THE IMPACT OF EQUAL SHARING AND MULTIPLE GROUPS PROBLEMS ON STUDENTS' UNDERSTANDING OF FRACTIONS

By

Roxanne R. Wright

A thesis

submitted in partial fulfillment of the requirements

for the degree of

Master of Education

FACULTY OF EDUCATION

LAKEHEAD UNIVERSITY

THUNDER BAY, ONTARIO

Abstract

The purpose of this qualitative case study was to examine the impact of division and multiplication word problems, namely Equal Sharing and Multiple Groups problems, on students' understanding of fractions. Thirteen middle school students, from two different middle schools, participated in this case study. Following a reform oriented approach to instruction, students were introduced to a series of Equal Sharing and Multiple Groups problems to solve and discuss. In addition, each student was given a preassessment, midassessment and postassessment. The data from these assessments along with the recordings of students' discussions were carefully analysed to determine the impact of Equal Sharing and Multiple Groups problems on students' understanding of fractions. Students' understanding of fractions was assessed through their ability to solve word problems related to the five different constructs of fractions: part-whole, measure, ratio, quotient and operator constructs. The results indicated that, for the most part, students' understanding of fractions did improve through the use of Equal Sharing and Multiple Groups word problems. The study also highlighted the need for further research in the area of naming fractions, in particular, the when and how to introduce the naming of fractions in order for students to have a deeper understanding of the multiplicative relationship between the numerator and denominator.

Keywords: Fractions, Equal Sharing problems, Multiple Groups problems

Acknowledgements

The completion of this thesis would not have been possible without the encouragement and support of many. I am forever grateful to all who offered support, advice and encouragement along the way.

I am sincerely grateful to the teachers and students who willingly participated in this research as well as Dr. Alex Lawson, my supervisor, who offered unwavering support and direction through the entire process. Thank you.

In addition, I would like to thank my family and friends who offered words of encouragement during crucial moments.

Finally, and most importantly, I would like to thank my heavenly Father, who without His strength, provision and wisdom none of this would be possible.

Table of Contents

Chapter 1: Introduction 1

Context 1

Reform-Oriented Instruction...2

Purpose of the Study 3

Research Question...3

Significance of the Study 4

Contribution to the Community 4

Chapter 2: Literature Review 6

Challenges to the Learning and Teaching of Fractions...7

Difficulties: The Multiple Meanings of Fractions 7

Difficulties: The Overemphasis of the Part-Whole Construct.....................................9

Difficulties: The Misapplication of Whole Number Concepts to Fractions 10

More Effective Teaching to Foster the Quotient Construct..11

Equal Sharing Problem 12

Effective Instruction Using the Quotient Construct: The Use of Children's Informal Knowledge 14

Effective Instruction Using the Quotient Construct: Application and Development of the Underlying Fraction Concepts 15

Effective Instruction Using the Quotient Construct: A Focus on the Multiplicative Relationship and Relational Thinking..18

Focusing on the Quotient Construct: Similarities Between Students' Progressive Strategies in Equal Sharing Problems and the Recursive Model of Mathematical Understanding 22

Non-Anticipatory Strategy 22
Emergent Anticipatory Strategy 23
Anticipatory Strategy 24
The Recursive Model of Mathematical Understanding 24
From the Quotient Construct to the Measure Construct Through Multiple Groups Problems .. 29
Effective Extension Using Multiple Groups Problems: Application and Development of the Underlying Fraction Concept of Partitioning 30
Effective Extension Using Multiple Groups Problems: Application and Development of the Underlying Fraction Concept of Equivalence 30
Effective Extension Using Multiple Groups Problems: Application and Development of the Underlying Fraction Concept of Unit Forming 32
The Impact of Equal Sharing and Multiple Groups Problems on Low Achieving Students' Understanding of Fractions 33
Summary 34
Chapter 3: Methodology 35
Research Questions 35
Propositions 35
Research Design 36
Research Sampling 37
Procedure 38
Data Collection 42
Data Analysis 43
Validity and Reliability 46
Construct Validity 47
Internal Validity 47

External Validity 47

Reliability 47

Limitations and Bias 48

Chapter 4: Findings 49

Classroom Instruction 49

Analysis of the Preassessment and the Postassessment 50

Division. 51

Fraction Constructs 58

Underlying Fraction Concepts 70

Chapter 5: Analysis of Students' Progression of Relational Thinking 76

Paul: No Coordination to Early Multiplicative Reasoning 76

Susan: A Later Trajectory Toward Multiplicative Reasoning 87

Angela: The Roadblock of Naming Fractions 98

Frank: The Roadblock of Procedural Knowledge 101

Chapter 6: Discussion and Conclusion 105

Summary of the Major Findings 105

Does the Development of Participants' Understanding Follow a General Progression With This Intervention? 106

Does Instruction Through Equal Sharing and Multiple Groups Problems Facilitate an Understanding in the Other Fraction Constructs? 107

Does Instruction Through Equal Sharing and Multiple Groups Problems Facilitate an Understanding of the Underlying Fraction Concepts? 110

How Does Students' Experiences With Whole Number Division Impact their Progression in Solving Equal Sharing Problems? 110

Conclusion 111

Future Consideration 112

References 114

Appendices 122

List of Tables

Table 1 A Comparison of Kieren's Model and Empson and Levi's Progressive Strategies 27

Table 2 Procedure Timeline 38

Table 3 Sequence of Equal Sharing Lessons and Assessments 40

Table 4 A Priori Codes 44

Table 5 Summary of Data Sources, Analyses and Propositions 45

Table 6 Case Study Tactics Used to Ensure Validity and Reliability (Yin, 2014) 46

Table 7 Sources of Data 49

Table 8 Summary of Preassessment and Postassessment for Question 1 51

Table 9 Summary of Preassessment and Postassessment for Question 2a 54

Table 10 Summary of Preassessment and Postassessment for Question 2b 54

Table 11 Summary of Preassessment and Postassessment for Question 3 58

Table 12 Summary of Preassessment and Postassessment for Question 4 58

Table 13 Students who Recognized the Shaded Region in Question 4 as Greater Than 2 59

Table 14 Summary of Preassessment and Postassessment for Question 5a 61

Table 15 Summary of Preassessment and Postassessment for Question 5b 62

Table 16 Summary of Preassessment and Postassessment for Question 6 66

Table 17 Summary of Preassessment and Postassessment for Question 7 67

Table 18 Summary of Preassessment and Postassessment for Question 8 69

Table 19 Summary of Preassessment – Underlying Fraction Concepts 70

Table 20 Summary of Postassessment – Underlying Fraction Concepts...................................... 70

List of Figures

Figure 1. Section from Kieren's framework for rational number knowing (1993) 15

Figure 2. A Grade 4 student's solution that demonstrates unit forming....................................... 18

Figure 3. A student's solution suggesting a non-relational understanding of fractions 21

Figure 4. An example of a non-anticipatory strategy... 22

Figure 5. An example of Additive Coordination — Sharing One Item at a Time 23

Figure 6. An example of Additive Coordination — Sharing Groups of Items 24

Figure 7. Model for the Recursive Theory of Mathematical Understanding............................... 25

Figure 8. An example of a fractional number curriculum using the recursive model of understanding 26

Figure 9. Example of using the recursive model of understanding to develop fractions as measures with quotient knowledge as primitive doing.. 28

Figure 10. Section from Kieren's framework for rational number knowing 30

Figure 11. Student's solution showing a relational understanding of fractions 32

Figure 12. Percentage of correct answers on preassessment and postassessment……………… 51

Figure 13. Sierra's attempt to make sense of the remainder 52

Figure 14. Sierra's postassessment partitive division solution. 53

Figure 15. Marilyn's preassessment partitive division solution.. 53

Figure 16. Marilyn's postassessment partitive division solution 54

Figure 17. Jessica's preassessment strategy for Question 2a. 56

Figure 18. Jessica's postassessment strategy for Question 2a 56

Figure 19. Anthony's ratio strategy solution to Question 2b 57

Figure 20. Horace's ratio strategy solution 57

Figure 21. Horace's incorrect strategy of counting four shaded and five unshaded to arrive at a fractional representation of 4/5 60
Figure 22. Angela's incorrect solution of counting the shaded fractional shaded and the five unshaded fractional pieces to arrive at the fractional representation of 2 2/5 60
Figure 23. Tasha's attempt to find a common denominator. 61
Figure 24. Jessica comparing dollar amounts without consideration of the amount of tickets being received 63
Figure 25. Marilyn demonstrating the emergence of multiplicative thinking when comparing... 63
Figure 26. Anthony demonstrating an emerging multiplicative thinking to compare ratios 65
Figure 27. Horace using his experience with Equal Sharing problems to solve Question 5b 65
Figure 28. Anthony uses his experience with Equal Sharing problems to solve Question 5b...... 65
Figure 29. Anthony's solution strategy for Question 6 68
Figure 30. Sierra demonstrating a common misconception that the divisor is always smaller than the dividend. 68
Figure 31. Maria uses repeated halving with no coordination at the end to facilitate her efforts to equally share the pizza 68
Figure 32. Susan uses repeated halving with no coordination at the end to facilitate her efforts to equally share the pizza 69
Figure 33. Susan's No Coordination between Sharers and Shares strategy 72
Figure 34. Susan's Additive Coordination strategy 72
Figure 35. Anthony demonstrating multiplicative thinking 73
Figure 36. Paul demonstrating additive thinking. 73
Figure 40. Paul demonstrating Additive Coordination Strategy 78

Figure 41. Paul's written solution for 6 brownies shared with 8 students 81

Figure 42. Paul's progress after five lessons with Equal Sharing problems 81

Figure 43. Paul's Direct Modeling strategy 83

Figure 44. Paul's Direct Modelling solution. 85

Figure 45. Paul's solution to a Multiple Groups: multiplication problem. 88

Figure 46. Susan and her partner's solution to 51 loaf cakes shared with 4 students 90

Figure 47. Susan transitional strategy 90

Figure 48. Susan's solution to five packages of clay shared with seven students. 91

Figure 49. Susan's Ratio Strategy to solve 12 clay sticks shared with 16 students 93

Figure 50. Susan's solution to first Multiple Groups problem 94

Figure 51. Demonstration of Susan's relational thinking 96

Figure 52. More examples of Susan's relational thinking 97

Figure 53. Susan's progression after four lessons with Equal Sharing problems 98

Figure 54. Angela's solution to a Multiple Groups problem 100

Figure 55. Frank's solution to six brownies shared with eight friends 102

Figure 56. Frank's attempt to solve an Equal Sharing problem with unfamiliar fractions. 103

Figure 57. Students' progression when solving Equal Sharing and Multiple Groups Problems. 107

List of Appendices

Appendix A: Progressive Strategies Used to Solve Equal Sharing Problems 123

Appendix B: The Fundamental Properties of Operations and Equality 124

Appendix C: Progressive Strategies Used to Solve Multiple Group Problems 125

Appendix D: Principal's Introductory Cover Letter 126

Appendix E: Principal's Consent Form 128

Appendix F: Parent(s)/Guardian(s)' Introductory Cover Letter 129

Appendix G: Parent(s)/Guardian(s) Consent Form 131

Appendix H: Potential Participant Introductory Cover Letter 133

Appendix I: Potential Participant Consent Form 134

Appendix J: Pre and Postassessment 135

Appendix K: Student's Continuum of Numeracy Development: Multiplication and Division .. 137

Appendix L: Midassessment — Equal Sharing Problems 138

Appendix M: Sequence of Equal Sharing and Multiple Groups Problems 144

Appendix N: Summary of Results from the Preassessment 146

Appendix O: Summary of Results from the Postassessment 147

Chapter 1: Introduction

Context

Learning with understanding is essential for the 21st century learner. It allows students to use what they have learned to solve the new kinds of problems they will inevitably face in the future (National Council of Teachers of Mathematics (NCTM), 2000). Recent theory and research from cognitive psychology suggest that knowledge is stored in the learner's head as a network of concepts. As such, learning occurs through the making of connections between new information and the learner's existing network of knowledge (Peterson et al., 1988). It is important to note that, this existing knowledge, also referred to as intuitive knowledge (Leinhardt, 1988) or as informal knowledge (Mack, 1990) does not have to be conceptually correct in order for connections to be made (Leinhardt, 1986). In addition, researchers consider learning with understanding as non-linear (Mack, 2001). Mack (2001) explains that, students require a frequent return to their initial understandings in order to stimulate connections between more complex ideas as well as to facilitate the restructuring of their knowledge. This approach to learning is succinctly summarized by researchers Franke and Kazemi (2001). They explain that learning with understanding goes beyond connecting new knowledge to existing knowledge; it also includes, they believe, a reorganization of knowledge to create rich integrated knowledge structures. This constructivist theory of learning that knowledge is built by the student rather than passively received whole from the teacher became the impetus for radical changes in mathematics education (Clements & Battista, 1990). These changes formed the basis for reform-oriented instruction and were outlined in the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional*

Standards for Teaching Mathematics (NCTM, 1991) and *Principles and Standards for School Mathematics* (NCTM, 2000).

Reform-Oriented Instruction

As researcher Sherin (2002) explains, there is no one definition or model for this approach to teaching. Instead, it is characterized by key phrases such as “teaching for understanding”, “building a community of inquiry” or “mathematics for all”. Nonetheless, there are two key components that are common in all reform-oriented instruction: problem solving and classroom discourse (A. Stylianides & G. Stylianides, 2007).

Problem Solving. Hiebert and Carpenter (1992) emphasize the importance of problem solving in developing mathematical understanding. They state that:

> Because the goal of mathematics education should be the development of understanding by all students, the majority of the curriculum should be composed of tasks that provide students with problem *situations.* Two reasons support this claim. The first is that mathematics that is worth learning is most closely represented in problem solving tasks. The second is that students are more apt to engage in the mental activities required to develop understanding when they are confronted with mathematics embedded in problem situations (p.87).

Classroom Discourse. The Principles and Standards for School Mathematics (NCTM, 2000) contends that learning with understanding can be enhanced by classroom interactions as students propose mathematical ideas and conjectures, learn to evaluate their own thinking and that of their peers and develop mathematical reasoning skills. Furthermore, it notes that, classroom discourse can be used to promote the recognition and connection among ideas as well as the reorganization of students’ knowledge. In other words, it promotes learning with understanding.

In summary, the underlining tenets of reform mathematics education is a underpinned by a constructivist theory of learning. This theory contends that through the connection of students' informal and formal knowledge, facilitated through problem solving tasks as well as classroom discourse, knowledge is constructed.

Purpose of the Study

The purpose of this qualitative case study is to investigate the impact of a reform-oriented intervention through division and multiplication word problems, specifically *Equal Sharing* and *Multiple Groups* problems on students' understanding of fraction concepts. A preassessment will be administered to assess students' understanding of fractions. After five weeks of reform-oriented instruction, a postassessment will be administered. The impact of these Equal Sharing and Multiple Groups problems on students' understanding will be assessed through a comparison of the results of both the pre and postassessments.

Research Question

How does the understanding of fractions for 14 middle school students, who struggle with fraction concepts, develop and progress after five weeks of reform-oriented instruction using Equal Sharing and Multiple Groups problems?

- Does the development of their understanding follow a general progression?
- Does instruction through Equal Sharing and Multiple Groups problems facilitate an understanding in the other fraction constructs (interpretations) (i.e., part-whole, measure, ratio, and operator constructs)?
- Does instruction through Equal Sharing and Multiple Groups problems facilitate an understanding of the underlying fractions concepts: partitioning, equivalence, and unit forming?

- How do students' experiences with whole number division impact their progression in solving Equal Sharing problems?

Significance of the Study

The results of this study will contribute to the growing evidence of research on the different increasingly sophisticated strategies students use when solving Equal Sharing and Multiple Groups problems. They will also provide insight on its impact on students' understanding of the other fraction constructs. In addition, the results may add to the evidence that supports learning through connections of students' informal and formal knowledge. Mack (2001) explains that although studies have suggested the effectiveness of building on students' informal knowledge, the extent of its effectiveness to all mathematical content domains is uncertain. Although, the effectiveness of reform-oriented instruction is not a primary focus of this study, the results may add to the evidence in support of its impact on academically struggling students. As noted in the Baxter et al. (2001) article, there are conflicting perspectives on the effectiveness of reform-oriented instruction for academically struggling students.

Contribution to the Community

My observation of local teachers' approach to the teaching and learning of fractions is that, for the most part, it is traditional, focusing on transmitting procedures for the various operations and equivalent forms. This study provided an opportunity to teach for understanding through the use of word problems, an approach that aligns more with current research as it builds on students' informal knowledge of sharing. Furthermore, the data collected from this study, specifically students' work and recorded discourse, will be shared at professional co-learning sessions. The sharing of students' work and discourse provide tangible evidence of the impact of

Equal Sharing and Multiple Groups problems on students' constructed knowledge and understanding of fractions.

Chapter 2: Literature Review

"Fractions have always been a considerable challenge for students – and adults" (Van de Walle, et al., 2016, p. 267). In fact, researchers Behr et al. (1993) assert that the learning of fractions is perhaps one of the most serious obstacles to the mathematical maturation of students. Such challenges interact with students' ability to solve problems, apply computational procedures, and engage in algebraic reasoning (Hunt & Empson, 2015). The American National Mathematics Advisory Panel report provides further affirmation of the importance of fractions in students' mathematical development. In their final report, the panelists assert that a fluency in fractions, including the ability to represent, compare and compute fractions efficiently, is one of the essential prerequisites for learning algebra (U.S. Department of Education, 2008). Although researchers Empson and Levi (2011) agree that the study of fractions is foundational to the study of algebra, their reasoning differs from those presented in the Mathematics Panel report. Empson, Levi and Carpenter (2011) argue that the relationship between fractions and algebra is rooted in students' relational thinking, that is, students' use of the fundamental properties of operations and equality (Empson & Levi, 2011). This is in contrast to the Mathematics Panel report that ascribes students' poor performance in algebra to their weak operational proficiency in fractions (Empson et al., 2011). Regardless of the differing points of view, many researchers believe that a shaky foundation in fractions can prevent students from advancing in mathematics and hence limit their career opportunities later in life (Bruce et al., 2013). For this reason, it is of pivotal importance, as an educator, to examine the factors contributing to the challenges of learning fractions and to implement effective teaching strategies for a more comprehensive understanding.

Challenges to the Learning and Teaching of Fractions

Some of the reasons cited for the challenges to learning and teaching fractions include, the multiple meanings and/or interpretations of fractions (Van de Walle et al., 2016), the overemphasis on the part-whole representation (Charles & Nason, 2000), and the misapplication of whole number thinking and its operations when introduced to fractions (Lamon, 2007).

Difficulties: The Multiple Meanings of Fractions

Kieren (1980) proposed five different yet interconnected constructs (or meanings) of fractions: part-whole relationships, measures, ratios, operators, and quotients. He argued that, in order to understand rational numbers, students require adequate experience with the different constructs (Kieren, 1976). The five constructs are defined as follows:

The Part-whole Construct. The part-whole construct of fractions, according to Marshall (1993), is defined as a situation in which a continuous quantity (usually a geometric shape) or a set of discrete objects, that are identical, are partitioned into parts of equal size. In this construct, a fraction represents a comparison between the number of parts of the partitioned unit (whole) to the total number of parts in which the unit (whole) is partitioned (Charalambous & Pitta-Pantazi, 2007).

The Measure Construct. According to Charalambous and Pitta-Pantazi (2007), the measure construct has two closely related notions. First, it is considered a unique number which conveys the quantitative personality of fractions with one location on the numberline. Secondly, it is associated with the measure assigned to some interval. More specifically, a unit fraction (i.e., $\frac{1}{b}$) is used repeatedly to determine a distance (Marshall, 1993). Given the fraction $\frac{a}{b}$, in the measure construct, the fraction takes the meaning of a instances (iterations) of the unit $\frac{1}{b}$. There is no limit on the size of a (Marshall, 1993). For example, in the fraction $\frac{5}{8}$, you can use the unit fraction $\frac{1}{8}$ as the selected unit length and then count, iterate, or measure to show that it takes five $\frac{1}{8}$ ths to reach $\frac{5}{8}$ (Van de Walle et al., 2016). As noted by Charalambos and Pitta- Pantazi (2007), this construct has systematically been associated with using number lines or other measuring devices, such as rulers, to determine the distance from one point to another in terms of $\frac{1}{b}$ units.

The Ratio Construct. Lamon (2011) defines a ratio as the comparison of any two quantities. Ratios compare measures of different types such as the ratio of cars to square kilometers. It can also compare measures of the same type. There are two types of ratios that compare measures of the same type: part-whole comparisons and part-part comparisons (Lamon, 2011). For example, in a carton of eggs containing 5 brown and 7 white eggs, all of the following ratios apply: 5 to 7 or $\frac{5\ brown\ eggs}{7\ white\ eggs}$ (part-part comparison) or 7 to 5 or $\frac{\text{7 white eggs}}{\text{5 brown eggs}}$ (part-part comparison) and 5 to 12 or $\frac{\text{5 brown eggs}}{\text{12 eggs}}$ (part-whole comparison) or 7 to 12 or $\frac{\text{7 brown eggs}}{\text{12 eggs}}$ (part-whole comparison) (Lamon, 2011, p.225). Lamon notes that, if the fraction notation is used, careful attention should be taken to include the quantities, as illustrated above, and not merely the numbers. She argues that, when ratios are written in fraction form, devoid of context and without careful note to label the quantity, the conceptual and operational differences between part to part ratio and the part whole ratio as a fraction can become fuzzy or unclear. For example, given the following problem:

> Yesterday Mary had 3 hits in 5 turns at bat. Today she had 2 hits in 6 times at bat. How many hits did she have for a two day total?
>
> Mary had 3:5 + 2:6 = 5:11 or 5 hits in 11 times at bat. If we were adding fractions, we would never write $\frac{3}{5} + \frac{2}{6} = \frac{5}{11}$ (Lamon, 2011, p.226).

The Operator Construct. The operator construct of fractions is seen as a function applied to some number, object or set (Behr et al., 1993; Marshall, 1993). These operators can be seen as a single composite function resulting from the combination of two multiplicative operations or as two discrete, but related functions applied consecutively (Charalambos & Pitta-Pantazi, 2007). That is, given the operator $\frac{2}{3}$, it can be viewed as a single operation on a quantity ($\frac{2}{3}$ of a dozen eggs) or as a multiplication performed on a division of a quantity (2 ($\frac{12\ eggs}{3}$)) or as a division performed on a multiplication of a quantity ($\frac{2(12\ eggs)}{3}$) (Lamon, 2011).

The Quotient Construct. The quotient construct, as the name suggests, represents a division. That is, given the fraction $\frac{a}{b}$, *a* is partitioned or divided into *b* parts (Marshall, 1993). Marshall importantly points out that although both the quotient and part-whole constructs rely on partitioning they are different in meaning. The numerator *(a)* and the denominator *(b)* in the quotient construct represent different items (e.g., three pizzas shared with four friends). In contrast, the numerator (a) and the denominator (b) in the part-whole construct are identical (e.g. three fourths of a pizza).

It is interesting to note that, despite the importance of each of these constructs in the development of a robust understanding (Boyce & Norton, 2016), the part-whole representation is often the only interpretation synonymous with fractions (Lamon, 2007).

Difficulties: The Overemphasis of the Part-Whole Construct

In Kerslake's (1986) summary report on students' difficulties with fractions, she highlighted their exclusive reliance on the part-whole construct. She observed students' inability to adjust their mental constructs to accommodate the notion of a fraction as a number. Similar reliance on the part-whole construct was noted by researcher Susan Lamon (2001). She observed that students' preparation for this complex domain of rational numbers consisted primarily of a brief introduction to the part-whole construct followed by years of procedural practice in fraction computations. She argued that without a multiple interpretation of fractions, students would not be able to develop a robust understanding (Lamon, 2001). So, as Charles and Nason (2000) argued, although the part-whole construct is central to a mature functioning with fractions, it is not sufficient.

Difficulties: The Misapplication of Whole Number Concepts to Fractions

"Children experience cognitive obstacles as they encounter fractions because they try to make connections with the whole numbers and operations with which they are familiar" (Lamon, 2011, p. 25). Researcher Susan Lamon explains that some of these ideas that students develop while working with whole numbers interfere with their ability to understand fractions and operations on and/or with them. Examples include:

1) Reasoning that because $7 > 3$, $\frac{1}{7} > \frac{1}{3}$. (Lamon, 2011)
2) Adding and subtracting across the numerator and the denominator to add or subtract fractions. Example: $\frac{1}{3} + \frac{2}{5} = \frac{3}{8}$ (Lamon, 2011)
3) Thinking the dividend (the number being divided) must always be larger than the divisor (Lamon, 2011). For example, given a word problem where 15 students share three cookies, students may reason that it is impossible or incorrectly write the division statement as $15 \div 3$.
4) An assumption that there are no numbers between fractions. For example, with whole numbers there is an exact counting sequence. If counting by ones, the next number in the sequence 1,2,3,4,5 is the number "6". However, when counting fractions, there are multiple equivalent fractions for every fraction, so counting in fraction sequence

> yields multiple answers (Parrish & Dominick, 2016). For example, if asked to count fractions in sequence from one half to one, the answers may vary because there are multiple equivalent fractions between the two fractional numbers. This is not the same for whole numbers.

The misapplication of whole number thinking to the learning of fractions results in a huge conceptual leap for students when initially introduced to fractions (Lamon, 2011).

In response to these challenges, many researchers have sought to find effective strategies and/or approaches to the teaching and learning of fractions that move beyond a procedural approach focused primarily on the part-whole construct. One such approach is a focus on the quotient construct.

More Effective Teaching to Foster the Quotient Construct

As discussed, a robust understanding of fractions depends on an understanding of all the different constructs (Behr et al., 1983; Freudenthal, 1973). In fact, Kieren (1976) argues that because each construct relates to particular cognitive structures and instructional strategies, a focus on one construct may lead to a lack of understanding of other cognitive structures. For example, if the focus of instruction was the measure construct with the use of the number line model, the multiplicative ideas of fractions would not be easily generated. The number line model would cognitively conflict with the other strategies and/or models afforded by the other constructs generating multiplicative ideas (Kieren, 1976). The question, therefore, becomes which construct would provide a foundation for the integration of the other constructs?

Some researchers (Freudenthal, 1973; Kieren, 1976,1993; Piaget et al., 1960; Streefland, 1993) consider the quotient construct as central to the development of fractional knowledge. This is partly due to the underlying cognitive structure of partitioning that the quotient construct affords (Kieren, 1976). Partitioning is the ability to divide an object or objects into a given number of equal parts (Kieren, 1976). In other words, the quotient construct of fractions is the result of a division that is five divided by seven is $\frac{3}{8}$. In fact, Empson and Levi (2011) reiterate the importance of partitioning to the development of an understanding of fractions. They insist that students — even children in kindergarten, first and second grades — need and can benefit from experience with whole number *partitive* and *quotative* division problem types as well as multiplication story problems prior to instructions in fractions. In a partitive division word problem, the number of groups is known but not the amount in each group. For example: Eric has 15 pieces of gum. He shares them equally into five packets. How many pieces of gum are in each packet? In this example, the number of groups (five packets) are known but not the amount in each group (three pieces of gum) (Empson & Levi, 2011). On the other hand, in quotative division word problems, the number in each group is known but not the number of groups. For example: Eric has 15 pieces of gum. He wants to put three pieces of gum in each packet, how many packets can he fill? In this example, the number of groups (five packets) is unknown but the number in each group (three pieces of gum) is known (Empson & Levi, 2011). These experiences, Empson and Levi explain, prepare students for solving Equal Sharing and Multiple Groups fractions problems.

Equal Sharing Problem

An Equal Sharing problem refers to equally sharing some number of same sized objects among some number of people or groups, where the result is a fractional quantity (Empson & Levi, 2011). In other words, it is a partitive division problem that results in a fractional quantity. This type of problem requires that each person receives the same sized share and that all of the material being shared is exhausted (Empson & Levi, 2011). An example of an Equal Sharing problem is: "Four children want to share 10 brownies so that everyone gets exactly the same amount. How much brownie can each child have?" (Empson & Levi, 2011, p. 6). Streefland (1993), in his seminal work, describes these activities as the building blocks to students' acquisition of an understanding of fractions. He explains that, in using fair sharing activities, "the concept of fraction and the informal *operating* with fractions are directly related to each other" (Streefland, 1993, p. 291). That is, fair sharing activities provide opportunities for the development of the concept of fraction along with rules anticipating operations (Streefland, 1993). This understanding is made available through students' varied solutions to fair sharing activities. He further explains that, in attaching a measure, weight or price to what is being shared, the operator construct can also be introduced (Streefland, 1993). For example: A person is given $\frac{3}{4}$ of a chocolate bar. An entire bar cost $1.20. How much does the portion given cost? (Streefland, 1993). Furthermore, as Streefland explains, by determining the measure and price of all sorts of sharing combinations, an indirect method of determining the sum, difference, product or quotient of fractions can be achieved. For example, given a pizza share of $\frac{1}{2} + \frac{2}{5}$ and a cost of $10.00, it can be determined that $\frac{1}{2}$ x $10 = $5 and $\frac{2}{5}$ x $10 = $4 for a total of $9 was consumed. In this regard, the student has determined the price of $\frac{9}{10}$ of a pizza.

So, although other constructs, such as measures and part-whole relationships, depend on partitioning, it is evident that instruction based on the quotient construct provides a basis for integration with the other constructs and operations of fractions. Moreover, the quotient construct, developed through Equal Sharing, allows teachers to draw on children's informal knowledge of partitioning (Empson, 1999; Kieren, 1993). It also provides opportunities to apply the underlying fraction concepts of partitioning, equivalence, and unit forming (Kieren, 1993) as well as facilitate the development of the multiplicative relationship of fractions (Empson et al.,

2006) and students' relational thinking (Empson et al., 2011). So, how does the quotient construct facilitate students' understanding of fractions?

Effective Instruction Using the Quotient Construct: The Use of Children's Informal Knowledge

Learning mathematics with understanding entails making connections between informal understandings and formal mathematical ideas (Ball, 1993). As it pertains to rational number concepts and procedures, there is growing evidence that students have a rich store of informal knowledge on which to draw (Mack,1990). In fact, Siemon (2003) argues that partitioning is the missing link between students' informal and formal knowledge of rational numbers. She cautions, however, that even though students may have some informal knowledge of fractions, particularly halving, it should not be assumed that the inherent relationships in fraction representations are understood.

Despite the documentation of a relationship disconnect between students' informal and formal knowledge of rational numbers concepts, Mack (1993) suggests that with appropriate instruction, students can make the necessary connections. One such suggestion is the use of real-life problems. Real-life problems, according to Mack (1993), should not only make sense to students but also be presented within a context that makes clear the critical features of the problem. This, she explains, allows students to think in terms of quantities represented in the problem rather than requiring them to reason with symbolic representations. That is, it allows students to draw upon their informal knowledge and, with instruction, make connection to more formal symbolic representations. Posing Equal Sharing problems provides the context and the sense making required to build on students' informal knowledge, particularly fair sharing and partitioning.

Effective Instruction Using the Quotient Construct: Application and Development of the Underlying Fraction Concepts

The quotient construct, through the posing of Equal Sharing problems, facilitates the application and, hence a deeper understanding of the underlying concepts of fractions. These basic thinking tools for understanding rational numbers are identified as: partitioning, order and equivalence, and unit forming (Behr et al., 1983; Kieren, 1993). It is interesting to note that, in Kieren's framework for rational number knowing, the quotient and measure constructs are the only two constructs that rely on these three underlying concepts (see Figure 1).

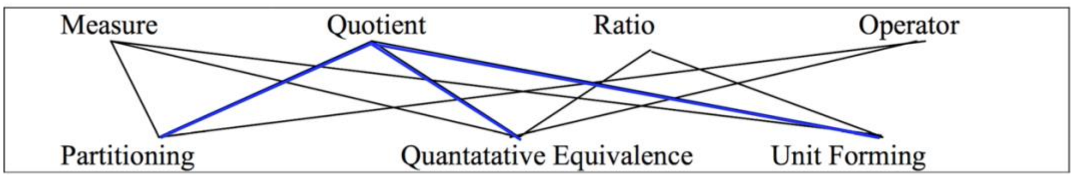


Figure 1. Section from Kieren's framework for rational number knowing (1993). Retrieved from *The Four-Three-Four Model: Drawing on Partitioning, Equivalence, and Unit Forming in a Quotient Sub-Construct Fraction Task*, by A. Mitchell and Mathematics Education Research Group of Australasia, 2012.

Underlying Fraction Concept: Partitioning. Halving, the most basic form of partitioning, is an intuitive process for most students (Simeon, 2003). Partitions other than by halving or successive halving, such as thirds or fifths, go beyond students' initial intuitive knowledge. Empson (1999) explains that partitions other than halving and/or successive halving require thinking about how a given number of partitions can fit into the unit. Equal Sharing problems facilitate the progressive development of students' partitioning strategies from halving and successive halving, to coordinating the number of sharers with the number of partitions, to a multiplicative coordination (Empson, 1999; Empson & Levi, 2011). Appendix A provides an outline, suggested by Empson and Levi, of the progressive strategies students use to solve Equal Sharing problems. It also includes a graphical summary developed by Lawson et al. illustrating the progression of students' use of increasingly sophisticated strategies when solving Equal Sharing problems.

Underlying Fraction Concept: Order and Equivalence. Equal Sharing problems also facilitate the development of students' concept of order and equivalence. Empson (1999) describes the results of a case study where Grade 1 students, through the use of Equal Sharing problems, compared fractional quantities. Students were asked to solve a problem where seven candy bars were shared by three children followed by a problem where nine candy bars were shared by four children. They were then asked to decide which amount was more: two and one third candy bars or two and one fourth candy bars. Initially, students in the case study, compared one third and one fourth by focusing on the whole number quantities. That is, they focused on how the bars were partitioned (the number of pieces) rather than the sizes of the pieces. This was evident through one student's reasoning. She reasoned that one fourth was the bigger amount because it had one more than one third. However, through students' discussions — a critical component in the development of their thinking of fractions (Empson & Levi, 2011) — they were able to refine their understanding of relative unit fraction size. They coordinated the size of the pieces with the number of sharers and therefore developed an understanding based on the size of the pieces (Empson, 1999). Post et al. (1985) highlight two phenomena that adversely affect students' understanding of ordering fractions. These were evident in Empson's (1999) case study. They include students' prior knowledge of whole numbers (e.g. one fourth is more than one third because four is more than three) and linguistics considerations. Post et al. (1985) explain that,

> [t]he words *more* and *greater* (and their counterparts *less* and *fewer*) cause difficulty for children because *more* can mean *more parts* in the partitioned whole or *more area* covered by *each* part. Similarly, *greater* can mean a greater number of parts in the

partitioned whole or a greater fraction size. A similar confusion exists with respect to *size* and *amount,* as illustrated by children who, when asked which of two fractions is less, reply, “Do you mean in size [e.g. size of each subdivision] or in amount [e.g. number of subdivisions]?” (p. 34)

In the same Empson (1999) case study, Equal Sharing problems were used to observe students’ use of their informal knowledge to develop the underlying concept of equivalence. Equal Sharing problems where the number of sharers and the number of items to be shared had a common factor were used. The following problem was discussed by students in the case study: “Six children have ordered blueberry pancakes at a restaurant. The waiter brings eight pancakes to their table. If the children share the pancakes evenly, how much can each child have?” Following the solving of this problem, the students engaged in discussions comparing two sixths and one third. It was through these discussions, along with their prior knowledge of fractions, that they were able to conclude the equivalence of two sixths and one third. Empson (1999) contends that students’ concept of equivalence deepens as they solve Equal Sharing problems that afford opportunities for equivalence.

