A Modification of the Effros-Handelman-Shen Theorem with $\mathbb{Z}_2$ actions

by

Bit Na Choi

Department of Mathematical Sciences
Lakehead University
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Contents

ABSTRACT

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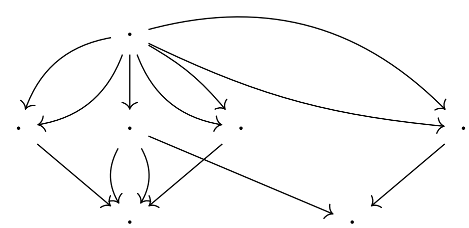
In this thesis, we show that if we have a $\mathbb{Z}_2$ action on a lattice-ordered dimension group, then it will arise as an inductive limit of $\mathbb{Z}_2$ actions on simplicial groups. This work was motivated by the range of the invariant problem in Elliott and Su's classification of AF type $\mathbb{Z}_2$ actions. In order to show this, we modify the proof of Effros-Handelman-Shen theorem to include $\mathbb{Z}_2$ actions at every stage of the arguments.

CHAPTER 1

Introduction

This is a thesis in Ordered Group Theory devoted to the study of actions on dimension groups. Dimension groups arise naturally as invariants in the classification of C^*-algebras.

The classification of AF-algebras started with Glimm's supernatural numbers, which he used to classify UHF-algebras [8]. Also, Bratteli showed how diagrams could be used to classify AF-algebras. The Bratteli diagram is a systematic way to write down inductive sequences of finite dimensional C^*-algebras [2]. For example, we encode a sequence $\mathbb{C} \to M_2 \bigoplus \mathbb{C} \bigoplus M_2 \bigoplus M_2 \to M_6 \bigoplus M_3$ as a Bratteli diagram. In the diagram below, the dots in each horizontal row represent the direct summands in the algebras in the inductive system, the number of arrows between dots counts the multiplicity of the partial embedding between those two summands. The Bratteli diagram below gives one sequence of maps such as above.



For AF algebras, the Bratteli diagram carries on downwards infinitely.

After that, George Elliott used K-theory to classify AF-algebras. He clarified the classification of these algebras by using invariants [5]. Elliott's invariant was the K_0 group of the C^*-algebras. The group K_0 of an AF algebra is an example of an ordered abelian group, a group with a partial order that is translation invariant.

An ordered abelian group (G, G^+) is said to be unperforated if every $x \in G$ for which $nx \geq 0$ for some $n \in \mathbb{N}$ satisfies $x \geq 0$. An ordered abelian group (G, G^+) is said to have the Riesz interpolation property if for every $x_1, x_2, y_1, y_2 \in G$ where $x_i \leq y_j$ for $i, j = 1, 2$, there exists $z \in G$ with $x_i \leq z \leq y_j$ for $i, j = 1, 2$ [14]. The Effros-Handelman-Shen theorem says that a countable ordered abelian group (G, G^+) is the K_0 group of an AF-algebra if and only if it is unperforated and has the Riesz interpolation property[4]. After the classification of AF-algebras was done, people began to add more things to the invariant besides K_0 in order to classify more algebras.

We mentioned the generalization of the classification of AF algebras above. Now, we would like to talk about the generalization of the classification of AF algebras to classifications of algebras with actions on them. First, Handelman and Rossmann assumed that the algebra was a UHF algebra and the action was product type in [10]. After that, they generalized the content of their previous paper [10] to locally representable action on an AF algebra in [11]. In [1], Blackadar showed that there were actions of $\mathbb{Z}_2$ that were not locally representable even on UHF algebras. Elliott and Su generalized the K-theoretic classification of Handelman and Rossman in [11] by removing locally representable. They still keep AF inductive limit type and they restricted the action to the group $\mathbb{Z}_2$ [7].

The range of the invariant that Elliott and Su used has still not been completely determined. In this thesis, we tried to generalize the Effros-Handelman-Shen theorem to apply to the invariant of Elliott and Su. Our main theorem is a step towards the generalization. This thesis is organized as follows. In chapter 2, we discuss classification, especially, using a functor. We discuss C^*-algebra facts in chapter 3. We describe the semi-groups $\mathcal{D}(A)$, the K_0 group of a unital C^*-algebra, and inductive limits in chapter 4. In chapter 5, we discuss Elliott's intertwining argument that is the pattern to prove Elliott's AF classification theorem. Chapter 6 contains the range of the invariant problem for the classification that goes with the theorem Elliott and Su found. Chapter 7 is our main result of this thesis. It contains a modification of the Effros-Handelman-Shen theorem where we restrict the dimension group to a lattice-ordered one but include $\mathbb{Z}_2$ actions.

CHAPTER 2

Classification

This chapter explains about classification, in particular, a classification by using a functor.
As we see in [6], there are lots of classifications as follows:

– A complete list
– A complete list using labels
– A functor

Here are examples of each kind of classification.

Example 2.1.

(1) An example of a complete list is the classification of finite simple groups. As we see in [3], the Hölder program is the project to classify those groups.
(2) An example of a complete list using labels is the classification of complex simple Lie algebras, of which there are four sequences A_n, B_n, C_n, D_n, and five exceptions(E_6, E_7, E_8, F_4, and G_2). These Lie algebras are classified by their Dynkin diagrams.
(3) The last example is the classifying functors that are used in the Elliott classification program.

We would like to explain about a functor in detail. Before explaining about a functor, we have to know what a category is. Here is the definition of a category.

Definition 2.1. [14, 3.2.1] A **category** C consists of a class $\mathcal{O}(C)$ of objects and for each pair of objects A, B in $\mathcal{O}(C)$ a set $\mathrm{Mor}(A, B)$ of morphisms from A to B with an associative rule of composition

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C),\ (\varphi, \psi) \mapsto \psi \circ \varphi$$

such that for each object X there is an element id_X in $\mathrm{Mor}(X,X)$ which satisfies $id_Y \circ \varphi = \varphi = \varphi \circ id_X$ for every φ in $\mathrm{Mor}(X,Y)$.

Here are various examples of categories.

EXAMPLE 2.2 (Category (Objects, Morphisms)).

- (Groups, Group homomorphisms)
- (Rings, Ring homomorphisms)
- (Vector spaces, Linear maps)
- (Sets, Functions)
- (Topological spaces, Continuous maps)
- (Pointed spaces, Pointed maps)
- (C^*-algebras, *-homomorphisms)
- (Partially ordered abelian groups, ordered group homomorphisms)

From now on, $\mathbf{C}^*$ denotes the category that consists of C^*-algebras and *-homomorphisms, and **AbG** denotes the category that consists of partially ordered abelian groups and ordered group homomorphisms.

Based on the definition of category, we define the two types of functors.

DEFINITION 2.2. [14, 3.2.1] Let C_1 and C_2 be categories. For each object A in C_1, we have an object $F(A)$ in C_2. Also, for each morphism $\varphi : A \to B$ in C_1,
We can define $F(\varphi)$ in two different ways.

(1) we have $F(\varphi) : F(A) \to F(B)$, $F(id_A) = id_{F(A)}$, and $F(f \circ g) = F(f) \circ F(g)$ where f and g are morphisms in C_1. A functor F such as this is called a **covariant functor**. This functor preserves identity morphisms and composition of morphisms.
(2) we have $F(\varphi) : F(B) \to F(A)$, $F(id_A) = id_{F(A)}$, and $F(f \circ g) = F(g) \circ F(f)$ where f and g are morphisms in C_1. A functor F such as this is called a **contravariant functor**. This functor reverses the direction of composition.

Here are examples of functors.

EXAMPLE 2.3.

(1) K_0 is a covariant functor from approximately finite dimensional $\mathbf{C}^*$ (AF $\mathbf{C}^*$) to $\mathbf{AbG}$.
(2) Fundamental group is a contravariant functor from pointed topological spaces to groups.
(3) A forgetful functor, which is covariant, forgets or drops some or all of the input's structure or properties before mapping to the output. For example, a mapping from vector space to set, and a mapping from linear maps to functions.

We will explain more about the first one of the above examples in chapter 4.

When we have classification by a functor, we have a class of objects that we want to classify with their homomorphisms, and a simpler category for the functor to take its values in. We want a functor to have the property that an isomorphism between invariants comes from an isomorphism between objects. In fact, if $\psi : F(A) \to F(B)$ is an isomorphism, then we want a $\widetilde{\psi} : A \to B$ that is an isomorphism with $\psi = F(\widetilde{\psi})$. In this case, trivial automorphisms should go to the identity maps. The advantage of using functors for classification is giving more information about the relationship between objects than isomorphism. We usually get a homomorphism theorem which tells us when one object embeds into another one.

In order to classify a category, we would like to have a functor which ignores certain automorphisms considered to be trivial. So, here are some definitions.

DEFINITION 2.3. Let R be a ring. Suppose x is an invertible element in R. The map $\varphi(y) = xyx^{-1}$ is an isomorphism of R with itself, also known as an automorphism. Such an automorphism is called an **inner automorphism** of R. With a C^*-algebra A, if u satisfies $uu^* = u^*u = 1$, then $x \mapsto uxu^*$ is called an **inner *-automorphism**.

There exist automorphisms that are not inner. For instance, consider the ring $\mathbb{C} \oplus \mathbb{C}$. Then, there is an automorphism $(x, y) \mapsto (y, x)$ where $x, y \in \mathbb{C}$. Since the ring is commutative, the only inner automorphism is the identity. In this case, the automorphism is not inner.

We would like to consider the case of inner automorphisms as a trivial one. Here is a brief example, a classification using a functor.

EXAMPLE 2.4. Consider the domain category, being classified, which consists of matrix algebras over complex number as objects and unital *-homomorphisms as morphisms, *i.e.*, (Matrix algebras(M_n), unital *-homomorphisms), and trivial automorphism in this case is inner automorphism.

An order unit for an ordered group G is any positive element u in G^+ such that for any element g in G, there is some positive integer n for which $|g| \leq nu$. Define the target category, (($\mathbb{Z}$, $\mathbb{Z}^+$, n), positive unital group homomorphism). Then, we can get a functor $(K_0, K_0^+, [1])$ from the domain category to the target category such that

$$(K_0(M_n), K_0(M_n)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, n).$$

In this case, each unital homomorphism looks like down below;

$$M_n \to M_{nk} \cong M_n \otimes M_k, \text{ by } x \mapsto u(x \otimes 1_k)u^* \text{ for some unitary } u$$
$$\mathbb{Z} \to \mathbb{Z} \text{ by } x \mapsto kx \text{ for some } k \geq 0$$

In particular, the map $x \mapsto kx$ is the image under the functor of the map $x \mapsto x \otimes 1_k$ from $M_n \to M_{nk}$. It means that K_0 ignores the inner automorphism part.

We explain more this example in the next chapter.

CHAPTER 3

C^*-algebras

As we mentioned above, we would like to describe C^*-algebras in this chapter. In particular, we define AF algebras, one of the most interesting classes of C^*-algebras. First of all, we define the category of C^*-algebras, *-homomorphism, and unital.

DEFINITION 3.1. [**14**, Definition 1.1.1]

(1) A **C*-algebra** A is an algebra over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$, $a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leq \|a\| \|b\|$ and $\|aa^*\| = \|a\|^2$ for every a, b in A.
An involution is a conjugate linear function that is its own inverse, *i.e.*, $a^{**} = a$.
(2) A ***-homomorphism** $\varphi : A \to B$ between C^*-algebras A, B is a linear and multiplicative map which satisfies $\varphi(a^*) = \varphi(a)^*$.
(3) A C^*-algebra A is called **unital** if it has a multiplicative identity, which will be denoted by 1 or 1_A.
(4) If A and B are unital and $\varphi(1_A) = 1_B$, then φ is called **unital**.

Here are some examples of C^*-algebras.

EXAMPLE 3.1.

(1) $C_0(X)$, the complex valued continuous function on X vanishing at infinity, where X is a locally compact Hausdorff space with pointwise multiplication $(fg)(x) = f(x)g(x)$, the involution $f^*(x) = \overline{f(x)}$, and the norm $\|f(x)\| = sup\,|f(x)|$
(2) $M_n(\mathbb{C})$, complex $n \times n$ matrices with the involution $A^* = \overline{A^{\intercal}}$ and the norm such that

$$\|A\| = sup\{\|Ax\| \mid x \in \mathbb{C}^n \text{ with } \|x\| = 1\}$$

(3) $B(H)$, the Banach space consisting of all bounded operators from H to H for H a complex Hilbert space, where the involution is the Hilbert adjoint, defined by $< x|Ty >=< T^*x|y >$. In this case, the norm is

$$\|T\| = sup\{\|Ty\| \mid y \in H \text{ with } \|y\| \leq 1\}.$$

In order to clarify the AF $\mathbf{C}^*$ that we mentioned in Chapter 2, we move on to the definition of AF-algebra.

Definition 3.2. [14, Definition 7.2.1] An **AF-algebra** A is a C^*-algebra that satisfies that there is an increasing sequence A_n of finite dimensional C^*-algebras such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$
The term “AF” is an abbreviation of Approximately Finite dimensional.

In order to fully understand the above definition, we need to know a bit about finite dimensional C^*-algebras.

In [12], it is shown that an arbitrary finite dimensional C^*-algebra A takes the form

$$M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$$

for some integers $n_1, n_2, \cdots, n_k$ where M_n is the algebra of $n \times n$ matricies over the complex numbers.

We can define a unital map between matrix algebras;

$$M_n \to M_k \cong M_n \otimes M_m, \text{ by } x \mapsto u(x \otimes 1_m)u^*$$

for some unitary $u \in M_k$. In fact, any unital $*-$ homomorphism between full matrix algebras is of this form.

By generalizing these unital maps, we can define *-homomorphisms between finite dimensional C^*-algebras. Define

$$\pi : M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k} \to M_l$$

by

$$\pi(x) = \begin{bmatrix} \pi_1(x) & & & \\ & \pi_2(x) & & \\ & & \ddots & \\ & & & \pi_k(x) \end{bmatrix},$$

where $\pi_i : M_{n_i} \to M_{n_ih}$ by $x \mapsto u(x \otimes 1_h)u^*$ for each $i = 1, 2, \cdots, k$ and for some unitary $u \in M_{n_ih}$, and there are, possibly, some zero maps, for example, $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$ by $(x, y) \mapsto x$ for any $x, y \in \mathbb{C}$.

This is what unital homomorphisms $M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k} \to M_l$ all look like, in other words, any *-homomorphism is conjugate to one of this forms, *i.e.*, $\varphi(x) = v\psi(x)v^*$ for some unitary v. In general, unital *-homomorphisms between finite dimensional algebras are direct sums of maps like above. Since the injective *-homomorphisms are norm preserving, we can define a norm for the union of an increasing sequence of finite dimensional algebras. In this way, we can take a completion. Therefore, we can get an AF-algebra.

CHAPTER 4

K-theory

In this chapter, we shall introduce the semi-group $\mathcal{D}(A)$ and the K_0 groups that arise from the semi-group. We will show that K_0 is a functor from AF $\mathbf{C}^*$ to $\mathbf{AbG}$ as we mentioned in Chapter 2.

1. The Semi-Groups $\mathcal{D}(A)$

In this section, we would like to describe a specific semi-group, $\mathcal{D}(A)$. First, we describe the set of projections, $\mathcal{P}_\infty(A)$.

Here are definitions of a projection and $\mathcal{P}_\infty(A)$.

Definition 4.1. [14, Definition 2.2.1] An element p in a C^*-algebra is a **projection** if $p = p^2 = p^*$. The set of all projections in a C^*-algebra A is denoted by $P(A)$.

Definition 4.2. [14, Definition 2.3.1] Put

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)),\ \mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A),$$

where A is a C^*-algebra and n is a positive integer.

Define the relation $\sim_0$ on $\mathcal{P}_\infty(A)$ as follows. Suppose that p is a projection in $\mathcal{P}_n(A)$ and q is a projection in $\mathcal{P}_m(A)$. Then $p \sim_0 q$ if there is an element v in $M_{m,n}(A)$ with $p = vv^*$ and $q = v^*v$.

Define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by

$$p \oplus q = diag(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Based on the $\mathcal{P}_\infty(A)$, we would like to introduce the definition of semi-group $\mathcal{D}(A)$.

