 1.3.6.

If $B \subset X$ then τ generates a topology τ' on B in the following way. A subset U of B is open relative to B if there is some open set $V \in \tau$ such that $U = B \cap V$. τ' is a topology on B and (B,τ') is called a subspace of (X,τ).

Definition 1.3.7.

A function f with domain X and range contained in a topological space Y is continuous if, for any set U open in Y, the set $f^{-1}[U]$ is open in X.

Definition 1.3.8.

$f : X \to Y$ is a homeomorphism if the following conditions are satisfied:

(1) f is a bijection;

(2) f is continuous;

(3) f^{-1} is continuous.

Definition 1.3.9.

Suppose $\{X_\alpha \mid \alpha \in I\}$ is an arbitrary collection of topological spaces where I is some index set. Then the product set $P = \prod\{X_\alpha \mid \alpha \in I\}$ of these spaces is the set of all functions f mapping I into $\bigcup\{X_\alpha \mid \alpha \in I\}$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in I$. Let $\alpha \in I$ and U_α be an open subset of X_α. Define W to be the set of all points $f \in P$ such that $f(\alpha) \in U_\alpha$ and let S be the set of all such sets W. From the remark following Definition 1.2.3 we see that S is a subbase for a unique topology on P. The product set with this topology is called the product space and the topology is called the product topology.

Definition 1.3.10.

A topological space is a T_1 space if, given distinct points a and b in it, there are open sets U and V such that $a \in U$, $a \notin V$, $b \in V$, and $b \notin U$. It is easy to see that a space is T_1 if and only if each set consisting of a single point is closed.

Definition 1.3.11.

A topological space is a Hausdorff space if, given distinct points a and b in it, there are disjoint open

sets U and V such that $a \in U$ and $b \in V$.

Definition 1.3.12.

A topological space is a regular space if, given a closed set B and a point $a \notin B$, there are disjoint open sets U and V such that $a \in U$ and $B \subset V$. An alternative characterization of regularity is the following: a topological space is regular if given an open set U and a point $a \in U$, there is an open set V such that $a \in V \subset \overline{V} \subset U$. It is easily seen that these two definitions are equivalent. Some authors use the term regular to imply a regular and T_1 space.

Definition 1.3.13.

A topological space is normal if, given disjoint closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. It can be shown that a topological space is normal if and only if, given a closed set A and an open set U such that $A \subset U$, then there is an open set V such that $A \subset V \subset \overline{V} \subset U$.

Definition 1.3.14.

A topological space is perfectly normal if it is normal and if every closed set A is a G_δ (that is, A is the intersection of a countable collection of open sets).

Definition 1.3.15.

A cover of a topological space X is a family C of subsets of X whose union is X. A cover is an open (closed) cover if all the sets in it are open (closed).

Definition 1.3.16.

If C and S are covers of X and $S \subset C$, then S is a subcover of the cover C of X.

Definition 1.3.17.

If C and R are covers of X and, given $A \in R$ there exists $B \in C$ such that $A \subset B$, then R is a refinement of C. If the sets in R are open (closed) then R is an open (closed) refinement.

Definition 1.3.18.

A collection of subsets of X is locally finite (locally countable) if, given $x \in X$ there is some open set U containing x such that U meets (has a non-empty intersection with) at most finitely (countably) many elements in the collection.

Definition 1.3.19.

A collection of subsets of X is point finite (point countable) if each point in X is in at most finitely (countably) many members of the collection.

Definition 1.3.20.

A topological space is compact if every open cover of it has a finite subcover.

Definition 1.3.21.

A topological space is Lindelöf if every open cover of it has a countable subcover.

Definition 1.3.22.

A topological space is paracompact if it is Hausdorff and every open cover has a locally finite open refinement.

Definition 1.3.23.

A collection L of open sets of X is a local base at $x \in X$ if, for any open set U containing x there exists $V \in L$ such that $x \in V \subset U$.

Definition 1.3.24.

A topological space is first countable if each point has a countable local base.

Definition 1.3.25.

A topological space is second countable if it has a countable base.

Definition 1.3.26.

A set A is dense in B if $B \subset \overline{A}$. Clearly, A is dense in X if and only if $X = \overline{A}$ or, equivalently, if and

only if any open set containing $x \in X$ contains a point $a \in A$. A is dense if it is dense in X.

Definition 1.3.27.

A topological space is separable if it contains a countable dense subset.

Definition 1.3.28.

A family of subsets of X is discrete if the closures of its members are pair-wise disjoint and if the union of any subcollection of the closures of the sets is closed.

Definition 1.3.29.

A collection C of sets is closure preserving if, for any subcollection D of C, the closure of the union of the sets in D is equal to the union of the closures of the sets in D.

Defintion 1.3.30.

A base B for a topological space X is a uniform base if, given $x \in X$, any infinite subcollection of elements of B, each of which contains x, is a local base at x.

Definition 1.3.31

A collection of sets is star finite (star countable) if each set in the collection meets at most finitely (countably) many members of the collection.

Definition 1.3.32.

A topological space is strongly paracompact if every open cover of it has a star finite open refinement.

Definition 1.3.33.

A topological space is locally separable if every point is contained in an open set that, when considered as a subspace, is separable.

Definition 1.3.34.

A topological space is locally second countable if each point is contained in an open set that, when considered as a subspace, is second countable.

Definition 1.3.35.

A base B for a topology on X is a locally countable base if each point in X is contained in an open set that meets at most countably many members of B.

Definition 1.3.36.

A property P of topological spaces is a hereditary property if, whenever a space X has P, then every subspace of X has P.

1.4 Metric Spaces.

Definition 1.4.1.

A metric on a set X is a real-valued function,

$D : X \times X \to R_1$ such that for arbitrary points x, y and z in X:

(1) $D(x,y) \geq 0$;

(2) $D(x,y) = 0$ if and only if $x = y$;

(3) $D(x,y) = D(y,x)$;

(4) $D(x,y) \leq D(x,z) + D(z,y)$ (triangle inequality).

If D is a metric for a set X, we refer to (X,D) as a metric space or, if D is clearly implied by the context, we refer to X as a metric space.

Definition 1.4.2.

Let (X,D) be a metric space. A subbase S for a topology on X is defined as follows: $U \in S$ if for some $x \in X$ and for some real number $\varepsilon > 0$, $U = \{y \in X \mid D(x,y) < \varepsilon\}$; and such a set U is denoted by $S(x,\varepsilon)$ and is called an open sphere of radius ε centred at x. We recall from Definition 1.2.3 that this topology, called the metric topology induced on X by D, is unique.

Theorem 1.4.1.

Let (X,D) be a metric space. Then X with the metric topology, is:

(1) perfectly normal;

(2) paracompact;

(3) first countable.

Proof.

(1) This result follows from Theorem 4.2.5 (page 58).

(2) Stone proves this in [22].

(3) The first countability of metric spaces follows from Definition 1.4.2 and the fact that S is a base.

Definition 1.4.3.

A topological space (X,τ) is metrizable if there is a metric D on X such that the metric topology induced on X by D is τ.

From Theorem 1.4.1 we see that if a topological space is metrizable then a great deal is known about the space. Consequently, much study has been concentrated upon the problem of finding necessary and sufficient conditions that a topological space possessing various properties be metrizable.

The basic approach to the metrization problem for a given class of spaces is usually to try to find some metric space into which each member of the class can be embedded by a homeomorphism. This turns out to be a worthwhile line of attack because if a given topological space (X,τ) is homeomorphic to a subspace of a topological space (Y,τ') and the space Y is metrizable, then X is metrizable. The proof is easy: let X and Y be such spaces, let $f : X \to Y$ be a homeomorphism from X onto a subspace of Y

and let $D : Y \times Y \to R_1$ be a metric for the topology on Y. Clearly $D^* : X \times X \to R_1$ defined by $D^*(x,y) = D(f(x),f(y))$ for $x,y \in X$ is a metric on X. To show that the topology induced by D^* on X coincides with the given topology τ on X we proceed as follows: let S be defined for D^* as in Definition 1.4.2. Then,

(1) every member $S(x,\varepsilon)$ of S is in τ since $S(x,\varepsilon) = f^{-1}[S(f(x),\varepsilon)]$;

(2) if $U \in \tau$ and $x \in U$, then, since f^{-1} is continuous, there exists some $\varepsilon > 0$ such that $S(x,\varepsilon) = f^{-1}[S(f(x),\varepsilon)] \subset U$.

From (1) and (2) it follows that S is a base for τ on X. Therefore D^* induces the original topology on X and (X,τ) is metrizable.

From the preceding theory, one is led immediately to the conclusion that problems of metrizability often reduce to problems of finding a "big" metric space and homeomorphisms that embed other spaces in the big space. In the following section, we soon become aware that putting the theory into practice can become a Herculean labour.

1.5 The Work of Tychonoff and Urysohn.

The first solution of an important metrization problem was due mainly to Tychonoff and Urysohn. One important phase of Tychonoff's research was devoted to the study of products of topological spaces. He proved that any product of compact spaces is compact. He also proved that a regular Lindelöf space is normal. Urysohn proved Lemma 1.5.4 which is the key to Theorem 1.5.1.

Theorem 1.5.1 (Urysohn's Metrization Theorem).

A second countable T_1 space is metrizable if and only if it is regular.

Proof.

If a topological space is metrizable then it is regular by Theorem 1.4.1.

