Engaging Multiple Representations in Grade Eight: Exploring Mathematics Teachers' Perspectives and Instructional Practices in Canada and Nigeria

by

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Abstract

This study was inspired by and utilises representations, one of the mathematical learning processes (NCTM, 2000), which is currently acclaimed as one of the reform-based instructional approaches to teaching and learning algebra. This concurrent mixed methods research project explored elementary in-service teachers' goals for, beliefs about and knowledge of representations, both in Ontario and Lagos. Data were collected through an online survey completed by 91 middle school in-service teachers concurrently with interviews with ten of them. Findings from the survey indicated that teachers from the Lagos subsample had weaker understandings about representations compared with their counterparts from Ontario. In the interviews, participants described to varying degrees their goals for and use of representations as opportunities for students to show connections, relationships, and reasoning, supporting students' confidence in problem-solving, and facilitation and opportunities for questioning and discussion. This research suggests that teachers generally, but particularly in Lagos, need a deeper understanding of representations and need to further develop the specialized mathematics content knowledge related to patterning and algebra. Other findings showed that: planning and sequencing instruction, use of contextual learning tasks, opportunities for students to generate their own representations, linking students' prior knowledge to new situations, and translation among multiple representations were reported as critical to teachers' use of representations. Recommendations are made to create more awareness among teachers, of the value, use and knowledge about representations. These findings would be relevant to school boards, teacher educators, researchers, and professional development providers wishing to improve teachers' use of representations, via enhanced beliefs, and knowledge.

Key words: Mathematics proficiency, Curriculum, Grade 8 patterning and algebra, Teachers, Representations, Goals, Beliefs, Pedagogy

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CHAPTER ONE: GENERAL INTRODUCTION

This study explores Grade 8 teachers' perspectives and their instructional practices in the use of and understanding of multiple representations as they teach patterning and algebra; how they generate and provide representations in Ontario, Canada and Lagos, Nigeria contexts. This chapter sets the context for the study, identifies the problem, lists the purpose and research questions, and explains the rationale and significance of the study. The last section states the overview of the study.

1.1 Background and Context

The Oxford English Dictionary defines 'algebra' as follows:

"Algebra is the department of mathematics which investigates the relations and properties of numbers by means of general symbols; and, in a more abstract sense, a calculus of symbols combining according to certain defined laws" (Simpson & Weiner, 1989, p. 311). Also, Cathcart et al. (2006) offered another definition of algebra as "the study of patterns, which forms the foundation for the logical connections in all of mathematics" (p. 394). Algebra continues to be a highly important and essential domain in mathematics, and it is fundamental for mathematical proficiency. Further, algebra is critically important to the success of students throughout middle school and college. As highlighted in WikiAnswers (2010), algebra has a number of uses in our modern world. Developing algebraic proficiency equips learners with required business skills, such as analyzing companies' annual budgets; for example, algebra plays a role in figuring out annual expenditures. Algebraic expressions and equations can be used to create models for interpreting and making decisions about data, and hence algebra is very important for further scientific study.

The development of algebraic proficiency is an ongoing priority for many countries. As such, curricular reforms in mathematics have made the algebra strand commonplace in elementary and middle schools. Curriculum reform raises several concerns, one of which is for all students to reach mathematics proficiency (Greenes & Rubenstein, 2008; Kieran, 2007; Nigeria Ministry of Education, 2008; Ontario Ministry of Education, 2005). Canada and Nigeria are no exception, with new approaches to teaching algebra and all the other strands. In Ontario, for example, one of the curriculum expectations is for students to model linear relationships graphically and algebraically, and solve and verify algebraic equations using a variety strategies. In Lagos, students are expected to solve simple equations and simplify algebraic expressions. However, topics in algebra are expected to be taught for its usefulness in other branches of mathematics and in the generalization of scientific truth, its power and verification of results in a simpler and more satisfactory manner, and its practical values in trade and industries (Odili, 2006; Sidhu, 2006).

Despite the benefits of learning algebra, there are some challenges associated with learning it and these include abstract reasoning and problem solving (Vogel, 2008), the language of mathematical symbols that seems completely foreign to students' previous experiences (Kilpatrick, Swafford & Findell, 2001), and the structural characteristics of algebra (Carraher & Schliemann, 2007; Kieran, 1992). Procedural transmission-style instruction may also make it harder for students to learn algebra. For example, Kieran (1992) reported that, traditional algebra instructions characterized by teacher explanation and student practice of routine symbolic manipulation skills. As such, students struggle to understand algebraic concepts (Greenes & Rubenstein, 2008), as it is one of the most poorly taught, widely hated and poorly understood strands of mathematics (Ali, Hukaindad, Akhter, & Khan, 2010). Student achievement in mathematics at the Grade 8 level internationally assessed in the Trends in International

Mathematics and Science Study shows that students tested weakest in algebra (TIMSS, 2011), an indication that students failed to achieve the minimum proficiency standard.

In response to challenges in access to quality teaching and learning, Hiebert and Wearne (1996) believed that conceptual understanding plays an important role in procedure adoption and generation. Educators and policy makers have placed increased emphasis on teaching the conceptual basis for problem-solving (NCTM, 1989) in hopes that increased conceptual understanding will lead to improved problem-solving performance. Educational reformers advocate using representations to improve students' conceptual understandings. Representations can help teachers to convey the intended mathematical meanings to students when properly introduced (Kamii, Kirkland, & Lewis, 2001) leading to conceptual understanding of any mathematical concepts. For the purpose of this study, *representation* is defined as a variety of forms, including pictures (e.g., drawings, charts, graphs), written symbols (e.g., numbers, equations, words), manipulative models, oral language, real-world situations (Van de Walle, 2004), and images on computers or calculators. It can also be the process of generating these forms.

Despite their interest in improving instruction, many middle school teachers report that they lack confidence when teaching mathematics and indicate that they do not understand some mathematical concepts and how to use representations in their mathematics teaching (Dreher & Kuntze, 2015; Mitchell, Charalambous, & Hill, 2014; Stylianou, 2010). Furthermore, teachers may lack understanding of students' conceptions and misconceptions needed to make the abstract concepts of algebra real and accessible for all students, and also need to find new ways of making classroom activities more engaging and rewarding. Their beliefs and practices may also have a strong impact on their instruction.

Recent research on teachers' engaging with representations in teaching mathematical concepts (Beatty, 2010; Dreher & Kuntze, 2015; Hiebert & Carpenter, 1992; Lawson, 2016; Mitchell, Charalambous, & Hill, 2014; Neria & Amit, 2004; Stylianou, 2010) offers a new approach to teaching which can help the teaching and learning of algebra. However, some research revealed that teachers grapple with how to integrate representation meaningfully in their instruction (Stylianou, 2010) in order to take up these new ideas. Little is generally known about how teachers generate and provide representations in mathematics classrooms beyond the United States. It is the gap that this study aims to fill by exploring Grade 8 mathematics teachers' use of representations and their instructional practices.

1.2 Statement of the Problem

Mathematics is not just about calculating, but has also been a part of the human search for understanding (Lewis, 2011), placing growing demands and expectations on the school systems across the globe. Mathematics is more valuable than ever before, as learning to think in mathematical terms is essentially likened to becoming a liberally educated person (Lewis, 2011). Part of the hope of mathematics education reform is to see more innovations in the teaching of mathematics to ensure every student reaches their mathematical potential. Recent mathematics curricular reforms brought with them greater challenges and responsibilities for teachers (Stigler & Hiebert, 2009). There is a paucity of published research, which provides an explanation on how teachers are using representations to illustrate and highlight key mathematics ideas (Stylinaou, 2010).

This study fills a gap in the literature as it represented, as far as I have been able to ascertain, the first scholarly attempt to compare the mathematical teaching methods in the grade eight classrooms in Canada and Nigeria using multiple representations. It is significant, because it will furnish a baseline of comparison for subsequent studies as the teaching and learning of algebra using multiple representations will be revealed in both countries.

1.3 Purpose of the Study

The purpose of this study is to explore how Grade 8 teachers in Ontario, Canada and Lagos, Nigeria generate and provide representations during the teaching of patterning and algebra.

1.4 Research Questions

In an attempt to explore teachers' instructional practices relating to engagement with multiple representations, this research seeks to address the following questions:

In what ways do Grade 8 teachers in Ontario, Canada and Lagos, Nigeria generate representations in their teaching of patterning and algebra? In what ways do Grade 8 teachers in Ontario, Canada and Lagos, Nigeria provide representations in their teaching of patterning and algebra?

The sub-questions are:

1. What are teachers' goals for and perspectives of using representations in Ontario and Lagos?
2. How do teachers' goals for and perspectives of using representations differ by region?

1.5 Rationale

Understanding how to appropriately use representations may help students to make meaning of algebra learning and make connections between the various concepts they have learnt. Students' proficiency in representation may support effective learning and flexibility in thinking about algebraic concepts. This kind of flexibility enhances and supports the ability of the students to move confidently across and between various representations in order to select appropriate ones as required in contextual situations. The curricular reforms in North America, as informed by the research in the revised *Principles and Standards for School Mathematics* document (National Council of Teachers of Mathematics [NCTM], 2000) underscores the importance of mathematics instruction, emphasizing the use of representations in presentation of mathematical concepts.

Algebraic thinking continues to be included in every grade level. Algebra is a precondition for achievement in mathematics education in general, and is reflected in curriculum frameworks at different levels of learning mathematics. Stacey, Chicks, and Kendal (2006) submit that students need algebra, but its abstract nature makes it hard to learn. As a result, teachers may need to expose students to problem solving context in which students would be able to see mathematical concepts in various forms. Teacher should create contexts that are accessible in relation to student developmental level.

If teachers find it challenging to use representations appropriately, it may limit how they encourage and expose their students to use them. In mathematics, beliefs and knowledge may pose challenges to teachers' instructional practices and there may be gaps in teachers' own ability to use mathematical representations when teaching patterning and algebra. More research needs to be done to explore why there is a narrow perspective to the use of representation in algebraic instruction (Dreher & Kuntze, 2015; Drijvers, Goddijn, Kindt, 2011; Kieran, 2007) and lack of this understanding may underlie many of the misconceptions that impede student learning of algebra.

1.6 Background of the Researcher

My choice of this topic was informed by my experiences. First, being a student in high school in Lagos, Nigeria, I was in a commercial mathematics stream (applied mathematics stream). We (applied math students) were seen as vulnerable students. As a result, most of our mathematics teachers often skipped topics such as bearing and distance, circle geometry, latitude and longitude, and some areas in algebra. These topics were not taught and were skipped because the math teachers always assumed we (applied math students) could not understand the topics. They claimed that the students in the science classrooms (academic classrooms) were still struggling to understand the topics. This was not fair to us as students because we all (both streams) sat the same mathematics examinations each semester, and also the same grade 12 examinations. The grade 12 examinations in Nigeria are conducted by the West African Examinations Council (WAEC) and National Examinations Council (NECO). Five West African countries (Gambia, Ghana, Sierra Leone, Liberia, and Nigeria) constitute the WAEC body and participate in this examination yearly.

As well, in Nigeria, students in academic classrooms take another mathematics course known as further mathematics, in addition to the general mathematics offered in all streams. According to Macaulay (2015), students in applied mathematics classrooms are a vulnerable population for mathematics teaching and learning. She further stated that teachers in her case study within Ontario expressed willingness to help students realize their mathematical potential.

One possible way for students to realize their mathematical potential is for mathematics teaching to change in schools. Success in mathematics education matters at the level of individual citizens because it opens options for college and career and increases prospects for future income (NAMP, 2008). I am hoping to see more innovations in the teaching of mathematics rather than assuming some students are vulnerable without attempting to address their needs.

1.7 Significance of the Study

This study has made a significant contribution towards existing knowledge about how teachers perceive representations, and its role in teaching patterning and algebra in Ontario and

Lagos. It appears, as recommended in *Principles and Standards for School Mathematics* (NCTM, 2000), that representation is an essential component of teaching and learning.

The outcome of the study may be beneficial to stakeholders in the teacher education sector. In particular, it may benefit pre-service teachers, as enhanced emphasis on representations will prepare them better for classroom use of these new strategies. In addition, this study provides new knowledge that can be shared through professional learning programs.

Teachers' views on this very important mathematical learning process may assist in the understanding of how representations may be used in promoting access to mathematics and subsequently improve students conceptual understanding. The study is also significant because the teaching and learning of algebra using multiple representations was revealed in two countries, furnishing a baseline of comparison for subsequent studies.

1.8 Overview of this Study

This study is organized into eight chapters. The chapters in the study are as follows:

Chapter one contains the background and context of the study, critical research questions, and the importance of the study.

Chapter two presents the review of literature pertaining to mathematical proficiency, representations in mathematics, specific case of patterning and algebra, mathematics teachers' beliefs, knowledge and perspectives.

Chapter three focuses on the theoretical framework for this study. The relevance of this theory to the study is clearly indicated.

Chapter four reports on the research methods and the rationale for choosing concurrent mixed methods design for the study. This chapter outlines the instruments employed.

Chapter five, six and seven includes the presentations of findings from the data obtained through the survey and interviews. These chapters aim to explore and respond to the critical questions of the study.

Chapter eight contains the conclusion and implications and verification of the research questions, and limitations of the study.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

My study focuses on how Grade 8 teachers in Ontario, Canada and Lagos, Nigeria generate and provide representations during teaching of patterning and algebra. In addition, it explores the perspectives of the teachers as they engage in the use of representations while teaching patterning and algebra. I examine relevant literature on various topics and issues that relate to this study. This includes a discussion of mathematical proficiency, algebra and patterning, representations in mathematics, the mathematics curriculum in Ontario and Nigeria (with a particular focus on patterning and algebra content), effective instructional practices, mathematics teachers' knowledge, perspectives, and practices, and the schooling systems in Canada and Nigeria in Grade 8 and 9.

2.2 Mathematical Proficiency

In order to examine instructional practice that has the potential to improve students' mathematical proficiency in algebra and patterning we must examine and describe proficiency in general. According to Kilpatrick, Swafford, and Findell (2001), mathematics proficiency is what is necessary for students to engage with "mathematics successfully" (p. 5). When it comes to the development of mathematics proficiency, procedural knowledge alone is not sufficient (Ghazali & Zakaria, 2011; McCormick, 1997; Star, 2007) – conceptual knowledge is also needed. Conceptual knowledge is knowledge that is rich in relationships and networks, while procedural knowledge might be thought of as knowledge of a sequence of actions (Hiebert & Carpenter, 1992). The National Council of Teachers of Mathematics (2000) recommends that the alliance of procedural understanding and conceptual understanding make these components usable in powerful ways. Although both procedural understanding and conceptual understanding are important, most classroom instruction is still based solely on procedural understanding (Boaler, 2014).

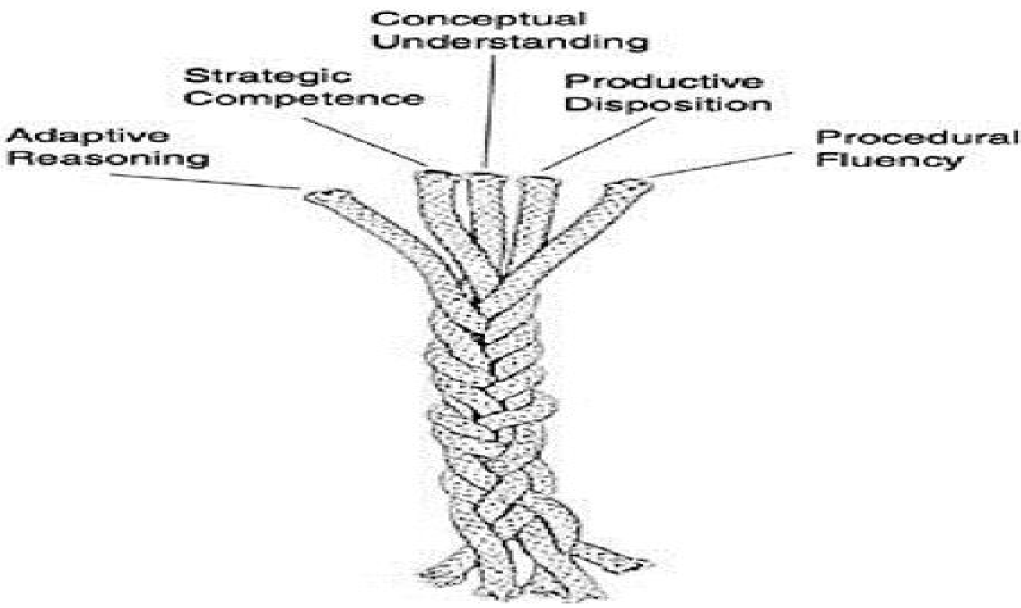
However, over the last 20 years in North America there has been an ongoing effort to move instruction beyond teaching strictly procedural mathematical knowledge (NCMT, 1991). According to Hiebert and Carpenter (1992), learning both concepts and procedures in problem solving contexts helps in making the connections needed for problem solving. Thus, conceptual knowledge and procedural knowledge are both imperative as students reason about mathematical tasks. Although conceptual and procedural understandings of any concept are important, according to Kilpatrick, Swafford, and Findell (2001), these alone are not sufficient, and there are even more factors to consider. Researchers often used the Kilpatrick et al.'s model of mathematical proficiency as a foundation for the design of instruction to improve students' knowledge, skills, abilities and beliefs (e.g., Samuelsson, 2010).

2.2.1 ***A Model of Mathematical Proficiency***

Kilpatrick et al. (2001) argue that there are five interwoven and interdependent strands involved in being mathematically proficient. The five strands provide a framework for discussing the knowledge, skills, abilities, and beliefs that build students' mathematical proficiency. These include conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (see Figure 1). In this model, conceptual understanding and procedural understanding form two of five strands in proficiency in mathematics.

Figure 1:

Intertwined strands of proficiency (Kilpatrick et al. 2001, p. 117)



Model describing components of mathematics proficiency.

In this model, conceptual understanding is an integration of mathematical ideas that students should know rather than isolated facts (Kilpatrick et al., 2001). The authors further note that with conceptual understanding, students have less to learn as they are able to see deeper similarities between seemingly unrelated contexts and appropriate use within such contexts. The procedural fluency strand implies the knowledge of procedures, including an awareness of when and how to use them flexibly to perform them accurately and efficiently (Kilpatrick et al., 2001) and thus includes procedural knowledge (Rittle-Johnson, Schneider, & Star, 2015).

Strategic competence is another strand of Kilpatrick's model necessary for mathematics proficiency, and has been examined in various studies (e.g., Khairan & Nordin, 2011; Samuelsson, 2010). Strategic competence is the ability to formulate, represent, and solve mathematical problems.

Khairan and Nordin (2011) examined three strands of mathematics proficiency, which include conceptual understanding, procedural fluency, and strategic competence among 14-year-old students. The findings of these authors revealed that students were most proficient in conceptual understanding followed by strategic competence and procedural fluency. In addition, there was a strong correlation between conceptual understanding and procedural fluency. Although conceptual understanding is a key for the basis of all other aspects of mathematical proficiency (Baroody, 2003), mathematics education researchers have argued that procedural fluency leads to strategic application of procedures and that both help and benefit conceptual understanding (Samuelsson, 2010).

Adaptive reasoning is another strand of the mathematical proficiency model, which refers to the capacity for logical thought, reflection, explanation, and justification. Productive disposition is the fifth strand, described as "the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 5). However, other literature on mathematical proficiency tends to focus primarily on conceptual understanding and procedural fluency in order to examine students' proficiency level (e.g., Khairan & Nordin, 2011). The depth of understanding of concept is strongly determined by the degree to which both procedural fluency and conceptual understanding are connected and the extent to which that knowledge is complete, well structured, abstract, and accurate (Baroody, Feil, & Johnson, 2007). Based on different beliefs about the teaching of procedural fluency as used in the model, and since other researchers used procedural understanding instead, I chose to use procedural understanding for the purpose of this study. Next, I present the relationships between procedural understanding and conceptual understanding as learners engage with mathematics.

2.2.2 *Conceptual Understanding and Procedural Understanding*

There are strong relationships between procedural understanding and conceptual understanding. Many researchers have found that when students' conceptual understanding is well grounded, it can further enhance their procedural understanding (Ghazali & Zakaria, 2011; McCormick, 1997; Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Star, 2007; Star, 2007). However, procedural understanding rather than conceptual understanding or both is still mostly used by students in solving mathematical tasks as revealed in their approaches (Boaler, 2014; Liljedahl, 2015; Siyepu, 2013; Skemp, 1986; Stein, Silvallard, & Smith, 2007). For example, in a study of 132 students, Ghazali and Zakaria (2011) examined secondary school students' procedural understanding and conceptual understanding in an algebra test. The authors found that students' level of procedural understanding was high whereas the level of conceptual understanding was generally low. Ghazali and Zakaria (2011) further found that a positive relationship between mathematics procedural understanding and conceptual understanding does exist.

Studies on procedural understanding and conceptual understanding present mixed results, hence a deeper understanding is necessary to further establish their relationships. According to Stein, Silvallard, and Smith (2007), when students develop conceptual understanding of a concept, they will be able to recognize its relationships with other concepts. It does not imply that conceptual understanding is better than procedural understanding, as both are important in the learning of mathematics.

What role can representations play in the development of students' development of mathematical proficiency as discussed above? In order to explore this, I will first define representations in mathematics.

2.3 Representations in Mathematics

The literature regarding representations in mathematics is the core literature of the current study. In this section, I focus on the history of the development of the use of representations, multiple representations, the concept of representations, and perspectives about teachers and representations. Before I delve into this section, I will give a brief illustration of representations as explained in the curricula of the two jurisdictions.

In the Ontario curriculum, *Grades 1-8: Mathematics, 2005,* representation is one of the learning processes. The term representation also refers to models such as “concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols” (Ontario Ministry of Education, 2005, p. 16). In the Nigeria mathematics curriculum, representation is less clearly defined (not a specific learning process). It is described only as written symbols, graphs, pictures, diagrams, and real world situations applied to solve problems (Federal Ministry of Education, 2008). Lagos uses a national curriculum: in all schools across Lagos. For the purpose of this study, I will refer to the national curriculum as the Lagos curriculum.

2.3.1 *History of The Development of Representations*

Mathematics is a system of related social practices, ways of doing things, and involves symbolizing, deriving, and analyzing (Lemke, 2003). Lemke explained that most mathematical writing before the modern times was integrated into verbal texts including ordinary words in which case symbolic expressions were rare. The challenges students often experience when engaging in mathematical knowledge acquisition may have brought about the need to access representations to ease such challenges. “The critical problem of mathematical comprehension for learners arises from the fact that the access to a mathematical object is possible only by means of representations and that these representations cannot be confused with the object itself”

(Duval, 2006, p. 107). Duval refers to this problem as “the cognitive paradox of access to knowledge of mathematical objects” (p. 107).

Students and teachers need to have insights into these historical contexts of representations (Lemke, 2003). The history of the development of mathematics may give insight towards understanding how the development of semiotic representations impacts the development of mathematical thought (Duval, 2006). Duval also explained that to illustrate mathematical treatment, for example calculation, “depends on the representation system” (p. 106). The representation system may need to be expressed in multiple ways to help learners access the mathematical objects involved.

Dienes’ (1977) multi-embodiment principle emphasized the role of representations in mathematical learning back in the 1960s. Dienes proposed the multiple embodiment principle theory, which emphasizes the importance of multiple representations in mathematics education, as a way to improve learning. Dienes argued that the same concepts could be represented in varying ways to provide learners with the opportunity to build abstractions about mathematics concepts. The multiple embodiment principle suggests that students’ conceptual learning is enhanced when they are exposed to a concept through a variety of representations. Dienes maintained that students need to go beyond thinking with a given embodiment (i.e., representation) to also thinking about it. For this to happen, Lesh and Zawojewski (2007) articulated that students need to experience a concept represented in multiple embodiments (i.e., representations) so that they will not solely attend to irrelevant features that are avoidably embedded in specific embodiments. So, in using multiple embodiments to represent a concept, students may be able to recognize the common abstract concepts that various embodiments are intended to suggest.

Similarly, it is important to stress that by adapting and comparing several structurally similar embodiments of a mathematical model (Lesh & Zawojewski, 2007; Nistal, Van Doren, Clarebout, Elen, & Verschaffel, 2009), students will be able to compare and contrast models to think about similarities and differences among them in order to investigate the relationships among alternative representations (Lesh & Zawojewski, 2007). Therefore, in thinking about school mathematics, many mathematicians, mathematics educators and teachers have encouraged the instructional path by which students are exposed to multiple approaches to solving mathematical problems. Next, I will examine multiple representations that are commonly used in the mathematics classrooms, in particular algebra classrooms for the purpose of supporting learning, interpreting representations, and constructing deeper understanding of situations.