Definition 4.3. [14, Definition 2.3.3] With $(\mathcal{P}_\infty(A), \sim_0, \oplus)$, set

$$\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0.$$

For each p in $\mathcal{P}_\infty(A)$, let $[p]_\mathcal{D}$ in $\mathcal{D}(A)$ denote the equivalence class containing p. Define addition on $\mathcal{D}(A)$ by

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}, \text{ for } p, q \in \mathcal{P}_\infty(A).$$

Then, $(\mathcal{D}, +)$ is an abelian semi-group.

Here are some examples about the semi-groups $\mathcal{D}(A)$.

EXAMPLE 4.1.

- $\mathcal{D}(\mathbb{C}) \cong \mathbb{N}$
- $\mathcal{D}(M_n(\mathbb{C})) \cong \mathbb{N}$
- $\mathcal{D}(B(H)) \cong \mathbb{N} \cup \infty$ when H is an infinite dimensional Hilbert space.

In each case, we get the isomorphism by taking the trace of any projection in an equivalence class.

2. The K_0 Group of a unital C^*-algebra

Before explaining the K_0 group, we should explain what the Grothendieck construction is.

DEFINITION 4.4. [14, 3.1.1] Let $(S, +)$ be an abelian semi-group. We say that the semi-group $(S, +)$ has the **cancellation property** if, whenever x, y, and z are elements in S with $x+z = y+z$, it follows that $x = y$. Define an equivalence relation $\sim$ on $S \times S$ by $(x_1, y_1) \sim (x_2, y_2)$ if there exists z in S such that $x_1 + y_2 + z = x_2 + y_1 + z$.
Let $G(S)$ be the quotient $(S \times S)/\sim$, and let $< x, y >$ denote the equivalence class in $G(S)$ containing (x, y) in $S \times S$. Then, the operation

$$< x_1, y_1 > + < x_2, y_2 > = < x_1 + x_2, y_1 + y_2 >$$

is well-defined and $(G(S), +)$ is an abelian group. The group $(G(S), +)$ is called the **Grothendieck group** of S. Define a map

$$\gamma_S \colon S \to G(S),\ x \mapsto < x + y, y >$$

for every y. This map γ_S is additive. It is called the **Grothendieck map** . It is injective if S has the cancellation property.

The Grothendieck construction generalizes how we obtain the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$.

EXAMPLE 4.2. Consider the abelian semi-group ($\mathbb{N}$,+). When we use the Grothendieck group construction, we obtain the formal differences between natural numbers as elements $n-m$. Since this semi-group has the cancellation property, we don't need the extra element added on the equivalence relation below

$$n - m \sim n' - m' \text{ if } n + m' = n' + m.$$

Now, define for all $n \in \mathbb{N}$,

$$\begin{cases} n := n - 0 \\ -n := 0 - n \end{cases}$$

This defines the integer $\mathbb{Z}$.

We consider another example. Consider the semi-group $\mathbb{N} \cup \infty$ with addition and $n + \infty = \infty$ for all $n \in \mathbb{N} \cup \infty$. In this case, every pair is equal to every other pair. There is only one equivalence class. So, the Grothendieck group of this is $\{0\}$ and the semi-group $(\mathbb{N} \cup \infty, +)$ does not have the cancellation property.

By using the semi-group and Grothendieck construction, we begin to introduce K_0 groups. K_0 groups are defined in two cases, unital C^*-algebras and non-unital C^*-algebras. In this thesis, we concentrate on unital C^*-algebras.

DEFINITION 4.5. [**14**, Definition 3.1.4] Let A be a unital C^*-algebra, and let $(\mathcal{D}(A), +)$ be the abelian semi-group. Define $\boldsymbol{K_0(A)}$ to be the Grothendieck group of $\mathcal{D}(A)$, in other words, $K_0(A) = G(\mathcal{D}(A))$. Define a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by

$$[p]_0 = \gamma([p]_{\mathcal{D}}) \in K_0(A), \text{ for } p \in \mathcal{P}_\infty(A),$$

where $\gamma : \mathcal{D}(A) \to K_0(A)$ is the Grothendieck map.

Here is the example when we apply K_0 to the example 4.1.

EXAMPLE 4.3.

- $K_0(\mathbb{C}) = \mathbb{Z}$

– $K_0(M_n(\mathbb{C})) = \mathbb{Z}$
– $K_0(B(H)) = 0$

Now, we would like to explain the functoriality of K_0.

Proposition 4.1. [14, Proposition 3.1.8] Let A and B be a unital C^*-algebra. Given a *-homomorphism $\varphi : A \to B$, we get a group homomorphism $K_0(\varphi)$ such that $K_0(\varphi)([p]) = [\varphi(p)]$ for every projection $p \in P_\infty(A)$.

With the definitions above we have a proposition below.

Proposition 4.2. [14, Proposition 3.2.4] [13, Proposition 9.151] The K_0 is a covariant functor from the category of unital C^*-algebras to the category of abelian groups, in other words,

(1) For each unital C^*-algebra A, $K_0(id_A) = id_{K_0(A)}$
(2) If A, B and C are unital C^*-algebras, and if $\varphi : A \to B$ and $\psi : B \to C$ are *-homomorphisms, then $K_0(\psi\circ\varphi) = K_0(\psi)\circ K_0(\varphi)$

If the C^*-algebras are AF-algebras, then the following statement is true. The Grothendieck map γ is injective and its image in K_0 is the positive cone for an order of the group.

The functor K_0 moves an inductive sequence to another inductive sequence and an inductive limit to another inductive limit, *i.e.*, $K_0(\varinjlim A_n) = \varinjlim K_0(A_n)$. In other words, K_0 is continuous with respect to inductive limits. This will be explained in the next section. Here is another property: $K_0(M_{n_1} \oplus \cdots \oplus M_{n_k})$ is isomorphic to $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-\text{times}}$ where the image of the Grothendieck map is the positive cone $\underbrace{\mathbb{N} \oplus \cdots \oplus \mathbb{N}}_{k-\text{times}}$.

3. Inductive Limits

The purpose of this section is to explain what an inductive limit is and what characteristics the inductive limit with actions have.

DEFINITION 4.6. An **inductive sequence** in a category C is consist of a sequence $\{A_n\}_{n=1}^{\infty}$ of objects in C and a sequence $\varphi_n : A_n \to A_{n+1}$ of morphisms in C. We write the inductive sequence like

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

For $m > n$, we consider the composed morphisms

$$\varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n : A_n \to A_m,$$

which are called the **connecting morphisms** (or connecting maps).

DEFINITION 4.7. [**14**, Definition 6.2.2]
An **inductive limit** of the inductive sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

in a category C is a system $(A, \{\mu_n\}_{n=1}^{\infty})$, where A is an object in C, where $\mu_n : A_n \to A$ is a morphism in C for each n in $\mathbb{N}$, and where the following two conditions hold.

(1) The diagram

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\mu_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & A & \end{tikzcd}$$

commutes for each n in $\mathbb{N}$.

(2) If $(B, \{\lambda_n\}_{n=1}^{\infty})$ is a system, where B is an object in C, $\lambda_n : A_n \to B$ is a morphism in C for each n in $\mathbb{N}$, and where $\lambda_n = \lambda_{n+1} \circ \varphi_n$ for all n in $\mathbb{N}$, then there is one and only one morphism $\lambda : A \to B$ making the diagram

$$\begin{tikzcd} & An \arrow[dl, "\mu_n"'] \arrow[dr, "\lambda_n"] & \\ A \arrow[rr, "\lambda"'] & & B \end{tikzcd}$$

commutative for each n in $\mathbb{N}$.

Here are examples of inductive limits. These examples clear up the definition and illustrate what inductive limits look like.

EXAMPLE 4.4.

(1) [14, Example 6.2.3] Let D be a C^*-algebra and let $\{A_n\}_{n=1}^{\infty}$ be an increasing sequence of finite dimensional subalgebras of D. Put
$$A = \overline{\bigcup_{n=1}^{\infty} A_n},$$
and for each n let $\iota_n : A_n \to A$ be the inclusion map. Then $(A, \{\iota_n\})$ is the inductive limit of the sequence $A_1 \to A_2 \to A_3 \to \cdots$ where the connecting maps are the inclusion maps.

(2) This is an example of a non-unital AF algebra.
Consider the sequence
$$\mathbb{C} \xrightarrow{\varphi_1} M_2(\mathbb{C}) \xrightarrow{\varphi_2} M_3(\mathbb{C}) \xrightarrow{\varphi_3} \cdots,$$
where the connecting map φ_n maps an $n \times n$ matrix into the upper left corner of an $(n+1) \times (n+1)$ matrix whose last row and last column are zero. The inductive limit of this sequence is isomorphic to $\mathcal{K}$, the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. Its K_0 group is $\mathbb{Z}$, with the usual order relation, but there is no class of the unit.

(3) This is an example of a UHF algebra called M_{2^∞}.
Consider the sequence
$$\mathbb{C} \longrightarrow M_2(\mathbb{C}) \longrightarrow M_4(\mathbb{C}) \longrightarrow \cdots,$$
with $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ in each case.
When we apply the functor K_0 to the above sequence, we get
$$(\mathbb{Z}, \mathbb{Z}^+, 1) \xrightarrow{\times 2} (\mathbb{Z}, \mathbb{Z}^+, 2) \xrightarrow{\times 2} (\mathbb{Z}, \mathbb{Z}^+, 4) \xrightarrow{\times 2} \cdots,$$

which has a inductive limit $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, 1)$,
where $\mathbb{Z}[\frac{1}{2}] = \{ \frac{k}{2^n} \mid n = 1, 2, 3, \cdots \text{ and } k \in \mathbb{Z}\}$.
We use that K_0 is a continuous functor in (2) and (3).

Now, we are going to show explicitely the construction of inductive limits for partially ordered abelian groups with actions of a fixed group and equivariant connecting maps.

Let $\{G_n, \varphi_{n,n+1}\}$ be an inductive system of abelian groups. (In other words, for each n, $\varphi_{n,n+1}$ is a group homomorphism and $\varphi_{n,n+1} : G_n \to G_{n+1}$. Write $\varphi_{n,m}$ for $\varphi_{m-1,m} \circ \cdots \circ \varphi_{n,n+1}$ when $m > n$.) Define $\prod_{n=1}^{\infty} G_n = \{(g_1, g_2, g_3, \cdots) \mid g_n \in G_n\}$ and $\bigoplus_{n=1}^{\infty} G_n = \{(g_1, g_2, g_3, \cdots) \in \prod_{n=1}^{\infty} G_n \mid g_k = 0 \text{ for all but finitely many } k\}$.
In this case, $\prod_{n=1}^{\infty} G_n$ is an abelian group with operation
$(g_1, g_2, g_3, \cdots) + (h_1, h_2, h_3, \cdots) = (g_1 + h_1, g_2 + h_2, g_3 + h_3, \cdots)$ and $\bigoplus_{n=1}^{\infty} G_n$ is a subgroup of $\prod_{n=1}^{\infty} G_n$.

Proposition 4.3. Define a map $i_n : G_n \to \prod_{n=1}^{\infty} G_n$ by

$$i_n(x) = (0, \cdots, 0, x, \varphi_{n,n+1}(x), \varphi_{n,n+2}(x), \varphi_{n,n+3}(x), \cdots)$$

where there are $n-1$ zeros at the beginning. Then i_n is a group homomorphism.

Proposition 4.4. Define a map

$$g_{n,\infty} := \pi \circ i_n : G_n \to \prod_{n=1}^{\infty} G_n / \bigoplus_{n=1}^{\infty} G_n$$

where $\pi : \prod_{n=1}^{\infty} G_n \to \prod_{n=1}^{\infty} G_n / \bigoplus_{n=1}^{\infty} G_n$ is the quotient map and let $G_\infty = \bigcup_{n=1}^{\infty} g_{n,\infty}(G_n)$.
Then, G_∞ is a subgroup of $\prod_{n=1}^{\infty} G_n / \bigoplus_{n=1}^{\infty} G_n$.

Straightforward calculations show proposition 4.3 and 4.4.

Theorem 4.1. If H is an abelian group and $\psi_n : G_n \to H$ is a collection of homomorphisms such that $\psi_{n+1} \circ \varphi_{n,n+1} = \psi_n$ for every n, then there exists a unique homomorphism $\psi_\infty : G_\infty \to H$ such that

$\psi_n = \psi_\infty \circ \varphi_{n,\infty}$ for every n.

$$\begin{tikzcd}
G_n \arrow[r, "\varphi_{n,n+1}"] \arrow[drrr, "\psi_n"'] & G_{n+1} \arrow[r] \arrow[drr, "\psi_{n+1}"] & \cdots \arrow[r] & G_\infty \arrow[d, "\psi_\infty"] \\
 & & & H
\end{tikzcd}$$

Proof. First of all, we need to show the existence of this homomorphism. Pick an element of $g_{n,\infty}(G_n)$,

$$\varphi_{n,\infty}(g) = (0, 0, \cdots, 0, g, \varphi_{n,n+1}(g), \varphi_{n,n+2}(g), \cdots),$$

where $g \in G_n$. We define $\psi_\infty(\varphi_{n,\infty}(g)) = \psi_n(g)$. We need to show that ψ_∞ is well defined. Suppose $\varphi_{m,\infty}(h) = \varphi_{n,\infty}(g)$ where $h \in G_m$. Since they represent the same class, they are eventually equivalent. By the commutativity of the diagram, we could go further along in the sequence before applying the maps ψ. Therefore, we could go past where the two agree. Since $G_\infty = \bigcup_{n=1}^{\infty} g_{n,\infty}(G_n)$ and the property of the commutative diagram determines what ψ_∞ must be on $g_{n,\infty}(G_n)$, uniqueness follows. Therefore, there exists the unique homomorphism. $\square$

This shows that G_∞ is the inductive limit of the system $\{G_n, \varphi_n\}$ in the category of abelian groups and group homomorphisms.

Proposition 4.5. Suppose that K is a group and that for each n, α_n is an action of K on G_n. Then, we get an action α of K on $\prod_{n=1}^{\infty} G_n$ defined by

$$\alpha(k)(g_1, g_2, g_3, \cdots) = (\alpha_1(k)(g_1), \alpha_2(k)(g_2), \alpha_3(k)(g_3), \cdots)$$

for each $k \in K$.

Proof. We need to show that $\alpha(k)$, which is an automorphism of the group for each $k \in K$, is an action. Let K be a group and $k_1, k_2 \in K$. Since α_n is action for each n, $\alpha_n(k_1 k_2) = \alpha_n(k_1)(\alpha_n(k_2))$

So,

$$
\begin{aligned}
&\alpha(k_1k_2)(g_1, g_2, g_3, \cdots)\\
=&(\alpha_1(k_1k_2)(g_1), \alpha_2(k_1k_2)(g_2), \alpha_3(k_1k_2)(g_3), \cdots)\\
=&(\alpha_1(k_1)(\alpha_1(k_2)(g_1)), \alpha_2(k_1)(\alpha_2(k_2)(g_2)), \alpha_3(k_1)(\alpha_3(k_2)(g_3)), \cdots)\\
=&\alpha(k_1)(\alpha(k_2)(g_1, g_2, g_3, \cdots))
\end{aligned}
$$

Suppose e is an identiy element of K. Then, $\alpha_n(e)(g_n) = g_n$ for each n. So,

$$
\begin{aligned}
\alpha(e)(g_1, g_2, g_3, \cdots) &= (\alpha_1(e)(g_1), \alpha_2(e)(g_2), \alpha_3(e)(g_3), \cdots)\\
&= (g_1, g_2, g_3, \cdots).
\end{aligned}
$$

□

PROPOSITION 4.6. With the action α of K on $\prod_{n=1}^{\infty} G_n$ defined above,we have $\alpha(k)(\bigoplus_{n=1}^{\infty} G_n) \subseteq \bigoplus_{n=1}^{\infty} G_n$. Also, we get an action $\tilde{\alpha}$ of K on $\prod_{n=1}^{\infty} G_n / \bigoplus_{n=1}^{\infty} G_n$ defined by $\tilde{\alpha}(k)((g_1, g_2, g_3, \cdots) + \bigoplus_{n=1}^{\infty} G_n) = \alpha(k)(g_1, g_2, g_3, \cdots) + \bigoplus_{n=1}^{\infty} G_n$ for $k \in K$.

