PRE-SERVICE TEACHERS' CONCEPTUAL UNDERSTANDING OF PERIMETER, AREA, AND VOLUME

Master of Education

by

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A thesis

submitted in partial fulfillment of the requirements

for the degree of

Master of Education

FACULTY OF EDUCATION

LAKEHEAD UNIVERSITY

THUNDER BAY, ONTARIO

April, 2007

Your file Votre référence
ISBN: 978-0-494-31160-8
Our file Notre référence
ISBN: 978-0-494-31160-8

Acknowledgements

Several people need to be mentioned here for their support during my research. The first people I need to thank are my wife Petra, and two daughters, Samantha and Meagan, whose patience, perseverance, and understanding throughout my studies have allowed me to finish this portion of my journey for wisdom. Without the knowledge, understanding, and editing skills of my supervisor, Dr. Alex Lawson, my research of this topic would not have been possible. Dr. Christina van Barneveld's knowledge of statistics has been greatly appreciated even though my thesis resulted in no inferential statistical data. Finally, I would like to thank the 110 participants in my research, for without them volunteering their time, my research would not have been successful.

Abstract

In this study, pre-service teachers' understanding of perimeter, area, the relationship between the two, and volume are explored. A questionnaire on these concepts was administered to 110 participants, comprising the three divisions of teacher candidates enrolled in a small northern Canadian Faculty of Education. This was followed by sample interviews to help clarify some of the responses found in the questionnaire. The results were surprising. Regardless of division or mathematics background while some had a conceptual understanding of the concepts many did not. Many students demonstrated only a procedural understanding including a sizable proportion of the I/S students. These findings are discussed as well as recommendations for further research and teacher training.

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CHAPTER ONE

Introduction

Statement of Problem

In 1991 Deborah Ball declared that, "Mathematics education is in trouble in this country and the signs of it are everywhere" (Ball, 1991, p. 63). At the root of her statement lies the contention that many people (Ball, 1990; Baturo & Nason, 1996) do not have a conceptual understanding of specific topics within mathematics. Yet researchers assert that teachers require a conceptual understanding of a topic before they can effectively teach children mathematics (Ball, 1991; Simon & Blume, 1994; Sloane, Daane & Giessen, 2002). Conceptual understanding in mathematics has been examined by a range of researchers (see for example Hiebert and Carpenter, 1992). For the purpose of this study, I will adopt the definition of conceptual understanding as defined by the Mathematics Learning Study Committee in their report *Adding it Up: Helping Children Learn Mathematics* as the "comprehension of mathematical concepts, operations, and relations" (Kilpatrick, Swafford, & Findall, 2001). They continue by stating "A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways... [and] to see how the various representations connect with each other" (p. 119).

While lack of teacher conceptual understanding may be a problem across the discipline of mathematics, I will concentrate on the topic of measurement, focussing on perimeter, area, and volume. Similar to mathematics in general, we find evidence of poor conceptual understanding of these specific topics in practicing teachers (Ball, 1990; Simon & Blume, 1994; Menon, 1998). Teachers with a weak conceptual understanding know how to apply rules in some typical situations but have difficulty using their knowledge in new situations or, relating the procedure to a connected concept (Hiebert & Carpenter, 1992). On the other hand, teachers with a strong conceptual understanding of mathematics have "knowledge that is rich in relationships" (Hiebert & Lefevre as cited in Hiebert & Carpenter, 1992, p. 78). In the area of measurement it may be that the mental ability to reason spatially is a key element in conceptual understanding (Battista, Wheatley & Talsma, 1982; Battista, 1994, 1999a & b; Outhred & Mitchelmore, 2000). If this is the case, then to teach children effectively, teachers would also need to have the ability to reason spatially.

While we have evidence of the lack of conceptual understanding of these measurement topics in the United States, what is the case in Ontario? What do pre-service teachers understand about the measurement topics of perimeter, area, and volume? Do pre-service teachers who have a more extensive background in mathematics have a better understanding of these related topics? Does spatial reasoning play a critical role in this understanding?

Purpose

The purpose of this mixed methods research is to determine what the three divisions of Ontario pre-service teachers, Primary/Junior (P/J), Junior/Intermediate (J/I) and Intermediate/Senior (I/S), understand about the measurement topics of perimeter, area, and volume and to compare the knowledge of these three groups.

Research Questions

1) What do pre-service teachers understand about the concepts of perimeter, area, the relationship between the two, and volume?

2) Do the three divisions of pre-service teachers (Primary/Junior, Junior/Intermediate, and Intermediate/Senior) have the same understanding of perimeter, area, the relationship between the two, and volume?

3) Is spatial reasoning a key element to solving problems on the topics of area, and volume for these pre-service teachers?

CHAPTER TWO

Literature Review

The Ontario Context

Ontario's Education Quality and Accountability Office (EQAO) administer yearly mathematics assessments at grades, 3, 6 and 9 to report on students' achievement. The EQAO (2005) reported in the 2003/2004 results that 68% of Grade 9 academic students achieved a Level 3 score or higher, while a mere 26% of the applied students achieved Level 3 or higher.[1] Grade 6 students' scores were somewhat better, with 57% achieving Level 3 or higher (EQAO, 2005). The Ontario Ministry of Education defines Level 3 as the provincial standard; and in secondary schools, levels are translated into percentages with Level 3 ranging between 70% and 79% (Ontario Ministry of Education, 1999). Unfortunately, only one overall mathematics score for all five strands of mathematics is given; without a strand analysis we do not know what role the strand of measurement plays in these results. Given the government's intention to have 75% of students working at the provincial standard, I will make the assumption that there is room for improvement in Ontario students' understanding of perimeter, area, and volume concepts. If we are to improve students' understanding of basic measurement concepts what role does prospective teachers' understanding play?

The Implications of Teachers' Weak Mathematics Content Knowledge

The EQAO test results hint at a more systemic problem than simply students' poor performance on a large scale assessment. If students are not developing a deep understanding of these concepts then it may be that their teachers do not have the necessary conceptual understanding to teach all, or parts, of the mathematics curriculum. There has been an ongoing

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[1] The terms *academic* and *applied* refer to the educational stream that a student has entered in high school. The academic level provides students with the education that allows for entry into university.

discussion in the research on elementary teacher's poor mathematical understanding and its potentially negative effect on instruction and student achievement (e.g. Ball, 1991; Battista, 1994, Manouchehri, 1998). In a longitudinal study Ball (1990) administered questionnaires to approximately 250 pre-service students and conducted interviews with a smaller core group of these participants in order to examine what they understood about mathematics. She concluded her analysis by stating, "the mathematical understandings [of] prospective teachers … are inadequate for teaching mathematics for understanding" (p. 464). She feels that while teachers may learn subject matter through teaching it, they will not learn the underlying concepts, principles, and connections of specific rules and "probably will not change their ideas about mathematics" (p. 465) once they are on the job. Similar to Ball, Manouchehri (1998) states that teachers entering Teachers College are "lacking the necessary basic mathematical skills and understandings" (p. 284) and, even more disconcerting, that in many cases professors who teach mathematics to pre-service teachers do not understand pedagogical theory, and therefore, do not provide vital information to the pre-service teachers. Her conclusions were developed from the findings of a study she conducted on 51 middle school mathematics teachers who took part in a wider study in Missouri simply called Project – M3. These findings were gathered through observation of teachers' classroom instruction, and individual and group interviews. If this is the case, what effect does a teacher's poor mathematical understanding have on teaching?

Ball (1991) has linked poor understanding to poor instruction. Based on previous literature about teaching mathematics and, contrasting it with examples from her third grade classroom of effective instruction, she claims that often teachers simply show children how to complete a task by following specific steps and then give the children practice problems. This leaves the children memorizing the procedures and provides them no background knowledge as to why they are doing what they are doing. The teacher is unable to connect the students' work to any related topics or to real life situations that may make mathematics interesting and more understandable to the children because the teacher does not know it themselves. More explicitly, researchers link this lack of conceptual understanding by teachers to the type of instruction Taylor (2003) identifies in her research on *Transforming Pre-service Teachers' Understandings of Mathematics* in which she claims mathematics instruction is generally traditional and systematic. The teacher checks homework from the previous night followed by a tediously pedantic introduction of a new concept and concludes with an individual practice assignment. The teacher does most of the talking with little student interaction. This style of teaching leads to poor understanding of mathematics for many students (Manouchehri, 1998; Taylor, 2003).

The Call for Teaching for Conceptual Understanding

Conceptual understanding can rarely occur in the type of environment delineated above: as Anghileri (1995) contends "[m]athematical knowledge is not something that is acquired by listening to teachers and reading textbooks but something that learners themselves construct by seeking out meanings and making mental connections in an active manner" (p. 3). Supporting this view, Manouchehri (1998) believes that there is a growing interest in an instructional approach that supports the necessity of understanding mathematical concepts. She summarizes that, "in this approach to teaching mathematics, the teacher facilitates knowledge acquisition and orchestrates learning environments conducive to authentic learning of mathematical ideas" (p. 276). For conceptual understanding to occur however, Ball (1991) states that:

> the outcomes of school mathematics teaching depend, at least in part, on a closer and more serious consideration of the mathematics that teachers need to understand, as well as how, when, and where they can acquire this kind of understanding and how we can

assess it. (p. 82)

The National Council of Teachers of Mathematics (NCTM) (2005) has recognized the importance of this issue by calling for instruction that focuses on conceptual understanding as part of teaching rather than simply computation and memorization of facts which can be classified as procedural understanding.

Conceptual Understanding versus Procedural Understanding

Teacher educators, Van De Walle and Folk (2005) theorize that knowledge "that is understood [conceptual knowledge] consists of logical relationships constructed internally and existing in the mind as part of a network of ideas" (p. 33). For example, to understand conceptually the area of a 10cm by 6cm rectangle a person might measure the length and multiply by the width to get $60cm^2$ while mentally understanding that they are actually finding how many 1 by 1cm square tiles fit into the rectangle. Procedural knowledge, on the other hand, is the knowledge a student gains from rules and procedures that they learn to solve "routine mathematical tasks" (p. 34) such as multiplying length by width to find the area of a rectangle without necessarily understanding why the product of two linear measures results in a square measure. Both types of knowledge are important in mathematics learning, although procedural rules should never be taught without an understanding of the concept. The authors however, state, "this happens far too often" (p. 34).

In the strand of measurement specifically, what are the concepts one must learn in order to understand conceptually perimeter, area, and volume? What understanding do elementary teachers have of these concepts?

Understanding Measurement

An overarching topic within mathematics is measurement; instruction of this topic begins early in a child's school career. A measurement is "the number that indicates a comparison between the attribute of the object being measured and the same attribute of a given unit of measure" (Van De Walle & Folk, 2005, p. 294). The Ontario Ministry of Education (2005) mathematics curriculum for grades one through eight states that:

> Measurement concepts and skills are directly applicable to the world in which students live...Students learn about important relationships...involved in calculating the perimeters, areas, and volumes of a variety of shapes and figures. Concrete experience in solving measurement problems gives students the foundation necessary for using measurement tools and applying their understanding of measurement relationships. (pp. 8-9)

In order for students to learn to measure they must carry out a number of steps. Typically, researchers believe that measurement can be broken down into three steps that students must follow. (Chapin & Johnson, 2000; Van De Walle & Folk, 2005). The first step is to "[d]ecide on the attribute to be measured" (Van de Walle & Folk, 2005, p. 295). For example, different attributes of a bucket can be measured such as the height, volume, or circumference. One must determine which attribute requires measuring.

When the attribute to be measured is determined the second step involves the selection of "a unit that has that attribute" (Van de Walle & Folk, 2005, p. 295). For example, if the child decided to measure the height of the bucket the child must select a measuring device (unit) that would appropriately measure the bucket's height. This 'unit' may be a toothpick, pen or a ruler that measures centimetres. In order to do this, students must physically compare similar objects to a measurable attribute through class activities. This allows the attribute to become the center of attention.

The third step in learning how to measure requires students to determine the number of units "by filling, covering, matching etc. the object" (Van de Walle & Folk, 2005, p. 295). If this step involves a measuring tool such as ruler, then students need to have the knowledge of how a ruler works and then measure the required surface obtaining the "numerical relationship (the measurement) between what is measured and the unit" (p. 294).

To understand how measurement works one needs to have achieved the three Big Ideas of *conservation*, *transitivity*, and *unit iteration* (Chapin & Johnson 2000). Conservation is "the principle that an object maintains the same size and shape if it is rearranged, transformed, or divided in various ways" (p. 178). As an example, a child has constructed conservation if he or she recognizes when pouring water from a full container into a second container which becomes completely full that the containers have the same volume. Strange and Kamii (2001) identify the comparing process as unit iteration in which a person measures repeatedly using the same unit of measure to measure a unit larger than the original unit. Young children cannot measure accurately since they have not yet determined that a smaller unit of measure is a part of a larger unit, for example; that four cups equals a quart. When first given the problem, young children do not fill the four cups to the proper level each time and the children then conclude that four cups does not equal a quart. The authors claim, that only when children have the ability to use unit iteration properly will they be able to measure accurately. Strange and Kamii state that indirect comparisons, as in this example, require the children to use unit iteration as well as the third big measurement idea, transitive reasoning. Students have constructed transitivity when they have "the ability to deduce a third relationship from two (or more) other relationships of equality or inequality" (p. 357). To reason in this way children can deduce that if a = b and b = c then a = c (Chapin & Johnson, 2000). With this understanding of the Big Ideas in measurement, students will have the basis for understanding other concepts involving measurement.

Teaching Measurement

To teach measurement effectively, Van De Walle and Folk (2005) suggest that teachers allow students to use concrete models of the unit being measured, any standard or non-standard units can be used, for example a 30cm ruler or a straw. They state that to understand the concept of a unit there needs to be as many same length units as required to cover the entire attribute being measured. For instance, if the students are measuring the length of the floor with straws there needs to enough same length straws to cover the whole length of the floor. This helps to avoid overlap or missed areas during the measuring process. It is also important that different units of measure be used to measure the same attribute as smaller units of measure creates a larger numerical measurement and this inverse relationship is hard for younger students to understand.

These authors also point out that by allowing students to make simple measuring units, with unit models; students will develop an understanding of how a measuring instrument works. By following this procedure and allowing students to reflect on what and how they measured, students will enhance their conceptual knowledge to understand "the attribute being measured … how filling, covering, matching, or making other comparisons of an attribute with units produces what is known as a measure … [and] the way measuring instruments work" (p. 296).

Each step involved in measurement: determining the attribute to be measured, selecting the correct unit for the attribute, comparing the attribute with the unit of measure, and the relationships amongst them pose particular problems for students when applied to perimeter, area, and volume.

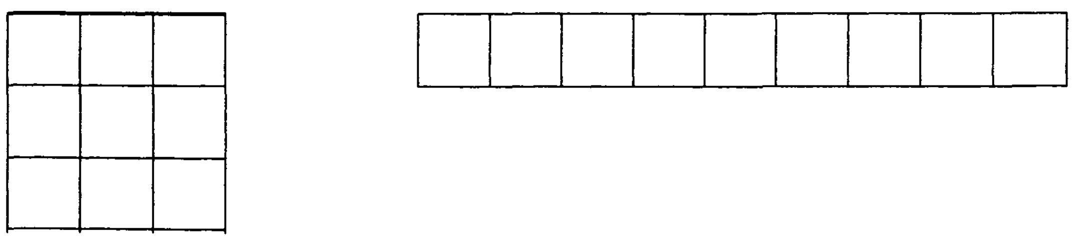
Students' Misconception of Perimeter and Area Measurement

Perimeter is "a measure of the length of the boundary of a figure" (Ma, 1999, p. 84). In the case of a rectangle, it is the sum of the lengths of the sides of a figure. Ma (1999) defines *area* as "a measure of the size of the figure" (p. 85). Generally, these two concepts are taught together (Ma, 1999), and it is the nature of the relationship between area and perimeter that is often poorly understood. In order to examine this challenge we must first look at how to measure area.