To prove that a regular second countable T_1 space is metrizable we will proceed along the following path. We show that a regular Lindelöf space is normal and that a second countable space is Lindelöf. Then we prove Urysohn's Lemma. This lemma allows us to construct certain functions that we can use to embed the space homeomorphically in a known metric space and thus conclude that the space is metrizable.

Lemma 1.5.1.

A second countable topological space is Lindelöf.

Proof.

Let X be a second countable topological space and let C be an open cover of X. We must show that C contains a countable subcover.

X has a countable base $B = \{B_1, B_2, \ldots\}$. For any $x \in X$ there exists $U \in C$ such that $x \in U$. Since U is open, there is some $B_i \in B$ such that $x \in B_i \subset U$. If B^* is the set of all such B_i, then B^* covers X. For every $B_i \in B^*$ there is at least one set $U \in C$ such that $B_i \subset U$. For each B_i choose one such set and denote it by U_i. Clearly $U_1, U_2, \ldots$ is a countable subcover of C. Therefore the space is Lindelöf.

Lemma 1.5.2 (Tychonoff).

A regular Lindelöf space is normal.

Proof.

Let A and B be disjoint closed sets in a regular Lindelof space X. By regularity we know that for every $a \in A$ there is an open set $G(a)$ such that $a \in G(a) \subset \overline{G(a)}$ and $\overline{G(a)} \cap B = \emptyset$. Consequently, the family composed of the set A^c and all the sets $G(a)$ such that $a \in A$ is an open cover of X. This cover must

contain a countable subcover $C = \{A^c, G_1, G_2, G_3, \ldots\}$. It follows that $A \subset G_1 \cup G_2 \cup G_3 \ldots$. Performing a similar construction for B, we obtain a sequence of open sets H_i, $i = 1, 2, \ldots$, such that:

$\overline{H_i} \cap A = \emptyset$ and $B \subset H_1 \cup H_2 \cup H_3 \cup \ldots$

We now define two sequences of open sets $\{U_i\}$ and $\{V_i\}$ as follows:

$U_1 = G_1$, $V_1 = H_1 \cap (\overline{U_1})^c$

and, for $n > 1$,

$$U_n = G_n \cap (\bigcup_{i=1}^{n-1} \overline{V_i})^c, \quad V_n = H_n \cap (\bigcup_{i=1}^{n} \overline{U_i})^c$$

Finally, we define the disjoint open sets G and H by:

$$G = \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad H = \bigcup_{i=1}^{\infty} V_i.$$

It is apparent that $A \subset G$ and $B \subset H$. Therefore the requirements for the normality of a space are satisfied and the lemma is proved.

Lemma 1.5.3.

The set of dyadic fractions is dense in $[0,1]$. (The dyadic fractions are the rational numbers of the form $k/2^n$ that lie in $[0,1]$, for non-negative integers k and n).

Proof.

From Definition 1.3.26 it follows that the theorem will be proved if for each $x \in (0,1)$ there is a dyadic fraction arbitrarily close to x. Evidently this is so, since x is between dyadic fractions that are arbitrarily close together.

Lemma 1.5.4. (Urysohn's Lemma).

Let F_1 and F_2 be disjoint closed subsets of a normal space X. Then there exists a continuous function $f : X \to [0,1]$ such that $f[F_1] = \{0\}$ and $f[F_2] = \{1\}$.

Proof.

Since $F_1 \cap F_2 = \emptyset$, $F_1 \subset F_2{}^c$ where F_1 is closed and $F_2{}^c$ is open. By Definition 1.3.13, there is an open set $G(1/2)$ such that $F_1 \subset G(1/2) \subset \overline{G(1/2)} \subset F_2{}^c$. Similarly, there are open sets $G(1/4)$ and $G(3/4)$ such that

$$F_1 \subset G(1/4) \subset \overline{G(1/4)} \subset G(1/2) \subset \overline{G(1/2)} \subset G(3/4) \subset \overline{G(3/4)} \subset F_2{}^c.$$

Proceeding in this way we construct, for every dyadic fraction a in $(0,1)$, a set $G(a)$ with the property that if a and b are two such dyadic fractions and $a < b$ then $\overline{G(a)} \subset G(b)$. We also define $G(1) = F_2{}^c$. We define $f : X \to [0,1]$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in F_2; \\ \inf \{t \mid x \in G(t)\}, & \text{if } x \notin F_2 \end{cases}$$

We note that $F_1 \subset G(t)$ for all dyadic fractions in $[0,1]$ and hence $f(x) = 0$ for $x \in F_1$. Therefore $f[F_1] = \{0\}$, $f[F_2] = \{1\}$ and $f : X \to [0,1]$.

The theorem is proved if we show that f is continuous. Since a function is continuous if the inverse images of sets in a subbase are open, we need only show that $f^{-1}[[0,a)]$ and $f^{-1}[(b,1]]$ are open for all $a,b \in (0,1)$.

Suppose $x \in f^{-1}[[0,a)]$. Then $f(x) \in [0,a)$, so there is some dyadic fraction t such that $x \in G(t)$ where $0 \leq f(x) < t < a$, since the dyadic fractions are dense in $[0,1]$ (Lemma 1.5.3).

Therefore $f^{-1}[[0,a)] \subset \cup\{G(t) \mid t < a\}$. (1)

Suppose $x \in \cup\{G(t) \mid t < a\}$ then $x \in G(s)$ for some $s < a$, so $f(x) = \inf\{s \mid x \in G(s)\} < a$ and $f(x) \in [0,a)$. Thus $x \in f^{-1}[[0,a)]$, and

$\cup\{G(t) \mid t < a\} \subset f^{-1}[[0,a)]$ (2)

From (1) and (2) we conclude that $f^{-1}[[0,a)] = \cup\{G(t) \mid t < a\}$; and, therefore $f^{-1}[[0,a)]$ is open.

To prove that $f^{-1}[(b,1]]$ is open we use an argument similar to the one used above to show that

$f^{-1}[(b,1]] = \cup\{\overline{G_t}^c \mid t > b\}$.

Therefore f is continuous and the lemma is proved.

Definition 1.5.1.

The Hilbert cube I is the set of all real sequences

$s = \{s_n\}$ such that $0 \leq s_n \leq 1/n$ for all n. The function $D : I \times I \to R_1$, defined for any $s,t \in I$ by

$$D(s,t) = \left[\sum_{n=1}^{\infty} | s_n - t_n |^2 \right]^{1/2}$$

is a metric on I.

Lemma 1.5.5.

If X is a normal second countable T_1 space, then there is a homeomorphism from X onto a subspace of the Hilbert cube.

Proof.

If X is finite then X is a discrete space and is homeomorphic to any subset of I that has the same number of points.

If X is infinite then consider a countable base $B = \{B_1, B_2, B_3, \ldots\}$. Clearly there are only a countable number of pairs (B_s,B_t) such that $\overline{B_s} \subset B_t$. We enumerate them (B_{s_n},B_{t_n}), $n = 1, 2, \ldots$. By Lemma 1.5.4, there is, for each of these pairs, a continuous function $f_n : X \to [0,1]$ such that $f[B_{s_n}] = \{0\}$ and $f[B_{t_n}{}^c] = \{1\}$.

Now we define $f : X \to I$ by:

$f(x) = (f_1(x)/2, f_2(x)/2^2, f_3(x)/2^3, \ldots, f_n(x)/2^n, \ldots)$

Since for every n,

$$0 \leq f_n(x) \leq 1$$

therefore $$0 \leq f_n(x)/2^n \leq 1/2^n < 1/n$$

and hence $f(x)$ is a point in I.

To show that f is an injection we need to demonstrate that distinct points x and y have distinct images $f(x)$ and $f(y)$. Let x and y be distinct points. Since X is T_1 there is a set B_i in the base such that $x \in B_i$ and $y \notin B_i$. Since X is normal and T_1 and hence regular, there is a B_j in the base such that $x \in B_j \subset \overline{B_j} \subset B_i$. Therefore (B_j, B_i) is one of the ordered pairs enumerated previously. Suppose it is the m-th ordered pair. Then $f_m[B_j] = \{0\}$ and $f_m[B_i{}^c] = \{1\}$. Thus $f_m(x) = 0$ and $f_m(y) = 1$ and $f(x) \neq f(y)$. Therefore f is an injection.

In order to prove that f is a homeomorphism it remains to be shown that both f and f^{-1} are continuous. First, we show that f is continuous.

Let $p \in X$ and $\varepsilon > 0$. There exists a positive integer m such that $\sum_{n=m+1}^{\infty} 1/2^n < \varepsilon^2/2$. Thus, for all $x \in X$

$$[D(f(x),f(p))]^2 = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(p)|^2}{2^n}$$

$$= \sum_{n=1}^{m} \frac{|f_n(x) - f_n(p)|^2}{2^n} + \sum_{n=m+1}^{\infty} \frac{|f_n(x) - f_n(p)|^2}{2^n}$$

$$\leq \sum_{n=1}^{m} \frac{|f_n(x) - f_n(p)|^2}{2^n} + \sum_{n=m+1}^{\infty} 1/2^n$$

$$< \sum_{n=1}^{m} \frac{|f_n(x) - f_n(p)|^2}{2^n} + \varepsilon^2/2.$$

For each n, $1 \leq n \leq m$, the function f_n is continuous, so there is an open neighbourhood G_n of p such that, if $x \in G_n$, $| f_n(x) - f_n(p) |^2 < \varepsilon^2/2m$.