Underlying Fraction Concept: Unit Forming. According to Kieren (1993), unit forming describes the additive nature of fractions. Just as eight could be made of seven and one, or six and two, or five and three, fractions can also be made from the sum of other non-equal fractional amounts. Empson and Levi (2011) describes solutions to Equal Sharing problems that facilitate unit forming as additive coordination. Given the following word problem: “Luz has 9 candy bars to share with her friends. Altogether, there are 12 children, including Luz. Everyone wants the same amount. How much candy bar can each child have?” (Empson & Levi, 2011, p. 18). A Grade 4 student used his knowledge of number facts to solve this problem. First, he drew 12 children and 9 candy bars. He then reasoned that since 6 x 2 = 12, he would split the first 6 candy bars into halves to achieve his goal of 12 pieces. He redrew these half pieces under each person (See Figure 2). He then split the next two into sixths, because 2 x 6 = 12. He redrew these sixths as a small piece under each person. The last candy bar was shared into twelfths. This provided a final answer of $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}$ (Empson & Levi, 2011). This solution is a combination of non-equal fractional amounts. As Empson and Levi explain, such strategies are rich with possible connections to addition of fractions with like and unlike denominators as well as equivalent fractions.

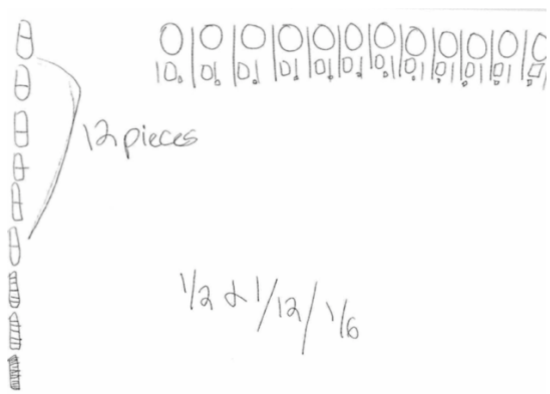


Figure 2. A Grade 4 student's solution that demonstrates unit forming. Adapted from *Extending Children's Mathematics: Fractions and Decimals (p.19*) by S. B. Empson and L. Levi, 2011, Portsmouth, NH: Heinemann. Copyright 2001 by the National Council of Teachers of Mathematics

Effective Instruction Using the Quotient Construct: A Focus on the Multiplicative Relationship and Relational Thinking

Researchers have emphasized the importance of the multiplicative relationship for the understanding of fractions (Lamon, 2007; Thompson & Saldanha, 2003; Vergnaud, 1988). Empson et al. (2006) argue that the "understanding of fractions as multiplicative structures involves the coordination of fractions with multiplication and division in a way that emphasizes mathematical relationships" (p. 2). Equal Sharing problems, according to Empson and Levi (2011), provide the initial rich opportunities for this understanding of fractions — a number whose value is determined by the multiplicative relationship between the numerator and the denominator or the result of its inverse, division. It is important to note that this multiplicative relationship is in contrast to the conceptual mapping approach suggested by researchers Charles and Nason (2000). For example, given the partitive quotient fraction $\frac{3}{4}$, three pizzas shared among four people, Charles and Nason would argue that for understanding, students would need to construct a conceptual mapping between the number of people (four) to the name of each share (fourths) as well as a conceptual mapping between the number of pizzas being shared (three) and the number of fourths in each share (three fourths). That is, a fraction as a quotient construct, is a conceptual mapping between the dividend and the numerator as well as between the divisor and the denominator. Fraction as a quotient, as Lamon (2011) succinctly explains, needs to go beyond the symbol $\frac{a}{b}$ meaning a÷b to a rational number resulting from a division, a quantity or a ratio. Students who know *why* a÷b = $\frac{a}{b}$ have a relational understanding of fractions (Empson & Levi, 2011). In fact, Empson et al. (2011) assert that before students can learn to operate on or with fractions, they need to understand fractions as quantities. Due to the multiplicative relationship between the numerator and denominator, this mature understanding of fractions is also relational in nature.

Relational thinking, according to Empson and Levi (2011), refers to students' use of the fundamental properties of operations and equality to solve problems. Appendix B provides a list of these properties. Empson and Levi state that students often demonstrate an intuitive understanding of these properties when solving problems. The solving and discussing of Equal Sharing problems facilitate the development of relational thinking (Empson & Levi, 2011). Empson and Levi describe the strategies of first graders who were able to add fractions with unlike denominators after five weeks of instruction focused on solving and discussing Equal

Sharing problems. The solutions demonstrated their intuitive understanding of the fundamental properties and thus relational thinking. The students were asked to calculate $\frac{1}{2}+\frac{3}{4}$. A few of the students used their knowledge of $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$, reasoned that $\frac{1}{2}+\frac{1}{2}=1$, and then added $\frac{1}{4}$ to arrive at a solution of $1\frac{1}{4}$ (Empson and Levi, 2011). Their thinking could be represented as follows: $\frac{1}{2}+\frac{3}{4}=\frac{1}{2}+(\frac{1}{2}+\frac{1}{4})=(\frac{1}{2}+\frac{1}{2})+\frac{1}{4}=1+\frac{1}{4}=1\frac{1}{4}$ (Empson et al., 2011). The students used their intuitive understanding of the fundamental properties to decompose $\frac{3}{4}$ and then regroup to add $\frac{1}{2}+\frac{1}{2}$. In other words, they implicitly used the associative property of addition (Empson & Levi, 2011). It can also be argued that students' experiences with Equal Sharing problems, specifically the application of the underlying concept of unit forming, facilitated their ability to decompose $\frac{3}{4}$.

This is in contrast to another third grader with a non-relational understanding of fractions. Given the following word problem: "Jeremy is making cupcakes. He wants to put $\frac{1}{2}$ cup of frosting on each cupcake. If he makes 4 cupcakes for his birthday party, how much frosting will he use to frost all of the cupcakes?" (Empson et al., 2011, p. 415). The third grader drew the picture shown in Figure 3 and decided the answer was "four halves". At that point, she was not able to see the relation between the "four halves". "For her, fractions existed separately from other numerical measures" (Empson et al., 2011, p. 415). In other words, halves were quantities to be counted (i.e., four halves) but not relational (i.e., $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=2$ or 4 x $\frac{1}{2}=2$).

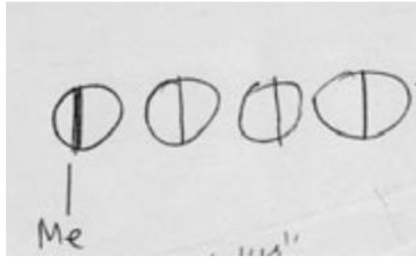


Figure 3. A student's solution suggesting a non-relational understanding of fractions. Adapted from *Extending Children's Mathematics: Fractions and Decimals (p. 5)* by S. B. Empson and L. Levi, 2011, Portsmouth, NH: Heinemann. Copyright 2001 by the National Council of Teachers of Mathematics

From these examples, it is clear that students require a relational understanding of fraction along with the fundamental properties of operations and equality in order to operate on or with fractions (Empson & Levi, 2011).

The relational understanding of fractions consists of a *relational understanding of unit fractions* as well as a *relational understanding of fractions as composite* (Empson & Levi, 2011). *A relational understanding of fractions as a unit* refers to the process of partitioning a whole unit in *n* equal parts resulting in $\frac{1}{n}$ size of a part. Therefore $1 \div n = \frac{1}{n}$. If all the shares are combined, the whole is reconstructed, that is $n \times \frac{1}{n} = 1$. *A relational understanding of fractions as composite* refers to an understanding that $\frac{m}{n}$ is *m* groups of $\frac{1}{n}$. It is important to note that in developing this relational understanding, students may first relate this composite relationship as additive (i.e., $\frac{1}{n} + \frac{1}{n} + \frac{1}{n} \ldots + \frac{1}{n}$ - m times $= \frac{m}{n}$) then as multiplicative (i.e., *m* groups of $\frac{1}{n} = m \times \frac{1}{n} = \frac{m}{n}$) (Empson & Levi, 2011).

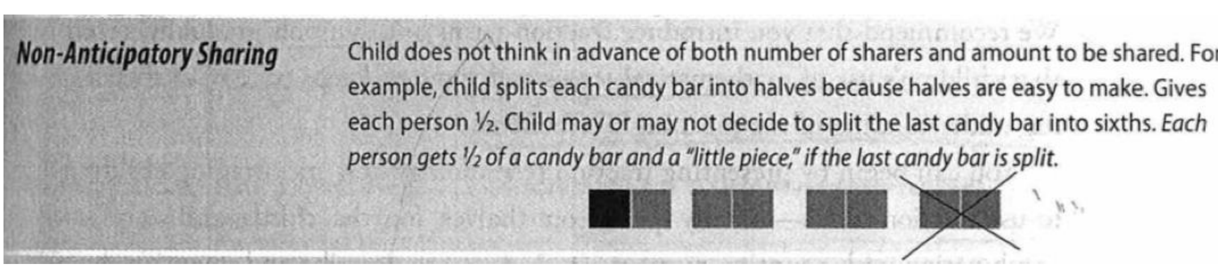
"As children come to understand fractions as relational, they start to use this understanding to decompose and recompose quantities for the purpose of transforming expressions and simplifying computations" (Empson et al., 2011, p. 416). These strategies are often implicit. Empson and Levi (2011) caution that for students to realize the full potential of relational thinking these implicitly used properties will need to be intentionally made explicit by the teacher. In fact, they contend that, without this intentionality by teachers, students' ability to think relationally discontinues and in many cases atrophies. When this occurs, students abandon making sense of mathematics.

Focusing on the Quotient Construct: Similarities Between Students' Increasingly Sophisticated Strategies in Equal Sharing Problems and the Recursive Model of Mathematical Understanding

As previously discussed, students' strategies to Equal Sharing problems follow a predictable developmental pattern (see Appendix A). Similarities between these evolving strategies and the recursive model of mathematical understanding (Kieren, 1993) are evident and interesting to note. The strategies used to solve Equal Sharing problems can be classified into three main progressively more sophisticated phases: *non-anticipatory, emergent anticipatory*, and *anticipatory* strategies (Empson & Levi, 2011).

Non-Anticipatory Strategy

With this strategy, both the number of sharers and the amount to be shared are not taken into consideration in children's partitioning strategies. For example, in an Equal Sharing problem where six children share four candy bars, a child may partition in halves, giving each person half a candy bar. The last bar may or may not be split into sixths (Empson & Levi, 2011, p. 25).


Non-Anticipatory Sharing

Child does not think in advance of both number of sharers and amount to be shared. For example, child splits each candy bar into halves because halves are easy to make. Gives each person ½. Child may or may not decide to split the last candy bar into sixths. *Each person gets ½ of a candy bar and a "little piece," if the last candy bar is split.*

Figure 4. An example of a non-anticipatory strategy. Adapted from *Extending Children's Mathematics: Fractions and Decimals (p. 25)* by S. B. Empson and L. Levi, 2011,

Portsmouth, NH: Heinemann. Copyright 2001 by the National Council of Teachers of Mathematics

Emergent Anticipatory Strategy.

This strategy, as the name implies, is the emergence of an anticipatory thinking strategy—a relational understanding of fraction. Students anticipate the relationship between the number of sharers and the amount to be shared (Hunt & Empson, 2015). Empson and Levi (2011) describe two strategies within this emerging stage: Additive Coordination—Sharing One Item at a Time and Additive Coordination—Sharing Groups of Items.

Additive Coordination—Sharing One Item at a Time. In this strategy, each of the four candy bars to be shared among six children are split into sixths, one at a time. Each person gets one sixth piece. Once the process is completed for all the candy bars, the one sixth piece each person received is added to arrive at $\frac{4}{6}$ (Empson & Levi, 2011). (See Figure 5).

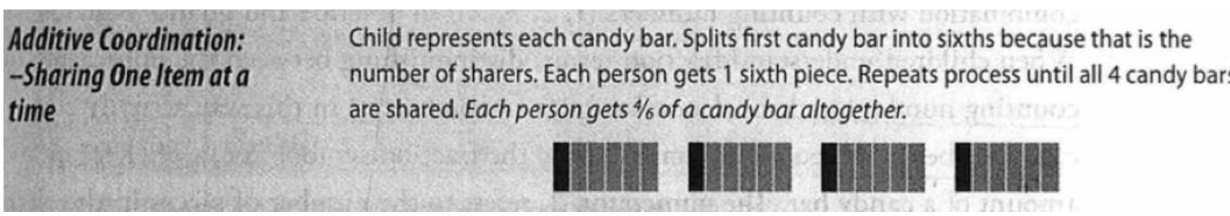
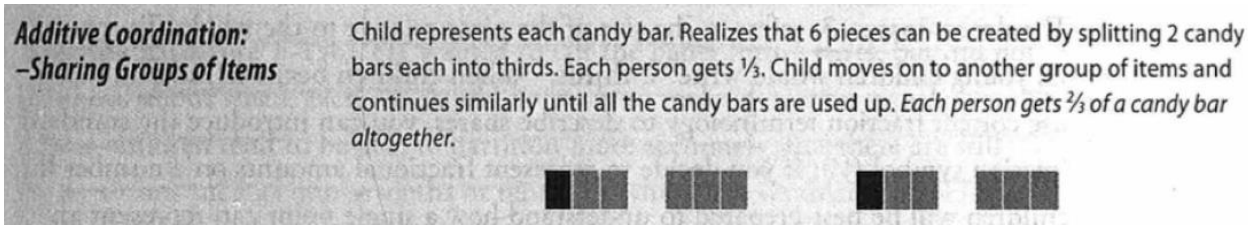


Figure 5. An example of Additive Coordination — Sharing One Item at a Time. Adapted from *Extending Children's Mathematics: Fractions and Decimals (p. 25)* by S. B. Empson and L. Levi, 2011, Portsmouth, NH: Heinemann. Copyright 2001 by the National Council of Teachers of Mathematics

Additive Coordination—Sharing Groups of Items. In this strategy, two of the candy bars are split into thirds, reasoning that in splitting two candy bars into thirds, the anticipated six equal pieces would be achieved. Each person receives one third. The other remaining two are partitioned in a similar manner, resulting in each person receiving two thirds altogether (Empson & Levi, 2011). (See Figure 6).


Additive Coordination: –Sharing Groups of Items

Child represents each candy bar. Realizes that 6 pieces can be created by splitting 2 candy bars each into thirds. Each person gets ⅓. Child moves on to another group of items and continues similarly until all the candy bars are used up. *Each person gets ⅔ of a candy bar altogether.*

Figure 6. An example of Additive Coordination — Sharing Groups of Items. Adapted from *Extending Children's Mathematics: Fractions and Decimals (p. 25)* by S. B. Empson and L. Levi, 2011, Portsmouth, NH: Heinemann. Copyright 2001 by the National Council of Teachers of Mathematics

Anticipatory Strategy

This strategy reflects students' relational understanding of fractions (Hunt & Empson, 2015). The coordination between the number of sharers and the amount to be shared becomes a mental recall. Students understand that *a* things shared by *b* people is $\frac{a}{b}$. (Empson & Levi, 2011). Empson and Levi refers to this strategy as Multiplicative Coordination.

These strategies clearly demonstrate a progressive development of relational thinking from intuitive partitioning of halving to a multiplicative understanding of fractions. How does it relate to Kieren's (1993) recursive model of mathematical understanding?

The Recursive Model of Mathematical Understanding

Kieren, in collaboration with the work of Pirie (1988), created the model of mathematical understanding shown in Figure 7. This model demonstrated mathematical understanding as dynamic, non-linear and recursive. That is, "it involves the use of the same sequence of processes, but at a new level, with new elements of action" (Kieren, 1993, p. 72).

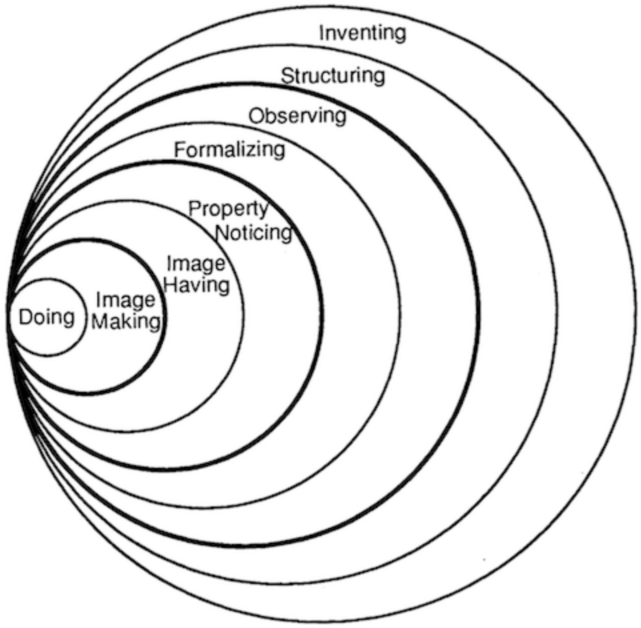


Figure 7. Model for the Recursive Theory of Mathematical Understanding. Adapted from *Rational and fractional numbers: From quotient fields to recursive understanding*. In T.P. Carpenter, E. Fennema, & T.A. Romberg, Rational numbers: An integration of research (p. 72), by T. E. Kieren, 1993, Hillsdale, NJ: Lawrence Erlbaum Associates. Copyright 1993 by Lawrence Erlbaum Associates.

Kieren suggested a curriculum for fractional numbers — at least in part— using this recursive model of understanding. In this suggested curriculum, partitioning would be the primitive doing upon which the outer levels would be based. He notes that, "in the actual implementation of such a curriculum, a teacher would provide activities or tasks that would allow children to make distinctions among their inner-level ideas and thus form ideas at a new, transcendent level" (p. 77). Equal Sharing problems are activities or tasks that may support this recursive model of understanding. In fact, Kieren (1993) mentions the use of children's activities from the work of Streefland (1984, 1987) to exemplify this curriculum. Streefland's work includes similar activities to Equal Sharing problems. Figure 8 shows an example of a fractional number curriculum reflecting the recursive model of understanding.

PD	IM	IH	PN	F
Partitioning; Unit Identification; Fraction language; Half fractions.	Sharing problems: record results using fraction language	Observe that fractions describe partitioned quantities; relate fractions showing the same amount	Properties of fractions as quantities: equivalences; addition; situation-oriented language to record properties.	Observe that fractions are a whole set of things that act like numbers. Reify properties using formal language.

Figure 8. An example of a fractional number curriculum using the recursive model of understanding. Adapted from *Rational and fractional numbers: From quotient fields to recursive understanding.* In T.P. Carpenter, E. Fennema, & T.A. Romberg, Rational numbers: An integration of research (p. 77), by T. E. Kieren, 1993, Hillsdale, NJ: Lawrence Erlbaum Associates. Copyright 1993 by Lawrence Erlbaum Associates.

Table 1 presents a summary of the similarities between Kieren's (1993) recursive model to Empson and Levi's (2011) progressive strategies.

Table 1

A Comparison of Kieren's Model and Empson and Levi's Progressive Strategies

Kieren's Model	Empson and Levi's Strategies
Primitive Doing (DM)	Non-anticipatory sharing
• partitioning	• no coordination
• half fractions	• repeat halving
Image Making (IM)	Additive coordination
• make a record of sharing activities	• anticipation
• coordination	• coordination
Image Having (IH)	
• identifying fractional names	
Property Noticing (PN)	
• equivalencies	
• addition	Multiplicative coordination
Formalizing (F)	• a shared with b people is $\frac{a}{b}$
• $\frac{a}{b}$ a fractional number where b $\neq$ 0	

Also of interest is Kieren's mention of the possibility of using the recursive model of understanding to extend to other fraction constructs. In fact, researcher Mack (1990) reiterates this possibility. She proposes that one can develop "a strand of rational number based on partitioning, and then … expand that conception to other strands once students can relate mathematical symbols and procedures to their informal knowledge and can reflect on the relations" (p. 30). Through the lens of a recursive model of understanding, two approaches are suggested by Kieren (1993) for an extension to the measure construct. These approaches, he cautions, are based on initial curricular activities involving partitioning. The first approach involves the provision of activities that prompt students to fold back to image making activities and extend their quotient image to a measure image through activities relating quotients and a measure such as length. The second approach, on the other hand, involves envisioning the formal knowledge of quotient as the primitive doing for the measure constructs. Figure 9 shows an example of using the recursive model of understanding to develop the measure construct with the quotient construct as primitive doing.

PD	**IM**	**IH**	**PN**	**F**
Figure 8 knowing and understanding	Given a dividable unit measure K; and record order, multiplicative and additive statements	Fractional numbers describe measures	Use quotient ideas and the "image" to build properties (order, density) in measure-oriented, fractional language	Recognize that measure numbers are the same formally as quotient numbers and are situation- or image-free.

Figure 9. Example of using the recursive model of understanding to develop fractions as measures with quotient knowledge as primitive doing. Adapted *from Rational and fractional numbers: From quotient fields to recursive understanding.* In T.P. Carpenter, E. Fennema, & T.A. Romberg, Rational numbers: An integration of research (p. 79), by T. E. Kieren, 1993, Hillsdale, NJ: Lawrence Erlbaum Associates. Copyright 1993 by Lawrence Erlbaum Associates

Empson and Levi (2011), it may appear, propose a similar opportunity for extension of the quotient construct to the measure construct through the use of Multiple Groups problems. Similar to Mack's (1990) proposition for possible opportunities for extension, they suggest the introduction of Multiple Groups problems once students are able to create the fraction quantities in their solutions to Equal Sharing problems. Although it is unclear as to which approach Empson and Levi's (2011) Multiple Groups problems aligns with, it demonstrates the recursive model of mathematical understanding. That is, a folding back to the quotient construct, developed through Equal Sharing problems to reconstruct the measure construct, extended through Multiple Groups problems. How do Multiple Groups problems facilitate the development of the measure construct?

From the Quotient Construct to the Measure Construct Through Multiple Groups Problems

Multiple Groups problems, as explained by Empson and Levi (2011), are division and multiplication word problems that involve a whole number of groups, with fractional amounts inside of each group. An example of a Multiple Groups division word problem is as follows: It takes $\frac{3}{4}$ meter of ribbon to make a bow. How many bows (groups) could you make with 9 meters of ribbon? (Empson & Levi, 2011). This is an example of a type of a quotative (measurement) division as the number of groups (i.e., bows) is unknown. An example of a Multiple Groups multiplication word problem is as follows: "I need to make peanut butter and jelly sandwiches for 12 children. I want to make $\frac{3}{4}$ of a sandwich for each child. How many sandwiches do I need to make?" (Empson & Levi, 2011, p. 51). Similar to Equal Sharing problems, Multiple Groups problems rely on partitioning, however, unlike Equal Sharing problems, the partitions (i.e., amount per group) are provided, iterated or counted. In the example above $\frac{3}{4}$ iterated 12 times. In this respect, not only do Multiple Groups problems reinforce students' understanding of fractions, they provide opportunities for connections to the measure construct of fraction through the iteration of a unit of measure.

As previously noted, only the quotient and measure constructs rely on all three underlying fraction concepts (see Figure 10, a replication of Figure 1, p. 20). As such, it can be argued that the development of students' robust understanding of fractions, through an extension of the quotient construct to the measure construct, seems logical. If this is the case, how do Multiple Groups problems reinforce and further develop the underlying fraction concepts of partitioning, equivalence and unit forming?

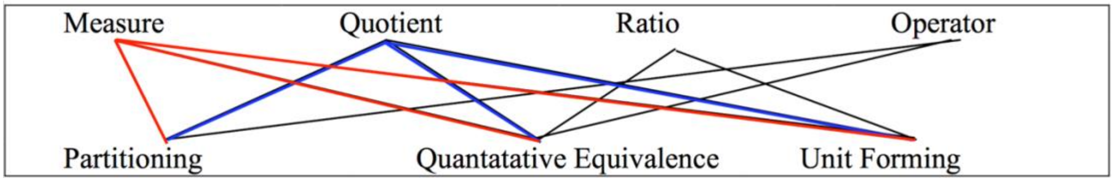


Figure 10. Section from Kieren's framework for rational number knowing. Retrieved from *The Four-Three-Four Model: Drawing on Partitioning, Equivalence, and Unit Forming in a Quotient Sub-Construct Fraction Task*, by A. Mitchell and Mathematics Education Research Group of Australasia, 2012.

Effective Extension Using Multiple Groups Problems: Application and Development of the Underlying Fraction Concept of Partitioning

Similar to Equal Sharing problems, Multiple Groups problems rely on students' intuitive knowledge of partitioning (Kieren, 1993). This was made evident in a study conducted by Kieren (1993) in which students were asked to find the number of quarters in $\frac{3}{2}$. In solving this Multiple Groups problem, one group "tiled" the $\frac{3}{2}$ area with $\frac{1}{4}$'s to arrive at an answer of 6. As Kieren explained, the students' ability to create a diagram to represent the problem, "tile", and then count the fractional amounts iterated demonstrated their use of their intuitive thinking tools, specifically of drawing and counting partitions. This initial intuitive strategy of direct modeling and repeated addition, Empson and Levi (2011) suggest, progress to grouping and combining strategies to eventually multiplicative strategies, as students solve and discuss their solutions to various Multiple Groups problems. Appendix C outlines this increasingly sophisticated development of strategies to solve Multiple Groups problems.

Effective Extension Using Multiple Groups Problems: Application and Development of the Underlying Fraction Concept of Equivalence

Experiences with Multiple Groups problems, according to Empson and Levi (2011), facilitate a relational understanding of fractions as well as a connection between students' intuitive thinking and mathematical notation. Students' ability to reason equivalence, they further explain, is based on this relational understanding. For example, the reasoning that if you cut a third of a pancake in half it will make 2 sixths of a pancake, Empson and Levi argue, is far more powerful a justification for equivalence than the demonstration that 2 sixths fits perfectly when laid on top of 1 third. Multiple Groups problems provide opportunities for students to move beyond, what Empson (1999) describes as, a reliance on an empirically based understanding of equivalence (i.e., how pieces look) towards a more relational understanding, connected to mathematical notation. For example, given the following Multiple Groups problem: "Mr. W has 10 cups of frog food. His frogs eat half a cup of frog food a day. How long can he feed his frogs before his food runs out?" (Empson et al., 2011, p. 415), a third grader, with relational understanding, represented the solution as shown in Figure 11. He used a relational understanding of the quantity $2 \times \frac{1}{2} = 1$ to arrive at a solution of 20 days. The student was able to decompose and recompose quantities for the purpose of simplifying his solutions. These strategies of decomposing and recomposing involve the fundamental properties of operation and equality.

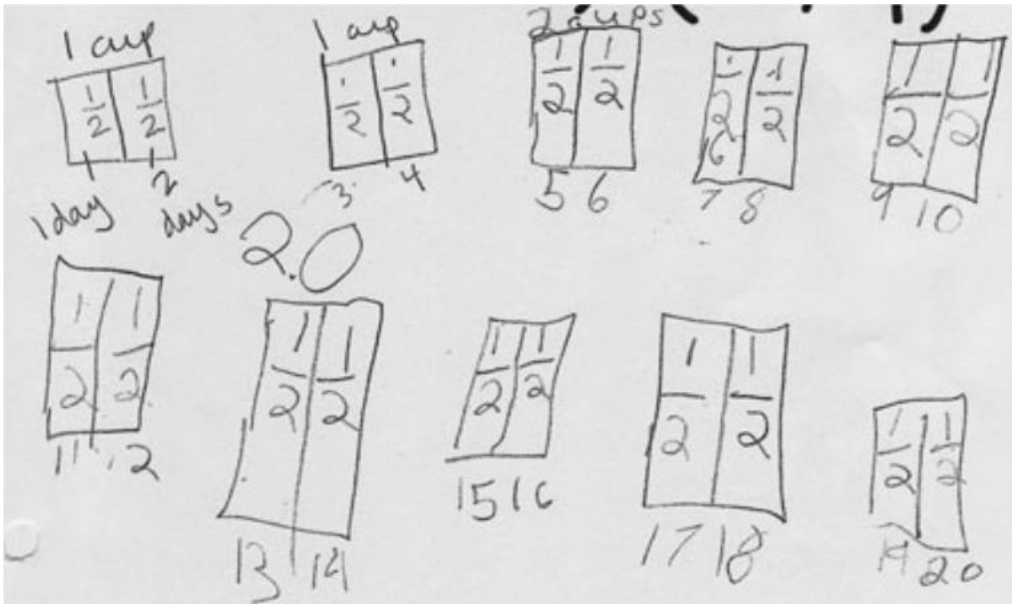


Figure 11. Student’s solution showing a relational understanding of fractions**.** Adapted from *Extending Children’s Mathematics: Fractions and Decimals (p. 416)* by S. B. Empson and L. Levi, 2011, Portsmouth, NH: Heinemann. Copyright 2001 by the National Council of Teachers of Mathematics

As students solve and discuss Multiple Groups problems similar to these, they develop a relational understanding and are able to not only reason that when you split one third into three pieces you will get one ninth but also make the connection relationally that three groups of one third is one ninth (i.e., $\frac{1}{3} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$ or $3 \times \frac{1}{9} = \frac{1}{3}$).

Effective Extension Using Multiple Groups Problems: Application and Development of the Underlying Fraction Concept of Unit Forming

As previously discussed, unit forming refers to the fact that fractions can be made from the sum of other non-equal fractional parts. Multiple Groups problems facilitate this underlying concept through the iteration of a unit of measure. For example, given the word problem “Nina has $10\frac{1}{2}$ yards of fabric to make pillows. If each pillow takes $\frac{3}{8}$ of a yard of material, how many pillows can Nina make before she runs out of fabric?”, a student used direct modeling and counting to solve the problem (Empson & Levi, 2011, p. 54). She drew rectangles to represent

$10\frac{1}{2}$ yards of fabric and then partitioned each rectangle into eighths. She then sectioned off $\frac{3}{8}$. The final half yard she partitioned into 4 equal segments, demonstrating her knowledge of equivalence, that is, $\frac{1}{2} = \frac{4}{8}$. Counting each section that she made by grouping eighths, she arrived at a solution of 28 pillows. In her counting, she implicitly understood that $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$ and $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1$ (Empson & Levi, 2011).

Given the aforementioned arguments for the effectiveness of understanding fractions through the quotient construct, could the use of Equal Sharing and Multiple Groups problems address the needs of students struggling with fraction concepts?

The Impact of Equal Sharing and Multiple Groups Problems on Low Achieving Students' Understanding of Fractions

Hunt et al. (2017) proposed that in order to increase the mathematics competence of low achieving students, instruction should begin by uncovering the competencies that already exist and can be built upon as they solve problems within their range. One such manner of building on students' informal knowledge, as previously discussed, is real-life, contextual word problems (Mack, 1993), such as Equal Sharing and Multiple Groups problems. In fact, in a case study by Empson (2003) on two low performing students, Empson reiterated the effectiveness of building on students' prior knowledge. Empson described how the "teacher's mathematics instruction revolved around posing story problems for children to solve using their own strategies. Discussion of problems was directed at understanding children's thinking, comparing strategies, and resolving disagreements" (p. 306) rather than direct instruction. Therefore, according to Empson, tasks that allow for the use of students' prior knowledge to generate new ideas along with students' participation in discussions, facilitate the learning for all students, including low achieving students. Equal Sharing as well as Multiple Groups problems provides opportunities, in context, to generate new ideas from students' prior knowledge as well as rich discussions.

Summary

Equal Sharing and Multiple Groups problems can provide a rich context upon which to incorporate and build upon students' informal knowledge. The context of these word problems relies on students' intuitive knowledge of fair sharing and partitioning. As students solve and discuss their solutions, their understanding of fractions progressively develops towards a relational understanding. In addition, the quotient construct may afford the opportunity to connect to the other fraction constructs (i.e., measure, part-whole, ratio and operator), a necessary connection for a robust understanding of fractions.

Chapter 3: Methodology

Research Questions

The aim of this research was to draw conclusions to the following research question: How does the understanding of fractions for 14 middle school students, who struggle with fraction concepts, develop and progress after 5 weeks of reform-oriented instruction using Equal Sharing and Multiple Groups problems? The following sub-questions were also addressed:

- Does the development of their understanding follow a specific progression?
- Does instruction through Equal Sharing and Multiple Groups problems facilitate an understanding in the other fraction constructs (i.e., part-whole, measure, ratio and operator)?
- Does instruction through Equal Sharing and Multiple Groups problems facilitate the understanding of the underlying fraction concepts of fractions: partitioning, equivalence, and unit forming?
- How do students' experiences with whole number division impact their progression in solving Equal Sharing problems?

Propositions

- Propositions are speculations as to what the findings of the research might conclude. These speculations are based on literature (Rowley, 2002). In fact, Baxter and Jack (2008) suggest that the more a study contains specific propositions, the more it will stay within feasible limits. On examining the relevant literature and based on the research question and sub-questions, I proposed the following:
    a) Equal Sharing and Multiple Groups problems support the development of an understanding of fractions. Solving Equal Sharing and Multiple Groups problems

relies on an intuitive understanding of partitioning—an underlying concept of fraction (Lamon, 2011; Kieren, 1976; Empson & Levi, 2011).

b) Empson and Levi (2011) developed a continuum of strategies students use to solve Equal Sharing and Multiple Groups problems. See Appendix A and C. It can therefore be proposed that the strategies used by students in this study will develop and progress in a similar manner as it relies on students' intuitive knowledge of fair sharing.

c) Instruction through Equal Sharing and Multiple Groups problems facilitates an understanding of the other constructs. According to Kieren's (1993) framework of rational number knowing, both the quotient and measure constructs are the only two constructs that rely on all three underlying concepts of fractions (i.e., partitioning, equivalence and unit forming).

d) Empson and Levi (2011) suggest that students' experiences with whole number multiplication and division problems prepare them to solve Equal Sharing problems. It can therefore be speculated that the more experiences students have with multiplication and division word problems, the more proficient their strategies will be for solving Equal Sharing problems.

Research Design

This research project was a qualitative case study. A qualitative case study facilitates the exploration of a phenomenon within its context using a variety of data sources (Baxter & Jack, 2008). Educational case studies are often guided by the work of two main researchers, Robert Stake and Robert Yin (Baxter & Jack, 2008). For this research, the approach adopted was guided by the work of Robert Yin which offers greater detail. Yin (2014) defines "a case study as an empirical inquiry that investigates a contemporary phenomenon (the "case") in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (p. 16). The phenomenon investigated, in this regard, was the development of students' understanding of fraction through the use of Equal Sharing and Multiple Groups problems within a classroom context. This was achieved through the collection of a variety of data, including students' pre, mid and postassessments, students' work samples, observations as well as video recordings of lessons and conversations over a five week period. The study was conducted in two middle schools.