To begin to learn to measure area students will often make use of a tiling technique where they cover the area of a shape with smaller square units (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996). This technique is an additive process; they need only count the tiles to determine the answer. It is a great leap from this process to the standard algorithm where students typically calculate area using the length by width formula which is a multiplicative process. Baturo and Nason (1996) contend that the standard algorithm "of multiplying the two linear measures … is conceptually far removed from the notion of area" (p. 239). This method results in a measurement that has no relationship to what was measured. The units of linear measure magically change to units of square area measure. Therefore, students must switch back and forth between addition and multiplication to understand the relationship between the two. Students' ability to use the area formula may mask their lack of understanding of the relationship between the linear dimensions and the square area measure of rectangular arrays.

Two of the most typical misconceptions found among students, in dealing with area and perimeter, are that rectangles with the same area have the same perimeter and that rectangles with the same perimeter have the same area (Chapin & Johnson, 2000). These statements are false. Rectangles with the same area can have perimeters which range from small to large. Those rectangles with the same area and smaller perimeters are more condensed and compact, closely resembling the shape of a square, while rectangles with the same area and larger perimeters are longer and thinner as shown in Figure 1. Examining the second statement, Figure 2 shows how the closer the rectangle is to a square shape the larger the area becomes.

Figure 1: Rectangles with the Same Area but Different Perimeter

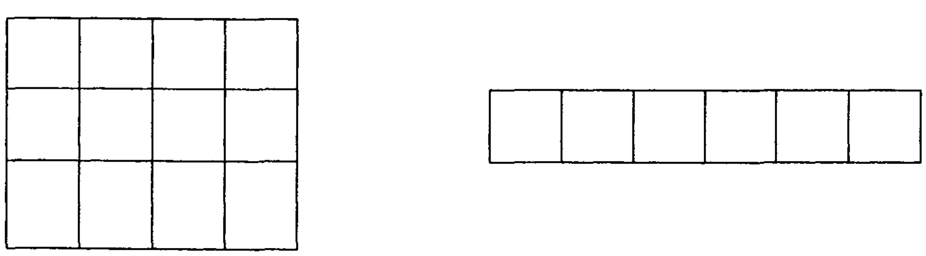


Area = $9u^2$ Area = $9u^2$

Perimeter = 12 units Perimeter = 20 units

Note: Figure 1 is adapted from Chapin and Johnson (2000)

Figure 2: Rectangles with the Same Perimeter but Different Area



Perimeter = 14 units Perimeter = 14 units

Area = $12u^2$ Area = $6u^2$

Note: Figure 2 is adapted from Chapin and Johnson (2000)

A third misconception with area is students' belief that when the dimensions of a rectangle are doubled and tripled the area also doubles and triples. This statement is also false. For example, a rectangle having dimensions of 4u by 5u has an area of $20u^2$. However, doubling the dimensions to 8u by 10u the area becomes $80u^2$, four times greater. By tripling the dimensions to 12u by 15u the area becomes nine times larger, $180u^2$ (Chapin & Johnson, 2000). These misconceptions reflect the students' lack of conceptual understanding. Based on these students' misconceptions, the question arises as to whether similar misunderstandings of measurement concepts are found among teachers as well.

Teachers' Misconceptions of Area and Perimeter Measurement

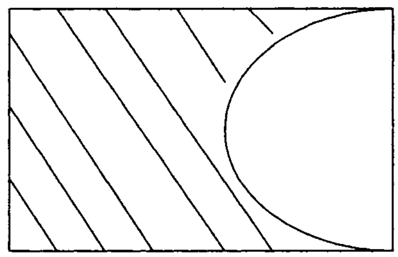
We find evidence that these same misunderstandings persist into adulthood. Adults also have difficulty understanding the multiplicative relationship in the area formula as well as the nature of the relationship between area and perimeter. For example, Simon & Blume (1994) examined pre-service teachers' understanding of multiplicative relationships within the topic of area. Incorporating a small group environment in their pilot program, 26 pre-service teachers attempted to solve the mathematical problems that the instructors gave them. Participants were given rectangular pieces and asked to determine how many of the pieces fit on the rectangular tables at which they were sitting. These pieces were not to be overlapped, cut or allowed to hang over the table's edge. Each group was given an allotted time to solve the problem and then asked to explain their solution to the rest of the class. During the small group discussions the researchers questioned participants on how they solved the given problem. In describing the strategies they used the participants could not explain why they did what they did to solve the problem, despite having obtained the correct answer through application of the formula. For this reason, the researchers concluded that pre-service teachers do not have the conceptual understanding of the multiplicative relationship of length and width of a rectangle to its area -- even though they had memorized the area formula over the years. They believe that this is a widespread problem that exists among teachers.

Baturo and Nason's (1996) study of sixteen pre-service teachers' understanding of area measurement had similar results. Through individual interviews, that consisted of eight measurement tasks on area, the authors concluded that in general the participants' subject matter knowledge "was rather impoverished in nature" (p. 260) and "would extremely limit their ability to help their learners develop integrated and meaningful understandings of mathematical concepts and processes" (p. 262).

In another study, Menon (1998) examined 54 elementary pre-service teachers' on their understanding of area and perimeter. They were given four tasks to complete halfway through their mathematics content course taught by Menon. Two tasks covered the topic of perimeter and two covered area. In the first task, participants were asked to create a question that would identify students' understanding of perimeter. This question would "indicate the pre-service teachers' pedagogical content knowledge of perimeter" (p. 361). She found that 83.5% of her participants had a superficial to minimal understanding about the underlying concept of perimeter. Typical questions participants created at this level, required students to find the perimeter of the given rectangle in which the dimension of one length and one width were given. At the highest level of understanding, 11% of the participants created a question such as "Given that a rectangle has a perimeter of 36 cm, list some possible dimensions of the rectangle" (p. 368). In task two, three, and four participants were asked if the questions given had enough information to solve, providing more information on the pre-service teachers' understanding of perimeter and area. In task two, the second perimeter question, 24% of the participants stated that the information given was insufficient to answer when the question was answerable. Menon states that these participants required a dimension for all the lengths of a figure in order to solve the question and reported, “they did not have an adequate understanding of perimeter” (p. 364). To understand perimeter conceptually, she states that one needs to involve “reasoning based on relationships among the sides of a given figure” (p.364). In task three and four, the two area questions, 28% and 85% of the participants respectively, thought that they were lacking sufficient information, when the tasks were answerable, resulting in Menon’s conclusion that pre-service teachers were not able to “see the relationships involved and were dependent on procedures that involved explicitly stated numbers” (p. 364). Most participants were attempting to solve the tasks through procedural methods rather than using solutions that could be found by non-algebraic methods including drawing lines and overlapping areas. Menon concluded in her research that although this group of participants had exposure to mathematics in high school and passed their mathematics public examinations, their conceptual understanding of mathematics was not satisfactory.

Coinciding with these studies, Reinke (1997) examined 76 elementary pre-service teachers’ understanding of perimeter and area. Her participants were given a diagram similar to that of Figure 3 and were asked to explain how they would find the perimeter and area of the shaded portion of Figure 3. Reinke reported that only nine of the participants answered the perimeter portion of the question correctly while 40 of participants were able to answer the area portion of the question correctly. In both cases the most incorrect strategy used to solve the problem was to ignore the semicircle completely. Although the researcher states that this problem was not an easy problem to solve she believes that “teachers must be able to think beyond the basic problems that they will give their students" (p. 77). How does weak conceptual understanding of area and perimeter affect teachers' capacity to teach these concepts?

Figure 3: Find the Perimeter and Area of the Shaded Portion

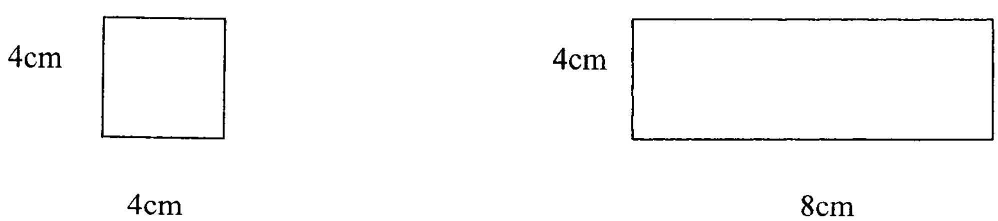


Note: Adapted from Reinke (1997)

Teaching Students for Conceptual Understanding of the Perimeter and Area Relationship

Ma (1999) claims that many teachers in the United States have difficulty in conceptually understanding the relationship between perimeter and area and she links this to inadequate instruction. Ma interviewed 23 American elementary school teachers to determine their response to different teaching scenarios. They were given a hypothetical situation where they had to determine if a student's theory, (if a perimeter increases in size so does the area), was true or false (see Figure 4). Only one of the American teachers could reach a proper solution for this statement demonstrating a conceptual understanding of the problem.

Figure 4: Example of Student's Theory: If a Perimeter Increases in Size so does the Area



Perimeter = 16cm | Perimeter = 24cm

Area = $16cm^2$ | Area = $32cm^2$

Note: Adapted from Ma (1999)

Similar to the solutions by children, most teachers could not see that the situation was dealing with two different numerical relationships. Ma identifies four ways in which the teachers responded to the student's theory as stated above. She classified the responses from a weak to a strong conceptual understanding. Eighty-seven percent of the teachers (20) responded to the scenario by saying that they would attempt to disprove the theory by finding counter-examples, comprising the first two responses. In the first method the teachers used figures that included a longer perimeter but smaller area or vice-versa. They provided no explanation for their examples. This method, of finding counter examples, could result in the teacher missing the truth, as was the case in Ma's study. The second method consists of exploring the possible relationships by using examples that supported and opposed the theory. In this method, teachers not only compared figures they also discussed the relationship between perimeter and area and examined how changes in area can change perimeter. Two teachers were evaluated at the third level of understanding which builds on the second method. Teachers clarified the conditions that each example presented and "explore[d] the numerical relationship between perimeter and area" (p. 94) through specific examples. For example, one teacher asked the students under what conditions the student's theory would hold true and when it would not hold true. The fourth method of understanding, and the highest level of conceptual understanding, involved teachers explaining the conditions that support and oppose the theory that were uncovered in the second and third methods. Only one American teacher was evaluated at this level. As a teacher passes through each of the four methods they achieve a more complete mathematical argument, providing clearer conceptual understanding for the students. She found that the instructional explanations teachers gave in the highest level were underpinned by a deep and clear conceptual understanding of the topic. It was in this area that the Chinese teachers (70%) outperformed the

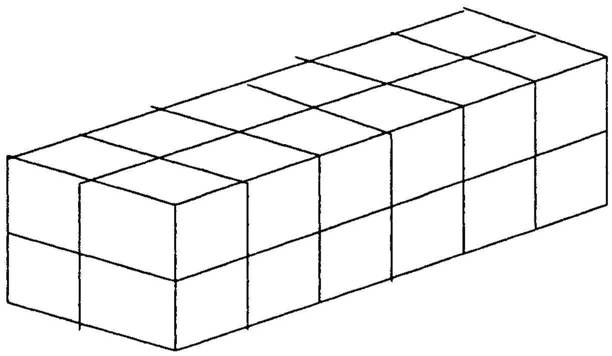
American teachers (4%). Teachers with a poor conceptual understanding of the topic gave incorrect or incomplete teaching explanations. It would seem likely that similar misconceptions and poor instructional explanations also exist in the more demanding concept of volume.

Misconceptions with Volume Measurement

Volume adds a third dimension and more challenges to understanding measurement for students and teachers. In the Ontario Ministry of Education's (2005) mathematics curriculum children begin to learn about three-dimensional objects as early as Grade 1 although measurement of volume of these objects do not occur until Grade 4. Volume is the measure of the "size of three-dimensional objects" (Van De Walle & Folk, 2005, p. 305) or the amount of space that an object fills and is much more challenging than measuring area or length (Chapin & Johnson, 2000). Battista (2003) states that most students who learn through traditional instruction of procedures do not understand "the spatial structuring that underlies" these procedures and therefore "they improperly apply these procedures to new problems" (p.132).

Even when instruction includes diagrams rather than simply procedures, students still experience difficulties. Ben-Haim, Lappan and Houang (1985) for example, argue that students often learn about three-dimensional objects, of which our world is comprised, from two-dimensional figures in text-books and on television and computers. These sources represent our world as two-dimensional when students should be using concrete models to deal with three-dimensional objects. An example of a three-dimensional object in a two-dimensional medium is shown in Figure 5.

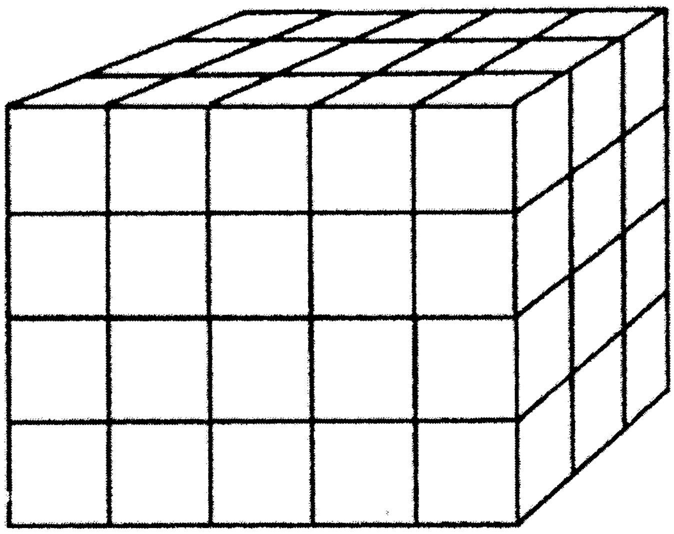
Figure 5: Picture of a Three-Dimensional Array of Cubes in a Two-Dimensional Medium



Generally, this becomes the students' introduction to spatial reasoning of solid objects which creates problems. For example, when asked 'How many cubes are in this solid?' the authors found four errors that eleven to fourteen year old students made: one, students counted the cube faces shown in the diagram; two, students doubled the amount of cube faces shown in the diagram; three, students counted the number of cubes shown in the diagram; and, four, students doubled the amount of cubes shown in the diagram. These errors stem from two main misconceptions that are related to spatial reasoning. The students saw the diagrams as two-dimensional rather than three-dimensional objects and the students could not visualize the cubes behind the shown cubes. Ben-Haim et al. conclude that students should encounter "concrete experiences with cubes – building, representing in two-dimensional drawings, and reading such drawings" (p. 407) in their classrooms to help improve their understanding.

Even when students work with a manipulative to solve problems their understanding of volume develops slowly over time. Battista and Clements (1996) provide a theoretical description of how students construct an understanding of three-dimensional cube arrays. They believe that students will develop an increasingly organized and complex understanding of volume over time. Their study focussed on 45 Grade 3 students and 78 Grade 5 students. During the interview students were given a 3 by 4 by 5 cube building made from centimetre cubes and were asked how many cubes made up the building.

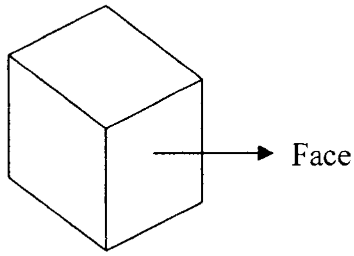
Figure 6: A 3 by 4 by 5 Cube Building Used During Interviews



Note: Taken from Battista and Clements (1996)

The authors identified four main strategies that students used. With the first strategy, students were able to "conceptualize the set of cubes [building] as a 3-D rectangular array organized into layers" (p. 262) and counted the cubes using addition or multiplication. With the second strategy, students conceptualized the building as being filled while counting the cubes, inconsistently organized into layers. The students using the third strategy visualized the building in terms of faces counting all or some of the faces around the building. With the final strategy, the students only used the formula for volume (L x W x H). Visualization was not used and there was no understanding of layering. For this study, an array is defined as a set of objects that are arranged in rows and columns (see Figure 5). A face is defined as a single surface or one side of a particular object as visualized in Figure 7.