Now, if we take $G = \bigcap_{n=1}^{m} G_n$, it follows that for any $x \in G$

$$[D(f(x), f(p))]^2 < \sum_{n=1}^{m} [(\varepsilon^2/2m)/2^n] + \varepsilon^2/2$$

$$< (\varepsilon^2/2m) + \varepsilon^2/2 = \varepsilon^2.$$

Therefore f is continuous.

Finally, to show that f^{-1} is continuous, relative to the induced topology on $f[X]$, it is sufficient to prove that, if V is an open subset of X, then $f[V]$ is open.

Suppose that V is an open subset of X and $f^{-1}(y) \in V$. Then there is an open set B_k from the countable base, such that $f^{-1}(y) \in B_k \subset V$ and an open set B_s such that

$$f^{-1}(y) \in B_s \subset \overline{B_s} \subset B_k \subset V$$

and hence the pair (B_s, B_k) has some enumeration r in the set of ordered pairs used to construct f.

Consider the open sphere

$$B = \{a \mid D(y,a)^2 < 1/2^{r+2}\}.$$

Let $M = B \cap f[X]$. Then M is open in the restricted topology on $f[X]$. Suppose $a \in M$. Then $a = f(x)$ for some $x \in X$ and so $a = (f_1(x)/2, f_2(x)/2^2, \ldots, f_r(x)/2^r, \ldots)$ by definition of f.

Also

$$D(y,a)^2 = \sum_{k=1}^{\infty} \frac{| f_k(y) - f_k(x) |^2}{2^k} < 1/2^{r+2}$$

and, in particular,

$$\frac{| f_r(y) - f_r(x) |^2}{2^r} < 1/2^{r+2}.$$

Hence, $| f_r(y) - f_r(x) |^2 < 1/2^2$. But $f_r(y) = 0$, hence, $| f_r(x) | < 1/2$, and it follows by the definition of f_r that $x \in B_k$. Therefore $f[B_s]$ is open and it follows that $f[V]$ is open. Therefore f^{-1} is continuous on $f[X]$.

By definition it follows that f is a homeomorphism of X onto a subspace of the Hilbert cube, and this concludes the proof of Lemma 1.5.5.

From the lemmas it follows that a second countable regular T_1 space X is normal and hence can be mapped homeomorphically onto a subspace of the Hilbert cube. Since the Hilbert cube is a metric space it follows that X is metrizable and Theorem 1.5.1 is proved.

We note that Example 2.4.2 (page 35) demonstrates that the regularity condition in Theorem 1.5.1 cannot be weakened to a Hausdorff condition.

1.6 The Concept of σ-structures.

Another very fruitful method of investigating problems of metrization has been to look at various metric spaces,

abstract properties that appear to be common to them, and then study the properties in a general setting to try to discover those properties that will imply metrizability.

Consider the Euclidean plane R_2 and define for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in R_2,

$$D(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Then D is a metric.

To determine one property that the metric topology on R_2 possesses we proceed as follows: let S be the class of all open spheres in R_2. For each positive integer n let

$$B_n = \{U \in S \mid \text{radius of } U \text{ is } 1/n\}$$

and let $B = \cup\{B_i\}$, $i = 1, 2, \ldots$. Then B is a base for the metric topology on R_2. This construction can be generalized to form a base for any given topological space where the topology is induced by a metric. We see that B is the union of a countable number of structures, structures that bear certain relationships to one another. For example, if $m > n$ then B_m is an open refinement of B_n.

A second property of R_2 can be demonstrated by forming sets T_m by selecting from each B_m (as defined above) all the open spheres centred at points having two rational co-ordinates. We see that each T_m contains countably many sets. It is also easy to see that $T = \cup\{T_i\}$, $i = 1, 2, \ldots,$ for a base for R_2 and, since T is a countable collection of countable classes, T is a countable base.

Another feature of the structure of R_2 is that we can reduce the B_n even further. Let M_i be the set of all open spheres of radius $1/2^{i-1}$ and centre $(m/2^{i-1}, n/2^{i-1})$ for arbitrary integers m and n. Then $M = \cup\{M_i\}$, $i = 1, 2, \ldots$, is a base and each M_i is locally finite. Theorem 2.2.1, which we shall study later, asserts that a regular T_1 space which has a base that is the union of countably many locally finite collections is metrizable. The converse of this theorem is also true. Roughly speaking, the topology of any metric space can be separated into countably many locally finite levels.

Thus the metric space R_2 possesses within itself, structures of a very specialized nature. Structures, such as these that are formed by the union of countably many, usually similar, structures, have attracted some attention from mathematicians. In topology, structures of this nature are often referred to as sigma structures or as σ-_____ where the blank is filled in with the word or words that describe the structures in the union.[2] In this thesis we propose to study some sigma structures that are of interest to topologists in their attack on problems on metrization. In Chapter II we study sigma structures that are bases. In Chapter III we concentrate on first countable spaces and, in Chapter IV on paracompactness and metrization of spaces.

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[2] For example, see Kuratowski [11], page 234.

Definition 1.6.1.

A σ-structure (sigma structure) is the union of countably many structures. If the structures are of the same nature the word "structure" is often replaced by words that in some way describe these structures and/or the σ-structure itself. We assume that the structures used in forming the σ-structure are put into one-to-one correspondence with the positive integers and we denote the structure corresponding to i as the i-th level in the σ-structure.

Chapter II

Sigma Bases

2.1 Introduction.

Definition 2.1.1.

A σ-base (sigma-base) for a topological space (X,τ) is a σ-structure B that is a base for τ on X. That is, $B = \cup\{B_i\}$ $i = 1, 2, \ldots$ where B_i is the i-th level of B. Also, we have the conventional understanding that in the term "σ-_____ base" the word(s) in the blank describes the levels of the σ-structure.

Every topological space has a σ-base: to display one for the space (X,τ) we set $B_i = \tau$ for $i = 1, 2, \ldots$ and $B = \cup\{B_i\}$. Consequently, to show that a particular topological space has a σ-base tells us little. However, to know that a topological space has a σ-base with various properties will be useful in determining the structure of the space.

In this chapter we study various cardinality restrictions on the levels of σ-bases. The line of attack is to consider local cardinality restrictions and global cardinality restrictions. If every point of a topological space X has at least one open neighbourhood that satisfies a particular property, say P, then P is called a local property of the

space. If the open set X satisfies P then P is called a global property.

To dispense with global cardinality restrictions on the levels is easy. We let (X,τ) be a topological space and B a base for τ on X. In the case that B is a finite base, the structure of the space is easily determined: we show later (page 32) that a T_1 space with a point finite base possesses a discrete topology, that is, every subset is open. In the case that B is a σ-finite base then, clearly, B is a σ-countable base, and, if B is a σ-countable base then B is a countable base. On the other hand, if B is a countable base, then it is easy to show that B is a σ-countable base: we note that $B = \cup\{B_i\}$ such that $B_i = B$ for $i = 1, 2, \ldots$. Also, using the method of Example 2.3.1 (page 35), we can construct a σ-finite base from any countable base. Therefore we arrive at the following equivalences for bases of τ on X:

countable $\leftrightarrow$ σ-countable $\leftrightarrow$ σ-finite.

Consequently, the problem of the metrizability of spaces with any of these bases, is solved in full by Urysohn's Metrization Theorem (Theorem 1.5.1).

2.2 Local Cardinality Restrictions of a Finite Nature on the Levels.

In the early 1950's a breakthrough occurred in the theory of metrization of topological spaces. In 1950, Nagata [17] published a proof of the first satisfactory solution of the problem

of metrizability of a space. Shortly thereafter, Smirnov [19] published his proof of the same theorem, which we denote Theorem 2.2.1. In 1951, Bing [3] proved an equivalent theorem, our Theorem 2.2.3.

Theorem 2.2.1 (Nagata-Smirnov Metrization Theorem)

A topological space is metrizable if and only if it is regular and T_1 with a σ-locally finite base.

Proof.

Suppose (X,D) is a metric space. Then according to Stone [22], X is paracompact. Therefore, every open cover of X has a locally finite open refinement. For every positive integer i, let R_i be the set of all open spheres of radius $1/i$. Then R_i is an open cover of X and has a locally finite open refinement L_i. We claim that L_i is the i-th level for a σ-locally finite base. Clearly the L_i are locally finite and we need only show that $B = \cup\{L_i\}$ is a base. If $x \in X$ and U is an open set containing x then there is an n such that the sphere of radius $1/n$ centred at x is contained in U. Since L_i covers X for every i then L_{2n} covers X and there is a set in L_{2n} that contains x, say $S(y,2n)$. This set must lie within the sphere of radius $1/n$ centred at x, as otherwise there is a point p that is in $S(y,2n)$ and not in $S(x,n)$. And then, since p is not in $S(x,n)$, $D(x,p) \geq 1/n$. But since x

and p are both in $S(y,2n)$, we have

$$D(x,p) \leq D(x,y) + D(y,p) < 1/2n + 1/2n = 1/n.$$

Thus $x \in S(y,2n) \subset S(x,n) \subset U$, and therefore B is a base, and hence a σ-locally finite base.