2.3.2 *The Concept of Representations*

Representation refers to "a range of meaning activities: steady and holistic beliefs about something, various ways to evoke and denote an object, how information is coded" (Duval, 1999, p. 2). The perspective of Duval on representations encompasses a lot of ideas; however, the aspect of the definition that resonates here is "how information is coded". Students may think differently with representations, and as a result, may have the possibility of working with and thinking about information in unique ways. The way information is coded is important for proper understanding, and could be used as a jumping off point to initiate discussions.

Bruner (1971) concluded that children demonstrate their understandings in three stages of representations: enactive (role of physical objects), iconic (image based), and symbolic (language-based). Further, he explained that the transition from an enactive representation to iconic or, from both to symbolic is effected when the device that renders a sequence of actions simultaneously, also renders it into an immediate representation. Bruner acknowledged that constructing an embodiment of some concepts is always a starting point for children, which may be followed by building a concrete model for the purposes of an operational definition.

Bruner notes that symbolic representation is crucial for cognitive development; however, he attaches great importance to language as a primary means of symbolizing the world. Bruner articulates that cognitive growth is a result of an interaction between basic human capabilities and culturally invented technologies that serve as amplifiers of these capabilities.

For Duval (2006), representations can be individuals' beliefs, conceptions, or misconceptions to which we understand the individual's verbal or schematic production. For example, semiotic representations could be images, or descriptions about some phenomena of the real external world, to which we can gain a perceptual and instrumental access (e.g., Duval, 1999). Friedlander and Tabach (2001) highlight the importance of verbal, numerical, graphical and algebraic representations as having the potential of making the process of learning algebra meaningful and effective.

Duval (2006) introduces the distinction between concepts of treatment and conversion. He refers to "treatment as transformations of representation that happen within the same register while conversion is the transformation of representations that consist of changing a register without changing the objects being denoted" (pp. 111-112). For example, in working on algebraic functions, the process of solving an equation belongs to treatment, while the transition from algebraic notations of a function to its graph is a way of using two different semiotic registers to illustrate the notion of conversion.

Goldin and Shteingold (2001) maintained that mathematical representations can not be understood in isolation. Researchers have distinguished between external representations and internal representations (Cai, 2005; Goldin & Shteingold, 2001). Goldin and Shteingold (2001) emphasized that the interaction between external and internal representations is important for effective teaching and learning of mathematics.

As noted earlier, research has shown that algebra may be difficult for students to learn and for teachers to teach. Since a beliefs system is related to practice, some kind of relationship exists between teacher perspectives about and use of representations. This relationship will be explored in the following section.

2.3.3 *Multiple Representations*

While explaining relational understanding of mathematical concepts, mathematics educators (e.g., Van de Walle, Karp, & Bay-Williams, 2013) note that the more ways students are allowed to think about and test an emerging idea, the more possibility there is of them forming and integrating correctly the ideas into a rich web of concepts. Students should be given the space to think about a problem and show their understanding so their emerging ideas can be represented in the form that best fits their understanding.

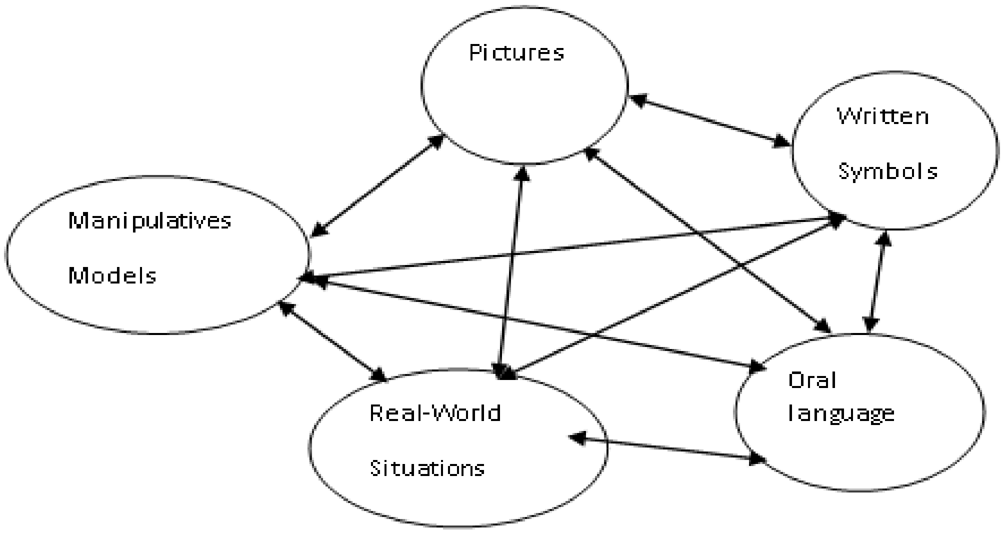
Multiple representations refers to the extent that external representations such as text, pictures, video, voice, graphs, and diagrams can be used to reinforce the messages designed to be conceived by the learners (Psomos & Kordaki, 2015). Using multiple representations may allow students to reflect deeply on concepts, as students are likely to visualise the structure by way of making comparisons and connections.

The ability to draw on multiple representations is an important aspect of students' mathematical understanding and problem-solving (Greeno & Hall, 1997; Hiebert & Carpenter, 1992; Stylianou, 2010). According to Mitchell, Charalambous, and Hill (2014), representations are often used in instruction to highlight key mathematical ideas and support student learning. The view of Mitchell and colleagues suggests that when teachers make use of representations during teaching, students experience meaningful and effective learning, as key mathematical ideas are emphasized.

According to Lesh, Cramer, Doerr, Post, and Zawojewski (2003), the understanding of any mathematics topic can be represented using five different types of representations (see Figure 2**Error! Reference source not found.**).

Figure 2:

Five representations of mathematical ideas (Lesh et al., 2003, p. 449)



Various representational system illustrated by Lesh and colleagues

Figure 2 is a representational system that is familiar in the Ontario K through 8 classrooms (Kieran, 2007; Lesh et al. 2003). Lesh et al. emphasize interactions within and among representations. The arrows connecting the different modes depict translations between modes. The Lesh et al. (2003) model suggests that the development of deep understanding of mathematical ideas requires experience in different modes, and experience in making connections between and within these modes of representation. Lesh et al. (2003) stress that understanding of elementary mathematical ideas is reflected in the ability to represent these ideas in multiple ways, together with the ability to make connections among the different embodiments. The Lesh et al. model further emphasises that translations within and between various modes of representation make ideas meaningful to students.

To translate between representations involves two modes of representations (Janvier, 1987). These include the source (initial representation) and the target (final representation). For example, consider algebraic and graphical modes as two translations such that, one translates from the graphical to the algebraic and from the algebraic to the graph. Janvier (1987) suggests that, to directly and correctly translate from one representation to another, teachers need to select and use the elements of the source that are important to achieve the target. Gagatsis and Shiakalli (2004) examined the translation ability of 195 university students by studying their ability to solve direct translation tasks from one representation of the concept of function to another. Gagatsis and Shiakalli (2004) administered two tests in two group sessions. In the first test, the source representation was verbal while the graphical and the algebraic were the target representations. In the second test, the source representation was graphical while the target representations were the verbal and algebraic representations. The authors revealed that students failed to realize that the graphical and verbal are different modes representing the same concept. Gatatsis and Shiakalli (2004) indicated that translation ability should be considered as an important factor in problem solving.

Stylianou (2010) highlights translation as a cognitive process of moving among different representations of the same mathematical concept by shifting as a way of encoding, reading, syntactic, elaborating and semantic elaboration while solving a problem. The Process Standards in the document, *Principles and Standards for School Mathematics* of NCTM (2000, p. 67) calls for all students to be able to “select, apply, and translate among mathematical representations to solve problems”, indicating the importance of this aspect in learning.

Moyer (2001) investigated the use of manipulatives models used as representations by 10 middle grades mathematics teachers during their teaching. Moyer contends that "the effective use of manipulatives for mathematics instruction is more complicated than it might appear" (p. 192), and suggests that having an understanding of mathematical relationships will inform the effective use of representations. Additionally, Moyer suggests that students should be able to make connections between their own internal representations and external representations or manipulatives. She mentioned that teachers need to create "mathematics environments that provide students with representations that enhance their thinking" (p. 178). A mathematics environment that provides students with a choice of multiple representations allows them to work with their preferred choice (Ainsworth, 1999). Students need to build understanding of how to effectively use representations as they work with their teachers who facilitate finding connections among concepts (Bransford, Brown, & Cocking, 2000).

Kilpatrick et al. (2001) suggest that students must first understand the situation in a problem including its key features before it can be represented accurately. As students try to understand the situation together with the key features of the problem, they may begin to flexibly negotiate between representations. According to Graham, Pfannkuch, and Thomas (2009), "flexible use of representations in particular is establishing meaningful links between and amongst representational forms and translating from one representation to another" (p. 682), and has been referred to by a number of terms, such as representational fluency (Lesh, 1999), representational competence (Shafrir, 1999), as well as representational flexibility. For the purpose of this study, I chose to use representational fluency. The flexibility with multiple forms of representation reflects a deep conceptual understanding of concept. If the use of multiple representations deepens students' understanding of mathematical concepts how can teachers effectively use them?

2.3.4 *Teachers' Effective Use of Multiple Representations in The Classroom*

Stylianou (2010) highlights three phases of representation in instruction: the launch phase, the exploration phase, and the discuss and summarize phase. The launch phase is when teachers use a variety of representations to present mathematical ideas or concepts. The exploration phase is when students get together in small groups using whatever tools are available to them to solve the problem. The last phase is when representations are used as tools in discussions to help students advance their argument and thinking. These are all important in the mathematics classroom, but the first and the third phase will be the focus of this study, since the second phase is focused more on the students than the teacher.

A good teacher continuously reflects on practice within the classroom context. According to Lamon (2001), "representations are for what the teacher already knows" (p. 155). In order to maximize the benefits for students to use representations, it is not a simple matter of using 'more' representations, rather consideration should be given to how the teachers and the students themselves can make the connections between different representations (Barmby, Bolden, Raine & Thompson, 2013; Ryken, 2009; Stylianou 2010). Mathematics educators suggest that teachers' responsibility of providing representation lies in providing appropriate contexts where students' ways of thinking develop naturally rather than in giving hints for correct solutions (Kajander, Fredrickson, Casasola, & Boland, 2013; Lesh, Lester, & Hjalmarson, 2003).

Stylianou (2010) examined 18 teachers' conceptions of representation in mathematics. In her analysis of the teachers' interviews, teachers argued that classroom discussion can be effectively impacted if appropriate choice of representations is made, be it teacher-generated or student-generated, with the teacher helping to focus students' attention on particular mathematical connections and concepts. So, how do teachers effectively use representations in the often- challenging strand of algebra?

2.4 The Specific Case of Patterning and Algebra

The above discussion was provided to clarify the need for using representations in supporting mathematics learning. The focus of this research is on a specific content area in mathematics –algebra and patterning—and how its learning might benefit from using multiple representations in instruction. Patterning and algebra has been chosen for study for two reasons: to focus on one content area of mathematics rather than mathematics in general, and because patterning and algebra is a gateway to higher mathematics. I discuss this in greater detail below.

First, the decision to focus on patterning and algebra was informed by the belief that focusing on one content area of mathematics in the classroom is more revealing than focusing on mathematics in general (Ball, Lubienski, & Mewborn, 2001). The choice of the topic area was informed by research findings that patterning and algebra is difficult for students to learn and for the teachers to teach (Grønmo, Lindquist, Arora, & Mullis, 2015; Kieran, 2007). While North American students typically know and can work with algebra in a procedural rule-based way (Carraher & Schliemann, 2007; Siegler & Alibali, 2005), many do not exhibit the flexibility to think accurately, fluently, and efficiently, and use conceptual knowledge, as Kilpatrick et al. (2001) described.

Similar to their North America counterparts, Lagos students use a traditional rule based approach to solve mathematics questions as they are often seen as passive listeners or information receivers (Nwoke, 2015). Algebra is one area of mathematics that is poorly taught, widely hated and abysmally understood in most Nigerian schools (Ladele, 2013; Nwoke, 2015).

Second, algebra is viewed as the gateway to higher mathematics (Stein, Kaufman, Sherman & Hillen, 2011) and it is a “gatekeeper” to studying other academic fields (Edwards, 2000, p. 26). According to Cathcart, Pothier, Vance, and Bezuk (2011), algebraic reasoning calls for representing, generalizing, and formalizing patterns and regularities that are found in all aspects of mathematics. Kaput (1998) commented that it is difficult to ignore the power of algebraic reasoning as there is no an area of mathematics in which it is not required. Siegler and Alibali (2005) articulated that the power of students' mathematical reasoning increases greatly as they learn algebraic concepts. These authors further explained that a single algebraic equation can be used to represent and reason about an infinite number of situations.

2.4.1 *The Challenges of Learning Algebra and Patterning*

In spite of the importance of and opportunities with algebraic learning, students often have difficulties learning algebra (Siegler & Alibali, 2005). Algebraic thinking continues to be included in every grade level, with emphasis on the use of patterns leading to generalizations, the study of change, and the concept of function (Van de Walle, Karp, & Bay-Williams, 2013). However, according to Carraher, Martinez, and Schliemann (2008), use of patterns is not an acknowledged and well defined concept in mathematics. Carraher et al. contend that there is no agreement among mathematicians about what patterns are, nor about their properties and operations.

This may be why as Carraher, Martinez, and Schliemann (2008) reported that many mathematics educators found that it may be challenging to get students from an understanding of the relationship of patterns to algebra (Moss, Beatty, McNab & Eistenband, 2006; Orton, 1999 as cited in Carraher et al. 2008). For example, in a study of 379 students of age 12 to 15, Warren (2000) tested students on tasks related to generalization of patterns and found that students have difficulty in doing such activities. On the other hand, Beatty (2010) contended that patterns build the students' confidence to be able to explore some fundamental algebraic concepts, as pattern use supports students' abilities to generalize.

The NCTM (2000) also recommends that students should participate in patterning activities, stating that they will be able to "make generalizations about geometric and numeric patterns, provide justifications for their conjectures, and represent patterns and functions in words, tables and graphs" (p. 223). The activities that show how students navigate from the understanding of patterns to algebra can be well observed in the activities of school algebra described by Kieran (2007). Kieran's framework (Kieran, 2004; 2007) which distinguishes between three types of algebra activities: generational, transformational, and global/meta level activities can be employed for the examination of the types of algebraic activities teachers make use of in their teaching. However, this framework focuses only on the type of algebraic activity and not on the cognitive level of activity (Eisenmann & Even, 2009), this current study will use Kieran's framework of algebraic activities to examine how teachers use representations to enhance the teaching and learning of algebra in two different jurisdictions. Next, I will discuss Kieran (2007)'s model of the activities of school algebra, as this model describes different mathematics activities that inform the use of various representations when solving problems in algebra.

2.4.2 *Effective Instruction of Patterning and Algebra Through the Lens of Kieran's Model*

Kieran (2007) developed a model that synthesizes the activities of school algebra into three types—generational, transformational, and global/meta-level. The generational activities of algebra involve the forming of the expressions and equations that are the objects of algebra. Transformational activities are the rule-based activities; these include collecting like terms, factoring, expanding, substituting one expression for another, solving equations and inequalities, and simplifying expressions, among others. Global/meta-level activities of algebra occur when students are engaged in mathematics exercises such as problem solving, modeling, working with generalization patterns, justifying and proving, making predictions and conjectures, studying change in functional situations, and looking for relationships or structures that don't involve any symbolic algebra. The Kieran model has served as a framework for presenting research findings by many mathematics educators on the teaching and learning of algebra, and so, it is important to examine here in detail.

2.4.2.1 Generational Activities. Kieran claimed that much of the building of understanding of algebraic objects occurs within the generational activity of algebra. Some of the areas examined in her work on learning and teaching algebra in middle schools for generational activity include letter-symbolic forms, multiple representations (tabular representations, graphical representations, and connections among representations), and the context of word problems. While examining the letter-symbolic forms of the objects of algebra, Kieran noted that most research studies concentrated on three major areas: variables, expressions, and equations; the negative numbers and the beginnings of structure sense. The Ontario and Lagos algebra strand of the curricula examined in this study focuses on the variables, expressions and equations as discussed in the next section.

2.4.2.2 Transformational Activities. The next algebra component examined in Kieran's model is transformational activities. Basically, transformational activity is concerned with changing the symbolic form of an expression or equation in order to maintain equivalence. Some of the areas examined include equivalence and theoretical control, expressions, equation and equations solving, and use of concrete manipulatives that support students' learning of algebra. These are areas that help to develop students' skills in simplifying expressions and solving equations. For example, in a study of 136 students, Tabach and Friedlander (2008) investigated students' work on a sequence of three different tasks designed as transformation activities. Although students first perceived the activities as generational, they gradually shifted towards transformational activity. The findings of these authors suggest that using appropriate tasks allow students to consider the same activities as generational, transformational, and global/meta-level activities.

Ainley, Bills, and Wilson (2005) examined how students focused on generational activity that takes the form of expressing calculations on spreadsheet formulae. These authors found that the students were able to use a wide range of semiotic means of objectification to construct meaning as they explore number patterns. Some of the semiotic means used include verbal, gesture and statements of calculations. Further, the authors acknowledged that the purpose of tasks drove the use of spreadsheets as tools used in the study for promoting equivalent expressions.

2.4.2.3 Global/Meta-Level Activities. The global/meta-level suggests more general mathematical processes and activity. Kieran (2007) noted that various research studies on learners' proficiency relating to global/meta-level activity in middle school focused on generalizing, proof and proving, and modelling. Some of the activities in this area have been integrated into generational and transformational activities. Kieran argues that due to the reform mathematics movement, the trend of frequent use of transformational activities in teaching and learning of algebra changed due to more emphasis on generational and global/meta-level activities as a result of the emergence of technological tools.

Kieran's model synthesized the main thrusts of various algebra studies by many researchers, such as research that involved analyzing relationships among quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving and predicting.

It may be that students' inability to use appropriate representations presents a challenge to solving mathematical problems—particularly algebraic problems. According to Kieran (2007), it is well known that students experience challenges in generating equations to represent relationships when solving algebraic tasks and in particular word problems. As the National Mathematics Advisory Panel [NMAP] (2008) reported, more precise measures are needed to specify in greater detail the relationships between the various representations used to help solve algebraic problems. The NMAP report further revealed that elementary and middle school teachers may benefit from instructional practices; in particular, using representations when teaching algebra in order for students to reach proficiency. This goal will be expanded in greater depth in the next section as presented below.

2.4.3 *Improving Algebraic Proficiency*

Algebra has its own notation and convention, and algebraic proficiency may be enhanced by exposing students to the use of appropriate representations, which can serve to deepen conceptual and procedural understandings. For example, functions can be expressed as different forms of representation, such as tables, graphs and algebraic expressions. To forge students' deep conceptual understanding, teachers may need to encourage students to explore multiple representations of linear relationships (Beatty & Bruce, 2012). Kajander and Boland (2014) contend that the mathematical power gained in developing a model to explain the reason behind using a rule is a worthwhile effort, and serves to deepen conceptual understanding. When students have a deeper conceptual understanding, they can comprehend and explain reasons for applying the right mathematical rules and be able to deal with similar situations in the future.

One of the commonly used forms of representations is manipulatives (Moyer, 2001), particularly in K-8 algebra classrooms. Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, and Mutch (1997) studied 21 students from Grade 8 in Australia. In the study they reported that students did not use the concrete manipulatives (cups, counters, and sticks) that were made available for solving linear equations, and suggested that these concrete representations increased the amount of mental activity required of the students. Moyer (2001) revealed that the teachers' choice to use manipulatives was often based on student behaviour, rather than the appropriateness of a representation to illustrate mathematical concepts. For example, one of the 10 middle school teachers in Moyer's (2001) study in the U.S. indicated that "behaviour played a crucial role in her decision about using the manipulatives" (p. 187) and that she would use them as a reward when students had behaved appropriately. Moyer further revealed that many teachers viewed the use of manipulatives for instruction primarily as playing, exploring, or as a change of pace, rather than to deepen conceptual understanding.

Vlassis (2002), who observed 40 students in Belgium over a period of 16 lessons on the use of a balance model for solving linear equations with one unknown, found that the balance model was an effective tool in conveying the principles of transformation. The balance model helped students to more successfully work with expressions and linear equations.

Hewitt (2003) reported preliminary results from 40 teachers and one particular class of Grade 7 students of age 11 to12 in the UK, and found that the inherent mathematical structure, and the visual impact of notation, had an effect on the way in which an equation was manipulated. The researcher argued that algebra instruction may impact the development of overall mathematical structure sense.

In summary, some evidence exists that students' mathematical proficiency, in particular, algebraic proficiency, would improve if teachers were able to facilitate learning using multiple representations. The construct of multiple representations appears to be an important aspect of students' mathematical understanding and problem-solving. In order to understand how representations are used by the teachers in the two jurisdictions, it is important to first look at the schooling system.

2.5 The Schooling Systems in Ontario and Lagos

To attempt to synthesize the schooling systems in two jurisdictions such as Ontario and Lagos (each with separate, well-defined schooling system) poses a challenge. A logical way forward is to look at the specific area of interest (patterning and algebra) in each separately, and then attempt to synthesize these. Before taking this step, however, it may be helpful to briefly look at the education systems overall in Ontario and Lagos. This section will outline the differences in the school systems in both jurisdictions. The basic structure of the education system, the mathematics curriculum, and professional learning for teaching may depend on the jurisdiction where a teacher works.

An education system provides the skills needed to support successful students by providing scholarships and mathematically rich environments. It is important to understand if the education system is meant to prepare students to ask questions about a phenomenon, develop and use models, or plan and carry out investigations in order to analyze and interpret data so that students can use mathematics and computational thinking. This section describes the education system in Ontario and Lagos in order to assess similarities and differences in the two jurisdictions.

2.5.1 *Education in Ontario*

The Ontario Ministry of Education establishes the policies and procedures that govern publicly-funded schools. Ontario has four publicly-funded school systems: English public, English Catholic, French-language public and French-language Catholic. These publicly funded schools are managed by district school boards. Apart from publicly-funded schools, there are also private schools that offer elementary and secondary education. These privately owned schools do not receive any government funding; students have to pay to attend them. Ontario public schools are free and offer good quality education (OMoE, 2005a). As of 2016, there are about 3000 public schools, 1600 public Catholic schools and 800 private schools in Ontario (Ontario Council of Agencies Serving Immigrants, 2016).

The Ontario Ministry of Education is responsible for establishing the provincial curriculum, stating and spelling out what students will learn in each grade. The Ministry of Education is also responsible for policies regarding assessment, evaluation, and reporting of students' achievement in Ontario schools from kindergarten to Grade 12. Another area the Ministry has accommodated is students with special needs. Basically, these regarding assessment, evaluation, and reporting applied to all students in connection with the achievement of curriculum expectations.

The Ontario Ministry of Education is responsible for curriculum and policy formulation to ensure that students within the province can compete globally, particularly in subjects such as mathematics, science, technology and language. A central principle of the predominant education policy in Ontario is the high priority of enabling all students fulfill their potential and succeed (O' Sullivan, 1999).

Although there are several school authorities that oversee schools in hospitals, treatment centres, and in remote regions, there are 72 school boards in Ontario in charge of implementing provincial policies. The school boards are responsible for deciding how to spend the funds they receive. The boards spend money on things like hiring teachers and other staff, building and maintaining schools and purchasing school supplies. Besides, the boards also work on developing local education policy such as safety in school, homework policies, and ensuring the schools follow the rules set out in the Education Act (Ungerleider & Levin, 2007). In an attempt to ensure proper implementation of the policies, the boards ensure ongoing professional learning for teachers (Ontario College of Teachers, 2016).

In *Many Roots, Many Voices* (2005), the OMoE articulates the common commitment of the province's teachers to ensure that students, regardless of culture, language and heritage are served effectively. The document further states that teachers are urged to orchestrate scholarly environments in order to ensure that students experience positive and enriching learning. The goal is that "the schools we create today will shape the society that we and our children share tomorrow" (OMoE, 2009, p. 6).