Figure 7: A 1 by 1 by 1 Cube Showing One of Its Six Faces



The authors reported that the students progressed from the third grade to the fifth grade from seeing an "uncoordinated medley of faces" (p. 290) to seeing these faces as layers of arrays however, the students had difficulty in counting cubes because of their incorrect spatial reasoning of arrays. Sixty-four percent of the third graders showed a lack of spatial reasoning while 21 % of the fifth graders showed a lack of spatial reasoning. Battista and Clements believe these results underestimate the limited understanding of the actual population as their sample of participants originated from a population of above average students. These authors strongly believe that students should not be taught the volume formula until they achieve the first and highest strategy, where they conceptually understand three-dimensional cube arrays using spatial reasoning to see a set of cubes as layers.

In a further study, Battista (1998) once again arrives at the conclusion, that students' "[s]patial [reasoning] plays a crucial role" (p. 405) in calculating the number of cubes in a three-dimensional rectangular array. He reaffirms his previous findings stating that:

> Students who spatially structure an array into columns or layers generally calculate the total by skip counting or multiplying by the number of cubes in a column or layer. Alternatively, many students structure an array as an unrelated set of rectangular-prism faces. They determine the number of cubes visible on all or some of these faces, usually counting cubes along the prism's edge more than once. (p. 405)

The root problem for students solving volume as well as area questions is the lack of spatial coordination (Battista, 2003). He suggests that "[p]roperly coordinating spatial information is extremely difficult for many students" (p. 130). Spatial reasoning, thus, arises as a crucial element in the understanding of measurement of area, perimeter, and volume.

Spatial Reasoning and the Connection to Understanding Mathematics: A Brief Overview

Connecting spatial reasoning to conceptual understanding of measurement has been a topic of discussion since the late 19th century (Smith, 1964; Bishop, 1980). Bishop (1980) identifies Galton, in 1883, as the first person to study spatial reasoning with his "systematic psychological inquiry" (p. 257). Smith (1964), in outlining the early research of this topic in his book *Spatial Ability*, states that researchers have agreed "that there is a fundamental difference between the abilities required for school mathematics and those required for higher mathematics" (p. 101). This difference, Smith identifies as spatial ability. In 1935, Hamely wrote, "[m]athematical ability is probably a compound of general intelligence, visual imagery, ability to perceive number and space configurations and to retain such configurations as mental patterns" (as cited in Smith, 1964, p. 104). Smith believes that this statement stresses the importance of spatial reasoning within mathematics. In 1994, Battista's article *On Greeno's Environmental/Model View of Conceptual Domains* reaffirms Hamely's statement by citing spatial reasoning as a key component in conceptually understanding mathematics. He contends that spatial reasoning and mathematics achievement are positively correlated and that by doing activities that involve spatial ability one can improve mathematical understanding. The link between spatial and mathematical thinking is "the [similar] operations performed while interacting with mental models" (p. 92).

The Ontario Ministry of Education (2005) also recognizes the importance of developing spatial reasoning skills in order to "support students' understanding of number and measurement" (p. 9) suggesting that:

> Spatial sense is necessary for understanding and appreciating the many geometric aspects of our world. Insights and intuitions about the characteristics of two-dimensional shapes and three-dimensional figures, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Students develop their spatial sense by visualizing, drawing, and comparing shapes and figures in various positions. (p. 9)

In *The Importance of Spatial Structuring in Geometric Reasoning*, Battista (1999a) claims that spatial reasoning is important for a student developing geometric knowledge. Battista concludes that teachers must help develop spatial reasoning in students to ensure that their geometric development is meaningful. To accomplish this goal, Battista believes that two things must occur. Teachers must realize that students initially see the world as unstructured and need to learn to structure spatially their world. For this to occur, teachers must also have strong spatial reasoning skills in order to instruct students effectively. Lastly, educators Chapin and Johnson (2000) state that if mathematical relationships are presented to the students spatially they "are better able to generalize and remember the underlying mathematical concepts" (p. 162). However, the authors continue by arguing that many teachers avoid spatial reasoning in their instruction of mathematics because they are unfamiliar with spatial reasoning and the association it has with mathematics.

In summary, teachers must have a conceptual understanding of measurement including the concepts of area, perimeter, and volume in order to teach effectively. This conceptual understanding of area, perimeter, and volume may be inextricably linked to teachers' spatial reasoning. Do prospective teachers enter the profession with a strong understanding of these concepts?

CHAPTER THREE

Methodology

Introduction

In this mixed methods research, I gathered both quantitative and qualitative data for analyses from pre-service teachers at Lakehead University. The quantitative data was collected through a questionnaire (see Appendix A) while the qualitative data was collected through taped interviews combined with written responses from the questionnaire.

Participants

The participants in this study were a *convenience sample* (Cohen, Manion, & Morrison, 2000) drawn from pre-service teachers at the Faculty of Education at Lakehead University during the 2005/2006 term. All participants were enrolled in the mathematics methodology course for their particular division, Primary/Junior (P/J), Junior/Intermediate (J/I) or Intermediate/Senior (I/S). Pre-service teachers who have a university degree in mathematics were enrolled in the I/S division. These students will teach secondary school mathematics. The pre-service teachers without a university mathematics degree, used in this study, are those students who were enrolled in the P/J or J/I divisions. These students will teach mathematics at the elementary school level.

Upon ethical approval from the university and from applicable professors, classes were selected to solicit participants. One P/J class was chosen to participate in this research based on access to the class. A total of 39 questionnaires were collected from the pre-service teachers of the P/J division. For the J/I division two classes were chosen that best fit the professor's schedule. These two classes provided a total of 46 questionnaires. Since the I/S division has only one mathematics methods class I arranged to administer the questionnaire during their lab time.

A total of 25 I/S pre-service teachers chose to participate. All questionnaires were administered in a one week period in early January. I introduced my thesis topic to each class and asked the pre-service teachers to read the questionnaire's cover page (see Appendix B). The pre-service teachers were advised that completing the questionnaire was voluntary. Students completed the consent form (see Appendix C) and the questionnaire. The administration of the questionnaire took 30 minutes to 45 minutes at the start of each class to complete. The pre-service teachers in each class who participated were provided with a small incentive.

Instruments

Two instruments were used to collect data, a questionnaire, and semi-structured video-recorded interview questions. The questionnaire (see Appendix A) was developed to determine what understanding the three divisions of pre-service teachers have in the measurement topics of perimeter, area, and volume and to compare the knowledge amongst the three divisions. A pilot questionnaire was administered to volunteering Master of Education students to ensure that each item functioned properly. The questionnaire was broken down into two parts. The first part included general questions concerning personal data such as gender and education background that was used to sort and classify the participants. The second part, involved questions dealing with perimeter, area, and volume. These questions were adaptations from the existing research (see Table 1 for the source of these questions). Of the six questions, one dealt with the concept of perimeter, one with area, two with the relationship between perimeter and area, and two with volume. These questions not only provided quantitative data for the analysis but also qualitative data which was collected through the explanation portion of each of the six questions. All six questions provided information about the participants' level of understanding in each of the three concepts as well as their spatial reasoning skills.

The semi-structured video-recorded interviews conducted on a sample of the participants, provided more qualitative data for this research. The interviews were used to clarify answers from the questionnaire to understand better the participants' problem solving techniques and, to enable the researcher to classify better the answers. Manipulatives such as rulers, square tiles, cubes, and rectangular boxes similar to those in the questionnaire were available to participants to help explain their methods and solutions. Questions used in the interviews included, but were not limited to:

1. Could you please explain how you came up with this answer?
2. Explain why you chose this method.
3. Can you think of another way to solve the problem?
4. What is the definition of perimeter, area, or volume?

Table 1

Questionnaire Item Source and Purpose

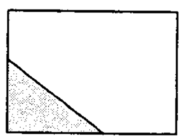
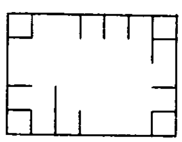
Questionnaire Item	Research Source	Purpose
1. Show how you would find the perimeter for the un-shaded area of the rectangle. 	This item was adapted from Rienke (1997).	This item was used to determine the level of participants' understanding of perimeter.
2. What is the area of the following figure?  Did you use a mental image when answering this question?	This item was taken from a larger set of tasks found in Battista and Clements et al (1998).	This item was used to determine the level of participants' understanding of area and the participants' ability to reason spatially.
3. Given the area of a rectangle is 36 square centimetres, can there be more than one set of dimensions for the rectangle?	This item was adapted from Baturo and Nason (1996), and misconceptions addressed by Chapin and Johnson (2000).	This item was used to determine the level of participants' understanding of perimeter and area as well as participants' understanding of the relationship between the two concepts.

Table 1 (cont'd)

Questionnaire Item	Research Source	Purpose
4. Can rectangles having the same total perimeter, but different dimensions, have the same area?	This item was adapted from Baturo and Nason (1996), and misconceptions addressed by Chapin and Johnson (2000).	This item was used to determine the level of participants' understanding of perimeter and area as well as participants' understanding the relationship between the two concepts.
5 a) What is the volume of a cube with the dimensions of 6cm by 5cm by 3cm? b) How many 1cm by 1cm by 1cm cubes does it take to make this cube complete? (see questionnaire item 5b in Appendix A for diagram) Did you use a mental image when answering this question?	This item was adapted from Battista & Clements (1996, 1998) and Battista (1999, 2003).	Part a) was used to determine if participants have a procedural understanding of volume. Part b) was used to determine if the participants have a conceptual understanding of volume and whether the participants have the capacity to reason spatially.
6. How many individual boxes with the dimension 2u by 2u by 2u (shown) fit into the larger rectangular box shown below? (see questionnaire in Appendix A for diagram) Did you use a mental image when answering this question?	This item was adapted from Battista and Clements (1996), and Outhred & Mitchelmore (2000).	This item was used to determine the level of the participants' understanding of volume. This item will also determine whether they have the capacity to reason spatially.

Data Collection

The questionnaires were administered in early January to the selected mathematics methodology course classes after the instruction of perimeter, area, and volume. The participants voluntarily completed the questionnaires. After the completion of the questionnaires I assigned pseudonyms to each participant. The pseudonym was written on the first and second page of each questionnaire and the first page was removed to ensure participant confidentiality. Once the questionnaires were scored I sorted them into two groups for each division: one, where answers were clear well described and no obvious misinterpretations could be made and two, where the answers to questions were unclear and further information as to the participants' meanings and knowledge would help evaluate the participants' understanding at the three levels. I then selected two participants from each division, one from the questionnaires that were answered well (likely with stronger understanding) and one from the questionnaires with errors (likely with weaker understanding) for a total of six who would participate in the fifteen to twenty minute video-taped interviews. This provided further insight into how and why they answered the questions in the manner they did. These participants were notified through e-mail of my interest in interviewing them. A total of twelve P/J, six J/I, and four I/S participants were asked to be interviewed so that I had two from each division. For the P/J division I was only able to interview two strong participants as no weaker participants chose to participate.

Scoring

I selected one student from the Masters of Education program at Lakehead University who was familiar with scoring similar questionnaires for conceptual understanding to help score the questionnaires. I conducted a short seminar to ensure the rating was consistent. The answers to each of the six questions were then sorted into three categories: *No demonstrated* *understanding* of the concept was rated as a zero, *some or procedural understanding* was rated as a one, and *conceptual understanding* of the concept was rated as a two. We worked together to score the papers and collaborated in classifying the answers to the three categories. The results of each questionnaire were reported on separate sheets of paper and identified with the participant's pseudonym (see Table 2).

Table 2

Individual Markers Rating

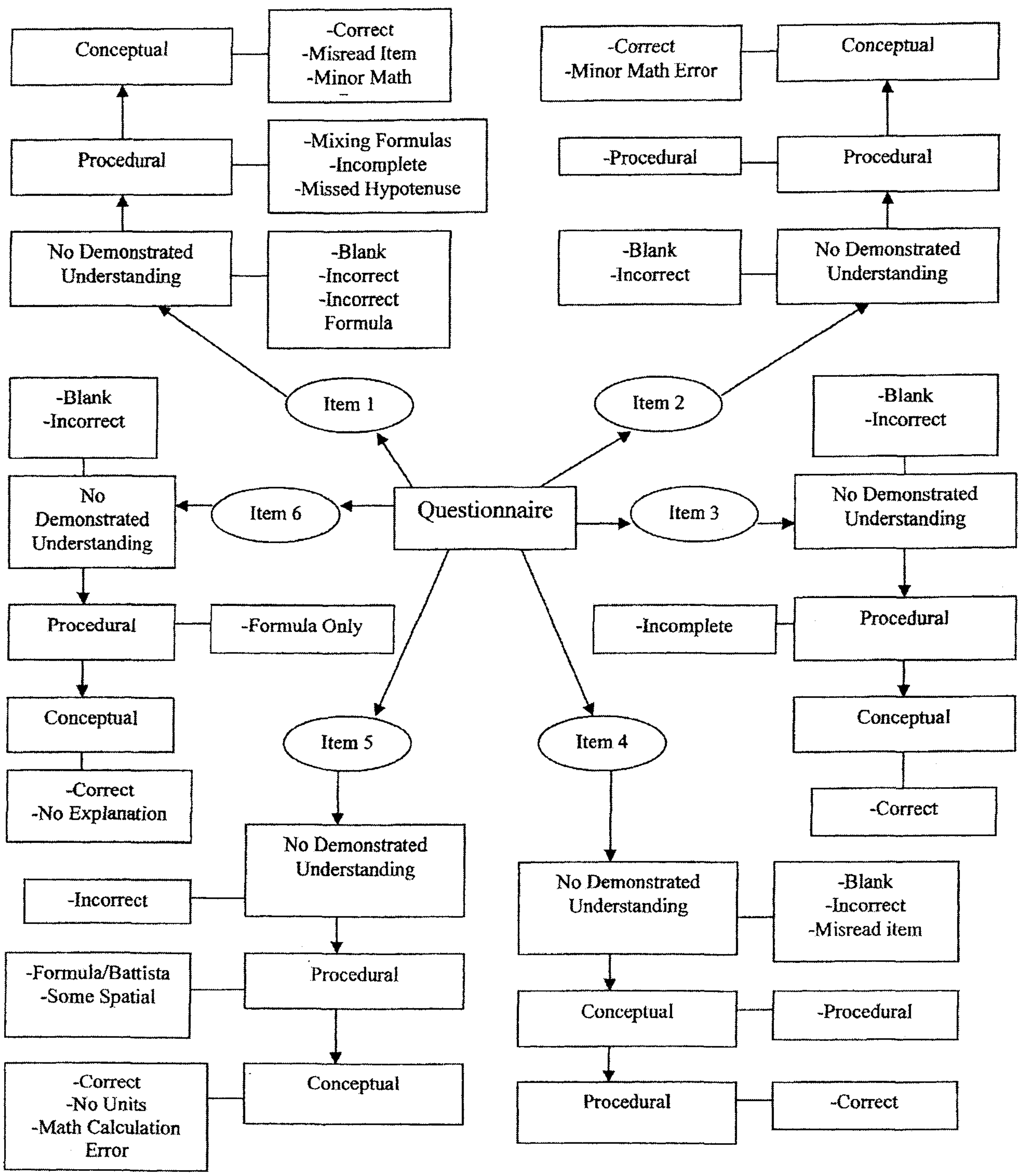
Questionnaire Item	No Understanding 0	Procedural or Some Understanding 1	Conceptual Understanding 2
1	☐	☐	☐
2	☐	☐	☐
3	☐	☐	☐
4	☐	☐	☐
5	☐	☐	☐
6	☐	☐	☐

Each questionnaire item was broadly evaluated as one of three levels of understanding: no demonstrated understanding, procedural or some understanding, and conceptual understanding. Solutions in each of these levels of understanding were then coded, typically as *blank*, *incorrect*, *incomplete*, and *correct*. The first two fall under the no demonstrated understanding level. If the students did not answer the item at all, the finding was identified as blank. Incorrect means the participants attempted to solve the item but their method reflected no understanding of the concept. An incomplete coding which falls under the procedural or some understanding level indicates that the participants began solving the problem in a correct fashion but could not finish the problem. Under the conceptual level of understanding findings were coded as correct when a solution to a problem showed a clear understanding of the concept. For all codes used for each questionnaire item and level of understanding see Figure 8.