We shall not prove here that a regular T_1 space with a σ-locally finite base is metrizable, but refer, for example, to Nagata [17] or Smirnov [19]. Rather we shall sketch a proof which is modelled upon the proof of Urysohn's Metrization Theorem. First perfect normality is proved, and then the space is embedded in the generalized Hilbert space H^{τ}, where τ is the cardinality of the σ-locally finite base.

We see that Theorem 2.2.1 is more general than Theorem 1.5.1. In fact, the proof that a regular space with a countable base is metrizable follows directly from Theorem 2.2.1. Theorem 2.2.1 and Theorem 2.2.3 (page 30) brought to a satisfactory conclusion decades of research on the metrizability of topological spaces. We note that Example 2.4.2 demonstrates that regular cannot be replaced by Hausdorff in Theorem 2.2.1.

Theorem 2.2.2.

A discrete collection of sets in a topological space is locally finite.

Proof.

Let X be a topological space and D a discrete collection of subsets of X. Let $x \in X$. Then we have two cases to consider.

Case i.

$x \notin \overline{\cup\{A \mid A \in D\}}$

If $x \notin \overline{\cup\{A \mid A \in D\}}$ then x is in the open set $(\overline{\cup\{A \mid A \in D\}})^c$ that meets no members of D.

Case ii.

$x \in \overline{\cup\{A \mid A \in D\}}$

If $x \in \overline{\cup\{A \mid A \in D\}}$ then, since by the definition of discrete it follows that $\overline{\cup\{A \mid A \in D\}} = \cup\{\overline{A} \mid A \in D\}$, $x \in \cup\{\overline{A} \mid A \in D\}$; and, $x \in \overline{A}$ for some $A \in D$.

Since $\overline{\cup\{B \mid B \neq A, B \in D\}} = \cup\{\overline{B} \mid B \neq A, B \in D\}$ and since $\overline{A} \cap \overline{B} = \emptyset$ for all $B \neq A$, such that $B \in D$, therefore $x \notin \overline{\cup\{B \mid B \neq A, B \in D\}}$. Consequently x is in the open set $(\overline{\cup\{B \mid B \neq A, B \in D\}})^c$ that meets only one member of D, namely A.

Therefore any point $x \in X$ is in an open set U that meets at most finitely members of D, and, thus, D is a locally finite collection of sets.

Theorem 2.2.3 (Bing [3]).

A topological space is metrizable if and only if it is regular and T_1 and has a σ-discrete base.

Proof.

Let X be a regular T_1 space with a σ-discrete base $B = \cup\{B_i\}$, $i = 1, 2, \ldots$. Then, by Theorem 2.2.2, B is a

σ-locally finite base and by Theorem 2.2.1, X is metrizable.

Let X be a metric space. In [2] Stone shows that for every positive ε, X has a cover $C = \cup\{ R_{\varepsilon i} \}$, $i = 1, 2, \ldots$, such that for each i, $R_{\varepsilon i}$ is a discrete collection of closed sets each of diameter less than ε.

For each i we will derive from $R_{\varepsilon i}$ a discrete collection of open sets $S_{\varepsilon i}$, such that $R_{\varepsilon i}$ is a refinement of $S_{\varepsilon i}$, and the diameter of each member of $S_{\varepsilon i}$ is less than ε.

Let $E \in R_{\varepsilon i}$ and let

$$D = \overline{\cup\{A \mid A \in R_{\varepsilon i},\ A \neq E\}}.$$

Then $D = \cup\{\overline{A} \mid A \in R_{\varepsilon i},\ A \neq E\}$

$= \cup\{A \mid A \in R_{\varepsilon i},\ A \neq E\}$.

Since $E \cap A = \emptyset$ for all $A \in R_{\varepsilon i}$ such that $A \neq E$. Therefore $E \cap D = \emptyset$.

For every $x \in E$ the distance from x to D is a positive number, say $d(x)$; as otherwise, $x \in \overline{D} = D$. We let $d(E)$ be the diameter of E, and note that $0 \leq d(E) < \varepsilon$. For every $x \in E$ we let $r(x) = \min \{d(x)/4, (\varepsilon - d(E))/3\}$. And we let $U(E) = \cup\{S(x,r(x)) \mid x \in E\}$. Then $S_{\varepsilon i} = \{U(E) \mid E \in R_{\varepsilon i}\}$ is a discrete collection of open sets each of which is of diameter less than ε.

Finally, the members of $\{ S_{\varepsilon i} \}$, where $\varepsilon = 1/k$, and i and k are arbitrary positive integers, can be used to form

levels in a σ-discrete base and the theorem is proved.

Example 2.4.2 (page 35) demonstrates that the regularity condition cannot be removed in Theorem 2.2.3.

If X is a T_1 space with a point finite base then for any $x \in X, \{x\}$ is the intersection of finitely many open sets and consequently is open. Therefore every subset of X is open and X possesses the discrete topology.

One may attempt to generalize Theorem 2.2.1 by considering regular T_1 spaces with σ-point finite bases. This attempt fails, since Example 2.4.4 (page 37) is a non-metrizable T_1 space that is regular and has a σ-point finite base.

2.3 Local Cardinality Restrictions of a Countable Nature on the Levels.

If a topological space X has a σ-point countable base then clearly, it must have a point countable base. Also, the converse implication obviously holds. For spaces with point countable bases, we have the following metrization theorems. For proofs, the reader is referred to Mischenko [16] and Stone [21].

Theorem 2.3.1 (Mischenko).

A compact Hausdorff space with a point countable base is metrizable.

Theorem 2.3.2 (Stone).

A regular T_1 space X that is the union of a point countable family of open sets S_α each of which is locally separable and metrizable, is metrizable (and locally separable).

We wish to show that X has a point countable base.

Since every $A \in S_\alpha$ is metrizable and hence paracompact, it follows that the collection of open spheres of radius $1/n$ has a locally finite, and hence, point finite open refinement. Thus each A has a point countable base and the union of all these point countable bases will be a point countable base for X since S_α is point countable.

Little is known about spaces with σ-locally countable bases. Clearly, a space with a locally countable base has a σ-locally countable base, and a space with a σ-locally countable base has a point countable base. Example 2.4.4 (page 37) displays a space with a point countable base but with no σ-locally countable base. Example 2.4.3 (page 36) describes a space with a σ-locally finite base, and hence a σ-locally countable base, but no locally countable base.

For locally second countable spaces we do have the following theorem due to Aleksandrov [1] which gives a result concerning the metrization of spaces with locally countable bases.

Theorem 2.3.3 (Aleksandrov).

For the metrizability of a regular T_1 space that is locally second countable, each of the following conditions is necessary and sufficient:

(1) paracompactness;

(2) strong paracompactness;

(3) the existence of a point countable base;

(4) the existence of a locally countable base;

(5) the existence of a star countable base.

Since a space with a locally countable base, clearly, must be locally second countable, Theorem 2.3.3 yields the following metrization theorem.

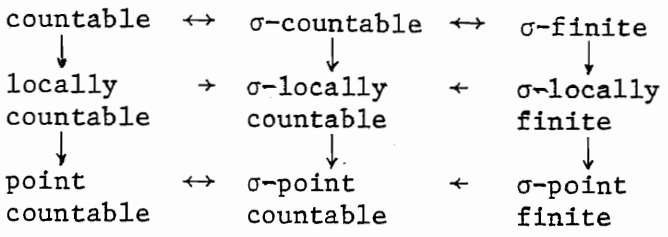
Theorem 2.3.4.

A regular T_1 space with a locally countable base is metrizable.

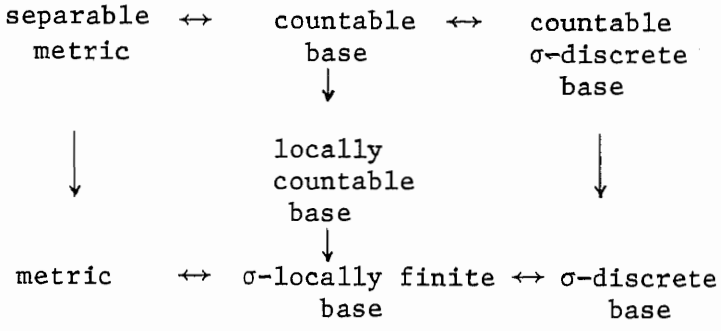
The converse implication, that every metric space has a locally countable base, is false, as is pointed out in Example 2.4.3 (page 36).

Before proceeding to the examples in section 2.4 we summarize some of the conclusions of this chapter:

If (X,τ) is a topological space we arrive at the following implication diagram concerning bases for τ on X.



If X is a regular T_1 space then we have the following implications concerning metric spaces and metrization:



2.4 Examples.

Example 2.4.1 A σ-Finite Base for R_1.

Consider R_1 with the usual topology. Clearly, the set of all open intervals with rational endpoints is a base for this topology. Therefore if we enumerate this countable set and take L_i to be the i th set in the enumeration, then $B = \cup\{L_i\}, i = 1, 2, \ldots,$ is a σ-finite base for R_1. Clearly, this construction can be used to form a σ-finite base for any space with a countable base.

Example 2.4.2 A Non-Regular Hausdorff Space with a Countable Base.

Let X be the open unit sphere in R_2.

If x is in X and x is not the origin then, as a local base at x we use open subsets of X (open in the usual topology of R_2) that contain x but not the origin. We construct a neighbourhood system about the origin in the following manner. From every open sphere with radius less than or equal to one, centred about the origin we delete all the points, except the origin, that are on or above the x-axis: the collection of the above sets is a base for a topology on X.