PROOF. By the definition of the action α, $\alpha(k)(g_1, g_2, g_3, \cdots) = (\alpha_1(k)(g_1), \alpha_2(k)(g_2), \alpha_3(k)(g_3), \cdots)$ where $(g_1, g_2, g_3, \cdots) \in \prod_{n=1}^{\infty} G_n$. If $(g_1, g_2, g_3, \cdots)$ belongs to $\bigoplus_{n=1}^{\infty} G_n$, then, since $\alpha_n(k)(0) = 0$ for all n, $(\alpha_1(k)(g_1), \alpha_2(k)(g_2), \alpha_3(k)(g_3), \cdots)$ becomes 0 when it passes some point. So, $\alpha(k)(\bigoplus_{n=1}^{\infty} G_n) \subseteq \bigoplus_{n=1}^{\infty} G_n$.

Now, we need to check that the action $\tilde{\alpha}$ is well defined. Take two sequences $g = (g_1, g_2, g_3, \cdots)$ and $h = (h_1, h_2, h_3, \cdots)$ in $\prod_{n=1}^{\infty} G_n$ such that $(g_1 - h_1, g_2 - h_2, g_3 - h_3, \cdots)$ belongs to $\bigoplus_{n=1}^{\infty} G_n$. Then, $\alpha(k)(g) - \alpha(k)(h) \in \bigoplus_{n=1}^{\infty} G_n$. □

PROPOSITION 4.7. If the connecting maps $\varphi_{n,n+1}$ are equivariant, that is to say $\alpha_{n+1}(k) \circ \varphi_{n,n+1} = \varphi_{n,n+1} \circ \alpha_n(k)$ for all $n \in \mathbb{N}$ and $k \in K$, then $\tilde{\alpha}(k)(G_\infty) \subseteq G_\infty$ for all $k \in K$, so we get an action of K on G_∞.

PROOF. Suppose $g \in G_\infty$. Then,

$$g = (0, 0, \cdots, 0, g, \varphi_{n,n+1}(g), \varphi_{n,n+2}(g), \cdots).$$

If we apply $\alpha(k)$ to the element g, then $\tilde{\alpha}(k)(g) =$
$(0, 0, \cdots, 0, \alpha_n(k)(g), \alpha_{n+1}(k)(\varphi_{n,n+1}(g)), \alpha_{n+2}(k)(\varphi_{n,n+2}(g)), \cdots)$.
If we have equivariant, we can interchange the order of the maps. So, we can replace $\alpha_{n+1}(k)(\varphi_{n,n+1}(g))$ with $\varphi_{n,n+1}(\alpha_n(k)(g))$, $\alpha_{n+2}(k)(\varphi_{n,n+2}(g))$ with $\varphi_{n,n+2}(\alpha_n(k)(g))$, and similarly for following elements in the sequence. So, $\tilde{\alpha}(k)(g) \in G_\infty$. $\square$

One can show that $(G_\infty, \tilde{\alpha})$ is the inductive limit of the (G_n, α_n) in the category of abelian groups with K actions and equivariant group homomorphisms.

Proposition 4.8. If each G_n is an ordered group, we get an order on G_∞ by defining $G_\infty^+ = \bigcup_{n=1}^\infty g_{n,\infty}(G_n^+)$ that makes G_∞ into an ordered group.

Proof. Since $g_{n,\infty}$ is a positive group homomorphism, it preserves the group structure. Since G_n is an ordered group for each n, G_n satisfies three conditions; (1) $G_n^+ + G_n^+ \subseteq G_n^+$, (2) $G_n^+ \cap G_n^- = \{0\}$, and (3) $G_n^+ + G_n^- = G_n$ for each n. Now, we need to show that G_∞ satisfies the three conditions as well.

(1) Suppose $g_1 = (0, 0, \cdots, 0, g_1, \varphi_{n,n+1}(g_1), \varphi_{n,n+2}(g_1), \cdots)$, $g_2 = (0, 0, \cdots, 0, g_2, \varphi_{n,n+1}(g_2), \varphi_{n,n+2}(g_2), \cdots)$ are in G_∞^+. Since each term of g_1 and g_2 is in G_n^+, the addition of each term is in G_n^+. So, $g_1 + g_2 \in G_\infty^+$. We may suppose that the elements g_1 and g_2 are in the same G_n^+. Therefore, $G_\infty^+ + G_\infty^+ \subseteq G_\infty^+$
(2) Suppose the element $g \in G_\infty^+ \cap G_\infty^-$.
Then, $g, -g \in \bigcup_{n=1}^\infty g_{n,\infty}(G_n^+)$. Since the images of $g_{n,\infty}$ are all compositions of these maps, $g, -g \in g_{m,\infty}(G_m^+)$ for some m. Then, there are $h_1, h_2 \in G_m^+$ such that $g_{m,\infty}(h_1) = g$ and $g_{m,\infty}(h_2) = -g$. Since $g_{m,\infty}(-h_2) = g = g_{m,\infty}(h_1)$, $g_{m,l}(-h_2) = g_{m,l}(h_1)$ for some $m > l$. Since $g_{m,l}$ is a positive map and h_2 is a positive element of G_m, $g_{m,l}(h_2) \geq 0$ and $g_{m,l}(-h_2) \geq 0$. So, the image of h_2 in G_l is 0. So, the image in G_∞ is also 0. Therefore, $g = 0$, and hence, $G_\infty^+ \cap G_\infty^- = \{0\}$.

(3) Suppose $g = (0, 0, \cdots, 0, g, \varphi_{n,n+1}(g), \varphi_{n,n+2}(g), \cdots) \in G_\infty^+$ and $-g \in G_\infty^-$. Then, each term of g is in G_n^+ and each term of $-g$ is in G_n^- for each n. Since $G_n^+ + G_n^- \subseteq G_n$, $G_\infty^+ + G_\infty^- \subseteq G_\infty$.

Now, we need to show that $G_\infty \subseteq G_\infty^+ + G_\infty^-$. Suppose the element $g \in G_\infty$. Since each term of g is in $G_n \subseteq G_n^+ + G_n^-$ for each n, the element g is in $G_\infty^+ + G_\infty^-$.

Therefore, G_∞ is an ordered group. $\square$

One can show that (G_∞, G_∞^+) is the inductive limit of (G_n, G_n^+) in the category of partially ordered abelian groups with positive group homomorphisms.

With the order defined in the above proposition, it is obviouse that an element $(0, 0, \cdots, 0, g_n, g_{n+1}, g_{n+2}, \cdots) \in G_\infty$ is positive if, and only if, for some $l \geq n, g_t \geq 0$ for every $t \geq l$. In other words, elements are positive if, and only if, they are eventually positive.

Proposition 4.9. If we have actions by positive automorphisms on the ordered groups, then these give an action by positive automorphisms on the ordered groups inductive limit.

Proof. Let $G_1 \to G_2 \to G_3 \to \cdots \to G_\infty$ be an inductive sequence where G_i is an ordered group for each i. Suppose that H is a fixed group and each α_n is an action by possitive automorphisms. Then we need to check that $\alpha_\infty(h)(G_\infty^+) \subseteq G_\infty^+$ for all $h \in H$. By the definition of action and the construction of an element in G_∞^+, each term of an element in $\alpha_\infty(h)(G_\infty^+)$ is in $\alpha_i(G_i)$, and hence, $\alpha_\infty(h)(G_\infty^+) \subseteq G_\infty^+$. So, it preserves group structures. Therefore, the action of H on G_∞ is by positive automorphisms on the ordered group inductive limit. $\square$

Define a category of H actions on partially ordered groups G. In this case, object is $(G, G^+, \alpha(h))$ and a morphism is an ordered group homomorphism $\varphi : G_1 \to G_2$ with $\varphi(G_1^+) \subseteq G_2^+$ and $\varphi(\alpha(h)(g)) = \alpha(h)(\varphi(g))$. Then, we have constructed an inductive limit which one can show satisfies the universal mapping property;

$$\begin{tikzcd}
(G_1, G_1^+) \arrow[loop above, "\alpha_1"] \arrow[r] \arrow[drrr] & (G_2, G_2^+) \arrow[loop above, "\alpha_2"] \arrow[r] \arrow[drr] & \cdots \arrow[r] & (G_\infty, G_\infty^+) \arrow[loop above, "\alpha_\infty"] \arrow[d] \\
 & & & \mathbb{G} \arrow[loop below, "\alpha"]
\end{tikzcd}$$

In other words, $(G_\infty, G_\infty^+, \alpha_\infty)$ is the inductive limit of the inductive system in the category of partially ordered abelian groups with H actions and equivariant group homomorphisms.

In this section, we have shown how the group G_∞ is defined, how its positive cone is defined, and how the action on it is defined.

CHAPTER 5

Elliott's Intertwining Argument

We will discuss three important theorems; Elliott's AF classification theorem, the Effros-Handelman-Shen theorem, and the Elliott-Su theorem that is the motivation for our main theorem.

In this chapter, we discuss the Elliott's AF classification theorem and Elliott's intertwining argument that is used to prove the theorem.

Theorem 5.1 (Elliott). [**14**, Theorem 7.3.4] If $(K_0(A), K_0(A)^+, [1_A]) \cong (K_0(B), K_0(B)^+, [1_B])$, then $A \cong B$ for AF-algebras. Moreover, if $\alpha : K_0(A) \to K_0(B)$ is an isomorphism that satisfies $\alpha(K_0(A)^+) = K_0(B)^+$ and $\alpha([1_A]) = [1_B]$, then $A \cong B$ and there is an isomorphism $\varphi : A \to B$ with $K_0(\varphi) = \alpha$.

Before we prove this theorem, there are a couple of lemmas to know.

Lemma 5.1. [**14**, Lemma 7.3.2]

(1) (Existence) Let A and B be AF-algebras. For each positive group homomorphism $\alpha : K_0(A) \to K_0(B)$ satisfying $\alpha([1_A]) \leq [1_B]$ there is a *-homomorphism $\varphi : A \to B$ with $K_0(\varphi) = \alpha$. If $\alpha([1_A]) = [1_B]$, then φ is necessarily unit preserving.

(2) (Uniqueness) Let A and B be finite dimensional algebras and let $\varphi, \psi : A \to B$ be *-homomorphisms. Then $K_0(\varphi) = K_0(\psi)$ if and only if there exists an unitary u in B such that $\psi(x) = \mathrm{Ad}u \circ \varphi(x)$ for every x.

We will prove (2), but only a special case of (1) when we go through the proof below.

Here is the proof of Theorem 5.1.

PROOF. This proof uses the Elliott's intertwining argument. It consists of four parts; pull back the invariant, existence lemma, uniqueness lemma, and intertwining. We will explain about each parts of the Elliott's intertwining argument.
Let

$$\begin{tikzcd} A_1 \arrow[r] & A_2 \arrow[r] & \cdots \arrow[r] & A & & InA \arrow[d, shift left, "\varphi"] \\ B_1 \arrow[r] & B_2 \arrow[r] & \cdots \arrow[r] & B & & InB \arrow[u, shift left, "\psi"] \end{tikzcd}$$

be sequences where A_i and B_i are finite dimensional algebras for all i, and φ, ψ are isomorphisms of invariants.

(1) (Pull back the invariant) Suppose an invariant is continuous with respect to inductive limits. When we apply the invariant to sequences, the invariant turns the inductive system into another inductive system. With a continuous invariant, such as K_0, the inductive limit of invariants is the invariant of the inductive limit. We want a commutative diagram:

$$\left.\begin{tikzcd} InA_1 \arrow[r] \arrow[dr, "\varphi_1"] & InA_2 \arrow[r] \arrow[dr, "\varphi_2"] & \cdots \arrow[r] & InA \arrow[d, shift left, "\varphi"] \\ InB_1 \arrow[r] & InB_2 \arrow[r] \arrow[u, "\psi_1"] & \cdots \arrow[r] & InB \arrow[u, shift left, "\psi"] \end{tikzcd}\right\} \quad (*),$$

where InC is invariant of C.

We need to show that there are, possibly after passing to subsequences, maps φ_i and ψ_i that make the above diagram commutative.

Since A_i and B_j are finite dimensional algebras, the values of invariants are finitely generated. Suppose $InA_1 = \mathbb{Z}^n$. Take the simplicial basis $x_1, x_2, \cdots, x_n$ in InA_1. We make x_1 go along the horizontal maps to InA and down to InB by φ. Since InB is the union of the images of InB_l's, there exists y_l which is the image along the bottom horizontal row of some element of InB_l such that $\varphi(x_1) = y_l$. So, we can do this for each of x_i's. Then, there exists a homomorphism φ_1 from InA_1 to InB_k for some k that makes a commuting diagram.

If we renumber B_k as B_2, then we get the homomorphism φ_1: $InA_1 \to InB_2$. We do the same thing to the simplicial basis of InB_2. Then we can get a map ψ_1 from InB_2 to some InA_n. Similarly, we renumber A_n as A_2. Finally, we get the map ψ_1: $InB_2 \to InA_2$.

Now, we need to show that we can make the first triangle that consists of the maps φ_1, ψ_1, and the horizontal map $InA_1 \to InA_2$ in $(*)$ commutative by going further along the top row and renumbering if necessary. The diagram that consists of φ_1, ψ, a map from InA_1 to InA, and a map from InB_2 to InB is commutative. Also, the diagram that consists of ψ_1, φ, a map from InA_2 to InA, and a map from InB_2 to InB is commutative. Therefore, the result when the element goes along the horizontal maps to InA is eventually same as the result when we take maps φ_1, ψ_1 and the horizontal maps to InA. Therefore, by moving out to some A_n further along the top row and renumbering, we can make the first triangle is commutative. Once we have done this, we can apply this to each triangle in turn. So, the diagram $(*)$ commutes. We can ensure all the maps are positive and preserve the class of the unit as well.

(2) (Existence Lemma) We stated the existence lemma in general above. We show explicitly a special case of the existence lemma here.

We have a map $\mathbb{Z}^k \to \mathbb{Z}^l$ such that

$$\begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} \mapsto \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_l \end{pmatrix}.$$

In this case, those two columns mean the class of the unit for each $\mathbb{Z}^n$. Since the homomorphism is positive and unital, we have a class of the unit which is the single vector in $\mathbb{Z}^k$ and our matrix $M = [a_{ij}]_{i,j}$ where $a_{ij} \in \mathbb{N}$ that preserves the unit. So, we need to show that there exists a unital *-homomorphism from A to B which induces the map $\mathbb{Z}^k \to \mathbb{Z}^l$ where $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$ and $B = M_{m_1} \oplus \cdots \oplus M_{m_l}$.

Consider the case $B = M_{m_1}$. In this case,

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1k} \end{pmatrix}.$$

Then, $M \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} = a_{11}n_1 + a_{12}n_2 + \cdots + a_{1k}n_k = m_1$

If we take a look an element $z \in A$ where

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{pmatrix} \in M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k},$$

then we can define a $*-$homomorphism by a map

$$z \mapsto diag(\underbrace{z_1, \cdots, z_1}_{a_{11}\text{times}}, \underbrace{z_2, \cdots, z_2}_{a_{12}\text{times}}, \cdots, \underbrace{z_k, \cdots, z_k}_{a_{1k}\text{times}})$$

and the size of this matrix is m_1. We can do this one direct summand at a time to get the expanded case, $B = M_{m_1} \oplus \cdots M_{m_l}$. This proves the existence lemma.

So, we get a diagram below that does not commute by applying existence lemma to $(*)$.

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\widetilde{\varphi_1}"'] & A_2 \arrow[r] \arrow[d, "\widetilde{\varphi_2}"'] & \cdots \arrow[r] & A \\ B_1 \arrow[r] \arrow[ur, "\widetilde{\psi_1}"] & B_2 \arrow[r] \arrow[ur, "\widetilde{\psi_2}"] & \cdots \arrow[r] & B \end{tikzcd}$$

Once we have got this diagram, this is not commuting. We need to fix this diagram so that it is commuting and induces the original one on the invariant. In order to fix the diagram, we need the uniqueness lemma.