Participants made errors at all three levels of understanding. Some of these errors were considered minor in nature while others were typical errors made due to a lack of conceptual understanding. Other error codes are more specific to each questionnaire item. These errors will be explained in more detail as each item is discussed. The full table of sub-codes that were combined into the main codes for each item can be found in Appendix D.

Interviews were conducted with two participants from each division of pre-service teachers to amplify or clarify item answers. One participant selected in each division generally answered the six items of the questionnaire correctly, demonstrating conceptual understanding, while the other participant had problems with some or all of the items. Comparing the two types of responses provided a clearer picture of the differences among conceptual, procedural, and no demonstrated understanding.

Figure 8: Diagram of Coding used for each Type of Understanding of each Questionnaire Item



Data Analysis

Incorporating the above scoring method, each of the three research questions were quantitatively analyzed using the computer programme SPSS based on the results of the six questions in the questionnaire (see Table 3).

Research question 1: What do pre-service teachers understand about the concepts of perimeter, area, and volume? To answer this question, descriptive statistics was used to describe the level of understanding of participants, for each of six items on the questionnaire. Qualitative data was also used from the written responses and interviews to clarify and support the quantitative data.

Research question 2: Do the three divisions of pre-service teachers P/J, J/I and I/S have the same understanding of perimeter, area, and volume? The same descriptive statistics and qualitative data that were used to answer question one was used to answer this question but was now classified by the division of pre-service teachers.

Research question 3: Is spatial reasoning a key element to solving problems on the topics of area, and volume for these pre-service teachers? This question was answered through an analysis of the qualitative data. Questionnaire Items 2, 5, and 6 had a spatial element involved in solving each of the problems and a specific question was asked as to whether the participants used a mental image to answer the questions. This information was analyzed to provide an answer to the research question.

Table 3

Identifying Data Source and Type of Analysis to the Research Question with Expected Outcomes

Research Question	Data Source	Type of Analysis	Expected Outcome
1	Item 1 -6	Descriptive statistics were used to describe the level of understanding of participants for each of the six items on the questionnaire. Qualitatively coded based on written responses to questions and interviews to clarify and support descriptive data.	A breakdown of what understanding the participants have for each concept as an entire group. The type of understanding was broken down into three levels, no understanding, some/procedural and conceptual.
2	Item 1 -6	Descriptive statistics were used to describe the level of understanding of participants for each of the six items on the questionnaire. Qualitatively coded based on written responses to questions and interviews to clarify and support descriptive data.	A breakdown of what understanding each division of pre-service teachers have for each concept. The type of understanding was broken down into three levels, no understanding, some/procedural and conceptual.
3	Items 2, 5 & 6	Based on qualitative data from written responses to questions and interviews to verify the effect of spatial reasoning.	Verify if spatial reasoning was a key element in solving area, and volume problems.

Note: Cramer's phi was removed from the table for analyzing research questions 1 and 2, as there was no significant degree of correlation between the nominal variables.

Ethical Issues

Ethical approval by Lakehead University was received prior to commencing the research. After the Ethics Committee approval I sought approval of each professor whose class I surveyed. All participants completed the consent form (see Appendix B). The participants were given the option to withdraw from the study at any time, for any reason. Participants were in no danger of the research causing harm.

Expected Outcomes

I expected to find that this sample of pre-service teachers had a limited understanding of perimeter, area, and volume beyond the memorizing of formulas, regardless of division. I expected the I/S group to have better spatial reasoning of the concepts of perimeter, area, and volume. Finally, I also expected to find that spatial reasoning would play a key role in solving problems related to area and volume.

CHAPTER FOUR

Findings

Introduction

In this section, I will discuss the major findings of each of the questionnaire items and provide amplification, where possible, from interviews conducted with a sample of the participants.

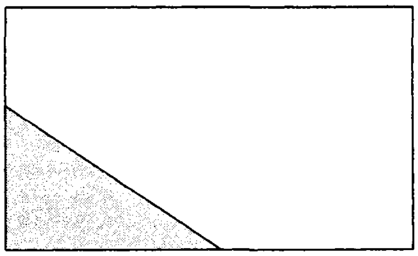
Interviews were conducted with two participants from each division of pre-service teachers to amplify or clarify item answers. One participant selected in each division generally answered the six items of the questionnaire correctly, demonstrating conceptual understanding, while the other participant had problems with some or all of the items. Comparing the two types of responses provided a clearer picture of the differences among conceptual, procedural, and no demonstrated understanding.

I report the analysis of each item in three different ways. First, I describe the levels of understanding of each item for all pre-service teachers. Second, I compare these ratings across divisions. Third, I explain the categorization of items within each level of understanding. In addition to individual item analyses, I also examined the use of spatial imagery to solve items 2, 5 and 6, reporting the proportion of its use at each level of understanding. Finally, I combined the results of some pairs of items into contingency tables in order to look at the strength of the relationship between items.

Questionnaire Item 1

Figure 9: Questionnaire Item 1

1. Show how you would find the perimeter for the un-shaded area of the rectangle. Explain your reasoning.



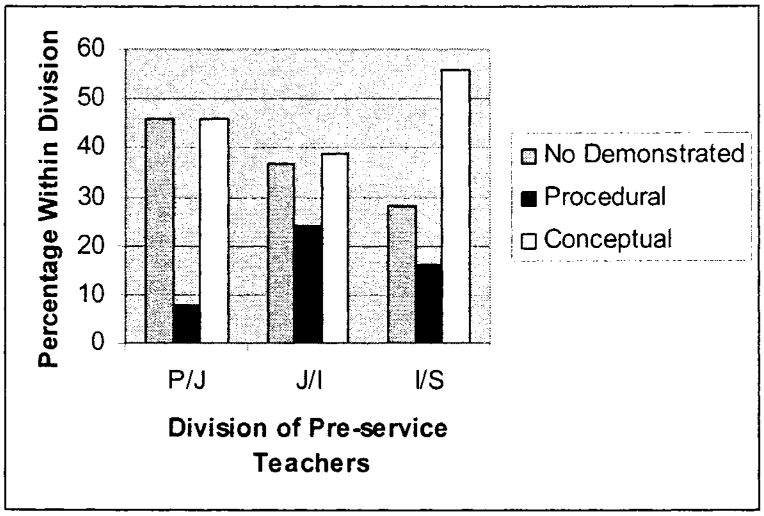
Pre-service teachers' solution of Item 1 (perimeter).

Most students and teachers know perimeter of a rectangle as 2L + 2W. This standard formula for perimeter cannot be used to solve this problem (see Figure 9). Similar to Reinke's (1997) geometric figure, this item was designed to require that students use more than the traditional perimeter formula (procedural method) for solution. Therefore, with a correct solution, it was likely that the pre-service teachers of this study had a conceptual understanding of perimeter. Since there is no way to solve this problem procedurally, pre-service teachers within all three divisions were evaluated as having a conceptual understanding or they had no demonstrated understanding of this item. Overall, about half (46%) of all participants demonstrated a conceptual understanding while 38% were evaluated as having no demonstrated understanding of perimeter, a small group (16%) demonstrating limited understanding.

Comparison across divisions.

Looking across divisions this item seemed more difficult for the P/J division (see Figure 10). Almost half (46%) of the P/J participants had no demonstrated understanding of perimeter with 37% in J/I and the smallest group, 28% in the I/S division.

Figure 10: Comparing Level of Understanding across Division, Item 1 (n = 110)

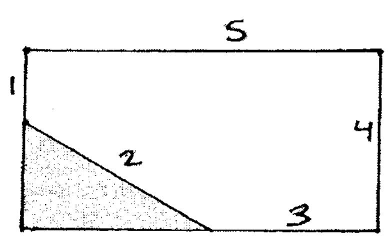


Conceptual understanding.

Thirty-eight percent of the P/J participants and 35% of J/I and I/S students solved the problem correctly (see Figure 11) and were evaluated at the conceptual level of understanding (see Table 4). A further 18% of the P/Js and 4% of the J /I and I /S participants respectively, misread the question but were nonetheless evaluated as having a conceptual understanding of the item. In this case, the participants misread the item and found the correct perimeter of the triangle. Therefore, a total of 46 % P/J, 39% J/I, and 56% I/S were evaluated as having a conceptual understanding of perimeter.

Figure 11: Evaluated as Conceptual and Coded as Correct (Cassidy P/J)

1. **Show how you would find the perimeter for the un-shaded area of the rectangle.** Explain your reasoning.



I would add up Side 1+2+ 3+ 4+5 to get my answer

Table 4

Level of Understanding by Division for (Perimeter) Item 1 (n = 110)

	Education Division					
Level of Understanding	P/J		J/I		I/S	
And Finding	n	%	n	%	n	%
No Demonstrated						
Blank	3	8	4	9	0	0
Incorrect	11	28	11	24	1	4
Incorrect Formula	4	10	2	4	6	24
Sub-total	18	46	17	37	7	28
Procedural						
Mixing Formulas	1	3	1	2	1	4
Incomplete	0	0	5	11	1	4
Missed Hypotenuse	2	5	5	11	2	8
Sub-total	3	8	11	24	4	16
Conceptual						
Correct	15	38	16	35	11	44
Misread Item	3	8	2	4	1	4
Minor Math Error	0	0	0	0	2	8
Sub-total	18	46	18	39	14	56
Total	39	100	46	100	25	100

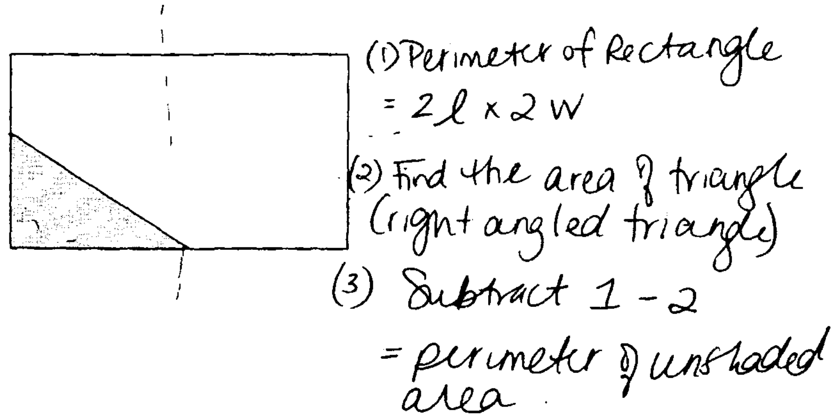
Note: Percentages have been rounded for clarity.

Procedural or some understanding.

The P/J division had 8%, the J/I 24% and the I/S 16% of the participants categorized into this level of understanding. Under the procedural level of understanding three types of errors became apparent. One participant in each division began answering the item by adding the perimeter of the rectangular region but then tried to subtract the area of the triangular region (see Figure 12). In this example, Ella[2] added the two lengths and then the two widths, multiplied them together, and then compounded her confusion by adding the area of the rectangle to her total.

Figure 12: Evaluated as Procedural or Some Understanding and Coded as Mixing Formulas (Ella I/S)

1. **Show how you would find the perimeter for the un-shaded area of the rectangle.** Explain your reasoning.



Ella (I/S) "really wasn't sure on this one" when answering the item on the questionnaire; although, during the interview and with some guidance she computed the correct solution. When asked if she could provide another way to solve the problem she responded after a short pause with "No … I can't". Grace (J/I) when interviewed repeatedly defined the question as finding the area until I read her the question "find the perimeter of the un-shaded area". She then showed

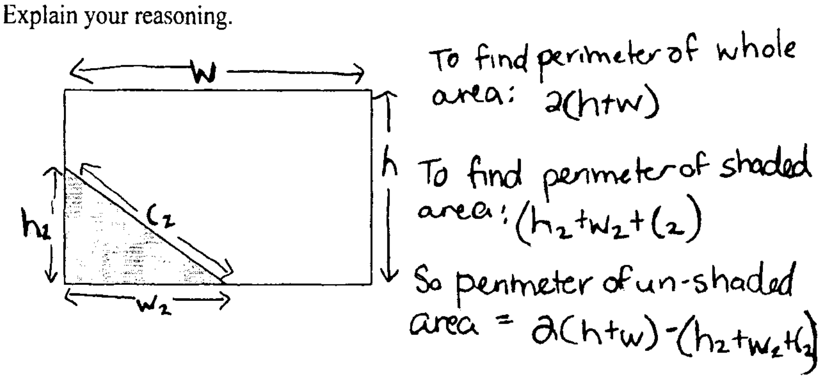
[2] All participant names have been changed to pseudonyms

how she would find the perimeter -- forgetting to add the hypotenuse -- which was coded into the next category of *missing the hypotenuse.*

Five percent of P/J, 11% of J/I and 8% of I/S participants did not add the hypotenuse of the triangle to the perimeter of the un-shaded area. These participants would add the perimeter of the entire rectangle, then subtract the length of the two sides of the triangle, and forget to add the length of the hypotenuse (see Figure 13). For example, Samantha (J/I) fell into this category and during the interview when asked about what should be done with the hypotenuse responded with "well, yeah, so we're going to have to add it". For this finding, I determined that they had started solving the problem correctly and therefore, I classified them as having some understanding of the problem.

Figure 13: Evaluated as Procedural or Some Understanding and Coded as Solution Missing Hypotenuse (Madeline P/J)

1. **Show how you would find the perimeter for the un-shaded area of the rectangle.** Explain your reasoning.



A third problem found at this level came mostly from the J/I division. Eleven percent of the participants started answering the question in a correct fashion by adding the sides of the rectangle but then seemed to get confused as to what to do with the triangle which resulted in an incomplete answer.

No demonstrated understanding.

In the no demonstrated understanding level, 8% P/J and 9% J/I participants left the item blank while 10% of P/J, and 4% of the J/I, and 24% of the I/S participants used the area formula to solve this problem (see Figure 14). In addition, approximately ¼ of the P/J and J/I and 4% of the I/S simply had no understanding of how to solve the problem (see Figure 15).

Figure 14: Evaluated as No Demonstrated Understanding and Coded as Incorrect Formula (Owen I/S)

1. **Show how you would find the perimeter for the un-shaded area of the rectangle.** Explain your reasoning.

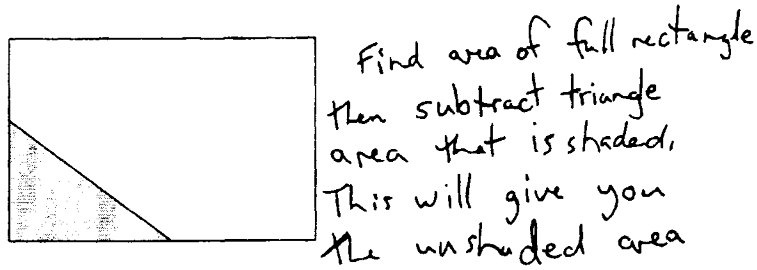
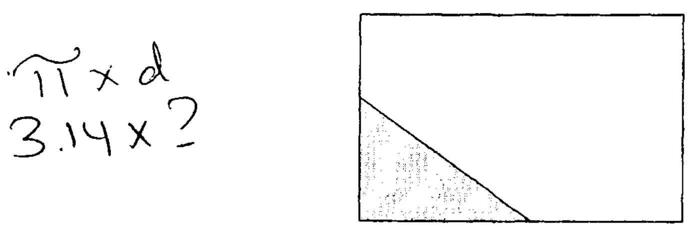


Figure 15: Evaluated as No Demonstrated Understanding and Coded as Incorrect (Alexandra J/I)

1. **Show how you would find the perimeter for the un-shaded area of the rectangle.** Explain your reasoning.