Research Sampling

Most qualitative research uses nonprobability sampling (Merriam & Tisdell, 2015), with purposeful sampling (Patton, 2015) being the most common (Merriam & Tisdell, 2015). According to researchers Merriam and Tisdell, this type of sampling is best suited for qualitative research as it provides an opportunity to select a sample from which to discover, understand and gain insight. For this reason, the sample selected for this study was a purposefully chosen sample. The sample was from two schools: Immaculate Middle School (pseudonym) and Campion Middle School (pseudonym). These two schools, both located in Ontario, are my assigned schools as a Grade 7/8 Student Success teacher. As a Grade 7/8 Student Success teacher, I provide support to target students who are performing below the provincial standard level in Mathematics. Due to my limited access to an entire class, the sample selected was a convenience sample (Merriam & Tisdell, 2015). Within this convenience sample, with the exception of one student who requested to be part of the study, 7-10 middle school students (Grade 6 and Grade 8) who struggle with fraction concepts were selected. This selection was done in collaboration with their mathematics teacher from each school. For the purpose of this study, 'students who struggle with fraction concepts' was defined as any student receiving a Level 2 or below in the Number Sense and Numeration strand for the 2017/2018 Term 1 reporting period. The term Level 2 refers to students performing below the Ontario provincial standards. Students participating in Special Education and/or English as a Second Language programs were not considered for this research. Binding the case in this regard ensured that the study remained within a reasonable scope (Baxter & Jack, 2008).

Procedure

Ethics approval was obtained from Lakehead University, Peel District School Board as well as from the principals of the two schools in which the case studies occurred (see Table 2). As the unit of analysis required the collection of student data, introductory letters along with consent forms were necessary. These forms were provided to the schools' principals (see Appendices D and E), parent(s) and/or guardian(s) (see Appendices F and G) as well as students (see Appendices H and I). The signed consent forms were collected and stored in their respective schools. Pseudonyms were used to preserve the anonymity as well as the confidentiality of the school board, the principals, and the students. The data from the research will be stored by Lakehead University for a minimum of five years.

Table 2

Procedure Timeline

Action Steps	Timeline
Ethics approval • Lakehead University • Peel District School Board • Principals	June 2018
Introductory letters and consent forms • Principals • Parents and/or guardians • Teachers • Students	March 2019

Action Steps	Timeline
Preassessment	April 2019
Equal sharing problem lessons	April–June 2019
Postassessment	June 2019

The reform-oriented instruction on the quotient construct of fractions was administered by myself over a period of five alternating weeks in each of the two schools for a total of 10 weeks. It is important to note that, in describing the instruction as reform-oriented, it is in reference to the type of instruction Empson and Levi (2011) suggest as being critical to the development of students' thinking of fractions. According to Empson and Levi:

> The basic teaching practices that support children to draw on what they understand to make sense of new content include:
>
> - posing problems to children without first presenting a strategy for solving the problems
> - choosing problems that allow children to craft a solution on their own
> - facilitating group discussions of children's strategies (p. 10).

For the purpose of this study, the aforementioned practices were used. In addition, students had access to a few simple tools to support them in solving the problems. These included "paper and pencil for drawing and notating, cutout pieces of paper for folding or cutting, coloured pencils or markers for allocating shares, and linking cubes for representing discrete quantities such as the people or candy bars in a problem" (Empson & Levi, 2011, p. 117). These lessons took place during the lunch period for a duration of 40 minutes. Students were given the opportunity to have lunch prior to the start or after the end of each session; it depended on their schedule. The lessons were taught on alternate weeks in order to accommodate my role as a Grade 7/8 Student Success teacher at two schools.

Prior to the instruction, a preassessment (see Appendix J) was administered. This assessment served as a diagnostic, assessing students' prior understanding of fractions. The assessment included questions related to the different fraction constructs. According to Van de Walle et al. (2016), students' understanding of fractions is dependent on their exposure to the various constructs. As the term, 'understanding of fractions' is open to various interpretations, the analysis of students' understanding will be based on the percentage increase of correct responses from the preassessment and postassessment as well as their improved strategy efficiency.

Reform-oriented instruction then began.These lessons were sequenced based on the literature provided primarily by Empson and Levi (2011) and include both Equal Sharing and Multiple Groups problems (see Table 3). It also included a midassessment (see Appendix L), assessing students' understanding of the underlying concepts of fractions as well as fraction as a quotient. Modifications to this sequence were made based on students' pre and midassessment results, daily observations, personal reflections and students' work samples.

Table 3

Sequence of Equal Sharing Lessons and Assessments

Sequence	Type of Equal Sharing Problems /Assessment
Preassessment	Assessment on all the fraction constructs (Anghileri, 2001; Empson, 1999; Kieren, 1993; Lamon, 1993, 2011; Marshall, 1993)
Lesson 1	Equal Sharing problems with solutions greater than one (i.e., mixed number). Focus given to problems with 4,8,3,6,10 and

Sequence	Type of Equal Sharing Problems /Assessment
	12 sharers (Empson & Levi, 2011)
Lesson 2	Equal Sharing problems with solutions less than one.
	Focus given to problems with 4,8,3,6,10 and 12 sharers (Empson & Levi, 2011)
	Multiple Groups problems where the number of groups is unknown (i.e., quotative division)
	Focus given to problems where the amount in each group is less than one and expressed with halves, fourths, tenths, eighths or sixths (Empson & Levi, 2011)
Midassessment	Two Equal Sharing problems (Empson & Levi, 2011)
Lesson 3	Multiple Groups problems where the total amount is unknown (i.e., multiplication).
	Focus given to problems where the amount in each group is less than one and expressed with halves, fourths, tenths, eighths or sixths (Empsom & Levi, 2011)
Lesson 4	Multiple Group problems where the number of groups is unknown (i.e., quotitive division)
	Focus given to problems where the amount in each group is a mixed number and expressed with halves, fourths, tenths, eighths or sixths (Empson & Levi, 2011)
Lesson 5	Multiple Groups problems where the total amount is unknown (i.e., multiplication).
	Focus given to problems where the amount in each group is a mixed number and expressed with halves, fourths, tenths, eighths or sixths (Empson & Levi, 2011)
Postassessment	Same as the preassessment

At the end of the five weeks' instruction in each school, students undertook a postassessment (see Appendix J). The postassessment followed the same parameters as the preassessment. Using the same parameters afforded the opportunity to assess the impact of Equal Sharing and Multiple Groups problems on students' understanding of fractions. Unfortunately a retention test was not administered due to changes in my role as a 7/8 Student Success teacher. These changes limited my access to students and the continuation of the research.

Data Collection

One of the great strengths of case studies is its use of multiple sources of evidence (Rowley, 2002). This enhances the credibility of the data (Yin, 2014). The following sources of data were used in this case study: students' pre, mid and postassessments, samples of students' work, field notes and video recordings. In order to organize and manage the data being collected, a computerized database was assembled (Baxter & Jack, 2008). Using a database improved the reliability of the study as it facilitated the tracking and organization of the data for easy retrieval (Baxter & Jack, 2008). It allowed for an audit trail to be created from data collection, through analysis to conclusion(s) (Baskarada, 2014). The organization of this database was facilitated by the use of *ATLAS.ti*, a computer-assisted qualitative data analysis software (Meriam & Tisdell, 2005).

Students' pre, mid and postassessments were entered into the database for subsequent coding. Similarly, samples of students' work were entered into the database. The field notes, consisting of direct observations, jot notes, and personal reflections were converted into formal field notes after each session (Yin, 2014). Each lesson was video recorded. The video was stationed at each of the students' desks recording each group separately. This allowed for an unobtrusive capture of students' discourse and class discussions. Any open-ended, unstructured interview was noted through jot notes. The use of video recording provided the additional evidence of students' understanding through their discussions. These video recordings were also included in the database resulting in 267 separate files. Due to the size of the data received from both schools, it was decided to analyse the pre, mid and postassessment from each school but only the lessons and instructions from Immaculate Middle School. The choice to analyse the findings from Immaculate Middle School was due to the fact that they were Grade 6 students with limited exposure to operation with fractions based on the learning expectations of the Ontario Math Curriculum.

Data Analysis

The data collection and analysis occurred concurrently (Baxter & Jack, 2008). The concurrent collection and analysis of data facilitated a chronological progression of students' understanding of fractions. The data collected daily were inputted into a database and coded with the use of ATLAS.ti. To facilitate a focused and transparent coding process, predetermined codes were created, also known as *a priori* codes. Table 4 provides a summary of the preliminary codes to be used. The pre, mid and postassessments as well as samples of students' work were coded as either correct or incorrect as well as according to the solution strategy and/or model used and the underlying fraction concepts addressed. Video recordings of students' thinking were summarized, analyzed and coded according to the strategies used, whether correct or incorrect and whether or not any of the underlying fraction concepts were addressed. The field notes and personal reflections were reviewed, summarized and recorded in the database. Table 5 provides a summary of the data sources, the types of analysis to be considered as well as links to the propositions.

Table 4

A Priori Codes

Category	A Priori Codes	Definition
Accuracy	ANSco	Answer Correct
	ANSinc.	Answer Incorrect
Division	DIVPh1	Direct modelling and counting
	DIVPh2	Counting efficiently
	DIVPh3	Working with numbers
	DIVPh4	Proficiency
Equal Sharing problems	EQSt1	Non-anticipatory sharing
	EQSt2a	Additive coordination—sharing one at a time.
	EQSt2b	Additive coordination—sharing groups of item
	EQSt3	Ratio
	EQSt4	Multiplicative coordination
Multiple group problems	MGSt1	Represents each group
	MGSt2	Grouping and combining strategies
	MGSt3	Multiplicative strategies
Underlying fraction concept	FCEq	Equivalence
	FCpart	Partitioning
	FCuf	Unit Forming

Table 5

Summary of Data Sources, Analyses and Propositions

Data Source	Type of Analysis	Linked Proposition
Pre and postassessments	Coded correct/incorrect, strategy used, underlying fraction concept, multiplication and division continuum	a, b, c and d
Samples of students' work	Coded correct/incorrect, strategy used, underlying fraction concept	a, b
Midassessment	Coded correct/incorrect, strategy used, underlying fraction concept	a, b
Video recordings	Summarized and coded correct/incorrect, strategy used, underlying fraction concept	a, b
Field notes	Interviews summarized and coded correct/incorrect, strategy used, underlying fraction concept All other field notes summarized and recorded in database	a, b

Note. Linked propositions can be found on p. 45

Once all the data was collected and coded, triangulation of the different sources of evidence was done in order to link data to the propositions (Yin, 2014). Yin suggests the use of any of the following techniques to analyze the evidence: pattern matching, explanation building, time-series analysis, logic models and cross-case analysis. For the purpose of this study, the pattern matching technique was employed. Yin (2014) describes this technique as the comparison of an empirically based pattern—one based on the findings in the case study—with a predicted one made before the collection of data. Pattern matching, also referred to as theory triangulation by some researchers, strengthens the validity of a study should the patterns appear to be similar

(Yin, 2014). Empson and Levi (2011) framework of progressive strategies was used in the comparison of the empirical data.

Validity and Reliability

In order to have any effect on the practice or the theory of a field, research studies must be rigorously conducted (Meriam & Tisdell, 2015). The trustworthiness of the study is therefore imperative. For this research, four design tests, as described by Yin (2014), were integrated throughout the study to ensure its validity and reliability. They included: construct validity, internal validity, external validity and reliability. Table 6 provides a summary of the case study tactics used along with the phase of research in which they were integrated.

Table 6

Case Study Tactics Used to Ensure Validity and Reliability (Yin, 2014)

Tests	Case Study Tactic	Phase of Research
Construct Validity	Multiple sources of data	Data collection
	Chain of evidence	Data collection
	Review of case study protocol by supervisor	Composition
Internal Validity	Data source triangulation	Data analysis
	Investigator triangulation	Data analysis
	Theory triangulation	Data analysis
External validity	Multiple case study	Research design
Reliability	Use of case study database	Data collection
	Case study protocol	Data collection

Construct Validity

Construct validity refers to the identification of correct operational measures for the concepts to be studied (Yin, 2014). This was achieved through the use of multiple data sources of evidence, the provision of a chain of evidence through sequential reporting of evidence and the review of the case study protocol and co-coding (on 10% of the files) by my supervisor to enhance accuracy and identify possible competing perspectives (Baskarada, 2014).

Internal Validity

Internal validity refers to the justification of causal relationships (Yin, 2014). This was achieved primarily through:

- data triangulation—the triangulation of the multiple sources of evidence
- investigator triangulation—the triangulation through the corroboration of coding and analyses between myself and my supervisor
- theory triangulation—also known as pattern matching (Baskarada, 2014).

External Validity

External validity refers to the extent to which the findings can be analytically generalized to other situations that were not part of the original study (Yin, 2014). This was achieved through the replication of the same case study at another school. The results of the findings were similar but due to the size of the data collected from both schools, a decision was made to report mostly on the findings from one of the schools, Immaculate Middle School.

Reliability

Reliability demonstrates that the operations of the study, such as the data collection, can be repeated, with the same results. This was achieved through the outlining of the operational steps of the research as well as the use of a database (Yin, 2014).

Limitations and Bias

There were a few limitations to this study. These limitations were mainly in regard to the design of the study. Firstly, the design sample size was small and purposely chosen with instruction occurring every other week. As such, the results of this study cannot be generalized to a regular classroom environment. Secondly, the students in the study were not familiar with a reform-oriented approach to instruction. In this regard, the students hesitated at times to share and discuss their solutions and strategies, with me, the researcher, as well as their peers. Despite these limitations, the necessary precautions were taken to ensure the validity and reliability of the study. This also included a design protocol to facilitate the replication of the exact study in a regular classroom setting.

Chapter 4: Findings

The impact of Equal Sharing and Multiple Groups problems on students' understanding of fractions was assessed through the observations and evaluation of students' responses to a variety of word problems related to fractions. The responses to students pre, mid, and postassessments were analysed and coded. A summary of the sources of data analysed and coded is presented in Table 7.

Table 7

Sources of Data

Test	No. Students	No. Questions	No. Primary Documents [1]
Preassessment	14	11	14
Midassessment	14	2	14
Postassessment	12	11	12

Classroom Instruction

In this case study, teaching and learning occurred through a series of word problems. Students worked in groups of two or three to solve these problems. They were encouraged to discuss and share their strategies with their peers. At times, whole class discussions occurred. These discussions, based on students' solutions, were orchestrated by me and served the purpose of extending students' understanding of new concepts and/or strategies. In order to deepen and/or extend students' understanding of new concepts or strategies, specific questioning techniques were employed, and connections made between the different strategies used. At the end of each day, the audio and video recordings, along with any field notes were reviewed. Depending on the findings, revisions were made to future lessons to facilitate learning or extend understanding.

The word problems consisted of a series of Equal Sharing and Multiple Groups problems. The sequence and type of word problems were based on the Empson and Levi (2011) text, Extending Children's Mathematics: Fractions and Decimal. Appendix M provides the sequence of lessons presented every other week to one of the schools in this case study.

Analysis of the Preassessment and the Postassessment

The preassessment was administered during students' lunch recess. After the distribution of the assessment, students were advised to answer as much as they were able without the use of a calculator. The purpose of this assessment was to assess students' understanding of fractions. The results of this assessment became the baseline for evaluating the impact of instruction through Equal Sharing and Multiple Groups problems on students' understanding of fractions. In order to capture the many interpretations and meanings of fractions (Van de Walle et al., 2016), the problems presented in the preassessment related to the five constructs (interpretations) of fractions: the part-whole construct, measurement construct, ratio construct, operator construct and quotient construct (Kieren, 1980). Also included were questions related to the underlying fractions concepts: partitioning, equivalence, and unit forming (Kieren, 1995) along with both partitive and quotative (measurement) division questions. The purpose of these division questions was to assess students' knowledge of division and how they make sense of the numbers in their division.

The postassessment, also administered during lunch, consisted of the same questions presented in the preassessment. In this section, an overall analysis of the pre and postassessment will be presented. The analysis will be provided per question to facilitate a more comprehensive analysis of the results. Appendix N and Appendix O provide a detailed summary of the pre and postassessment results. Figure 12 below shows a comparison of the percentage of correct answers in both the preassessment and postassessment.

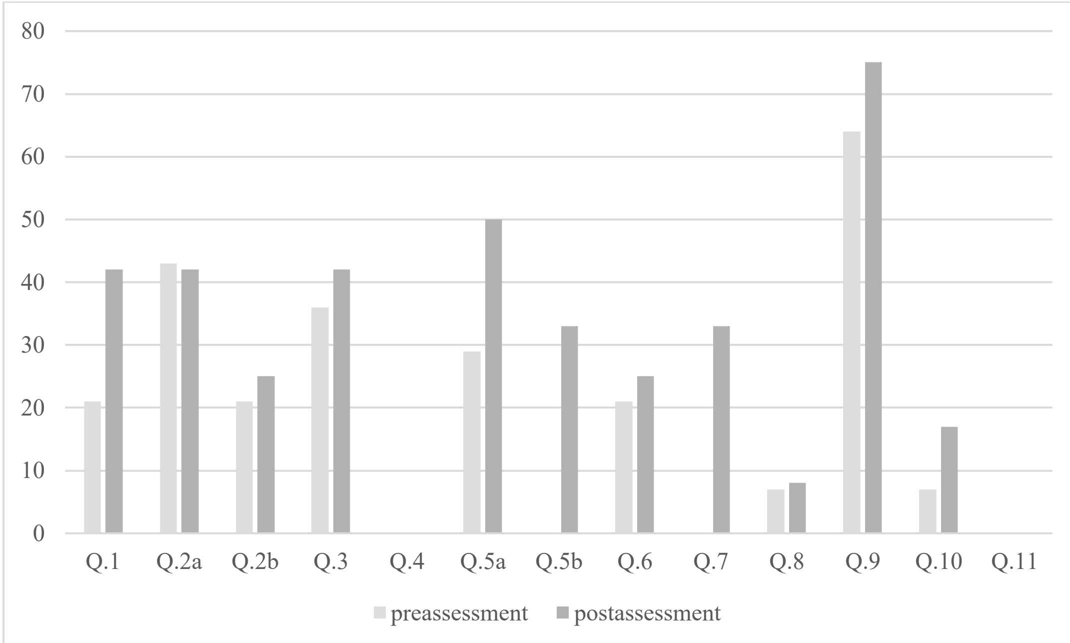


Figure 12: Percentage of correct answers on preassessment (*N = 14)* and postassessment *(N = 12)*

Division.

Partitive Division.

Table 8

Summary of Preassessment and Postassessment for Question 1- Partitive Division

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	3	5	6
Postassessment	12	5	5	2

Note. Question 1- Partitive Division: 256 apples are divided among 7 Grade 6 classes. How many apples will each Grade 6 class get?

In the preassessment, as shown in Table 8, 21% of the students answered Question 1 correctly with almost half the students (43%) choosing not to answer the question. It is also interesting to note that they were all Grade 6 students (See Appendix N). The Grade 8 students either did not attempt to answer or answered incorrectly. During the preassessment, these Grade

8 students expressed disappointment in not being able to use a calculator. Their reliance on the calculator was clearly reflected in their solutions. All the six Grade 6 students who answered the partitive word problem correctly used the same division algorithm. However, they differed in their response to the remainder of 4 apples. For example, after correctly executing the algorithm, Jessica stated, "36.4 apples will be given" (P 244)[1]. Jessica's solution was therefore coded as incorrect. This is in contrast to Sierra, who, in my opinion, is making as much sense as she can of the remaining 4 apples. She stated, "Every class will get 36 whole apples with half an apple" (P 247). Her diagram, in Figure 13, demonstrates her attempt to share the remaining 4 amongst the 7 classes. The manner in which students approached the remainder highlighted their heavy reliance on the memorization of an algorithm without making sense of the problem or the numbers being used. Anghileri et al. (2002) argues the perspective that the application of taught methods can become mechanical and unthinking where students are unclear about the links between a taught procedure and the meanings they can identify.

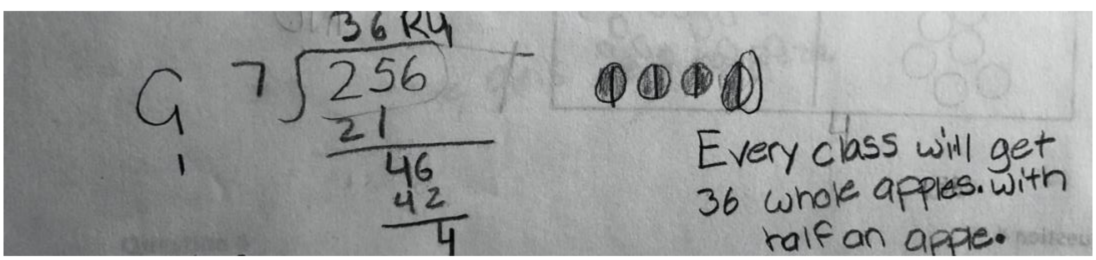


Figure 13. Sierra's attempt to make sense of the remainder. Retrieved from Atlas.ti, P 247.

In the postassessment, there was only a slight percentage increase in the number of students answering correctly. There were, however, some noteworthy improvements in their responses to the remainder. One such example came from student Sierra, who was previously

[1] P means Primary Document within the *ATLAS.ti* program. The number indicated the specific document

discussed. Figure 13 shows her preassessment response. In the postassessment, she responded, “Each class will get 36 apples. There will be 4 apples left over” (P 265). See Figure 14.

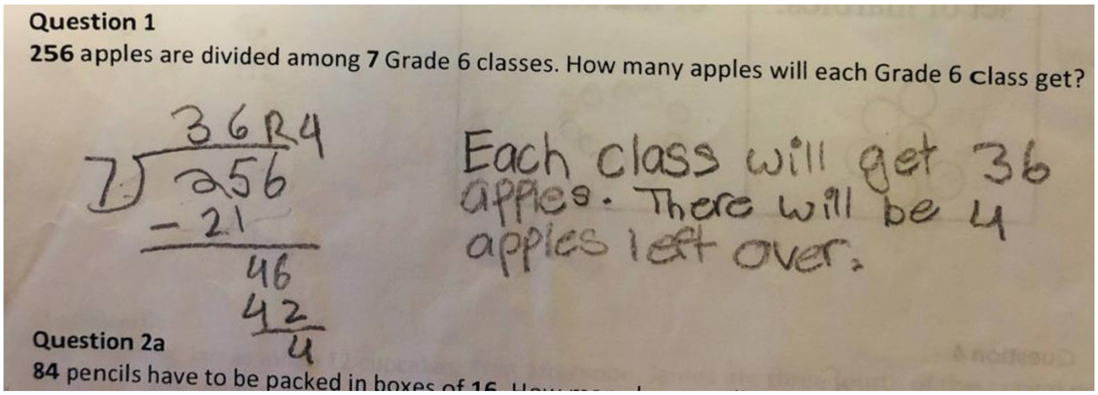


Figure 14. Sierra’s postassessment partitive division solution. Retrieved from Atlas.ti, P 265.

In addition, there were a few improvements in students’ strategies. It would appear that some students took advantage of their experience solving Equal Sharing problems. Marilyn, during the preassessment, was not able to successfully answer the division question. There was evidence of some effort made to solve the question, however, in the end, she opted to leave the question blank. This is in stark contrast to the solution presented in the postassessment. She uses a trial amount and build up to her answer, expressing the remainder as a fractional amount. See Figure 15 and Figure 16.

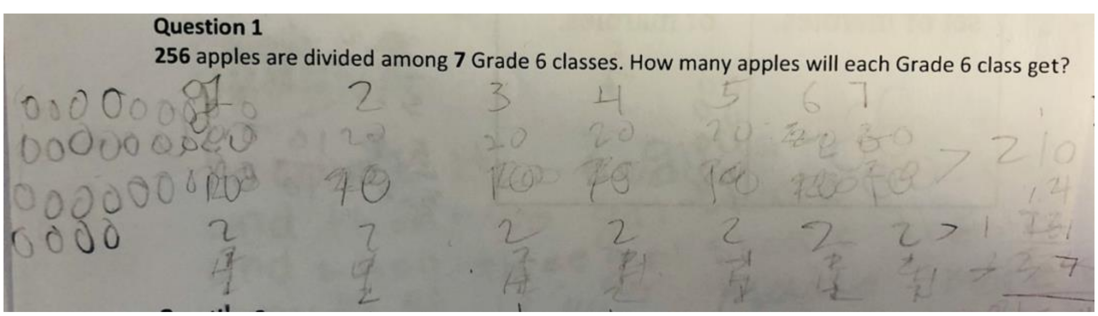


Figure 15. Marilyn’s preassessment partitive division solution. Retrieved from Atlas.ti, P 258.

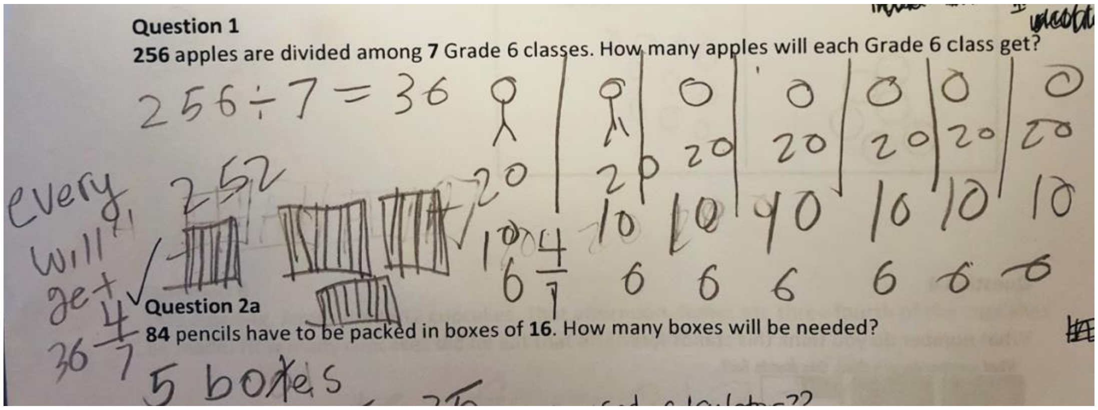


Figure 16. Marilyn's postassessment partitive division solution. Retrieved from Atlas.ti, P 263.

Measurement Division.

Table 9

Summary of Preassessment and Postassessment for Question 2a - Measurement Division

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	6	4	4
Postassessment	12	5	3	2

Note. Question 2a – Measurement Division: 84 pencils have to be packed in boxes of 16. How many boxes will be needed?

Table 10

Summary of Preassessment and Postassessment for Question 2b - Measurement Division

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	3	8(4)[1]	3
Postassessment	12	3	8(6)[1]	1

Note. Question 2b – Measurement Division: A carton of apple juice fills 8 glasses. How much apple juice (in carton) do you need to fill 20 glasses?

[1] Indicates, in brackets, the number of students out of the total whose strategy was correct but did not present a sense making solution

The results of the preassessment highlighted further the ‘mechanical and unthinking’ (Anghileri et al., 2002) approach students bring to solving division problems. As shown in Table 10, question 2b had a greater percentage of incorrect answers. The reason for this greater percentage of inaccuracy might be attributed to the fact that although the solution to the division statement $20 \div 8$ is 2 remainder 4, students needed to make sense of numbers and the context of the problem. So, although the division yielded 2 remainder 4, the sense making solution would be 3.

Both Question 2a and 2b saw only a slight improvement in the postassessment. (See Table 9 and Table 10). This was mainly due to students not making sense of their solutions. Despite the lack of improvement in the correct answer for Question 2a, a few students’ strategies did improve. One such student was Jessica. She incorporated her learning from solving Equal Sharing and Multiple Groups problems. Figure 17 shows Jessica’s preassessment solution. Her solution clearly shows her misapplication of her procedural knowledge of fractions along with misconstrued knowledge of decimals. However, as shown in Figure 18, her approach to solving the division changed. It would appear that she used her knowledge of multiplication and expressed the remainder fractionally but was not able to produce the sense making solution that 6 boxes would be needed. Although explicit instructions were not provided on division, especially division with two digits divisors, the beneficial impacts of reform-oriented instruction, I would argue, were evident, especially the benefits of classroom discourse where students made sense of their own and each other’s thinking. As noted previously, the Principles and Standards for School Mathematics (NCTM, 2000) argues that learning with understanding can be enhanced by classroom interactions. This claim is supported by Piaget (1995). He argued that construction of knowledge is facilitated by cooperative relationships. The students had multiple opportunities to share their strategies for partitioning when solving Equal Sharing and Multiple Groups problems. The strategies shared during classroom discussions, in my opinion, facilitated Jessica's success in acquiring a strategy that made sense to her and could further be extended to divisions with two digits divisors. A similar result was also observed by this same student when answering Question 2b. This confirmed further for me her increased comfort level in solving division questions.

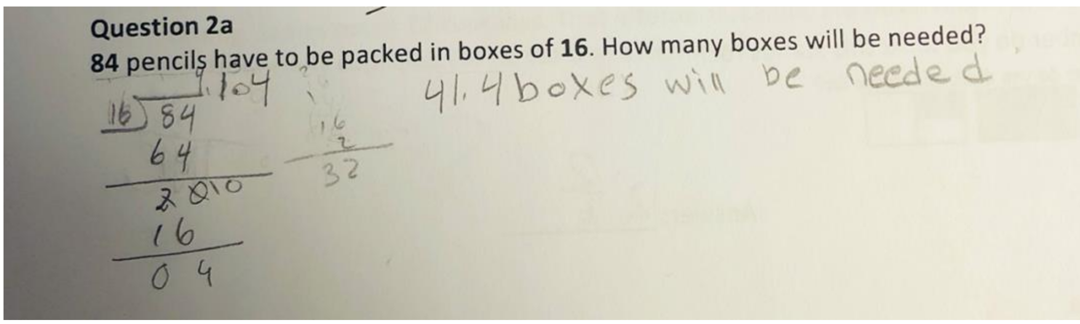


Figure 17. Jessica's preassessment strategy for Question 2a. Retrieved from Atlas.ti, P 244.

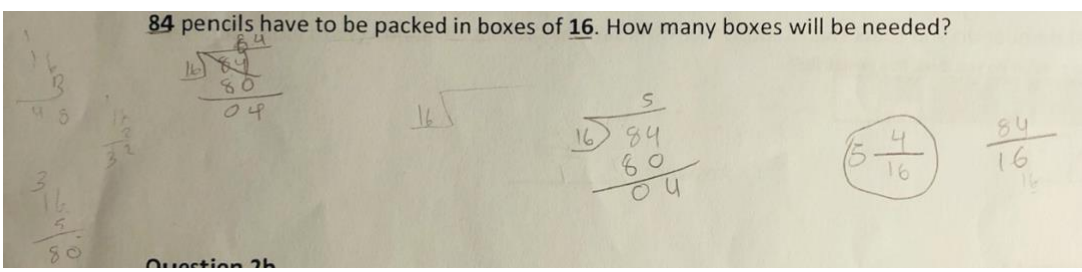


Figure 18. Jessica's postassessment strategy for Question 2a. Retrieved from Atlas.ti, P 252.

Question 2b also saw an improvement in strategies used. Of interest to note, was the use of multiplicative thinking, to solve this measurement division question. Figure 19 and Figure 20 shows the ratio strategy used by two students despite the fact that during the preassessment they were not able to answer this question correctly or even attempt the question.

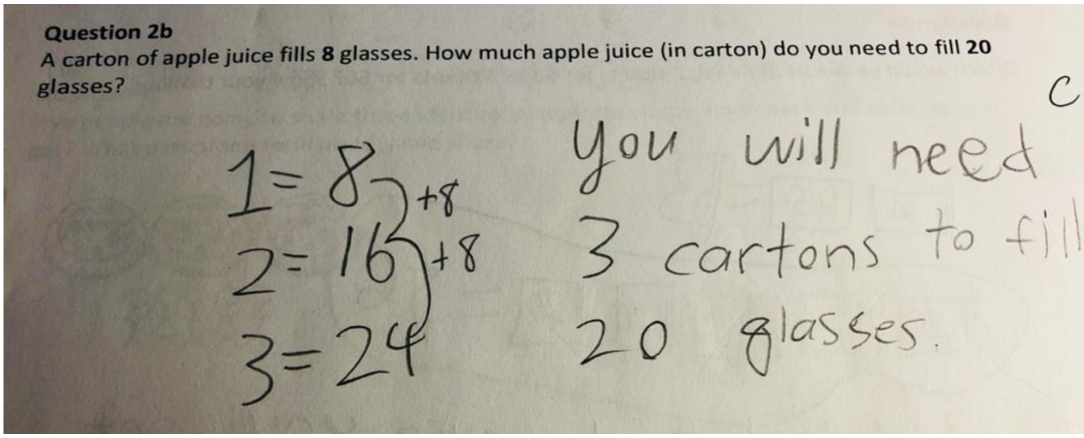


Figure 19. Anthony’s ratio strategy solution to Question 2b. Retrieved from Atlas.ti, P 260.

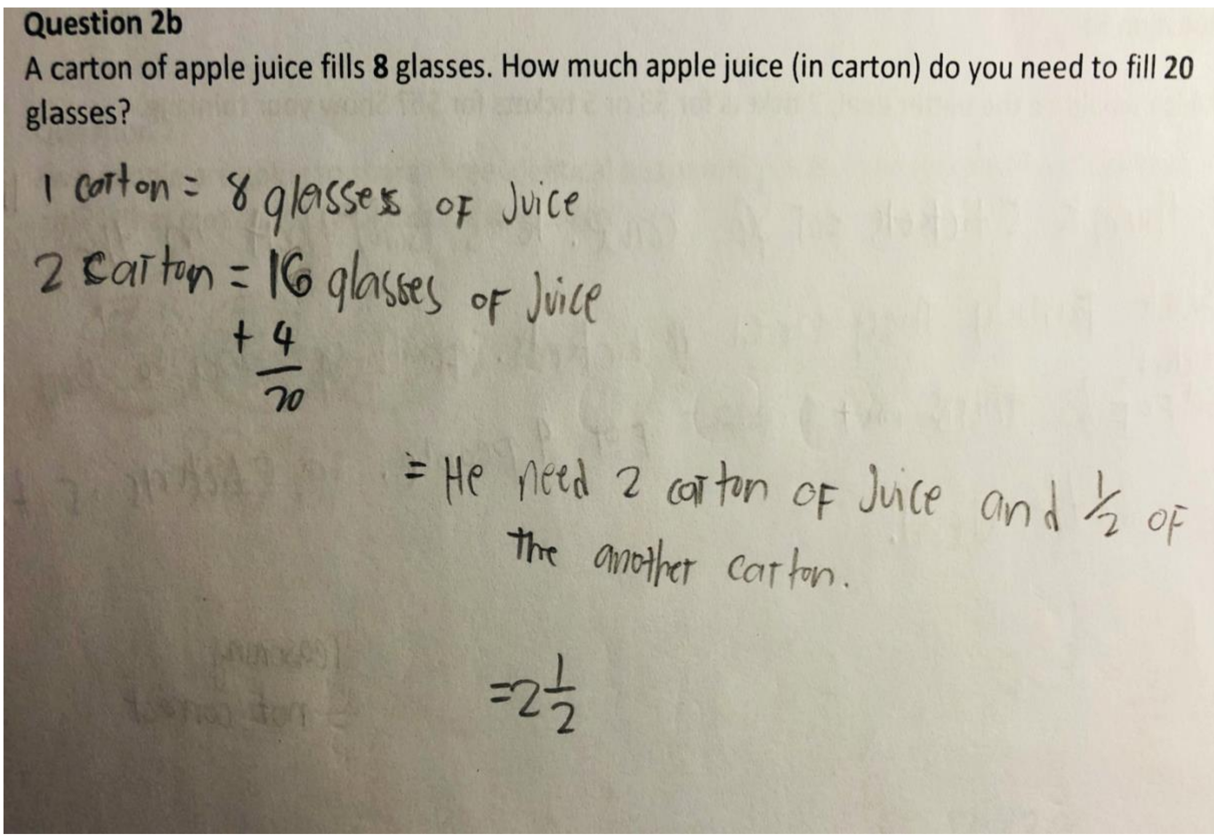


Figure 20. Horace’s ratio strategy solution. Retrieved from Atlas.ti, P 262.