2.5.2 *Education in Lagos*

Education in Nigeria is administered by the federal, state and local governments. The federal ministry of education is responsible for overall policy formation and ensuring quality control. Although the federal ministry of education is expected to fund public schools, the education system remains underfunded (World Education Service [WES], 2017). As a result of the underfunding, facilities are often poor, teachers inadequately trained, and participation rates are low by international standards. School education is largely the responsibility of state government (WES, 2017).

The Lagos Ministry of Education influences and reshapes the educational system in the state. The Ministry operates the national curriculum on education as stipulated by the federal government of Nigeria (Lagos Ministry of Education [LMoE], 2016). The Lagos Ministry of Education administers education policies and funds to two categories of schools referred to as public schools and model schools (Sanni, 2012). The public schools are the least-resourced schools with very high teacher-student ratios, ranging between 1:46 and 1:95, while the model schools are well-resourced and provide full boarding facilities (Sanni, 2012). There are 1001 primary schools, 339 junior secondary schools (middle schools) and 319 senior secondary schools in the state (LMoE, 2016).

There are various departments and units saddled with various responsibilities. The Lagos Ministry of Education has six district school boards that monitor and administer the day to day running of schools. The teacher's establishment and pensions office has the responsibility of recruiting, training, manpower development, the welfare of teachers in the education districts, and registrations of professional teachers in the state (LMoE, 2016). The state universal basic education board (SUBEB) is in charge of primary education up to the junior secondary schools. The SUBEB is responsible for policy guidelines for implementing universal basic education programs, prescribing minimum standards, and builds and also identifies areas of intervention in provision of adequate basic educational facilities.

2.5.3 *Ontario Mathematics Curriculum*

Curriculum reform in Ontario has been on-going since 1997 in mathematics. The reform is directed at improving students' achievement and is potentially a major influence on teachers' work. The goal of the curriculum is improving students' achievement in core skills in the area of mathematics, and increasing emphasis on skills that are transferable to meet the demands of today and tomorrow. In Lagos, reforms implemented in 2014 have led to a restructuring of the national curriculum. The new Lagos curriculum has a stronger focus on vocational training than the previous one. It is intended to increase the employability of high school graduates.

As mentioned earlier, Ontario's Ministry of Education establishes a provincial curriculum. The Ontario mathematics curriculum is designed to equip students with "knowledge, skills, and habits of mind that are essential for successful and rewarding participation" (OMoE, 2009, p. 3) in an information and technology-based society. In Ontario, the Ministry of Education launched the current document, *The Ontario Curriculum, Grades 1-8: Mathematics,* in *2005* which covers mathematics programs for Grades 1 to 8. In many Ontario schools, Grade 8 is the last elementary school grade; after completing Grade 8, students transition to secondary schools. According to *The Ontario Curriculum, Grades 1-8: Mathematics, 2005*, “the transition from elementary school mathematics to secondary school mathematics is very important for students’ development of confidence and competence” (OMoE, 2005b, p. 4).

The choice of Grade 8 for this study was prompted in part by the dramatic changes that occur between grades 8 and 9, taking into consideration the transitional needs of students in both jurisdictions. In Ontario for instance, typically students with strong mathematical foundations at this grade would choose to take academic mathematics courses after the transition into secondary school. Students with weaker mathematics foundations may choose to take applied mathematics or the locally developed mathematics courses. According to Macaulay (2015), applied mathematics students are vulnerable, as they are “more likely to not reach the provincial standard on the Grade 9 mathematics assessment than they are to reach it” (p. ii).

In Ontario, Grades 9 and 10 academic mathematics courses are designed to “develop students’ knowledge and skills through the study of theory and abstract problems. These courses focus on the essential concepts of a subject and explore related concepts as well. They incorporate practical applications as appropriate” (OMoE, 2005c, p. 6). The expectations in Grade 9 and Grade 10 academic mathematics courses indicate that students’ mathematical skills may not be fully developed if they are not exposed to appropriate theory and abstract problems using practical applications that would eventually lead them to the mastering of the essential concepts of the subject. On the other hand, the applied courses “focus on the essential concepts of a subject, and develop students’ knowledge and skills through practical applications and concrete examples. Familiar situations are used to illustrate ideas, and students are given opportunities to experience hands-on applications of the concepts and theories they study”

(OMoE, 2005c, p. 6). The assumption is that the applied mathematics students may not adequately develop their mathematical skills if familiar situations are not used to illustrate concepts and theories.

Macaulay (2015) noted that Grade 9 applied students consistently do not attain proficiency in mathematics. Grade 8 is a crucial school year for students and teachers in Ontario due to the major transition to high school the following year. Even though there are efforts to bridge the gaps and increase collaboration between Grade 8 teachers and Grade 9 teachers in Ontario (Holm, 2014), there is still a big disconnection between the two panels, which may be impacting students' academic achievements (Holm & Kajander, 2015).

It should be noted that Ontario recently unveiled a new mathematics curriculum, *The Ontario Curriculum, Grades 1 – 8: Mathematics—Curriculum Context, 2020*. The expectations in the new mathematics curriculum are organized into six strands: A. Social Emotional Learning (SEL) Skills in Mathematics and the Mathematical Processes, B. Number, C. Algebra, D. Data; E. Spatial Sense; and F. Financial Literacy. Although, the Ontario government launched this new curriculum in June 2020, my research was carried out between 2017 and 2019 based on *The Ontario Curriculum, Grades 1-8: Mathematics, 2005.* Hence, while this current study is of relevance to the implementation of the new curriculum, the data was gathered from teachers using the 2005 version. As stated in the new algebra strand, "students develop algebraic reasoning through working with patterns, variables, expressions, equations, inequalities, coding, and the process of mathematical modelling" (OMoE, 2020, p. 34). Two of the newer topics in this new algebra strands include coding and mathematical modelling, and the current data contributes particularly to the latter of these.

2.5.4 *Lagos Mathematics Curriculum*

The Lagos mathematics curriculum is designed to discourage memorization of facts but is interspersed with skills for lifelong learning and emphasizes daily use of mathematical knowledge. In Nigeria in 2007, the Federal Ministry of Education through the National Council on Education released the 9-year basic education mathematics curriculum. According to Nigeria's National Policy on Education (2004), basic education covers nine years of compulsory formal schooling consisting of six years of elementary and three years of junior secondary education. Lagos adopts the same schooling system.

In Lagos, Grade 8 students are in their second year of secondary education, part of the last three years of a nine-year compulsory education. Students move to secondary school upon completion of primary education Grades 1-6, on the basis of continuous assessment since 2004, except for students who get into unity schools. A unity school is a federal government owned school. Students must pass the National Common Entrance Examination (NCEE) conducted by the National Examination Council (NECO) to be invited into a unity school. NECO is an examination body that conducts exams at three points for the purpose of certification including NCEE for Grade 6, Basic Education Certificate Examination (BECE) for Grade 9, and Senior School Certificate Examination for Grade 12.

Students in Grade 8 are next expected to take their first national external examinations (BECE) at Grade 9 and must pass at least six subjects (including English and Mathematics) before they can be awarded a BECE certificate. Students with a solid mathematical foundation in Grade 8 are expected to be successful in Grade 9, and this success determines whether to repeat Grade 9, drop out of school, move to a technical college, or proceed to Grade 10. It is in Grade 10 that students are moved to sciences (academic) or humanities (applied) depending on their performance in the Grade 9 examinations.

In summary, in both school systems, there is a high demand on the Grade 8 students to prepare for external examinations. As discussed, students in Ontario will sit for EQAO after transition from Grade 8 (although it is not high-stakes), and Lagos students similarly must take national exams during their Grade 9 year.

2.5.5 A Brief Examination Reports

The examination results for both jurisdictions revealed that students typically lack proficiency in using appropriate representations while solving problems. For example, the Grade 9 scoring rubric (EQAO, 2015) revealed that Ontario students were typically unable to interpret word problems and correctly draw required diagrams that would help in solving questions. In Lagos, the examiners' reports (West African Examinations Council, 2014) observed students' weaknesses in factorization, functions, and interpretation and solution to word problems.

Student achievement may be improved if teachers are exposed to effective instructional practices in which use of representations may be encouraged. This instructional practice may be enhanced through professional development programs. "Professional development should in theory, produce changes in teacher practice and, ultimately, improvements in student achievement" (Hill, 2004, p. 217).

2.6 Patterning and Algebra Within the Wider Context of School Mathematics

In order to provide common ground for understanding the concepts of patterning and algebra as they relate to this study, it may be helpful to examine the Grade 8 mathematics curricula in both Ontario and Lagos. The challenges students face in algebra may be derived from curricular issues (Arcavi, 1995), however, Ball (2003) claims that "no curriculum teaches itself" (p. 1). Teachers are expected to implement a curriculum that promotes the use of representations (OMoE, 2005b) in the teaching and learning of mathematics and this study seeks to investigate the extent to which teachers are sufficiently informed about this approach. In particular, this current study sought to determine how the teachers enacted the process of transition between perspectives and instructional practices as they utilized representations when teaching patterning and algebra. Swafford and Langrall (2000) concluded that the emphasis in any mathematics curriculum should be more on developing and linking multiple representations than generalizing problem situations. I will now clarify the curriculum objectives of teaching patterning and algebra in both Ontario and Lagos.

2.6.1 *Objectives of Teaching Patterning and Algebra in Ontario*

The Ontario Curriculum, Grades 1-8: Mathematics, 2005 is organized into five content areas/strands, listed as number sense and operation, measurement, geometry and spatial sense, patterning and algebra, and data management and probability (OMoE, 2005b). Each content area includes both overall and specific learning outcomes for each grade. The overall learning expectations for Grade 8 focus on:

- representing linear growing patterns (where the terms are whole numbers) using graphs, algebraic expressions, and equations;
- model linear relationships graphically and algebraically, and solve and verify algebraic equations, using a variety of strategies, including inspection, guess and check, using a "balance" model (p. 116).

The specific expectations of Grade 8 patterning and algebra teaching are as follows (see Table 1).

Table 1:

Curriculum expectations adapted from Ontario Curriculum. (OMoE, 2005b, pp. 116-117).

Area	Specific expectations
Patterns and Relationships	• Represent, through investigation with concrete materials, the general terms of a linear pattern, using one or more algebraic expressions • Represent linear patterns graphically (i.e., make a table of values that shows the term number and the term, and plot the coordinates on a graph), using a variety of tools (e.g., graph paper, calculators, dynamic statistical software) • Determine a term, given its term number, in a linear patter that is represented by a graph or an algebraic equation
Variable, Expressions, and Equations	• Describe different ways in which algebra can be used in real-life situations • Model linear relationships using tables of values, graphs, and equations through investigation using a variety of tool (e.g., algebra tiles, pattern blocks, connecting cubes, base materials) • Translate statements describing mathematical relationship into algebraic expressions and equations • Evaluate algebraic expressions with up to three terms, by substituting fractions, decimals, and integers for variables • Make connections between solving equations and determining the term in a pattern, using the general term • Solve and verify linear equations involving a one-variable term and having solutions that are integers, by using inspection, guess and check, and a balance model

Specific expectations of the Ontario's patterning and algebra strand as contained in the Grades 1-8 mathematics curriculum.

These objectives comprise only an outline of the overall expectations and specific expectations of patterning and algebra in Ontario. For more details, reference should be made to *The Ontario Curriculum, Grades 1-8: Mathematics*, *2005* document (OMoE, 2005b).

2.6.2 *Objectives of Teaching Patterning and Algebra in Lagos*

The *9-year Basic Education Curriculum, Mathematics for Upper Basic Education JSS 1-3* document use in Lagos is a national curriculum. It is organized into five content areas/strands, namely as number and numeration, measurement and geometry, basic operations, algebraic processes, and everyday statistics (Federal Ministry of Education [FMoE], 2008). Each content area includes general and specific (teacher's activities and student's activities) learning outcomes for each grade. The general learning expectations for Grade 8 algebra emphasize mastery of four basic mathematical skills. The following are the general expectations (see Table 2).

Table 2:

Curriculum expectations adapted from Nigeria Curriculum (FMoE, 2008, pp. 12-15).

Area	General expectations
Algebraic expressions	Expand a given algebraic expression Factorize simple algebraic expressions Apply the use of quadratic equation box in expanding and factorizing algebraic expressions Solve quantitative reasoning problem Simplify algebraic expressions of fractions with monomial denominators Interpret and solve word problems involving algebraic fraction:
Simple equations	Solve problems of simple equations such as 3n – 4 =2n + 1
Linear inequalities	Identify linear inequalities in one variable Solve linear inequalities in one variable Represent solution of linear inequalities in one variable on num line Solve word problems involving linear inequalities in one varial
Graphs	Identify the x-axis and y-axis Plot points on the Cartesian plane Prepare tables of values Plot the graphs of linear equations in two variables Interpret the plotted graph Plot linear graphs from real life situations Solve quantitative aptitude problems

Specific expectations of the Nigeria's algebra strand as contained in the Grade 8 mathematics curriculum.

Comparing the two jurisdictions' lists of expectations suggests there are sufficient parallels between the two jurisdictions to constitute a basis for this research, considering the similarities in mathematics curricula contents in both countries. A further description of the differences and similarities noticed in the curricula is discussed in the next section.

2.7 A Synthesis of the Expectations of Patterning and Algebra Teaching in Ontario and Lagos

It would seem that the specific expectations of algebra in Ontario are mostly consistent with those of Lagos. The specific expectations of algebra teaching in Ontario could be drawn from two interrelated headings for Grade 8 outlined in *The Ontario Curriculum, Grades 1-8: Mathematics*, *2005*. These are “Patterns and relationships” and “Variables, expressions, and equations” (OMoE, 2005b, p. 116). The Lagos *9-year Basic Education Curriculum, Mathematics for Upper Basic Education JSS 1-3* document for Grade 8 highlights algebraic expressions, simple equations, linear inequalities, and graphs (FMoE, 2008) as specific expectations of focus.

There appear to be many similarities between Ontario and Lagos in terms of the expectations of algebra teaching in Grade 8 mathematics. The algebra general expectations of both, for example, emphasize the development of students’ abilities to model linear relationships graphically and algebraically. In both settings, there also appears to be commitment to teaching students problem-solving skills in algebra with the aim of describing and applying these skills in real-life situations. Both curricula also emphasize that learners should be able to evaluate algebraic expressions by substituting fractions for variables, and the use of different representations.

Despite these similarities however, there are distinct zones of mutual exclusivity between the Ontario and Lagos mathematics curricula in terms of relative emphasis on approaches and expectations. In Ontario, the patterning and algebra strand emphasizes the need for learners to represent linear patterns in different ways and establish one or more algebraic expressions and equations.

Another dissimilarity between the Ontario and Lagos algebra curricula concerns the relative emphasis on the extent of connections between patterning and algebra. The Ontario curriculum appears to have emphasized the need to link patterning with algebra more explicitly than the Nigerian one. For example, the OMoE (2005b) states that "problem solving provides students with the opportunities to develop their ability to make generalizations and deepen their understanding of the relationship between patterning and algebra" (p. 9) effectively indicating that patterning and algebra are two separate topics. In contrast, the Lagos curriculum places emphasis on formal algebra requirements for higher algebra learning (Odili, 2006). For example, students in Lagos are supposed to begin a formal study of algebra that involves development of algebraic reasoning and generalization, factoring, and use of algebraic symbols in solving of equations, with no mention of patterning.

In identifying these discrepancies, it must be pointed out that the Ontario provincial Grades 1-8 mathematics curriculum outlines the mathematical process expectations associated with all the strands in greater detail than does the Lagos State one. In Lagos, details pertaining to mathematical processes are stated in the curriculum as essential elements necessary to understand major ideas of mathematical concept (FMoE, 2008). In contrast, the curriculum in Ontario, (OMoE, 2005b) states that:

> students represent mathematical ideas and relationships and model situations using concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols. Learning the various forms of representation helps students to understand mathematical concepts and relationships, communicate their thinking, arguments, and understandings, recognize connections among related mathematical concepts, and use mathematics to model and interpret realistic problem situations. (p. 16)

In Lagos, the student activities portion of the mathematics curriculum states that students represent mathematical ideas and relationships using graphs, tables, and a quadratic equation box to interpret and solve realistic problems.

Given these similarities and differences, the objectives of patterning and algebra teaching in Ontario and Lagos schools may be summarized as including the following ideas:

- Development of students’ ability to visualize, represent pictorially and apply algebraic ideas to describe and answer questions about a variety of patterns
- Development of students’ mathematical reasoning skills which relate to mathematicians consistent interest in arguing, conjecturing, identifying, investigating, justifying, and generalizing
- Development of students’ understanding of algebraic expressions and equations in real life situations

The expectations of patterning and algebra teaching in Ontario and Lagos as outlined above appear to be consistent with what NCTM (2000) suggests is suitable for Grades 6-8, namely:

- Understand patterns, relations, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships
- Analyze change in various contexts

The NCTM descriptors appear to be consistent with the recommended reasons for paying attention to algebraic thinking mentioned in OMoE (2014, p. 5), which are:

- Exploring properties and relationships
- Exploring equality as a relationship between quantities
- Using symbols including letters as variables

Indeed, central to the internationally recognized standards of the National Council of Teachers of Mathematics (2000) document, the *Principles* and *Standards for School Mathematics* for grades pre-K-12 emphasizes “relationships among quantities, including functions, ways of representing mathematical relationships and the analysis of change” (NCTM, 2000, p. 37). Thus, it seems that generating and providing appropriate representations are crucial for the attainment of the patterning and algebra expectations. Teachers generate and provide representations, and they decide on the learning experiences that learners will go through in class. So, teachers determine how the expectations are implemented in schools. Teachers' knowledge of how to introduce and apply the use of representations may have an effect on how the curriculum expectations are implemented. The teachers' instructional practices may also be influenced by the school system when implementing the curriculum expectations. My research was set in two jurisdictions, so professional learning is very important for understanding teachers' knowledge, beliefs and practice in this study. Teachers acquire these practices, knowledge and beliefs from experience at school, in-service training and professional learning programs (Ladele, 2013). Next, I will examine professional learning.

2.8 Professional Learning for Teaching

Professional learning may allow teachers to gain an understanding of how students think as they engage in mathematical tasks, as well as modifying teachers' practices to improve students' understanding (Krebs, 2005; Sowder, 2007). Effective professional development may enhance teachers' pedagogical and content knowledge in mathematics (Lee, 2007). Mathematics researchers (e.g., Ball & Cohen, 1999; Cwikla, 2004) argue that there needs to be a focus on teacher's practices, in particular, teachers' thinking, and student thinking and learning.

2.8.1 *Professional Learning in Ontario*

In Ontario, it is noted that professional development should aim to empower teachers to make their own changes (Ontario Ministry of Education, 2007). Sinclair and Bruce (2015) note that "another area of fertile, future work is certainly that of teacher preparation and professional learning" (p. 327).

In a call to strengthen school district capacity to enhance mathematics teaching and learning across Ontario, the Ontario Ministry of Education launched a professional learning program in Kindergarten to Grade 6 (Bruce, Esmonde, Ross, Dookie & Beatty, 2010). Bruce et al. (2010) investigate the relationship between teacher efficacy and student achievement of two school districts, one with history of professional learning and the other without. The pretest mean score of teacher efficacy in district A (exposed to professional learning, mean = 4.67) in their commitment to standard-based teaching was lower compared to district B (not exposed to professional learning, mean = 5.22). Interestingly, the posttest mean score of district A (mean = 5.00) was higher than district B (mean = 4.88) after the professional learning program, even though it started out as lower. One important thing that is of interest in the study was change in student achievement of district A. Bruce et al. (2010) examined student mathematics content areas and their use of different mathematical learning processes (problem-solving, communication, reasoning, representation, and connections). There was significant improvement in the post-test mean scores of district A students in all the areas as compared to district B, suggesting that the professional development did translate to higher student achievement.

Many of the strategies of teaching through problem solving and analyzing student mathematical thinking were initially unfamiliar to the majority of the participants in district B. Teachers in this school district before the intervention focused relatively more on professional learning in literacy than mathematics (Bruce et al., 2010): This corroborates one of the findings of Holm (2014) as she reported that some of the elementary teachers expressed concern that compared to the funding, time, and energy allocated to literacy professional learning, there have not been sufficient funds for mathematics, and this lack has accounted for student scores falling below the expectation. Kajander (2010) noted that professional development in mathematics is currently not readily available in Ontario to all practising teachers.

Bruce, Esmonde, Ross, Dookie and Beatty (2010) provide a series of inspiring arguments for examining growth in professional learning… “sustained professional learning that are collaborative and classroom-embedded, support effective professional learning that leads to substantial student achievement gains and the related gains in teaching quality" (p. 1609). Borko (2004), in her review of research on teacher professional development, concluded that effective professional learning programs should emphasize subject content knowledge involving teachers working through specific problems and tasks. Participants in professional development programs should be given the opportunity to decide what they prioritize as important for the success of their development and not the priority interest of the researcher (Kajander & Mason, 2007). For effective teaching to be visible in our classrooms, teachers require sufficient content knowledge, positive attitudes and confidence in teaching mathematics (Moss, Bruce & Bobis, 2016).

2.8.2 *Professional Learning in Lagos*

In Lagos, a major realization from the national policy on education is emphasis on teacher development as the key to effectively implement policy and curriculum, foster teachers' thinking and raise educational standards (Federal Ministry of Education, 2009). A major impediment to realizing this goal is the lack of qualified teachers, hence, the need for professional development at the elementary and secondary levels of education. Professional development of teachers therefore remains a key factor in ensuring quality teaching at any level (Ladele, 2013). Teacher needs continuing professional learning in order to be informed of new developments in curriculum and pedagogical brought about as a result of changing and evolving educational, social and cultural context (Ladele, 2013; Olaleye, 2012). Ladele (2013) argues that professional learning may provide teachers with an understanding of students' thinking on how they solve mathematical problems.

In Ladele's (2013) study, 12 intermediate grade teachers participated in a professional learning program that focused on teacher awareness of students' misconceptions and errors in beginning algebra and language-based teaching strategies particularly using the Newman interview protocol. According to Ladele (2013), Newman interview protocol consists of five structured questions students are asked in relation to a given problem that they have previously solved incorrectly. One of the teachers in Ladele (2013) acknowledged that her algebra knowledge was limited as she was not aware of some of the misconceptions about a letter as an object or label. Ladele (2013) reports that before the intervention, in one of the lessons, the teacher explained "the letter in algebra stand for something. You will understand it better when you attach the number to something" (p. 130). As a result of the professional learning intervention, the same teacher who was observed over a six-week period opines that teacher should be trained in the discipline (mathematics) and should have strong knowledge in order to teach mathematics effectively (Ladele, 2013). After the professional learning, stronger beliefs that communication skills, feedback, and language-based approaches were effective strategies for teaching and learning mathematics emerged.

2.9 Critique of Teaching and Learning in Ontario and Lagos

The attempt to synthesize the objectives of patterning and algebra teaching in two jurisdictions like Ontario and Lagos, each with its separate, well-articulated set of objectives, poses a challenge.

Table 3:

A comparison of selected features of school systems in Ontario and Lagos.

Basis of comparison	Ontario	Lagos
Jurisdiction for education	Education is a residual power of the province	Education is the residual power of both federal and state
Influence of state/provincial department of education	Strong	Moderate
Federal presence in education	No federal department of education, little funding, no federal law	High federal influence the education system; federal department of education, indirect funding through the state, federal laws mostly influencing education
School board autonomy	Relatively dependent	Relatively dependent, directors are appointed by the state
Funding equality	Relatively modest inequalities among school boards, and schools	Relatively high inequality
Professional development	School board	Initiative of the state
Status of professional development	Ongoing professional training	Not continuous; once in every five years or not at all; a lot of bureaucracy involved

Note. Some features of school systems noticeable in both settings.

gives a summary of some selected features of schooling in Ontario and Lagos. One of the major goals of this study as stated earlier is to describe and assess teachers' patterning and algebra instructional practices using representation in Ontario and Lagos. The purpose of this review of literature has therefore been, in part, to establish a theoretical framework in terms of how to evaluate teachers' perspectives in these jurisdictions.

Mathematics teaching in Ontario should include a focus on conceptual understanding (Kajander, 2007). In contrast, the focus of mathematics teaching in Lagos is towards preparing students for examinations (Okereke, 2016). For example, in Lagos, many teachers cling to traditional methods of teaching in which answers to the previous day's home work are first given, then teacher directed explanations are used to present materials for the new lesson (Odili, 2006). The power of thinking and understanding are thus not developed in the students. Conceptual understanding appears not to be a focus.