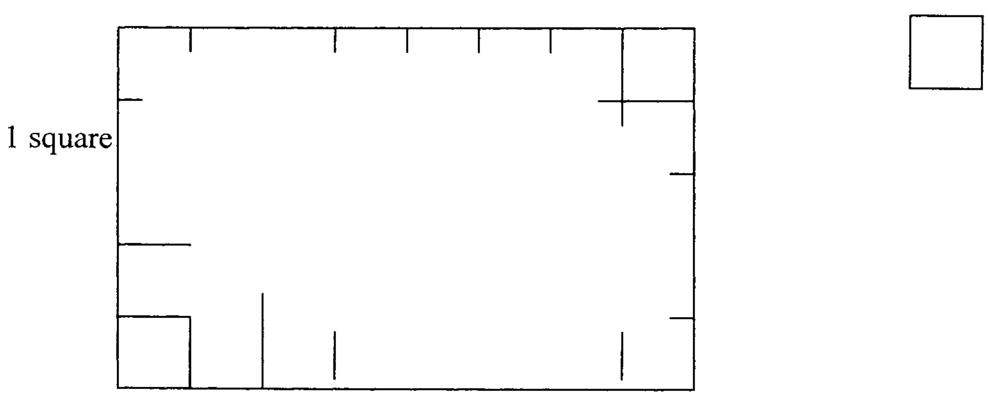


The results of this item may be exaggerated for the levels of no understanding and procedural understanding due to an error identified in the wording of the item. The item reads, "Show how you would find the perimeter of the un-shaded area of the rectangle". It is possible some participants did not complete the problem correctly because of the word 'area' after the word 'un-shaded' which may have been confusing. This could be the case for those participants whose answers were coded as incorrect formula and mixing formulas. For example, David (P/J) responded on the questionnaire that he "would find the overall area, and the area of the shaded triangle. I would then subtract the triangle from the rectangle" showing that he felt he had to find the area, not the perimeter. Another participant, Abigail (J/I), attempted to solve the problem by writing: "find 'P[erimeter]' for entire area, find 'A[rea]' for triangle, not sure what to do next", clearly mixing the formulas together.

Questionnaire Item 2

Figure 16: Questionnaire Item 2

2. What is the area of the following figure?



Did you use a mental image when answering this question?

Y ☐ N ☐

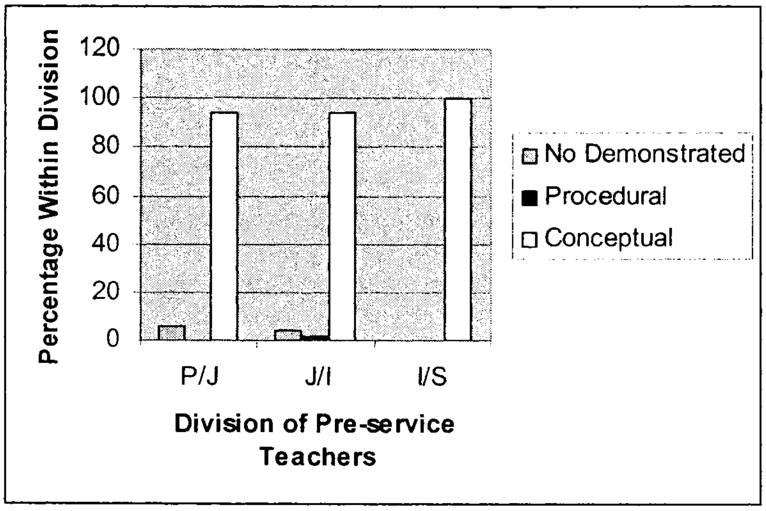
Pre-service teachers' solution of Item 2 (area).

This question (see Figure 16), which was adapted from Battista et al (1998), was used to determine if the pre-service teachers had a conceptual understanding of the concept of area. This problem cannot be solved by using a formula until the participant understands that the large rectangle contains a total of eight squares along the base and five squares along the height. For this reason, a correct answer can only be conceptual in nature. It is interesting to note that 96% of all participants had a conceptual understanding of this problem (see Figure 17). A mere 4% were categorized with having a procedural or no demonstrated understanding.

Comparison across divisions.

All three divisions mastered this problem, with the I/S division scoring 100% and the P/J and J/I divisions each scoring 94% in the conceptual understanding level. The few errors made, 5 in total, came from the P/J and J/I divisions.

Figure 17: Comparing Level of Understanding across Division, Item 2 (n = 110)



Conceptual understanding.

Most of the students, 94% or higher in each division answered this item of the questionnaire correctly with conceptual understanding (see Table 5). It should be noted that a large percentage of the participants in each division, P/J 38%, J/I 22% and I/S 12% were coded as not providing a unit of measure to their answer but were nonetheless classified as having a conceptual understanding of the problem as I was looking for the method pre-service teachers used to solve the problem.

Table 5

Level of Understanding by Division for (Area) Item 2 (n = 110)

	Education Division					
Level of Understanding	P/J (n = 39)		J/I (n = 46)		I/S (n = 25)	
And Finding	n	%	n	%	n	%
No Demonstrated						
Blank	1	3	1	2	0	0
Incorrect	1	3	1	2	0	0
Sub-total	2	6	2	4	0	0
Procedural						
Incomplete	0	0	1	2	0	0
Sub-total	0	0	1	2	0	0
Conceptual						
Correct	22	56	33	72	22	88
Minor Math Error	15	38	10	22	3	12
Sub-total	37	94	43	94	25	100
Total	39	100	46	100	25	100

Note: Percentages have been rounded for clarity.

Procedural or some understanding.

Only one participant in the J/I division fell into this level of understanding. This person started to answer the question correctly but could not finish it sufficiently and was coded as incomplete.

No demonstrated understanding.

A total of two participants each from the P/J and J/I division were evaluated at this level with one from each division under the code of blank and incorrect.

Use of spatial images.

This questionnaire item had an additional question regarding the participants' use of a mental image to help them answer the question. My intent was to see if spatial ability played a key role in pre-service teachers' understanding of area and volume. In this case, I was then able to categorize the participants' use of mental images, for area, with this item (see Table 6) by collapsing the divisions and categorizing the results according to use of mental image. Of the 95% of the pre-service teachers who had a conceptual understanding of this area problem, 64% identified that they had used a mental image to solve the problem while 31% stated that they had not used a mental image. Of the six participants interviewed, five connected the lines on the figure to clarify the image and stated that they used a mental image to solve the problem. For example, during the interview, Cassandra (P/J) stated that she "assumed that the larger spaces were … evenly divided" by the lines and then she "counted them" which was a typical way in which the participants' answered the item. Liam who stated that he did not use a mental image wrote that he could not answer it with the given information because he cannot conclude that the larger spaces could be equally divided into the smaller one square unit. During the interview he was asked to assume that the lines can be divided equally into the smaller one square unit, he was then able to determine the correct answer. He was classified into the procedural level of understanding.

Table 6

Use of Mental Image to Solve Questionnaire Item 2 Across All Divisions

	Level of Understanding					
	None		Procedural		Conceptual	
Spatial Ability	n	%	n	%	n	%
No	3	3	0	0	34	31
Yes	1	1	1	1	71	64
Total	4	4	1	1	105	95

Note: Percentages have been rounded for clarity.

Questionnaire Item 3

For this item the participants were asked whether there could be more than one set of dimensions for a rectangle that has an area of 36 square centimetres.

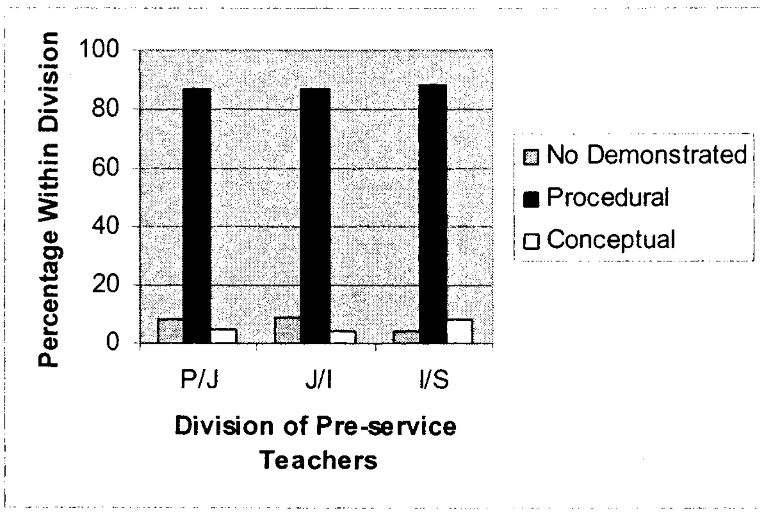
Pre-service teachers' solution of Item 3 (relationship between perimeter and area).

As several researchers (Baturo & Nason, 1996; Ma, 1999) have determined, students and teachers have difficulties understanding the relationship between perimeter and area; this question was created in an attempt to reproduce these findings. After reviewing the results of the questionnaire, I realized that this item could be answered correctly without necessarily having a conceptual understanding of the relationship. A participant could answer the item correctly by simply knowing the formula for area and substituting in numbers that are factors of the total area, in this case 36, which many have done. This does not mean that these students lack a conceptual understanding of the relationship, only that we do not know. We can see that the large majority of participants (86%) used a procedural method to solve this problem.

Comparison across divisions.

Once again, across divisions we can see that the I/S division is marginally stronger. When solving this relationship, the I/S participants had the lowest percentage within the no demonstrated understanding level while having the highest percentages within the procedural and conceptual level (see Figure 18).

Figure 18: Comparing Level of Understanding across Division, Item 3 (n = 110)



Conceptual understanding.

Despite the weakness of the item there are a few ways that students could demonstrate an understanding of the concept in their explanation. They could, for example, respond by explaining that, given a fixed area you could create rectangles with a variety of dimensions. The rectangles with smaller perimeters will be more compact, while larger perimeters create rectangles that are more elongated. Of the 110 participants, two participants within each division answered the item providing a statement that shows a conceptual understanding of the topic (see Table 7). Answers range from analytical to simple, even within a division. For instance, Dylan (I/S) wrote, "there are more than one possible side lengths. The desired area is known, therefore, one length can be written as a function of the other. If this were plotted, a line of possible answers would be found." A simple response, that nonetheless provides insight to a participant's conceptual knowledge of the problem, comes from Noah (I/S) who stated that he thinks of "squishing it [the rectangle]" to achieve a different size perimeter but maintain the same area.

Table 7

Level of Understanding by Division for (Relationship between Perimeter and Area) Item 3 (N = 110)

	Education Division					
Level of Understanding	P/J (n = 39)		J/I (n = 46)		I/S (n = 25)	
And Finding	n	%	n	%	n	%
No Demonstrated						
Blank	0	0	1	2	0	0
Incorrect	3	8	3	6	1	4
Sub-total	3	8	4	9	1	4
Procedural						
Incomplete	34	87	40	87	22	88
Sub-total	34	87	40	87	22	88
Conceptual						
Correct	2	5	2	4	2	8
Sub-total	2	5	2	4	2	8
Total	39	100	46	100	25	100

Note: Percentages have been rounded for clarity.

Procedural or some understanding.

Over 80% of the participants in each division were categorized into procedural or some understanding with, 87% P/J, 87% J/I, and 88% I/S respectively. The majority of participants evaluated at this level were able to identify that any combination of multiplies of 36 would produce a correct answer. An example of this comes from Emily (J/I) who wrote, "[b]y finding the various multiples of 36 you can find a variety of different rectangles". She then listed all the multiples and wrote that you then "[f]ind which pairs equal 36 when multiplied". As discussed earlier, without providing a statement to explain their reasoning, I could only classify them as having a procedural understanding.

Some, such as Ella (I/S), excluded a six by six square as a possibility, stating, "it is not a rectangle". While this is a misunderstanding of the definition of rectangle it does not indicate a misunderstanding of the concept under scrutiny, therefore they were categorized in this level.

No demonstrated understanding.

A small proportion, 8% P/J, 9% J/I, and 4% of the I/S participants could not answer the item correctly. Typical attempts at solving the problem involved discussions on cubing and the inclusion of three dimensional drawings (see Figures 19a and 19b).They seemed to be trying to think about volume, possibly confused by the word 'dimensions' as seen in Figure 18a.

Figure 19a Evaluated as No Demonstrated Understanding and Coded as Incorrect (Ryan J/I)

3. Given the area of a rectangle is 36 square centimetres, can there be more than one set of dimensions for the rectangle? Explain how you got your answer.

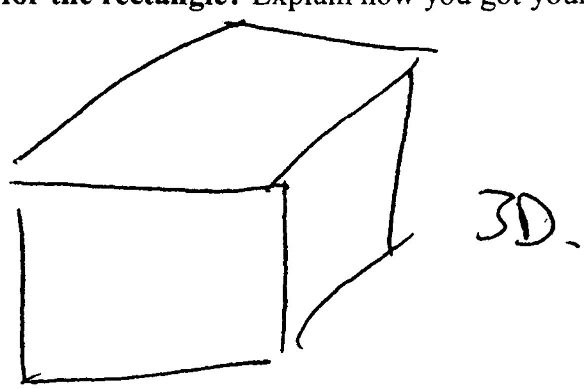
~~Yes it can~~

No I don't think so because the result in an area question is squared.

More than one dimension would be cubed.

Figure 19b Evaluated as No Demonstrated Understanding and Coded as Incorrect (Daniel J/I)

3. Given the area of a rectangle is 36 square centimetres, can there be more than one set of dimensions for the rectangle? Explain how you got your answer.



Questionnaire Item 4

Similar to Item 3, participants were to determine if rectangles with different dimensions, but the same total perimeter, could have the same total area (as detailed by Chapin and Johnson, 2000).

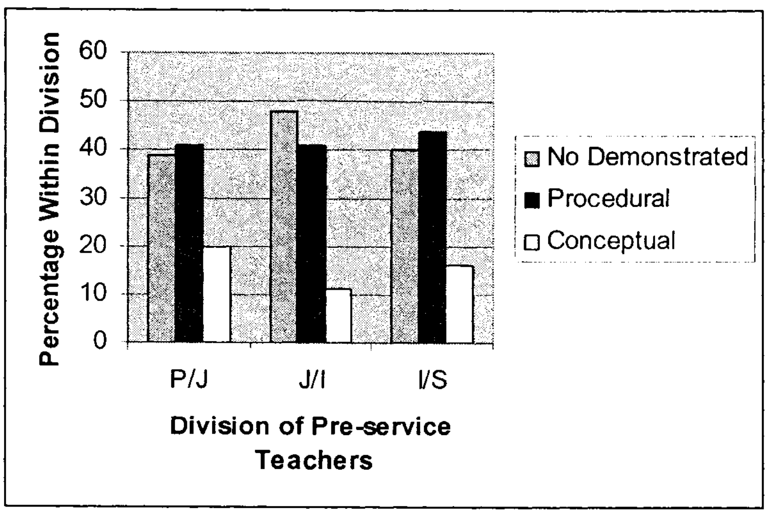
Pre-service teachers' solution of Item 4 (relationship between area and perimeter).

This item was far more difficult for the participants than the previous one. Similar to Item 3, a conceptual understanding of this problem requires that participants explain how the area of a rectangle changes when the dimensions change, regardless of the size of the total perimeter. Different from Item 3, this is a more sensitive and accurate measure of conceptual understanding as it is difficult to solve correctly using rote methods. Overall, a surprisingly small group (15%) of the participants had a conceptual understanding while 42% and 43% had a procedural or no demonstrated understanding respectively (see Figure 20).