(3) (Uniqueness Lemma) Now, we show the uniqueness lemma stated above. Consider the case $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$ and $B = M_{m_1}$. Assume that φ and ψ are unital homomorphisms. We have $K_0(\varphi) = K_0(\psi)$. Consider $\varphi(M_{n_1}) \subseteq B$, $\psi(M_{n_1}) \subseteq B$, and $e_{11} \in M_{n_1}$. Since $K_0(\varphi(e_{11})) = K_0(\psi(e_{11}))$, $\psi(e_{11}) \sim \varphi(e_{11})$. There exists an element $v \in B$ such that $vv^* = \psi(e_{11})$ and $v^*v = \varphi(e_{11})$. Let $u_1 = \sum_{j=1}^{n_1} \psi(e_{j1})\, v\, \varphi(e_{1j})$. Then $u_1 u_1^* = \psi(\sum_{s=1}^{n_1} e_{ss})$ and $u_1^* u_1 = \varphi(\sum_{s=1}^{n_1} e_{ss})$. By direct calculations, $u_1 \varphi(e_{kl}) u_1^* = \psi(e_{kl})$ if $e_{kl} \in M_{n_1} \subseteq A$. Now, we do the same for $M_{n_2}, \cdots, M_{n_k}$, and let $u = u_1 + u_2 + \cdots + u_k$. We can do this one summand at a time for the general case on B. Then we get $u\varphi(x)u^* = \psi(x)$ for all $x \in A$. This proves the uniqueness lemma.

Now, we use the uniqueness lemma one at a time to adjust all of the maps to make the diagram commute. Apply inner automorphims $A_i \to A_i$ and $B_j \to B_j$ where $i, j = 1, 2, 3, \cdots$ to make the diagram commutative.

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\widetilde{\varphi_1}"] & A_2 \arrow[r] \arrow[d, "\widetilde{\varphi_2}"] \arrow[loop above] & \cdots \arrow[r] & A \\
B_1 \arrow[r] \arrow[ur, "\widetilde{\psi_1}"] \arrow[loop below] & B_2 \arrow[r] \arrow[ur, "\widetilde{\psi_2}"] \arrow[loop below] & \cdots \arrow[r] & B
\end{tikzcd}
$$

Since it does not change the image at the level of the invariant, the diagram in (2) commutes and we get the diagram $(*)$ when we apply the invariant.

(4) (Intertwining) If we have the commutative diagram like the diagram in (2), then we get maps between A and B that make the whole diagram commutative [15].

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\widetilde{\varphi_1}"'] & A_2 \arrow[r] \arrow[d, "\widetilde{\varphi_2}"'] & \cdots \arrow[r] & A \arrow[d, dashed, shift left, "\tilde{\varphi}"] \\
B_1 \arrow[r] \arrow[ur, "\widetilde{\psi_1}"] & B_2 \arrow[r] \arrow[ur, "\widetilde{\psi_2}"] & \cdots \arrow[r] & B \arrow[u, dashed, shift left, "\tilde{\psi}"]
\end{tikzcd}
$$

The maps $\widetilde{\psi}$ and $\widetilde{\varphi}$ induce the maps ψ and φ when we apply the invariant. Since this diagram is commutative, the maps are isomorphisms.

$\square$

The same pattern of Elliott's intertwining argument has been used in other classification arguments. In particular, Elliott and Su used this pattern for classification with $\mathbb{Z}_2$ actions.

CHAPTER 6

The Range of Invariant Problem

We discussed that the functor K_0 sends AF algebras to partially ordered abelian groups. What partially ordered abelian groups arise as K_0 groups of AF C^*-algebras is the natural question of the range of invariant problem. The Effros-Handelman-Shen theorem gives the answer to the range of invariant problem for AF C^*-algebras. In order to precisely understand the Effros-Handelman-Shen theorem, there are some terms to know: a dimension group and a simplicial group.

To clarify the definition of dimension group, we would like to define new terminologies that help to understand the meaning. We have previously mentioned the definition of these words earlier in the thesis.

DEFINITION 6.1. [9]

- **Directed** means that every element has the form $x - y$ for $x, y \in G^+$.
- **Unperforated** means that if $x \in G$, $n \in \mathbb{N} \backslash 0$, $nx \geq 0$, then $x \geq 0$.
  For instance, if $G = \mathbb{Z}$ and $G^+ = \mathbb{Z}^+ = \{0, 1, 2, \cdots\} = \mathbb{N}$, then (G, G^+) is directed and unperforated. Consider the different example. If $G = \mathbb{Z}$ and $G^+ = \{0, 2, 3, 4, \cdots\}$, then (G, G^+) is directed but not unperforated because $1+1 \in G^+$, but $1 \notin G^+$.
- **Interpolation** means that for every $x_1, x_2, y_1, y_2 \in G$, where $x_i \leq y_j$ for all i, j, there exists an element $z \in G$ with $x_i \leq z \leq y_j$ for all i, j.

DEFINITION 6.2. [9] A dimension group (G, G^+) is any directed, unperforated, interpolation group.

Also, we want to define a simplicial group. Here is the definition.

DEFINITION 6.3. [9] A **simplicial group** is any partially ordered abelian group that is isomorphic (as an ordered group) to $\mathbb{Z}^n$ for some nonnegative integer n. A **simplicial basis** for a simplicial group G is any basis $\{x_1, \cdots, x_n\}$ for G as a free abelian group such that also $G^+ = \sum \mathbb{Z}^+ x_i$. By convention, the empty set is considered to be a simplicial basis for the zero simplicial group.

THEOREM 6.1 (Effros-Handelman-Shen). [9, Theorem 3.19]
Any dimension group is isomorphic to a direct limit (or inductive limit) of a direct system of simplicial groups (in the category of partially ordered abelian groups).

This theorem points out that simplical groups are dimension groups and inductive limits of a sequence of simplicial groups are dimension groups.

Now, we would like to move on the range of invariant problem for the Elliott-Su classification of actions. In [7] , Elliott and Su generalized the Elliott AF classification theorem to classifying the inductive limit of dynamical systems where the actions are by the group $\mathbb{Z}_2$.

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r] \arrow[loop above, "\mathbb{Z}_2"] & \mathcal{A}_2 \arrow[r] \arrow[loop above, "\mathbb{Z}_2"] & \cdots \arrow[r] & \mathcal{A} \arrow[loop above, "\mathbb{Z}_2"] \end{tikzcd}$$

We need to know what certain crossed products are. We shall only be concerned with the special case of crossed product where the group is $\mathbb{Z}_2$. Here is the definition of a crossed product ($\rtimes$), dual action , and special element that we will use in this thesis.

DEFINITION 6.4. Let A be a unital C^*-algebra and let α be a $\mathbb{Z}_2$ action on A. There is a canonical embedding of A into $A \rtimes_\alpha \mathbb{Z}_2$. There is also a dual action of $\mathbb{Z}_2$ on $A \rtimes_\alpha \mathbb{Z}_2$. In general, $(A \rtimes_\alpha \mathbb{Z}_2) \rtimes_{\hat{\alpha_*}} \mathbb{Z}_2 \cong M_2(A)$ and with the inclusions

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & \alpha(a) \end{pmatrix}, \; g \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So,

$$\begin{aligned} A \rtimes_\alpha \mathbb{Z}_2 &= \{a + bg \mid a, b \in A\} \\ &= \{\begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix} \mid a, b \in A\} \end{aligned}$$

and the dual action is given by γ, $\hat{\alpha_*}(x) = \gamma x \gamma^*$. The special element mentioned below is $\frac{1+g}{2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The K-theory data we need here is the following:

- $(K_0(A),\ K_0(A)^+,\ [1_A],\ \alpha_*)$
- $(K_0(A \rtimes_\alpha \mathbb{Z}_2),\ K_0(A \rtimes_\alpha \mathbb{Z}_2)^+$, the special element, $\hat{\alpha_*})$
- The map $K_0(A) \to K_0(A \rtimes_\alpha \mathbb{Z}_2)$

Consider the following special cases of the invariant in [7].

Example 6.1.

(1) Consider $A = M_n$ and $\alpha(x) = uxu^*$ where $u = diag(\underbrace{1, 1, \cdots, 1}_{k}, \underbrace{-1, -1, \cdots, -1}_{l})$ with $k + l = n$. Then, $M_n \rtimes_\alpha \mathbb{Z}_2 \cong M_n \oplus M_n$. In this case, the special element is, by the definition,

$$[\tfrac{1+g}{2}] = ([\tfrac{1+u}{2}], [\tfrac{1-u}{2}]) = (k, l)$$

So, the invariant is
  - $(K_0(A),\ K_0(A)^+,\ [1_A],\ \alpha_*) \cong (\mathbb{Z}, \mathbb{Z}^+, n, id)$ where n is a dimension.
  - $(K_0(A \rtimes_\alpha \mathbb{Z}_2),\ K_0(A \rtimes_\alpha \mathbb{Z}_2)^+$, the special element, $\hat{\alpha_*})$ $\cong (\mathbb{Z} \oplus \mathbb{Z},\ \mathbb{Z}^+ \oplus \mathbb{Z}^+,\ (k, l),\ (a, b) \mapsto (b, a))$
  - The map $k \in \mathbb{Z} \mapsto (k, k) \in \mathbb{Z} \oplus \mathbb{Z}$

(2) A UHF-algebra("Uniformly Hyper-Finite C^*-algebra") is a C^*-algebra which is isomorphic to the inductive limit of the sequence

$$M_{k_1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{k_2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{k_3}(\mathbb{C}) \xrightarrow{\varphi_3} \cdots$$

for some natural numbers $k_1, k_2, k_3, \cdots$ and some unit preserving connecting *-homomorphisms $\varphi_1, \varphi_2, \varphi_3, \cdots$ ([14, Definition 7.4.1]).

Consider UHF-algebra A and $\alpha(x) = vxv^*$,

where $v = diag(\underbrace{1, 1, \cdots, 1}_{p}, \underbrace{-1, -1, \cdots, -1}_{q})$ with $p + q = r$.

Suppose $r = k_1$. Then, $v \in M_{k_1}(\mathbb{C})$

In this case, the invariant is

- $(K_0(A),\ K_0(A)^+,\ [1_A],\ \alpha_*) \cong (G, G^+, r, id)$ where G is a subgroup of $\mathbb{Q}$ and r is a class of unit.
- $(K_0(A \rtimes_\alpha \mathbb{Z}_2),\ K_0(A \rtimes_\alpha \mathbb{Z}_2)^+$, the special element, $\hat{\alpha}_*)$ $\cong (G \oplus G, G^+ \oplus G^+, (p, q),\ (s, t) \mapsto (t, s))$
- The map $x \mapsto (x, x)$

(3) Consider $A = M_n \oplus M_n$, $\alpha(x, y) = (y, x)$

The invariant is

- $(K_0(A),\ K_0(A)^+,\ [1_A],\ \alpha_*)$ $\cong (\mathbb{Z} \oplus \mathbb{Z},\ \mathbb{Z}^+ \oplus \mathbb{Z}^+,\ (n, n),\ (x, y) \mapsto (y, x))$
- $(K_0(A \rtimes_\alpha \mathbb{Z}_2),\ K_0(A \rtimes_\alpha \mathbb{Z}_2)^+$, the special element, $\hat{\alpha}_*)$ $\cong (\mathbb{Z},\ \mathbb{Z}^+,\ n,\ id)$
- The map $(x, y) \mapsto x + y$

In [7], Elliott and Su used this invariant to classify certain actions by using the pattern of the Elliott intertwining argument. As we would like to get an Effros-Handelman-Shen theorem for this invariant from the type of action Elliott and Su classified, we try to show what kind of action we can get on a dimension group arising from a $\mathbb{Z}_2$ action of inductive limit type. What we were able to show is that if the dimension group is a lattice-ordered group, then any action of $\mathbb{Z}_2$ comes from the direct limit of $\mathbb{Z}_2$ actions on simplicial groups.

CHAPTER 7

A Modification of the Effros-Handelman-Shen Theorem

The purpose of this chapter is to check that the Effros-Handalman-Shen theorem, any countable dimension group is isomorphic to a direct limit of a countable sequence of simplicial groups, is still valid if we restrict the dimension group to lattice-ordered groups but with $\mathbb{Z}_2$ actions added. Before starting to prove a modificaton of the theorem, we need to check a few propositions that support our main theorem.

First of all, we would like to talk about lattice-ordered groups, the relation between lattice ordered groups and dimension groups, and a few examples.

DEFINITION 7.1. [9, pp. xxi & 5]

- If every finite subset of a partially ordered set X has a least upper bound and a greatest lower bound in X, then X is called a **lattice** .
- A **lattice-ordered abelian group** is any partially ordered abelian group which, as a partially ordered set, is a lattice.

EXAMPLE 7.1. A group $\mathbb{Z}^n$ with the usual order is a lattice-ordered group. Such groups are called **simplicial groups.**

PROPOSITION 7.1. [9, Proposition 1.22 & pp.44] Any lattice-ordered abelian group is a dimension group.

EXAMPLE 7.2. [9, pp.44] The group $\mathbb{Q}^2$ equipped with the strict ordering is a dimension group which is not lattice-ordered.

LEMMA 7.1. If G is lattice-ordered dimension group, and α is an action of $\mathbb{Z}_2$ on G, then the fixed-point subgroup G^α is also a lattice-ordered dimension group.

Proof. Suppose $x, y \in G^\alpha$, i.e., $\alpha(x) = x$ and $\alpha(y) = y$. In G, there exists an element $x \wedge y$ such that $x \wedge y \leq x, y$ and if $z \leq x, y$, then $z \leq x \wedge y$ for any $z \in G$. If $x \wedge y \in G^\alpha$, then this will do.
We will show that in general $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$ for an ordered automorphism α of period 2. Suppose $z \leq \alpha(x)$ and $z \leq \alpha(y)$. Then $\alpha(z) \leq x$ and $\alpha(z) \leq y$. So, $\alpha(z) \leq x \wedge y$ and $z \leq \alpha(x \wedge y)$. In particular, since $\alpha(x) \wedge \alpha(y) \leq \alpha(x)$ and $\alpha(x) \wedge \alpha(y) \leq \alpha(y)$, $\alpha(x) \wedge \alpha(y) \leq \alpha(x \wedge y)$. Since $x \wedge y \leq x$, we have $\alpha(x \wedge y) \leq \alpha(x)$. Similarly, $x \wedge y \leq y$, so $\alpha(x \wedge y) \leq \alpha(y)$. Thus, $\alpha(x \wedge y) \leq \alpha(x) \wedge \alpha(y)$. Therefore, $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y) = x \wedge y$. $\square$

The following propositions and theorem are modifications of Effros, Handelman and Shen's original [4].

Proposition 7.2. Let G be a lattice-ordered dimension group, and α be an action of $\mathbb{Z}_2$ on G. Suppose $x_1, \cdots, x_n$ are elements of G^+ such that α acts on $\{x_1, \cdots, x_n\}$ by a permutation σ_n. Suppose $p_1, \cdots, p_n$ are integers such that $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = 0$. Then there exist elements $y_1, \cdots, y_t$ in G^+ such that α acts on $\{y_1, \cdots, y_t\}$ by permutation σ_t and nonnegative integers q_{ij} (for $i = 1, \cdots n$, and $j = 1, \cdots, t$) such that

$$x_i = q_{i1} y_1 + \cdots + q_{it} y_t \text{ and } p_1 q_{1j} + \cdots + p_n q_{nj} = 0$$

for all $i = 1, \cdots, n$ and $j = 1, \cdots, t$, and $M_n Q = Q M_t$, where M_n, M_t are the permutation matrices giving σ_n, σ_t respectively, and Q is the matrix of the q_{ij}'s.

Proof. The proof closely follows Goodearl's treatment [9, pp. 51-53]. We consider the relationship between $\mathbb{Z}^n$ and G. From the hypothesis of this proposition, we get a diagram below.

$$\begin{tikzcd} \mathbb{Z}^n \arrow[loop above, "\alpha_n"] \arrow[d, "\varphi"] \\ G \arrow[loop below, "\alpha"'] \end{tikzcd},$$

where φ is a positive homomorphism sending the simplicial basis for $\mathbb{Z}^n$ to the elements x_i and α_n is given by the permutation σ_n. The map φ sends an element $p = p_1e_1 + \cdots + p_ne_n \in \mathbb{Z}^n$ to $p_1x_1 + \cdots + p_nx_n$ which is 0.