Amazigo (2000) and Okereke (2016) have identified teaching problems and shortage of qualified mathematics teachers as major factors responsible for poor performance in algebra in Nigeria. In a wider study across Africa, Bassey, Joshua, and Asim (2007), blamed the colonizers of Africa for applying direct transfer of western science curriculum, examinations, and teaching methods which have failed to address the continental challenges of Africa. The direct effect of the transfer of western curriculum is evidenced in de-contextualized knowledge being transmitted by poorly trained teachers in under-resourced and sometimes overcrowded classrooms. For example, although in most schools in Lagos, teachers are expected to be specialists in their subject areas, there are many non-professional and inexperienced teachers who present topics of mathematics to students in such a way that students find it difficult to grasp (Iji, 2002; Onose, 2007).

In Ontario, although many elementary teachers are generalists with little specialist expertise in mathematics education and sometimes have mathematics phobia (Adeyemi, 2015), professional learning programs may have been used to strengthen their confidence (Holm, 2014). In general, my study may uncover some findings as a result of the context in which the school systems in both jurisdictions operate. As this study examines how Grade 8 teachers in Ontario, Canada and Lagos, Nigeria generate and provide representations during teaching of patterning and algebra, the differences between the two jurisdictions may serve as a caution against applying generalizations. The next section of this literature review focuses on mathematics teachers' knowledge, perspectives and practices.

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Note. Some features of school systems noticeable in both settings.

We now have an overview of the definition of a representation, its central role in algebra and patterning, effective instructional practice, the schooling systems and the curricular expectations in the two jurisdictions under study. How do these translate into the classroom? What is the role of teachers' beliefs, knowledge, and practice in the effective use of representation in algebraic instruction? The last section of this review therefore considers literature about teachers' beliefs, knowledge and how they impact teachers' use of representations and about practice.

2.10 Mathematics Teachers' Beliefs, their Knowledge, Perspective, and Practices

According to Artzt, Armour-Thomas and Curcio (2008), the instructional practices of the teacher occur in the classroom where teachers' goals, knowledge, and beliefs play a central role in their instructional efforts to guide learners in their search of knowledge. In this section, I will give an overview of the different domains of teachers' knowledge, and describe teachers' belief systems about the use of representations when teaching algebra and patterning. Additionally, I report what the influence of teachers' knowledge and beliefs is on their instructional practices as they engage with representations.

2.10.1 *Knowledge in Relation to Representations*

In 2008, Ball, Thames, and Phelps introduced the notion of mathematical knowledge for teaching (MKT) in an elaboration of Shulman's categorization of teacher knowledge. Ball et al. (2008) hypothesized some refinements to the concept of pedagogical content knowledge and to the broader concept of content knowledge for teaching. Ball et al. (2008) focused on the domain of the mathematical knowledge and skills needed by teachers. The mathematical knowledge as conceptualized by Ball et al. (2008) is in two categories, namely subject matter knowledge and pedagogical content knowledge (PCK). The subject matter knowledge was further divided into common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge while PCK includes knowledge of content and teaching, knowledge of content and students, and knowledge of content and curriculum.

CCK involves knowing central facts, concepts, and principles within a relationship and using terms and notation correctly. SCK is to know more than just explaining the content; teachers must be able to explain why a concept works, why it is worth knowing, and how to relate it to other learning outcomes and other disciplines both in theory and in practice. Kajander et al. (2010) talked about SCK as "other" mathematical understanding which could be seen "as facility with appropriate mathematical models, alternate approaches to concepts and ways of thinking and reasoning conducive to students" (p. 50). For example, other researchers (Baumert et al. 2010; Ma, 1999) maintained that a teacher with profound conceptual understanding is able to explain why procedures work and under what conditions they do not work, recognize and select representations, and link them to underlying mathematics ideas. Research has shown that the knowledge required for teaching mathematics is different from the knowledge possessed by other professionals who use mathematics in solving problems in different areas (Ma, 1999). Ball et al.'s (2008) 'mathematical knowledge for teaching' is similar to Ma's conception, although there are some fundamental differences.

While making a comparison of the knowledge of basic mathematical concepts among American and Chinese teachers in her study, Ma (1999) revealed that the Chinese teachers' knowledge of mathematical content included knowledge of how the content might be comprehensible to learners. Therefore, for the Chinese teachers, pedagogical content knowledge is fundamentally interwoven with content knowledge. In contrast, Ma revealed that U.S. teachers hardly indicate any sort of connections among the topics discussed in her study. Further, she observed that with the U.S. teachers, there was lack of interaction between pedagogical content knowledge and content knowledge. Ball et al. (2008) refer to aspects of this interwoven knowledge as specialized content knowledge.

For the purpose of the current study, SCK is applicable as teachers are expected to demonstrate particular skills in explaining the process of using representations, in a way that those in other professions or fields would not be expected to do. Ball (2003) argues that it is essential for teachers to know more than what other educated members of the society are required to know when illustrating mathematical concepts. CCK is also important in this study as it is expected that the application of the subject as knowing central facts, concepts and representations will help transitions to problem solving and vice-versa, particularly in the case of algebraic concepts.

The focus of research on MKT has revealed the importance of teacher knowledge particularly in the era of reform teaching (Baumert et al. 2010; Hill, Ball & Schilling, 2008; Holm & Kajander, 2012; Shechtman, Roschelle, Haertel & Knudsen, 2010). However, Shechtman el at. (2010) found that MKT did not correlate with instructional decision-making (topic coverage, choice of teaching goals, and use of technology). Shechtman et al. suggest that investigating how MKT influences student learning within the context of the full classroom instructional system will reveal greater insights. On the other hand, Baumert et al. reported that teachers' pedagogical content knowledge was theoretically and empirically distinguishable from their content knowledge. Baumert et al. revealed that a substantial positive effect of pedagogical content knowledge on students' learning gains was mediated by the provision of cognitive activation and individual learning support. Also, their findings revealed deficits in CCK are to the detriment of PCK, limiting the scope for PCK development.

According to Ball, Lubienski, and Mewborn (2001), pedagogical content knowledge underlies the choice of representations and explanations. Studies revealed that an insufficient understanding of mathematical content may limits teachers' capacity to explain and represent such content for better understanding to students (Even, 1993; Stein, Baxter, & Leinhardt, 1990). In addition, many teachers may lack knowledge of how mathematical ideas are transformed into representations (Ball, 1999; Moyer, 2001; Stylianou, 2010). For example, Stein et al. (1990) examined an experienced fifth grade teacher as he taught a lesson sequence on functions and graphing. These authors found that the teacher lacked knowledge for fostering meaningful connections between key concepts and representations. Moyer (2001) in her study of 10 teachers revealed that some of the teachers had difficulty following the students' thinking and their use of representations. Molenje and Doerr (2006) argue that the teacher's knowledge of mathematics is fundamental to how he or she articulates and balances the use of different representations.

Ball (1999) contends that teachers must develop fruitful representational contexts to help students' mathematical thinking. Cobb, Yackel and Wood (1992) suggest that teachers should not use representations in a rigid way as this could encourage "algorithmatization of mathematics" (p. 14) among students, and could result in students not applying their mathematical understanding gained in school in other similar situations, particularly in out-of-school settings. Appropriate use of representations should be one of the teachers' ways of facilitating instruction; and furthermore, specialized content knowledge should help teachers in this regard. Yet even if teachers have adequate knowledge of using representation as well as knowledge of patterning and algebra, they may sometimes be influenced by their belief system. Next, I will discuss the teachers' belief systems.

2.10.2 *Beliefs in Relation to Representations*

Teachers' beliefs about representations may also play a role in how they use them. Artzt et al. (2008) define beliefs as integral systems of personalized assumptions that include the nature of the subject, the students, learning and teaching. Beliefs can be conceptualized as mental representations that describe the subjective probability that an object has particular characteristics (Wyer & Albarracín, 2005), while epistemological beliefs can be conceptualized as a system of more-or-less independent beliefs (Hofer & Pintrich, 1997; Schommer & Walker, 1997). By "a system of beliefs" it is meant that there is more than one belief to consider. In other words, individuals are influenced by different beliefs, ranging from family, social beliefs, cultural assumptions, and peer beliefs. For instance, some families might believe that a young female child cannot outperform a male child in mathematics and a male child cannot outperform a female child in English literature.

Researchers cannot assume that because “one belief or set of beliefs is logically incompatible with others that these beliefs cannot co-exist” (Fives & Buehl, 2014, p. 443). There is no belief that can stand alone or exist in isolation, and there are situations when we see connections between our beliefs and those of others, in terms of what we do, and how we do it. Thus, as our beliefs systems differ, teachers must be equipped with multiple teaching skills and abilities so as to understand the impact of such beliefs on classroom activities. For example, to use instructional strategies that foster communication and get students to engage in mathematical reasoning, teachers must be facilitators of students’ learning (Artzt et al., 2008).

Like other instructional practices, use of representations is related to teachers’ pedagogical beliefs, mathematical disposition, and pedagogical understanding of what and how a teacher could be thinking in a given situation. According to Ball (1990), teachers’ beliefs about mathematics are powerful as they tend to influence their representations of mathematics. Teachers’ beliefs about representations often affect students’ use as well. For example, some teachers may believe that representations are helpful for students to show their thinking after the fact, while others may believe that representations are useful tools to support the actual thinking process. Teachers’ beliefs about the use of mathematical representations may inform what is displayed and how it is displayed as the teacher tend to make appropriate selections relating to their beliefs during problem solving (Elia, Gagatsis, & Demetriou, 2007; Niemi, 1996; Panaoura, Gagatsis, Deliyianni, & Elia, 2009; Speer, Smith, & Horvath, 2010). So, teachers should emphasize to students why they must understand the meaning of what they are learning and be able to express the ideas in different ways, and representations are useful in this regard.

Stigler and Hiebert (1997) articulate that “teachers should be engaged in improvement because they are the only ones who can ensure students’ learning improves” (p. 136). As the gatekeeper and authority figure in classrooms, the teacher is expected to recognize that within a given classroom, students could engage in multiple ways of interpreting a problem situation and have multiple paths for refining and revising their ideas (Lesh & Doerr, 2003). In general, belief systems are related to teachers' use of representations. According to Ball (1990), teachers' beliefs about mathematics are powerful as they impact their uses and choices of representations of mathematics. In Ball's study, many teachers were challenged in terms of how they often used representations unknowingly in class, for example, when they resorted to the use of representations with unique characteristics. Next, I will focus on the description of teachers' practices on the use of representations.

2.10.3 *Practices: Use of Representations*

Mathematics education researchers (e.g., Izsák & Sherin, 2003; Knuth, 2002; Stylianou, 2010) suggest that teachers may have gaps in their ability to use mathematical learning processes when doing and teaching mathematics, and that their learners may also experience difficulties with the same processes. Barmyby, Bolden, Raine, and Thompson (2013), for example, observed eight primary school teachers on the use of diagrammatic representations of mathematics concepts in their classrooms after three one-day professional development training sessions. Barmby, Bolden, Raine, and Thompson (2013) found different levels of sophistication of classroom teachers' use of diagrammatic representations. Barmby et al. revealed that some of the teachers included as many visual representations as they could without a great deal of thought about how the children might make the links between the different representations. Representation emerged but teachers were not able to consider how pupils could make the necessary connections between different representations.

In Dreher and Kuntze's (2015) study, about 100 teachers from two different secondary school types in Germany participated in a study about how teachers handle the double role of representations. The double role of representations as pointed out in the study stated that: on one hand changing between representations is essential for mathematical understanding, but on the other hand such changes can involve excessive demands that often hinder learning. Dreher and Kuntze (2015) studied how teachers evaluate the learning potentials of tasks which make use of multiple representations. Dreher and Kuntze (2015) found that teachers may have low awareness of the double role of multiple representations for students' learning. Dreher and Kuntze (2015) concluded that teachers did not fully understand the key role of multiple representations for learning mathematics.

David, Tomaz, and Ferrira (2014) observed 28 lessons of 90 minutes each on how visual representations for Grade 9 students are introduced, in order for students to transform ideas into specific algebraic procedures. The authors illustrate how a teacher's use of a visual representation display on the board, accompanied by the metaphor of a "shower" in illustrating the use of the distributive law in early algebra, can become over-generalized as well as used incorrectly by students. David et al. (2014) found a difference between the teachers' way of signifying the algebraic procedure and the students' overuse of a visual display they associated with it. The teacher became aware of students' inappropriate use, and worked hard to correct it. David and colleagues further revealed that these tensions impel changes in the classroom activity and further point out that there could be cases where a teacher is not aware of possible misinterpretation on the part of students. as an alternative to students using representations provided by teachers, encouraging and supporting students in constructing their own representations and using them for their own benefit in learning may provide better learning.

Bill (2000) observed a teacher and his 33 pupils to explore how the teacher used a variety of external representations to communicate mathematical ideas to his pupils. Bill found that pupils seldom spontaneously visualized teachers' representations or attempted mental manipulation of visual images to help with calculation. He noted, however, that pupils had mental representations that reproduced some aspects of the teachers' representations. In summary, teachers' use of multiple representations is a potential strength of instruction aimed at improving students' problem solving but could cause the possibility of over-generalization and allow students to make inappropriate connections between different representations; thus, it is important for teachers to be aware that representation is not self-explanatory, and they must ensure a shared understanding.

Ferrini-Mundy, Lappan and Phillips (1997) maintained that verbal descriptions, tables, graphs, and symbolic expressions are all reasonable ways of expressing relationships that aim toward generalization. For example, the translation from the verbal expressions such as "the set of points for which the y coordinate is five times the x coordinate" to the algebraic representation of "y = 5x" needs to be better understood to make the appropriate connections and relationships. Ferrini-Mundy et al. (1997) recommended PCK with respect to representations that need to be studied in greater depth for a better understanding of how representations are used by the teachers. In order to contribute to such a greater understanding, my study explores teachers' beliefs, knowledge and practices, based on their ways of thinking about multiple representations in the classroom.

CHAPTER THREE: THEORETICAL FRAMEWORK

3.1 Introduction

Any researcher approaches a study field “with some orienting ideas” (Doll, 2012, p. 17). One of the main contributions of the theoretical framework is that it enables a new and different perspective on the seemingly familiar and ordinary (Jansen, 2013). According to Maxwell (2005), the theoretical framework of a study is “the systems of concepts, assumptions, expectations, beliefs, and theories that support and inform research” (p. 33). In the current study, constructivism theory was used as a theoretical framework to analyze teachers’ perspectives about using multiple representations, and their use of them. This section explains constructivism as a framework that underpins my research as follows. First, there is an overview of the study, including methods employed as a basis for the discussion of the framework. Second, an explanation is provided on how the theoretical framework was selected, along with a discussion of why this particular framework is appropriate. Subsequently, the theoretical framework is described in detail.

3.2 Rationale for Using Constructivism as A Framework

This current study sought to understand how teachers generate and provide representations during patterning and algebra teaching. It also examined how teachers’ perspectives and instructional practices impact or contribute to their instructional practice of algebra using representations. A mixed methods approach was used. An embedded multiple case study using quantitative and qualitative analysis was selected.

Two research questions were posed: *In what ways do grade eight teachers in Ontario and Lagos generate representations in their teaching of patterning and algebra?* and *In what ways do grade eight teachers in Ontario and Lagos provide representations in their teaching of patterning and algebra?*

The theoretical framework of the current study draws generally on constructivism with a particular focus on Vygotsky's (1978) notion of social constructivism. Research in mathematics education contributes to the description and understanding of the context, situation, and practice, and for this reason, I chose to use social constructivism. The goal for the use of this theory is to understand desirable instructional practices in the context of two settings (Ontario and Lagos), specifically with respect to teachers' perceptions of teaching patterning and algebra using representations in order to improve students' conceptual understandings.

3.3 Constructivism

Constructivism is basically a metaphor of learning likening the acquisition of knowledge to a process of building or construction (Tobin, Tippins, & Gallard, 1994). The theory suggests that learning involves constructing, creating, inventing, and developing one's own knowledge and meaning. Perkins (1992) notes that constructivism has multiple roots in psychology and philosophy, such as the developmental perspectives of Piaget (1969), and that of cognitive psychology. Constructivism is a set of beliefs about knowledge that begins with the assumption that reality exists but cannot be known as a set of truths (Tobin, Tippins, & Gallard, 1994). A key idea of constructivism is that knowledge cannot be transmitted in any direct way to students; instead, they construct knowledge themselves based on their experiences and social environment (Clement & Battista, 1990). Teachers can facilitate this process by playing the role of guidance as they help students in navigating through learning new ideas.

Although, constructivism describes the way that students make sense of materials and also how the materials can support learning, it can also be used to understand how teachers make sense of representations and use them in teaching patterning and algebra. The ways in which teachers develop their classroom practice is tied to their understanding of how their students learn (Jofili, Geraldo & Watts, 1999). The theory of constructivism covers a wide range of forms chosen from the literature, such as Piagetian constructivism, radical constructivism, social constructivism, and critical constructivism.

3.3.1 Piagetian Constructivism

Piaget emphasises the idea that individuals construct knowledge for themselves through construing the repetition of events, and that knowledge is individual and adaptive rather than objective (Geelan, 1997). Piaget's theory of constructivism describes what students are interested in, and able to achieve, at different stages of their development. Piaget believed that knowledge is not information teachers deliver expecting the students to encode, memorize, retrieve, and apply in the future.

3.3.2 Radical Constructivism

von Glaserfeld, the defender of radical constructivism, emphasizes the ability of human beings to use the understandings they create to help them navigate life, regardless of whether or not such understandings match an external reality (von Glaserfeld, 1993). Von Glaserfeld draws from Darwinian evolutionary and Piagetian cognitive developmental theory, to point out that human perception is adaptive. From the point of view of constructivism, the process of knowing is that the learner dynamically adapts to a variable interpretation of experience and he/she doesn't need to construct knowledge related to the real world. In a radical constructivist approach the emphasis is on discovery learning, and learning in complex situations. "For radical constructivists, mental representations—evaluated in terms of their viability, empirical adequacy, and goodness of fit with experience—are central" (Shotter, 1995, p. 54). From the perspective of radical constructivism, communication is not necessary to involve sharing meaning among participants. Von Glaserfeld (1990) points out that only if the learner does everything exactly and meets the expectation of others, then shared meaning is kept. "Thus, radical constructivists believe that students learn through a uniform sequence of internal reorganizations, each more encompassing and integrative than its predecessor" (Prawat & Floden, 1994, p. 43).

3.3.3 Critical Constructivism

This theory involves a synthesis of constructivist interest in the interaction of students' knowledge with new knowledge. Critical theory promotes self-reflection. Kincheloe (1995) describes critical constructivism as follows:

> Critical constructivism [..] ask what are the forces which construct the consciousness, the ways of seeing of the actors who live in it. […] critical constructivism concerns attempts to move beyond the formal style of thinking which emerges from empiricism and rationalism, a form of cognition that solves problems framed by the dominant paradigm, the conventional way of seeing. (p. 88).

Critical constructivism provides teachers with the opportunity to contextualise that thinking within a broader social historical and political context.

3.3.4 Social Constructivism

Social constructivist theorists draw on Leontev's and Vygotsky's work and note that an individual develops her/his reasoning in line with the patterns of the society (Cobb, 2007). Social constructivist theory recognizes the role of the teacher and the need for the teacher's own knowledge as an important aspect of teaching. Although Vygotsky's ideas centred mostly around the learning of children, his ideas also provide the basis for exploring adult learning, which is important since the current study focuses solely on teachers' perspectives. It is therefore a useful perspective to understand how the teacher may scaffold, support, and create opportunities for students to appropriately use and develop multiple representations. Social constructivism was used here as the predominant theory for developing the framework for understanding the teachers' instructional practice, since teachers typically operate in a variety of social settings.

Teaching is such a complex activity that it must be analyzed in many ways to study it and to share what is learned (Hiebert, Gallimore, & Stigler, 2002; Stigler & Hiebert, 1997). Greeno and Hall (1997) argued that "learning to construct and interpret representations involves learning to participate in the complex practices of communication and reasoning in which the representations are used" (pp. 361-362). Further, researchers contend that the process in which mathematical concepts are learned is important and that learning occurs when proper communication and interaction using mathematical terms are explored (Campbell, Davis, & Adams, 2007; Lim & Presmeg, 2011). Such learning should encourage and support actions, group activity, creativity, diversity, mediated meaning, critical thinking, and interaction. Vygotsky (1978) maintained that human learning takes place in the form of interactions among signs, mediating artifacts/tools, and the individual, and according to von Glasersfeld (1995), "the human mind can know only what the human mind has made" (p. 21).

The theory of social constructivism suggests that we are not isolated individuals interacting with our environment on a purely biological basis, but rather that our relationship with the world is mediated by other people, through the use of signs or symbols of language (Tytler, 2012). According to van Oers (2000), "social assistance" (p. 141) is offered to students when complex ideas and solutions to problems are constructed on the social plane of the classroom and made available to support each individual as they internalize and construct knowledge. Vygotsky, who examined the tools of psychology (maps, language, and writing), claimed that since the tools were social, they were contrived. Vygotsky's interest focused on the development of human consciousness through mediation by the use of psychological tools such as language, but also social influences. Vygotsky maintained that our mental functions are social in origin and are incorporated in the context of the sociocultural setting.

Uden (2006) articulates that the learner, the material to be learned, and the context in which the learning occurs cannot be separated. Using a familiar context may help learners to interact with materials intended to be used in order to foster learning outcomes. More so, the nature of mathematics learning involves teaching that may involve various activities, and a mathematics teacher may want to use different representations such as diagrams, pictures, tables, symbols, textbooks, manipulatives, calculator and a computer to teach her/his students. Thus, the support of constructivist teaching may be enhanced by certain types of knowledge, as well as tools in the environment, and the availability, acceptance, and awareness of the best use of these resources could influence their effectiveness. Hence, the differences between the teachers' cultural settings in Ontario and Lagos may result in differences to their access to math knowledge and resources that could affect the use representations during teaching.

Researchers (e.g., Rogoff, 1998; Simon, 1995) in social constructivist theory contend that teaching should be dynamic, and a classroom that is scripted and solely controlled by the teacher deprives students of being able to co-construct their knowledge. The use of multiple representations may help to initiate, monitor, and encourage mathematical development within each student if the students are well supported and understood. For students to develop a mathematical sense, it is important to understand how teachers may use cultural and social factors to provide a safe place for taking mathematical risks through different mathematical tasks, providing rich problem contexts, and artefacts for illustrating ideas. Hence, having a good knowledge of what representation means and how to use it, may help teachers to see relationships between the human knowledge and artefacts. My research is based on the idea that teachers should learn to encourage multiple ways of solving problems, and plan mathematics activities that engage the students in such explorations, in particular in patterning and algebra.

3.4 A Social Constructivism Theory Perspective on Embodied Knowledge

Embodied knowledge is manifested in different ways by different types of representations. Some researchers (e.g., Alibali & Nathan, 2012) have argued that mathematical knowledge is embodied. Alibali and Nathan (2012) explained that mathematical knowledge is embodied in perception and action, and grounded in the physical environment. An embodiment of meaning can develop along different reconstructions of the symbol, diagram, model and so on. Additionally, the reconstruction of the context in which such embodiment is used gives powerful meaning, providing insights into the problem solving process (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). For example, symbols or models only become embodiments of a given mathematical system after a child has coordinated the relevant meaning (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Research in the field of mathematics education recognises the powerful influence that social context has on how students come to make meaning and make sense of the mathematical concepts and processes (Cole, 1996; Stylianou, 2011). Meanings are related to the embodiments of mathematical constructs (Alibali & Nathan, 2011). For instance, if students who are beginning to study algebraic concepts are given a task, that required them to explore and understand the nature of mathematical concepts without any form of embodiments up front, they tend to apply knowledge acquired within their cultural settings.

However, it is worth mentioning that learners bring different identities to the school context following some social and cultural factors and the embodiment of mathematical practices relevant to the learners' identity (Chionaki, 2011). Pape and Tchoshanov (2001) emphasized that "representation is inherently a social activity. Students come to understand both the process of representation and its product through social activity" (p. 126). Learners may be required to work with others, negotiate meanings, seek support when needed as well as share their experiences with the teachers and peers.