Comparison across divisions.

We can see that all three divisions had difficulties understanding the relationship between area and perimeter. All three divisions had slightly more than 40% in the procedural understanding level while the P/J division had 39%, the J/I 48%, and the I/S 40% in the no demonstrated understanding level (see Figure 20).

Figure 20: Comparing Level of Understanding across Division, Item 4 (n = 110)



Conceptual understanding.

Surprisingly, 20% of the P/Js answered this item correctly, while only 11% and 16% of the J/I and I/S participants, respectively answered correctly (see Table 8). Typical written responses to this problem include, "[b]ecause the fatter the shape, the more space occupied hence larger area. Therefore, area depends on the shape" (Alyssa, J/I) and "[n]o the area would be different because as dimensions change the total area changes" (Courtney P/J).

Procedural or some understanding.

Almost half (41% P/J and J/I and 44 % I/S) of the participants in each division were categorized into this level. These participants generally provided one or two examples of rectangles with the same total perimeter but different dimensions that did not have the same area, which is not a sufficient proof that there is no case in which this might occur. What they failed to describe, to show conceptual understanding, was how area becomes larger as the rectangle becomes more compact (less difference between length and width) or some other explanation of the relationship. For example, Jade (P/J) provided three examples of rectangles with the same perimeter but different dimensions that have different total areas (see Figure 21).

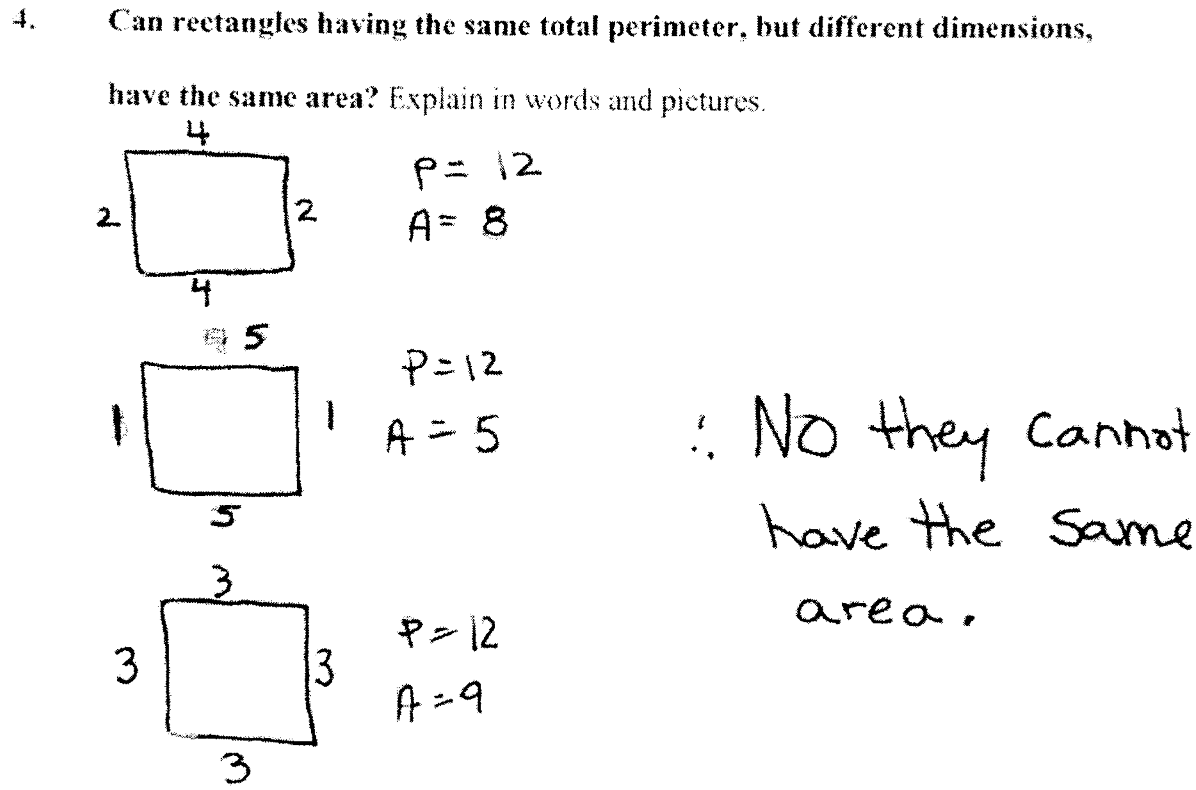
Table 8

Level of Understanding by Division for (Relationship between Area and Perimeter) Item 4 (n = 110)

	Education Division					
Level of Understanding	P/J (n = 39)		J/I (n = 46)		I/S (n = 25)	
and Finding	n	%	n	%	n	%
No Demonstrated						
Blank	4	10	5	11	0	0
Incorrect	8	21	14	30	5	20
Misread Item	3	8	3	7	5	20
Sub-total	15	39	22	48	10	40
Procedural						
Procedural	16	41	19	41	11	44
Sub-total	16	41	19	41	11	44
Conceptual						
Correct	8	20	5	11	4	16
Sub-total	8	20	5	11	4	16
Total	39	100	46	100	25	100

Note: Percentages have been rounded for clarity.

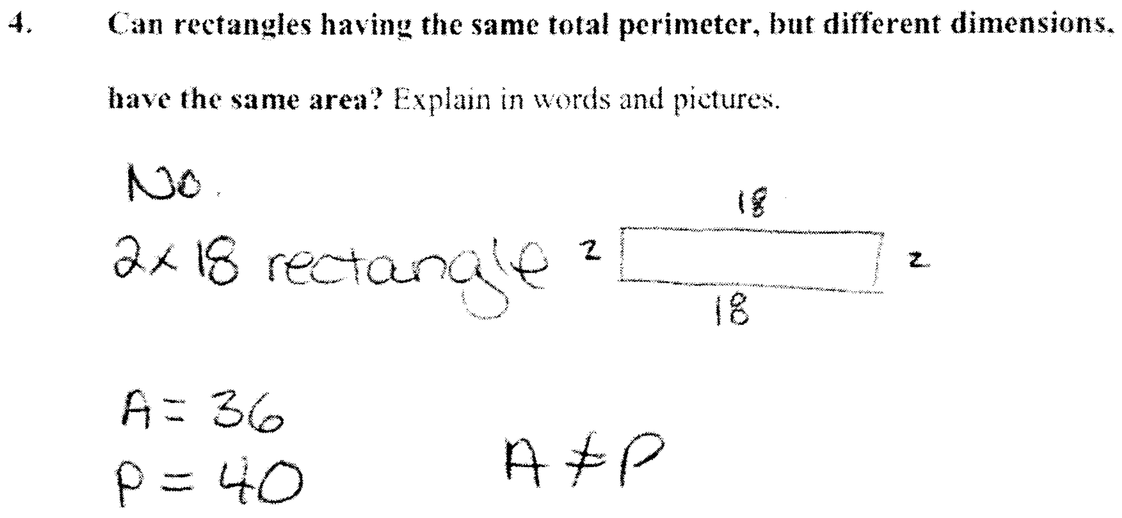
Figure 21: Evaluated as Procedural or Some Understanding and Coded as Procedural (Jade P/J)



Although Jade may have a conceptual understanding of the problem, without showing that she understands by providing her reasoning, I could not categorize her answer as demonstrating conceptual understanding. Instead I categorized this as correct procedural understanding. Ma (2000) makes a similar decision in her categorization of teachers' solutions to area and perimeter problems labelling this as a solution through cases or *more examples* rather than *mathematical insight* (p. 87)

Other students answered correctly but seem to have misread the problem as 'Can you have a rectangle with the same perimeter and area?' Jenna (J/I) provided the example of a 2 by 18 rectangle stating that the area is 36 and the perimeter is 40 (see Figure 22). She then stated that the area does not equal the perimeter concluding 'no'.

Figure 22: Evaluated as Procedural or Some Understanding and Coded as Procedural (Jenna J/I)

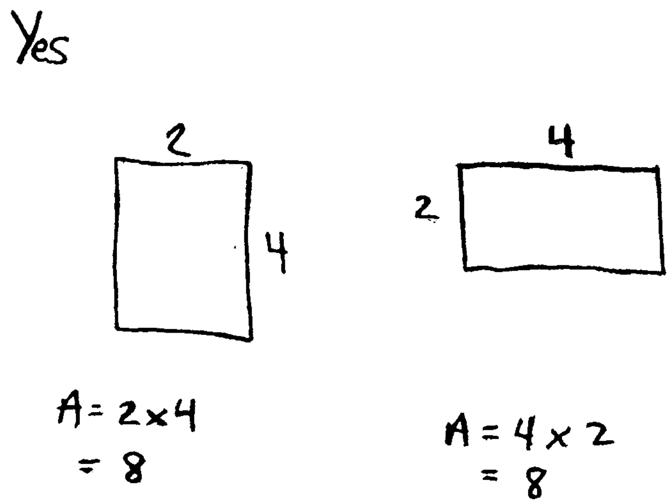


No demonstrated understanding.

A sizeable proportion of the participants were classified in the level of no demonstrated understanding, 39% of the P/J, 48% of the J/I, and 40% of the I/S because they could not provide correct answers. Students attempted various solutions for this item such as trying to create a 3D solution similar to Item 3, or stating that they knew the answer was "yes" because they were taught this in class, or inverting the dimensions such as Justin (I/S) did, (see Figure 23) resulting in the same dimensions. In the latter case, an argument could be made that a 4 x 2 is different than a 2 x 4 rectangle in some real life situations. If a student indicated that this was the only case in which 'different dimensions' would result in the same area this was coded as correct (procedural or some understanding). There were three other students however who, like Justin, gave no further explanation or any indication that this was the only case in which 'different dimensions' would give the same area. I therefore, classified these four as no demonstrated understanding. This lack of understanding was confirmed during interviews with three of the four.

Figure 23: Evaluated as No Demonstrated Understanding and Coded as Incorrect (Justin I/S)

4. **Can rectangles having the same total perimeter, but different dimensions, have the same area?** Explain in words and pictures.



Relationship between perimeter and area.

To evaluate the type of understanding pre-service teachers have of the relationship between the perimeter and area, I explored the relationship between questionnaire items three and four, used to measure this concept. In Chapter 3, I had proposed to use Cramer's phi to measure the degree of correlation between the nominal variables however, I determined that this statistical procedure was untenable, as the items could not be appropriately combined into composite scores. The sample size was not large enough to provide a significant correlation; therefore a contingency table was used.

In order to examine the strength of the relationship between Item 3 and Item 4 I created a contingency table to compare the two items (see Table 9). In this table, I categorized pre-service teachers' scores for both items into the three levels of understanding to see into which level of understanding the participants' solutions fell. We can see that 55% of the participants' scores agree with the measure of association while there are no scores in strong disagreement. This tells us that there is a strong relationship between the two items. Only 10% of the participants had a conceptual understanding of Item 4 when they had a procedural understanding of Item 3; however, 35% who had a procedural understanding of Item 3 had no demonstrated understanding of Item 4. We can infer that there were students who correctly solved Item 3 by finding factors of 36 without more deeply understanding the relationship between area and perimeter. Without this understanding they could not solve Item 4, showing that Item 4 is indeed more difficult. Forty-two percent of the participants had a procedural understanding of both items while only 5% had a conceptual understanding of both items.

Table 9

Measurement of the Concept of Relationship between Perimeter and Area (n = 110)

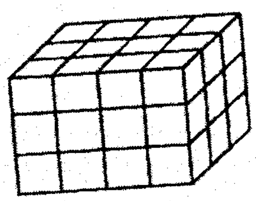
Item 3	Item 4		
Level of Understanding	None	Procedural	Conceptual
None	8	0	0
Procedural	39	46	11
Conceptual	0	0	6

Questionnaire Item 5

Figure 24: Questionnaire Item 5

5. a) What is the volume of a rectangular prism with the dimensions of 6cm by 5cm by 3cm?

b) How many 1cm by 1cm by 1cm cubes does it take to make this cube complete? Explain your answer.



Note: Taken from Battista and Clements, 1998.

Pre-service teachers' solution of Item 5 (volume).

In this two-part item the participants had to determine the volume of a rectangular prism with the dimensions 6cm by 5cm by 3cm and then determine how many 1cm cubes were in the figure (see Figure 24). The answers generally fell into the conceptual or procedural understanding levels (see Table 10). Part a is a traditional word problem requiring the use of the volume formula. If answered correctly the participant would have a procedural understanding of the item. On the other hand, by adding Part b to the item a participant's conceptual understanding of the item could be determined. In this case, participants needed to know that volume is the space inside the object and by counting the tiles and knowing that these tiles are actually small cubes they can determine the quantity of cubes within the larger object. It may be that they eventually use the volume formula to solve the problem but they needed the conceptual understanding first. It is interesting to note that for this volume problem, the majority of the participants (73%) were able to understand the concept (see Figure 25). A mere 3% had no demonstrated understanding while the remaining 24% were rated as having a procedural understanding.

Table 10

Level of Understanding by Division for (Volume) Item 5 (n = 110)

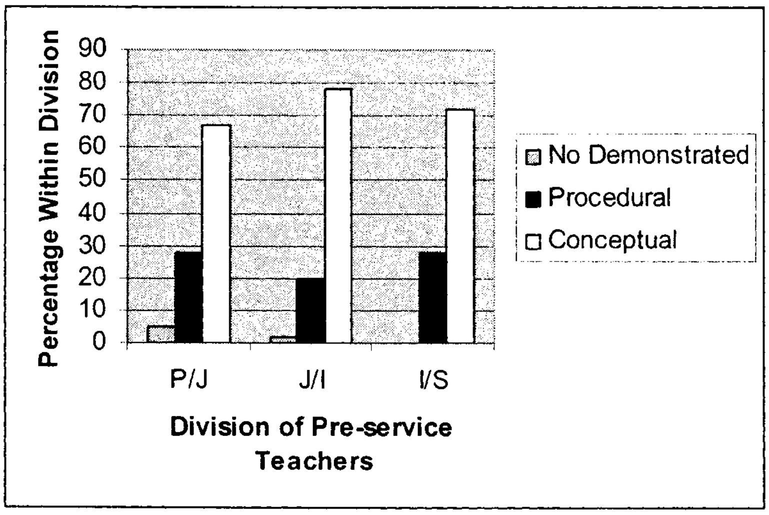
	Education Division					
Level of Understanding	P/J (n = 39)		J/I (n = 46)		I/S (n = 25)	
And Code	n	%	n	%	n	%
No Demonstrated						
Incorrect	* 2	5	1	2	0	0
Sub-total	2	5	1	2	0	0
Procedural						
Formula/Battista	6	15	6	13	4	16
Some Spatial	5	13	3	7	3	12
Sub-total	11	28	9	20	7	28
Conceptual						
Correct	24	62	29	63	17	68
No Units	2	5	6	13	0	0
Math Calculation Error	0	0	1	2	1	4
Sub-total	26	67	36	78	18	72
Total	39	100	46	100	25	100

Note: Percentages have been rounded for clarity.

Comparison across division.

Most participants had a conceptual understanding of this item with the lowest percentage of participants (67%) found in the P/J division, while the J/I division had the highest with 78%. The I/S division had no participants in the no demonstrated understanding level although they had 28% in the procedural understanding level as did the P/J participants (see Figure 25).

Figure 25: Comparing Level of Understanding across Division, Item 5 (n = 110)



Conceptual understanding.