Fraction Constructs

Part-Whole Construct.

Table 11

Summary of Preassessment and Postassessment for Question 3-Part-Whole Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	5	9	0
Postassessment	12	5	9	0

Note. Question 3 – Part-whole construct: This is $\frac{3}{5}$ of a set (See Appendix J for diagram). Draw the set of marbles.

Table 12

Summary of Preassessment and Postassessment for Question 4 - Part-Whole Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	0	12	2
Postassessment	12	0	11	1

Note. Question 4 – Part-whole construct: What number do you think this stands for? (See Appendix J for diagram)

Both Question 3 and Question 4 assessed students understanding of the part-whole construct of fraction. Question 3 focused on a discrete representation whereas Question 4 focused on a continuous representation. The results of the preassessment for both questions were quite surprising. (See Table 11 and Table 12). The part-whole construct is the most represented construct in the teaching and learning of fractions (Charles & Nason, 2000) and as such, a greater percentage of correct answers was expected. Question 4 had all students answering incorrectly. Upon further reflection, perhaps the additional challenge of adding unfamiliar fractional amounts brought some additional complexity to Question 4. In light of this reflection, I decided to assess based on students' ability to recognize that the number represented was greater than 2. Table 13 summarizes the results based on this new criterion.

Table 13

Students who Recognized the Shaded Region in Question 4 as Greater Than 2

Question 4 Part-Whole Construct (continuous)	Greater than 2 %	Less than 2 %	No Response %
preassessment	64	14	14
postassessment	75	8	17

Examining further the responses of the students who thought the answer was less than 2, a common misapplication of whole number knowledge to fractions (Van de Walle et al., 2016) was observed. Horace, a Grade 8 student, incorrectly stated the answer as $\frac{4}{5}$. He counted the number of shaded parts (four) and then the number of unshaded parts (five). No consideration was taken for equal sized fractional parts when counting. See Figure 21. This misapplication was also noticed with students whose answers were greater than 2. Thirty-three percent of the students who answered greater than 2, stated that the answer was $2\frac{2}{5}$. These students, one Grade 6 and two Grade 8, recognized accurately that two wholes were shaded but then counted the remaining fractional parts shaded (two) and the number of fractional parts unshaded (five) to arrive at a solution of $2\frac{2}{5}$. Figure 22 shows one such solution. These results demonstrated what Empson and Levi (2001) describe as a novel idea for many children, that the value of a fraction is determined by the multiplicative relationship between the numerator and the denominator (p. 4). Furthermore, they explain that students' weak understanding of fractions, in this regard, will further inhibit their understanding of equivalency and computation with fractions.

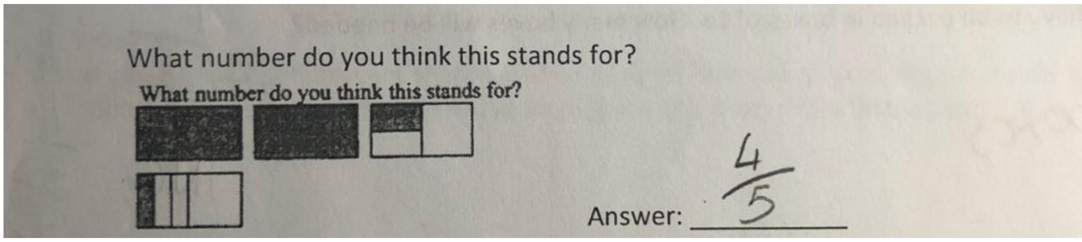


Figure 21. Horace's incorrect strategy of counting four shaded and five unshaded to arrive at a fractional representation of $\frac{4}{5}$. Retrieved from Atlas.ti, P 257.

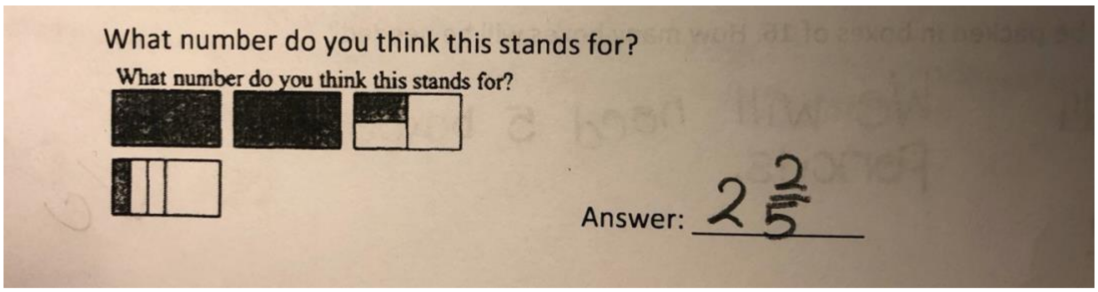


Figure 22. Angela's incorrect solution of counting the shaded fractional shaded and the five unshaded fractional pieces to arrive at the fractional representation of $2\frac{2}{5}$. Retrieved from Atlas.ti, P 242.

As shown in Table 13, 75% of the students were able to recognize that the most likely solution was greater than 2 in the postassessment compared to the 64% in the preassessment. The challenge, of course, was finding the value of the shaded fractional amount. Tasha, who did not answer the question during the preassessment, was the only student who attempted to find a common denominator for the fractional shaded piece. She divided each remaining fractional part into 12 equal pieces. See Figure 24. This was, in my opinion, a significant change in her learning and understanding of the part-whole construct of fraction.

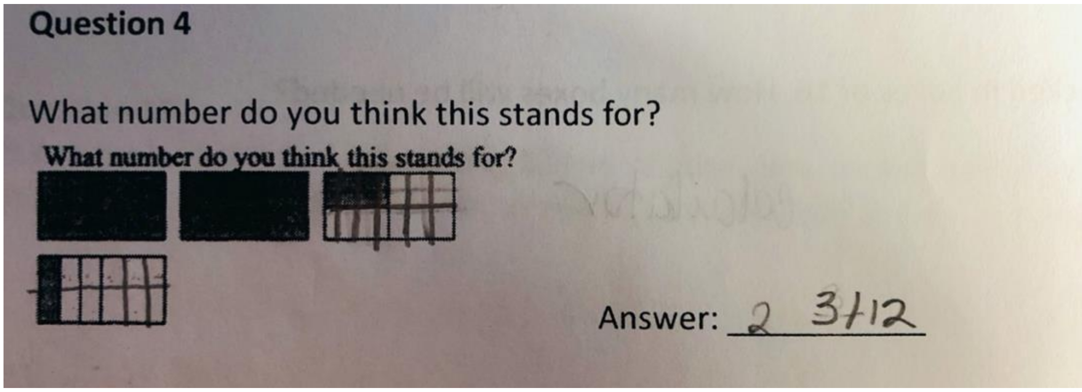


Figure 23. Tasha’s attempt to find a common denominator. Retrieved from Atlas.ti, P 264.

Ratio Construct.

Table 14

Summary of Preassessment and Postassessment for Question 5a – Ratio Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	4	8	2
Postassessment	12	6	4	2

Note. Question 5a – Ratio Construct: Which would be the better deal, 2 tickets for $3 or 5 tickets for $6? Show your thinking.

Table 15

Summary of Preassessment and Postassessment for Question 5b – Ratio Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	1	10	3
Postassessment	12	4	6	2

Note. Question 5b – Ratio Construct: Who gets more pizza, the boys or the girls? Show your thinking (See Appendix J for diagram).

The preassessment results for both Question 5a and 5b, as shown in Table 14 and Table 15, indicated students' limited knowledge and/or exposure to the ratio construct of fraction. In fact, Van de Walle et al. (2016) suggest that students' understanding of ratios depends on their prior understanding of multiplicative reasoning. It can therefore be argued that these students' thinking aligned more with an additive approach to reasoning versus a multiplicative approach. This argument was evident in students' solutions. All of the students who incorrectly answered that 2 tickets for $3 was a better deal used their additive reasoning, comparing $3 to $5 with no thought to the amount of tickets being received. Figure 24 shows one such solution. This is in contrast with another student, Marilyn, who compared the dollar amounts but also took into consideration the amount of tickets. As seen in Figure 25, she used her emerging multiplicative thinking to reason that 5 tickets for $6 was a better deal than 4 tickets for $6.

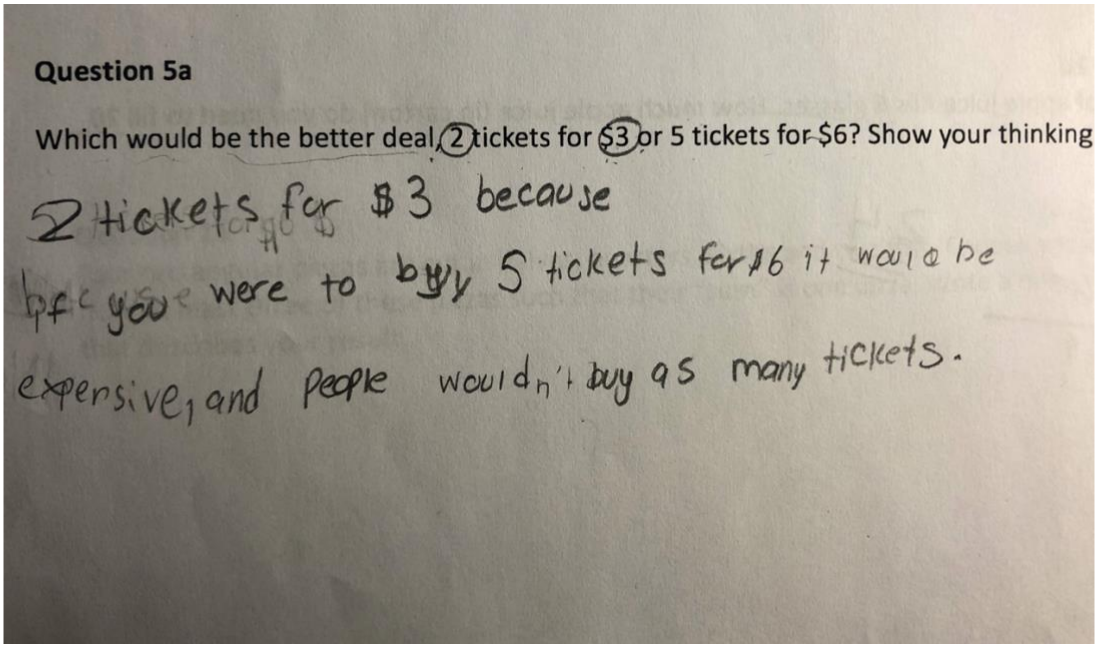


Figure 24. Jessica comparing dollar amounts without consideration of the amount of tickets being received. Retrieved from Atlas.ti, P 244

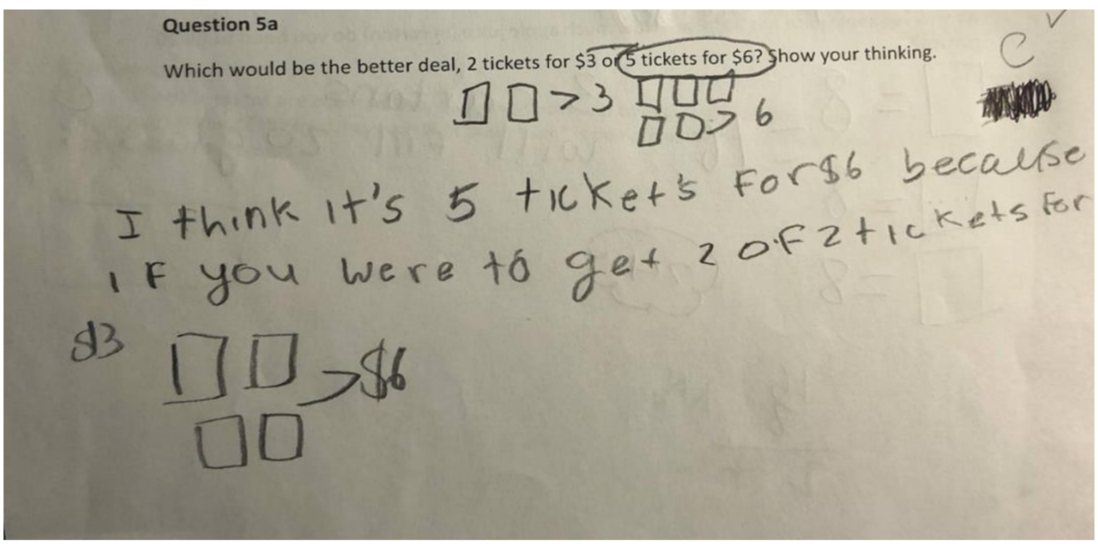


Figure 25. Marilyn demonstrating the emergence of multiplicative thinking when comparing. Retrieved from Atlas.ti, P 258.

The postassessment showed a slight improvement in the number of students answering Question 5a and 5b correctly. In fact, a perusal of Figure 12 shows that this construct had the second largest increase in percentage of correct answers. Based on my own observations of the development of students' multiplicative reasoning, through exposure to Equal Sharing and Multiple Groups problems, this result was not surprising. We already observed two students, Horace and Anthony, applying a ratio strategy to solve a measurement division type question. (See Figure 19 and Figure 20) Based on Anthony's successful application of the ratio strategy in Question 2b, I was curious if a similar strategy was applied in Question 5a. Although, he was not successful during the preassessment, he was successful in the postassessment. He used a ratio strategy in which he found the unit price for each ticket. See Figure 26.

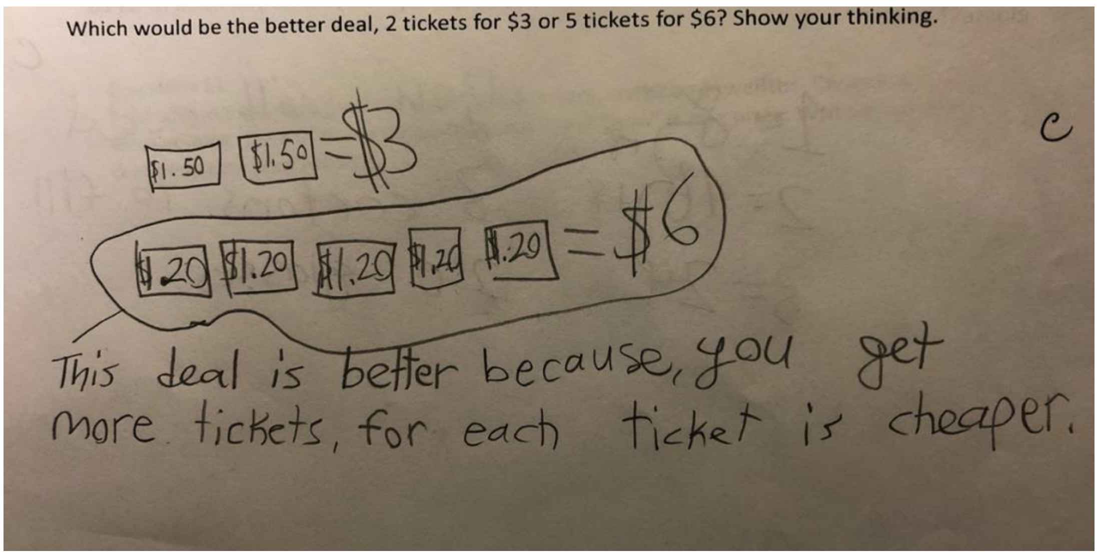


Figure 26. Anthony demonstrating an emerging multiplicative thinking to compare ratios. Retrieved from Atlas.ti, P 260.

Question 5b saw a greater percentage increase in correct answers than Question 5a. During the preassessment, only two students responded correctly compared to the six students that answered correctly during the postassessment. I would, however, argue that the strategy applied was more aligned with students' experience with solving Equal Sharing problems than the use of an explicit ratio strategy. Figure 28 shows Horace's solution strategy. He evidently used his experience with Equal Sharing problems to solve the question. Figure 29 shows a similar strategy but in this solution, Anthony was not able to express the fractions or compare them in order to accurately answer the question of who would receive more pizza. It was evident, however, that his experience with Equal Sharing Problems also facilitated his attempt.

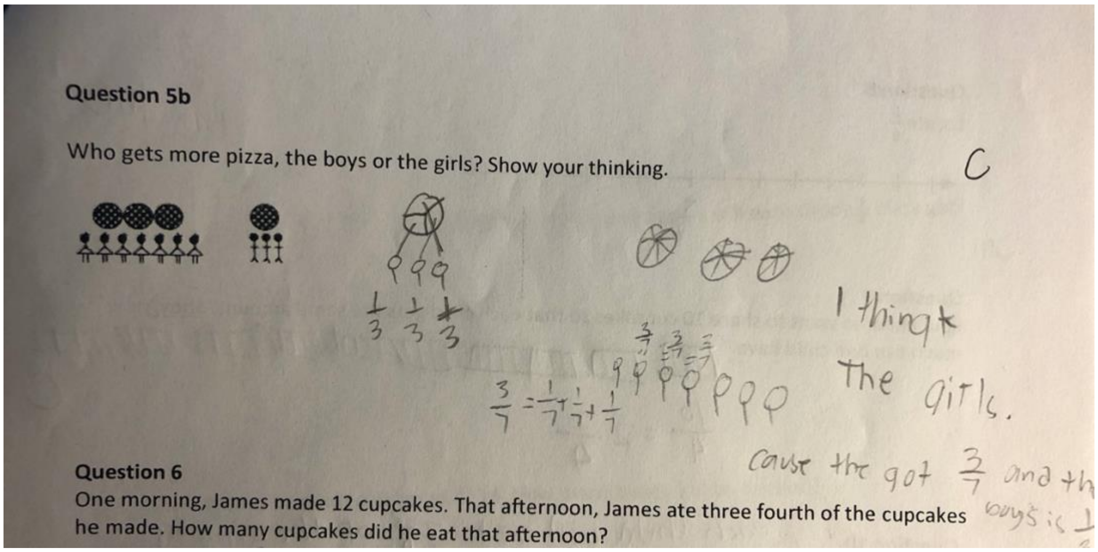


Figure 27. Horace using his experience with Equal Sharing problems to solve Question 5b. Retrieved from Atlas.ti, P 262

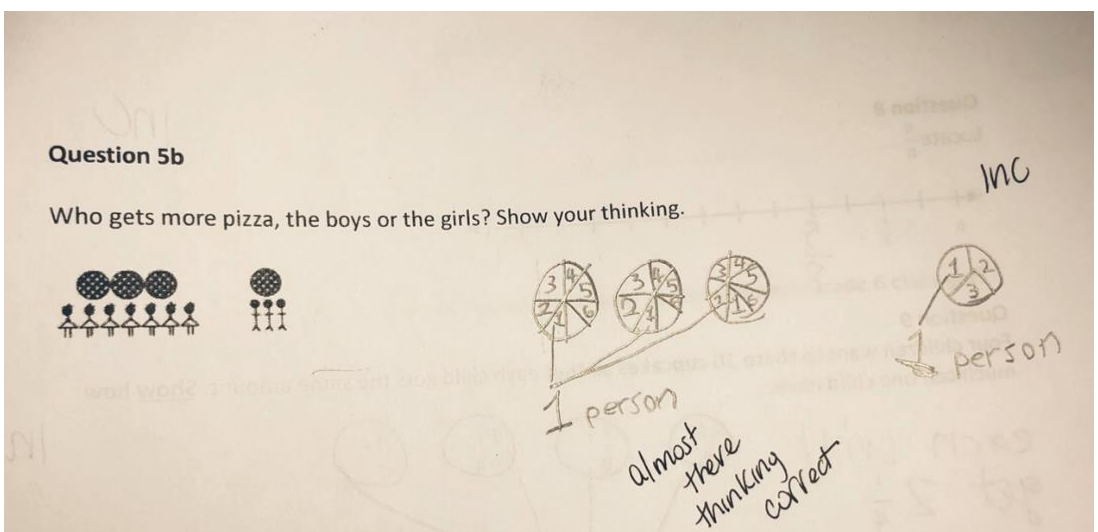


Figure 28. Anthony uses his experience with Equal Sharing problems to solve Question 5b. Retrieved from Atlas.ti, P 260.

Operator Construct.

Table 16

Summary of Preassessment and Postassessment for Question 6 – Operator Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	3	5	6
Postassessment	12	3	8	1

Note. Question 6 – Operator Construct: One morning, James made 12 cupcakes. That afternoon, James ate three fourth of the cupcakes he made. How many cupcakes did he eat that afternoon?

Almost half the students opted not to answer this question with only three being successful in the preassessment. This result aligns with Usiskin (2007) observations. He noted that the operator construct is not emphasized enough in school curricula. In fact, perusal of the Ontario Mathematics Curriculum confirms this perspective. In comparing students' pre and postassessment solutions, it was interesting to note that there was not much improvement in their solutions to this question. It was, however, observed that students' exposure to Equal Sharing and Multiple Groups problems facilitated their ability to at least attempt the problem. For example, Anthony applied his knowledge gained from solving Equal Sharing and Multiple Groups problems to solve the question, albeit, with a minor error. In the preassessment, he expressed, via a question mark, that he had no idea how to solve the question. In the postassessment, however, he had a strategy. As seen in Figure 30, he drew 12 cupcakes, shaded three quarters of each cupcake and then proceeded to count the amount of whole cupcake eaten. Although his strategy was correct, a miscount unfortunately prevented him from arriving at the correct solution. The fact that the student chose to partition each cupcake rather than taking the quantity of 12 as the new whole provided for me evidence of a lack of exposure to questions relating to the operator construct and perhaps some part-whole misconceptions. This solution also presented some evidence for the argument that students' exposure to Equal Sharing and Multiple Groups problems have the possibility to facilitate their understanding of the operator construct.

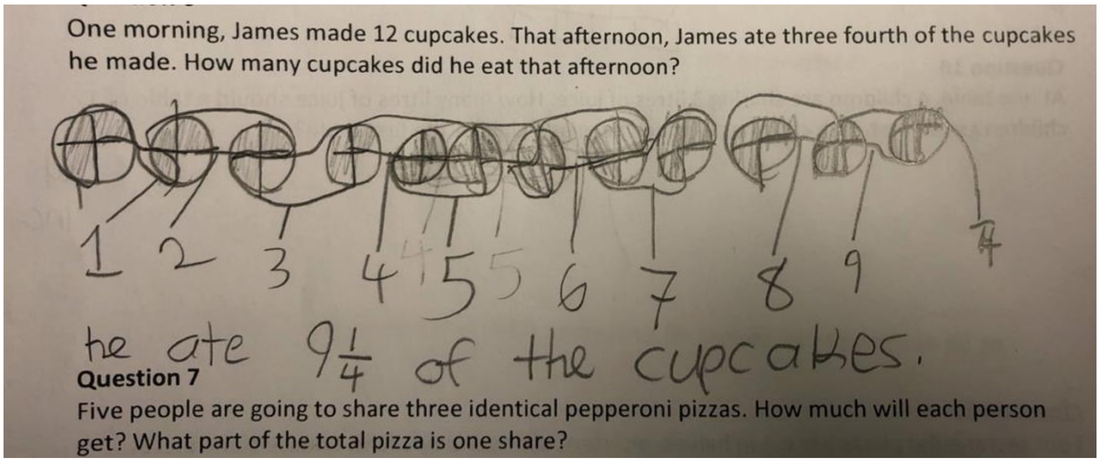


Figure 29. Anthony's solution strategy for Question 6. Retrieved from Atlas.ti, P 260.

Quotient Construct.

Table 17

Summary of Preassessment and Postassessment for Question 7 – Quotient Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	0	6	8
Postassessment	12	5	5	2

Note. Question 7 – Quotient Construct: Five people are going to share three identical pepperoni pizzas. How much will each person get? What part of the total pizza is one share? (See Appendix J for diagram)

During the preassessment, none of the students were able to answer this question correctly. In fact, almost 60% did not attempt the problem. In review of students' solutions, I made a few observations. One such observation was the evidence of students' misapplication of whole number knowledge to fractions, primarily they thought that the dividend (the number being divided) must always be larger than the divisor (Lamon, 2011) (see Figure 30). I also found that students struggled to solve problems that could not be solved through repeated halving (see Figure 31 and 32). Finally, there was overemphasis on the part-whole construct through diagrams, in particularly the continuous model, in student's learning of fraction. Students, such as Susan, in Figure 32 determined the fraction by counting shaded and unshaded sections.

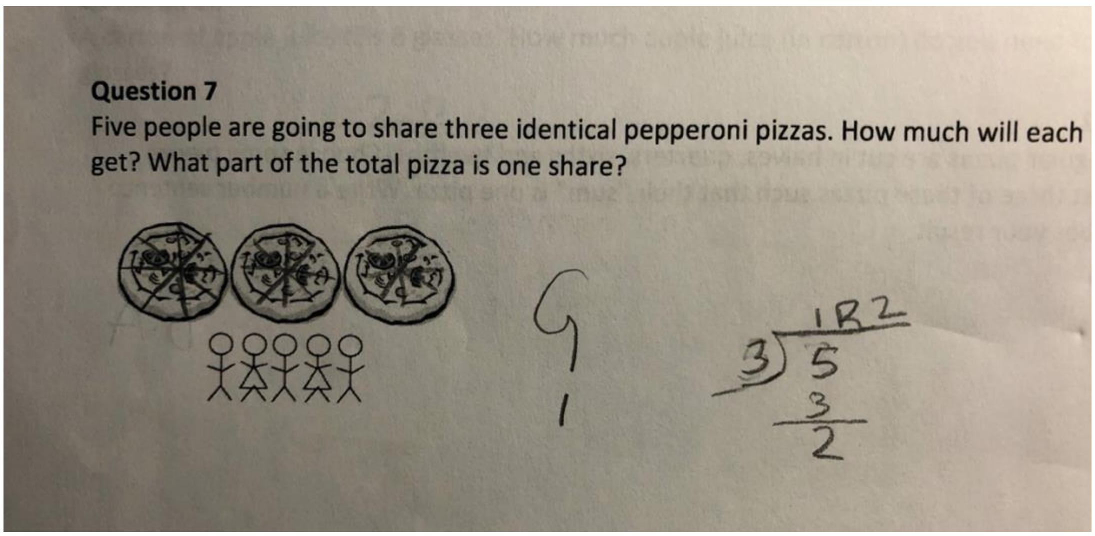


Figure 30. Sierra demonstrating a common misconception that the divisor is always smaller than the dividend. Retrieved from Atlas.ti, P 247.

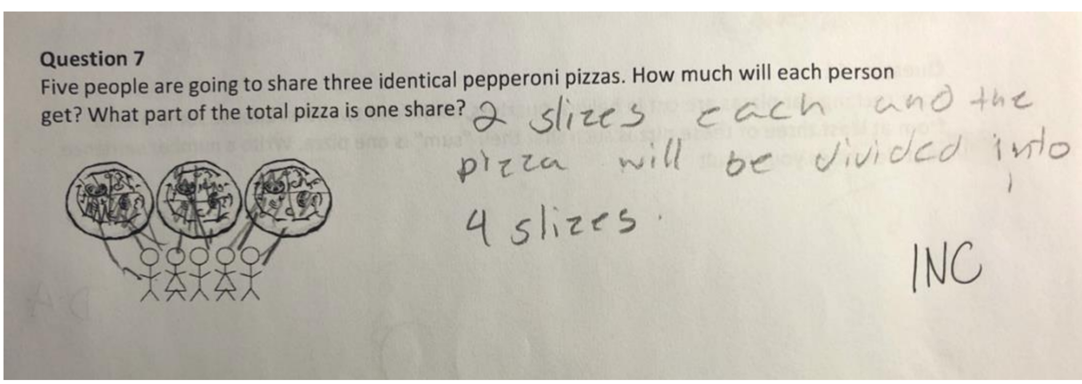


Figure 31. Maria uses repeated halving with no coordination at the end to facilitate her efforts to equally share the pizza. Retrieved from Atlas.ti, P 245.

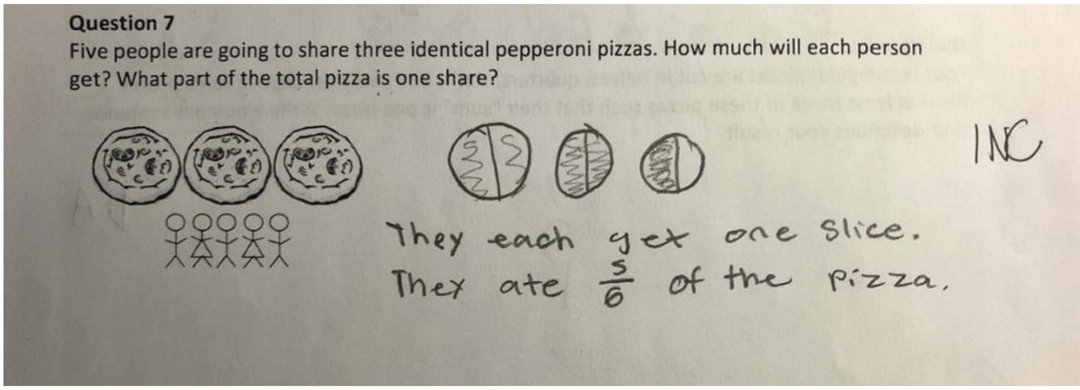


Figure 32. Susan uses repeated halving with no coordination at the end to facilitate her efforts to equally share the pizza. Retrieved from Atlas.ti, P 248.

As expected, there was an increase in the percentage of students accurately responding to this question in the postassessment. Both Equal Sharing and Multiple Groups problems relate to the quotient construct. Forty two percent of the students were successful in their responses. Based on the fact that the case study focused on the quotient construct, I was quite surprised that the success rate was not higher and even more so that two students did not attempt the problem. Perhaps students required more practice in order to retain their new learnings with Equal Sharing and Multiple Groups problems.

Measure Construct.

Table 18

Summary of Preassessment and Postassessment for Question 8 – Measure Construct

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Preassessment	14	1	8	5
Postassessment	12	1	10	1

Note. Question 8 – Measurement Construct: Locate $\frac{5}{8}$. (See Appendix J for the given number line)

During the preassessment, 64% of the students attempted this question with only one of them answering correctly. This demonstrated their unfamiliarity with this fraction construct and/or this type of question. Upon further analysis of students' solutions, the underlying factor appeared to be their inability to identify one-eighth as a unit and then iterate. There was no change in students' success rate in the postassessment. It was notable, however, that, compared to the preassessment where 36% of the students did not attempt the question, only one student opted not to attempt to solve the question in the postassessment. This provided, as noted before, additional evidence that students' experience within this case study boosted their confidence to make the effort when problem solving.

Underlying Fraction Concepts

Table 19

Summary of Preassessment – Underlying Fraction Concepts

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Partitioning (Question 9)	14	9	4	1
Equivalence (Question 10)	14	1	4	9
Unit Forming (Question 11)	14	0	1	13

Table 20

Summary of Postassessment – Underlying Fraction Concepts

Test	No. Students	No. Correctly Answered	No. Incorrectly Answered	No. Not Attempted
Partitioning (Question 9)	12	9	3	0
Equivalence (Question 10)	12	2	4	6
Unit Forming (Question 11)	12	0	0	12

Note. Question 9 – Partitioning: Four children want to share 10 cupcakes so that each child gets the same amount. Show how much can one child have.
Question 10 – Equivalence: At one table, 4 children are sharing 3 litres of juice. How many litres of juice should a table of 12 children get so that each child has as much juice as a child at the first table?

Question 11 – Unit Forming: Four rectangular pizzas are cut in halves, quarters, sixths and twelfths. Choose some pieces from at least three of these pizzas such that their "sum" is one pizza. Write a number sentence that describes your result.

Partitioning. As shown in Table 19 and Table 20, the underlying concept of partitioning had the highest success rate in both the pre and postassessment. Majority of the students who answered correctly did so with the support of a diagram. The success, in my opinion, was due to students' own familiarity with sharing items and therefore they could make connections and solve accurately. Empson and Levi (2011) affirm this point of view. They state that children learn mathematics by using what they know to make sense of new material. There was an 11% increase in the number of students answering this question correctly in the postassessment. One solution was worthy of mention. It demonstrated the progression of strategies described by Empson and Levi (2011) when solving Equal Sharing problems. They explain that children's strategies for Equal Sharing problems follow a predictable pattern. The most important feature of this pattern involves how children relate the two quantities in the problem – the people sharing, and the items being shared – to make an equal share. In this example, Susan progressed from a 'no coordination between sharers and shares' strategy during the preassessment to a 'Additive Coordination – one item at a time' strategy during the postassessment. Empson and Levi (2011), go on further to explain that during this preliminary stage of no coordination, students either create equal shares but not use up everything to be shared or use up everything to be shared but do not create equal shares. In this example, the student used the more common strategy of not creating equal shares. See Figure 33. In the postassessment, however, we see the student's progress, as predicted, to an 'Additive Coordination' strategy, sharing one item at a time (see Figure 34).

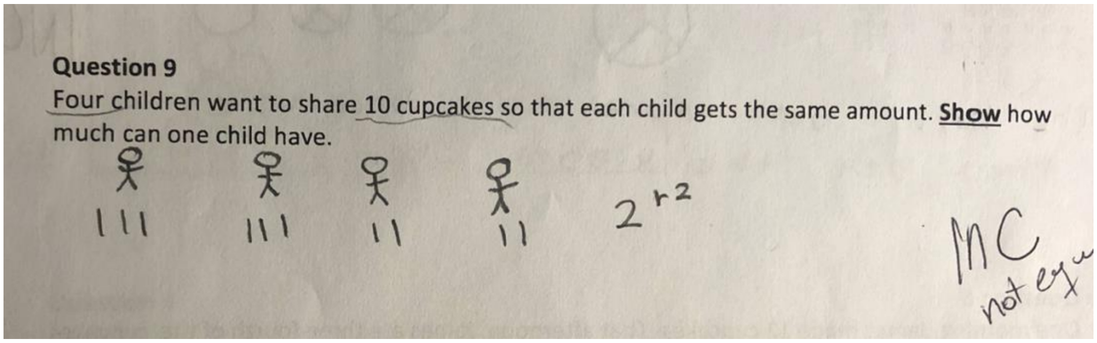


Figure 33. Susan's No Coordination between Sharers and Shares strategy. Retrieved from Atlas.ti, P 248.

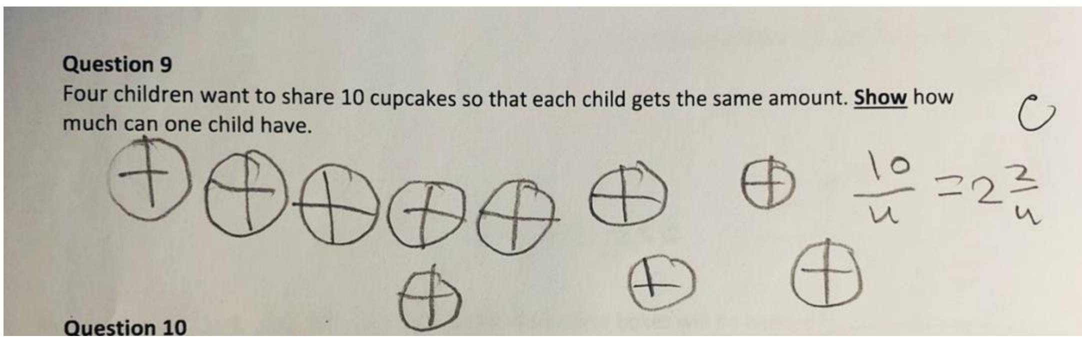


Figure 34. Susan's Additive Coordination strategy. Retrieved from Atlas.ti, P 266.