The conclusion of the proposition is that we can construct a commuting diagram below.

$$\left.\begin{tikzcd} \mathbb{Z}^n \arrow[loop above, "\alpha_n"] \arrow[r, "\psi"] \arrow[d, "\varphi"] & \mathbb{Z}^m \arrow[loop above, "\alpha_m"] \arrow[dl, "\varphi_2"] \\ G \arrow[loop below, "\alpha"] & \end{tikzcd}\right\} \quad (*)$$

where the map ψ is given by the matrix $Q^\intercal$ in the statement of the proposition. We get new maps ψ and φ_2, a new action α_m, and $\psi(p) = 0$. Two new maps also intertwine the action; $\psi \circ \alpha_n = \alpha_m \circ \psi$ and $\varphi_2 \circ \alpha_m = \alpha \circ \varphi_2$.

We will show that we may assume that for each i, either $\alpha(x_i) = x_i$ or $\alpha(x_i) \wedge x_i = 0$. Since G is a lattice-ordered group, we can consider

$$\begin{aligned} x_i \wedge \alpha(x_i) &= r_i \\ x_i - (x_i \wedge \alpha(x_i)) &= s_i \\ \alpha(x_i) - (x_i \wedge \alpha(x_i)) &= t_i \end{aligned}$$

Then, we get a new set of variables;

$$\begin{aligned} \alpha(r_i) &= r_i \\ \alpha(s_i) &= t_i \\ \alpha(t_i) &= s_i \\ s_i \wedge t_i &= 0 \\ x_i &= s_i + r_i \\ \alpha(x_i) &= t_i + r_i \end{aligned}$$

We are going to show that we can replace our original x_i with the new list r_i, s_i, and t_i the variables of which satisfy the conditions.

We consider two cases; $x_i = \alpha(x_i)$ and $x_i \neq \alpha(x_i)$. We look at the first case, $x_i = \alpha(x_i)$. Two variables s_i and t_i are 0. In this case, we use only r_i. Otherwise, we use variables r_i, s_i, and t_i.

Now, we need to check that the proposition still works if we use the new lists. First of all, we need to show that there is a commutative diagram like (*) above with $\mathbb{Z}^n$, $\mathbb{Z}^m$, where m is the number of variable in our new list, and G. From the hypothesis of the proposition, we get the relationship between $\mathbb{Z}^n$, and G. We can consider two cases, one is e_i is fixed by α_n and the other is not. Suppose $e_1, \cdots, e_l$ are fixed and $e_{l+1}, \cdots, e_n$ are flipped in pairs by the action α_m. Then, we take a simplicial basis of $\mathbb{Z}^m$, $\{a_1, \cdots, a_n, b_{l+1}, \cdots, b_n, c_{l+1}, \cdots, c_n\}$. We let

$$a_i \mapsto r_i,\ \alpha_m(a_i) = a_i$$
$$b_i \mapsto s_i,\ \alpha_m(b_i) = c_i$$
$$c_i \mapsto t_i,\ \alpha_m(c_i) = b_i$$

for each i. Now, we consider the map $\psi : \mathbb{Z}^n \to \mathbb{Z}^m$. We look at an element $e_i \in \mathbb{Z}^n$ such that e_i goes to $x_i \in G$ for each i. Since $x_i = r_i$ when $i = 1, \cdots, l$ and $x_i = r_i + s_i$ when $i = l+1, \cdots, n$, we can send e_i to a_i when $i = 1, \cdots, l$ and e_i to $a_i + b_i$ when $i = l+1, \cdots, n$. If we adapt this process to all basis, then we can get the map ψ. Then, we get a commutative diagram below.

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[loop above, "\alpha_n"] \arrow[r, "\psi"] \arrow[d, "\varphi"] & \mathbb{Z}^m \arrow[loop above, "\alpha_m"] \arrow[dl, "\varphi_1"] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}$$

By the proposition, with our new assumption, applied to our new variables and the image of p under ψ, we get a commutative diagram below.

$$\begin{tikzcd} \mathbb{Z}^m \arrow[r, "\psi_2"] \arrow[d, "\varphi_1"] \arrow[loop above, "\alpha_m"] & \mathbb{Z}^{m'} \arrow[dl, "\varphi_2"] \arrow[loop above, "\alpha_{m'}"] \\ G \arrow[loop below, "\alpha"] & \end{tikzcd}$$

We put two diagrams together.

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "\psi"] \arrow[d, "\varphi"] \arrow[loop above, "\alpha_n"] & \mathbb{Z}^m \arrow[r, "\psi_2"] \arrow[dl, "\varphi_2"] \arrow[loop above, "\alpha_m"] & \mathbb{Z}^{m'} \arrow[dll, "\varphi_3"] \arrow[loop above, "\alpha_{m'}"] \\ G \arrow[loop below, "\alpha"] & & \end{tikzcd}$$

where $p \in \mathbb{Z}^n$ and $\varphi : e_i \to x_i$. This whole diagram commutes. In this diagram, the maps $\psi_2 \circ \psi$ and φ_3 solve the problem with original variables. From now on, we may assume two conditions; either $\alpha(x_i) = x_i$ or $\alpha(x_i) \wedge x_i = 0$ for each i.

First of all, we need to consider the case in which all the $p_i \geq 0$. If any $p_i > 0$, then

$$0 \leq x_i \leq p_i x_i \leq p_1 x_1 + \cdots + p_n x_n = 0,$$

and hence $x_i = 0$. In particular, if all the $p_i > 0$, then all the $x_i = 0$. In case all the $p_i \leq 0$, we apply the same process to the relation $(-p_1)x_1 + \cdots + (-p_n)x_n = 0$.

For the general case, we assign a degree to the coeficient list $p_1, \cdots, p_n$, and proceed by an induction on degree. The degree of a list $p_1, \cdots, p_n$ means the ordered pair (p, λ) where p is the maximum of the values $|p_i|$, and λ is how many times p appears in the list $|p_1|, \cdots |p_n|$.

Next, we show that we can divide the problem into two special cases. If $\varphi(p) = 0$, then $\varphi(\alpha_n(p)) = 0$. So, $\varphi(p + \alpha_n(p)) = 0$ and $\varphi(p - \alpha_n(p)) = 0$. Conversely, If $\varphi(p + \alpha_n(p)) = 0$ and $\varphi(p - \alpha_n(p)) = 0$, then $\varphi(2p) = 0$. Since G is torsion free, $\varphi(p) = 0$. Here, we devide two special cases; $\alpha_n(p) = -p$ and $\alpha_n(p) = p$. Now, we look at $q_1 =$

$p + \alpha_n(p)$ and $q_2 = p - \alpha_n(p)$. Then, $\alpha_n(q_1) = q_1$ and $\alpha_n(q_2) = -q_2$. If we use the first special case, then, we get two maps $\varphi_2 : \mathbb{Z}^{m_1} \to G$ and ψ_1, both equivariant, such that we have a commuting diagram:

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[loop above, "\alpha_n"] \arrow[r, "\psi_1"] \arrow[d, "\varphi"] & \mathbb{Z}^{m_1} \arrow[loop above, "\alpha_{m_1}"] \arrow[dl, "\varphi_2"] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}.$$

Since ψ_1 and φ_2 are equivariant, they intertwine the two actions. By the first case, $\psi_1(q_2) = 0$. We get $\varphi_2(\psi_1(q_2)) = \varphi(q_2) = 0$.

If we use the second special case, we can replace q_2 with $\psi_1(q_2)$. Then we can define a new map $\psi_2 : \mathbb{Z}^{m_1} \to \mathbb{Z}^{m_2}$, and a new commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{m_1} \arrow[loop above, "\alpha_{m_1}"] \arrow[r, "\psi_2"] \arrow[d, "\varphi_2"] & \mathbb{Z}^{m_2} \arrow[loop above, "\alpha_{m_2}"] \arrow[dl, "\varphi_3"] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}.$$

Then, we get $\psi_2(\psi_1(q_2)) = 0$.

Put these diagrams together.

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[loop above, "\alpha_n"] \arrow[r, "\psi_1"] \arrow[d, "\varphi"] & \mathbb{Z}^{m_1} \arrow[loop above, "\alpha_{m_1}"] \arrow[r, "\psi_2"] \arrow[dl, "\varphi_2"] & \mathbb{Z}^{m_2} \arrow[loop above, "\alpha_{m_2}"] \arrow[dll, "\varphi_2"] \\
G \arrow[loop below, "\alpha"] & &
\end{tikzcd}$$

We need to check what result we can get about p. Let $\psi = \psi_2 \circ \psi_1$. Then,

$$\begin{aligned}
&\psi(2p) \\
&= \psi(q_1 + q_2) \\
&= \psi_2(\psi_1(q_1) + \psi_1(q_2)) \\
&= \psi_2(\psi_1(q_1)) \\
&= 0
\end{aligned}$$

Therefore, we can consider two cases,

Case 1 $\alpha_n(p) = -p$ and
Case 2 $\alpha_n(p) = p$

Now, we look at Case 1, $\alpha_n(p) = -p$.
Suppose $e_1, e_2, \cdots, e_j$ are fixed and $e_{j+1}, \alpha_n(e_{j+1}), e_{j+2}, \alpha_n(e_{j+2}), \cdots,$ $e_k, \alpha_n(e_k)$ are flipped by α_n. Then

$$p = p_1e_1 + p_2e_2 + \cdots + p_je_j + p_{j+1}e_{j+1} + p'_{j+1}\alpha_n(e_{j+1}) + \cdots + p_ke_k + p'_k\alpha_n(e_k)$$

If $\alpha_n(p) = -p$, then $p_1 = p_2 = \cdots = p_j = 0$ and $p_l = -p'_l$ where $l = j+1, \cdots, k$. Our relation becomes $p_{j+1}x_{j+1} + \cdots + p_nx_n = p_{j+1}\alpha(x_{j+1}) + \cdots + p_k\alpha(x_k)$ where all $p_l \geq 0$, and we may assume p_{j+1} is the largest coefficient. We may assume $x_l \wedge \alpha(x_l) = 0$ for all $l = j+1, \cdots, k$. With the relation like above, we have $p_{j+1}x_{j+1} \leq p_{j+1}\alpha(x_{j+1}) + \cdots + p_k\alpha(x_k)$. From lattice-ordered and the condition, $p_{j+1}x_{j+1} \leq (p_{j+1}\alpha(x_{j+1}) + \cdots + p_k\alpha(x_k)) \wedge p_{j+1}x_{j+1}$ and $p_{j+1}x_{j+1} \wedge (p_{j+2}\alpha(x_{j+2}) + \cdots + p_k\alpha(x_k)) = 0$. Then, it implies

$$\begin{aligned}
p_{j+1}x_{j+1} &\leq p_{j+2}\alpha(x_{j+2}) + \cdots + p_k\alpha(x_k) \\
&\leq p_{j+1}\alpha(x_{j+2}) + \cdots + p_{j+1}\alpha(x_k) \\
&= p_{j+1}(\alpha(x_{j+2}) + \cdots + \alpha(x_k)) \\
x_{j+1} &\leq \alpha(x_{j+2}) + \cdots + \alpha(x_k)
\end{aligned}$$

By Riesz decomposition, $x_{j+1} = z_{j+2} + \cdots + z_k$ for some elements $z_i \in G^+$ such that $z_i \leq \alpha(x_i)$ for each $i = j+2, \cdots, k$. Also, $\alpha(x_{j+1}) =$

$\alpha(z_{j+2}) + \cdots + \alpha(z_k)$, $\alpha(z_i) \leq x_i$, and $\alpha(z_i) \wedge z_i = 0$ for each $i = j+2, \cdots, k$.

Observe that,

$$\begin{aligned}
&p_{j+1}x_{j+1} + \cdots + p_k x_k = p_{j+1}\alpha(x_{j+1}) + \cdots + p_k\alpha(x_k)\\
\Rightarrow\ & p_{j+1}(z_{j+2} + \cdots + z_k) + p_{j+2}x_{j+2} + \cdots + p_k x_k\\
&= p_{j+1}(\alpha(z_{j+2}) + \cdots + \alpha(z_n)) + p_{j+2}\alpha(x_{j+2}) + \cdots + p_k\alpha(x_k)\\
\Rightarrow\ & p_{j+1}(z_{j+2} + \cdots + z_k) + p_{j+2}x_{j+2} + p_{j+2}z_{j+2} - p_{j+2}z_{j+2}\\
&\quad + \cdots + p_k x_k + p_k z_k - p_k z_k\\
&= p_{j+1}(\alpha(z_{j+2}) + \cdots + \alpha(z_k)) + p_{j+2}\alpha(x_{j+2})\\
&\quad + p_{j+2}\alpha(z_{j+2}) - p_{j+2}\alpha(z_{j+2}) + \cdots + p_k\alpha(x_k) + p_k\alpha(z_k) - p_k\alpha(z_k)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & (p_{j+1} - p_{j+2})z_{j+2} + \cdots + (p_{j+1} - p_k)z_k + p_{j+2}(x_{j+2} - \alpha(z_{j+2}))\\
&\quad + \cdots + p_k(x_k - \alpha(z_k))\\
&= (p_{j+1} - p_{j+2})\alpha(z_{j+2}) + \cdots + (p_{j+1} - p_k)\alpha(z_k)\\
&\quad + p_{j+2}(\alpha(x_{j+2}) - z_{j+2}) + \cdots + p_k(\alpha(x_k) - z_k)\\
\Rightarrow\ & \sum_{i=j+2}^{k}(p_{j+1} - p_i)z_i + \sum_{i=j+2}^{k} p_i(x_i - \alpha(z_i))\\
&= \sum_{i=j+2}^{k}(p_{j+1} - p_i)\alpha(z_i) + \sum_{i=j+2}^{k} p_i(\alpha(x_i) - z_i)
\end{aligned}$$

We label the collection of the new variables; $\underbrace{z_{j+2}, \cdots, z_k}_{\vec{z}}$, $\underbrace{x_{j+2} - \alpha(z_{j+2}), \cdots, x_k - \alpha(z_k)}_{\vec{x}-\alpha(\vec{z})}$, $\underbrace{\alpha(z_{j+2}), \cdots, \alpha(z_k)}_{\alpha(\vec{z})}$, $\underbrace{\alpha(x_{j+2}) - z_{j+2}, \cdots, \alpha(x_k) - z_k}_{\alpha(\vec{x})-\vec{z}}$ This relation has smaller degree and satisfies condition 1. If p_1 still occurs, it occurs one time less. By induction hypothesis, there exist elements $y_1, \cdots, y_t$ in G^+ such that α permutes these with a permutation σ' and nonnegative integers r_{il}, s_{il}, r'_{il}, and s'_{il} for $i = j+2, \cdots, k$ and $l = 1, \cdots t$ such that

$$z_i = r_{i1}y_1 + \cdots + r_{it}y_t$$

$$x_i - \alpha(z_i) = s_{(i)1}y_1 + \cdots + s_{it}y_t$$
$$\alpha(z_i) = \alpha(r_{i1}y_1 + \cdots + r_{it}y_t) = r'_{i1}y_1 + \cdots + r'_{it}y_t$$
$$\alpha(x_i) - z_i = \alpha(x_i - \alpha(z_i)) = s'_{i1}y_1 + \cdots + s'_{it}y_t$$

for $i = j+2, \cdots, k$.