If the participants answered Part a and Part b correctly they were evaluated as having a conceptual understanding of volume. Sixty-seven percent of the P/J, 78% of the J/I and 72% of I/S participants fell into this level. Of those interviewed four had a conceptual understanding. For example, Naomi (P/J) explained that:

> I looked at it as pancakes. So here is one pancake, two pancakes, three pancakes [looking at layering by stacking pancakes] so you have three levels of pancakes and each level has 12 … so because it's a cube it has to be 36cm cubed.

Liam (I/S) explained that he looked at the figure and counted how many blocks were along the length, width, and height and then used the volume formula to solve the problem.

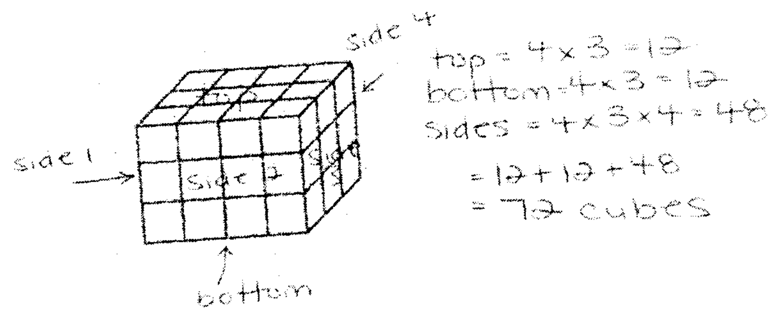
Errors such as missing units of measure and multiplication errors were nonetheless included as part of the conceptual understanding level, as the participants were able to show that they understood the concept of volume. This however, was a relatively small number of the 110 participants: 10 in total.

Procedural or some understanding.

Two scenarios formed the procedural level of understanding. Participants who were able to complete Part a correctly but had an incorrect answer for Part b comprised slightly more then half of this group; overall the P/J division had 15%, the J/I 13%, and the I/S 16%. These participants were coded as having a procedural 'formula/Battista' understanding. Interestingly, for Part a, they all used the standard length by width by height formula and for part b they were unable to visualize the formula from the diagram as in Figure 26 and answered in typical fashion as identified by Battista & Clements (1998). We can see how Samantha (J/I) could not visualize the cubes in the diagram; rather she counted the faces to solve the item. Here the participant counted how many squares were on each face (12), which in this case is not true; however, she then multiplied this by the six sides of the prism not realizing she is counting some cubes more than once and others not at all.

Figure 26: Typical Battista and Clements (1998) Response to Item 5 (Samantha J/I)

h) **How many 1cm by 1cm by 1cm cubes does it take to make this cube complete?** Explain your answer.



Note: Taken from Battista and Clements, 1998.

I multiplied 4x3 (l x w) to find out the number of cubes on each face and then since there are 6 sides/faces, I multiplied 4x3 = 12 by 6 which gave me 72.

On the other hand, participants who could not answer Part a correctly, but were able to get the right answer for Part b, had only some understanding of the concept and were coded as having some 'spatial' understanding. For this code there were 13% P/J, 7% J/I, and 12% I/S pre-service teachers. Based on the response of one participant, Maria (P/J), it could be argued that the word 'prism' in the first part of the question confused these participants into believing the item was asking about the volume of a triangular prism. Answering Part b correctly, Maria used the correct formula for the volume of the rectangular prism in Part a however, chose to divide her answer by two (see Figure 27) which would have given her the volume of the triangular prism. Some of the participants who were classified with this code could not answer Part a although answered Part b using their spatial reasoning (see Figure 28). These participants therefore, were evaluated at the procedural or some understanding level.

Figure 27: Evaluated as Procedural or Some Understanding and Coded as Some Spatial (Maria P/J)

5. a) What is the volume of a rectangular prism with the dimensions of 6cm by 5cm by 3cm?

$V = l \times w \times h$

$V = \frac{6 \times 5 \times 3}{2}$

$= \frac{90}{2}$

$= 45 cm^3$

the volume of a rectangular prism is the dimensions of 6cm by 5cm x 3cm is $45cm^3$

Figure 28: Evaluated as Procedural or Some Understanding and Coded as Some Spatial (Courtney P/J)

5. a) What is the volume of a rectangular prism with the dimensions of 6cm by 5cm by 3cm?

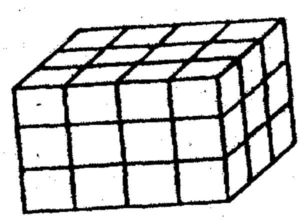
?

b) How many 1cm by 1cm by 1cm cubes does it take to make this cube complete? Explain your answer.

$3 \times 3 = 9$

$9 \times 4 = 36$

36



Note: Taken from Battista and Clements, 1998.

No demonstrated understanding.

Two P/J participants and one J/I participant were evaluated as having no demonstrated understanding of this problem. These three participants' attempt at solving the problem was incorrect.

Use of spatial images.

This item deals with mental imagery involving volume and three dimensional figures. Once again, the participants had to identify whether they used a mental image to help solve the problem. Of the 72% who had a conceptual understanding of the problem, 47% used mental imagery and 25% did not (see Table 11). In the procedural understanding level (25%), 16% had used a mental image while 9% did not. Only 3% were evaluated at the no understanding level.

Table 11

Use of Mental Images to Solve Questionnaire Item 5 Across All Divisions (n = 110)

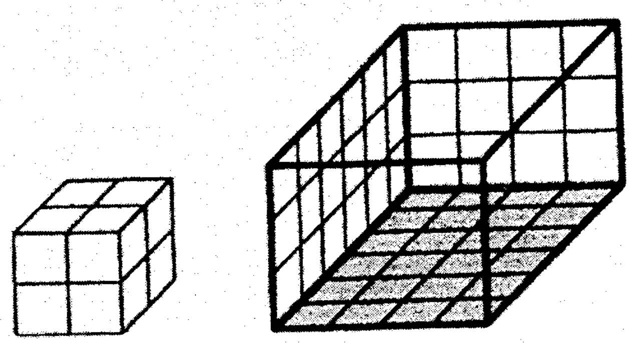
	Level of Understanding					
	None		Procedural		Conceptual	
Spatial Ability	n	%	n	%	n	%
No	2	2	10	9	27	25
Yes	1	1	17	16	53	47
Total	3	3	27	25	80	72

Note: Percentages have been rounded for clarity.

Questionnaire Item 6

Figure 29: Questionnaire Item 6

6. **How many individual rectangular prisms with the dimension 2cm by 2cm by 2cm (shown) fit into the larger rectangular box shown below?** Explain your answer.



Note: Taken from Battista and Clements, 1998.

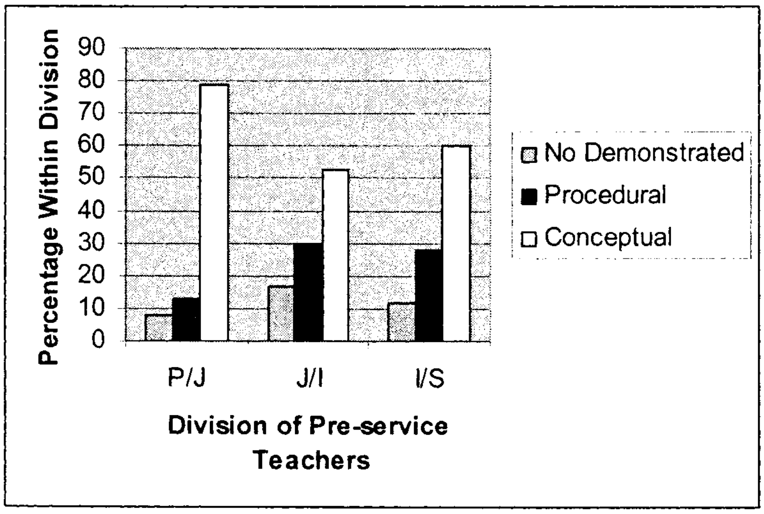
Pre-service teachers' solution of Item 6 (volume).

This item, taken from Battista and Clement (1998) was used to determine whether participants had a conceptual understanding of the concept of volume (see Figure 29). If participants solved the problem using traditional formula method, the result would be nine, a number that only works if three smaller boxes were cut in half. This would show that they are using a procedure without full understanding of the concept. On the other hand, if the participants were able to solve the problem they would conclude that only six smaller boxes would fit into the larger box. In this final item of the questionnaire the majority of participants were evaluated as having a conceptual understanding of volume while almost all the others had a procedural understanding. Overall 64% were categorized as having conceptual understanding while 24% and 12% were categorized into the procedural and no demonstrated understanding levels respectively.

Comparison across divisions.

Interestingly a much larger proportion of P/J participants' understanding of the item was rated at the conceptual level than was the case in the other two divisions (see Figure 30). Across divisions we can see that 79% P/J, 53% J/I, and 60% I/S had a conceptual understanding of the problem.

Figure 30: Comparing Level of Understanding across Division, Item 6 (n = 110)



Conceptual understanding.

A large percentage of participants from each division were able to answer the problem with a conceptual understanding (see Table 12). Seventy-nine percent of the P/J, 53% of the J/I, and 60 % of the I/S participants were in this level. Of these participants 13% of the P/J and 30% of the J/I pre-service teachers gave no explanation for their correct answer. Once again I assumed they had a conceptual understanding of the problem since they provided a correct answer. Manipulatives played a key role in solving this item during the interviews. For example, Cassandra (P/J) who answered with a conceptual understanding on the questionnaire could not

Table 12

Level of Understanding by Division for (Volume) Item 6 (n = 110)

	Education Division					
Level of Understanding	P/J (n = 39)		J/I (n = 46)		I/S (n = 25)	
And Code	n	%	n	%	n	%
No Demonstrated						
Incorrect	3	8	6	13	3	12
Blank	0	0	2	4	0	0
Sub-total	3	8	8	17	3	12
Procedural						
Formula Only	5	13	14	30	7	28
Sub-total	5	13	14	30	7	28
Conceptual						
Correct	26	67	21	46	15	60
No Explanation	5	13	3	7	0	0
Sub-total	31	79	24	53	15	60
Total	39	100	46	100	25	100

Note: Percentages have been rounded for clarity.

reproduce her answer during the interview until she used the manipulatives to help her visualize the six cubes in the diagram. Although Grace's (J/I) solution was conceptually answered on the questionnaire she expanded her answer by stating that if the smaller cubes were split in half two more could fit. During the interview, she changed her answer correctly to three after using the manipulatives to visualize the problem.

Procedural or some understanding.

Thirteen percent, 30%, and 28% of P/J, J/I, and I/S participants respectively, had only a procedural understanding of this item. Here, they simply calculated the volume of the small and large boxes using the volume formula and then divided the volume of the smaller box into the volume of the larger box not realizing that the entire amount of smaller boxes could not fit in as a whole.

No demonstrated understanding.

There were 8% P/J, 17% J/I/, and 12% I/S participants who could not answer the item. On the questionnaire, Ella (I/S) visualized 12 smaller cubes fitting into the larger one, two layers of six (obviously incorrect); however, during the interview, while playing with the manipulatives, she responded by saying "there is no way that is 12" and then carried on, solving the problem correctly. She was categorized at this level of understanding because manipulatives were not used while the participants completed the questionnaire.

Use of spatial images.

Once again, mental images played a role in answering this problem on volume. A total of 63% of the participants had a conceptual understanding of the problem (see Table 13). Of these most participants (56%) stated that mental imagery was used to help solve this problem while 7% identified that they did not use mental imagery. Most (but not all) of the students who solved the item in a procedural way did not use mental imagery. Interestingly of the 12% who were classified as having no demonstrated understanding 7% reported using mental imagery.

Table 13

Use of Mental Image to Solve Questionnaire Item 6 (n = 110)

	Level of Understanding					
	None		Procedural		Conceptual	
Spatial Ability	n	%	n	%	n	%
No	5	5	19	18	8	7
Yes	8	7	8	7	62	56
Total	13	12	27	25	70	63

Note: Percentages have been rounded for clarity.

Concept of Volume.

As with the relationship between perimeter and area, I combined volume Items 5 and 6 to determine whether there was a relationship or overlap between the items. Once again, Cramer's phi was initially proposed as a way to measure the correlation between the variables but became untenable. Fifty-four percent of the participants have a measure of association while only 5% do not, that is, 54% of the participants were evaluated at the same level of understanding in both items whereas in only 5% of the cases was a participant evaluated as having polarized levels of understanding (see Table 14). Therefore, we have some confirmation that there is a relationship between the two items. Of the 110 pre-service teachers, 48% had a conceptual understanding of the two items, 4% had a procedural understanding of the two items, and only 2% with no understanding in either case. Twenty-one percent of the participants had a conceptual understanding of Item 5 and only a procedural understanding of item 6, while 15% had a conceptual understanding of Item 6 and a procedural understanding of Item 5.

Table 14

Measurement of the Concept of Volume (n = 110)

Item 5	Item 6		
Level of Understanding	None	Procedural	Conceptual
None	2	0	1
Procedural	7	4	16
Conceptual	4	23	53

CHAPTER FIVE

Discussion

In this chapter I will answer the three research questions in the following order:

1) What do pre-service teachers understand about the concepts of perimeter, area, the relationship between the two, and volume?

2) Is spatial reasoning a key element to solving problems on the topics of area, and volume for these pre-service teachers?

3) Do the three divisions of pre-service teachers (Primary/Junior, Junior/Intermediate, and Intermediate/Senior) have the same understanding of perimeter, area, the relationship between the two, and volume?

Overview of Research

The pre-service teachers in this research generally have a poor understanding of three of the four concepts in question. Disconcertingly, approximately 38%, 42%, and 25% had a procedural understanding of perimeter, the relationship between perimeter and area, and volume items respectively. The only concept the pre-service teachers excelled in was the area item, in which 95% had a conceptual understanding. I also found that little difference existed amongst the three division's understanding of each concept. Based on the results of the questionnaire, these participants can apply the rules but not explain their reasoning about one or more of the concepts.

Pre-service Teachers' Understanding of the Four Concepts

Perimeter.

Just under half of the pre-service teachers were able to solve the perimeter item of the questionnaire. The majority of the remaining participants are typical of Menon's (1998) research in which her participants felt there was insufficient information to answer the question or they used a procedural method incorrectly in an attempt to answer the problem. Similarly, my participants had difficulties resulting in incomplete solutions or the mixing of perimeter and area formulas. Reinke's (1997) study, from which this item was adapted, found that the most common incorrect answer her participants provided (22%) was ignoring the semicircle. Similarly, although not as large of a percentage (8%), participants in this study simply forgot to add on the hypotenuse to their answer.

Area.

Given the difficulties with perimeter it was surprising that 96% of the participants in this study demonstrated a conceptual understanding of area, based on Questionnaire Item 2. This item taken from Battista et al. (1998) was geared to students in grade 2 to 5 who assessed the area of regular shapes and perhaps this item was far too easy for the pre-service teachers in this research. While almost all participants had a conceptual understanding of area, many had more difficulty with the relationships between perimeter and area.

Relationship between perimeter and area.

Comparing Items 3 and 4 of the questionnaire we can see that both were problematic for the participants. In particular, only 6% and 15% of the participants were able to explain how area and perimeter were related on both Item 3 and Item 4. The majority of the participants for both items can be compared to Ma's (1999) American participants who had difficulties conceptually understanding the relationship between perimeter and area. As with Ma's participants, these participants could only provide examples in an attempt to disprove the statement within the two items (as explained in Ma's book), falling into Ma's first category of having a weak conceptual understanding of this relationship.

Volume.

Although many of the participants conceptually understood the volume items there were still 27% and 36% who showed a difficulty with the concept of items 5, and 6, respectively. Of these participants who did not have a conceptual understanding, many had similar difficulties in solving problems related to volume as the participants in Ben-Haim, Lappan & Houang's (1985) and Battista's (2003) research. These participants would count faces on the rectangular prism rather than recognizing them as cubes. This led then, to counting the cubes more than once on the edges and corners of the rectangular prism.