X, with this topology, is a Hausdorff space since it is evident that any pair of points can be separated by disjoint open sets.

X is not regular: denote the set of points (the origin excluded) that are on or above the x-axis by A. Clearly $X - A$ is open. Hence A is closed. If we attempt to separate A and the origin by disjoint open sets we see that any open set, U, containing A must contain at least one point directly below every point, except the origin, on the x-axis in X. And every open set containing the origin therefore must contain a point of U.

Clearly, X has a countable base, say $B = \{B_i\}$, $i = 1, 2, \ldots$. The σ-base formed by putting only the set B_i in the i th level is a σ-locally finite base and also a σ-discrete base.

Example 2.4.3 A Metric Space with No Countable Base.

Consider the open unit sphere U in the plane. We define a

distance function as follows: if a and b are on the same radius of U then the distance from a to b is the usual Euclidean distance. If a and b are on different radii, then the distance from a to b is the Euclidean distance from a to the origin plus the Euclidean distance from the origin to b. It is easy to show that this distance function is a metric and therefore gives rise to a metric topology on U. Since the space is metrizable we conclude, using Theorem 2.2.1, that the space has a σ-locally finite base. If B is any base for the space then any element U of B containing the origin will contain uncountably many points of the space.

It is clear then because of the structure of the space that U must contain uncountably many elements of B. Therefore it is impossible for the space to have a locally countable base. It follows from this that the space does not have a countable base.

Example 2.4.4 A Non-Metrizable Paracompact Space with a Point Countable Base.[1]

Let X be the set of all points in the open unit square as well as those points of the forms $(r,0)$ or $(r,1)$ where r is any rational between 0 and 1. We order the points of X lexicographically, that is, $(x_1,x_2) < (y_1,y_2)$ if $x_1 < y_1$ or if $x_1 = y_1$ and $x_2 < y_2$. We define a base for a topology on X

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[1] The author is grateful to Professor S. Willard for communicating this example to him.

as follows:

If $x, y, z \in X$ and $x < y < z$, under the defined order relation, then $\{u \mid u \in X \text{ and } x < u < z\}$ is an open set containing y.

If $Y = [0,1] \times [0,1]$ is given the order topology used above then Y is a compact Hausdorff space, and is therefore regular. Since the properties of being Hausdorff and regular are hereditary, and since X is a subspace of Y, therefore X is a regular Hausdorff space.

To show that X is paracompact we let C be any open cover of X and $B = \{x = (x_1, x_2) \mid x \in X, x_2 = 0 \text{ or } x_2 = 1\}$. Then, since B is countable, $B = \{x_i\}$, $i = 1, 2, \ldots$. For some $U_1 \in C$, $x_1 \in U_1$, and there is some open set V_1 such that $x_1 \in V_1 \subset U_1$ where $V_1 = \{x \in X \mid (r_1, r_2) < x < (r_3, r_4)$ and r_1, r_2, r_3 and r_4 are rationals between 0 and 1$\}$. If $x_2 \notin V_1$ then $x_2 \notin \overline{V_1}$ and if $x_2 \in U_2 \in C$ then there is a V_2 of the same form as V_1 such that $x_2 \in V_2 \subset U_2$ and $\overline{V_2} \cap \overline{V_1} = \emptyset$. If $x_2 \in V_1$ then we define $V_2 = V_1$. Proceeding in this manner we define a V_i for each x_i. Therefore $V = \{V_i\}$, $i = 1, 2, \ldots$ is a pairwise disjoint countable cover of B. If $W = \cup\{V_i\}$ and if $x \notin W$ then, clearly x has a neighbourhood meeting at most two elements of V. If $x \in W$ then x is in only one element of V. Therefore V is a locally finite family of subsets of X.

$X - W$ is a collection of disjoint metric, and therefore paracompact, spaces. It is evident that $X - W$ has a cover C' such that:

(1) C' is a locally finite collection of open subsets of X;

(2) each element of C' is contained in some element of C.

We conclude that $V \cup C'$ is a locally finite refinement of C, and, since X is Hausdorff, it follows that X is paracompact.

X has a point countable base. We define B_1 to be the set of all open intervals of X with end points (x_1, x_2) and (y_1, y_2) such that x_1, x_2, y_1 and y_2 are rationals. We define B_2 to be the set of all open intervals of X with end points (x_1, x_2) and (x_1, x_3) such that x_1 is irrational and x_2 and x_3 are rational. It follows that $B = B_1 \cup B_2$ is a point countable base. Also B_2 is clearly the union of countably many disjoint collections of open sets and therefore by suitable selection of levels from subsets of B a σ-point finite base can be formed.

Finally X is not metrizable. If X were metrizable then every closed set would be a G_δ, (Theorem 1.4.1 (page 11)), and $A = \{(x_1, x_2) \mid 0 < x_1 < 1, x_1 \text{ is rational}, x_2 = 0 \text{ or } x_2 = 1\}$ would be a G_δ. But if this were true it could be shown that

the set of rational points in $(0,1)$ is a G_δ in R_1 with respect to the usual topology. This is not true and therefore X is not metrizable.

Chapter III

First Countable Spaces

3.1 Introduction.

In Theorem 1.4.1, it was pointed out that all metric spaces are first countable. In this chapter various first countable spaces are studied by investigating σ-bases that give rise to their topologies. We begin by introducing a σ-base characterization of first countable spaces.

Theorem 3.1.1.

A topological space X is first countable if and only it has a σ-base such that:

(1) each level covers X;

(2) for every $x \in X$ there exists some selection of sets, one taken from each level, that is a local base at x.

Proof.

Assume that X is a first countable space. Then for each point $x \in X$ there exists a countable collection $\{g(n,x)\}$, $n = 1, 2, \ldots,$ that is a local base at x. We form levels for a σ-base B by setting $B_i = \{g(i,x) \mid x \in X\}$. Clearly, $B = \cup\{B_i\}$, $i = 1, 2, \ldots,$ is a σ-base that satisfies the required conditions.

Assume that X has a σ-base that satisfies the given

Chapter III

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3.1 Introduction.

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(2) for every $x \in X$ there exists some selection of sets, one taken from each level, that is a local base at x.

Proof.

Assume that X is a first countable space. Then for each point $x \in X$ there exists a countable collection $\{g(n,x)\}$, $n = 1, 2, \ldots,$ that is a local base at x. We form levels for a σ-base B by setting $B_i = \{g(i,x) \mid x \in X\}$. Clearly, $B = \cup\{B_i\}$, $i = 1, 2, \ldots,$ is a σ-base that satisfies the required conditions.

Assume that X has a σ-base that satisfies the given

conditions. Then, clearly, each point x has a countable local base and the theorem is proved.

3.2 Semi-metric Spaces.

One method of studying metric spaces is to vary the definition of the distance function on a space. Such a method was used to derive a class of spaces called semi-metric spaces by K. Menger (see [18] for pertinent references). These spaces were intensively studied in the 1920's and 1930's by Frechet [7], (he called them E spaces), Wilson [25], and Chittenden [5]. More recently, interest in semi-metric spaces and their relationships to metric spaces, has been renewed, primarily as a result of the efforts of F. B. Jones and his students (see for example Seminar on Semi-metric Spaces in [2]).

Definition 3.2.1.

A semi-metric on a set X is a real valued function $D : X \times X \to R_1$ such that for any $x, y \in X$:

(1) $D(x,y) \geq 0$;

(2) $D(x,y) = 0$ if and only if $x = y$;

(3) $D(x,y) = D(y,x)$.

In the original definition, as formulated by K. Menger, a semi-metric space was a set X with a semi-metric D, however such a space is not necessarily a topological space, and consequently, the definition has been modified to pertain to topological spaces.

Definition 3.2.2.

A topological space X is a semi-metric space if there exists a semi-metric D on X such that for every $A \subset X$,

$$\overline{A} = \{x \mid \inf\{d(x,y) \mid y \in A = 0\}\}.$$

There are several characterizations of semi-metric spaces. For example, see Heath [8] and Ceder [4]. In the next theorem we develop a σ-base characterization for semi-metric spaces, that will be useful in proving some of the theorems that follow.

Theorem 3.2.1.

A T_1 space X is semi-metric if and only if it has a σ-base $B = \cup\{B_i\}$ such that:

(1) for each i and for each $x \in X$ there is a particular element $b(i,x)$ of B_i containing x;

(2) $\{b(i,x)\}$, $i = 1, 2, \ldots,$ is a local base at x;

(3) for every $x \in X$ and every open set U containing x, there is an n such that, if $m \geq n$ and $y \in U^c$, $x \notin b(m,y)$.

Proof.

Suppose X is a semi-metric space with a semi-metric D. We define for every $x \in X$ and for each positive integer i,

$$c(i,x) = \{y \mid D(y,x) < 1/i\} \text{ and } b(i,x) = (c(i,x))^{\circ}.$$

We now show that the σ-structure B formed by taking $B_i = \{b(i,x) \mid x \in X\}$ as the i th level satisfies the requirements of the theorem.

(1) $x \in b(i,x)$.

Consider the set $c(i,x)^c$. For every $y \in c(i,x)^c$, $D(y,x) \geq 1/i$. Therefore $\inf\{D(y,x) \mid y \in c(i,x)^c\} \geq 1/i > 0$. By Definition 3.2.1 $x \notin \overline{c(i,x)^c}$ and there is some open subset of $c(i,x)$ containing x. Therefore $x \in c(i,x)^\circ = b(i,x)$.