We get a matrix $R = \begin{pmatrix} r \\ s \\ r' \\ s' \end{pmatrix}$ where r, s, r', and s' are matrices. Then there exists a permutation matrix $M_\sigma = \begin{pmatrix} 0 & 0 & E_k & 0 \\ 0 & 0 & 0 & E_k \\ E_k & 0 & 0 & 0 \\ 0 & E_k & 0 & 0 \end{pmatrix}$ which gives a permutation of the generators $\vec{z}, \vec{x} - \alpha(\vec{z}), \alpha(\vec{z})$, and $\alpha(\vec{x}) - \vec{z}$ such that

$$\begin{pmatrix} 0 & 0 & E_k & 0 \\ 0 & 0 & 0 & E_k \\ E_k & 0 & 0 & 0 \\ 0 & E_k & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{x} - \alpha(\vec{z}) \\ \alpha(\vec{z}) \\ \alpha(\vec{x}) - \vec{z} \end{pmatrix} = \begin{pmatrix} \alpha(\vec{z}) \\ \alpha(\vec{x}) - \vec{z} \\ \vec{z} \\ \vec{x} - \alpha(\vec{z}) \end{pmatrix}$$

where E_k is a $k \times k$ identity matrix.
Also, we get a permutation σ' that y_t's undergo by the action α. Then there exists a permutation matrix $M_{\sigma'}$ of y_t's. $M_\sigma R = RM_{\sigma'}$ follows by induction hypothesis.
By using a matrix $\begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix}$, we get $\begin{pmatrix} \vec{x} \\ \alpha(\vec{x}) \end{pmatrix}$ from $\begin{pmatrix} \vec{z} \\ \vec{x} - \alpha(\vec{z}) \\ \alpha(\vec{z}) \\ \alpha(\vec{x}) - \vec{z} \end{pmatrix}$, i.e., $\begin{pmatrix} \vec{x} \\ \alpha(\vec{x}) \end{pmatrix} = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{x} - \alpha(\vec{z}) \\ \alpha(\vec{z}) \\ \alpha(\vec{x}) - \vec{z} \end{pmatrix}$
Also, there exists a matrix $M_{\sigma''} = \begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix}$ that gives a permutation from $\vec{x}$ and $\alpha(\vec{x})$ such that $\begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \alpha(\vec{x}) \end{pmatrix} = \begin{pmatrix} \alpha(\vec{x}) \\ \vec{x} \end{pmatrix}$

Then, $M_{\sigma''}\begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} M_\sigma$
Now, we need to check that $M_{\sigma''}Q = QM_{\sigma'}$
where $Q = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} R$.
We get

$$\begin{aligned} M_{\sigma''}Q &= M_{\sigma''}\begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} R \\ &= \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} M_\sigma R \\ &= \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} RM_{\sigma'} \\ &= QM_{\sigma'} \end{aligned}$$

Finally, suppose $Q = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} R$. Then

$$\begin{aligned} Q\vec{y} &= \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} R\vec{y} \\ &= \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{x} - \alpha(\vec{z}) \\ \alpha(\vec{z}) \\ \alpha(\vec{x}) - \vec{z} \end{pmatrix} \\ &= \begin{pmatrix} \vec{x} \\ \alpha(\vec{x}) \end{pmatrix} \end{aligned}$$

Next, we look at Case 2, $\alpha_n(p) = p$.

In this case, coefficient of e_i is equal to coefficient of $\alpha_n(e_i)$. Suppose $e_1, \cdots, e_h$, and $e_{m+1}, \cdots, e_s$ are fixed, and $e_{h+1}, \alpha_n(e_{h+1}), \cdots, e_m, \alpha_n(e_m)$ and $e_{s+1}, \alpha_n(e_{s+1}), \cdots, e_l, \alpha_n(e_l)$ are flipped by α_n. Then,

$$\begin{aligned} p = p_1e_1 + \cdots + p_he_h + p_{h+1}e_{h+1} + p_{h+1}\alpha_n(p_{h+1}) + \cdots + p_me_m + p_m\alpha_n(e_m) \\ - p_{m+1}e_{m+1} - \cdots - p_se_s - p_{s+1}e_{s+1} - p_{s+1}\alpha_n(e_{s+1}) - \cdots - p_le_l - p_l\alpha_n(e_l) \end{aligned}$$

with all $p_i \geq 0$. Then our relation becomes

$$p_1x_1 + \cdots + p_hx_h + p_{h+1}x_{h+1} + p_{h+1}\alpha(x_{h+1}) + \cdots + p_mx_m + p_m\alpha(x_m)$$

$$= p_{m+1}x_{m+1}+\cdots+p_sx_s+p_{s+1}x_{s+1}+p_{s+1}\alpha(x_{s+1})+\cdots+p_lx_l+p_l\alpha(x_l)$$

in G. We label the collection of variables in the original relation; $\underbrace{x_1, \cdots, x_h}_{\mathcal{X}_1}$, $\underbrace{x_{h+1}, \cdots, x_m}_{\mathcal{X}_2}$, $\underbrace{\alpha(x_{h+1}), \cdots, \alpha(x_m)}_{\alpha(\mathcal{X}_2)}$, $\underbrace{x_{m+1}, \cdots, x_s}_{\mathcal{X}_3}$, $\underbrace{x_{s+1}, \cdots, x_l}_{\mathcal{X}_4}$, $\underbrace{\alpha(x_{s+1}), \cdots, \alpha(x_l)}_{\alpha(\mathcal{X}_4)}$. So, we put these together in one vector $\vec{x} = \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \alpha(\mathcal{X}_2) \\ \mathcal{X}_3 \\ \mathcal{X}_4 \\ \alpha(\mathcal{X}_4) \end{pmatrix}$, and we get a permutation

$$M_{\sigma''} = \begin{pmatrix} E_{k_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_{k_2} & 0 & 0 & 0 \\ 0 & E_{k_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{k_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{k_4} \\ 0 & 0 & 0 & 0 & E_{k_4} & 0 \end{pmatrix}$$

where $E_{k_1}, E_{k_2}, E_{k_3}$ and E_{k_4} are the square identity matrices for $k_1 = h, k_2 = m - h, k_3 = s - m$, and $k_4 = l - s$

In the case 2, we have two situations to consider: whether the biggest coefficient is one of the fixed ones or one of the flipped ones.

Suppose it is one of the fixed ones, p_1. We have $p_1x_1 \leq q$ with $q = p_{m+1}x_{m+1} + \cdots + p_sx_s + p_{s+1}(x_{s+1} + \alpha(x_{s+1})) + \cdots + p_l(x_l + \alpha(x_l))$. Suppose $x_i + \alpha(x_i) = w_i$. We have $x_1, x_{m+1}, \cdots, x_s, w_{s+1}, \cdots, w_l$ are all in G^α, the fixed point subgroup. Since G^α is a lattice-ordered dimension group, we can write $x_1 = z_{m+1}+\cdots+z_s+y_{s+1}+\cdots+y_l$ with $0 \leq z_i \leq x_i$ and $0 \leq y_j \leq w_j$ where $z_i, y_j \in G^\alpha$. We have $w_j = x_j + \alpha(x_j)$ with $x_j \wedge \alpha(x_j) = 0, y_j \leq w_j$, and write $y_j = z_j + z'_j$ with $z_j \leq x_j$ and $z'_j \leq \alpha(x_j)$. Since $x_i \wedge \alpha(x_i) = 0$, we get $z_j = y_j \wedge x_j, z'_j = y_j \wedge \alpha(x_j)$.

Also, since $\alpha(y_j) = y_j$, we get $\alpha(z_j) = z'_j$. Then, we get $x_1 = z_{m+1} + \cdots + z_s + z_{s+1} + \alpha(z_{s+1}) + \cdots + z_l + \alpha(z_l)$

Observe that,

$$\begin{aligned}
&p_1 x_1 + \cdots + p_h x_h + p_{h+1} x_{h+1} + p_{h+1}\alpha(x_{h+1}) + \cdots \\
&\quad + p_m x_m + p_m \alpha(x_m) \\
&\quad = p_{m+1} x_{m+1} + \cdots + p_s x_s + p_{s+1} x_{s+1} + p_{s+1}\alpha(x_{s+1}) + \cdots \\
&\quad + p_l x_l + p_l \alpha(x_l) \\
&\Rightarrow p_1(z_{m+1} + \cdots + z_s + z_{s+1} + \alpha(z_{s+1}) + \cdots + z_l + \alpha(z_l)) + p_2 x_2 \\
&\quad + \cdots + p_h x_h + p_{h+1} x_{h+1} + p_{h+1}\alpha(x_{h+1}) + \cdots + p_m x_m + p_m \alpha(x_m) \\
&\quad = p_{m+1} x_{m+1} + \cdots + p_s x_s + p_{s+1} x_{s+1} + p_{s+1}\alpha(x_{s+1}) \\
&\qquad + \cdots + p_l x_l + p_l \alpha(x_l) \\
&\Rightarrow (p_1 - p_{m+1}) z_{m+1} + \cdots + (p_1 - p_s) z_s \\
&\quad + (p_1 - p_{s+1})(z_{s+1} + \alpha(z_{s+1})) + \cdots + (p_1 - p_l)(z_l + \alpha(z_l)) \\
&\quad + p_2 x_2 + \cdots + p_h x_h + p_{h+1} x_{h+1} + p_{h+1}\alpha(x_{h+1}) + \cdots \\
&\quad + p_m x_m + p_m \alpha(x_m) \\
&\quad = p_{m+1}(x_{m+1} - z_{m+1}) + \cdots + p_s(x_s - z_s) \\
&\quad + p_{s+1}((x_{s+1} - z_{s+1}) + (\alpha(x_{s+1}) - \alpha(z_{z+1}))) + \cdots \\
&\quad + p_l((x_l - z_l) + (\alpha(x_l) - \alpha(z_l))) \\
&\Rightarrow \sum_{i=m+1}^{s} (p_1 - p_i) z_i + \sum_{i=s+1}^{l} (p_1 - p_i)(z_i + \alpha(z_i)) + \sum_{i=2}^{h} p_i x_i \\
&\quad + \sum_{i=h+1}^{m} p_i(x_i + \alpha(x_i)) \\
&\quad = \sum_{i=m+1}^{s} p_i(x_i - z_i) + \sum_{i=s+1}^{l} p_i((x_i - z_i) + (\alpha(x_i) - \alpha(z_i)))
\end{aligned}$$

$$\Rightarrow \sum_{i=m+1}^{s}(p_1-p_i)z_i + \sum_{i=s+1}^{l}(p_1-p_i)z_i + \sum_{i=s+1}^{l}(p_1-p_i)\alpha(z_i)$$
$$+\sum_{i=2}^{h}p_ix_i + \sum_{i=h+1}^{m}p_ix_i + \sum_{i=h+1}^{m}p_i\alpha(x_i)$$
$$= \sum_{i=m+1}^{s}p_i(x_i-z_i) + \sum_{i=s+1}^{l}p_i(x_i-z_i) + \sum_{i=s+1}^{l}p_i(\alpha(x_i)-\alpha(z_i))$$

We label the collection of the new variables; $\underbrace{z_{m+1},\cdots,z_s}_{\mathcal{Z}_1}$, $\underbrace{z_{s+1},\cdots,z_l}_{\mathcal{Z}_2}$, $\underbrace{\alpha(z_{s+1}),\cdots,\alpha(z_l)}_{\alpha(\mathcal{Z}_2)}$, $\underbrace{x_2,\cdots,x_h}_{\mathcal{X}_1'}$, $\underbrace{x_{h+1},\cdots,x_m}_{\mathcal{X}_2}$, $\underbrace{\alpha(x_{h+1}),\cdots,\alpha(x_m)}_{\alpha(\mathcal{X}_2)}$, $\underbrace{x_{m+1}-z_{m+1},\cdots,x_s-z_s}_{\mathcal{X}_3-\mathcal{Z}_1}$, $\underbrace{x_{s+1}-z_{s+1},\cdots,x_l-z_l}_{\mathcal{X}_4-\mathcal{Z}_2}$, $\underbrace{\alpha(x_{s+1}-z_{s+1}),\cdots,\alpha(x_l-z_l)}_{\alpha(\mathcal{X}_4)-\alpha(\mathcal{Z}_2)}$. This relation has smaller degree and satisfies condition 2, i.e., the relation is invariant under the automorphism. If p_1 still occurs, it occurs one time less. By induction hypothesis, there exists $\vec{y} = \begin{pmatrix} y_1 \cdots y_t \end{pmatrix}$ for $y_i \in G^+$ such that α permutes these with a permutation σ' and matrices $r, s, t, r', s', t', r'', s''$ and t'' whose entries are nonnegative integers such that

$$\begin{aligned}
\mathcal{Z}_1 &= r\vec{y}^{\top} \\
\mathcal{Z}_2 &= r'\vec{y}^{\top} \\
\alpha(\mathcal{Z}_2) &= \alpha(r'\vec{y}^{\top}) = r''\vec{y}^{\top} \\
\mathcal{X}_1' &= s\vec{y}^{\top} \\
\mathcal{X}_2 &= s'\vec{y}^{\top} \\
\alpha(\mathcal{X}_2) &= \alpha(s'\vec{y}^{\top}) = s''\vec{y}^{\top} \\
\mathcal{X}_3-\mathcal{Z}_1 &= t\vec{y}^{\top} \\
\mathcal{X}_4-\mathcal{Z}_2 &= t'\vec{y}^{\top} \\
\alpha(\mathcal{X}_4)-\alpha(\mathcal{Z}_2) &= \alpha(s'\vec{y}^{\top}) = t''\vec{y}^{\top}
\end{aligned}$$

We get a matrix $R^\top = \begin{pmatrix} r & r' & r'' & s & s' & s'' & t & t' & t'' \end{pmatrix}$.
Then there exists permutation matrix

$$M_\sigma = \begin{pmatrix} E_{k_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{k'_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{k_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{k_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 \end{pmatrix}$$

where E_{k_i} is the square identity matrix for $k'_1 = h - 1$,
$k_2 = m - h, k_3 = s - m$, and $k_4 = l - s$ gives permutation from the generators $\mathcal{Z}_1, \mathcal{Z}_2, \alpha(\mathcal{Z}_2), \mathcal{X}'_1, \mathcal{X}_2, \alpha(\mathcal{X}_2), \mathcal{X}_3 - \mathcal{Z}_1, \mathcal{X}_4 - \mathcal{Z}_2$, and $\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2)$ such that

$$M_\sigma \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \alpha(\mathcal{Z}_2) \\ \mathcal{X}'_1 \\ \mathcal{X}_2 \\ \alpha(\mathcal{X}_2) \\ \mathcal{X}_3 - \mathcal{Z}_1 \\ \mathcal{X}_4 - \mathcal{Z}_2 \\ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_1 \\ \alpha(\mathcal{Z}_2) \\ \mathcal{Z}_2 \\ \mathcal{X}'_1 \\ \alpha(\mathcal{X}_2) \\ \mathcal{X}_2 \\ \mathcal{X}_3 - \mathcal{Z}_1 \\ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) \\ \mathcal{X}_4 - \mathcal{Z}_2 \end{pmatrix}.$$

Also, we get a permutation σ' that y_t's undergo by the action α. Then there exists a permutation matrix $M_{\sigma'}$ of y_t's. By induction hypothesis, we get a relationship such that $M_\sigma R = R M_{\sigma'}$. So, we get a commutative diagram below.

$$\begin{tikzcd}
\mathbb{Z}^m \arrow[loop above, "M_\sigma"] \arrow[r, "R^\top"] \arrow[d] & \mathbb{Z}^{m'} \arrow[loop above, "M'_\sigma"] \arrow[dl] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}$$

We denote

$$\mathcal{X}_1 = \begin{pmatrix} x_1 \\ \mathcal{X}'_1 \end{pmatrix} = \left(\begin{array}{c|c} (1, \cdots, 1) & 0 \\ \hline 0 & E_{k'_1} \end{array}\right) \left(\begin{array}{c} \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \alpha(\mathcal{Z}_2) \\ \hline \mathcal{X}'_1 \end{array}\right)$$

where $E_{k'_1}$ is the $(h-1) \times (h-1)$ identity matrix. From the above block matrix, we can define the new matrices

$$P_{k_i} = \left(\begin{array}{c} 1, \cdots, 1 \\ \hline 0 \end{array}\right) \text{ and } \widetilde{E_{k'_1}} = \left(\begin{array}{c} 0, \cdots, 0 \\ \hline E_{k'_1} \end{array}\right)$$

for $i = 3, 4$. By using a matrix

$$R' = \begin{pmatrix}
P_{k_3} & P_{k_4} & P_{k_4} & \widetilde{E_{k'_1}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_{k_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_{k_2} & 0 & 0 & 0 \\
E_{k_3} & 0 & 0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 \\
0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 \\
0 & 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & E_{k_4}
\end{pmatrix},$$

we get old variables $\begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \alpha(\mathcal{X}_2) \\ \mathcal{X}_3 \\ \mathcal{X}_4 \\ \alpha(\mathcal{X}_4) \end{pmatrix}$ from new ones $\begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \alpha(\mathcal{Z}_2) \\ \mathcal{X}_1' \\ \mathcal{X}_2 \\ \alpha(\mathcal{X}_2) \\ \mathcal{X}_3 - \mathcal{Z}_1 \\ \mathcal{X}_4 - \mathcal{Z}_2 \\ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) \end{pmatrix}$.