In order to develop the mathematical constructs that underlie the study of algebraic concepts, students may investigate structural similarities among activities involving different embodiments. For example, in Stylianou's (2011) study of grade-sixth students working on the Party problem, she revealed that although: the students were not offered any representations, algorithms, or worked-out examples up front, they were able to work on the task moving from concrete to abstract. Stylianou (2011) revealed that, “the students’ representations moved from showing tables with people, to actions on the people (slashes to show eliminations), to a row of tables without people, and finally to the abstract rectangle” (p. 11) with the teacher providing appropriate prompts in order to guide their thinking. Dienes used the term embodiment to refer to concrete manipulatable materials (e.g., arithmetic blocks) that are useful in helping students develop elementary but powerful constructs that provide powerful foundations for elementary reasoning. “Concrete materials only become embodiments of a given mathematical system after a child has coordinated the relevant actions to function as a system as a whole in the context of these materials” (Lesh & Carmon, 2003, p. 38).

Artefacts gain relevance when we seek to understand learning as a phenomenon emergent from participation in social practices (Holland & Cole, 1995). Artefacts are always considered in relation to use within a system of activity (Lantolf, 2000). Artefacts, whether physical or symbolic, are modified as they are passed on from one generation to another (Lantolf, 2000). Artefacts are collective tools with histories and functions that are continually modified within social practices in order to mediate human cognitions (Holland & Cole, 1995). We use symbols, tools, or signs to mediate and regulate our relationships with others and with ourselves and thus change the nature of these relationships (Cole, 1996). Artefacts together with their social structure are a part of the historical trace left by the reproduction cycles, and they reveal the production character of these cycles and the contribution to the constitution and re-construction of the practice over time.

It is common to see people characterising artefacts in two ways: (i) as tools and signs; (ii) external artefacts and internal artefacts. Engestrom (1999) proposes a further differentiation regarding the use of artefacts:

> The first type is *what* artefacts are used to identify and describe objects. The second type is *how* artefacts are used to guide and direct processes, and procedures on, within or between objects. The third type is *why* artefacts are used to diagnose and explain the properties and behaviour of objects. Finally, the fourth type is *where* are artefacts used to envision the future state or potential development of objects, including institutions and social systems (Engestrom, 1999, p. 382 Italics included in the original document).

In the current study, I focus on representations that teachers generate and provide during the teaching of patterning and algebra. Applying the idea of Engestrom (1999) will impact on my study in the area of what, how and why artefacts are involved. Teachers may need to identify and describe the kind of representations they use in their teaching of algebra. It is also important that the teacher may need to explain how these representations are used to guide and direct students' thinking process. The *why* of artefacts will further be used to illustrate the reason for choosing relevant representations that best fit the solution of a problem.

Through this current study, I explored the literature on how teachers come to make meaning from their mathematical experiences, and how social constructivist positions may be supporting or hindering their own way of using representations during the teaching of patterning and algebra. Representations, one of the reform-based practices of teaching and learning, may be required in order to redress poor achievement in mathematics among students in Lagos, in particular.

3.5 The Implications of Constructivism Theory on the Use of Representations in the Classroom

Representation is often understood to be a product; a static picture or set of symbols. These static representations or products are often used to aid instruction or illustrate a mathematical idea. Representation is also a process—the path that one follows while developing mathematical understanding. Diagrams and symbols evolve dynamically during problem solving, assuming different roles and providing insights into this process (Stylianou, 2011).

Some theorists suggest that what becomes critical is teachers providing a learning environment that enables students to build a deep understanding of mathematics. A constructivist perspective implies that teachers will guide and support students as they learn to construct their understanding of the culture and communities of which they are a part (Cobb, 1994; Jorgensen, 2014). Representations enable teachers to structure learning activities that address student misconceptions, seek student elaboration of their work, and pose questions. Social constructivist teaching practices focus on how teachers demonstrate problem steps, provide hints, prompts, and cues for successful problem completion and how they encourage and enable the use of appropriate materials and models. Teachers are supposed to provide explanations, elaborations and clarifications where requested, foster explanations, examples and multiple ways of understanding of a problem or difficult material (Bonk & Cunningham, 1998).

In the current study, focused on how teachers generate and provide representations while teaching patterning and algebra, I theorized that the more flexible teachers are, particularly in encouraging students in recognizing alternative ways to represent mathematical ideas, the more likely it is that the students will be successful in mathematics. As explained, I employed a social constructivist lens to analyze teachers' ideas and thinking about using representations in algebra instruction in their classroom.

CHAPTER FOUR: RESEARCH DESIGN AND METHODOLOGY—MIXED METHODS

4.1 Introduction

This chapter of the thesis provides an overview of the research questions, research approach, describes the proposed methodology of the study, and then articulates the design and the sampling strategy, instrumentation, methods of data collection and analysis, and ethical considerations. Every research project requires major methodological decisions (McMillan & Schumancher, 2010) in relation to the areas highlighted above.

4.2 Research Questions

The current study aimed to explore how Grade 8 teachers generate and provide representations. It sought to identify the representations they claim to use when teaching patterning and algebra. The research questions are:

1. What are teachers' goals for and perspectives of using representations in Ontario and Lagos?
2. How do teachers' goals for and perspectives of using representations differ by region?

4.3 Research Design-Justification for the Concurrent Mixed Methods Design

According to Creswell (2012), research design can be classified under quantitative, qualitative and mixed method research studies. The mixed methods approach was used here to answer the study research questions. Tashakkori and Creswell (2007) suggest that a researcher is involved in mixed methods when the researcher collects and analyses data, integrates the findings, and draws inferences using both quantitative and qualitative approaches in a single study. According to Johnson and Onwuegbuzie (2004), mixed methods research is "the class of research when the researcher mixes or combines qualitative and quantitative research techniques, methods, approaches, concepts or language into a single study" (p. 17). A mixed method is appropriate for my study, as it was used concurrently (quantitative and qualitative) to collect data through a survey in order to gain a deeper understanding of the participants' responses by conducting interviews so that inferences can be drawn on the two sets of data. I used the questions of the survey to gather data from Grade 8 teachers on their perspectives of using representations. I also used interviews to answer questions on how they use representations during instructions.

The mixed method is necessitated when the use of either qualitative or quantitative is inadequate to provide possible data to accomplish the purpose of a study (McMillan & Schumacher, 2010). The rationale for combining two types of data is that using the single approach designs of qualitative or quantitative is insufficient to understand the trends and details of situations (Creswell, 2012), such as teachers generating and providing representations while teaching patterning and algebra.

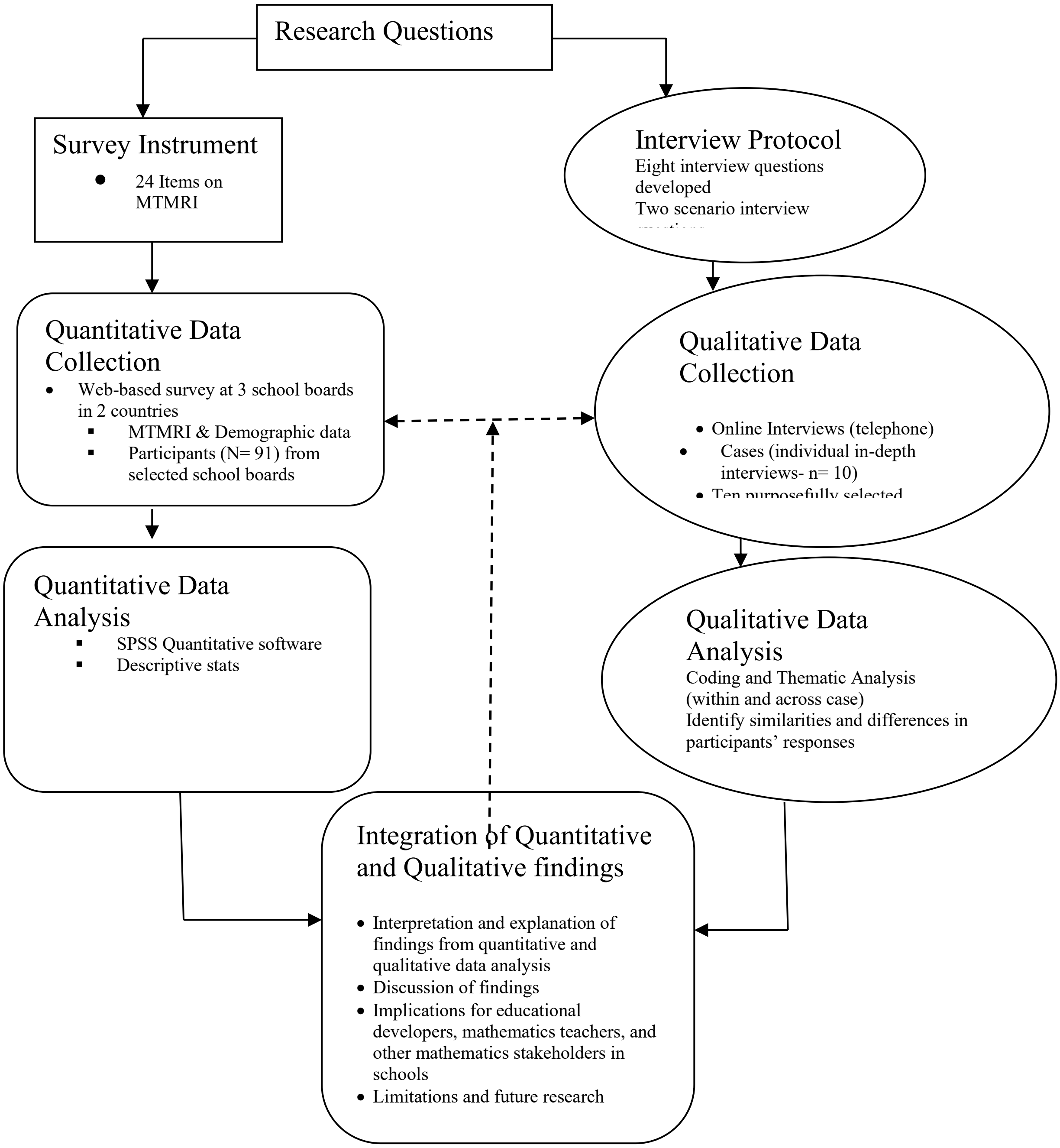
Ivankova, Creswell, and Stick (2006) note that there are approximately forty mixed-methods research designs reported in literature. Ivankova et al. (2006) further say that, out of these, the six designs that are highly popular and most frequently used by researchers are in two categories, called concurrent and sequential. Specifically, for the purpose of this study, I used the concurrent triangulation mixed-methods design (Tashakkori & Teddlie, 1998), in which the researcher "attempts to confirm, cross-validate, or corroborate findings within a single study" (Tashakkori & Teddlie, 1998, p. 229). This type of design is characterized by the collection and analysis of data at the same time. The use of the concurrent data collection approach results in a shorter data collection time period when compared to sequential approaches (Tashakkori & Teddlie, 2003). The quantitative data collection was done through the use of a web-based survey, and qualitative data were collected through in-depth interviews. Both processes were concurrent, happening during one phase of the research study. The survey was used to corroborate the interviews and vice versa.

This study aims to understand how teachers generate and provide representations when teaching algebra. I used a survey and quantitative analysis to determine teachers' perspectives on the use of representations when they teach algebra. The qualitative aspect was focused mainly on interviews of Grade 8 teachers. I describe below the specific purposes of the quantitative and qualitative approaches in this study. The design is illustrated in

Figure 3

Figure 3:

Visual model for concurrent triangulation mixed-methods design (Adapted from Tashakkori & Teddlie, 2003)



4.4 Research Design

4.4.1 Quantitative: Survey

The quantitative aspect of my study responds to the first research question: What are teachers' goals for and perspectives of using representations in Ontario and Lagos? According to Creswell (2012), quantitative researchers are able to approach research problems by observing trends or giving explanations of the phenomena. I used this approach to determine the perspectives of teachers with respect to the use of representations in relation to patterning and algebra. Fraenken and Wallen (2009) note that quantitative research seeks to establish relationships among variables and to look for and sometimes explain the causes of such relationships. This approach helps to separate facts and feelings as a researcher looks at the world as a single entity made up of facts that can be discovered. In this part of my study I used a survey, in which the views, opinions and perceptions of teachers on representations was determined.

Martella, Nelson, Morgan, and Marchand-Martella (2013), articulate that

> "a survey is used to identify how people feel, think, act, and vote; it is useful for collecting information from a relatively large number of dispersed groups of people rather than a small number, as in the case of other research methods" (p. 257).

I used the survey to reach out to a wide range of individuals to gather their views about the use of representations as one of the mathematical learning processes. Survey design allows the collection of data that involve direct observation based on the self-report of individuals' knowledge, attitudes or behaviors (Mertens, 2010). For the purpose of my study, the survey is designed to provide information about teachers' views and perceptions about use of representations—in the area of patterning and algebra focusing on goals, beliefs and knowledge as teachers' cognition (Artzt, Armour-Thomas, & Curcio, 2008). Next, I discussed the qualitative approach.

4.4.2 Qualitative: Case study

In the qualitative approach, several possible options exist, each with its advantages and disadvantages. In the literature, three important factors for choosing a design emerged and these include (a) type of research questions, (b) the amount of control that the researcher has over actual events, and (c) the focus on contemporary as opposed to historical phenomena (Yin, 1994). A case study best fit with these factors. Therefore, the research design for my study will be embedded multiple case studies consisting of ten embedded cases (Yin, 2003). A single case study was neither adequate nor useful in my study. The reason being that the selected case was unique (Yin, 1994) considering the phenomenon being studied. Also, a single case study cannot possibly show differences or similarities in teachers' perceptions and in how representations are generated and provided while teaching patterning and algebra across the two jurisdictions.

A case study is an exploration of a bounded system or a case over time through detailed, in-depth data collection involving multiple sources of information and rich in context (Merriam, 1998). A multiple case study design includes more than one case, and the analysis is performed at two levels: within each case and across the cases. I used a multiple case study as it allows an in-depth study of a particular phenomenon (Cohen et al., 2007, Fraeken & Wallen, 2009, Savin-Baden & Major, 2013). An embedded multiple-case design supports an understanding of similarities and differences across contexts and how this relates to the various phenomena to be studied (Yin, 1994). According to Creswell and Plano Clark (2007), the term *embedded design* refers to a study in which one set of data is used as supportive or secondary in another set of data.

In this study, qualitative and quantitative data were combined to expand an understanding from one data set to another (Creswell, 2003). For the purpose of my study, individual teachers and teachers from each of the two research settings were viewed as a single case study.

According to Yin (2014), “within the single case study a subunit of analysis may be incorporated so that embedded design is developed” (p. 56), and he further explains that, the subunit will allow for extensive analysis, revealing and enhancing more insights into the single case.

Yin (2006) highlights five procedures to tighten the use of mixed methods so that it could be seen as part of a single case study: the research question, unit of analysis, samples of the study, instrumentation and data collection, and analytic strategies. Yin suggests that, in considering the research questions, it is important the researcher address both the outcome question (quantitative) and the process questions (qualitative) in an integrated form. The research question in my study covered the “what” (outcomes) and the “how” (process) of teachers’ use of representations when teaching patterning and algebra. The unit of analysis is another idea that suggests that researchers should consistently maintain the same point of reference when data is analysed. Yin (2006) articulates that persistent reference to the same unit of analysis allows a force of integration that blends the different methods into a single study, so researchers can deliberately cover the same questions in different methods. I employed the use of similar questions covered in both the survey and the case study. This enabled me to integrate the different methods using one form of analysis. While describing the samples procedure, Yin (2006) suggests that the samples should be nested within the different methods. To achieve this in my study, the five teachers from each research location in the case studies were samples of teachers that took part in the survey. Next, I describe the selection of research participants, instrumentation and data collection methods that I employed.

4.5 Population

The targeted population in this study were in-service Grade 8 teachers from Ontario, Canada and Lagos, Nigeria. These teachers are mainly mathematics teachers teaching Grade 8 in Lagos, and mainly generalist Grade 8 teachers in Ontario. Since Lagos teachers are subject-specific teachers, I anticipated that the teachers in Lagos would have more mathematics background. I also anticipated that teachers in Ontario would use representations more fluently because they have ongoing professional learning programs to update their pedagogical content knowledge.

4.6 Sampling

Recruiting participants without incentive is a challenging part of a study; therefore, I adopted convenience sampling (Cohen, Manion, & Morrison, 2007; Creswell, 2012; Fraenkel & Wallen, 2009) to select school boards in both Northwestern Ontario and Western Lagos for the research. According to Creswell (2012), the convenience sample can provide useful information for answering questions. Convenience sampling is carried out by selecting a group of individuals (volunteers) who are willing and available to the researcher for study (Creswell, 2012; Fraenkel & Wallen, 2009). Although convenience samples cannot be considered as representative of the population, nevertheless, it may be argued that they will produce a snapshot of the nature of mathematics education in Ontario and Lagos. In convenience sampling, caution must be taken to include gender, years of teaching, highest educational level, or access to technology and other characteristics of the sample being studied (Fraenkel & Wallen, 2009), therefore these factors will be recorded so that they can be controlled for in the analysis.

Newby (2010) suggests that convenience sampling could be used for preliminary studies or when time is limited. For the purpose of this study, both the survey and the interviews were conducted at the same time as time was of essence. The survey is not aimed at making any statistical predictions as the sample is not demonstratively representative of a larger population. Relatability (Opie, 2004), rather than generalisation, was intended in this study. The findings were related to what is happening in the classrooms. Cohen, Manion, and Morrison, (2007) suggest that a "convenience sample may be a sampling strategy selected for a case study or series of case studies" (p. 114). As mentioned above, I employed embedded multiple-case studies in this current study.

After obtaining approvals from the Lakehead University Research Ethics Board, two school boards in Northwestern Ontario, and one school board in Western Badagry, Lagos were contacted. The approval from the Lakehead University Research Ethics Board depended on getting formal approval from the school boards before I was able to conduct my research with their boards. The schools used in these school boards were English-speaking as I cannot communicate in the French language. A school board in North Western Ontario was contacted after which principals were contacted and requested to email teachers in Grade 8. The choice of Ontario and Badagry was due to the proximity of where I reside across the two jurisdictions. All those who responded were part of the survey. Consent was secured from those who completed the survey. Participants interested in the one-to-one interviews provided their personal contact information (e.g., name, the last four digits of their phone number and email) when they completed the surveys.

Although Internet access is growing (Mertens, 2010), the access to Internet did not affect the rate of response in the online survey. In Nigeria only 46.1% of the population have access (Internet Live Stats, 2016). To avoid low response rates for the online survey, Mertens (2010) recommends a mixed method survey. For the purpose of this study, I used web-based survey so that I did not have to go into schools and talk to Grade 8 teachers within the time frame of the study to encourage participation. One of the advantages of using a web survey is that "persons with low incidence of disabilities may be able to respond more effectively to a web survey" (Mertens, 2010, p. 203).

A total of 91 in-service Grade 8 teachers responded to the online survey, out of which 20 of them were from Ontario and 71 from Lagos. The difference in the number of respondents in the two jurisdictions is due to number of schools and population.

In Table 4 below, I provide a brief description of the qualifications and experience of the ten teachers who participated in the survey and interviews, and whose data I examined in this section.

Five Ontario-based in-service elementary school teachers participated in the one-to-one interviews (one male, four females). The teachers in Ontario were generalist trained elementary school teachers, with similar qualifications, teaching backgrounds and Grade 8 teaching experiences. Three of the five Ontario teachers had more than four years' experience in teaching Grade 8 mathematics. Sara and Susan had more than ten years of Grade 8 teaching experience as well as a non-teaching role supporting students with disabilities in the classroom. The sample was drawn from two different school boards. The majority of the teachers in Ontario reported that they had professional development training in order to use representations effectively in the classroom. Each participant was assigned a pseudonym for confidentiality purposes.

Five Lagos-based in-service teachers participated in the one-to-one interviews (one female, four males). In contrast to the Ontario teachers the teachers in Lagos were subject-specific trained teachers. The teaching experience of all of the Lagos teachers ranged between 4 years to over 10 years. The least experienced teacher, Bryce, together with Ben, had some professional development training background but three of the teachers, Bola, Beth, and Baker did not have any professional development training background. Bola and Bryce had a science background. Bola took up a teaching appointment with the Lagos State government. He then undertook the one-year professional qualification course to qualify as a mathematics teacher.

Table 4:

Profile of the ten case-study teachers

Descriptor	**Scott**	**Silva**	**Susan**	**Sonia**	**Sara**	**Bola**	**Beth**	**Ben**	**Baker**	**Bryce**
Gender	Male	Female	Female	Female	Female	Male	Female	Male	Male	Male
Degree	Bachelor of Outdoor Recreation	Bachelor of Arts	Bachelor of Science in Nursing	Bachelor of Arts	Bachelor of Arts	Bachelor of Science in Physics & Electronics	NCE Bachelor of Science	NCE Bachelor of Science	Bachelor of Science	NCE Bachelor of Science
Qualifications	Mathematics Education (P/J), Part 1 Reading (Part 1)	Mathematics Education (Intermediate)	Special Education, Part 1 & 2, Specialist Mathematics Education (P/J), Part 1 & 2, Specialist Principal's Qualification, Part 1 & 2	Mathematics Education (P/J), Part 1 English (I/S) History (I/S)	Mathematics Education (P/J), Specialist Visual Arts (J/I)	Science Education	Mathematics Education	Mathematics Education	Science Education	Mathematics Education
Grade 8 teaching experience	4 – 6 years	4 – 6 years	More than 10 years	4 – 6 years	More than 10 years	7 – 10 years	7– 10 years	More than 10 years	7 – 10 years	4 – 6 years
Professional development background	Yes	Yes	Yes	Yes	Yes	No	No	Yes	No	Yes

Ben was the only teacher who taught in a private school. The majority of the Lagos teachers taught mathematics in a grade 7 or 8 classroom.

4.7 Research Instruments

The two instruments that were used in this present study included a survey questionnaire and semi-structured interview. The association between the instruments and the research questions are summarised in Table 5Table 5 and this is followed by a description of each instrument.

Table 5:

Relationship between research questions and data gathering instruments.

	Research question	Instrument to be used
1.	What are teachers' goals for and perspectives of using representations in Ontario and Lagos?	Teacher survey questionnaire
2.	How do teachers' goals for and perspectives of using representations differ by region?	Teacher survey questionnaire Interview protocol

4.8 Instrument Development Process

As discussed above, the survey instrument was used to explore teachers' view of their representational practices in the algebra classroom. In this section, an outline of six steps used for the development of the instrument is described in detail (see

Figure 4

Figure 4).

4.8.1 Step 1: Construction of Measures for Algebra Teaching Using Representations

In designing the questionnaire, I conducted an extensive literature review of instruments that focus on measuring teachers' knowledge and perceptions about representation in relation to algebra instruction (See

Figure 4). I examined the databases ERIC, Google Scholar, and PsycINFO. Key terms such as patterning, algebra, multiple representations, representational taxonomies, teacher beliefs, teacher knowledge, student characteristics, task characteristics, and teaching practices, among others, were used, as well as relevant combinations of these terms. In addition, a variety of mathematics books, middle school mathematics textbooks and teaching materials was screened in order to investigate which modes of representations were commonly used (e.g., Bassarear, 1997; Carpenter, Franke, & Levi, 2003; Cathcart, Pothier, Vance, & Bezuk, 2011; Hatfield, Edwards, & Bitter, 1997; Huetinck & Munshin, 2008; Kajander & Boland, 2014; OMoE, 2005b; NCTM, 2000; Van de Walle& Lovin, 2006; Van de Walle et al., 2013; Van de Walle et al. 2014).

A possible pool of 24 items was created by identifying instructional practices used by teachers in previous research studying the teaching of mathematics using multiple representations in middle grades (e.g., Barmby, Bolden, Raine, & Thompson, 2013; Cai, 2005; Coleman, McTigue, & Smolkin, 2011; David & Tomaz, 2012; Izsàk & Sherin, 2003; Mitchell, Charalambous, & Hill, 2014; Moyer, 2001; Stylianou, 2010; Watanabe, 2015; Zazkis & Sirotic, 2004). After generating a pool of items to be used, it was necessary to categorize these items into components of teachers' thinking. In Step 2, I describe the process of this categorization.

Figure 4:

The instrument development process

Step 1	Item pool from literature review
Step 2	Categorize items into components
Step 3	Selection/construction of specific items of representations to measure each component from literature review
Step 4	Survey design Likert-scale, layout, and refinement of items following suggestions from faculty
Step 5	Validity assessment Refinement of items based on feedback from undergraduate students
Step 6	Testing for internal consistency—Reliability pilot test of the measurement items undergraduate students
	Instrument finalization

The various stages of the instrument development process.