Equivalence. During the preassessment, 64% of the students did not attempt this question. Of those who did, only Anthony was successful, demonstrating his multiplicative thinking. See Figure 36. It is interesting to note that this multiplicative thinking was not demonstrated when attempting the ratio construct questions. This leads me to speculate that perhaps his answer was based on familiarity with other questions like this in the past. Of further interest was Paul's solution shown in Figure 37. His solution demonstrated an incorrect additive approach to thinking, reasoning that since 4 children shared 3 litres, then 12 children will share 11 litres. Paul noticed a difference of 1 and maintained that difference when reasoning how much would be needed for 12 children.

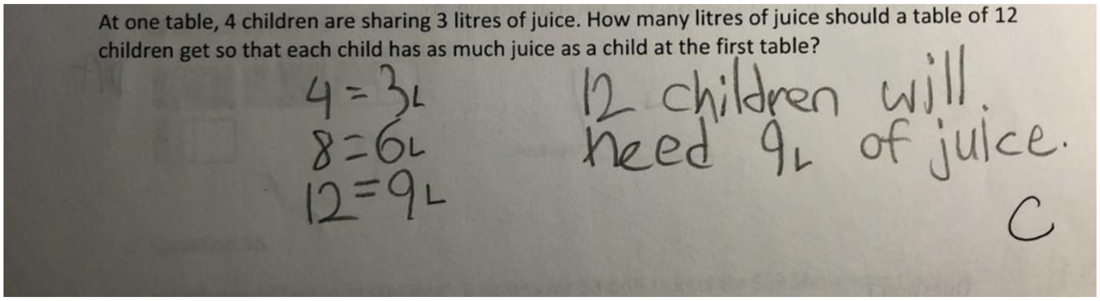


Figure 35. Anthony demonstrating multiplicative thinking. Retrieved from Atlas.ti, P 254

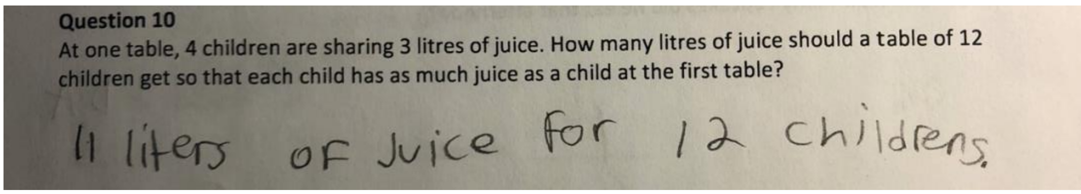


Figure 36. Paul demonstrating additive thinking. Retrieved from Atlas.ti, P 246.

The postassesment did not show much improvement in the percentage of students answering this question correctly. It was interesting to note, however, that Paul, whose preassessment solution, shown in Figure 36 changed. Although, he did not show his work, he stated, “9 litres of juice for 12 children” (P 253).

Unit Forming. In both the preassessment and postassessment no student was successful in answering this question. The reason for such a large number of students not attempting this question is unclear. It could be speculated that perhaps they were fatigued or their unfamiliarity with adding fractions such as sixths and twelfths. According the Ontario Mathematics Curriculum, the addition of fractions with like and unlike denominators is not introduced until Grade 7. The students in this study were Grade 6 students.

Based on the analysis of the pre and postassessment data, it can be summarized that it is unclear if students’ experience with Equal Sharing and Multiple Groups problems had some impact on their understanding of fractions. For the most part, the data demonstrated an improvement, minimal as it maybe, in students’ success when answering questions related to the different constructs of fractions. These improvements, although minimal, furthered my belief that exposure to word problems related to the quotient construct supports students' understanding of the other constructs. Of interest to me is that the results seem to indicate an increase in an understanding of the ratio construct or use of a ratio strategy. In Equal Sharing problems it is the relationship between the number of items being shared and the number of people sharing that leads to a fractional amount (Empson & Levi, 2011). That is, a fraction is a multiplicative relationship between the numerator and the denominator (Empson & Levi). The use of Multiple Groups problems reinforces and extends this understanding of fractions (Empson & Levi). This same multiplicative relationship is required for ratio construct type questions as well as measurement construct type questions, where a unit fraction is counted (Van de Walle et al., 2016).

Results of the Midassessment and Analysis

Between the administration of the pre and postassessment, a midassessment was administered. As this assessment did not shed light on the overall student development the results are discussed in Appendix L.

Chapter 5: Analysis of Students' Progression of Relational Thinking

In order to examine change over time, four students' records were selected for an in-depth examination of students' general development as explored in Chapter 4. The four grade six students selected were chosen for different reasons. To preserve their anonymity, they will be referred to as Paul, Susan, Angela and Frank. Paul, Susan and Angela, according to their Mathematics teacher and reporting records, were all performing at Level 2. Frank, who requested to participate in the case study, was performing at a Level 3. I selected Paul and Susan in order to look at two different trajectories of development over time. Paul was at the earlier stages of the development of multiplicative reasoning while Susan was at the most advanced. I selected Angela and Frank in order to explore, in greater depth, some of the significant roadblocks to their learning, as a few of the students exhibited. As the development of their relational thinking progressed at different rates, it highlighted certain factors that could possibly impact the progression of multiplicative thinking and thereby, students' understanding of fractions.

Paul: No Coordination to Early Multiplicative Reasoning

From the very first introduction to Equal Sharing problems, Paul exemplified the benefit afforded through these types of problems. Empson and Levi (2011) explained that students are able to build on their informal knowledge of fractions through their intuitive knowledge of sharing. In solving his first Equal Sharing problem, 'Ms. Wright has 29 brownies to share with 4 friends. How much will each friend receive?', Paul recognized that it was a division question. He solved the division, stating that each student would receive 7 brownies with one left over. He then concluded, with the support of his peer and his diagram, that each student would receive seven and one fourth brownies. When asked how he knew each would receive one fourth of the remaining brownie, he quickly responded matter-of-factly, "because there is four kids" (P 6.10).

His response demonstrated to me that he understood that one item shared with four friends, resulted in a quantity of one fourth. Did his understanding transfer to unfamiliar fractions?

The following Equal Sharing problem was given immediately after. ‘Ms. Wright has 27 brownies to share with 4 friends. How much will each friend receive?’ Paul provided an incomplete strategy. His solution demonstrated evidence that perhaps his previous correct answer for 29 brownies shared with four friends was possibly due to familiarity with the fraction one fourth. Paul was unable to share the three remaining brownies equally. He reasoned “only three people will get and the last person won’t get” (P 3.7). He had not generalized his understanding of one shared by four is one fourth to three shared by four is three fourths. Empson and Levi (2011) describes this strategy as the ‘No-Coordination between Sharers and Shares’ strategy, a strategy some students use when first introduced to Equal Sharing problems. This incomplete strategy was again observed the next day when asked to solve the following word problem, ‘4 children want to share 51 loaf cakes so that everyone gets the same amount. How much will each person get?’. Paul, in the video, could be heard sharing out the loaf cakes to all four students. He explained after, that three students would get 13 loaf cakes and one will get 12 (P 30.6). After debriefing the question with Paul and his peers, I asked how much each student would receive. Paul, answered confidently, 15 (P 29.2). It appears that Paul added the 12 whole loaf cakes and the 3 fractional pieces. He is not differentiating the whole pieces from the fractional pieces. Five non-instructional days after, the following Equal Sharing word problem was given, ‘Melissa has 17 cupcakes that she wants to share equally with 5 friends. How much will each friend receive?’. Paul was able to share evenly and explain his strategy. In his video recording, Paul counted out loud as he equally shared the cupcakes among five friends. He stopped at fifteen and stated, “I’m only going up to fifteen cause then there are two cupcakes left” After discussions with his peers, he concluded that “everyone gets one fifth of 2” (P 36.2). Paul demonstrated his first step toward multiplicative thinking. He was not yet able to articulate the fractional quantity of two fifths but recognized, as seen in Figure 40, that because there are two cupcakes left, each student would receive one fifth of each of the remaining cupcakes. It can also be noted that Paul partially solved the problem using one of the predictable strategies described by Empson and Levi (2011). This particular strategy is called the 'Additive Coordination: Sharing One Item at a Time'.

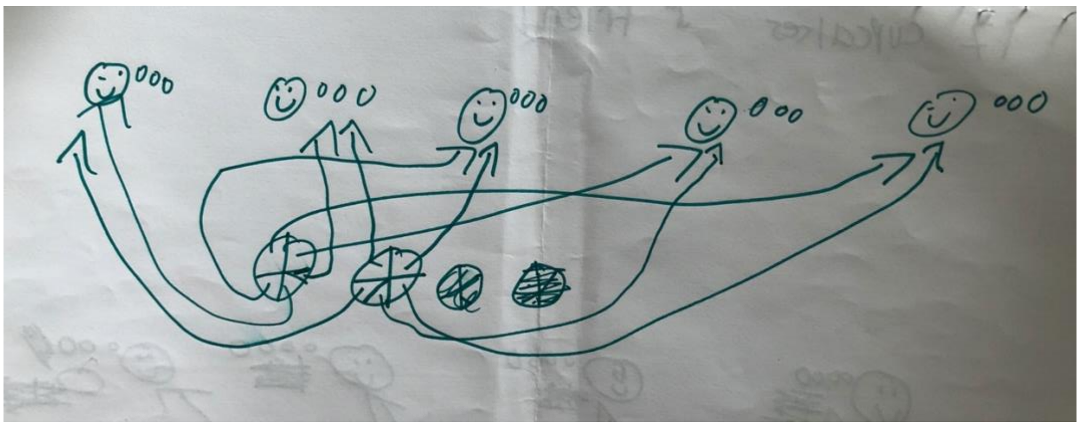


Figure 40. Paul demonstrating Additive Coordination Strategy. Retrieved from Atlas.ti, P 37.

The next day, as we transitioned from Equal Sharing problems with solutions greater than one to solutions less than one, Paul's progress toward multiplicative thinking was observed. With the support of scissors and construction paper, Paul solved the Equal Sharing problem, '7 children in art class have to share 5 packages of clay equally. How much clay will each child receive?' accurately. He remarked to his peers, when thinking about the problem "I will cut every piece in sevenths…doesn't that make sense guys? Cut seven pieces from every package. Should I do that?" (P 59.2). After cutting and sharing his fractional pieces, he concluded that each student would receive five sevenths. Three days later, while in discussion with Paul over his solution to eight students sharing six brownies, I observed yet again what I considered a verbally preliminary stage of multiplicative thinking. When I asked Paul how he arrived at an answer of $\frac{6}{8}$, he explained as follows "cause in one brownie I divided by eighths then every student gets six brownies" (P 95.5). I interrupted his explanation to ask if it was six whole brownies and he responded "no, one eighth". Paul was able to articulate his relational thinking in his explanation. He demonstrated his ability to recognize the multiplicative relationship in the fractional quantity $\frac{6}{8}$. He recognized that six eighths is actually six groups of eighths. What is interesting to note is the disparity in the strategy displayed in his written answer and the strategy he explained verbally. Figure 41 shows his written answer. In his written answer, he presented an 'Additive Coordination: Sharing One Item at a Time' strategy, however, in his verbal explanation he shared only one brownie, then reasoned that since there are 6 brownies it would be six eighths. It may be that his concrete modelling gave him the foundation to make the leap to a 'Transitional Multiplicative Coordination' strategy. Empson and Levi (2011) describe the multiplicative coordination as a strategy whereby students are able to synthesize, after much practice, that when a items are shared with b students the result is a fractional quantity of $\frac{a}{b}$. Paul's multiplicative thinking and hence his ability to apply a multiplicative coordination strategy was not completely solidified but if he had had further practice with Equal Sharing problems, I could see this thinking developing further.

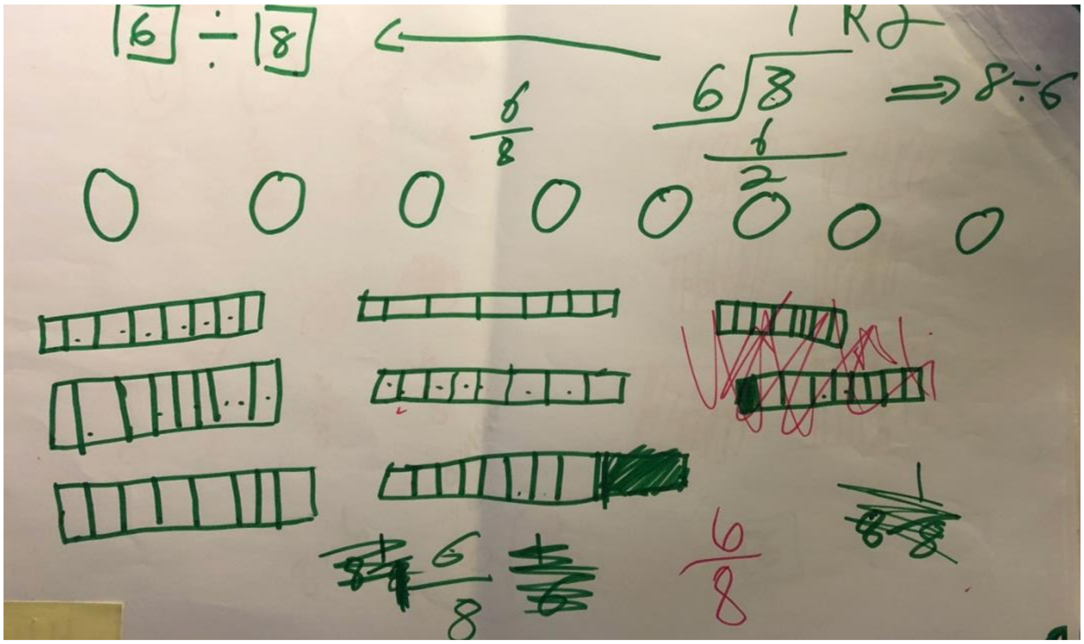


Figure 41. Paul's written solution for 6 brownies shared with 8 students. Retrieved from Atlas.ti, P 97.

Observing Paul's thinking to this point reinforced for me the effectiveness of Equal Sharing problems in developing the idea that the value of a fraction is determined by the multiplicative relationship between the numerator and the denominator. In summary, he progressed over five lessons from a 'No Coordination between Sharers and Shares' strategy to a 'Transitional Multiplicative Coordination' strategy as seen in Figure 42.

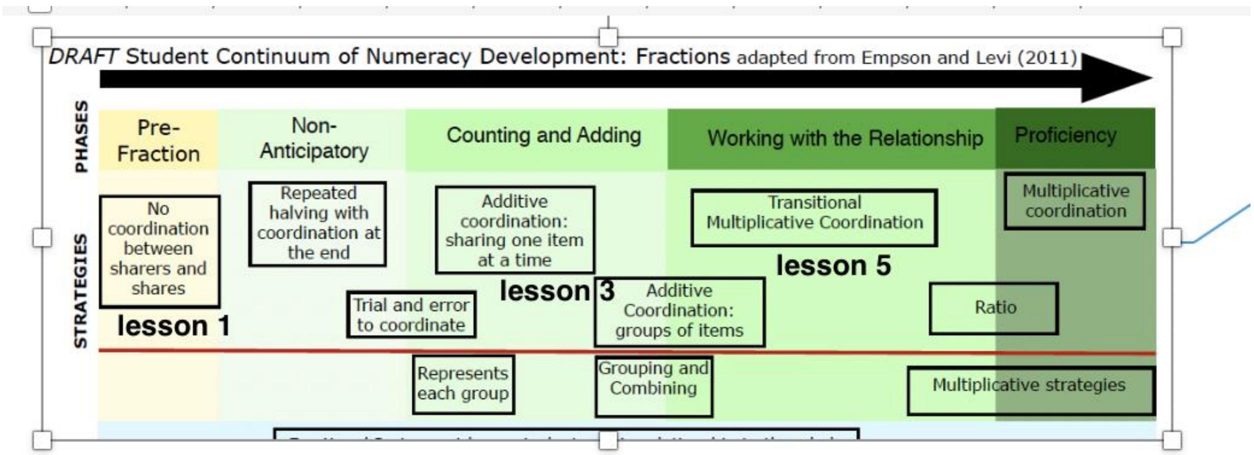


Figure 42. Paul's progress after five lessons with Equal Sharing problems. Adapted from Lawson, et al. (2019) Adaptation of Empson, S. and Levi, L. (2011) Extending Children's Mathematics. Portsmouth, NH: Heinemann.

Empson and Levi (2011) suggest a departure from Equal Sharing problems to posing Multiple Groups problems once students are able to create and name fractional quantities. I hesitated to shift as I was not quite confident with Paul's ability to consistently name correctly fractional quantities. He was able to explain verbally but, at times, with hesitation. However, as the majority of my students were able to create and name fractional quantities and given the limitation in time, my sequence of word problems transitioned to posing Multiple Groups problems. We first transitioned to Multiple Groups: Measurement division problems, where the number of groups is unknown but the amount per group and total is known. From there, we then practiced Multiple Groups: Multiplication problems, where the number of groups and the amount per group is known but the total is unknown.

When Paul was introduced to his first Multiple Groups problem three weeks later, he misunderstood the question. The question presented was 'Ms. Wright wants to feed each of the children she babysits one quarter of a KitKat. If she babysits 7 children, how many KitKats should she buy?' It is my thought that part of Paul's misunderstanding stemmed from the fact that the problem structure was different than that of Equal Sharing problems. After sitting and rereading the question with him he attempted the problem. Through discussions with his classmate as well as myself, Paul was able to create a model for his solution. Paul used a strategy called Direct Modeling. Empson and Levi (2011) describes this as one of the basic strategies for solving Multiple Groups problems. In this strategy, students represent all of the quantities in the problem and count or add to figure the answer. Figure 43 shows Paul's Direct Modeling strategy.

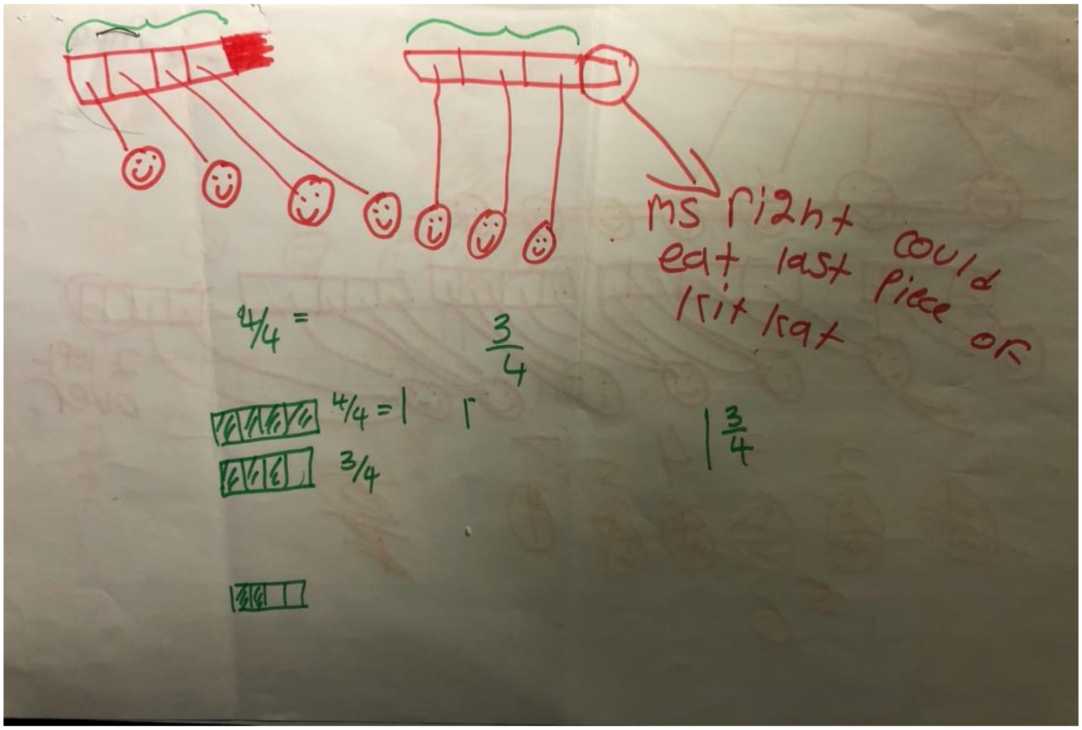


Figure 43. Paul's Direct Modeling strategy. Retrieved from Atlas.ti, P 135.

The conversation with Paul revealed the importance noted by Empson and Levi (2011) of students being able to name fractional quantities prior to attempting Multiple Groups problems. I noticed that Paul did not express his answer fractionally but with the use of a diagram (his work in red). I decided to ask him about his diagram.

> Me (pointing to the three fourths section): What fraction is this?
>
> Paul: ahhh…one third? No. three fourths
>
> Me (pointing to the whole): what fraction is this?
>
> Paul: four fourths
>
> Me: four fourth is the same as saying what?
>
> Paul: three fourths? (P 135).

Were I given the opportunity to repeat this conversation, I would have asked him how much of a KitKat four fourths was. Nonetheless, it was clear to me that Paul required additional support in naming fractional quantities. Researchers distinguish between, first, verbally naming a unit fraction as one third or one fourth and later, supporting students to write the name using fractional notation such as $\frac{1}{3}$ or $\frac{1}{4}$. I believe Paul was still grappling with the earlier understanding of simply naming rather than referring to his sub-divisions as pieces. (While it is the case that his initial answer of one third could be argued as correct from the perspective of the ratio construct none of his other work found him using a ratio interpretation, so this is unlikely). Paul's responses, both during Equal Sharing problems and now Multiple group problems, highlighted the importance of students' ability to name the 'pieces' as well as count the 'pieces' in their partitions, prior to introduction to Multiple Groups problems. In fact, Empson and Levi (2011) argue that presenting students with Equal Sharing problems provides rich mathematical meaning to which the fraction notation could later be attached. This approach they believe is more productive in the long run than the traditional approach of introducing the fraction notation and then presenting the meaning for the numerator and denominator, primarily through the part-whole construct.

For the second Multiple Groups problem presented, Paul was successful with his 'Direct Modeling' strategy but again he was not able to name or express his answer using fraction words or symbols. Given the following Multiple Groups problem 'Two thirds meters of fabric is needed to make a pillow. How many meters of fabric would it take to make 15 pillows?', Paul attempted to express his answer using fractional notation. However, when prompted to explain his written fraction he was unable to do so erroneously writing $\frac{30}{3} = \frac{10}{3}$. It was my assumption that he was getting his fractional answer from his peers. His ability to express his correct thinking through fractional notation was limited. Figure 44 shows his solution to this problem.

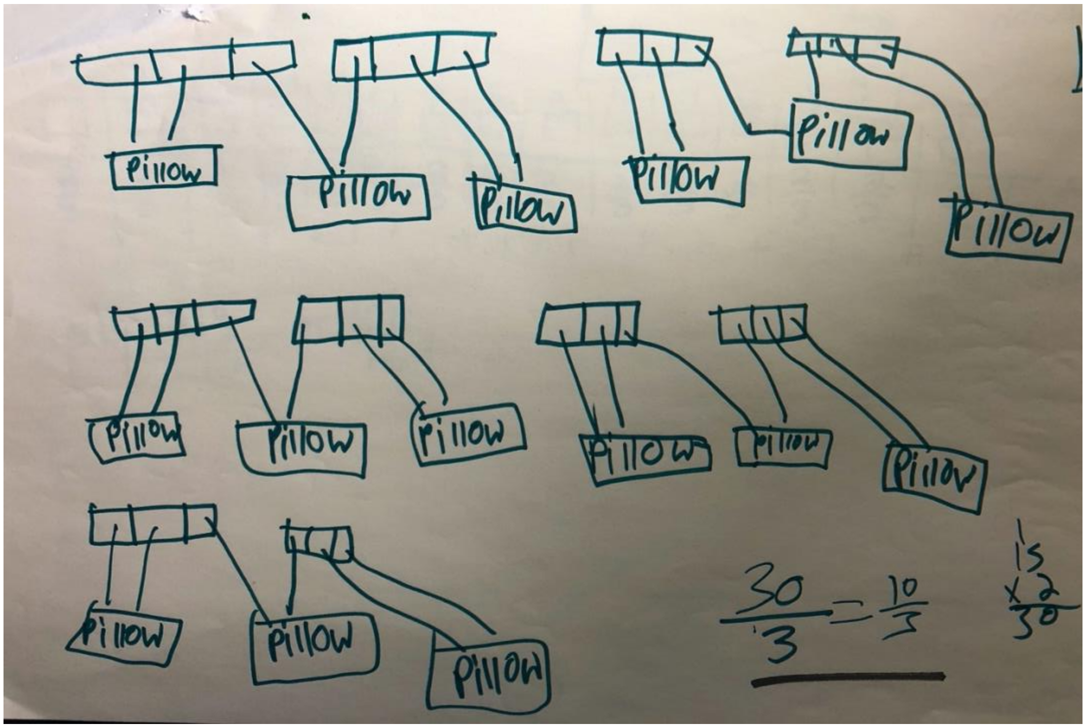


Figure 44. Paul's Direct Modelling solution. Retrieved from Atlas.ti, P 197.

This further reinforced for me the importance of fostering students' ability to name fractional quantities prior to introduction of Multiple Groups problems. As touched on earlier, the introduction of fraction symbols and terms have been given much thought and consideration by Empson and Levi (2011). They suggest that students should be introduced to the fraction notation $\frac{a}{b}$ once they are able to use the fraction terminology correctly. That is, they are able to recognize the numerator as the counting number and the denominator as the fractional term. For example, for the fraction $\frac{2}{3}$, the denominator 3 refers to the size of piece relative to the whole (thirds) and the numerator 2 represents the quantity of thirds. Equal Sharing problems facilitate this introduction (Empson & Levi, 2011). In fact, the researchers believe that students' progression in their use of fraction terms and symbols, when solving Equal Sharing problems, follows a trajectory from pictorial to symbolic. Students begin with diagrams, then progress to the use of numbers and words, such as 2 eighths then eventually to symbols, example $\frac{2}{8}$. It is my thought that Paul was still grappling with the number and words stage. He was sometimes able to accurately express his fractional quantity verbally using numbers and words or visually but not yet with fractional notation. Perhaps more practice was needed with Equal Sharing problems to further develop his ability to name fractions with understanding.

The next day, the focus turned towards Multiple Groups: multiplication problems. Paul was somewhat more successful in his solution strategies for these types of Multiple Group problems. He, at times, opted to not use a diagram to solve the question. Given the following problem, 'Each small cupcake takes three quarters cups of frosting. If Saida wants to make 20 cupcakes, how much frosting does she need?', Paul wrote ' $\frac{3}{4}$ x 20 = $\frac{60}{4}$ = $\frac{15}{4}$'. I was surprised he went straight to multiplication of fractions. Perhaps this was an interference (coaching) from some of his peers who had a procedural understanding of multiplying fractions. When asked why he multiplied three quarters by 20, he explained accurately that each cupcake required three fourths and there were 20 cupcakes (P 160.2). Figure 45 shows his solution and attempt to simplify his fraction.

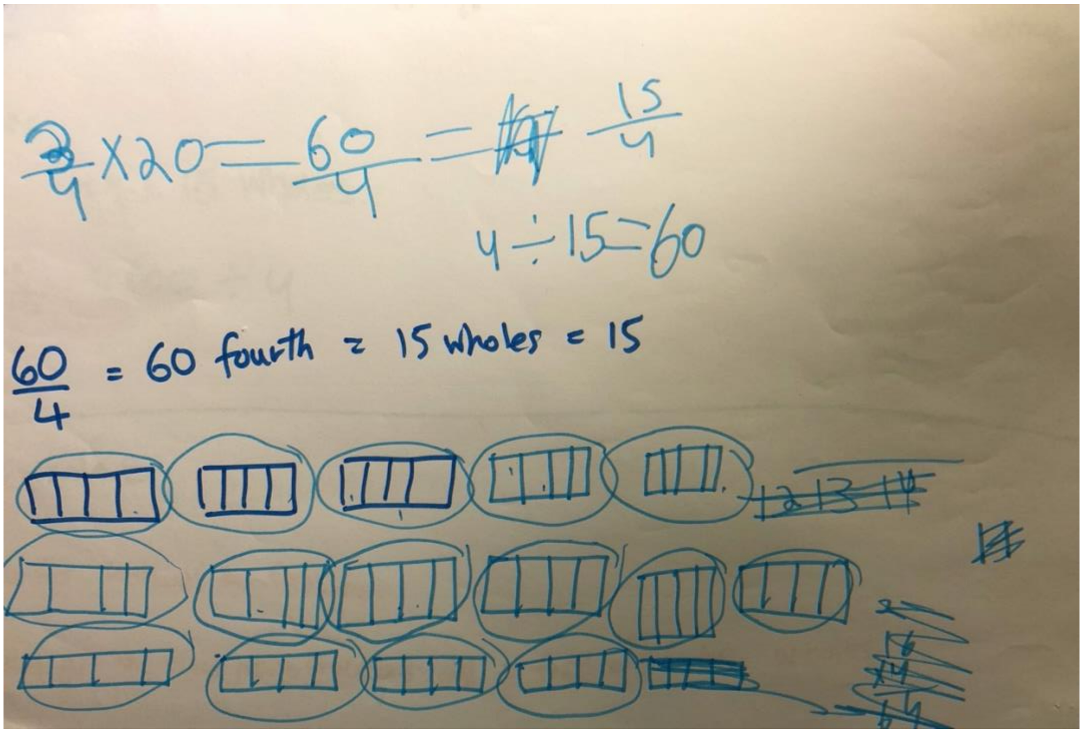


Figure 45. Paul’s solution to a Multiple Groups: multiplication problem. Retrieved from Atlas.ti, P 161.

He simplifies $\frac{60}{4}$ as $\frac{15}{4}$ the same error he made earlier. With more practice, Paul was able to solve and simplify his solutions with accuracy. However, I would hesitate to suggest that his initial stage of multiplicative thinking demonstrated with Equal Sharing problems was reinforced through Multiple Groups problems. Instead I think it was perhaps inhibited by interference from peers’ procedural knowledge of multiplying fractions. In other words, he understood that $\frac{3}{4}$ x 20 meant $\frac{3}{4}$ of a cup of icing per cupcake but was unsure of multiplication beyond that. He was also not it would seem really using his direct modelling as he did not circle the $\frac{3}{4}$ and count them.

Susan: A Later Trajectory Toward Multiplicative Reasoning

Susan’s trajectory reflects a much stronger understanding of fractions and perhaps the other end of the spectrum of learning in the group. After two days of introduction to Equal

Sharing problems, the following word problem was presented. '4 children want to share 51 loaf cakes so that everyone gets the same amount. How much will each person get?'. Susan recorded her final answer as $12\frac{3}{4} = \frac{51}{4}$. However, prior to arriving at this answer, in collaboration with her partner, they wrote the answer as 12.3. In the video recording, she explained as follows "what we did is four divided by 51 which we got 12.3 and everyone children get 12.3 loaf cakes" (P 27.1). The progression from 12.3 to the correct answer of $12\frac{3}{4}$ demonstrated the power of group discussions and visual representations in her learning process. After reviewing their solution of 12.3 with them, I asked the pair to represent their solution visually. They drew mini circles to represent 51 loaf cakes, 4 students and circled the remaining three as seen in Figure 46. She then stated the following. "then maybe we can split up tha ..ahhh the three parts which would be two twelfths then they would get the same amount basically" (P 27.2). In response, Susan responded, "twelve and one fourth". Susan then took the marker from her partner. She drew 12 mini loaf cakes below each student and then divided each of the remaining three loaf cakes into fourths, allocating a fourth to each student. She then wrote the solution $12\frac{3}{4} = \frac{51}{4}$. See Figure 46. Susan changed her thinking as she discussed and listened to her partner's reasoning. Unfortunately, I was not able to return to discuss the new solution. It was, however, interesting to note that she shared one fourth to each student from only one loaf cake and then wrote $12\frac{3}{4}$. How did she arrive at three fourths from sharing only one of the remaining three loaf cakes? Could this be preliminary evidence of multiplicative thinking?

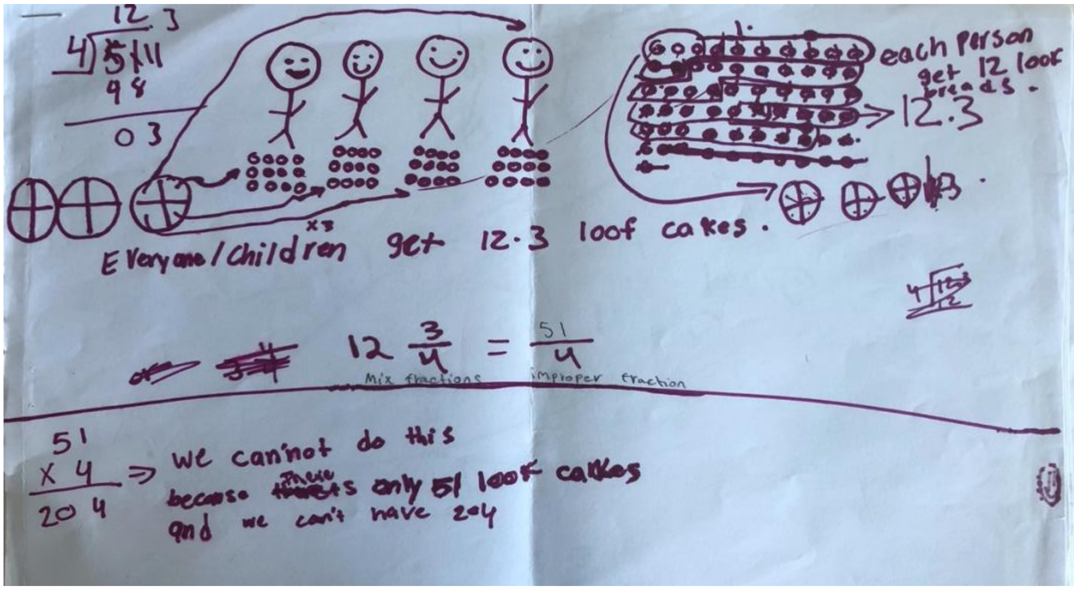


Figure 46. Susan and her partner's solution to 51 loaf cakes shared with 4 students. Retrieved from Atlas.ti, P 28.

The next day, Susan was successful in solving the word problem 'Melissa has 17 cupcakes that she wants to share equally with 5 friends. How much will each friend receive?' She responded, as seen in Figure 47, $3\frac{2}{5}$ or 3.2. Although her decimal representation was incorrect, she was successful in equally sharing the cupcakes. Once again, it was observed that she did not have to divide the two remaining to arrive at the fractional quantity $\frac{2}{5}$. She demonstrated a transitional strategy between the more common strategy of Additive Coordination and Multiplicative Coordination.