(2) $\{b(i,x)\}$ is a local base at x.

Let U be an open set containing x. Then it follows from part 4 of Definition 3.2.1 that for some i, $c(i,x) \subset U$. By (1) above, $x \in b(i,x) \subset c(i,x) \subset U$. Therefore $\{b(i,x)\}$ is a local base at x.

(3) Given $x \in X$ and an open set U containing x, there is an n such that, if $m \geq n$ and $y \in U^c$ then $x \notin b(m,y)$. If for each i, $i = 1, 2, \ldots$, there is some $y_i \in U^c$ such that $x \in b(i,y_i)$, then $\inf\{D(x,y) \mid y \in U^c\} = 0$ and, by part 4 of Definition 3.2.1, $x \in \overline{U^c}$. This contradiction yields the conclusion that, for some k, $x \notin b(k,y)$ for all $y \in U^c$. Clearly if $m \geq k$ the conclusion also holds.

Finally, from (2) above, we see that the σ-structure B is a σ-base, and therefore the first part of the proof is complete.

Suppose, on the other hand, that a T_1 space X has a σ-base that satisfies conditions (1) - (3) in the statement of the theorem. Then we define the function $D : X \times X \to R_1$ as follows. If $x,y \in X$ then $D(x,y) = \inf\{1/i \mid x \in b(i,y) \text{ or } y \in b(i,x)\}$.

It is easy to show that D satisfies Definition 3.2.2. Therefore X is a semi-metric space and the theorem is proved.

It follows from condition (2) in Theorem 3.2.1 that every semi-metric space is a first countable space. The non-metrizable space of Example 2.4.4 (page 37) is first countable. It is also paracompact and has a point countable base. It cannot be a semi-metric space since Theorems 3.4.3 and 3.4.2 would then imply that the space is metrizable. Therefore we draw the conclusion that first countable spaces are not necessarily semi-metric spaces.

3.3 Developable Spaces.

Another result of the search for characterizations of metric spaces which was undertaken during the early part of this century, was the development of the theory of Moore spaces, named after R. L. Moore. For an account of this work the reader is referred to Jones [9]. In this section we study a class of spaces, called developable, that includes as a sub-class the family of Moore spaces.

Definition 3.3.1.

A topological space X is <u>developable</u> if there is a sequence $G_1, G_2, \ldots$ of open covers of X such that if $x \in X$ and U is an open set containing x then there is an n such that every member of G_n containing x is a subset of U. The sequence

G_1, G_2, ... is called a development for X.

Definition 3.3.2.

A topological space X is a Moore space if it is regular, T_1, and developable.

Theorem 3.3.1.

A topological space X is developable if and only if it has a development which is a σ-base such that:

(1) each level covers X;

(2) for any $x \in X$, any selection of sets containing x, where one set is chosen from each of the levels, is a local base for the topology at x.

(The reader is invited to compare the statement of this theorem with the statement of Theorem 3.1.1 and with the definition of uniform base, Definition 1.3.30.)

Proof.

Suppose X is developable and that G_1, G_2, ... is a development for X.

Let x be any point of X and let $\{U_i\}$, $i = 1, 2, \ldots$, be any collection of sets containing x such that $U_i \in G_i$ for each i. Let V be any open set containing x. By definition there exists n such that every element of G_n that contains x is a subset of V. Therefore $x \in U_n \subset V$ and it follows that $B = \cup\{G_i\}$, $i = 1, 2, \ldots$, is a σ-base that satisfies the requirement of the theorem.

On the other hand, suppose that $B = \cup\{B_i\}$ is a σ-base that satisfies the hypothesis of the theorem. Then to show that $B_1, B_2, \ldots$ is a development for the topology on X, let U be any open subset of X and let x be any point in U. Then we wish to show that for some i, every member of B_i that contains x is a subset of U. Assume that for every j there is an element of B_j that contains x and meets U^c. Clearly the collection of these element cannot be a local base at x. And therefore there exists j such that every member of B_j containing x is contained in U. Consequently, $B_1, B_2, \ldots$ is a development for the topology on X and the theorem is proved.

Theorem 3.3.2.

A developable T_1 space is a semi-metric space.

Proof.

Let X be a developable T_1 space and let $G_1, G_2, \ldots$ be a development for X. For every $x \in X$ and $i = 1, 2, \ldots,$ we can choose $g(i,x)$ such that $x \in g(i,x) \in G_i$.

We define:

$$b(i,x) = \bigcap_{j=i}^{i} g(j,x)$$

$B_i = \{b(i,x) \mid x \in X\}$, and

$B = \cup\{B_i\}$, $i = 1, 2, \ldots$.

Let U be an open subset of X and $x \in U$. It follows that,

if $y \notin U$ then $x \notin g(k,y)$ and, clearly, $x \notin b(j,y)$ for $j \geq k$. Therefore by Theorem 3.2.1 X is a semi-metric space.

From the preceeding theorem we see that every T_1 developable space is semi-metric. The converse implication, that every semi-metric space is developable, is false, (see Example 3.5.1).

3.4 Metrizability of First Countable Spaces.

In this section we discuss some of the ways in which semi-metric spaces, developable spaces and metric spaces are related.

Theorem 3.4.1.

A metric space is a developable space.

Proof.

Let (X,D) be a metric space. We recall that $S(x,\varepsilon)$ is the open sphere of radius ε, centred at x. For each positive integer i we define $G_i = \{S(x,1/i) \mid x \in X\}$. Clearly, $G_1, G_2, \ldots$ is a development and the theorem is proved.

Corollary.

A metric space is a Moore space.

Example 3.5.2 demonstrates that there are Moore spaces that are not metrizable.

Theorem 3.4.2 (Bing [3]).

A topological space is metrizable if and only if it is developable and paracompact.

Proof.

If a topological space X is metrizable then by Theorem 3.4.1 it is developable. Also, by Theorem 1.4.1, it is paracompact.

Let X be developable and paracompact. Since a paracompact space is normal and Hausdorff (Dieudonné [6]), X is regular and T_1, and we need only show that X has a σ-locally finite base in order to use the Nagata-Smirnov metrization theorem (page 28).

Let $G_1, G_2, \ldots$ be a development for X. For each i, let B_i be a locally finite open refinement of G_i.

Let U be open and $x \in U$. Then for some j, every element of G_j containing x is contained in U. Since B_j refines G_j, there is some V in B_j and some W in G_j such that $x \in V \subset W$. But $W \subset U$, therefore $x \in V \subset U$ and so $B = \cup\{B_i\}$, $1, 2, \ldots$, is a σ-base, and a σ-locally finite base. Therefore X is metrizable, and the theorem is proved.

Theorem 3.4.3 (Heath [8]).

A semi-metric space with a point countable base is developable.

Proof.

Let X be a semi-metric space with a σ-base B, as in Theorem 3.2.1; and a point countable base C.

For each $x \in X$ we let $c(1,x), c(2,x), \ldots$ be a simple ordering of the elements of C containing x, and, for each n,

we let $h(n,x) = b(n,x) \cap c(n,x)$. We well order X, and, for each $x \in X$ and each positive integer n we let $y[n,x]$ be the least $z \in X$ such that $x \in h(n,z)$. Finally, for each $x \in X$ and each n we let

$$g(n,x) = b(n,x) \cap (\bigcap\{h(i,y[i,x]) \mid i \leq n\})$$
$$\cap (\bigcap\{c(j,y[i,x]) \mid x \in c(j,y[i,x]),\ j \leq n,\ i \leq n\}).$$

We wish to show that the σ-structure whose n-th level is $G_n = \{g(n,x) \mid x \in X\}$ satisfies Theorem 3.3.1.

If x is any point in X and $\{x_i\}$, $i = 1, 2, \ldots,$ is any sequence of points of X, such that for each i, $x \in g(i,x_i)$ then the theorem will be proved if we show that $\{g(i,x_i)\}$, $i = 1, 2, \ldots,$ is a local base at X.

If U is any open set containing x then there exists m such that $c(m,x) \subset U$. Also, by Theorem 3.2.1 there exists n such that if $i \geq n$ and $x \in b(i,y)$ then $y \in c(m,x)$. Therefore, in particular, if $i \geq n$ then $x_i \in c(m,x)$. Also, if $z = y[n,x]$ then $z \in c(m,x)$ and it follows that for some p, $c(p,z) = c(m,x)$.

Since $x \in c(p,z) \cap h(n,z)$ by Theorem 3.2.1 there exists q such that if $i \geq q$ then $x_i \in c(p,z) \cap h(n,z)$. Therefore, for all $i \geq q$, $y[n,x_i] \prec z$ with respect to the well ordering of X. But, for all $i \geq n$, $x \in g(i,x_i) \subset h(n,y[n,x_i])$, so that $z = y[n,x] \prec y[n,x_i]$. Therefore, for all $i \geq n + q$, $y[n,x_i] = z$, so that for $i \geq n + q + p$, $g(i,x_i) \subset c(p,y[n,x_i]) = c(p,z) = c(m,x) \subset U$. Therefore,

$\{g(i,x_i)\}$, $i = 1, 2, \ldots$ is a local base at x and the theorem is proved.

Corollary.