4.8.2 Step 2: Categorization of Items Into Components

Artzt et al. (2008) suggest three overarching aspects of teacher cognition, which include teachers' goals, knowledge, and beliefs. I provide below a description of how these three components provide a framework for the purpose of this current study.

- Goals are teachers' expectations relating to the intellectual, social, and emotional outcomes for students as a result of their classroom experiences.
- Teacher knowledge involves knowing central facts, concepts, and principles about the pupils, content, and pedagogy acquired over time.
- Beliefs are personalized assumptions of the teacher relating to the nature of the subject, the pupils, learning, and teaching.

The three overarching aspects of teachers' cognition Artzt et al. (2008) described above were used to categorize the items on the survey. These are also the general areas of the use of representations common in the literature. Table 6Table 6 describes each component in greater detail, as well as indicating how many survey questions relate to each component. The component labels are: teacher goals (C1); teacher knowledge (C2) which was subdivided into content (C2KC), learners (C2KL), and teaching (C2KT); and teacher beliefs (C3) which was subdivided into content (C3BC), learners (C3BL), and teaching (C3BT).

Table 6:

Teachers' Cognition Components of Using Representations Adapted from Artzt et al. (2008).

C1: Goals	The expectation about the use of representations is to model and interpret physical, social, and mathematical phenomena (NCTM, 2000). The teachers' goals become clear as they observe their own instructional practices as a result of experiences (3 items).
C2: Knowledge	
C2KC-Content	The teacher engages with representations to foster conceptual and procedural understandings of the content, and is aware of and appreciates the effective connections between these when teaching algebra (1 item).
C2KL-Learners	The teacher has specific knowledge of how representations can be used to support and motivate learners to effectively communicate mathematical ideas in algebra (5 items).
C2KT-Teaching	The teacher has understandings of how to generate and provide representations to effectively explain difficult areas in algebraic concepts (5 items).
C3: Beliefs	
C3BC-Content	The teacher perspectives about patterning and algebra and how different representations are used to explain, illustrate, and make connections between representations (3 items).
C3BL-Learners	The teacher views her/his role as ensuring students actively engage and discuss their thoughts as they share solutions to problems using different representations (2 items).
C3BT-Teaching	The teacher views her/his role as a facilitator of how representations are selected during problem solving and communicated when sharing mathematical ideas (5 items).

A detailed description of the subcomponents of the survey instrument.

4.8.3 Step 3: Selection of Specific Items in Each Component

In order to categorize specific items for each component, I analyzed literature relevant to teachers' cognition that contributes to the teaching of algebra using representations. The items in each component are described below.

4.8.3.1 C1: Goals of teaching with representations

One of the goals of using representations is to model and interpret physical, social and mathematical phenomena (NCTM, 2000). Much of the literature concerning the goals of using representations places importance on developing conceptual understanding and allowing students to value mathematics and feel confident in their own abilities (e.g., Bills, 2000; Boaler, 2014; Moyer, 2000; Lesh & Doerr, 2003; Philip, Johnson &Yezierski, 2014; Hubber, Tytler & Haslam, 2010; Lesh, 1999; Nitz, Prechtl, & Nerdel, 2014; Ryken, 2009; Wang & Siegler, 2013). It is important that teachers encourage and help students develop representational competency (Hubber, Tytler, & Haslam, 2010). In response to the literature that describes the essential goals of teachers using representations in order to help students construct their own meaning, I included items 2, 20, and 23 on the survey asking teachers to clarify their goals (see Table 7).

Table 7:

Goals of Teaching with Representations.

Item number	Item	SA	A	N	D	SD	Don't know
2	Providing representations to support reasoning is something I will often do to explain difficult concepts to students.						
20	Appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings.						
23	It is necessary that teachers should assist in choosing appropriate representations for students.						

Items containing teachers' goals of using representations in the classroom

While teachers may have the goal of using representation effectively to teach algebra they sometimes may not have the necessary knowledge. Next, I will discuss the teachers' knowledge in relation to content, learners, and teaching.

4.8.3.2 C2: Knowledge Regarding Content, Learners, and Teaching

Ball et al. (2008) describe the mathematical knowledge for teaching as that needed to carry out the work of teaching mathematics. According to Hill, Schilling, and Ball (2004), the mathematical knowledge needed for teaching is multidimensional—general mathematical ability is not sufficient for the knowledge and skills entailed in teaching. Often times the information, directions, and messages that teachers communicate are not understood by all students in the exact way that teachers intend them to be heard (Gordon, Kane, & Staiger, 2006). Teachers who know more about content, teaching, and learners' abilities and learning styles, and their interest and attitudes, will select tasks that are motivational and fit students' difficulty levels (Gardner, 1999; Artzt et al., 2008). I discuss briefly in the next section each type of knowledge regarding content (C2KC), learners (C2KL), and teaching (C2KT).

4.8.3.2.1 *C2KC—Knowledge of Mathematical Content*

Teachers are expected to use representation in a mathematically accurate and understandable manner for students as they engage with mathematical content. Studies have revealed that an insufficient understanding of mathematical content may limits teachers' capacity to explain and represent such content for better understanding to students (Even, 1993; Stein, Baxter, & Leinhardt, 1990). In addition, many teachers are said to lack knowledge of how mathematical ideas are transformed into representations (Ball, 1999; Moyer, 2001; Stylianou, 2010). For example, Stein et al. (1990) examined an experienced fifth grade teacher as he taught a lesson sequence on functions and graphing. These authors found that the teacher lacked the necessary knowledge for fostering meaningful connections between key concepts and representations. Molenje and Doerr (2006) contend that the teacher's knowledge of mathematics is fundamental to how he or she articulates and balances different representations. Item 16 (see

Table 8:

Knowledge of Mathematical Content

Item number	Item	SA	A	N	D	SD	Don't know
16	A specialized understanding of elementary mathematics is not necessarily needed on the part of the teacher in order to use representations effectively in teaching patterning and algebra.						

) was included to examine how teachers' knowledge of mathematics will contribute to effective use of representations.

Table 8:

Knowledge of Mathematical Content

Item number	Item	SA	A	N	D	SD	Don't know
16	A specialized understanding of elementary mathematics is not necessarily needed on the part of the teacher in order to use representations effectively in teaching patterning and algebra.						

4.8.3.2.2 ***C2KL—Knowledge of Learners***

Teachers' knowledge of learners' characteristics may inform their use of representations. A teacher needs adequate knowledge of the learners in areas such as needs, interests, prior knowledge, ability, learning difficulties, and misconceptions. The teacher who knows his or her subject well and also know how to make it accessible to learners will more likely use representations to highlight mathematical concepts in order to foster students' work (Cohen, Raudenbush, & Ball, 2003). Having experience with what students know and are struggling with may help the teachers' awareness and evaluations of students' errors relating to representations.

The teachers in Lee and Luft's (2008) study articulate that the knowledge of students can only be acquired through classroom experience. The knowledge of students is essential as the teacher chooses teaching strategies and makes connections to content knowledge. I included items 10, 11, 13, 14, and 21 (see Table 9) to find out how teachers' specific knowledge of students' ability to manipulate symbols or some forms of representation in different contexts supports the development of abstract understanding.

Table 9:

Knowledge of Learners

Item number	Item	SA	A	N	D	SD	Don't know
10	Representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.						
11	Representations are less effective when suggested to students by teachers, rather than being generated by students.						
13	Knowing which representation to use is sometimes confusing for students.						
14	Encouraging students to use representations can improve their problem-solving skills.						
21	Including a lot of representations within a lesson could add confusion for students.						

4.8.3.2.3 ***C2KT—Knowledge of Teaching***

Speer, Smith, and Horvath, (2010) contend that the way teachers generate and provide representations is influenced by their chosen (or assigned) textbooks. Speer et al. (2010) revealed that teachers select representations likely because of how their students have worked with and understood these representations on similar topics. Items 6, 12, 15, 17, and 18 (see Table 10) were included to examine how teachers' knowledge of teaching with representations are used to explain difficult areas in algebraic concepts.

Table 10:

Knowledge of Teaching

Item number	Item	SA	A	N	D	SD	Don't know
6	I would need to learn more about teaching-related mathematics before using representations in my teaching.						
12	Graphical representations are the most important kind of representations to illustrate algebraic concepts.						
15	Representations are hard to use in teaching.						
17	The use of manipulatives is only good for teaching patterning but not for algebra.						
18	Patterning and algebra is one of the more difficult strands to teach as a lot of representation is involved.						

Items describing teachers' knowledge of teaching with regards to representations

Giving the importance of teachers' knowledge as they use representation in their teaching, investigators have also found that teachers' interpretation and use of representations are influenced by their beliefs. I will next discuss beliefs in relation with content, learners, and teaching.

4.8.3.3 C3: Beliefs Regarding Content, Learners, and Teaching

Beliefs refer to a viewpoint or a way of thinking or even a preconceived idea a person holds. I define beliefs in relation to Schoenfeld (1998) "as mental constructs that represent the codification of people's experience and understanding" (p. 19). According to Beswick (2007), mathematics teachers' beliefs that underpin their practice are beliefs about the content, mathematics learning, students and their capabilities and teachers' beliefs about themselves. Researchers (e.g., Swars, Hart, Smith, Smith, & Tolar, 2007) contend that there is a disconnection between teachers' specialized content knowledge and their belief in the skills and abilities required to teach mathematics effectively. In response to the literature that indicates the importance of teachers' beliefs regarding content, learners, and teaching, items were included on the survey. These items were further categorized and are discussed below.

4.8.3.3.1 *C3BC—Beliefs About Content*

According to Ball (1990), teachers' beliefs about mathematics are powerful as they tend to influence their use of representations of mathematics. A teacher's knowledge about the content has a strong impact on the content taught (Ball et al. 2008; Drageset, 2010). Therefore, it is important that to teach mathematics effectively, a teacher must understand and know the content that is to be taught. Although Philip (2007) remarked that what happens in the classroom may differ from that which is expected, Driscoll (1999) contends that when the teacher uses appropriate questioning to engage learners, relevant connections with a mathematics concept are achieved as students experience a balance in their use of verbal, tabular, graphical and symbolic representations. Teaching with representations requires that the teacher re-examine and reflect on the way in which the artifacts are presented. I included items 1, 5, and 7 (see Table 11) to assess teachers' views about what representations are.

Table 11:

Beliefs About Content

S/N		SA	A	N	D	SD	Don't know
1	The use of multiple representations is not clearly explained in the curriculum.						
5	Representations can be mental images.						
7	Representations are usually not physically visible.						

4.8.3.3.2 *C3BL—Beliefs About Learners*

Ollerton (2009) argues that teachers cannot force students to have a positive relationship with their subject but they need to realize that they have a "massive impact" (p. 2) on their students. It is therefore important that teachers provide students with a positive learning atmosphere where sufficient opportunities are given in order to access different representations.

Teachers believed that students learn from using symbolic notation to build their algebra reasoning. For example, one of the teachers in Blanton and Kaput's (2011) study reported that t-charts and function tables were important representations that foster students' mathematical reasoning. Blanton and Kaput (2011) argued that symbols are vital tools by which we mediate and communicate mathematical ideas broadly. Research suggests that asking students to restate problems in their words help them to translate among representations, and also enable them to learn abstract ideas rooted in meaningful concrete models. Items 4 and 9 (see Table 12) are included to examine teachers' views about how students learn from representational use.

Table 12:

Beliefs About Learners

Item number	Item	SA	A	N	D	SD	Don't know
4	Allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.						
9	Representations help in moving students from using concrete models to abstract representations.						

4.8.3.3.3 ***C3BT—Beliefs About Teaching***

According to Wilkins (2008), teachers' beliefs have a strong effect on their practices. Teachers' beliefs about the use of mathematical representations may inform what is displayed and how it is displayed as they tend to make appropriate selections during problem solving (Elia, Gagatsis, & Demetriou, 2007; Niemi, 1996; Panaoura, Gagatsis, Deliyianni, & Elia, 2009; Speer, Smith, & Horvath, 2010). Differences in representations can affect learning and how ideas are communicated when one representation is easier to comprehend than another, or when one representation elicits more reliable and meaningful solution strategies than another (Koedinger &

Nathan, 2004). Items 3, 8, 19, 22, and 24 (see Table 13) were included to understand the teachers' beliefs about how representations should be selected during problem solving and communicated effectively.

Table 13:

Beliefs About Teaching

S/N		SA	A	N	D	SD	Don't know
3	The use of representations is not particularly useful in teaching and learning patterning and algebra.						
8	One specific representation of a pattern may not be enough in a patterning and algebra lesson.						
19	Selecting a worthwhile task determines what representation to use.						
22	Teachers should flexibly shift among different representations as they are generated by students.						
24	The effective use of representations requires a lot of planning.						

Items describing teachers' beliefs about teaching with regards to representations.

4.8.3.4 Open Ended Item

In addition to the closed ended questions discussed above, an open ended question was included in the survey to ask for a written opinion by teachers as to "what do you really think representations mean?" to them. The open-ended question was included to make comparisons between different subgroups in Ontario and Lagos, the two settings where the research was conducted. The open-ended question was included to permit greater depth of response and insight into the reasons for responses (Mills & Gay, 2016).

Participants may have to think harder before responding to open-ended questions due to the need to think more about composing an answer or in some cultural settings due to personal inadequacy in being able to answer in the language required (Opie, 2004), and may sometimes avoid them. Gorard (2001) articulates that open-ended questions are best used in two situations:

where it is clear how the responses will be analyzed, or where the responses will be used not only to create a statistical pattern, but to help explain it. Gorard (2001) and Newby (2010) suggest mixing the type of questions in any instrument since there may be so little similarity between responses to closed-choice and open ended questions. Gorard recommends that the background questions (respondents' personal characteristics) come last as they can appear intrusive. He further maintains that having them at the end encourages people to start the questionnaire, even if they drop out at this section, as substantive responses would have already been collected. I placed the open-ended question at the end of the survey to encourage response to the earlier Likert scale questions.

According to Gorard, three common problems with open-ended questions are lack of clarity, lack of knowledge and intrusiveness. One of the aims of the study is to establish if the teachers have knowledge of representations. Muijs (2004) articulates that open-ended questions allow a researcher to discover opinions that the researcher had not thought about before. According to Newby (2010), open-ended questions provide an authentic voice, a richer picture of some aspect under investigation, and enable the researcher to convey in powerful ways the perspectives that are important to the interpretations and explanations of the issue under investigation.

4.8.4 Step 4: Survey Design

In step 4, the items were then assembled in the survey using the following design.

4.8.4.1 Likert Scale

Likert scale items are commonly self-reporting instrument used to investigate the attitudes, opinions, or beliefs of individuals to a series of written or verbal statements by indicating the extent of agreement. According to Fraenken and Wallen (2009), each choice of a Likert scale is given a numerical value, and the total score is presumed to indicate the attitude or belief in question. A Likert scale is appropriate for this instrument as it is a useful and reliable way of collecting attitudinal data (Maurer & Pierce, 1998), and the primary focus of interest is on teachers' use of representations in algebra. Turner (1993) contended that Likert scale questionnaires are not the only option available to a researcher measuring respondents' characteristics, attitudes, opinions, or beliefs, questionnaires or interviews but well-planned open-ended questions also allow respondents to express any opinion or attitude on a topic. Although teachers' use of representations could be better understood with data from classroom observations, survey items organized in both close-ended and open-ended questions form may provide for greater information than only one of these. I therefore chose a Likert scale.

Although, according to Jamieson (2004), there are five categories of response a Likert scale provides, ranging from 5 = strongly agree to 1 = strongly disagree with a 3 = neutral. A "don't know" category at the end of the scale was added giving respondents who do not have an answer or opinion a chance to make a choice and also to interpret whether respondents do not understand the question, or don't have an opinion (Muijs, 2004). It is important to give the option "don't know" to some respondents who may be genuinely neutral and as such their views might be misrepresented. I therefore chose a 6 point Likert scale.

4.8.4.2 Survey Layout

There were two parts to the questionnaire. The first part of the questionnaire contains 24 items that asked for the teachers' perceptions about teachers' overarching cognitions, use of representations and teaching patterning and algebra. All items asked for responses on Likert scales except for the second part which had one open-ended question.

A third part was later added that consisted of questions relevant to various demographic and socio-cultural factors (e.g., ethnicity, years of experience of teaching) that was used to determine sample characteristics and compare the sample to the population.

4.8.5 Step 5: Evidence of Face and Content Validity

Validity and reliability are common terms used largely to describe quality in quantitative research. Validity refers to the degree to which a method, test or research actually measures what it was supposed to measure (Opie, 2004). According to Fraenken and Wallen (2009), validity refers to the "appropriateness, correctness, meaningfulness and usefulness of the specific inferences researchers make based on the data they collect" (p. 148). Fraenken and Wallen further articulate that validity depends on the amount and type of evidence there is to support the interpretations researchers wish to make concerning the data they collected. This has implications for the researcher, as well as the research instruments, the research contexts, and the participants.

My graduate student colleagues, four Faculty members (in educational psychology, educational foundations, and two mathematics education faculty members) reviewed the instrument for face validity. The consultation was carried out in order to critique, suggest, give feedback and see possibilities of formatting, modifying, and laying out the survey questions so that they were easily accessible to teachers.

Face validity and content validity of the items were established by education faculty members. To establish content validity, the aforementioned colleagues were also consulted to critically review items for each component of the instrument. After eliminating and rewording certain items based on suggestions and feedback from the aforementioned, a modified draft of the survey instrument was subjected to a pilot test. The intermediate/senior prospective teachers commented on the appropriateness of the items, critiqued their ease of comprehension and suggested changes to improve their wording, all of which were incorporated to create a final version (see Appendix) of the survey to use in the study. This was the final stage of the development process. Construct validity of the questionnaire was examined by principal component analysis (Steenekamp, Van der Merwe, & Athayde, 2011) during the main study.

4.8.6 Step 6: Evidence of Reliability

In order to interpret the results of the survey for reliability, a pilot test was conducted. The pilot test was carried out in the winter term of 2016. Seven intermediate/senior prospective mathematics teachers participated in the pilot study. According to Kajander (2010), intermediate/senior prospective mathematics teachers encompass middle and high school level teachers with a solid mathematics foundation and broad mathematics background knowledge. By the time of the pilot test, they had developed some foundation in specialized content knowledge and pedagogical knowledge preparation. The pilot test carried out with the prospective teachers helped in determining the appropriateness of the instrument. Participants were encouraged to comment on the items, hence some items were reworded.

Reliability refers to the consistency of measure—the extent to which the results are similar over different forms of the same instrument or occasion of data collection (McMillan & Schumacher, 2010; Thomas, 1998). Ross et al. (2003) noted that if a survey does not have reliability, then “it is no more than a rubber ruler” (p. 348). Using SPSS 20, an internal consistency analysis was performed for all the items and also separately for the items of each component of the teachers’ overarching cognition of using representations. The Cronbach alpha measure yielded a result of 0.69 and of the subscale components; C1= 0.74, C2KC= 0.69, C2KL= 0.66, C2KT= 0.60, C3BC= 0.53, C3BL= 0.72, and C3BT= 0.59 of the instrument from the pilot test with preservice teachers, which was an indication of the reliability of the items in testing the underlying construct. According to Mertens (2010), reliability obtained will differ after every use because of differences in groups, settings, and other factors. The responses solicited from the teachers taking the survey are not intended to be used for any prediction with regards to teachers' tendencies in relation to multiple representations. Instead, what is important for use of the survey is the generation of responses that could be used to launch further investigations. The final version of the survey is included in the Appendix A.

4.9 Procedures for Data Collection

4.9.1 Survey

The survey was the only source of quantitative data collection. An online version of the survey was created with Fluid Survey software and kept at the University of Windsor secured server. The University of Windsor survey platform was made possible because of the Joint PhD program. The address to the server was sent to the school boards for onward distribution to their teachers using their email list. The invitation by the Boards to the teachers contained a letter, access to the link that provided the survey instrument, and information about the researcher for those who were willing to participate in the interview. The instrument was administered online through the web-based survey tool Fluid Survey. No hard copy of the survey instrument was mailed or sent as an attachment to teachers as none of them requested it. Surveys delivered online have the advantages of automating the data collection process, but may also experience lower response rates. Completion of the survey required about 10-13 minutes and participants were asked to click 'submit' as a way of authorising their participation. It was also indicated that those who were interested in the one-to-one interviews should give their personal contact information (e.g., name, last four digits of their cell number, and email), and they were then contacted to further participate.

4.9.2 Interviews

A semi-structured interviews and scenario interviews were conducted with ten conveniently selected Grade 8 mathematics teachers who volunteered to participate in the interviews. Interviews are an important means for a researcher to check the consistency of his/her interpretations (Fraenkel & Wallen, 2009). Interviews enable the researcher to "explore complex issues in detail, they facilitate the personal engagement of the researcher in the collection of data, and they also allow the researcher to provide clarification, to probe and to prompt" (p. 72). I conducted interviews with ten Grade 8 teachers. I conducted one interview with each teacher in my study and used these interviews to address issues including how teachers generate and provide representations, and teachers' experience about teaching with representations. Interviews lasted between forty to fifty minutes and were immediately transcribed.

According to Mills and Gay (2016), conducting interviews is data collection, but recognizing the discrepancies between the two sources (in this case survey and interviews) is data analysis. Interviews constitute a very important technique and can yield useful information to answer research question (Creswell, 2016). The purpose of interviews in this study is not only to gather additional information regarding teachers' responses in the questionnaire, but more importantly, to understand the processes that teachers use to generate and provide representations to support key mathematical concepts. For the purpose of this study, I conducted interviews with each of the selected teachers. I describe below the specific purposes of interviewing each of these teachers.

The focus of the interviews was to find out from each teacher, what types of representations they are using and how they use the selected representations during teaching. The interview focused on their instructional practices. The interview was scheduled on a convenient day for them, to encourage a relaxed atmosphere.

I developed an interview protocol (see Appendix B) that contained 8 open-ended questions and 2 scenario interview questions in order to corroborate the quantitative results. The interview protocol was designed to focus on goals, beliefs, content knowledge, and pedagogical content knowledge issues around algebraic representations. The scenarios were presented to the teachers who were asked to describe their approaches to handling them in their classrooms. In constructing the scenarios, I wanted something that would provide insight into teachers' knowledge on algebraic thinking relating to use of representations. I set up the scenarios to assess (i) how the teachers themselves saw the incorrect ideas of representing algebra problems, and (ii) how they might react to the learners who came up with the initial ideas. I sent out the scenario questions before the end of the interview. This was to avoid making the teachers feel that I was testing or assessing their knowledge or performance. I emphasized that there was no right or wrong answers or approach to untangling each of the scenarios. I was interested in possible approaches as I could not get into different classroom contexts. Table 14Table 14 below provides a sample of the nature of the interviews in each case.

Table 14:

Sample Interview Questions

Interview questions	Scenario question
1. How can you explain your experience with the use of representations during teaching? 2. What informs your use of representations during mathematics teaching? 3. How did you plan to approach an algebraic lesson in order to bring the learners to understand the content and context? Will you give examples of how you generate representations for your students?	1. Sam has x bananas and Codi has p bananas. Collin counts the number of bananas each of them have and finds they are the same. Sam said you write as x= p, but Codi said that x and p are different letters and so cannot be the same. What would you say to these students?

All of the interviews were audio recorded, and I started the interview session by seeking the participants' permission to record, even though they had consented to it before-hand.

According to Creswell (2016), interviews give the ability to probe and open up an issue in order to explain it further. Interviews have limitations as do other means of data collection. First, similar to observations, interview data may be deceptive and provide the perspectives the participants want the researcher to hear (Creswell, 2012). The second is getting participants to speak and talk about the central phenomenon (Creswell, 2016). Various authors (Gay, 2016; Creswell, 2012; Savin-Baden & Major, 2013) provide some suggestions on how to minimize the tendency of not being able to capture detailed information through other sources of data collections. To follow the suggestions of the authors I tried to (i) be as neutral as I could (ii) be observant of the reactions as I would be transcribing (iii) be a good listener and avoid interruption, and (iv) be non-judgmental. The third issue is that audio taped interviews do not capture actions that accompany respondents' talk. The action of the respondents is not of importance as much in my analysis, except in cases when respondents use a form of gesture relating to representations.

4.10 Analysis of Data

4.10.1 Quantitative

Descriptive statistics were presented. This included overall means and percentages, graphs, response rates, and reliability (Cronbach alpha).