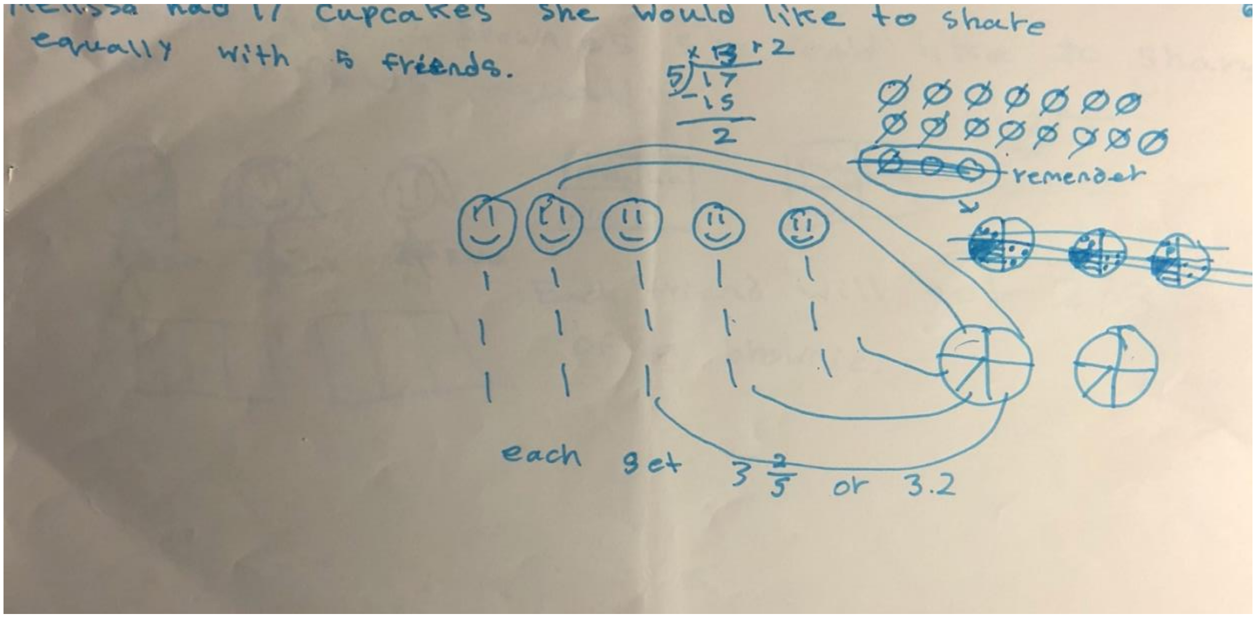


Figure 47. Susan transitional strategy. Retrieved from Atlas.ti, P 34.

At this point, our Equal Sharing problems changed from Equal Sharing problems where the solutions were greater than one to solutions less than one. In solving the Equal Sharing problem where two brownies are shared with three friends, Susan demonstrated clearly her multiplicative thinking. This was observed in a conversation with a peer.

> Cindy: One third, one third, one third…one third, one third, one third of both brownies. Of each brownie.
>
> Susan: Wouldn't that make them get one third twice, so that's two thirds?
>
> Cindy: Of one brownie everyone gets one third. That's what I mean. Each person will get one of each (she stresses) brownie.
>
> Susan: It's the same thing.
>
> Cindy: Yes, it is the same thing (P 32.6).

From this conversation, Susan demonstrated clearly her multiplicative thinking. She understood the multiplicative relationship between the numerator and the denominator. She understood that two thirds is the same as two groups of one third. Given a similar Equal Sharing problem where seven students shared five packages of clay, Susan expressed confusion. When her classmate asked how she could help her, Susan responded “There are five packages but seven students.” (P 65.3). It appears that Susan realized for the first time that the number of items to be shared was less than the number of sharers. In Figure 48, you can see that she drew five packages of clay under each student. She had successfully solved a similar situation the day before. What was the difference? Was it a familiarity with thirds and not sevenths? I found this surprising. Her partner then explained to her that she had to divide the clay packages into sevenths (P 65.3).

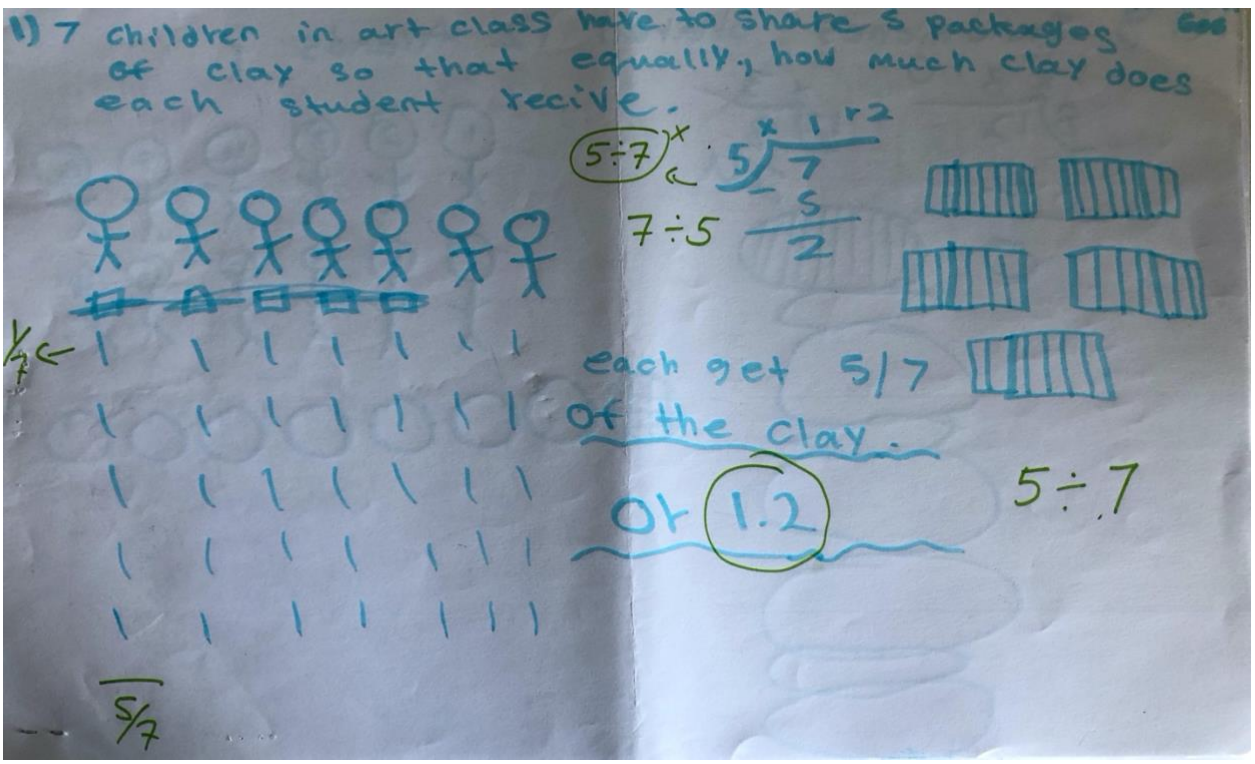


Figure 48. Susan’s solution to five packages of clay shared with seven students. Retrieved from Atlas.ti, P 34.

After solving the word problem, her partner asked her to explain why she thought the answer was five sevenths. Susan accurately explained that “when you split the five clays into sevenths, each of them gets five” (P 65.6). Also, of interest to note, was the fact that Susan’s division statements for both Equal Sharing problems were incorrect. See Figure 48 for an example. This was a common error amongst the majority of the students in the case study. I wondered if this inhibited the students’ ability to relate fractions as a division. Empson and Levi (2011) stressed the importance of students having experience with multiplication and division story problems with whole numbers prior to introduction of Equal Sharing problems. I would argue that this experience included the proper notation of the division statement. This would, in my opinion, facilitate the progression of recognizing a fraction as a quotient.

The following day, given the following question, ‘16 students need to share 12 sticks of clay. If they share the clay equally, how much clay would each student get?’, Susan demonstrated her relational thinking. That is, she used her knowledge of multiples and factors to arrive at an efficient strategy for solving the question. My conversation with Susan went as follows:

> Susan: I think the answer is twelve sixteenth
>
> Me: I think you are correct. Now tell me how you got to that solution?
>
> Susan: If there was 2, there would be one, two sixteenth pieces but there are twelve students so twelve sixteenths (P 83.1).

Susan’s strategy can be seen in Figure 49. Her strategy demonstrated her understanding that fractions are a unit that can be counted. She stated “one, two sixteenth pieces”, a concept relatable to the measure construct of fraction. Empson and Levi (2011) describes Susan’s strategy as a Ratio Strategy because Susan was able to reason that “If there was 2, there would be …two sixteenth pieces but there are twelve students so twelve sixteenths” (P 83.1). She recognized the 1:1 ratio between the number of sixteenths and the number of students.

.

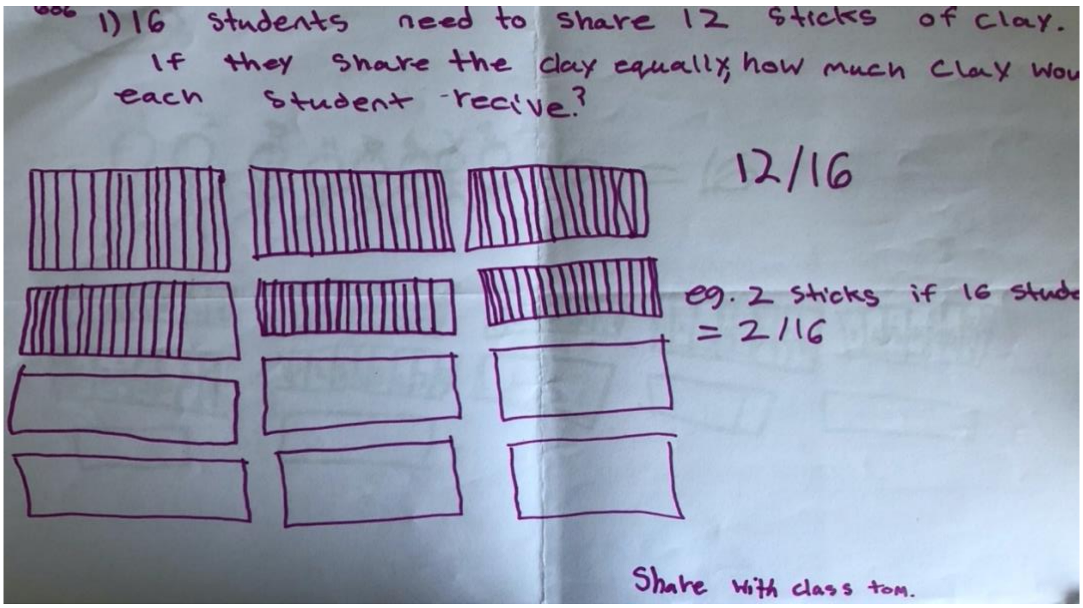


Figure 49. Susan’s Ratio Strategy to solve 12 clay sticks shared with 16 students. Retrieved from Atlas.ti, P 74.

After one more day of Susan showing her multiplicative thinking through Equal Sharing problems, the focus turned to Multiple Groups problems. Similar to Paul, exposure to Multiple Groups problems, particularly measurement type, highlighted the importance of naming fractions to be successful. Given the word problem, ‘Ms. Wright wants to feed each of the children she babysits one quarter of a KitKat. If she babysits 7 children, how many KitKats should she buy?’ Susan eventually answered correctly but not before correcting her thoughts about how many ‘pieces’ made up the whole. Figure 50 shows Susan’s solution to the word problem.

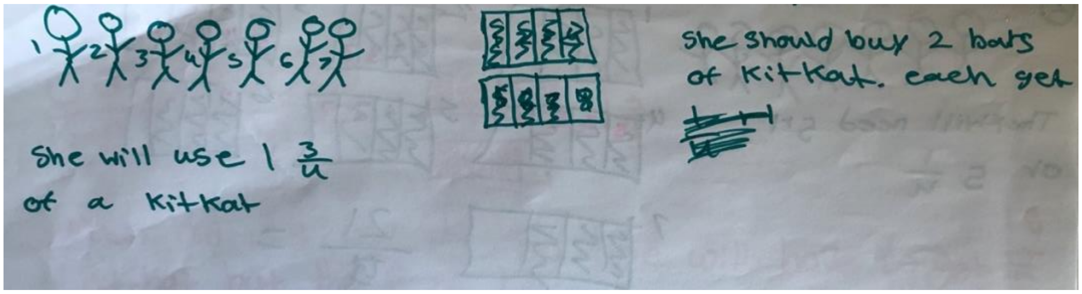


Figure 50. Susan's solution to first Multiple Groups problem. Retrieved from Atlas.ti, P 132.

She assigned each of the seven students a fractional piece of the KitKat and labelled them one through eight. She concluded that Ms. Wright would need to purchase 2 KitKats. In response to my question of how much of the two KitKats would be eaten, the following discussion ensued:

Susan: One fourth

Me: Only one fourth?

Susan: Seven eighths

Me: How did you get seven eighths? Can you shade in for me the amount of KitKat that will be used out of the 2 KitKat

She shades in one whole kitkat and three fourth of the second KitKat.

Me: What fraction does that represent?

Susan: Seven eighths

Me: Can you explain?

Susan: There are seven shaded and there are eight pieces. Oh! Wait! No! One whole and three fourth

Me: Why did you change from seven eights to one whole and three fourths?

Susan: Because there is one whole bar shaded.

Me: How many parts are in the whole?

Susan: Four (P 125.1).

Susan's discussion reinforced what Empson and Levi (2011) believe Multiple Groups problems afford —– the ability for students to reason explicitly about the relationship between the unit fraction and its whole. Reinforcing this relationship will further enhance students' relational thinking (Empson & Levi, 2011) as well as the measure construct of fraction (Van de Walle et. al. 2016). This is a construct Susan has begun to explore as she identifies unit fractions and counts. In fact, it was at this point that I realized that benefits afforded through Multiple Groups problems to facilitate an understanding of fraction through the measure construct. Four days later, the word problems focused on Multiple Groups: Multiplication problems. The majority of the students found it easier to solve Multiple Group: Multiplication problems than Multiple Groups: Measurement problems. Susan presented relational thinking that was not observed by any of the other students. Figure 51 illustrates an example of her relational thinking when solving the following question 'I am making sub sandwiches for friends. There will be 12 friends eating sub sandwiches. Each friend will get $\frac{3}{4}$ of a sub. How many sub sandwiches do I need?" Susan direct modelled the number of subs needed for 6 friends and then doubled the amount to arrive at her answer of $\frac{36}{4}$. She then simplified her answer to 9 wholes.

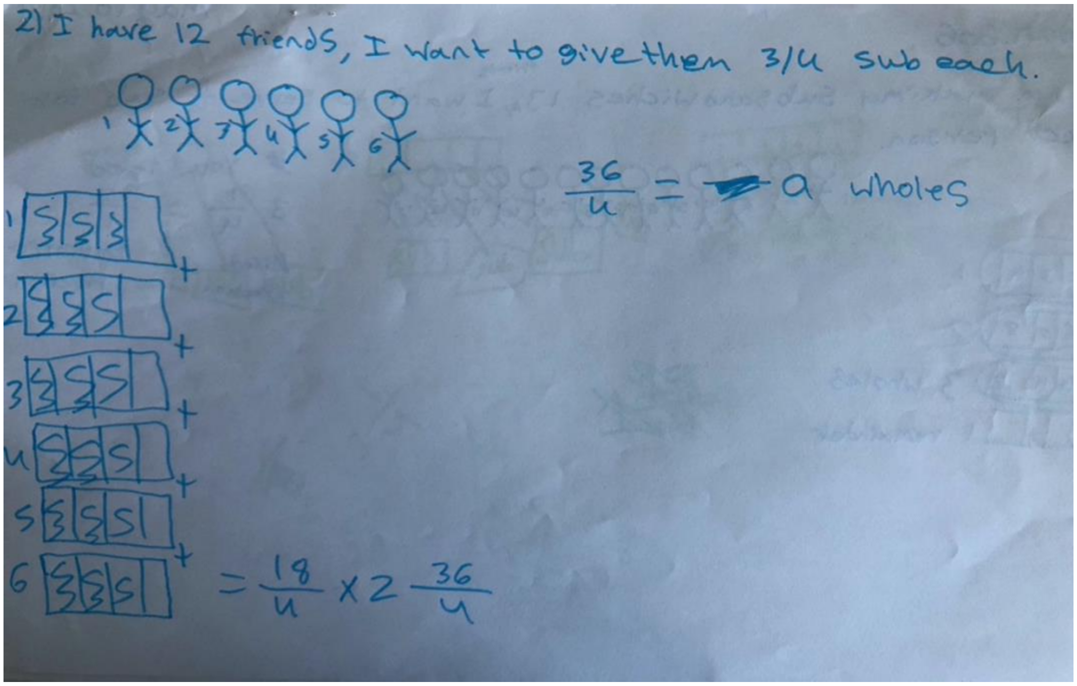


Figure 51. Demonstration of Susan's relational thinking. Retrieved from Atlas.ti, P 174.

After a week of no lessons, Susan's relational thinking continued to develop. Figure 52 shows her solutions to a series of word problems presented to help develop students' relational thinking. The word problem presented was 'It takes __ m of fabric to make a pillow. How much meter of fabric would it take to make ___ pillows? [$\frac{1}{3}$,3], [$\frac{2}{3}$,3], [$\frac{2}{3}$,9], [$\frac{2}{3}$,15]'.

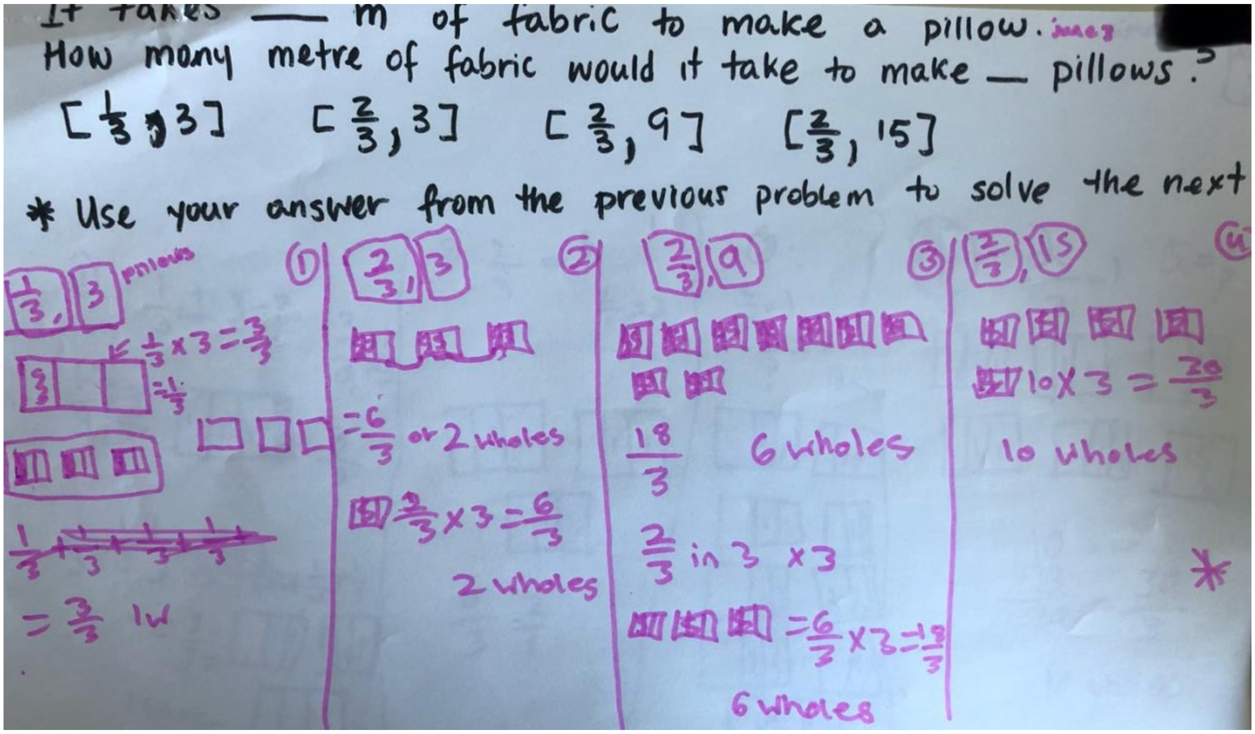


Figure 52. More examples of Susan's relational thinking. Retrieved from Atlas.ti, P 184.

In her solutions, Susan demonstrated her knowledge that $\frac{1}{n}$ x $n = \frac{n}{n}$ as well as her intuitive knowledge of the associative property when she reasons that $\frac{2}{3}$ x 9 = $\frac{2}{3}$ x 3 x 3. Susan definitely demonstrated the effectiveness of Multiple Groups problem in extending students' understanding of fractions in terms of mathematical relationships. In contrast to Paul's progression, Susan progressed from a Transitional Multiplicative Coordination strategy to a Multiplicative Coordination strategy over a series of three lessons as shown in Figure 53.

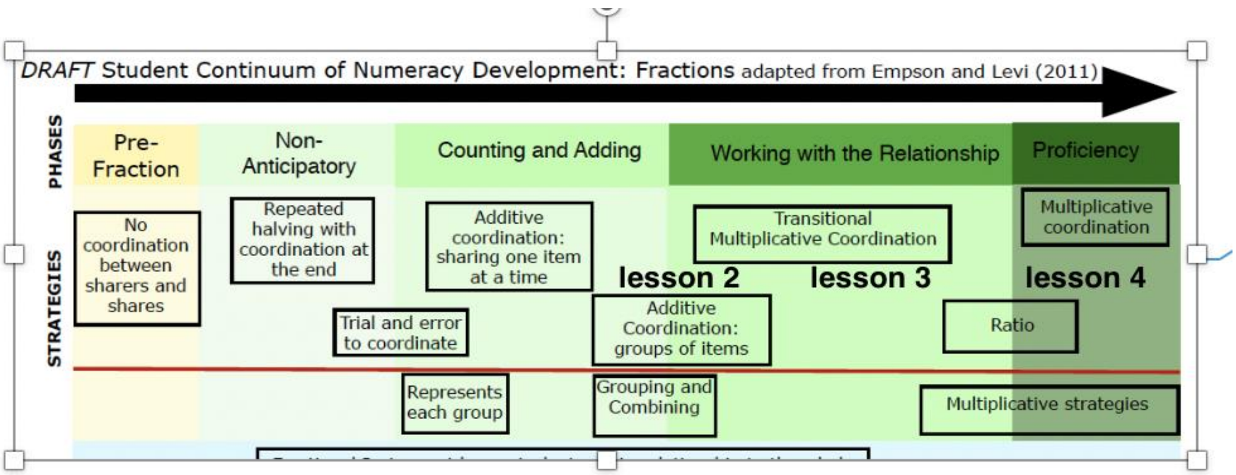


Figure 53. Susan's progression after four lessons with Equal Sharing problems. Adapted from Lawson, et al. (2019) Adaptation of Empson, S. and Levi, L. (2011) Extending Children's Mathematics. Portsmouth, NH: Heinemann.

Angela: The Roadblock of Naming Fractions

The analysis of Angela's development will be focused mainly on the role of recognizing and naming of fractions in developing her multiplicative thinking. Given the Equal Sharing problem where 2 brownies are shared amongst 3 friends, Angela and her partner were given scissors and construction paper to facilitate their thinking and problem solving. The following discussion was observed after Jessica, her classmate, decided that they should cut each brownie, represented by the construction paper, into three equal parts.

Angela: So, I think it is quarters. Did we cut them into quarters?

Jessica: I don't know

After cutting the two brownies into 3 pieces, Jackie labelled them 1, 1, 2, 2, 3, 3 and proceeded as follows.

Jessica: For number one for the first person they get 2 pieces and for number two they get 2 pieces for the second person and for the third person you get also 2. So that is equal. So, each person gets three parts of one brownie.

Me (pointing to one of the pieces): What fraction is this?

Jessica (looking to Angela): One over ……

Angela: Four?

Jessica: Three? three. Two over three I think

Angela: Two thirds?

Me: Just this one piece

They both pause

Angela: One third? (P 49.7).

This conversation highlighted both Jessica's and Angela's inconsistencies in naming fractions. These inconsistencies were observed frequently with Angela throughout the case study. Another example brought to the forefront the importance of the language used when developing students understanding of fractions. Given the Multiple Groups problem, 'I am making sub sandwiches for friends. There will be 13 friends eating sub sandwiches. Each friend will get one quarter of a sub. How many sub sandwiches do I need?' Angela went straight to multiplication. She multiplied one quarter by thirteen to arrive at an answer of thirteen fourths. See Figure 54.

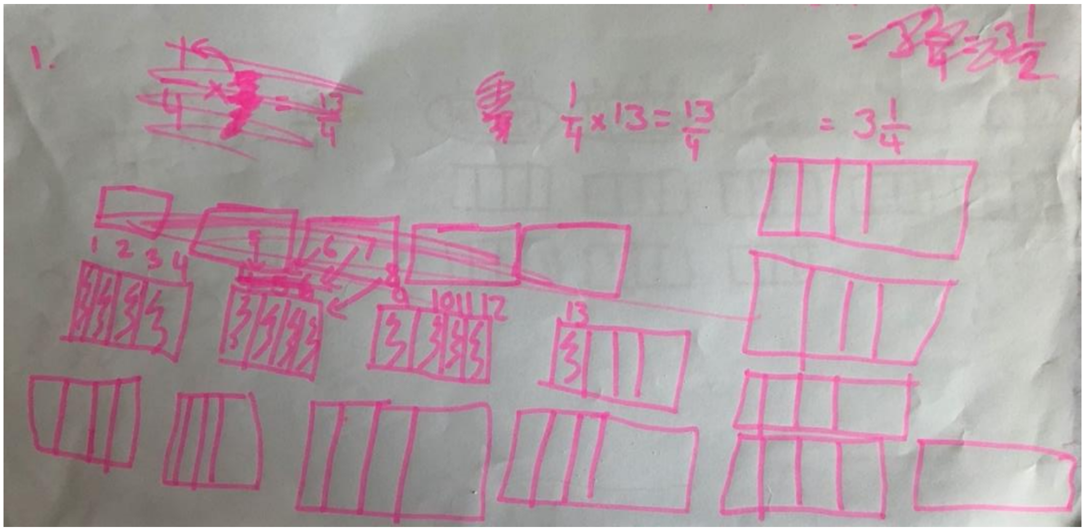


Figure 54. Angela’s solution to a Multiple Groups problem. Retrieved from Atlas.ti, P 164.

In explaining to her partner, Emma, why the denominator stays the same when multiplying, Angela informs her that ‘there is also a one at the bottom’, referring to the 13 wholes. She further explains that ‘thirteen is still thirteen by one’. Angela frequently referred to fraction in this manner. For any fraction $\frac{a}{b}$ she referred to it as *a* by *b*. When I asked how many whole subs there would be, both Angela and Emma were not able to answer. Emma decided to draw thirteen subs and divide them into fourths. Angela followed suit. After discussions with myself, Angela concluded as follows ‘so its thirteen one by four, three one by four’. It is my thought that Angela’s naming of fractions in this manner reflected her inability to see fractions as a quantity. For example, stating $\frac{13}{4}$ as ‘thirteen by four’ deprived her of the ability to recognize the fraction as an iteration of one fourth thirteen times, thirteen one fourths. Seeing it in this manner would have facilitated a more conceptual understanding of the multiplication rather than the procedural explanation provided to her classmate, Emma. Kent et al. (2015) argue that ‘if students do not see fractions as quantities, they have difficulty making sense of operations on quantities such as adding and multiplying’ (p. 89). This was quite evident in Angela’s case. She was not able to make sense of the multiplication she did procedurally. My belief was reinforced when I asked her how many whole subs would be needed. It is my assumption that if I had asked her to convert the improper fraction to a mixed fraction, she would have been more readily able to answer. As I reflect on Angela's progression of multiplicative thinking compared to her classmates, the difference was startling. Throughout the case study, the development of Angela's multiplicative thinking was not as evident as her peers. She relied heavily on her procedural knowledge and found it challenging to explain her thinking behind her procedures. It is my thought that her communication mathematically, such as stating $\frac{13}{4}$ as 'thirteen by four', reflected her lack of understanding of the relationship between the numerator and the denominator – that is, seeing fractions as a quantity. By stating thirteen fourths as thirteen by four, Angela was prevented from connecting her model (see Figure 54) to an iteration of the unit fraction, one fourth, thirteen times, hence thirteen fourths. So, not only did it reflect her lack of understanding of the relationship between the numerator and the denominator but inhibited the progression of this understanding afforded through Equal Sharing and Multiple Groups problems.

Frank: The Roadblock of Procedural Knowledge

Similar to Angela, Frank experienced difficulty with developing his relational thinking. He was able to identify and name fractions but had difficulty explaining his procedural strategies. I believe that Frank, like Angela, was not yet able to see a fraction as a quantity. In fact, Frank often resorted to decimals when solving Equal Sharing problems and found Multiple Groups: Measurement problems challenging. Figure 55 shows a typical solution from Frank when solving Equal Sharing problems. He would solve the problem as a decimal and then express his answer as a fraction. He refused to use the provided scissors and construction paper to facilitate his thinking and further his conceptual understanding.

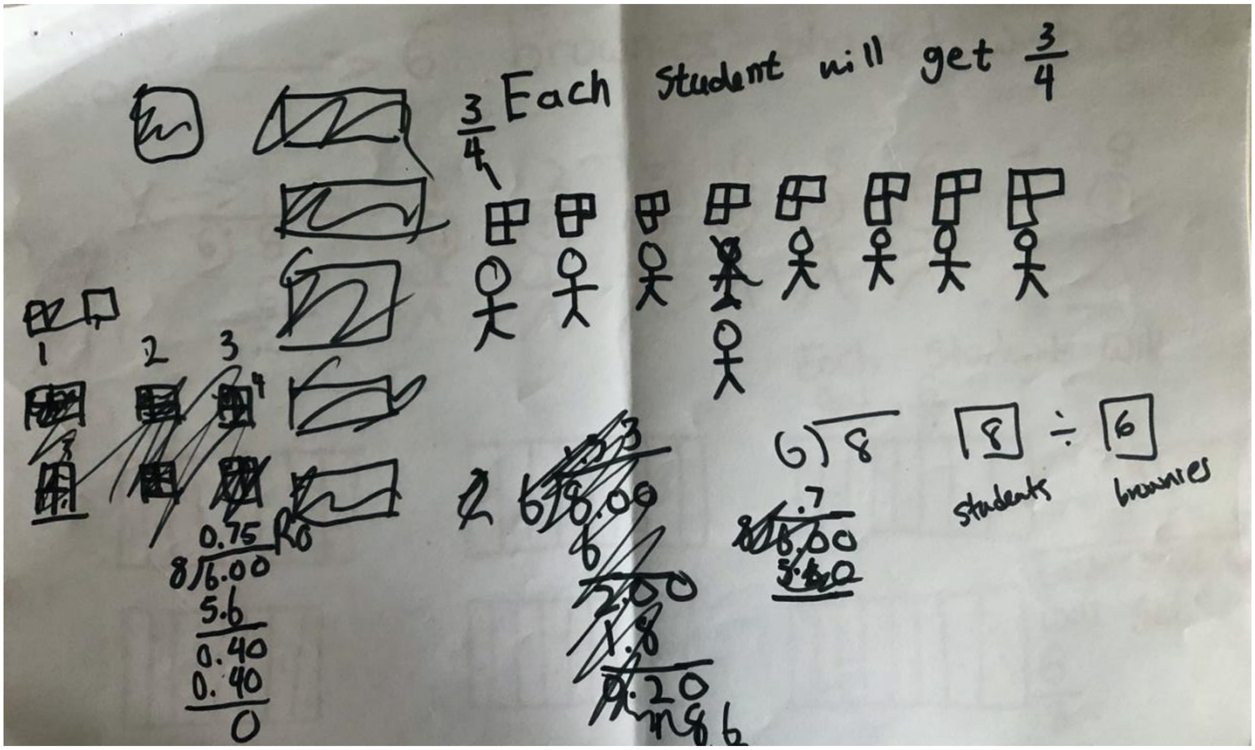


Figure 55. Frank’s solution to six brownies shared with eight friends. Retrieved from Atlas.ti, P 94.

Notice that his solution is three fourths and not the common answer provided by most students of six eighths in the study. He is not really using his model to think with, instead they are a drawing after the fact. Frank relied heavily on his procedural understanding of division and conversion of a decimal to a fraction. He recognizes that his decimal quotient of 0.75 is equivalent to three fourths and so expressed his answer as such. When Frank was asked to explain his thinking, his frustration was visible (P 93.1). After numerous attempts to make sense of the three fourths when given eight brownies for six students, he settled with “it just popped into my head”. In response to my request to visually represent his solution Frank further demonstrated his challenges in making sense of his strategy and connecting it to the Equal Sharing problem. I was curious as to what the result would have been had the fraction not be familiar. Figure 56 shows one such example.

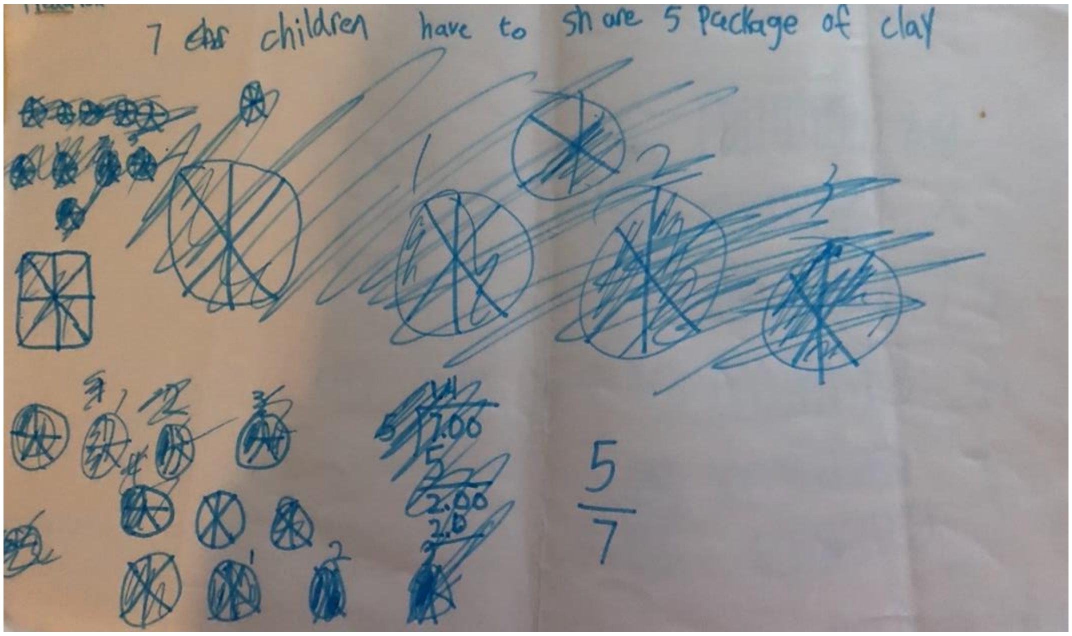


Figure 56. Frank's attempt to solve an Equal Sharing problem with unfamiliar fractions. Retrieved from Atlas.ti, P 60.