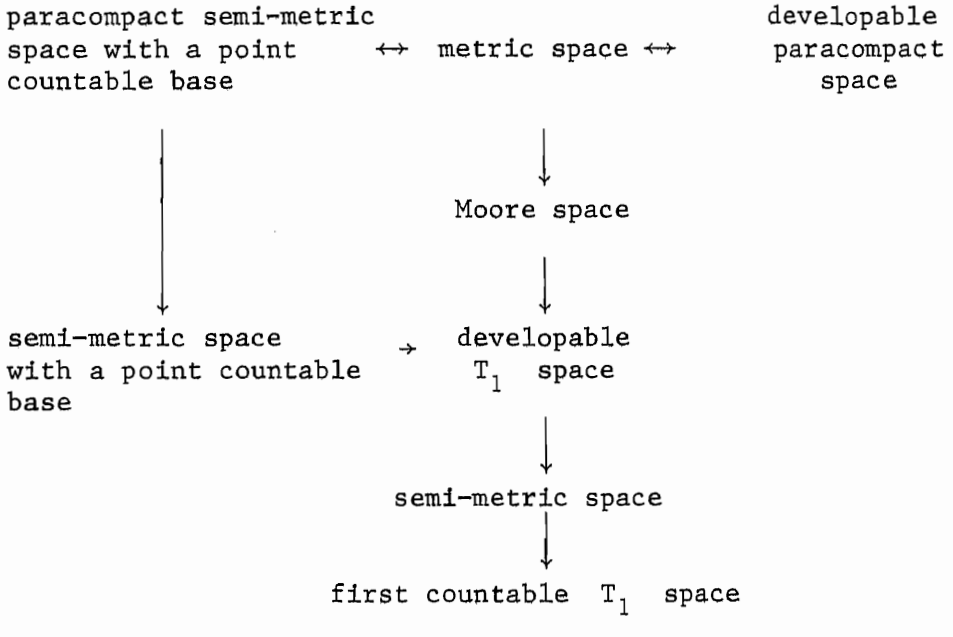
A topological space X is metrizable if and only if it is a paracompact, semi-metric space with a point countable base.

Proof.

Let X be metrizable. Then, by Theorem 1.4.1, it is paracompact, by Theorem 3.4.1 it is developable and by Theorem 2.2.1 it has a σ-locally finite base which is clearly a point countable base.

Let X be a paracompact, semi-metric space with a point countable base. Then by Theorem 3.4.3, X is developable and therefore by Theorem 3.4.2, X is metrizable.

The conclusions of this chapter are summarized by the following implication diagram for topological spaces.



3.5 Examples.

Example 3.5.1 A Semi-metric Space that is not Developable.

This example is due to L. McAuley [12]. Consider R_2 with the usual metric d. We let X be the set of points in R_2, and Y the set of points along the x-axis of R_2.

If $x \in X - Y$ and k is the first positive integer such that $S(x, 1/k) \cap Y = \emptyset$ then we define for each positive integer n,

$$b(n,x) = \begin{cases} \overline{S}(x,1/k) & \text{if } n \leq k \\ S(x,1/n) & \text{if } n > k. \end{cases}$$

If $x = (x_1,0) \in Y$ and n is any positive integer then, $b(n,x) = \{x\} \cup \{y = (y_1, y_2) \mid y \in X, y_1 \neq x_1$ and $d(y,x) + |y_2 / (y_1 - x_1)| < 1/n\}$.

If we define $B_i = \{b(i,x) \mid x \in X\}$ then it is evident that $B = \cup\{B_i\}, i = 1, 2, \ldots,$ is a σ-base for a topology on X. We also see that the conditions of Theorem 3.2.1 are satisfied and that therefore the space is semi-metric.

A straight forward calculation shows that this space is regular. Also, the space is separable, since $R = \{x = (x_1, x_2) \mid x \in X - Y$ and x_1, x_2 are rational$\}$ is a countable dense subset. A separable metric space must have a countable base. Therefore, since the topology on X cannot have a countable base, X is not metrizable.

To show that X is not developable we proceed as follows. For each positive integer n we define

$$G_n = \{b(n,x) \mid x \in Y \text{ or } x = (x_1,x_2) \text{ and } |x_2| \geq 2/n\}.$$

Clearly, $G = \cup\{G_i\}$, $i = 1, 2, \ldots,$ is a σ-closure preserving base for X. Therefore, since X is regular and T_1, X is paracompact and perfectly normal (Definition 4.2.1 (page 55), Theorem 4.2.3 (page 58), Corollary to Theorem 4.2.4 (page 58), and Theorem 4.2.5 (page 58)). Therefore X cannot be developable, since Theorem 3.4.2 (page 48) tells us that a developable paracompact space is metrizable.

Example 3.5.2 A non-metrizable Moore space.

We modify example 3.5.1 by defining for each $x = (x_1,0) \in Y$ and each positive integer n, $b(n,x) = S((x_1, 1/n), 1/n \cup \{x\}$. If $x \in X - Y$, then $b(n,x)$ is as defined in Example 3.5.1. If $B_i = \{b(i,x) \mid x \in X\}$ then $B = \cup \{B_i\}$, $i = 1, 2, \ldots,$ is a σ-base for a topology on X. Clearly, the conditions of Theorem 3.3.1 are satisfied and therefore X is a developable space.

X is a regular Hausdorff space and therefore it is a Moore space. X is separable, but not second countable and therefore not metrizable.

Chapter IV

Paracompactness and Metrization.

4.1 Paracompactness and σ-structures.

In 1944, J. Dieudonné [6] introduced the notion of a paracompact space: a Hausdorff space for which every open cover has a locally finite open refinement. Dieudonné also gave some of the topological properties of paracompact spaces. In particular, he showed that paracompact spaces are normal.

Beginning in 1953, E. Michael published a series of papers [13, 14, 15] in which he gave several characterizations of paracompact spaces. Several of Michael's theorems are stated below.

Theorem 4.1.1.

A regular T_1 space X is paracompact if and only if every open cover of X has a σ-locally finite open refinement. (That is, an open refinement $U = \cup\{U_i\}$, $i = 1, 2, \ldots$, such that each U_i is a locally finite collection of sets.)

Theorem 4.1.2.

A regular T_1 topological space X is paracompact if and only if every open cover of X has a σ-closure preserving open refinement.

Definition 4.1.1.

If U and V are collections of subsets of a topological space X then U is cushioned in V, if for every $G \in U$ there is $H(G) \in V$ such that, if U^* is any subset of U,

$$\overline{\cup\{G \mid G \in U^*\}} \subset \cup\{H(G) \mid G \in U^*\}.$$

Definition 4.1.2.

If V is a cover of a topological space X, and $U = \cup\{U_i\}$, $i = 1, 2, \ldots$, is a refinement of V such that U_i is cushioned in V for all i, then U is a σ-cushioned refinement of V.

Theorem 4.1.3.

A T_1 space X is paracompact if and only if every open cover of X has an open σ-cushioned refinement.

4.2 M_i-spaces.

The Nagata-Smirnov Theorem shows that regular T_1 spaces with σ-locally finite bases are metrizable. This fact and Michael's Theorems of the previous section led J. Ceder to the study of spaces with σ-bases that have the properties of the σ-refinements in Michael's Theorems. Ceder calls these spaces M_i spaces, and studies them in [4].

Definition 4.2.1.

An M_1 space is a regular T_1 space that has a σ-closure preserving base.

Definition 4.2.2.

A collection B of subsets of X is a quasi-base for the topology on X if, for any open set U and $x \in U$, there exists $V \in B$ such that $x \in V^{\circ} \subset V \subset U$.

Definition 4.2.3.

An M_2 space is a regular T_1 space that has a σ-closure preserving quasi-base.

Definition 4.2.4.

Let B be a collection of ordered pairs $P = (P_1, P_2)$ of subsets of a topological space X, such that $P_1 \subset P_2$ for all $P \in B$. Then B is called a pair-base for the topology on X if P_1 is open for all $P \in B$ and if, for any $x \in X$ and open neighbourhood U of x, there exists a $P \in B$ such that $x \in P_1 \subset P_2 \subset U$. B is called cushioned if for every $B^* \subset B$,

$$\overline{\cup\{P_1 \mid P \in B^*\}} \subset \cup\{P_2 \mid P \in B^*\}.$$

B is σ-cushioned if it is the union of countably many cushioned subcollections.

A σ-cushioned point countable (point finite) pair-base is a σ-cushioned pair base, such that each $x \in X$ appears in at most a countable (finite) number of P_1's in the ordered pairs in a given level.

Definition 4.2.5.

An M_3 space is a T_1 space with a σ-cushioned pair-base.

The M_i spaces are part of a hierarchy of topological spaces that includes metric spaces and paracompact spaces. We develop, now, relationships between members of this hierarchy.

Theorem 4.2.1.

A locally finite collection of subsets of a space X is closure preserving.

Proof.

Let B be any locally finite collection of subsets of a space X, and let $B^* \subset B$.

We wish to prove that

$$\overline{\cup\{A \mid A \in B^*\}} = \cup\{\overline{A} \mid A \in B^*\}.$$

It is apparent that

$$\cup\{\overline{A} \mid A \in B^*\} \subset \overline{\cup\{A \mid A \in B^*\}}.$$

Suppose $x \notin \overline{A}$ for all $A \in B^*$. Since B is locally finite, there exists an open set U, such that $x \in U$ and U has a nonempty intersection with at most, finitely many elements of B, and, hence, with at most finitely many elements of B^*, say $A_1, A_2, \ldots, A_n$.