The wording of Item 5 Part b was problematic as well. Labelling the figure a cube may have led participants astray. The figure is a rectangular prism, not a cube. However, if this was the case only one participant would have been affected. Dylan (I/S) identified the cube as being 4 by 4 by 4 therefore, concluding that 28 more cubes would be required to 'complete a cube'. The majority of these participants who had difficulties with the items had difficulty spatially visualizing the item as discussed in the next section.

Is Spatial Reasoning a Key Element in Solving Problems Regarding Area and Volume?

Educators Chapin and Johnson (2000) claim that having the skill to reason spatially provides people with the ability to "remember the underlying mathematical concepts" (p. 162). Based on my findings I would have to agree, particularly with the concept of volume.

Area.

Of the participants who have a conceptual understanding of area, 68% stated that they used a mental image to solve the problem; the remainder were able to solve the problem without visualization. As stated earlier none of the participants had misunderstandings of the concept similar to those found by other researchers studying children such as Battista et al., 1998.

Volume.

Ben-Haim, Lappan & Houang (1985) and Battista (2003) identified in their research that spatial ability plays a key role in understanding problems related to volume. My research had similar results. We can see that spatial reasoning does play a key role, when attempting to solve problems related to volume. For instance, of the 70 participants who conceptually understood Item 6, 89% of them used spatial reasoning when solving the item. Only 11% who conceptually understood were able to solve the item without using spatial reasoning. Battista (1998) argued that students who spatially reason can visualize the smaller cubes within a prism and proceed to "skip counting or multiplying by the number of cubes in a column or layer" (p. 405). This was quite evident in my study; although there were another 16 participants (15%) categorized into procedural or no demonstrated understanding who stated that they used spatial reasoning in solving the item. Therefore, spatial reasoning plays a key role but does not guarantee an accurate answer. Presumably their visualization was inaccurate. Many of those who had stated they did not use a mental image to solve the problem were those who counted cubes more than once along the edges because they were counting the faces of the cubes, similar to students in Battista's (1998) study.

Additionally, I should have asked the participants, on the questionnaire, to explain how they spatially reasoned in each of the three items rather than only asking if they spatially reasoned. This may have clarified some of the discrepancies noted with participants' answers to the items.

Understanding of the Four Concepts by Each of the Divisions of Pre-service Teachers

A concern that I had when beginning this research was that pre-service teachers do not have the conceptual understanding of mathematics to be able to teach it. This is supported by many researchers such as Ma (1999), Menon (1998), and Simon and Blume (1994). In particular, I was concerned about the capacity of non-math majors (i.e. the P/J and J/I pre-service teachers) to teach mathematics to young students. Therefore, I was looking for differences, or similarities, in the way the three divisions of prospective teachers were able to complete my questionnaire items.

Perimeter.

The I/S division sample shows that there are twice as many participants with a conceptual understanding of perimeter than there are participants with no demonstrated understanding. This is a sizable difference when comparing the results of the other two divisions. The P/J and J/I participants have the same amount in each of the two levels, 46% and approximately 38% respectively. While this is a stronger finding, in favour of I/S understanding, there are over one quarter I/S, two-thirds J/I, and almost half of the P/J pre-service teachers who do not have a clear understanding of the concept of perimeter. I was surprised at the results, as I had assumed through personal experience that perimeter would be an easier concept to understand than area.

Area.

All three divisions of pre-service teachers in this study have a better understanding of area than they have of perimeter as measured by the test items. This result differs from the results of researchers such as Simon & Blume (1994) and Baturo & Nason (1996) who found that their participants had difficulty with the concept of area. This may be in part due to the fact that the perimeter item was more difficult then the area item. The difference between the three divisions (6%) is not sufficient to justify the conclusion that one division had a better understanding than another.

Relationship between perimeter and area.

Based on questionnaire Items 3 and 4, the three divisions have similar results in regard to the type of understanding the participants have of this concept. Similar to Ma (1999), we can see that there are high percentages in each division with only a procedural understanding of the relationship. Although 20% of the P/J participants had a conceptual understanding of Item 3, the highest of the three divisions, it is not reasonable to say that they have an overall better understanding of the concept based on the high percentages within the procedural and no demonstrated understanding levels.

Volume.

All three divisions have teachers with a conceptual understanding of the volume items. It seems however, that the P/J division have an overall better understanding than the other divisions. This may be due to the fact that the P/J division wrote a content exam at the end of the fall semester that included questions regarding the concept of volume. Therefore, while they may not have had a greater understanding coming into the programme, a sizable number attended tutoring or worked in study groups prior to this study, unlike the I/S or J/I divisions.

General Discussion

Ben-Haim, Lappan & Houang's (1985) found it difficult for people to learn about three dimensional objects solely from paper drawings. It is interesting to note that the participants who had trouble answering the questionnaire items were able to reach a correct solution and understand where they went wrong in solving particular problems, by using manipulatives during the interview.

Mitigating Circumstances

There are three important issues that must be addressed while reviewing the findings that may have played a role into the outcome of this research. First and foremost is the context in which this research took place. By administering the questionnaire halfway through the year each division had different experiences related to the items on the questionnaire. For instance, the pre-service teachers in the P/J write a content exam in which some of these concepts are addressed and have had the opportunity to enrol in a tutoring class to help prepare them for the exam. On the other hand, the J/I division participants had no teachings regarding these topics except for small group lessons in which a few students may have addressed one or all of these topics. The I/S division students were not taught any of the basic concepts and did not review them; although as math majors one would assume understanding. The situation may explain the poorer JI results.

A second concern is the questionnaire itself. Two items may have influenced the outcome, as the wording may have been unclear and perhaps misled the participants. For instance the wording of Item 1 may have influenced the results of this portion of the study, as discussed in the previous chapter.

Thirdly, the questionnaire did not contain enough items of each concept to provide a thorough study and analysis of the participants' understanding. Providing a more extensive questionnaire and/or administering a pre and post questionnaire, would have offered a more thorough investigation of this group of pre-service teachers' understanding of the four concepts.

CHAPTER SIX

Conclusion

Outcome

As mentioned in the Methodology section of this paper, I expected several outcomes to occur. I expected that this sample of pre-service teachers would have a limited understanding of the concepts and that the I/S participants would perform slightly better then the other two divisions. I also anticipated that the I/S division's better understanding would be attributed to their better use of spatial reasoning. Lastly, I expected that spatial reasoning would play a key role when solving problems related to the concepts of area and volume.

Based on this research, these pre-service teachers, as a whole, have a limited understanding of the four concepts in question, except for area. These participants are mostly using their procedural skills of formula memorization to enable them to compute an answer to the questionnaire items. Surprisingly, the I/S participants did not perform better in any of the concepts, except for perimeter, even though they have a mathematics background.

As for spatial reasoning, I found that most participants, regardless of division, who stated that they used spatial reasoning to help solve the items, had a better understanding of the concept. Since, the I/S division did not have a better understanding of area and volume than the other two divisions I could not determine if their spatial reasoning helped them to better understand the concepts.

Limitations

Although the study had its limitations, particularly with the wording of several of the questionnaire items and when the questionnaire was administered, it is clear that the P/J and J/I participants have a similar understanding of the concepts as do the I/S division participants. For the P/J participants, I believe the content exam which they must write as part of their Mathematics Methodology course may have played a large role with the strength of their results in this research. Without the exam I suspect that the P/J's would have fared more poorly as the exam had similar volume items to those appearing in this study. The J/I participants' performance may be attributed to what was taught in their Mathematics Methodology course, or their previous experiences with Mathematics in general, although this could not be determined.

Recommendations

I believe there should be more research that provides data on the similarities and differences that each division of pre-service or in-service teachers have in their mathematics understanding. Based on these results, I believe that Faculties of Education and School Boards alike should focus on providing pre-service and in-service teachers direct instruction of math content to enhance their ability to conceptually understand topics in mathematics such as the P/J division seems to be doing at this university. This would give them the necessary (although not sufficient) foundation to learn to teach effectively.

Appendix A: Measurement Questionnaire

SECTION 1. Demographic Information

Complete each part of this section by filling in the blanks or putting a check mark in the appropriate box. Please print clearly.

Name: ____________________ **Gender:** M: ☐ F: ☐

Contact Info: E-mail: ____________________________ @lakeheadu.ca

Which level are you enrolled in: P/J: ☐ J/I: ☐ I/S: ☐

Mathematical Background: (Please check the most appropriate box)

University Major: ☐ Graduated with Grade 13 or Academic Math: ☐

University Minor: ☐ Graduated with Grade 12 or Applied Math: ☐

Some University Math: ☐ Grade 11 Math or Lower: ☐

Other (Please Specify): ____

Are you enrolled in the Concurrent Education Program at Lakehead University?

Y: ☐ N: ☐

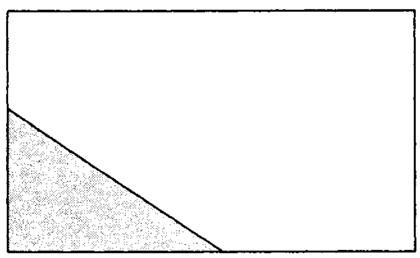
Appendix A: Measurement Questionnaire (cont'd)

SECTION 2. Math concepts

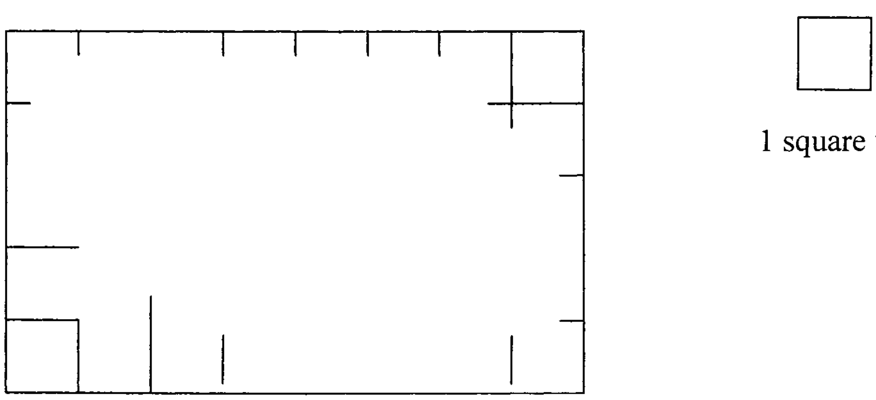
For the following 6 questions please show all your work and explain your answers in words and pictures when prompted: (use the back page if necessary but please identify the question)

1. **Show how you would find the perimeter for the <u>un-shaded</u> area of the rectangle.**

   Explain your reasoning.



2. **What is the area of the following figure?**



1 square unit

Did you use a mental image when answering this question?

Y ☐ N ☐

Appendix A: Measurement Questionnaire (cont'd)

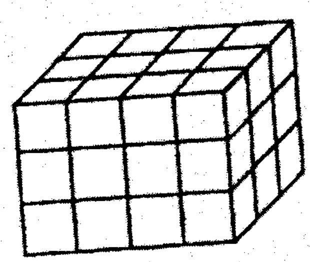
3. **Given the area of a rectangle is 36 square centimetres, can there be more than one set of dimensions for the rectangle?** Explain how you got your answer.

4. **Can rectangles having the same total perimeter, but different dimensions, have the same area?** Explain in words and pictures.

Appendix A: Measurement Questionnaire (cont'd)

5. **a) What is the volume of a rectangular prism with the dimensions of 6cm by 5cm by 3cm?**

b) How many 1cm by 1cm by 1cm cubes does it take to make this cube complete? Explain your answer.



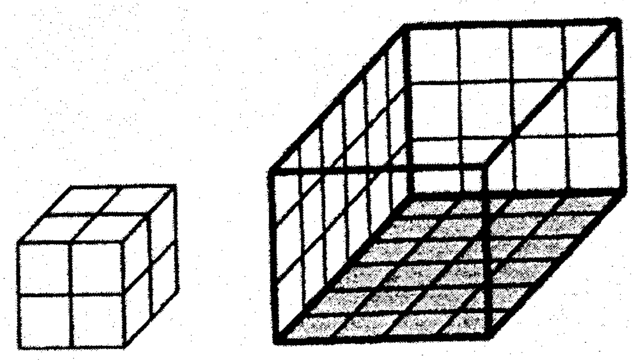
Note: Taken from Battista and Clements, 1998.

Did you use a mental image when answering this question?

Y ☐ N ☐

Appendix A: Measurement Questionnaire (cont'd)

6. **How many individual rectangular prisms with the dimension 2cm by 2cm by 2cm (shown) fit into the larger rectangular box shown below?** Explain your answer.



Note: Taken from Battista and Clements, 1998.

Did you use a mental image when answering this question?

Y ☐ N ☐

Your overall comments please:

Appendix B: Informed Consent Cover Letter

Hello,

My name is Walter Latt and I am in the second year of my Masters of Education program. I am interested in studying the knowledge prospective teachers have of measurement. My thesis is titled *Pre-service Teachers Conceptual Understanding of Perimeter, Area, and Volume.* For this research, my purpose is to determine what the three groups of Ontario pre-service teachers, Primary/Junior (P/J), Junior/Intermediate (J/I) and Intermediate/Senior (I/S), understand about the measurement topics of perimeter, area, and volume and to compare the knowledge of these three groups. Similarly, I am also interested in comparing the knowledge pre-service teachers with a major in mathematics have to those pre-service who do not have a mathematics majors. For this reason you are being invited to participate. Please note that your participation is voluntary and you may refuse to participate at any time throughout the study or refuse to answer a particular item within the questionnaire. Your participation requires you to answer items on the questionnaire and you may be asked to participate in a follow-up video-recorded interview which will take no more than one hour of your time.

Your name will remain confidential and pseudo names will be used in reporting findings and discussions with my supervisor or committee members. The data collected will be reviewed by me and one or two other education graduate students and data will be stored during the research period in the education graduate lounge in a private filing cabinet. Upon completion of my thesis the data will be stored at Lakehead University for seven years. If requested, I will provide a summary of the research following the completion of my thesis.

Although there may be no direct benefit to you from this research, it will contribute to our understanding of conceptual knowledge within mathematics and specifically in the topic of measurement. Another key component of this research is to study whether spatial reasoning plays a role in conceptually understanding mathematics.

If you have any questions regarding this research please contact me at welatt@lakeheadu.ca, or my supervisor Dr. Alex Lawson at alex.lawson@lakeheadu.ca. For general research questions please contact the Office of Research at 343-8283.

Thanking you in advance for your participation,

____

Walter Latt

Appendix C: Participant Consent Form

I the undersigned (print name) ______________________ agree to participate in a study by Walter Latt, Master of Education student at Lakehead University, on *Pre-service Teachers' Conceptual Understanding of Perimeter, Area, and Volume.* I have read and understand the cover letter, and received instructions in regards to the purpose and procedures of the study. I understand that:

1) As a volunteer, I may withdraw from the study at any time.
2) I will remain anonymous in any publication/public presentation of research findings.
3) I may be asked to partake in a one hour video-recorded interview based on the questionnaire but that I may freely choose not to partake.
4) The collected data I provide will be confidential and stored securely at Lakehead University for 7 years.
5) There is no apparent risk of physical or psychological harm to me.
6) The study will have no impact whatsoever on my course grade, and that my instructor will not be able to connect my name to the survey responses I provide.
7) Upon request, I will receive a summary of the research following the completion of the study.

Participant Signature, **Date,**

____________________ ____________________

Appendix D: Sub-Codes Combined into Main Codes for Each Questionnaire Item

	Type of Understanding	Main Code	Sub-Code
Item 1			
	Conceptual	Correct	Analytical
Item 2			
	Conceptual	Correct	Filled in Figure
Item 4			
	No Demonstrated	Blank	'Don't Know'
Item 5			
	No Demonstrated	Incorrect	Typical Battista
	Procedural	Formula Only	No Spatial
Item 6			
	Procedural	Formula Only	Minor Math Error

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