In this example, Frank attempts to solve the Equal Sharing problem of 5 packages of clay shared with 7 children. Throughout the video recording (P 59) Frank expresses his frustration and challenges in expressing his work. His other two partners suggested to try drawing rectangles and he quickly rejected the idea stating he liked circles. His evident frustration provided further evidence of Frank's reliance on procedures and not making sense of the strategies he used or the word problem. Frank definitely is not able, as yet, to see a fraction as a quantity or a relationship between the numerator and the denominator. It is my thought that his refusal to use the scissors and paper or apply his peers' suggestions to use rectangles further inhibited his opportunity to see the multiplicative relationship and further develop his multiplicative thinking.

In summary, these students' discussions and solutions demonstrated the effectiveness of Equal Sharing and Multiple Groups problems in the development of students' understanding of fractions. The three main common observations, examining students' discussions and solutions, were: the significant importance of understanding the multiplicative relationship between the numerator and the denominator and not just an ability to express a quantity fractionally; the importance of delaying the introduction of naming fractions until students are able to identify fractions as quantities, such as stating 2 thirds; the effectiveness of Multiple Groups problems in developing students' understanding of fraction through the measure construct. Multiple Groups problems allow students, given problems involving unit fractions, an opportunity to count and iterate, a key factor of the measure construct (Van de Walle et al., 2016).

Chapter 6: Discussion and Conclusion

Summary of the Major Findings

The purpose of this research was to examine the impact of Equal Sharing and Multiple Groups problems on students' understanding of fractions. To facilitate the evaluation of the impact, the following questions were considered:

- Does the development of participants' understanding follow a general progression with this intervention?
- Does instruction through Equal Sharing and Multiple Groups problems facilitate an understanding in the other fraction constructs (i.e., part-whole, measure, ratio and operator)?
- Does instruction through Equal Sharing and Multiple Groups problems facilitate an understanding of the underlying fraction concepts (i.e., partitioning, equivalence and unit forming)?
- How do students' experiences with whole number division impact their progression in solving Equal Sharing problems?

Along with these questions the following propositions were put forth (see Chapter 3).

- Equal Sharing and Multiple Groups problems support the development of an understanding of fractions.
- Participants' strategies for solving Equal Sharing and Multiple Groups problems progress in a similar manner suggested by Empson and Levi (2011).
- Instruction through Equal Sharing and Multiple Groups problems facilitates an understanding of the other constructs
- Students' experiences with multiplication and division problems have a positive relationship to the strategies used to solve Equal Sharing problems.

The major findings to each of these questions will be addressed. A conclusion will then be made on the impact of the Equal Sharing and Multiple Groups problems on students' understanding of fractions and examining how closely they align to the propositions put forth.

Does the Development of Participants' Understanding Follow a General Progression with this Intervention?

Empson and Levi (2011) observed that students' strategies for solving Equal Sharing and Multiple Groups problems follow a general and predictable progression. This was observed in students' strategies when solving both Equal Sharing and Multiple Groups problems. For example, I observed Paul progressing from a 'Non-Anticipatory Sharing' strategy to an 'Additive Coordination: Sharing One Item at a Time' strategy. This was then followed by a transitional strategy where he partitioned only one item and then generalized across all items. This strategy is considered a transitional strategy as it is an intermediary step between the 'Additive Coordination' strategy and the 'Multiplicative Coordination' strategy. A similar progression was observed while students were solving Multiple Groups problem. Students progressed from 'Direct Modelling' to 'Multiplicative Strategies'. Only one student was observed using the intermediary 'Grouping and Combining' strategy. This observation aligns perfectly with the proposition that student strategies to solving Equal Sharing and Multiple Groups problems follows a progression as observed by Empson and Levi (2011). Figure 57 shows a draft of the trajectory students progress through as they solve the progression through both Equal Sharing and Multiple Groups problems

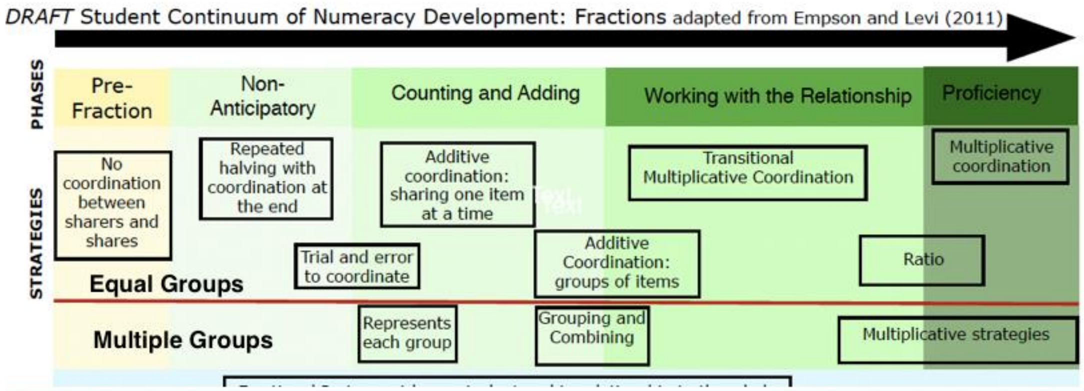


Figure 57. Students' progression when solving Equal Sharing and Multiple Groups Problems. Adapted from Lawson et al.'s draft figure of Empson & Levi (2011)

What does this progression tell us? This progression allows us to appreciate the opportunity provided when teachers pose Equal Sharing and Multiple Groups problems (without giving instructions for solutions) in order to develop students' relational thinking, in particular their multiplicative thinking. Multiplicative thinking is central to students' understanding of a fraction as a quantity (Empson & Levi, 2011). For example, it allows students to understand that the fraction quantity two thirds is in reality a multiplicative relationship between two, the numerator, and three, the denominator. In other words, they are able to see two thirds as two one-thirds rather than as a region of two whole shaded parts embedded within a larger whole (Hackenberg & Lee, 2012).

Does Instruction Through Equal Sharing and Multiple Groups Problems Facilitate an Understanding in the Other Fraction Constructs?

Empson and Levi (2011) take advantage of students' intuitive understanding of sharing to introduce students to fractions through Equal Sharing problems without direct instruction. Equal Sharing problems primarily focus on the development of an understanding of fraction through the quotient construct. Multiple Groups problem, on the other hand, focus on deepening students' understanding of fractions in terms of mathematical relationships (Empson & Levi, 2011). It could further be argued that Multiple Groups problems facilitate the development of students' understanding of fraction through the measure construct. Van de Walle et al. (2016) explain that in the measure construct, a unit fraction is selected and counted. So, for the fraction $\frac{5}{8}$, the unit fraction $\frac{1}{8}$ would be counted 5 times. Multiple Groups problems, in particular the measurement type, facilitate this type of understanding. They facilitate students' ability to see fractions as a multiple of a unit fraction (Steffe & Olive, 2010). However, as researchers such as Van de Walle et al. (2016) and Kieren (1980) have noted, a deep understanding of fraction requires an understanding of all the different fraction constructs. The five different constructs of fraction are: part-whole, measure, quotient, ratio and the operator. From analysis of the pre and post assessment, improvement in all constructs were observed with the exception of the measure construct. This was observed through the percentage increase in students' correct answers in the postassessment, as seen in Figure 12. Some students' ability to solve problems with a ratio strategy increased. Upon reflection, this aligns well with students' progression in their relational thinking through Equal Sharing and Multiple Groups problems. As students' progress in their strategies to solve these types of problems, their ability to think relationally and hence multiplicatively, improved. This ability to think multiplicatively, I believe, improved their performance in solving problems related to the ratio construct.

In contrast to the ratio construct, the measure construct saw little to no apparent improvement. One possible reason could be the limited amount of time students had to practice Multiple Groups problems, specifically the measurement type. Multiple Groups problems provide opportunities for students to iterate or count partitions. This type of iteration is required to solve measurement type problems when a unit is identified and then iterated and counted. What if students are not able to name or identify the unit to be iterated or counted? This became an important finding in this case study. From my observation, it would appear that the manner in which students name fractions impacts the development of their multiplicative thinking.

In fact, Hackenberg (2013) identifies three multiplicative phases that students progress through towards a multiplicative understanding of fraction. These three multiplicative conceptual phases he abbreviates as MC1, MC2, and MC3. Students' progression from the basic MC1 to the more advanced MC3 is based on their ability to partition, iterate, and disembed. Disembedding is the ability to take a part out of a whole without mentally destroying the whole (Steffe & Olive, 2010). In general, Hackenberg (2013) describes that students at MC1 are able to partition but not make a connection between the part and the whole. Students in MC2 go through a variety of stages but eventually are able to partition, disembed, and iterate, whereas the MC3 students are able to split (a combination of partitioning and iterating) as well as disembed. He explains further that it is at MC2 stage that students are able to see fractions as a measurable quantity. It is at this stage that naming of fraction is developed. Students are able to identify a unit fraction and iterate, thereby developing the idea of fraction as a quantity. It is my thinking that students in this case study had previously worked with the standard fraction notation without an opportunity to develop completely within the MC2 stage. This missed opportunity to develop completely within the MC2 stage inhibited their ability to think multiplicatively about fractions and hence fraction as a measurable quantity. For example, as shown in Figure 44 and explained, Paul was not able to explain his solution of $\frac{30}{3}$. He was able to partition the fabric into thirds and identify two thirds. He then direct modelled his iteration of two thirds fifteen times writing thirty thirds but was not able to make the connection between his iterations of two thirds fifteen times with his written solution of $\frac{30}{3}$. Paul demonstrated the preliminary stages of MC2. It is my belief that with more practice and problem solving, he would be able to make the connection between his iterations and his written fraction demonstrating his ability to think of, in this case, thirty thirds as a measurable quantity and multiple of the disembeded two thirds.

It may also be that the measure construct assessment items were not sufficiently well written to discern any growth in this topic. If one of the questions was a **unit** fraction multiple groups division problem such as, ‘I have 5 brownies. I am going to give $\frac{1}{3}$ of a brownie to each friend. How many friends can I feed?’ I would likely have seen some improvement from the pre to the postassessment.

Despite these limitations, specifically the impact of Equal Sharing and Multiple Groups problem on the measure construct, I would agree with the proposition that these word problems facilitate an understanding of the other constructs of fractions.

Does Instruction Through Equal Sharing and Multiple Groups Problems Facilitate an Understanding of the Underlying Fraction Concepts?

Researchers, such as Behr et al. (1983) and Kieren (1993), identified partitioning, order and equivalence, and unit forming as basic thinking tools for understanding rational numbers. The results of the pre and postassessment demonstrated some improvement in the underlying concepts of partitioning. However, the same could not be observed with both the underlying concepts of order and equivalence as well as unit forming. It is my thought that order and equivalence as well as unit forming, required more practice with Equal Sharing and Multiple Groups problems, in particular the measurement type. Empson (1999), as previously stated, argues that students' concept of equivalence deepens as students solve Equal Sharing problems that lend themselves to discussions on equivalence. In addition, the underlying concept of unit forming could be further developed had more time be given to discussing solution strategies where students solved Equal Sharing problems using an 'Additive Coordination' strategy requiring the addition of a variety of unit fractions (Empson & Levi, 2011). This strategy was not common within this case study as students opted to verbally explain but not represent their thinking fractionally. An opportunity inhibited, perhaps, by their inability to accurately name fractions.

How do Students' Experiences with Whole Number Division Impact their Progression in Solving Equal Sharing Problems?

Empson and Levi (2011) stress the importance of students having experiences with both multiplication and division story problems prior to any instructions in fractions. This importance became very evident as I observed students trying to make sense of the remainder in division word problems as well as their attempt to write a division statement when the divisor was smaller than the dividend. Some students in this case study expressed their remainder as a decimal. For example, for a division question resulting in an answer of 3 remainder 2, a few students wrote incorrectly the answer as 3.2. In another instance, a student proficient in division answered 0.75 then expressed it as $\frac{3}{4}$, when solving a problem where six brownies were shared with eight students. He was not able to make the connection between six eighths and three fourths and had difficulty expressing his solution with a diagram. See Figure 55. This lack of experience in solving multiplication and division word problems, as more than procedures, with whole numbers in context, inhibited students' development of the multiplicative thinking necessary to see fraction as a quantity. So, although noted in the proposition, students' experiences with multiplication and division word problem impact their solution strategies, it is more their ability to make sense of their procedure and the numbers in the word problem. This lack of experience, beyond a procedure, was further exacerbated by their confusion when writing a division statement. Students believe and may have perhaps been erroneously taught that when writing division statements, the larger number always goes first. So, in the Equal Sharing problem where six brownies are shared with eight students, their diagram might illustrate six brownies shared with eight students but express it as $8 \div 6$. This misconception further inhibited students transition to a 'Multiplicative Coordination' strategy when solving Equal Sharing problems. In fact, Empson and Levi (2011) state that students who understand why $6 \div 8 = \frac{6}{8}$ have a relational understanding of fractions and division. The students in this case study were not yet afforded this opportunity to make this connection as they often expressed their division statement incorrectly.

Conclusion

This case study, despite its limitations of small sample size and length of time, demonstrated, as noted in the propositions, that Equal Sharing and Multiple Groups problems may be effective in developing students' understanding of fractions. Some key factors are vital for the success of this development of understanding. It is important that students, prior to their introduction to Equal Sharing problems have had experiences with multiplication and division contextual word problems (Empson & Levi, 2011) with a focus on the interpretation of the remainder as well as notation for division. In addition, Equal Sharing and Multiple Groups problems should be presented in a manner where students are able to discuss and problem solve with their partners, and careful attention is given to listening to their reasoning and thinking. Without careful attention to students' verbal explanations the development of students' relational thinking might be lost. It is difficult to ascertain the development of students' multiplicative thinking when a fraction quantity such as two thirds is written versus when we hear students articulate how they counted the number of thirds. Finally, it is important to delay the need for students to express their solutions using a fractional notation and focus more on their thinking and strategies as they progress from an additive approach to a multiplicative approach. Once students are able to demonstrate their multiplicative thinking verbally, the naming of fractions can be meaningfully introduced (Empson & Levi 2011).

In summary, evidence was gathered to suggest that the introduction to fractions through the quotient construct facilitates students' conceptual understanding of fractions. Equal Sharing and subsequently Multiple Groups problems appear to provide an effective approach to the development of both the quotient and measure construct. In fact, Kieren (1993) in her framework of rational number knowing (See Figure 10) highlights that both the quotient and measure constructs rely on the three underlying concepts for the understanding of fraction: partitioning, order and equivalence and unit forming. These underlying concepts provide the basic thinking tools for understanding rational number (Kieren, 1993). Therefore, it could be argued that Equal Sharing problems, focusing on fair sharing problems (the quotient construct), and Multiple Groups problems, focusing on the measure construct, provides opportunity to develop, in a meaningful manner, the foundation for an understanding of fractions.

Future Consideration

There is still much research needed on the development of students' understanding of fraction as a multiplicative relationship between the numerator and the denominator. There were quite a few limitations in this research. Despite these limitations, the effectiveness of Equal Sharing and Multiple Groups problems was observed. Some of the limitations of this study were time, sequencing of lessons and the selection of word problems for the pre and postassessment as well as for the Equal Sharing and Multiple Groups problems within the case study.

The case study was abruptly ended due to changes in my role as a Student Success teacher. These changes affected the sequence of lessons as well as prevented the administration of a retention test. In addition, due to the nature of my teaching role, lessons were provided on alternative weeks disrupting the learning process every other week. Upon reflection of the word problems chosen for the pre and postassessments as well as those within the case study, these problems could have been more carefully selected, sequenced and worded to support the development of students' multiplicative thinking and avoid the possibility of ambiguity.

Despite the challenges of my role as Student Success teacher and the choice of some weak word problems, the study highlighted the effectiveness of Equal Sharing and Multiple Groups word problems on students' understanding of the multiplicative relationship between the numerator and denominator and hence the development of students relational thinking – a key factor in not only the understanding of fractions but algebra as well (Empson & Levi, 2011). In addition, it highlighted, through students' discussions of their solutions, the importance of when and how to introduce the standard fraction notation in order for students to have a richer understanding of fractions, as it pertains to the quotient and measure construct of fractions.

References

Anghileri, J., Beishuizen, M., & van Putten, K. (2002) From informal strategies to structured procedures: mind the gap! *Educational Studies in Mathematics, 49,* 149-170. https://doi.org/10.1023/A:1016273328213.

Ball, D. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-196). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Baskarada, S. (2014). Qualitative case study guidelines . *The Qualitative Report, 19*(40), 1-18. Retrieved from https://nsuworks.nova.edu/tqr/vol19/iss40/3.

Baxter, J. A., Olsen, D., & Woodward, J. (2001). Effects of reform-based mathematics instruction on low achievers in five third-grade classrooms. *The Elementary School Journal , 101*(5), 529-547. doi:10.1086/499686.

Baxter, P., & Jack , S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The Qualitative Report , 13*(4), 544-559. Retrieved from https://nsuworks.nova.edu/tqr/vol13/iss4/2.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis - Emphasis on the operator construct. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-125). New York: Academic Press.

Boyce, S., & Norton, A. (2016). Co-construction of fractions schemes and units coordinating structures. *The Journal of Mathematical Behavior, 41*, 10-25. doi:10.1016/j.jmathb.2015.11.003.

Bruce, C., Chang, D., Flynn, T., & Yearley , S. (2013). *Foundations to Learning and Teaching Fractions: Addition and Subtraction*. Retrieved from Engage Math website: http://www.edugains.ca/resourcesDP/Resources/PlanningSupports/FINALFoundationstoLearningandTeachingFractions.pdf

Hiebert, J. & Carpenter, T. P., (1992). Learning and teaching with understanding. In D. Grouws, *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York , NY: Macmillan.

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understanding of fractions. *Educational Studies in Mathematics*, 64, 293-316. doi:10. 1007/s10649-006-9036-2.

Charles, K., & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics, 43*, 191-221. doi:10. 1023/A:1017513716026.

Clements , D. H., & Battista, M. T. (1990). Constructivist learning and teaching. *The Arithmetic Teacher*, *38*(1), 34-35.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first grade classroom. *Cognition and Instruction*, *17*(3), 283-342. Retrieved from http://www.jstor.org/stable/3233836.

Empson, S. B. (2003) Low-performing students and teaching fractions for understanding: An interactional analysis. *Journal for Reasearch in Mathematics Education, 34*(4), 305-343. https://doi.org/10.2307/30034786

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth, NH: Heinemann.

Empson, S. B., Levi , L., & Carpenter , T. P. (2011). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai & E. Knuth (Eds.), *Early algebration (pp.* 409-428). doi:10. 1007/978-3-642-17735-4_22

Empson, S., Junk, D., Dominguez, H., & Turner, E. (2006). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63*(1), 1-28. Retrieved from http://www.jstor.org/stable/25472109

Franke, M. L., & Kazemi, E. (2001). Learning to teach mathematics: Focus on student thinking . *Theory into Practice, 40*(2), 102-109. doi:10. 1207/s15430421tip4002_4.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht: Reidel

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior, 32(*2013), 538-563. doi:10.1016/j.jmathb.2013.06.007

Hackenberg, A. J., & Lee, M. (2012). Pre-fractional middle school students' algebraic reasoning. L.R. Van Zoest, J.J. Lo, & J.L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 943-950). Kalamazoo, MI: Western Michigan University.

Hunt, J. H., & Empson, S. B. (2015). Exploratory study of informal strategies for equal sharing problems of students with learning disabilities. *Learning Disability Quarterly, 38*(4), 208-220. doi:10. 1177/0731948714551418.

Hunt, J., Westenskow, A., & Moyer-Packenham, S. (2017). Variations of reasoning in equal sharing of children who experience low achievement in mathematics: Competence in Context. *Education Science, 7*(37), 1-14. doi:10. 3390/educsci7010037.

Kent, L.B., Empson, S. B., & Nielsen, L., (2015). The richness of children's fraction strategies. *Teaching Children Mathematics, 22*(2), 84-90. doi: 10.5951/teacchilmath.22.2.0084.

Kerslake, D. (1986). *Fractions: Children's strategies and errors. A report of the strategies and errors in secondary mathematics project.* Windsor, Berkshire: NFER-NELSON.

Kieren, T. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. Lesh, *Number and measurement* (pp. 101-144). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Kieren, T. E. (1980). The rational number construct, its elements and mechanisms. In T. E. Kieren, *Recent research on number learning* (pp. 125-150). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T.P. Carpenter, E. Fennema, & T.A. Romberg, *Rational numbers: An integration of research* (pp. 49 - 84). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lamon, S. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *Roles of Representations in School Mathematics* (pp. 146-165). Reston, V.A.: National Council of Teachers of Mathematics.

Lamon, S. J. (2011). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers.* New York: Routledge.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-666). Charlotte, NC: Information Age Publishing.

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. *Educational Psychologist, 23*(2), 119-144. doi:10.1207/s15326985ep2302_4.

Mack, N. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P. Carpenter, E. Fennema, & T. A. Romberg, *Rational numbers: An integration of research* (pp. 85-105). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mack, N. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267-295. doi:10. 2307/749828.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*(1), 16-32. doi:10. 2307/749454.

Marshall, S. P. (1993). Assessment of rational number understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 261-288). Hillsdale, New Jersey: Lawrence Erlbaum Associates .

Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative research: A guide to design and implementation.* San Francisco, CA: John Wiley & Sons.

Mitchell, A., & Mathematics Education Research Group of Australasia. (2012). T*he Four-Three-Four Model: Drawing on Partitioning, Equivalence, and Unit-Forming in a Quotient Sub-Construct Fraction Task.* Mathematics Education Research Group of Australasia.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics.* Reston, VA.: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics* . Reston, VA: National Council of Teachers of Mathematics.

Paiget, J., Inhelder, B., Szeminska, A. (1960). *The child's conception of geometry.* London: Routledge and K. Paul.

Parrish, S., & Dominick, A. (2016). *Number talks. Fractions, decimals, and percentages: A mulitmedia professional learning resource.* Sausalito, California: Math Solutions.

Patton, M. Q. (2015). *Qualitative research & evaluation methods: Integrating theory and practice.* Thousand Oaks, California: SAGE Publications, Inc.

Peterson, P. L., Fennema, E., & Carpenter, T. (1988). Using knowledge of how students think about mathematics. *Educational Leadership, 46*(4), 42-46.

Piaget, J. (1995). Sociological studies London ; Routledge.

Post, T. R. Wachsmuth, I., Lesh, R., & Behr, M. J. (1985). Order and equivalence of rational n umbers: A cognitive analysis. *Journal for Research in Mathematics Education, 16*(1), 18-36. https://doi.org/10.2307/748970

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*(5), 307-317. doi:10. 2307/748675.

Rowley, J. (2002). Using case studies in research. *Management Research News, 25*(1), 16-26. Retrieved from https://pdfs.semanticscholar.org/4e18/426cc8767b4141c924236612aafaef75fa75.pdf

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119-150. doi:10. 1207/S1532690XCI2002_1.

Siemon, D. (2003). Partitioning: The missing link in building fractional knowledge and confidence. *Mathematics ~ making waves* (pp. 22-24). Adelaide: The Australian Association of Mathematics Teachers Inc.

Steffe, L., & Olive, J. (2010). *Children's fractional knowledge.* New York, NY: Springer Science and Business Media.

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 289-325). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Stylianides, A., & Stylianides, G. (2007). Learning mathematics with understanding: A critical consideration of the learning principle in the Principles and Standards for School Mathematics. *Montana Mathematics Enthusiast, 4*(1), 103-114. Retrieved from https://scholarworks.umt.edu/tme/vol4/iss1/8.

Thompson, P. W., & Saldanha, L. A. (n.d.). Fractions and mulitiplicative reasoning. In J. Kilpatrick , W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston: National Council of Teachers of Mathematics.

U.S. Department of Education (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel.* Retrieved from https://www2.ed.gov/about/bdscomm/list/mathpanel/index.html

Van de Walle, J. A., Karp, K. S., Bay-Williams, J. M., & McGarvey, L. M. (2016). *Elementary and middle school mathematics: Teaching developmentally* (5th ed.). Don Mills, Ontario, Canada: Pearson Canada Inc.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert, & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Hillsdale, NJ: Lawrence Erlbaum.

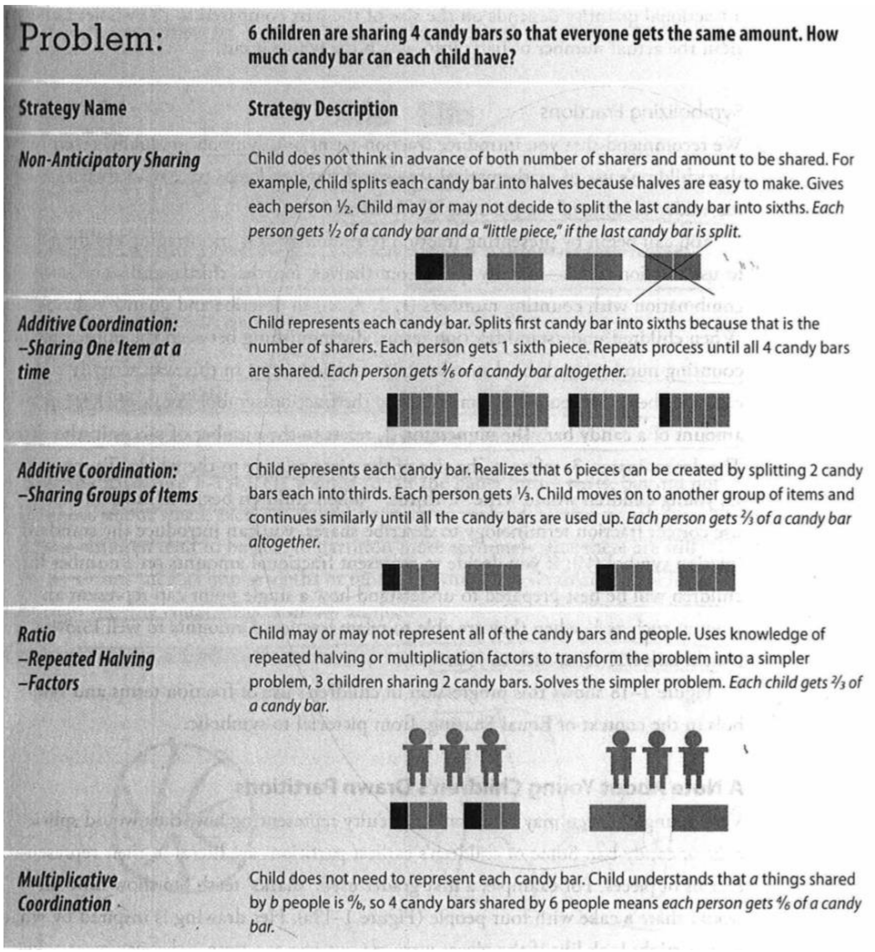
Yin, R.K. (2014). *Case study research design and methods* (5th ed.). Thousand Oaks, CA: Sage.

Appendices

Appendix A

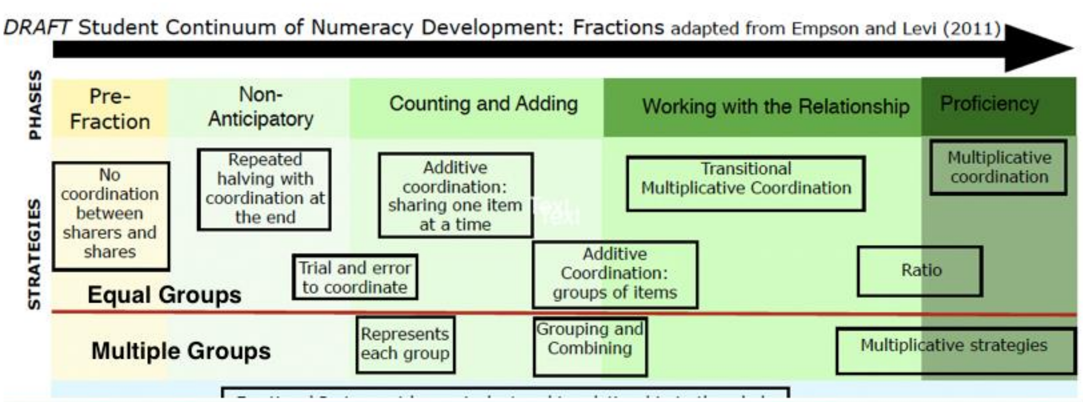
Progressive Strategies Used to Solve Equal Sharing Problems (Empson & Levi, 2011, p.25)

Figure A1



Problem: 6 children are sharing 4 candy bars so that everyone gets the same amount. How much candy bar can each child have?

Strategy Name	Strategy Description
Non-Anticipatory Sharing	Child does not think in advance of both number of sharers and amount to be shared. For example, child splits each candy bar into halves because halves are easy to make. Gives each person $\frac{1}{2}$. Child may or may not decide to split the last candy bar into sixths. *Each person gets $\frac{1}{2}$ of a candy bar and a "little piece," if the last candy bar is split.*
Additive Coordination: –Sharing One Item at a time	Child represents each candy bar. Splits first candy bar into sixths because that is the number of sharers. Each person gets 1 sixth piece. Repeats process until all 4 candy bars are shared. *Each person gets $\frac{4}{6}$ of a candy bar altogether.*
Additive Coordination: –Sharing Groups of Items	Child represents each candy bar. Realizes that 6 pieces can be created by splitting 2 candy bars each into thirds. Each person gets $\frac{1}{3}$. Child moves on to another group of items and continues similarly until all the candy bars are used up. *Each person gets $\frac{2}{3}$ of a candy bar altogether.*
Ratio –Repeated Halving –Factors	Child may or may not represent all of the candy bars and people. Uses knowledge of repeated halving or multiplication factors to transform the problem into a simpler problem, 3 children sharing 2 candy bars. Solves the simpler problem. *Each child gets $\frac{2}{3}$ of a candy bar.*
Multiplicative Coordination	Child does not need to represent each candy bar. Child understands that *a* things shared by *b* people is $\frac{a}{b}$, so 4 candy bars shared by 6 people means *each person gets $\frac{4}{6}$ of a candy bar.*



Appendix B

The Fundamental Properties of Operations and Equality (Empson & Levi, 2011, p.91)

Figure B2

Properties of Addition		**Examples**
Identity	$a + 0 = a$	$\frac{3}{8} + 0 = \frac{3}{8}$
Inverse	For every real number a there is a real number $-a$ such that $a + (-a) = 0$	$\frac{1}{3} + (-\frac{1}{3}) = 0$
Commutative	$a + b = b + a$	$\frac{1}{6} + \frac{1}{2} = \frac{1}{2} + \frac{1}{6}$
Associative	$a + (b + c) = (a + b) + c$	$\frac{4}{5} + (\frac{1}{5} + \frac{1}{2}) = (\frac{4}{5} + \frac{1}{5}) + \frac{1}{2}$
Properties of Multiplication		**Examples**
Identity	$a \times 1 = a$	$\frac{4}{3} \times 1 = \frac{4}{3}$
Inverse	For every real number a, $a \neq 0$, there is a real number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$	$8 \times \frac{1}{8} = 1$
Commutative	$a \times b = b \times a$	$9 \times \frac{2}{3} = \frac{2}{3} \times 9$
Associative	$a \times (b \times c) = (a \times b) \times c$	$(5 \times 4) \times \frac{3}{4} = 5 \times (4 \times \frac{3}{4})$
Distributive Property of Multiplication over Addition		**Example**
	$a \times (b + c) = (a \times b) + (a \times c)$	$6 \times 2\frac{1}{3} = 6 \times (2 + \frac{1}{3}) = (6 \times 2) + 6 \times \frac{1}{3})$
Other Properties of Operations		**Examples**
Addition and Subtraction are Inverse Operations	If $a + b = c$, then $c - b = a$	$\frac{3}{4} + \frac{1}{4} = 1$, so $1 - \frac{1}{4} = \frac{3}{4}$
Multiplication and Division are Inverse Operations	If $a \times b = c$, then $c \div b = a$	$8 \times \frac{3}{4} = 6$, so $6 \div \frac{3}{4} = 8$
Properties of Equality		**Examples**
Addition Property of Equality*	If $a = c$, then $a + b = c + b$	$\frac{1}{3} = \frac{1}{6} + \frac{1}{6}$, so $\frac{1}{3} + \frac{1}{6} = (\frac{1}{6} + \frac{1}{6}) + \frac{1}{6}$
Multiplication Property of Quality*	If $a = c$, then $a \times b = c \times b$	$3 \times \frac{2}{3} = 2$, so $5 \times (3 \times \frac{2}{3}) = 5 \times 2$

Appendix C

Progressive Strategies Used to Solve Multiple Group Problems (Empson & Levi, 2011, p.51)

Figure C3

Multiple Groups Problem	**I have 12 peanut butter and jelly sandwiches. How many children could I feed with these sandwiches if I give ¾ sandwich to each child? (Measurement Division)**	
Represents Each Group —Direct Modeling —Repeated Addition	Child represents each fractional group, either by drawing or by a fraction. For example, draws 12 sandwiches. Partitions each into fourths. Groups 3 fourths portions together and counts as 1 child. Continues this process until all 12 sandwiches are used up. Can feed 16 children.	
Grouping and Combining Strategies	Child does not represent each fractional group. Creates a more efficient grouping by combining fractional groups. Uses these groupings to count number of groups. For example:	
	$\frac{3}{4} + \frac{3}{4} = 1\frac{1}{2}$	2 kids
	$1\frac{1}{2} + 1\frac{1}{2} = 3$	4 kids
	$3 + 3 = 6$	8 kids
	$6 + 3 = 9$	12 kids
	$9 + 3 = 12$	16 kids
Multiplicative Strategies	Child does not represent each fractional group. Relates fractional group or a grouping to total, using multiplication. For example, "4 groups of ¾ is 3. There are 4 threes in 12 so there are 4 times 4 or 16 groups of ¾ in 12 so you could feed 16 children."	

Appendix D

Principal's Introductory Cover Letter

(to be printed on university letterhead)

April 2019

Dear [Principal's Name],

Thank you for considering participation in this study. This study is part of the fulfillment of my Master of Education degree. The focus of this study is to investigate a different approach to the teaching and learning of fractions. From my observations in the classroom, the teaching and learning of fractions is challenging for both students and teachers. This challenge led to a desire to investigate effective strategies to support students' understanding. The title of my research is *The Impact of Equal Sharing and Multiple Groups Problems on Students' Understanding of Fractions through the Quotient and Measure Constructs*.

In order to gather the information needed for the study, I would like to conduct reform-oriented lessons of 40 minutes duration every other week for five weeks. These lessons will occur during recess and therefore not take away from the students' normal learning environment in the classroom. A pre- and postassessment will be administered to evaluate the effectiveness of Equal Sharing and Multiple Groups problems on students' understanding of fractions. Students' work will be photographed or photocopied with the original returned to students for appropriate filing. With permission, students' classroom discourse will be videotaped. The video recordings will allow for a careful assessment and examination of students' thinking through classroom discourse when solving word problems. Edited classroom footage as well as students' work samples may be used by myself or my supervisor, Dr. Lawson for professional development purposes for other teachers.

Parents and their children will be given permission forms requesting their willingness to participate in this research. There is no apparent risk in participating in this study except that students may not learn more about fractions and feel frustrated. The risk is no greater than what a student may experience in a regular classroom setting. It is therefore my hope that the students will participate for the duration of the study, however, as Principal, you may withdraw your permission at any time, for any reason, without penalty, as participation is entirely voluntary.