Also, there exists a permutation matrix $M_{\sigma''}$ that gives a permutation from the original variables. One can check that $R'M_\sigma = M_{\sigma''}R'$. So, we get a commuting diagram below.

$$\begin{tikzcd} \mathbb{Z}^n \arrow[loop above, "M_\sigma''"] \arrow[r, "R'^\top"] \arrow[d] & \mathbb{Z}^m \arrow[loop above, "M_\sigma"] \arrow[dl] \\ G \arrow[loop below, "\alpha"] & \end{tikzcd}$$

If we put together two above diagrams, then we get the result that $M_{\sigma''}Q = QM_{\sigma'}$ where $Q = R'R$.

Now, suppose the largest coefficient is one of the flipped ones, p_{h+1}. We have $p_{h+1}(x_{h+1} + \alpha(x_{h+1})) \leq q$ where $q = p_{m+1}x_{m+1} + \cdots + p_s x_s + p_{s+1}x_{s+1} + p_{s+1}\alpha(x_{s+1}) + \cdots + p_l x_l + p_l \alpha(x_l)$. Suppose $x_j + \alpha(x_j) = v_j$. Then $v_j \in G^\alpha$ for each j. Since G^α is a lattice-ordered dimension group, we can write $x_{h+1} + \alpha(x_{h+1}) = z_{m+1} + \cdots + z_s + r_{s+1} + \cdots + r_l$ with $0 \leq z_i \leq x_i$ and $0 \leq r_j \leq v_j$ where $z_i, r_j \in G^\alpha$. We have $v_j = x_j + \alpha(x_j)$ with $x_j \wedge \alpha(x_j) = 0$, $r_j \leq v_j$, and write $r_j = z_j + z_j'$ with $z_j \leq x_j$ and $z_j' \leq \alpha(x_j)$. Since $x_i \wedge \alpha(x_i) = 0$, we get $z_j = v_j \wedge x_j$ and $z_j' = v_j \wedge \alpha(z_j)$. Also, since $\alpha(v_j) = v_j$, we get $\alpha(z_j) = z_j'$. Then, we get

$$x_{h+1} + \alpha(x_{h+1}) = z_{m+1} + \cdots + z_s + (z_{s+1} + \alpha(z_{s+1})) + \cdots + (z_l + \alpha(z_l)).$$

Observe that,

$$\begin{aligned}
&p_1x_1+\cdots+p_hx_h+p_{h+1}x_{h+1}+p_{h+1}\alpha(x_{h+1})+\cdots\\
&+p_mx_m+p_m\alpha(x_m)\\
&=p_{m+1}x_{m+1}+\cdots+p_sx_s+p_{s+1}x_{s+1}+p_{s+1}\alpha(x_{s+1})+\cdots\\
&+p_lx_l+p_l\alpha(x_l)\\
\Rightarrow\ &p_1x_1+\cdots+p_hx_h+p_{h+1}(z_{m+1}+\cdots+z_s+(z_{s+1}+\alpha(z_{s+1}))+\cdots\\
&+(z_l+\alpha(z_l)))+p_{h+2}(x_{h+2}+\alpha(x_{h+2}))+\cdots+p_m(x_m+\alpha(x_m))\\
&=p_{m+1}x_{m+1}+\cdots+p_sx_s+p_{s+1}x_{s+1}+p_{s+1}\alpha(x_{s+1})\\
&\quad+\cdots+p_lx_l+p_l\alpha(x_l)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &p_1x_1+\cdots+p_hx_h+(p_{h+1}-p_{m+1})z_{m+1}+\cdots+(p_{h+1}-p_s)z_s\\
&+(p_{h+1}-p_{s+1})(z_{s+1}+\alpha(z_{s+1}))+\cdots+(p_{h+1}-p_l)(z_l+\alpha(z_l))\\
&+p_{h+2}(x_{h+2}+\alpha(x_{h+2}))+\cdots+p_m(x_m+\alpha(x_m))\\
&=p_{m+1}(x_{m+1}-z_{m+1})+\cdots+p_s(x_s-z_s)\\
&+p_{s+1}((x_{s+1}-z_{s+1})+(\alpha(x_{s+1})-\alpha(x_{z+1})))\\
&+\cdots+p_l((x_l-z_l)+(\alpha(x_l)-\alpha(z_l)))
\end{aligned}$$

$$\Rightarrow\quad \left.\begin{aligned}
&\textstyle\sum_{i=1}^h p_ix_i+\sum_{i=m+1}^s(p_{h+1}-p_i)z_i+\sum_{i=s+1}^l(p_{h+1}-p_i)z_i\\
&\textstyle+\sum_{i=s+1}^l(p_{h+1}-p_i)\alpha(z_i)+\sum_{i=h+2}^m p_i(x_i+\alpha(x_i))\\
&\textstyle=\sum_{i=m+1}^s p_i(x_i-z_i)+\sum_{i=s+1}^l p_i(x_i-z_i)\\
&\textstyle+\sum_{i=s+1}^l p_i(\alpha(x_i)-\alpha(z_i))
\end{aligned}\right\}\qquad(\dagger)$$

We need to split the $x_{h+1}+\alpha(x_{h+1})$ to x_{h+1} and $\alpha(x_{h+1})$. Define the new variables;

$$\left.\begin{array}{l}
t_{m+1},\cdots,t_s \text{ where } t_i=z_i\wedge x_{h+1}\\
r_{m+1},\cdots,r_s \text{ where } r_i=z_i\wedge\alpha(x_{h+1})
\end{array}\right\}\quad \alpha(t_i)=r_i$$

$$\begin{gathered}
v_{s+1},\cdots,v_l \text{ where } v_j=z_j\wedge x_{h+1}\\
\alpha(v_{s+1}),\cdots,\alpha(v_l)\\
w_{s+1},\cdots,w_l \text{ where } w_j=z_j\wedge\alpha(x_{h+1})\\
\alpha(w_{s+1}),\cdots,\alpha(w_l)
\end{gathered}$$

Then, $x_{h+1} = t_{m+1} + \cdots + t_s + v_{s+1} + \cdots + v_l + \alpha(w_{s+1}) + \cdots + \alpha(w_l)$, $\alpha(x_{h+1}) = r_{m+1} + \cdots + r_s + \alpha(v_{s+1}) + \cdots + \alpha(v_l) + w_{s+1} + \cdots + w_l$, $z_i = t_i + r_i$ for $i = m+1, \cdots, s$, $z_j = v_i + w_i$ for $j = s+1, \cdots, l$.

We label the collection of the new variables; $\underbrace{x_1, \cdots, x_h}_{\mathcal{X}_1}, \underbrace{t_{m+1}, \cdots, t_s}_{\mathcal{T}}$, $\underbrace{r_{m+1}, \cdots, r_s}_{\mathcal{R}}$, $\underbrace{v_{s+1}, \cdots, v_l}_{\mathcal{V}}$, $\underbrace{\alpha(v_{s+1}), \cdots, \alpha(v_l)}_{\alpha(\mathcal{V})}$, $\underbrace{w_{s+1}, \cdots, w_l}_{\mathcal{W}}$, $\underbrace{\alpha(w_{s+1}), \cdots, \alpha(w_l)}_{\alpha(\mathcal{W})}$, $\underbrace{x_{h+2}, \cdots, x_m}_{\mathcal{X}_2'}$, $\underbrace{\alpha(x_{h+2}), \cdots, \alpha(x_m)}_{\alpha(\mathcal{X}_2')}$, $\underbrace{x_{m+1} - z_{m+1}, \cdots, x_s - z_s}_{\mathcal{X}_3 - \mathcal{Z}_1}$, $\underbrace{x_{s+1} - z_{s+1}, \cdots, x_l - z_l}_{\mathcal{X}_4 - \mathcal{Z}_2}$, $\underbrace{\alpha(x_{s+1}) - \alpha(x_l), \cdots, \alpha(x_l) - \alpha(z_l)}_{\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2)}$. This relation that we get from (†) has smaller degree and satisfies condition 2. If p_{h+1} still occurs, it occurs one time less. By induction hypothesis, there exists $\vec{y} = \left(y_1 \cdots y_t\right)$ for $y_i \in G^+$ such that α permutes these with a perutation σ' and matrices $a, a', a'', b, c, d, d', e,$
e', f, f', and f'' whose entries are nonnegative integers such that

$$\begin{aligned}
\mathcal{X}_1 &= a\vec{y}^\top \\
\mathcal{X}_2' &= a'\vec{y}^\top \\
\alpha(\mathcal{X}_2') &= a''\vec{y}^\top \\
\mathcal{T} &= b\vec{y}^\top \\
\mathcal{R} &= c\vec{y}^\top \\
\mathcal{V} &= d\vec{y}^\top \\
\alpha(\mathcal{V}) &= d'\vec{y}^\top \\
\mathcal{W} &= e\vec{y}^\top \\
\alpha(\mathcal{W}) &= e'\vec{y}^\top \\
\mathcal{X}_3 - \mathcal{Z}_1 &= f\vec{y}^\top \\
\mathcal{X}_4 - \mathcal{Z}_2 &= f'\vec{y}^\top \\
\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) &= f''\vec{y}^\top
\end{aligned}$$

We get a matrix

$$R^\top = \begin{pmatrix} a & a' & a'' & b & c & d & d' & e & e' & f & f' & f'' \end{pmatrix}$$

Then there exists a permutation matrix

$$M_\sigma = \begin{pmatrix}
E_{k_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_{k_2'} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_{k_2'} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_{k_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0
\end{pmatrix}$$

where E_{k_i} is the square identity matirix for $k_1 = h, k_2' = m-h-1, k_3 = s-m$, and $k_4 = l-s$ which gives the permutation from the generators such that

$$M_\sigma \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2' \\ \alpha(\mathcal{X}_2') \\ \mathcal{T} \\ \mathcal{R} \\ \mathcal{V} \\ \alpha(\mathcal{V}) \\ \mathcal{W} \\ \alpha(\mathcal{W}) \\ \mathcal{X}_3 - \mathcal{Z}_1 \\ \mathcal{X}_4 - \mathcal{Z}_2 \\ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) \end{pmatrix} = \begin{pmatrix} \mathcal{X}_1 \\ \alpha(\mathcal{X}_2') \\ \mathcal{X}_2' \\ \mathcal{R} \\ \mathcal{T} \\ \alpha(\mathcal{V}) \\ \mathcal{V} \\ \alpha(\mathcal{W}) \\ \mathcal{W} \\ \mathcal{X}_3 - \mathcal{Z}_1 \\ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) \\ \mathcal{X}_4 - \mathcal{Z}_2 \end{pmatrix}.$$

Also, we get a permutation σ' that y_t's undergo by the action α. Then there exists a permutation matrix $M_{\sigma'}$ of y_t's. By induction hypothesis, we get a relationship such that $M_\sigma R = R M_{\sigma'}$. So, one can check that we get a commutative diagram below.

$$\begin{tikzcd}
\mathbb{Z}^m \arrow[loop above, "M_\sigma"] \arrow[r, "R^\top"] \arrow[d] & \mathbb{Z}^{m'} \arrow[loop above, "M'_\sigma"] \arrow[dl] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}$$

We denote

$$\mathcal{X}_2 = \begin{pmatrix} x_{h+1} \\ \mathcal{X}'_2 \end{pmatrix} = \left(\begin{array}{c|c} (1, \cdots, 1) & 0 \\ \hline 0 & E_{k'_2} \end{array}\right) \left(\begin{array}{c} \mathcal{T} \\ \mathcal{V} \\ \alpha(\mathcal{W}) \\ \hline \mathcal{X}'_2 \end{array}\right)$$

and

$$\alpha(\mathcal{X}_2) = \begin{pmatrix} \alpha(x_{h+1}) \\ \alpha(\mathcal{X}'_2) \end{pmatrix} = \left(\begin{array}{c|c} (1, \cdots, 1) & 0 \\ \hline 0 & E_{k'_2} \end{array}\right) \left(\begin{array}{c} \mathcal{R} \\ \alpha(\mathcal{V}) \\ \mathcal{W} \\ \hline \alpha(\mathcal{X}'_2) \end{array}\right)$$

From the above block matrix, we can define the new matrices

$$P_{k_i} = \left(\begin{array}{c} 1, \cdots, 1 \\ \hline 0 \end{array}\right) \text{ and } \widetilde{E_{k'_2}} = \left(\begin{array}{c} 0, \cdots, 0 \\ \hline E_{k'_2} \end{array}\right)$$

for $i = 3, 4$. By using a matrix

$$R' = \begin{pmatrix}
E_{k_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \widetilde{E_{k'_2}} & 0 & P_{k_3} & 0 & P_{k_4} & 0 & 0 & P_{k_4} & 0 & 0 & 0 \\
0 & 0 & \widetilde{E_{k'_2}} & 0 & P_{k_3} & 0 & P_{k_4} & P_{k_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_{k_3} & E_{k_3} & 0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & E_{k_4} & 0 & 0 & E_{k_4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & E_{k_4} & 0 & 0 & E_{k_4}
\end{pmatrix},$$

$$\text{we get old variables } \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \alpha(\mathcal{X}_2) \\ \mathcal{X}_3 \\ \mathcal{X}_4 \\ \alpha(\mathcal{X}_4) \end{pmatrix} \text{ from new ones } \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2' \\ \alpha(\mathcal{X}_2') \\ \mathcal{T} \\ \mathcal{R} \\ \mathcal{V} \\ \alpha(\mathcal{V}) \\ \mathcal{W} \\ \alpha(\mathcal{W}) \\ \mathcal{X}_3 - \mathcal{Z}_1 \\ \mathcal{X}_4 - \mathcal{Z}_2 \\ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) \end{pmatrix}.$$

Also, there exists a permutation matrix $M_{\sigma''}$ that gives a permutation from the original variables. So, we get a commuting diagram below.

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[loop above, "M''_\sigma"] \arrow[r, "R'^\top"] \arrow[d] & \mathbb{Z}^m \arrow[loop above, "M_\sigma"] \arrow[dl] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}$$

If we put together two above diagrams, then we get the result that $M_{\sigma''}Q = QM_{\sigma'}$ where $Q = R'R$.

$\square$

The following Lemma will aid to prove the proposition 7.3.

LEMMA 7.2. [**13**, Corollary 4.96]All subgroups of a finitely generated abelian group are finitely generated.

PROPOSITION 7.3. Let G_1 be a simplicial group with simplicial basis $\{e_1, \cdots, e_n\}$ and $\mathbb{Z}_2$ action α_1 given by the permutation σ_n. Let G be a lattice-ordered dimension group with $\mathbb{Z}_2$ action α, and let $g_1 : G_1 \to G$ be a positive equivariant homomorphism. Then there exist a simplicial group G_2 with an $\mathbb{Z}_2$ action α_2, and positive equivariant homomorphisms h and g_2 such that $g_1 = g_2 h$, $ker(g_1) = ker(h)$.

$$\begin{tikzcd}
G_1 \arrow[loop above, "\alpha_1"] \arrow[r, "h"] \arrow[d, "g_1"] & G_2 \arrow[loop above, "\alpha_2"] \arrow[dl, "g_2"] \\
G \arrow[loop below, "\alpha"] &
\end{tikzcd}$$

Proof. The proof closely follows Goodearl's treatment [9, pp.53–54]. We would like to show that finite generators $a_1, \cdots, a_k$ in $ker(g_1)$ also lie in $ker(h)$ by induction hypothesis. First, we need to check when $k = 1$. If G_1 is a zero group, then G_2 is also zero group and we define homomorphisms h, g_2 are zero maps. Now, we assume that G_1 is nonzero. Let $\{e_1, \cdots, e_n\}$ be the simplicial basis for G_1, and let σ_n be the permutation of the simplicial basis that gives the action α_1. Let $\{x_1, \cdots, x_n\} \in G^+$ be the images of $\{e_1, \cdots, e_n\}$ and $ker(g_1)$ be finitely generated by a_1. Set $x_i = g_1(e_i)$ for each $i = 1, \cdots, n$. Since g_1 is an equivariant homomorphism, it follows that $\alpha(x_i) = \alpha(g_1(e_i)) = g_1(\alpha_1(e_i))$ for each $i = 1, \cdots, n$. Write $a_1 = p_1 e_1 + \cdots + p_n e_n$ for some integers p_i, and observe that

$$p_1 x_1 + \cdots + p_n x_n = g_1(a_1) = 0.$$

According to proposition 7.2, there exist elements $y_1, \cdots, y_t$ in G^+ such that α acts on $\{y_1, \cdots, y_t\}$ by permutation σ_t and nonnegative integers q_{ij} (for $i = 1, \cdots n$, and $j = 1, \cdots, t$) such that

$$x_i = q_{i1} y_1 + \cdots + q_{it} y_t \text{ and } p_1 q_{1j} + \cdots + p_n q_{nj} = 0$$

for all i and j, and $M_n Q = Q M_t$, where M_n, M_t are the permutation matrices giving σ_n, σ_t respectively, and Q is the matrix of the q_{ij}'s.