Scores of each participant were computed by adding the item values on the MTMRI. The negative items (11) on MTMRI were reversed-coded before the total scores for participants were calculated. In reporting the results, the data from the two columns of "strongly agree" and "agree" were combined and the data for "strongly disagree" and "disagree" were combined as well. These data were analysed using methods of descriptive statistics such as means and percentages.

4.10.2 Qualitative

Data analysis was done using both content analysis and thematic analysis for qualitative data using ATLAS.ti. Content analysis is an analysis of frequency and patterns of use of terms, phrases, and visual artefacts (Savin-Baden & Major, 2013). Content analysis was used to code the text and categorise for further analysis using thematic analysis. Thematic analysis involves familiarizing oneself with data, generating codes, searching for themes, reviewing themes, defining and naming themes, and producing the report (Braun & Clarke, 2006 in Savin-Baden & Major, 2013). The data analysis followed from first reading the written data and going through it numerous times until I became very familiar with the details thereof. Triangulation is considered to be one of the best ways to enhance validity and reliability in qualitative research (Miles & Huberman, 1994).

I transcribed the recordings from the ten Grade 8 teachers that I interviewed. The transcriptions were given to the teachers to member check as a measure of trustworthiness. The transcription was then analyzed with the use of ATLAS.ti, a computer based system for analysis of qualitative data. Data gathered from the interviews were useful for understanding and giving insights into teachers' instructional practices in terms of how they used representations in their teaching.

4.11 Ethical Considerations

Before starting my research, I obtained ethics clearance from the Lakehead University Research Ethics Board and the relevant school boards. Informed consent was obtained from all participant teachers in the qualitative study before their interviews were analyzed. Also, the teachers who participated in the qualitative aspect of the study were contacted via email, and were informed that their participation was voluntary, and that they had the right to withdraw from the study at any time without penalty.

Participants were advised that taking part in my study would be neither an advantage nor disadvantage to them and that there will be no foreseeable risks in participating. Although teachers might fear repercussions from administration, their reputation, and fear of data use, participants were assured not to entertain any form of fear. Teachers participating may have benefited from the study as it may have given them the opportunity to reflect on their own teaching of patterning and algebra with the use of representations, which may have had an impact on their students. Participants were assured that their names and identities were kept confidential at all times and in all academic writing emanating from the study. I informed potential participants that they would have an opportunity to verify the information I obtained through the use of different data-gathering strategies before reporting the research. The names of the participating mathematics teachers as well as the names of the participating schools that appeared in this research report were all pseudonyms.

Lastly, I assured potential participants of the safekeeping of confidential documents locked up for a period of seven years at Lakehead University.

4.12 Integration of Quantitative and Qualitative Data

The mixed methods permit the integration of two types of data that might occur at several stages in the research process (Creswell, 2003). This includes the data collection stage, analysis, interpretation or some combination of these stages. In my study, integration of qualitative and quantitative data occurred first at the data collection stage and then, largely at the interpretation stage. For example, during the data collection stage, an open-ended question was combined with the closed-ended questions in the MTMRI. These were aimed at achieving the same goal—an understanding of teachers' perspectives of using representations. According to Creswell (2003), "mixing" the data at the collection stage enables the researcher to gather a richer and more comprehensive data set, making possible more detailed description and a deeper understanding of the phenomenon being studied.

Creswell (2003) suggests that qualitative and quantitative data may be combined and interpreted to corroborate, cross-validate or complement results from either data source. In this study, qualitative and quantitative data were combined to achieve a combination of these results. For example, the results from the MTMRI served both to cross-validate and complement data from the interviews. I now turn to these results.

CHAPTER FIVE: SURVEY DATA ANALYSIS, RESULTS AND DISCUSSION

5.1 Introduction

The purpose of this study was to explore how Grade 8 teachers in Ontario, Canada and Lagos, Nigeria generate and provide representations during their teaching of patterning and algebra. A quantitative analysis using multiple techniques including descriptive statistics was used to categorize, summarize, and visually present results.

5.2 Results from the Online Survey

The online survey completed by participants had three main parts: (i) a questionnaire that contained 24 questions about teachers' self report of their goals, beliefs and knowledge about using representations during the teaching of patterning and algebra; (ii) an open-ended question that permitted a greater depth of responses on what representations mean; (iii) a question about demography that could be associated with years of experience teaching Grade 8.

5.3 Research Participants

School boards in Ontario, Canada and Lagos, Nigeria, were selected for the study and research participants were drawn from the two jurisdictions. There were 91 in-service middle school teachers. Most (78%), of the teachers, were drawn from Lagos as compared to the number of participants (22%), from Ontario. In relation to mathematics teaching experience, the majority (60%), of the teachers in Ontario had between one and ten years of experience (Table 15). However, only a few (12.7%), of the teachers had more than ten years of teaching experience in Lagos (Table 16). Three percent did not respond.

Table 15:

Teachers by Jurisdiction (n=91)

Jurisdiction	Number	Percent
Ontario	20	22
Lagos	71	78
Total	*91*	*100*

Source: Online survey

Table 16:

Teachers' Years of Grades 1-8 Teaching Experience (n=91)

Years of experience	*Number (percent) of teachers*	
	Ontario	Lagos
1 – 3 years	6 (30)	25 (35.2)
4 – 6 years	5 (25)	19 (26.8)
7 – 10 years	1 (5)	16 (22.5)
More than 10 years	7 (35)	9 (12.7)
Unspecified	1 (5)	2 (2.8)
Total	*20(100)*	*71 (100)*

5.4 Results

The results are organized around the first research question of the study: What are teachers' goals for and perspectives of using representations?

5.4.1 Teachers' Perspectives on Representations

Overall, teachers who participated in the survey showed evidence that they perceive the use of representation in different ways based on their goals for, beliefs about and knowledge of representations.

The survey data analysis was conducted in two stages. First, I analyzed the data from the Likert scale component of the survey by reporting the frequency counts of each of the questions. Examining the findings suggests that some teachers may have interpreted the survey items differently than intended (Sullivan & Artino, 2017). The interpretation of why the differences exist between the researcher and respondents may be multidimensional (Krosnick, 1999) including: differences based on how some of the items were worded, and the respondent's own knowledge and cultural nuances. As such, I focused on the most helpful items and, I drew more strongly on these items to further guide the analysis of the interviews. This will be further explained in the next chapter. For example, one cannot neglect the likelihood that respondent's own content knowledge may have influenced their responses. Many researchers (e.g., Krosnick & Milburn, 1990) have found that people who are more knowledgeable about a topic are better equipped to form relevant opinions. Teachers who were less clear on what representations mean may not have interpreted questions about their use in the same way as more knowledgeable teachers.

For the Likert scale responses on the survey, I looked at the percentages of each answer. Some of the items were reverse coded, and the original data appears in the Appendix (See Appendix A). However, for the purpose of clarity, all reversed items in Table 17 have been re-worded to the positive. In reporting the results, the data from the two columns of "strongly agree" and "agree" were combined and the data for "strongly disagree" and "disagree" were combined as well, for the purpose of streamlining the discussion. The data reveal that participants have different goals for, beliefs about and knowledge of the use of representations as it relates to patterning and algebra. In a series of strongly agree/agree statements goals, beliefs and knowledge statements, the percentage of teachers who agreed with goals statements was high. For example, respondents (98%) felt that teachers should use appropriate representations to highlight important mathematical ideas, and the majority of all the teachers (97%) agreed that providing representations to support reasoning is something they would often do to explain difficult concepts to students.

In statements relating to knowledge, the majority of the teachers (95%) perceived that representations could help students draw on their conceptual understandings. About 96% of the teachers believed that students should be encouraged to use representations in order to improve their problem solving skills.

The participants believe that: teachers should flexibly shift among representations as they are generated by students (90%), and not surprisingly, that effective use of representations requires planning (94%), and also that representations help in moving students from using concrete models to abstract representations (95%). The majority of all the teachers were more likely to feel that selecting a worthwhile task determined what representations to use (91%) and perhaps more interestingly that (93%) believed that allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.

Overall, teachers' perspectives on these items not only revealed some possible ways they used representations in their teaching in both Ontario and Lagos, but also that they had a general positive attitude towards representations helping students' understanding. I will return to this in the next chapter. Given that this study involved teachers from two separate but similar social contexts (Ontario and Lagos), it was necessary to consider differences in their views separately for each setting.

5.4.1.1 Ontario Teachers' Responses to MTMRI

Participants reported a variety of views about their use of representations (See Table 18). For example, respondents (95%) felt that teachers should use appropriate representations to highlight important mathematical ideas, and participants (90%) indicated that they were willing to support students' reasoning with representations if a concept seems difficult.

When asked, 95% of the teachers indicated that teachers need specialized understanding of elementary mathematics in order to effectively use representations, 89.5% agreed that representations could help students draw on their conceptual understandings. About 90% of the teachers believed that students should be encouraged to use representations in order to improve their problem solving skills.

Results indicated some beliefs about students and teaching teachers considered in order to use representations. These beliefs were: representations help in moving students from using concrete models to abstract representations (85%); selecting a worthwhile task determined what representations to use (85%) and when students generate their own representations, their understanding will be developed (85%). About 85% participants agreed that teachers should flexibly shift among representations as they are generated by students, and that effective use of representations requires planning (95%).

Overall, Ontario teachers' perspectives on these items revealed that they had a strong positive attitude towards using representations.

Table 17:

Teachers' Responses to MTMRI

	Percent of teachers						
	Strongly agree	Agree	Undecided	Disagree	Strongly disagree	Don' t know	Mean
Goals of teaching with representations							
Providing representations to support reasoning is something I will often do to explain difficult concepts to students.	56.0	41.8	1.1	0.0	1.1	0.	4.52
Appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings.	36.3	62.6	1.1	0.0	0.0	0.0	4.35
Teachers do not need to assist students to choose appropriate representations. (R)	0.0	7.7	3.3	59.3	29.7	0.0	4.11
Knowledge regarding content							
Teachers need to have a specialized understanding of elementary mathematics in order to use representations effectively in teaching patterning and algebra. (R)	5.5	14.3	3.3	60.4	16.5	0.0	2.32
Knowledge regarding learners							
Representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.	50.5	45.1	3.3	0.0	0.0	0.0	4.48
Representations are less effective when suggested to students by teachers, rather than being generated by students.	12.1	30.8	19.8	30.8	4.4	1.1	3.12
Students usually know which representation to use. (R)	23.1	65.9	5.5	4.4	1.1	0.0	4.06
Encouraging students to use representations can improve their problem-solving skills.	63.7	33.0	2.2	0.0	0.0	1.1	4.57
Students are able to work with multiple representations without confusion. (R)	26.4	62.6	3.3	6.6	1.1	0.0	4.07
Knowledge regarding teaching							
I understand enough about teaching-related mathematics to use representations in my teaching. (R)	14.3	61.5	5.5	16.5	2.2	0.0	3.69
Graphical representations are not the most important kind to illustrate algebraic concepts. (R)	3.3	42.9	18.7	28.6	3.3	2.2	3.10
Representations are not hard to use in teaching. (R)	-	22.0	15.4	54.9	7.7	0.0	2.52
The use of manipulatives is good for teaching patterning and also algebra. (R)	7.7	40.7	15.4	25.3	7.7	0.0	2.78
The use of representations does not make patterning and algebra a difficult strand to teach. (R)	-	47.3	6.6	32.5	9.9	1.1	2.89
Beliefs regarding content							
The use of multiple representations is clearly explained in the curriculum. (R)	1.1	2.2	6.6	69.2	20.9	-	1.89
Representations can be mental images.	22.0	65.9	3.3	7.7	0.0	1.1	3.99
Representations are usually physically visible. (R)	7.7	29.7	3.3	44.0	14.3	1.0	3.24
Beliefs about learners							
Allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.	45.1	48.4	6.6	0.0	0.0	0.0	4.39
Representations help in moving students from using concrete models to abstract representations.	54.9	39.6	4.4	0.0	1.1	0.0	4.47
Beliefs about teaching							
The use of representations is useful in teaching and learning patterning and algebra. (R)	5.5	17.6	5.5	42.8	28.6	0.0	2.29
One specific representation of a pattern may not be enough in a patterning and algebra lesson.	18.7	67.0	7.7	5.5	1.1	0.0	3.97
Selecting a worthwhile task determines what representation to use.	20.9	70.3	4.4	3.3	0.0	0.0	4.10
Teachers should flexibly shift among different representations as they are generated by students.	22.0	68.1	6.6	3.3	0.0	0.0	4.09
The effective use of representations requires a lot of planning.	34.1	60.4	3.3	2.2	0.0	0.0	4.26

Weighted average 3.65

Note: Confidence was scored on a 6 point scale Strongly agree=5; Agree=4; Undecided=3; Disagree=2; Strongly disagree=1; Don't know=0, R = negative items reworded as positive (See Appendix C for the original version)

Table 18:

Ontario Teachers' Responses to MTMRI (n=20)

	Percent of teachers						
	Strongly agree	Agree	Undecided	Disagree	Strongly disagree	Don' t know	Mean
Goals of teaching with representations							
Providing representations to support reasoning is something I will often do to explain difficult concepts to students.	60.	40	0	0	0	0	1.4
Appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings.	25	70	5	0	0	0	1.80
Teachers do not need to assist students to choose appropriate representations. (R)	0	10	5	65	20	0	2.05
Knowledge regarding content							
Teachers need to have a specialized understanding of elementary mathematics in order to use representations effectively in teaching patterning and algebra. (R)	20	75	5	0	0	0	4.15
Knowledge regarding learners							
Representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.	47.4	42.1	10.5	0	0	0	1.63
Representations are less effective when suggested to students by teachers, rather than being generated by students.	10.5	26.3	26.3	26.3	5.3	5.3	3.05
Students usually know which representation to use. (R)	0	0	5	80.0	15.0	0	1.90
Encouraging students to use representations can improve their problem-solving skills.	35.0	55.0	5.0	0	0	5.0	1.90
Students are able to work with multiple representations without confusion. (R)	1.4	20	10	55	15	0	2.35
Knowledge regarding teaching							
I understand enough about teaching-related mathematics to use representations in my teaching. (R)	5.0	35.0	10.0	45	5	0	2.90
Graphical representations are not the most important kind to illustrate algebraic concepts. (R).	5.3	52.6	21.1	15.8	0	5.3	3.63
Representations are not hard to use in teaching. (R)	5	55	15	25	0	0	3.40
The use of manipulatives is good for teaching patterning and also algebra. (R)	30	35	10	10	15	0	3.55
The use of representations does not make patterning and algebra a difficult strand to teach. (R)	0	50	5	40	5	0	3.00
Beliefs regarding content							
The use of multiple representations is clearly explained in the curriculum. (R)	5.0	75	15	0	5.0	0	3.75
Representations can be mental images.	0	75	10	10	0	5	2.85
Representations are usually physically visible. (R)	15	50	5	20	5	5	3.65
Beliefs about learners							
Allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.	35	50	15	0	0	0	1.8
Representations help in moving students from using concrete models to abstract representations.	45	40	15	0	0	0	1.70
Beliefs about teaching							
The use of representations is useful in teaching and learning patterning and algebra. (R)	40	30	5	25	0	0	3.85
One specific representation of a pattern may not be enough in a patterning and algebra lesson.	50	45	5	0	0	0	1.55
Selecting a worthwhile task determines what representation to use.	10	75	10	5	0	0	2.10
Teachers should flexibly shift among different representations as they are generated by students.	25	60	10	5	0	0	1.95
The effective use of representations requires a lot of planning.	35	60	5	0	0	0	1.70

Note: Confidence was scored on a 6 point scale Strongly agree=SA; Agree=A; Undecided=U; Disagree=D; Strongly disagree=SD; Don't know=DK; X=unanswered question, R= negative items reworded as positive (See Appendix C for the original version)

5.4.1.2 Lagos Teachers' Responses to MTMRI

As with the Ontario subsample, participants from the Lagos subsample reported a fairly strong view than the Ontario about their use of representations on at least some of the items (see Table 19). For example, all the participants felt that teachers should use appropriate representations to highlight important mathematical ideas, and over 97% (based on those who strongly agreed or agreed) of the participants indicated that they were willing to support students' reasoning with representations if a concept seemed difficult.

When asked, less than three quarters of participants (71.8%) indicated that teachers need specialized understanding of elementary mathematics in order to effectively use representations, and 98.6% agreed that representations could help students draw on their conceptual understandings. Almost all the teachers believed that students should be encouraged to use representations in order to improve their problem solving skills.

The results of the study indicated some important beliefs about students and teaching teachers considered in order to use representations. These beliefs were: representations help in moving students from using concrete models to abstract representations 97.1% (85%); selecting a worthwhile task determined what representations to use 94.3% (85%) and when students generate their own representations, their understanding will be developed 95.8% (85%). About 91.5% (85% participants agreed that teachers should flexibly shift among representations as they are generated by students, and that effective use of representations requires planning 94.4% (95%).

Table 19:

Lagos Teachers' Responses to MTMRI (n=71)

	Percent of teachers						
	Strongly agree	Agree	Undecided	Disagree	Strongly disagree	Don' t know	Mean
Goals of teaching with representations							
Providing representations to support reasoning is something I will often do to explain difficult concepts to students.	54.9	42.3	1.4	0	1.4	0	1.51
Appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings.	39.4	60.6	0	0	0	0	1.61
Teachers do not need to assist students to choose appropriate representations. (R).	0	7	2.8	57.7	32.4	0	1.85
Knowledge regarding content							
Teachers need to have a specialized understanding of elementary mathematics in order to use representations effectively in teaching patterning and algebra. (R)	15.5	56.3	2.8	18.3	7	0	3.55
Knowledge regarding learners							
Representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.	52.1	46.5	1.4	0	0	0	1.49
Representations are less effective when suggested to students by teachers, rather than being generated by students.	12.7	32.4	18.3	32.4	4.2	0	2.83
Students usually know which representation to use. (R)	1.4	5.6	5.6	60	25.4	0	1.96
Encouraging students to use representations can improve their problem-solving skills.	71.8	26.8	1.4	0	0	0	1.30
Students are able to work with multiple representations without confusion. (R)	1.4	2.8	1.4	64.8	29.6	0	1.82
Knowledge regarding teaching							
I understand enough about teaching-related mathematics to use representations in my teaching. (R)	1.4	11.3	4.2	66.2	16.9	0	2.14
Graphical representations are not the most important kind to illustrate algebraic concepts. (R)	2.8	22.5	18.3	50.7	4.2	1.4	2.73
Representations are not hard to use in teaching. (R)	8.5	54.9	15.5	21.1	0	0	3.51
The use of manipulatives is good for teaching patterning and also algebra. (R)	5.6	42.3	16.9	29.6	5.6	0	3.13
The use of representations does not make patterning and algebra a difficult strand to teach. (R)	11.3	33.8	7	46.5	0	1.4	3.14
Beliefs regarding content							
The use of multiple representations is clearly explained in the curriculum. (R)	25.4	67.6	4.2	0	0	2.8	4.10
Representations can be mental images.	28.2	63.4	1.4	7.0	0	0	1.87
Representations are usually physically visible. (R)	5.6	23.9	2.8	50.7	16.9	0	2.51
Beliefs about learners							
Allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.	47.9	47.9	4.2	0	0	0	1.56
Representations help in moving students from using concrete models to abstract representations.	57.7	39.4	1.4	0	1.4	0	1.48
Beliefs about teaching							
The use of representations is useful in teaching and learning patterning and algebra. (R)	25.4	46.5	5.6	15.5	7	0	3.68
One specific representation of a pattern may not be enough in a patterning and algebra lesson.	9.9	73.2	8.5	7	1.4	0	2.17
Selecting a worthwhile task determines what representation to use.	24.3	70	2.9	2.9	0	0	1.84
Teachers should flexibly shift among different representations as they are generated by students.	21.1	70.4	5.6	2.8	0	0	1.90
The effective use of representations requires a lot of planning.	33.8	60.6	2.8	2.8	0	0	1.75

Note: Confidence was scored on a 6 point scale Strongly agree=SA; Agree=A; Undecided=U; Disagree=D; Strongly disagree=SD; Don't know=DK; X=unanswered question, R= negative items reworded as positive (See Appendix C for the original version)

It should be noted that, while a number of the items were negatively worded, Table 19 attempts to present all of the items positively, by reversing the wording (and scores) of the negatively worded items. The goal of doing this was to allow the calculation of a "mean" for each item, with the idea that a larger mean score would suggest more alignment with the ideas of mathematics reform principles.

Overall, Lagos teachers' perspectives on the MTMRI revealed that they tended to hold a marginally stronger positive attitude towards using representations than their Ontario colleagues, as drawn from responses in the areas of goals and beliefs. For example, about 91% of the Lagos teachers strongly agreed/agreed that teachers should flexibly shift among different representations as they are generated by students while 85% of the Ontario teachers strongly agreed/agreed. However, there was a relatively higher percentage difference about the need for specialized knowledge in favour of the Ontario teachers. That means that Ontario teachers, unlike their Nigeria counterparts, agreed that specialized understanding is needed to use representations effectively. Further analysis of teachers' individual understanding of representation may shed some light on how teachers use representation, how they think about representation and which types of representations teachers use when teaching.

5.4.2 Representation Described by The Teachers

To answer the research question regarding what representations teachers in Ontario and Lagos use when teaching algebra, I will present some summarized data from the open-ended survey question about the terms used to describe representation. The purpose of this sub-section is to provide a detail description of representation as the teachers understood it.

5.4.2.1 Meaning of Representation Given by Participants

I used the open-ended question to analyze the meaning teachers attached to representation. Sixty-nine (75.82%) of the 91 participants who responded were from Lagos, 19 (20.88%) who responded were from Ontario, and 3 (3.30%) of teachers did not respond to the survey's open-ended question. The following are illustrative of teachers' description of representations as they understood it. These comments revealed typical responses of seven teachers from each location that were randomly selected.

Lagos teachers commented:

Representations mean the use of charts, symbols, and diagrams to explain any math concepts.

Representations are thinking tools such as symbols used for doing mathematics.

It is process of presenting algebraic, scientific or mathematical concepts, especially those that are abstract in nature with concrete, symbols, visible and simplified ideas that can help in aiding the understanding of the concept been taught.

They are diagrams or symbols use for presenting data that are to be solved or solutions to some questions in mathematics. Examples graphical, bar chart, histogram etc.

They are used to develop and to communicate different mathematical features of the same object or operations.

They are used to understand different mathematical concepts

Representations means teaching Mathematical concepts using diagrams, graphs, symbols etc.

As can be seen, many of the Lagos teachers included the word "symbol" in their explanation, and some explanations lack clarity.

Ontario teachers responded:

A picture, symbol representing an equation or number statement

A way to show mathematical thinking and concepts; a model. A tool to solve mathematical problems.

I believe that it is our ability to symbolize the math we are doing. If you are able to see the questions more abstract then what is on the paper is a good beginning.

… representation in math to me is about using visuals such as math diagrams, pictures, symbols, charts tables etc or concrete materials to explain or communicate an understanding of different mathematical concepts and how they relate to each other. - math representations can also be used to help students interpret and uncover their own understanding of a math concept.

Representations are a way of modeling a mathematical relationships. In patterning and algebra representations can take on various forms including tables, graphs, geometric models, symbols, words, algebraic expressions or real world examples.

> Representations are a way for students to make sense of a wide range of mathematical concepts in a variety of strands. They can show student understanding and thinking but can also be effective tools for working through problem-solving situations and develop an understanding of complex problems.

> To show the relationships in a more tangible way…I talk about translating between different representations/different languages for showing a relationship.

While the Ontario teachers had varying responses, they tended to use words like "model" more often than the Lagos teachers.

From the variety of comments above, it is hardly surprising that teachers did not provide one clear meaning of representation. This is because unlike some of the other mathematical learning processes, representation is one that does not have a one, uncontested (Stylianou, 2011), definition in mathematics education. My findings can therefore be regarded as consistent with Stylianou's (2011) findings in which all the teachers in her study gave different definitions of representation.

However, equally well worth quoting under the definition of representation, is a teacher from Ontario who pointed out that, it is a way of explaining mathematics including representation as a process and a product. For example, one teacher noted that representation is

"Using a variety of ways to teach/explain mathematical concepts and relationships. It can include manipulatives, graphs, number sentences".