The [Name of] School Board, [Name of] School, and the students will not be identified in any written publication, including my master's thesis, possible journal articles or conference presentations. If edited video footage is used for professional development purposes, the students will be identified by first name only. In this regard, anonymity and confidentiality will not be maintained. The raw data that is collected will be securely stored at Lakehead University for a minimum of five years after completion of the project. A report of the research will be available upon request.

The research project has been approved by Lakehead University Research Ethics Board as well as the Peel District School Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca.

Should you have any questions in regard to this research project, it would be my great pleasure to speak with you. You are most welcome to contact me at 416-797-3676 or rwright@lakeheadu.ca.

If you are in agreement with participating in this study, please sign the attached letter of consent and return to me. I would suggest that you keep this letter in the case you would like to contact any of us.

Sincerely,

Roxanne Wright

Ms. Roxanne Wright
Master of Education Student
Lakehead University
416-797-3676
rwright@lakeheadu.ca

Dr. Alex Lawson, Ph.D.
Thesis Supervisor
Lakehead University
807-343-8720
alawson@lakeheadu.ca

Ms. Sue Wright
Research Ethics Board
Lakehead University
807-343-8283
research@lakeheadu.ca

Appendix E

Principal's Consent Form

(to be printed on letterhead)

Consent Form

I, ____, agree to [Name of] School's participation
(Principal's Name/please print)

in the study with Ms. Roxanne Wright as described in the attached letter.

I understand that:
1. Students will be videotaped in the classroom environment as part of the research.
2. Students' participation is entirely voluntary and they are able to withdraw permission at any time, for any reason, with no penalty during the research
3. There is no apparent risk except the possibility that a student may not learn more about fractions and feel frustrated. This is the same risk they would experience in any classroom.
4. In accordance with Lakehead University policy, the raw data will remain confidential and securely stored at Lakehead University for a minimum of five years.
5. The [Name of] School Board and [Name of] School will remain anonymous in any written publication resulting from the research project.
6. The edited video footages of classroom discourse and students' work may be included in professional development for teachers conducted by Roxanne Wright or Dr. Lawson. Should students appear in edited video footages, they will be identified by their first name only.

☐ I initial this box to give permission for students to appear in video footages which might be used for professional development purposes, as outlined above in 6.

Your signature below confirms [Name of] School's consent to participate in this study. Once completed please sign and return to me.

____
Name of Principal (please print)

____ ____________________________

Signature of Principal Date

Appendix F

Parent(s)/Guardian(s)' Introductory Cover Letter

(to be printed on letterhead)

April 2019

Dear Parent(s)/Guardian(s) of Potential Participant,

My name is Roxanne Wright and I am a Grade 7/8 Student Success Teacher with the [Name of] School Board. In this role, I support academically struggling students in two schools. I am also working on my Master of Education degree at Lakehead University. As part of my degree, I am conducting a research study. The purpose of this study is to investigate a new approach to the teaching and learning of fractions. The title of my study is *The Impact of Equal Sharing Problems and Multiple Groups Problems on Students' Understanding of Fractions through the Quotient and Measure Constructs*. This is an invitation for your child to participate in this study. In participating in this study your child will benefit from additional teaching time using research based strategies for understanding fractions as well as contribute to the teaching community a new resource for the teaching and learning of fractions.

Your child was recommended to be part of this study by their Mathematics teacher, however, the decision to particpaite will soley be dependent on you, the parent/guaridan as well as your child. The lessons in this study will take place during the lunch hour after all participating students have had an opportunity to have their lunch. Each lesson will be 40 minutes in duration. During this time, your child will be requested to solve and discuss their solutions to a variety of fraction word problems. The research will be five weeks in duration, occuring on alternating weeks to accommodate my schedule in supporting students in two schools. In other words, lessons will occur 5 days a week every second week. During these alternating 5 weeks of lessons, there will be a pre, mid and postassessment administered. These assessments will be used to assess your child's progress. I would kindly request that during this period your child refrain from additional support related to fractions.This will allow for an accurate evaluation of the effectiveness of these word problems on students' understanding of fractions. The study will begin in April 2018.

During the 40 minute period, I will collect samples of your child's work as well as video recordings of their math talk as they solve word problems together. The video recordings will provide me the opportunity to carefully analyse how they solve the problems. These conversations may be transcribed and quoted anonymously in my final project in order to demonstrate their understanding of fractions. Myself or my supervisor, Dr. Lawson, may also make use of some of the edited classroom footages and work samples for professional development purposes for other teachers. Upon completion of the project, you are welcome to obtain a summary of the research by contacting me at the school or by providing your mailing address on the consent form.

Your child will not be identified in any written publication, including my master's thesis or possible journal articles. Should your child's video footage be used for professional development purposes, they will be identified by their first name. In this respect, their anonymity and confidentiality will not be maintained. The raw data that is collected will only be accessible to myself and Dr. Lawson and will be securely stored at Lakehead University for a minimum of five years. Participation in this study is voluntary.You or your child may withdraw the use of their data at any time during the research, for any reason, without penalty. This includes participation in the math sessions during lunch as well as any discussions during the research. The Lakehead University Research Ethics Board, the [Name of] School Board, and the Principal of [Name of] School have approved the research project. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at Lakehead University Research Ethics Board at 807-343-8283 or research@lakeheadu.ca.

There is no apparent risk for your child except that they may not learn more about fractions and feel frustrated. The risk is no greater than what your child may experience in a regular classroom setting. Should you give permission for your child to participate this study, he or she will also be asked of their willingness to participate. It is important to note that whether or not you and/or your child decides to participate in this study it will not have any impact on their regular education, marks, relationship with their teacher, the principal or the School Board.

Should you have any questions concerning this research project, please do not hesitate to contact me at [School's phone number] or rwright@lakeheadu.ca. I would be very pleased to speak with you.

If you are in agreement to your child's participation in this study, please sign the attached letter of consent and return it to [Teacher] at the school. Please keep this letter in the event that you would like to contact any of the listed persons.

Sincerely,

Roxanne Wright

Ms. Roxanne Wright
Master of Education Student
Lakehead University
[School's phone number]
rwright@lakeheadu.ca

Dr. Alex Lawson, Ph.D.
Thesis Supervisor
Lakehead University
807-343-8720
alawson@lakeheadu.ca

[Name of Principal]
[Name of] School
[School's phone number]
[Principal's e-mail address]

Ms. Sue Wright
Research Ethics Board
Lakehead University
807-343-8283
research@lakeheadu.ca

Appendix G

Parent(s)/Guardian(s) Consent Form

(to be printed on letterhead)

Parent/Guardian Consent Form

I DO give permission for my son/daughter, ____,
(Student's Name/please print)

to participate in the study with Ms. R. Wright as described in the attached letter.

I understand that:

1. My child will be videotaped in the classroom environment as part of the research.
2. My child's participation is entirely voluntary, and he/she can withdraw permission at any time, for any reason, with no penalty during the research.
3. There is no apparent risk except for the possibility that my child may not learn more about fractions and feel frustrated. This is the same risk they would experience in any regular classroom.
4. In accordance with Lakehead University policy, the raw data will remain confidential and securely stored at Lakehead University for a minimum of five years.
5. All participants will remain anonymous in any publication resulting from the research project.
6. The video footages of the classroom or student work may be included in professional development for teachers conducted by myself or Dr. Lawson. Should my child appear in video footages for professional development purposes, he or she will be identified by their first name only, thereby removing anonymity and confidentiality.

☐ I initial this box to give permission for my child to appear in video footages which may be used for professional development purposes, as outlined above in 6.

7. Upon completion of the project, I can receive a summary, upon request, by calling or writing, or by providing my address or email below.

Please keep the introductory letter on file. It has the relevant contact information should you have further questions at a later date. If you are in agreement to your child's participation in this study, please complete this form and have your child return it to [Teacher].

____

Name of Parent/Guardian (please print)

____ ______________________________

Signature of Parent/Guardian Date

Address or email address (if you would like a summary of the findings):

Appendix H

Potential Participant Introductory Cover Letter

(to be printed on letterhead)

April, 2019

Dear Potential Participant,

My name is Ms. Wright and I am a Grade 7/8 Student Success Teacher at [Name of School]. I am currently researching how certain types of division and multiplication word problems help students to understand fractions. Thank you for considering to be part of my research project.

The research will be conducted in April 2018 during the lunch hour for a duration of 40 minutes. You will first have time to eat. The lessons will be five weeks in duration but will be on alternating week as I support two schools. Therefore, the week I am at [Name of School] we will have lessons at lunch, then the following week while I am at my other school there will be no lessons, then I return the following week to continue and so on until we have done five weeks of lessons. The lessons will include a variety of division and multiplication word problems that you will solve together with your peers and discuss. There will be a video camera present in the classroom. This is to help me with my research. It will record your discussions and thinking while solving problems.

My supervisor, Dr. Lawson or myself, may want to use some of the video footages from our lessons as well as samples of your work at conferences to help other teachers learn more about teaching fractions. If you are in a video that will be seen by teachers, I will use your first name only, not your last name.

The unit of lessons will start with a preassessment. This preasssessment will tell us what you know about fractions before the start of the research. We will then start our resarch lessons. During this time, you will be recorded, your work photographed or photocopied so that I can review it later. You will keep the originals. At the end of the unit, you will be given a postassessment to see how the use of multiplication and division word problems helped you understand fractions. There will also be a midassessment, to assess your progress. Although there is no harm in participating in this research, you may experience frustration when solving word problems. This is no different from the frustation you experience, at times, in a regular class. Please ask me any questions you have about my research project. Thank you for thinking about being part of my research project.

I have included a consent form. If you would like to be part of my research project, please sign and return to your Mathematics teacher.

Sincerely,

Ms. R. Wright

Appendix I

Potential Participant Consent Form

(to be printed on letterhead)

Potential Participant Consent Form

I, ____, would like to participate in this research
…….. (Student's Name/please print)

with Ms. R. Wright as described in the attached letter.

I understand that:
8. I will be videotaped in the classroom as part of the research.
9. I don't have to participate in the research and if I change my mind about participating at anytime during the research I can do so with no consequences.
10. Although I might experience frustration when solving word problems, it is safe for me to be a part of this study. This frustration is similar to that experienced in any regular classroom.
11. All of the information Ms, Wright collects for this research will be kept safe at Lakehead Univeristy for five years and then it will be destroyed.
12. My real name will never be used in any of Ms. Wright's writings about this research.
13. Ms. Wright or Dr. Lawson may use some of the videos or samples of my work to help other teachers learn about teaching fractions. My first name will be used in the videos.

☐ I put my initials in this box to indicate that I agree to appear in video footages that will be used to help other teachers learn about teaching fractions.

If you are in agree to be part of this research project, please sign this form and return to me.

____

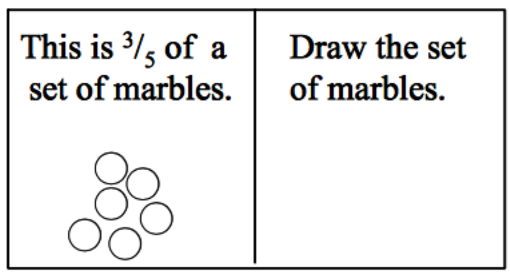
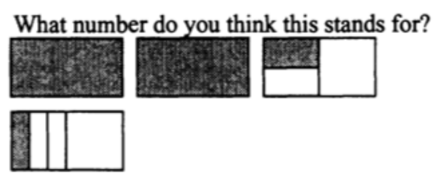
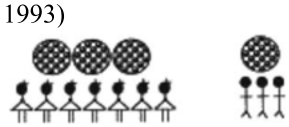
Name of Student (please print)

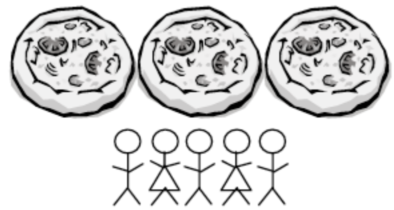
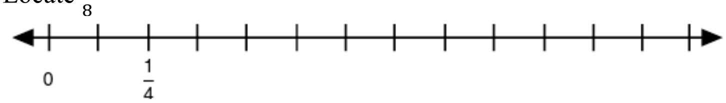
____ __________________________

Signature of Student Date

Appendix J

Pre and Postassessment

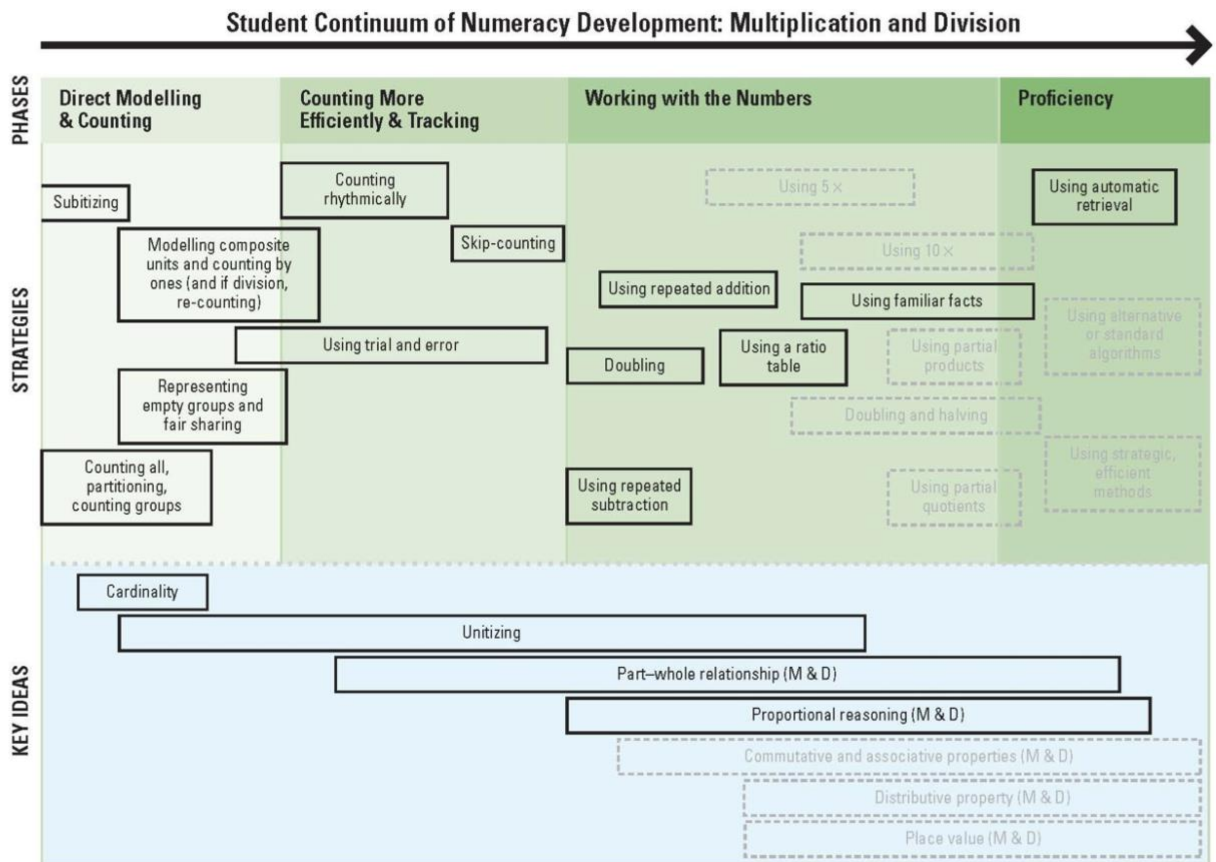
Problem Type	Word Problem
Division	
• Partitive division	256 apples are divided among 7 Grade 6 classes. How many apples will each Grade 6 class get? (Anghileri, 2001)
• Measurement division	84 pencils have to be packed in boxes of 16. How many boxes will be needed? (Anghileri, 2001) A carton of apple juice fills 8 glasses. How much apple juice do you need to fill 20 glasses? (Empson, 1999)
Fraction Constructs	
• Part-whole (discrete)	 (Empson, 1999)
• Part-whole(continuous)	 (Empson,1999)
• Ratio	Which would be the better deal, 2 tickets for $3 or 5 tickets for $6? (Lamon, 2011) Who gets more pizza, the boys or the girls? (Lamon, 1993, Marshall, 1993) 
• Operator	One morning, James made 12 cupcakes. That afternoon, James ate three fourth of the cupcakes he had made. How many cupcakes did he eat that afternoon? (Empson, 1999)

Problem Type	Word Problem
• Quotient	Five people are going to share three identical pepperoni pizza. How much will each person get? What part of the total pizza is one share? (Lamon, 2011) 
• Measure	Locate $\frac{5}{8}$ 
Underlying Fraction Concepts	
• Partitioning	Four children want to share 10 cupcakes so that each child gets the same amount. Show how much can one child have. (Empson, 1999)
• Equivalence	At one table, ___children are sharing ___ litres of juice. How many litres of juice should a table of ___ children get so that each child has as much juice as a child at the first table? (4, 3 and 12) and (8, 6 and 12)
• Unit Forming	Four rectangular pizzas are cut in halves, quarters, sixths and twelfths. Choose some pieces from at least three of these pizzas such that their "sum" is one pizza. Write a number sentence that describes your result. (Kieren, 1993)

Appendix K

Student's Continuum of Numeracy Development: Multiplication and Division

Figure K3



Appendix L

Midassessment — Equal Sharing Problems

Question 1
Prince and his 4 friends would like to share 3 brownies equally. How much brownie would each of them receive?

Question 2
Melanie brought 12 Mexican candies to school. She wants to share them equally with 5 students in her class. How much will each student receive?

(Adapted from Empson & Levi, 2011)

The midassessment consisted of two Equal Sharing word problems. The first Equal Sharing word problem resulted in an answer less than one whereas the second resulted in an answer greater than one. The purpose of the midassessment was to assess students' strategies when solving Equal Sharing problems and the impact that exposure to these types of problems have had on developing their understanding of the multiplicative relationship between the numerator and the denominator, that is, fraction as a division. In addition, I purposefully placed the Equal Sharing problem resulting with an answer less than 1 prior to the answer greater than 1. During the intervention lessons, according to Empson and Levi (2011) sequence of lessons, Equal Sharing problems with answers greater than 1 were presented prior to problems where the solutions were less than 1. Equal Sharing problems where the answer is greater than 1 are easier to solve and help bridge their understanding of whole numbers and fractions. The purpose of this was to observe if students were making sense of the problems or simply going through a procedure. 12 students wrote the midassessement. In this section, I will present the findings and a summary of the results.

Of the 12 students who wrote the midassessment, 8 of the students got the first question correct. All of the students used an 'Additive Coordination' strategy. Empson and Levi (2011) describe this strategy as a fairly common strategy for solving Equal Sharing problems. Furthermore, they explain that it is the initial strategy used by middle-grades students who have had minimal exposure to Equal Sharing problems. The results of this midassessment, in my opinion, appears to support their findings. One student, Sierra, initially attempted to solve the problem through division. However, after some difficulty with her division, she resorted to an 'Additive Coordination' strategy. It is evident that at first, she had some challenges with sharing but was eventually able to illustrate the three brownies shared in five equal parts. Of interest to me, was the expression of her solution. See Figure 38. She wrote "Each person will get $\frac{1}{5}$ of each brownie but in total they will get $\frac{3}{5}$" (P 269). This demonstrates, in my opinion, the beginning stages of an understanding of the multiplicative relationship between the numerator and denominator. This multiplicative relationship is believed to be necessary for the understanding of fractions (Lamon, 2007; Thompson & Saldanha, 2003; Vergnaud, 1988).

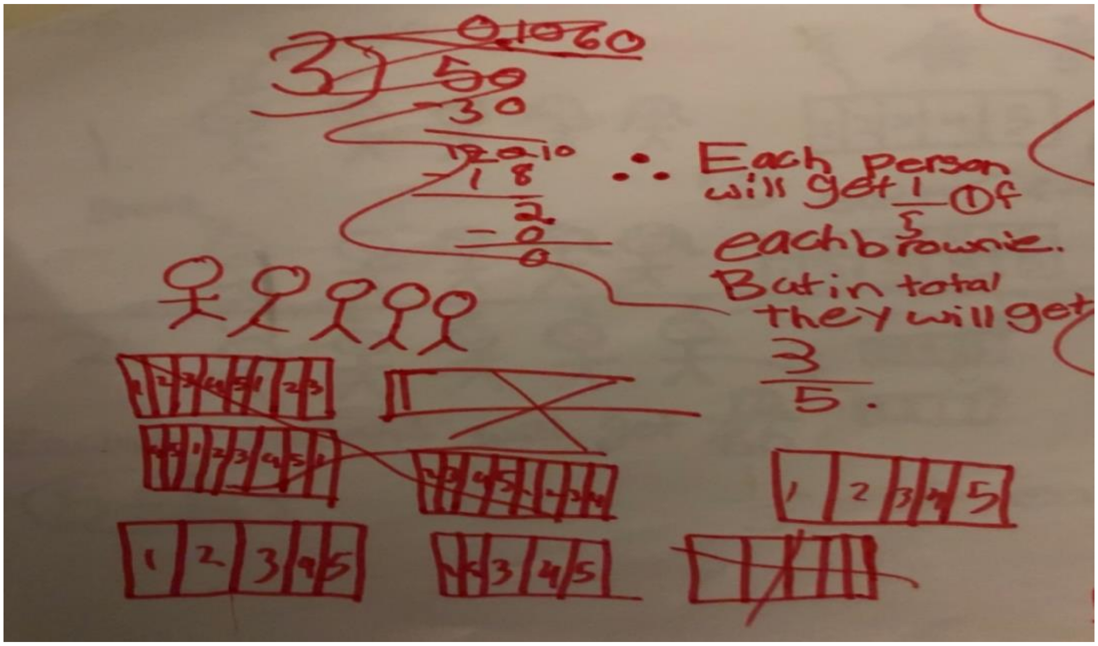


Figure 38. Sierra demonstrating the multiplicative relationship between numerator and denominator. Retrieved from Atlas.ti, P 269.

Figure 39 shows another example of this beginning stage by Denise.

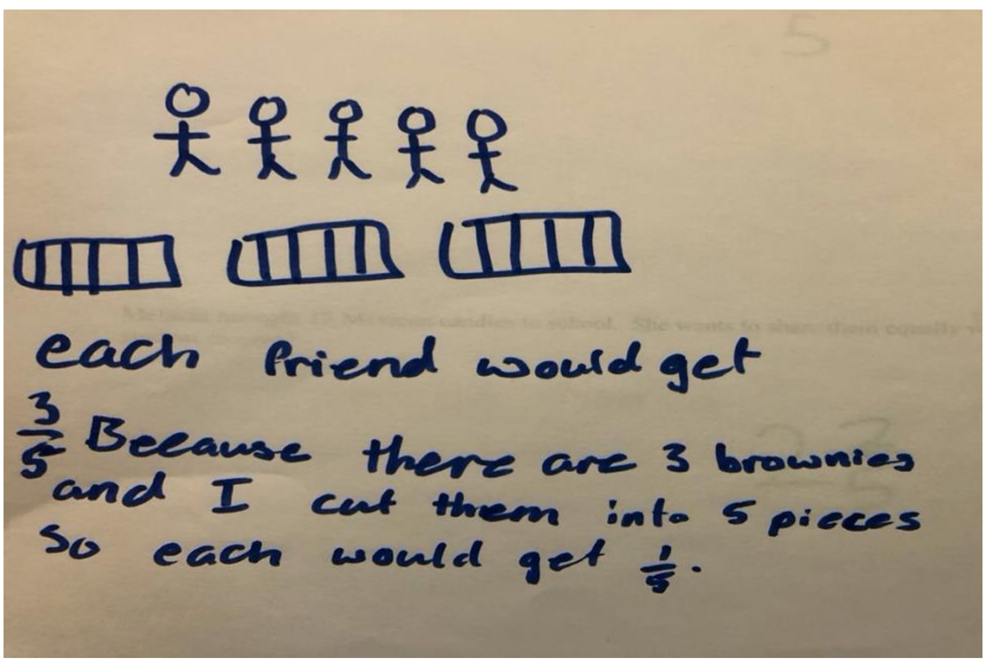


Figure 39. Denise demonstrating her understanding of the multiplicative relationship between numerator and denominator. Retrieved from Atlas.ti, P 271.

A somewhat similar result was observed in the second question, with seven out of the 12 students answering correctly. Most of the students, in solving this question, divided each of the 12 candies into five equal parts rather than reasoning that each student would get two whole candies and then divide the remaining two into five equal parts, a more efficient strategy. This more efficient strategy was previously observed in students' solutions to Question 9 of the preassessment and their intervention lessons prior to the midassessment. Although surprising, it provided, begrudgingly, the evidence I was seeking. Were students reasoning and making sense of their solutions or going through procedures? Two of the students expressed their solutions by stating that each student will receive $\frac{1}{5}$ of each candy. One such example is shown in Figure 40. They did not demonstrate an ability to express the solution as $\frac{12}{5}$ or $2\,\frac{2}{5}$. However, in discussing their solutions with them, they were able to express that the answer was $\frac{12}{5}$. Sierra, as seen in Figure 40, crossed out her initial answer of $\frac{1}{5}$ and inserted $\frac{12}{5}$ after. It's also interesting to note that these solutions both came from the Grade 6 classes whereas the Grade 8 classes, for the most part, expressed their solutions as $2\,\frac{2}{5}$ showing some reasoning that each child would get 2 whole candy and a fraction of the two remaining.

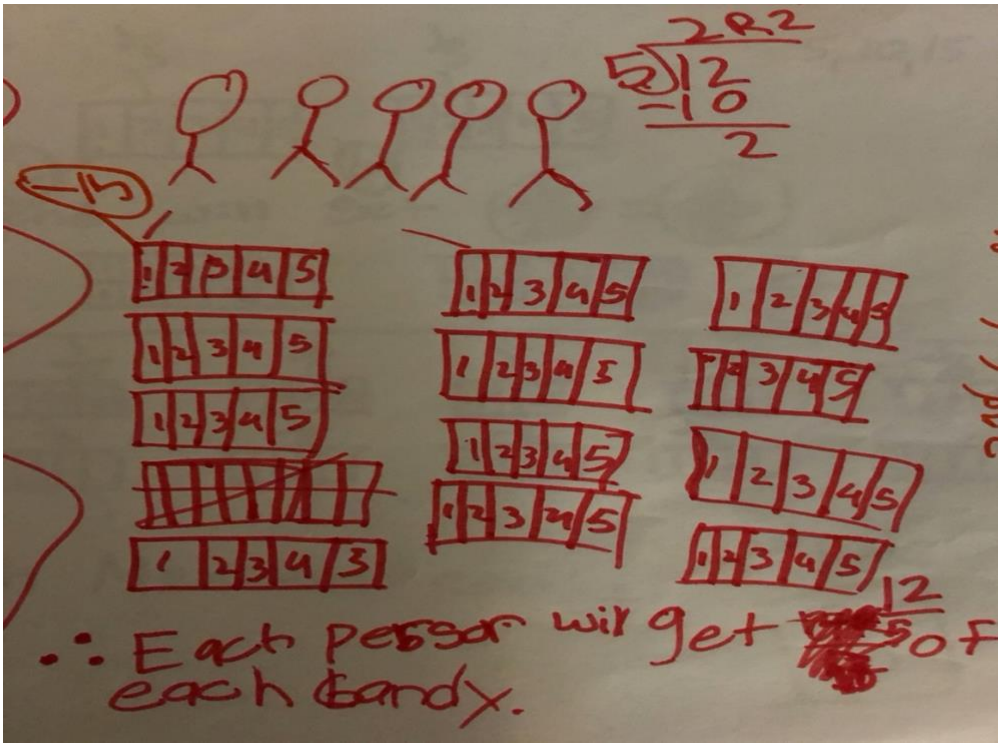


Figure 40. Sierra demonstrating her understanding of the multiplicative relationship between numerator and denominator. Retrieved from Atlas.ti, P 269.

In summary, the midassessment demonstrated the beginning stages of an understanding of the multiplicative relationship between the numerator and denominator. This conclusion was determined through the students' use of the 'Additive Coordination' strategy to solve the questions and their written solution statement of counting the number of fifths in each of the questions. I would describe students' inefficient strategy, in light of what I knew they were capable of from past experience, in solving question 2 of the midassessment as cognitive dissonance. That is, students are beginning to make sense of their new knowledge and connecting it to their previous knowledge. For example, in writing as their solution that each student would receive $\frac{1}{5}$ of each candy bar but not express the solution as $\frac{12}{5}$ or $2\,\frac{2}{5}$ demonstrates multiplicative thinking to some degree but some disconnect to the fraction $\frac{12}{5}$ or $2\,\frac{2}{5}$ which in the past, perhaps, was only seen represented as a part-whole construct, through diagrams. This observation and conclusion, in my opinion, is supported by the evidence that majority of the Grade 8 students were able to express the solution as $2\frac{2}{5}$, who perhaps have had more exposure and experience with mixed fractions outside of the part-whole construct.

Appendix M

Sequence of Equal Sharing and Multiple Groups Problems

Sequence	Word Problem
Lesson 1	Ms. Wright has 29 brownies to share with 4 friends. How much will each friend receive?
	Ms. Wright has 27 brownies to share with 4 friends. How much will each friend receive?
Lesson 2	4 children want to share 51 loaf cakes so that everyone gets the same amount. How much will each person get?
Lesson 3	Melissa has 17 cupcakes that she wants to share equally with 5 friends. How much will each friend receive?
	Ms. Wright has 2 brownies and she would like to share those two brownies with 3 friends equally. How much brownie would each student receive
Lesson 4	7 children in art class shares 5 packages of clay equally. How much clay does each child receive?
Lesson 5	16 students need to share 12 sticks of clay. If they share the clay equally, how much clay would each student get?
Lesson 6	Ms. Wright wants to feed each of the children she babysits one quarter of a kitkat. If she babysits 7 children how many kitkats should she buy?
	Ms. Wright has 7 sticks of clay to share. If each student received three quarter of a stick of clay. How much clay does each student receive?

Lesson 7	Two third metre of fabric is needed to make a pillow. How many metres of fabric would it take to make 15 pillows.
Lesson 8	Each small cupcake takes three quarter cup of frosting. If Saida wants to make 20 cupcakes, how much frosting does she need?
Lesson 9	I am making sub sandwiches for friends. There will be [13,12] friends eating sub sandwiches. Each friend will get [one quarter, three quarter] of a sub. How many sub sandwiches do I need?
Lesson 10	It takes ____ m of fabric to make a pillow. How many metres of fabric would it take to make __ pillows? [$\frac{1}{3}$, 3], [$\frac{2}{3}$, 3], [$\frac{2}{3}$, 9] [$\frac{2}{3}$,15]
Lesson 11	Tanesha wants to share with each of her classmates $\frac{1}{4}$, of a sandwich for lunch. If there are 24 students, how many sandwiches should she make? [$\frac{3}{4}$,24], [$\frac{1}{8}$, 24], [$\frac{5}{8}$,,24]
Lesson 12	A kitten eats $\frac{1}{10}$ cup of kitten food. How much kitten food would you need if you wanted to feed 30 kittens [$\frac{1}{5}$, 25], [$\frac{3}{10}$, 30], [$\frac{3}{5}$, 25]

Appendix N

Summary of Results from the Preassessment

	Q.1	Q.2	Q.2b	Q.3	Q.4	Q.5a	Q.5b	Q.6	Q.7	Q.8	Q.9	Q.10	Q.11
Gr6													
Emma	—	—	—	—	—	—	—	—	—	—	X	—[1]	—
Angela	X	X	—	—	—	—	—	—	—	—[1]	X	—	—[1]
Frank	X	X	—[b]	X	—	X	—[1]	X	—	—[1]	X[c]	—[1]	—[1]
Jessica	X[a]	—	—	—	—	—	—	—[1]	—[1]	—[1]	—	—[1]	—[1]
Maria	—	X	X	—	—	X	—	—	—	—[1]	X	—[1]	—[1]
Paul	—	X	—[b]	—	—	X[c]	—	—[1]	—[1]	—	X	—	—[1]
Sierra	X	X	—[b]	—	—	—	X[c]	X	—	X	—	—[1]	—[1]
Susan	—[1]	—[1]	—[b]	—	—	—	—	—	—	—	—	—	—[1]
Gr8													
Anthony	—[1]	—[1]	—	X	—[1]	—	—	—[1]	—[1]	—	X	X	—[1]
Denise	—[1]	—[1]	X	X	—	—[1]	—	—[1]	—[1]	—	X	—[1]	—[1]
Ethan	—	—	—[1]	—	—	—[1]	—[1]	—[1]	—[1]	—	X	—	—[1]
Horace	—[1]	—	—[1]	—	—	—	—	—[1]	—[1]	—	—	—[1]	—[1]
Marilyn	—[1]	X	X	X	—	X	—	—	—[1]	—	X	—[1]	—[1]
Tristan	—[1]	—[1]	—[1]	X	—[1]	—	—[1]	X	—[1]	—[1]	—[1]	—[1]	—[1]
Total	3	6	3	5	0	4	1	3	0	1	9	1	0

Note. Q = question; Gr6 = Grade 6 students; Gr8 = Grade 8 students.
[1] Didn't attempt the question and coded as incorrect. [a] Strategy correct but written answer express remainder as decimal. [b] Strategy correct but answer incorrect (not a sense-making answer). [c] No thinking shown just an answer.

Appendix O

Summary of Results from the Postassessment

	Q.1	Q. 2a	Q. 2b	Q. 3	Q. 4	Q. 5a	Q. 5b	Q. 6	Q. 7	Q. 8	Q. 9	Q.10	Q.11
Gr6													
Emma	—	—	—	—	—	—	—	—	—	—	X	—[1]	—[1]
Angela	X	X	—[b]	—	—	X[c]	X[c]	—	—	—[1]	—	—	—[1]
Frank	X	X	—[b]	X	—	X	X	X	X	X	X[c]	X	—[1]
Jessica	X	X	—[b]	—	—	X[c]	—	—	—	—	X	—	—[1]
Maria	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS
Paul	—[1]	X	—[b]	—	—	—	—	—[1]	—	—	X	X[c]	—[1]
Sierra	X	X	—[b]	X	—	—[1]	—[1]	—	—[1]	—	X	—[1]	—[1]
Susan	—	—	—	—	—	—	—	—	X	—	X	—[1]	—[1]
Gr8													
Anthony	—[1]	—[1]	X	X	—[1]	X	—	—	X	—	—	—[1]	—[1]
Denise	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS	ABS
Ethan	—	—[1]	—[1]	—	—	—[1]	X	—	X	—	X	—[1]	—[1]
Horace	—	—	—[b]	—	—	—	X	—	X	—	X	—[1]	—[1]
Marilyn	X	—[b]	X	X	—	X	—[1]	X	—[1]	—	X	—	—[1]
Tristan	—[b]	—[b]	X	X	—	X[c]	—	X	—	—	—	—	—[1]
Total	5	5	3	5	0	6	4	3	4	1	9	2	0

Note. Q = question; Gr6 = Grade 6 students; Gr8 = Grade 8 students; ABS = Absent
[1] Didn't attempt the question and coded as incorrect. [a] Strategy correct but written answer express remainder as decimal. [b] Strategy correct but answer incorrect (not a sense-making answer). [c] No thinking shown just an answer.