Set $G_2 = \mathbb{Z}^t$, and let $\{f_1, \cdots, f_t\}$ be simplicial basis for G_2. Define group homomorphisms $h : G_1 \to G_2$ and $g_2 : G_2 \to G$ so that

$$h(e_i) = q_{i1} f_1 + \cdots + q_{it} f_t$$

for $i = 1, \cdots, n$ and $g_2(f_j) = y_j$ for $j = 1, \cdots, t$. Define an $\mathbb{Z}_2$ action α_2 on G_2 by the permuation matrix M_t. Then it follows that

$$h(\alpha_1(e_i)) = q_{i1}\alpha_2(f_1) + \cdots + q_{it}\alpha_2(f_t) = \alpha_2(h(e_i))$$

for $i = 1, \cdots, n$ and $\alpha(g_2(f_j)) = \alpha(y_j) = g_2(\alpha_2(f_j))$ for $j = 1, \cdots, t$. So, the maps intertwine the actions.

As each $q_{ij} \in \mathbb{Z}^+$ and each $y_j \in G^+$, we see that h and g_2 are positive homomorphisms. Since

$$g_2h(e_i) = g_2(q_{i1}f_1 + \cdots + q_{it}f_t) = q_{i1}y_1 + \cdots + q_{it}y_t = x_i = g_1(e_i)$$

for all $i = 1, \cdots, n$, we obtain $g_2h = g_1$. Also,

$$h(a_1) = h(\sum_{i=1}^{n} p_i e_i) = \sum_{i=1}^{n}\sum_{j=1}^{t} p_i q_{ij} f_j = \sum_{j=1}^{t}(\sum_{i=1}^{n} p_i q_{ij}) f_j = 0,$$

so that $a_1 \in ker(h)$.

Now, we show that the induction step. Let $k > 1$. Assume that there exist a simplicial group G_3 with $\mathbb{Z}_2$ action α_3 by the permutation matrix M_s, positive homomorphisms $h_1 : G_1 \to G_3$ and $g_3 : G_3 \to G$ such that $g_1 = g_3h_1$. Assume that $a_1, \cdots, a_{k-1}$ lie in $ker(h_1)$. Since $g_3h_1(a_k) = g_1(a_k) = 0$, the element $h_1(a_k)$ lies in $ker(g_3)$. Hence, by the above result, there exist a simplicial group G_2 and positive equivariant homomorphisms $h_2 : G_3 \to G_2$ and $g_2 : G_2 \to G$ such that $g_3 = g_2h_2$ and $h_1(a_k) \in ker(h_2)$. Set $h = h_2h_1$, which is a positive equivariant homomorphism from G_1 to G_2 such that $g_2h = g_2h_2h_1 = g_3h_1 = g_1$. Since $a_1, \cdots, a_{k-1}$ lie in $ker(h_1)$, they also lie in $ker(h)$. Because $h_1(a_k)$ lies in $ker(h_2)$, the element a_k lies in $ker(h)$. This completes the induction step.

Since G_1 is finitely generated as an abelian group, all its subgroups are finitely generated. Hence, we may choose generators of $ker(g_1)$ as a group. By the result of above induction step, there exist a simplicial group G_2 and positive equivariant homomorphisms $h : G_1 \to G_2$ and $g_2 : G_2 \to G$ such that $g_1 = g_2h$ and $a_1, \cdots, a_k$ all lie in $ker(h)$. Thus $ker(g_1) \subseteq ker(h)$. The reverse inclusion, $ker(g_1) \supseteq ker(h)$, follows from the factorization $g_1 = g_2h$. Therefore, $ker(g_1) = ker(h)$.

□

With aid of previous proposition, we prove our main theorem.

Main Theorem. If there exists a $\mathbb{Z}_2$ action on a countable lattice-ordered dimension group, then it can be expressed as an inductive limit of $\mathbb{Z}_2$ actions on simplicial groups.

Proof. This proof closely follow Goodearl's treatment[9, pp.54-55].

Let G be a countable lattice-ordered dimension group with the action α and $\{x_1, x_2, \cdots\} = G^+$. We construct a countable sequnce with simplicial groups $G_1, G_2, \cdots$ with the actions $\alpha_1, \alpha_2, \cdots$ and positive equivariant homomorphisms $g_n : G_n \to G$ and $h_n : G_n \to G_{n+1}$ for all $n \in \mathbb{N}$ such that $x_n \in g_n(G_n^+)$, $g_{n+1} \circ h_n = g_n$, and $ker(g_n) = ker(h_n)$ for all $n \in \mathbb{N}$. Also, we define the limit of the sequence that we construct, G_∞ with a positive equivariant homomorphism $g_\infty : G_\infty \to G$, and a $\mathbb{Z}_2$ action on G_∞, α_∞.

$$\begin{tikzcd}
G_1 \arrow[loop above, "\alpha_1"] \arrow[r, "h_1"] \arrow[drrrr, "g_1"] & G_2 \arrow[loop above, "\alpha_2"] \arrow[r, "h_2"] \arrow[drrr, "g_2"] & G_3 \arrow[loop above, "\alpha_3"] \arrow[r, "h_3"] \arrow[drr, "g_3"] & \cdots \arrow[r, "h_\infty"] \arrow[dr] & G_\infty \arrow[loop above, "\alpha_\infty"] \arrow[d, "g_\infty"] \\
& & & & G \arrow[loop below, "\alpha"]
\end{tikzcd}$$

First of all, we set $G_1 = \mathbb{Z}^2$ with a $\mathbb{Z}_2$ action α_1 that flips the elements, i.e., $\alpha_1(e_1) = e_2$ and $\alpha_1(e_2) = e_1$. We define a positive homomorphism $g_1 : G_1 \to G$ so that $g_1(e_1) = x_1$ and $g_1(e_2) = g_1(\alpha_1(e_1)) = \alpha(x_1)$. Suppose that we have constructed $g_1, G_1, \alpha_1, \cdots, g_n, G_n, \alpha_n$ which meet the requirements. We would like to construct the next one; $g_{n+1}, G_{n+1}, \alpha_{n+1}$. The direct product $H = G_n \oplus \mathbb{Z}^2$ is a simplicial group and we define a positive homomorphism $g : H \to G$ by the rule $g(a, k, l) = g_n(a) + kx_{n+1} + l\alpha(x_{n+1})$, and a $\mathbb{Z}_2$ action α' such that $g_n(\alpha_n(e_i)) = \alpha'(g(x_i))$ for all i.

By the proposition 7.3, there exist a simplicial group G_{n+1} with a $\mathbb{Z}_2$ action α_{n+1}, positive homomorphisms $h : H \to G_{n+1}$ and $g_{n+1} : G_{n+1} \to G$ such that $g = g_{n+1}h$, and $ker(g) = ker(h)$. By the rule, $g_{n+1}h(0,1) = g(0,1) = x_{n+1}$ with $h(0,1) \in G_{n+1}^+$. So $x_{n+1} \in g_{n+1}(G_{n+1}^+)$. Since $G_n \subseteq H$, we can construct the map from G_n to G_{n+1} which composes with the inclusion map from G_n to H and h.

Let G_∞ be the direct limit and $q_n : G_n \to G_\infty$ be the canonical map. Since the condition $g_{n+1}h_n = g_n$, there exists a positive homomorphism $g_\infty : G_\infty \to G$ such that $g_\infty q_n = g_n$ for all $n \in \mathbb{N}$. Given $x \in ker(g_\infty)$, write $x_n = q_n(y)$ for some $n \in \mathbb{N}$ and some $y \in G_n$. Then

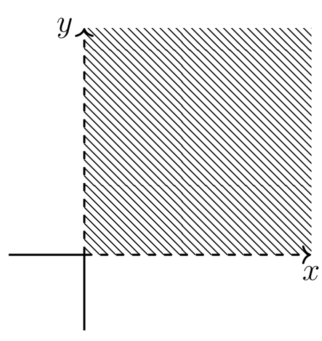
$$g_n(y) = g_\infty q_n(y) = g_\infty(x) = 0,$$

and $h_n(y) = 0$ becasue $ker(g_n) = ker(h_n)$. Thus, g_∞ is injective.

Next, we would like to show that g_∞ is surjective. Since $x_n \in g_n(G_n^+)$, we obtain $x_n \in g_n(G_n^+) = g_\infty q_n(G_n^+) \subseteq g_\infty(G_\infty^+)$ for all n, and hence $g_\infty(G_\infty^+) = G^+$. It follows that g_∞ is surjective. Therefore g_∞ is a group isomorphism and it is equivariant. $\square$

Here is an example related to our main theorem.

EXAMPLE 7.3. Let $G = \mathbb{Q}\times\mathbb{Q}$ with strict order, i.e., $G^+ = \{(0,0)\}\cup \{(x,y) \mid x > 0 \text{ and } y > 0\}$.



Let $\alpha(x,y) = (y,x)$. Suppose we have $(x,y) \in G$ and $n(x,y) \geq 0$ where $n \in \mathbb{N}\backslash 0$. Then,

$$0 \leq n(x,y) = (nx, ny) \Rightarrow \begin{cases} nx = ny = 0 \text{ or} \\ nx > 0,\ ny > 0 \end{cases}$$

So, $(x, y) \in G^+$. Therefore, (G, G^+) is unperforated. Also, $G^+ \cap G^- = \{(0,0)\}$ and $G^+ + G^- = G$.

Suppose $x_1, x_2, y_1, y_2 \in G$ such that $y_j \leq x_i$ for $i, j = 1, 2$. Then, we get a figure below.

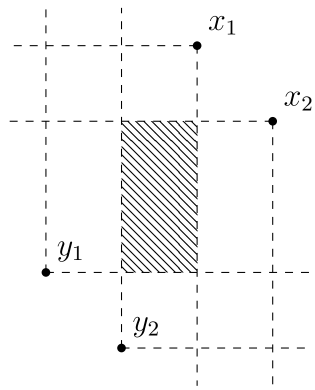
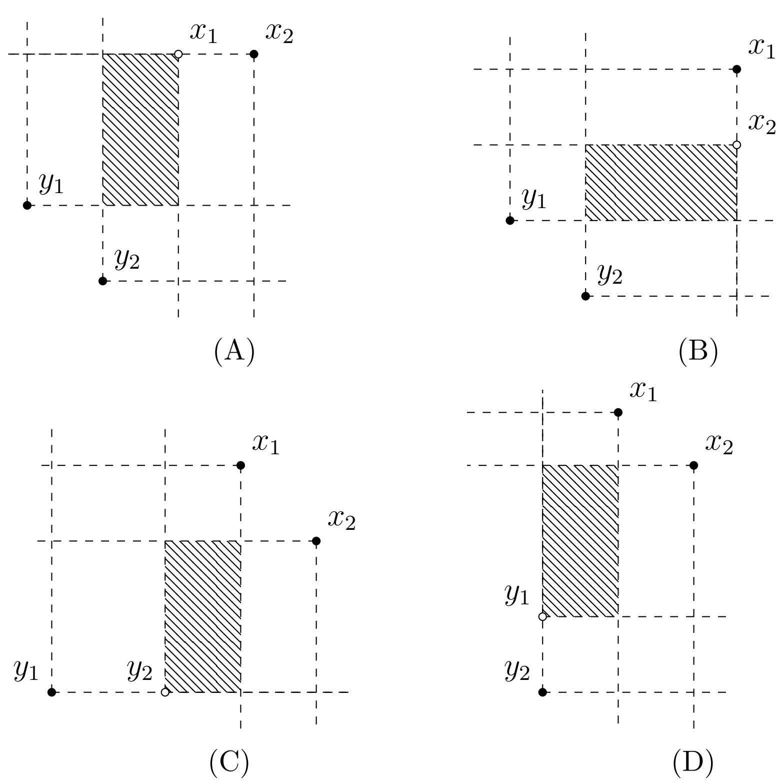


FIGURE 1

The points in the dashed rectangle are the interpolation points.

We consider speical cases, (A) x_i's are in the same horizontal line, (B) x_i's are in the same vertical line, (C) y_j's are in the same horizontal line, and (D) y_j's are in the same vertical line.



We also consider $x_i = y_j$. In this case, $x_i = y_j = z$. So, (G, G^+) is an interpolation group. Therfore, (G, G^+) is dimension group. Since the above dashed rectangle in figure 1 is open, there is no biggest element in the rectangle. So, there is no greatest lower bound for x_1 and x_2 in figure 1. Therefore, (G, G^+) is not a lattice-ordered group.

Now, define $A_k = \begin{pmatrix} 2^{m_k} & 1 \\ 1 & 2^{m_k} \end{pmatrix}$ such that $A_k : \mathbb{Z}^2 \to \mathbb{Z}^2$ commutes with the actions α that flip the coordinates. Then, we get an inductive system, $\{(\mathbb{Z}^2, \alpha), A_k\}$. We need to check the inductive limit of this inductive system is a dimension group but is not a lattice-ordered dimension group. We will construct maps φ_k that will give us a diagram:

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[loop above, "\alpha"] \arrow[r, "A_1"] \arrow[d, "\varphi_1"] & \mathbb{Z}^2 \arrow[loop above, "\alpha"] \arrow[r, "A_2"] \arrow[d, "\varphi_2"] & \mathbb{Z}^2 \arrow[loop above, "\alpha"] \arrow[r] \arrow[d, "\varphi_3"] & \cdots \arrow[r] & \lim\{\mathbb{Z}^2, \psi_n\} \arrow[loop above, "\alpha"] \arrow[d, "\varphi_\infty"] \\
G_1 \subseteq & G_2 \subseteq & G_3 \subseteq & \cdots \subseteq & G_\infty = \bigcup_{k=1}^\infty G_k \subseteq G
\end{tikzcd}$$

Now, we define the x_k and y_k in G as follows. Since A_k is invertible, we get $B_k = A_k^{-1} = \frac{1}{2^{2m_k}-1} \begin{pmatrix} 2^{m_k} & -1 \\ -1 & 2^{m_k} \end{pmatrix}$. Define a sequence $(x_k, y_k) \in G \times G$ by $x_1 = (2,1)$, $y_1 = \alpha(x_1) = (2,1)$ and $\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = B_k \begin{pmatrix} x_k \\ y_k \end{pmatrix}$ where $x_k,\ y_k \in G$. Define G_k to be the subgroup of G generated by x_k and y_k. Since $\begin{pmatrix} x_k \\ y_k \end{pmatrix} = A_k \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix}$, $G_k \subseteq G_{k+1}$. Consider the homomorphism φ_k from $\mathbb{Z}^2$ with the flip automorphism α to G_k given by $e_1 \mapsto x_k$ and $e_2 \mapsto y_k$. When $k = 1$, if $ax_1 + by_1 = 0$, then $a = 0$ and $b = 0$. So, φ_1 is injective. Suppose φ_k is injective. Then $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} = 0$ implies $(a, b) = (0, 0)$. Now, we need to check φ_{k+1} is injective. Suppose $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} B_k \begin{pmatrix} x_k \\ y_k \end{pmatrix} = 0$. Since B_k is invertible, $a = 0$ and $b = 0$. Therefore, the map φ_{k+1} is injective.

Next, we would like to check the image in each G_k^+ is in the first quadrant. By the definition, x_1 and y_1 are in G^+. Suppose x_k and y_k are in G^+. We choose m_k so that x_k and y_k lie between two lines made by the column of A_k in the first quadrant. Then we apply B_k to the coordinates. Then, x_{k+1} and y_{k+1} are still in the first quadrant. At every stage, the angles between the lines made by x_k and y_k are getting wider. However, these lines converge to some lines between the positive x-axis and positive y-axis.

Finally, we need to check the union of the images is dense. The lengths of x_k and y_k is tending to zero and they are linearly independent. This implies that the union of the images is dense. The positive cone is an open wedge in the first quadrant, so the same argument as for G above shows that it is not lattice-ordered.

According to the above example, the hypothesis of the main theorem that the group is lattice-ordered is not necessary. Therefore, we suggest that there would be more research needed to generalize the theorem.

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