Clearly $V = U \cap (\bigcup_{i=1}^{n} \overline{A}_i)^c$ is an open set that contains x and $V \cap (\cup\{A \mid A \in B^*\}) = \emptyset$. Therefore, $x \notin \overline{\cup\{A \mid A \in B^*\}}$; and we have $\overline{\cup\{A \mid A \in B^*\}} \subset \cup\{\overline{A} \mid A \in B^*\}$. Thus, B is closure preserving and the theorem is proved.

Theorem 4.2.2.

A metrizable space is an M_1 space.

Proof.

Let X be a metrizable space. Then, X is regular and T_1 and by Theorem 2.2.1, X has a σ-locally finite base. Therefore, by Theorem 4.2.1, X has a σ-closure preserving base and the theorem is proved.

Theorem 4.2.3.

An M_1 space is an M_2 space.

Proof.

The theorem follows, since a σ-closure preserving base is clearly a σ-closure preserving quasi-base.

Theorem 4.2.4.

If a regular space X has a σ-closure preserving quasi-base then X has a σ-cushioned pair-base.

Proof.

Let $B = \cup\{B_i\}$, $i = 1, 2, \ldots,$ be a σ-closure preserving base quasi-base for X. For each i we define $P_i = \{(U^\circ, \overline{U}) \mid U \in B_i\}$. Then $P = \cup\{P_i\}$, $i = 1, 2, \ldots,$ is σ-cushioned. Also, since X is regular, P is a σ-cushioned pair-base; and the theorem is proved.

Corollary.

An M_2 space is an M_3 space.

Theorem 4.2.5.

An M_3 space is paracompact and perfectly normal.

Proof.

Let X be an M_3 space with a σ-cushioned pair-base $B = \cup\{B_i\}$, $i = 1, 2, \ldots$, where each B_i is a cushioned collection of pairs, (P_1, P_2).

Let C be an open cover of X. For each $x \in X$ there exists some $U(x) \in C$ such that $x \in U(x)$. Also, there is some B_j and some $(P_1(x), P_2(x)) \in B_j$ such that

$$x \in P_1(x) \subset P_2(x) \subset U(x).$$

We define $R_j = \{P_1(x) \mid x \in X \text{ and } (P_1(x), P_2(x)) \in B_j\}$.

If $R = \cup\{R_j\}$, $i = 1, 2, \ldots$, then R is an open σ-cushioned refinement of C and since X is T_1, by Theorem 4.1.3, X is paracompact.

Since every paracompact space is normal, to prove that X is perfectly normal, it is sufficient to show that every closed subset of X is a G_δ set (the intersection of a countable collection of open sets).

Let F be a closed subset of X.

Let $G_n = (\overline{\cup\{P_1 \mid (P_1, P_2) \in B_n \text{ and } P_2 \subset F^c\}})^c$.

It follows from the definition of cushioned that $F \subset G_n$ for $n = 1, 2, \ldots$. Also each G_n is open. It is easy to prove that $F = \cap\{G_n\}$, $n = 1, 2, \ldots$. Therefore, X is perfectly normal and the theorem is proved.

The preceeding theorems give us the following implication diagram for topological spaces:

Metrizable space
↓
M_1 space
↓
M_2 space
↓
M_3 space
↓
paracompact and perfectly normal space.

With respect to the converse of the above implications, we note that every M_1 space is not metrizable, as Example 3.5.1 (page 52) demonstrates. Also, there are paracompact, perfectly normal spaces that are not M_3. For example, Sorgenfrey's half open interval space [20] is paracompact and perfectly normal, but the product of the space with itself is not paracompact. Ceder [4] proves that a countable product of M_i spaces is M_i. Therefore, since an M_3 space is paracompact, the half open interval space cannot be M_3. Finally, whether or not any of the implications regarding M_i spaces can be reversed is not known, but the author guesses that M_1, M_2, and M_3 spaces are all equivalent.

4.3 Metrization of M_i Spaces.

In this section, we develop a metrization theorem that is equivalent to the Nagata-Smirnov Theorem and to Bing's Theorem. The type of σ-base used imposes fewer restrictions than the σ-bases of the other two theorems. In Theorem 2.2.3, we showed that a

metric space has a σ-discrete base. From Theorem 2.2.2 it follows that a σ-discrete base is a σ-locally finite base. Theorem 4.2.1 shows that a σ-locally finite base is a σ-closure preserving base. In a regular space, a σ-closure preserving base can be used to construct a σ-cushioned pair-base (Theorem 4.2.4). In the metrization theorem of this section we require the existence of a σ-cushioned pair-base. Also, local finiteness of the levels in the two original theorems is replaced by point countability of levels in this theorem. Before we can prove the theorem, we must introduce a new class of spaces.

Definition 4.3.1.

A Nagata space X is a T_1 space such that for each $x \in X$ there exist sequences of neighbourhoods of x, $\{U_n(x)\}$, $n = 1, 2, \ldots,$ and $\{S_n(x)\}$, $n = 1, 2, \ldots,$ such that:

(1) for each $x \in X$ and every open set U containing x there exists i such that $x \in U_i(x)^\circ \subset U_i(x) \subset U$;

(2) for every $x,y \in X$, $S_n(x) \cap S_n(y) \neq \emptyset$ implies that $x \in U_n(y)$.

(We recall that a neighbourhood of x need not be open, but that it must contain an open subset that contains x.)

Theorem 4.3.1.

A Nagata space is a semi-metric space.

Proof.

Let X be a Nagata space as in Definition 4.3.1.

For each $x \in X$ we define:

$$b(n,x) = (\bigcap_{i=1}^{n} S_i(x))^{\circ}.$$

For each n we define:

$$B_n = \{b(n,x) \mid x \in X\}.$$

Then $B = \cup\{B_i\}, i = 1, 2, \ldots,$ is a σ-base for the topology on $\overset{*}{X}$ since each $b(n,x)$ is open and since each $S_n(x) \subset U_n(x)$.

If $x \in X$ and U is any open set containing x, then there is some k such that $x \in U_k(x) \subset U$. If $j \geq k$ and $x \in b(j,y)$ for some $y \in X$, then $x \in S_k(y)$, and by the definition of Nagata space $S_k(y) \subset U_k(x)$. Therefore, $b(j,y) \subset U$, and $y \in U$. Thus, the conditions of Theorem 3.2.1 are satisfied and it follows that X is a semi-metric space.

Theorem 4.3.2 (Ceder [4]).

A topological space is a Nagata space if and only if it is first countable and M_3.

Proof.

Let X be a Nagata space; then X is first countable. We define $P_n = \{(S_n(x)^{\circ}, U_n(x)) \mid x \in X\}$ for $n = 1, 2, \ldots$. Then $P = \cup\{P_n\}, n = 1, 2, \ldots,$ is a σ-cushioned pair-base for X.

Let X be M_3 and first countable. For each $x \in X$, we let $\{W_n(x)\}$, $n = 1, 2, \ldots$, be a local base at x. We also let $B = \cup\{B_n\}$, $n = 1, 2, \ldots$, be a σ-cushioned pair-base for X. We assume for each n that $(X,X) \in B_n$.

For any positive integers m and n and for any $x \in X$, we define:

$$U_{m,n}(x) = \cap\{\overline{P}_2 \mid W_m(x) \subset P_1,\ (P_1, P_2) \in B_n\}$$

and

$$S_{m,n}(x) = \cap\{P_1 \mid W_m(x) \subset P_1,\ (P_1, P_2) \in B_n\} - \cup\{\overline{P}_1 \mid x \notin P_2,\ (P_1, P_2) \in B_n\}.$$

It is readily verified that upon suitable indexing, we get $\{S_k(x)\}$, $k = 1, 2, \ldots$ and $\{U_k(x)\}$, $k = 1, 2, \ldots$ that satisfy Definition 4.3.1.

Therefore the theorem is proved.

We are now in a position to state and prove the main theorem of this section.

Theorem 4.3.3.

A topological space is metrizable if and only if it is T_1 and has a σ-cushioned point countable pair-base.

Proof.

Let X be metrizable. Then X is T_1.

By Theorem 2.2.1, X has a σ-locally finite base B. By Theorem 4.2.1, B is a σ-closure preserving base. If B is used to construct a σ-cushioned pair-base as in Theorem 4.2.4,

then we see that the base so constructed is a σ-cushioned point finite pair-base. Therefore a metrizable space is T_1 and has a σ-cushioned point countable pair-base.

On the other hand, assume that X is T_1 and has a σ-cushioned point countable pair-base. Then X is an M_3 space. Clearly, X is first countable and from Theorem 4.3.2, X is a Nagata space. Therefore, it follows from Theorem 4.3.1 that X is a semi-metric space. From this, and the fact that X has a point countable base, by Theorem 3.4.3, X is developable. Since X is an M_3 space, by Theorem 4.2.5, X is paracompact. Finally, by Theorem 3.4.2, we have that X is metrizable.

Therefore the theorem is proved.

The following corollaries are direct consequences of this theorem.

Corollary 1 (Ceder [4]).

A topological space X is metrizable if and only if it is an M_1 space with a σ-closure preserving point finite base.

Corollary 2 (Heath [8]).

A regular T_1 space is metrizable if and only if it has a σ-closure preserving base with point countable levels.

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