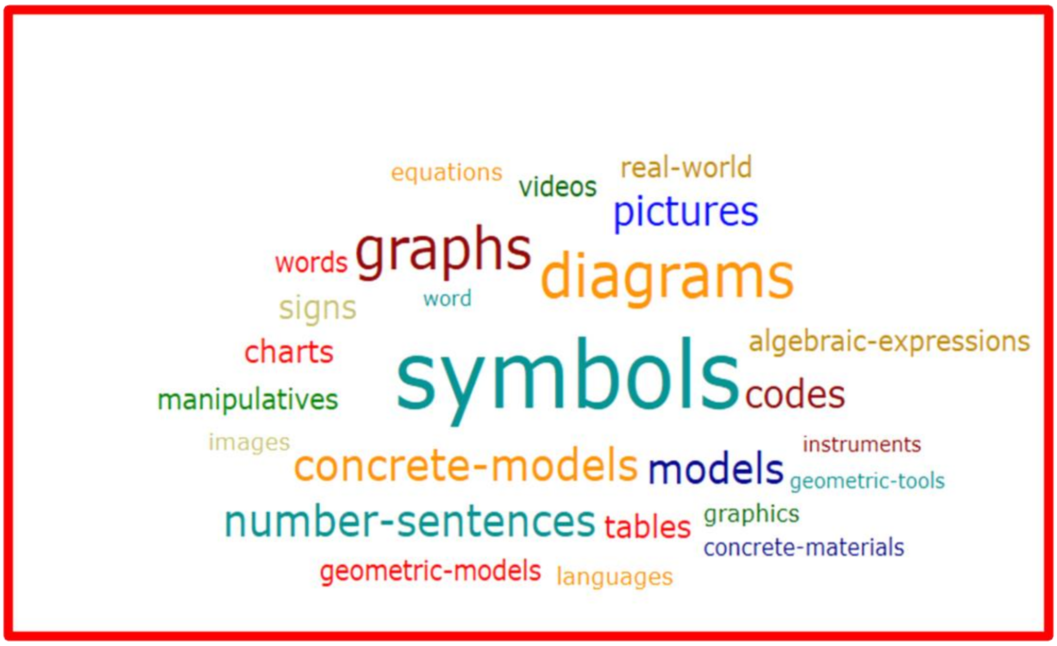
This finding strongly confirms NCTM's (2000) position in their national standards document, which stated that "the term representation refers both to process and product—to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67). In the teachers' definitions, many of the teachers focused on both the product and process aspect of representation. In the definitions, teachers discussed representation as a product as something that is created to show one's reasoning process or result after the fact, while others mentioned representations as a process, that is used as a tool for thinking about a problem.

5.4.2.2 Terms Used to Describe Representation by Participants

I analyzed the frequency counts of teachers' responses to the open-ended question from the MTMRI online survey, and a number of terms were used to describe representation emerged from the analysis of their responses. The result is represented as a word cloud in **Error! Reference source not found.** as generated from ATLAS. ti 8. The larger a word appeared in the word cloud shows the more the word is used and, it does not matter where a word is positioned in the word cloud. As **Error! Reference source not found.**: illustrates, the participating teachers in this study showed a rather strong usage of *symbols* compared to other modes of representations, according to their given meaning of representations in the MTMRI survey. However, in examining Figure 5 it must be remembered that the sample size of the Lagos teachers was much larger than Ontario, and hence this data weighted more towards teachers in that region.

Figure 5

Terms Used by Participants to Describe What Representations Are



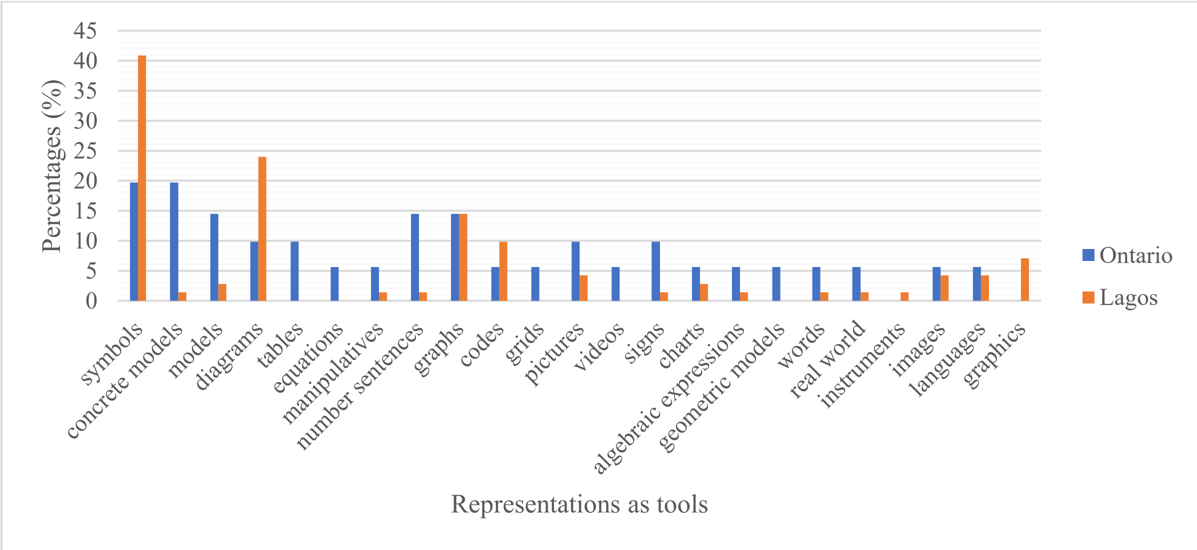
The different types of representations indicated by the teachers.

The teachers demonstrated that they are familiar with representations associated with the mathematics concepts, in particular, patterning and algebra, to varying degrees. *Diagrams, graphs, concrete materials, algebraic expressions, manipulatives, pictures, models, number sentences,* and *words* were the modes of representations the teachers mentioned in their responses to the open-ended question. These results were found to be consistent with the Ontario curriculum, *Grades 1 – 8: Mathematics, 2005*, which mentions "concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols" (Ontario Ministry of Education, 2005, p. 16).

The results were further analyzed in order to see how the teachers from the Ontario subsample compared with those of the teachers from the Lagos subsample (see Figure 6). Frequency counts for each of the modes of representations were first levelled and then, calculated separately for the Ontario participants and the Lagos participants. This was done because the difference in sample sizes would make it difficult for such comparison otherwise. Figure 6 presents the data.

Figure 6:

Terms Used to Describe Representations by Jurisdiction



There was a wide difference in the frequency counts between the Lagos teachers (40.9%) and the Ontario teachers (19.7%) on thinking of representations as *symbols* when teaching patterning and algebra. The Ontario teachers' frequency counts of the use of the *concrete models* (19.7%), *models* (14.5%), *tables* (9.8%), *equations* (5.6%), *signs* (9.7%), *videos* (5.6%), *grids* (5.6%), *manipulatives* (5.6%) and *number sentences* (14.5%) were higher than that of Lagos teachers. This could mean that teachers in Lagos did not regard them as modes of representations or did not often use them. Similarly, *codes* (9.9%) and *diagrams* (24%) were higher in favour of Lagos. There was only a marginal difference between the frequency counts of Ontario and Lagos teachers concerning *models* and *signs* in favour of the Ontario teachers. They mentioned *graphs* in equal numbers. This finding draws our attention to the fact that there are multiple ways of solving mathematical problems, so instead of teachers focusing only on one method, students should be allowed to explore many other possible ways. Regarding the uses and benefits of using various representations such as *symbols, signs, colour, diagrams, gestures and pictures*, Naidoo

(2011) suggested that these representations could serve as an alternative to the traditional approach to teaching. For example, Hanna (2000), clearly stated that diagrams may efficiently be used to facilitate students' understanding. A representation such as a diagram is capable of showing precisely what we are trying to express, which in verbal statements is not easily understood (Skemp, 1976).

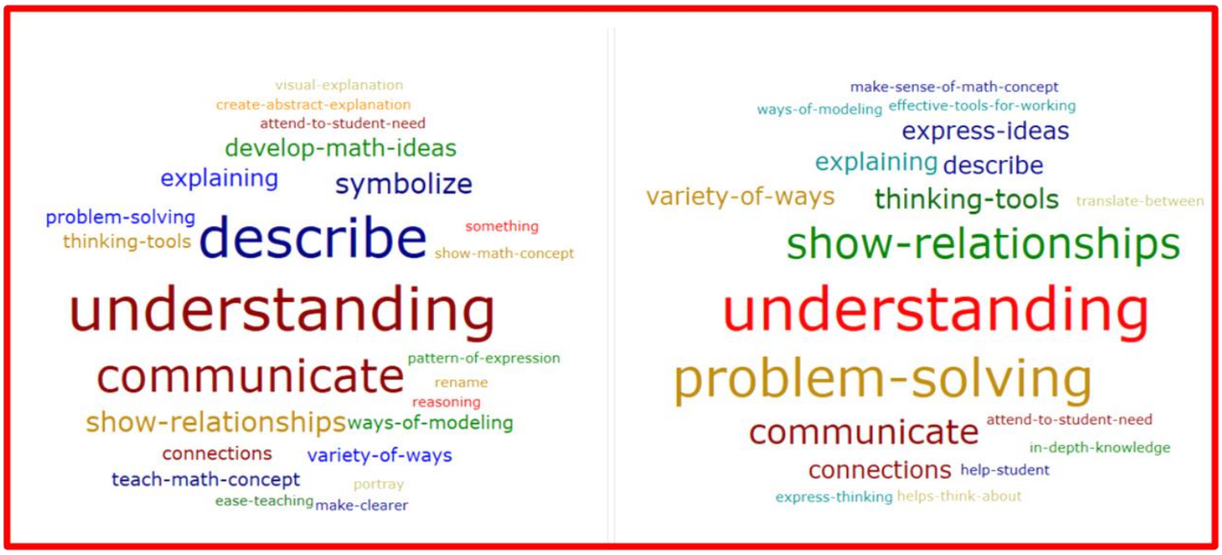
Finally, I created a word cloud comparison of participants' responses to the open-ended question, which showed different words teachers used to describe the roles of representation. Although the open-ended question did not require teachers to provide the explicit or implicit roles of representation, about 90% of all the teachers mentioned their purpose in using representations to teach patterning and algebra. In order to establish the comparison, frequency counts for each of the representations provided by the teachers were first levelled, and then, calculated separately for the Ontario participants and the Lagos participants, as before. The result, illustrated in Figure 7, showed that in each case, the *symbols, models, diagrams, graphs, signs, number sentences, codes, concrete materials, equations, tables* are useful for *understanding, describing, explaining, connecting*, *communicating, problem-solving, thinking tool, modeling*, and *showing relationships* with other systems. For example, one of the Ontario participants responded:

> representation means many things. In my practice representation means to show students both concrete models and the connection they have to the math concepts or number sentences they describe. In addition, representation with various models depends on the students need at the particular time. Using models effectively requires teachers to have in-depth knowledge of the math they are teaching so that they can pull out the kind of representation needed for their students to help make connections and build understanding. (ON8, Interview)

I found that the teachers who reported that they used representations to communicate and build understanding provided data consistent with the research results obtained by Stylianou (2011) who found that all the teachers in her study used representation as a communication tool. In addition, teachers from both jurisdictions mentioned that they use representation as a thinking tool. These results were found to be consistent with Johnson and Lesh (2003), who stated that, a student's way of thinking put into a given graph or diagram would often lead to new ways of thinking that are read out of these graphs and diagrams.

Figure 7:

Teachers' Mention of Different Types of Roles for Representations in Lagos and Ontario



The left half of Figure 7 reflects responses obtained from the Lagos teachers while the right side of Figure 7**Error! Reference source not found.Error! Reference source not found.** reflects words used by the Ontario teachers. More responses tending towards math reform approaches were obtained on the open-ended question from Ontario teachers than from the Lagos teachers. For instance, there were higher frequencies found in the Ontario sample on the use of representation to build *understanding* (15.4% ~ 29.6%), *relationships* (5.6% ~ 19.7%), and *problem-solving* (2.8% ~ 25.4%). Other areas included *way of modeling* (2.8% ~ 4.2%), *variety of ways* (2.8% ~ 9.9%), *explaining* (4.2% ~9.9%), *connections* (2.8% ~ 9.9%), and *communicate* (11.3% ~ 15.4%) also in favour of Ontario teachers. With regards to using representations to *describe* (14.1% ~ 9.9%), Lagos teachers indicated a slightly higher response, which may also be associated with a more traditional teaching style. Representations have the power to mold, shape, amplify, and generate ideas (Bruner, 1969). Bruner further stated that different representational tools are powerful, such that, they help students to simplify complex patterns and relationships, go beyond the information given and develop skills of prediction and explanations that is observed in a given information.

5.5 Summary

The chapter described the findings from the data collected through the online survey. The survey results suggest that, in general, teachers are open to the use of representations in learning and believe in their value.

Of the different representations, participants, particularly Lagos participants were more comfortable with the use of symbols than other types of representations, while Ontario teachers tended to refer more to models, including concrete models. This suggests that, while teachers may claim to believe in, know and use representations, they may not be fully effectively doing so. The next chapter focuses on the data obtained from the ten teachers who were individually interviewed, referred to as the case study teachers.

CHAPTER SIX: FINDINGS—CASE STUDY TEACHERS

6.1 Introduction

Five teachers from each country were individually interviewed, and the data obtained from these teachers is provided next. In the first part of this chapter, some background information regarding the scenario interview questions is provided as well as biographical information regarding the participants in Ontario (Scott, Sara, Silva, Susan and Sonia) and Lagos (Bryce, Ben, Beth, Bola and Baker). In the second part, I thematically present and discuss the vignettes from each participant, and relate the vignettes to the MTMRI survey data as well as the scenario interview questions. Pseudonyms were used to protect their identities.

In this section and throughout the thesis I have used double quotation marks where I have used the case study teachers' actual words. The rest of the text is a paraphrasing of their words and a filing out and connecting of data in other instances.

6.2 Scenario Interview Questions

Question one required the students' knowledge of letters as quantities instead of processing letters alphabetically as the question did not involve any mathematical operations. The second scenario question presented a real-world problem requiring the students to write linear equations and also model different cell phone plans. Students were expected to look at the graphs of the lines in the context of the cell phone and make a connection to the meaning of intersection points of two lines with the simultaneous solution of the two linear equations. It was expected that the second problem be solved graphically (see Table 20). The Ontario Grades 1- 8 mathematics curriculum in its specific expectations requires Grade 8 students to "describe different ways in which algebra can be used in real-life situations", "evaluate algebraic expressions with up to three terms by substituting fractions, decimals and integers for the variables" and "model linear relationships using tables of values, graphs and equations" (2005, p.

116 – 117). Similarly, the Lagos Grade 8 mathematics curriculum in its specific expectations required students to solve quantitative reasoning problems, interpret word problems involving algebraic expressions and plot the graph of linear equations in two variables as well as from real life situations.

Table 20:

Scenario Questions

Question	Scenario
1	Sam has x bananas and Codi has p bananas. Collin counts the number of bananas each of them have and finds they are the same. Sam said you write as x= p, but Codi said that x and p are different letters and so cannot be the same. What would you say to these students?
2	Olamide just arrived in Canada and needed a phone in order to communicate. Olamide met Tyler who visually displays three plans and points out the advantages of each plan to Olamide. Plan A costs a basic fee of $29.95 per month and 10 cents per text message Plan B costs a basic fee of $90.20 per month and has unlimited text messages Plan C costs a basic fee of $49.95 per month and 5 cents per text message All plans offer unlimited calling Calling on nights and weekends are free Long distance calls are included Olamide wants to know how to decide which plan will save him the most money. Your students were to determine which plan has the lowest cost, depending on the number of text messages Olamide is likely to send. Explain this to Olamide. For example, you could explain by defining variables, writing equations, making tables, constructing graphs, finding slopes and intercepts, and finding points of intersection. Imagine that two of your students came to you with initial solutions as below: S1: $f(x) = 29.95x + 0.10y$, $f(x) = y + 90.20x$, $f(x) = 49.95x + 0.05y$ [adding an extra variable to the equations] S2: $x = 29.85 + 10y$, $x = 90.20$, $x = 49.95 + 5y$. What would you say to these students?

6.3 Presenting Ontario Teachers

In the next section, biographical information regarding the five Ontario case studies participants Scott, Silva, Susan, Sonia and Sara is provided as well as a brief description of the qualification and experience of the five teachers. I present and discuss the findings from the survey and interviews. All discussions on the sub-themes goals, knowledge and beliefs are structured according to the specific order of the different survey dimension descriptors. A summary is provided at the end of this section in Table 21.

6.3.1 Scott

6.3.1.1 Background

Scott, in his late 20s, teaches mathematics at the elementary level in northern Ontario. He has more than five years of mathematics experience, but less than that in grade 8. He has his Bachelor of Outdoor Recreation degree and a Bachelor of Education degree. His basic qualifications are in Environmental Science (I/S divisions) and English (I/S divisions), and his additional qualifications are in reading (Part 1) and mathematics education (Primary and Junior, Part 1).

6.3.1.2 Teachers' Goals for and Use of Representations in the Classroom

One of Scott's goals for using representations in his classroom was to help his students develop a sense of the mathematical language that they would be using in the real-world problems in future math and physics. He reported that he worked with representations such as manipulatives, smart cubes, pattern blocks, and algebra tiles by "attaching the terminology and the language that should be used to describe the patterns and then transform[ed] them over into a table of values and eventually build the graph". Scott mentioned that he used the language of *input equal to output and multiplying it* to help his students to build the mathematical language that could be attached to the algebra down the road. Furthermore, Scott explained that "the trajectory of instruction with representation will be starting with the word form and the physical representations, and then making sure that as we use the word form, we are using the language to describe the pattern".

Another goal for using representations in Scott's classroom was to let students show the relationships among the numerical, graphical, and algebraic representations, and enable them to understand difficult concepts. Scott believed that teachers should support students' use of representations to show relationships, develop mathematical ideas, and switch between representations fluently. He felt these were important skills, which he tried to build in his students. For Scott, developing good representational skills of the students in his class meant building strong confidence in solving any given algebra task using different representations. He said that "Some students are comfortable building them geometrically in table form, and then graphing it; other students are very comfortable going straight to using variables, expressions and equations". Scott felt that instruction also depended on the particular students and the type of representations they are able to access.

A third goal of using representations based on the survey response and interview with Scott, was that he used representations in his classroom as a scaffold to help students learn, including from their misconceptions and to clarify any misunderstandings. This was also apparent in his survey response in which he strongly agreed with the statement that suggested that providing appropriate representations during classroom discussions could be used to explain difficult concepts as well as clarify misunderstandings for students. It became very clear talking with Scott that he believed that, when a teacher noticed misconceptions, the teacher should be able to support their students in dealing with such misconceptions through a responsive use of representations. This was evident based on what Scott said during the scenario interview.

Scott quickly noted that the first scenario question (Table 20) was a misconception. Furthermore, he knew how to help his students uncover misconceptions. "I was thinking that Codi seems to have a misconception of what variables are and what they mean". In order to uncover the student's misconception, he thought he would need to ask some questions. Some of the questions Scott intended to ask to uncover Codi's misconceptions were: "What are variables? What are the variables used for? What does the variable represent?".

In summary, one of Scott's goals for and use of representations in the classroom is for students to learn math language to solve future real-world problems. A second goal was to have students understand the relationships between different ways to think about and show concepts. Scott also felt students learn more, particularly from misconceptions, clarify misunderstandings, and solve math task confidently when they develop representational skills.

6.3.1.3 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.3.1.3.1 *Content*

Scott stated that there is a strong connection between the curriculum content and the use of different representations. He mentioned that his choice of representations was informed by the focus of his planning cycle, the learning goals he intended to achieve that were connected to the curriculum, and the tools or representations that are described in the curriculum. This was corroborated in his survey responses in which he strongly agreed with the statement that suggested that effective use of representations requires a lot of planning. During the interview, Scott reflected on the way in which he used different representations and talked about using the *systems in action unit*, in which students were taught how to calculate work and mechanical advantage in order to practice their patterns skills in the real-world context.

Scott believed that deep content knowledge is needed to teach effectively and to apply the appropriate representations. He said, "If you have that deep content knowledge, you are going to notice some misconceptions right away and be able to ask questions that might guide students towards uncovering that misconception or realizing that there is a mistake". He further explained that a teacher with in-depth content knowledge could make a quick assessment of his/her students and be able to predict the readiness of the students for a new representation. This was corroborated during his survey response, as Scott disagreed with the statement that suggested that a teacher does not necessarily need a specialized understanding of elementary mathematics in order to use representations effectively.

Scott's explanations showed he has a good knowledge of what representations mean and how they could be used in math class. For example, Scott said, "representations are something that can show a mathematical relationship similar to a model". He further explained that "representations can be a physical representation; can be represented algebraically; in numbers; you could do representations through graphing, a chart". According to the survey, Scott strongly disagreed that representations are usually not physically visible. During the interview, he explained that "So, representations are really something to show the mathematical relationship of a concept". Scott also talked about having great experiences with the use of representations in his classroom. Scott mentioned, From Patterns to Algebra (2012) by Ruth Beatty, as one of the resources that helped him not just to gain confidence teaching with representations, but also gave him focus to "unwind what types of representations we should be using in patterning and algebra".

6.3.1.3.2 ***Students and Pedagogy***

Scott expressed some beliefs that indicated that he felt the use of representations was important in the math classroom. He believed that representations build on each other. For example, Scott said, "I taught Grade 8 last year, and I'm teaching Grades 3/5 this year, and it's kind of interesting how representations build on each other year by year". He stated that one of his beliefs about using representations was to encourage students to move from concrete to abstract representations. Not surprisingly, on the survey, he strongly agreed with the statement in the survey that suggested that representations help in moving students from using concrete models to abstract representations. Scott also felt that students' ability to work with different representations was important to learning. He explained, “I decide on that scaffolding of how can I make it as concrete as possible and build either the language or the actual concepts onto those concrete models”.

Scott had a specific pedagogical approach on how one could most effectively encourage students to transition to other representations rather than just relying on one. When asked if Scott and his students had learned anything new from using representations to teach patterning and algebra, he talked about teaching students to move from using a table to using a graph while working on linear and exponential growth rate. For example, Scott noted that “they may just see it on the table of values as an increasing pattern, but by graphing it they may notice these patterns are growing in very different ways”. Not surprisingly, Scott strongly agreed with the statement in the survey that suggested that teachers should flexibly shift among different representations as they are generated by students. He explained that “they may notice that certain patterns create points in a graph and some patterns don’t, they won’t easily notice that relationship unless they are using the graph as a representation”. For Scott, an excellent transition to other representations meant his students would be better prepared to switch among representations irrespective of the patterns they are working on.

Scott believed that students learn by being guided developmentally. He started by using the word forms and then moved to algebraic expressions with variables. Scott explained that first, he started with his own representation and then, he allowed students to start developing their own representations using different manipulatives. Scott believed that it is easier for him to generate representations for his students, than for them to describe his representations. Scott for instance said, “I guess they want to be comfortable interpreting my representations before they feel comfortable building their own”. Perhaps this was why Scott agreed with the statement in the survey that suggested that teachers should assist in choosing appropriate representations for students. He felt that accessing the language was important for the students to feel comfortable in generating their own representations “unless they’ve seen the model a few times already”.

Scott discussed his instructional methods for encouraging students’ deep thinking of the material. For example, he said “If I intended to use a particular representation and then decided not to, it would be that I was either trying to change the question from a certain representation to potentially uncover some deep thinking…”. This practice manifested itself in his survey response. According to the survey, Scott strongly agreed that providing representations to support reasoning is something he would do to explain difficult concepts to his students. He elaborated, “I potentially noticed a misconception and that, they are applying specific representation incorrectly, and I feel like we need to figure this out before we move to a new representation”.

While responding further to the first scenario question, in dealing with the task and what students needed to learn from this task, Scott predicted that he would use questioning and “a little bit of flexibility about what a variable could represent and how we can use variables to represent different numbers”. He concluded that “if that still wasn’t getting through, then I may be planning a specific intervention for Codi or some different problems that allow him to express those possibilities with variables”.

Based on Scott’s experience teaching Grade 8, I asked him how he usually covered the gaps between Grades 8 and 9 in order to help his students prepare for Grade 9 mathematics. Ultimately Scott believed, “I think it depends overall on the students, and it depends on what streams they will be going into next year”. Scott further explained, “one of the biggest things is, this is just from my experience of the students, was how those linear patterns were taught, that is $y = mx + b$”. Scott frequently worked his students through problems involving a lot of linear growth patterns. Scott believed that once the students had exposure to constant linear growth patterns, it built their confidence before they were introduced to algebra in Grade 9.

Scott felt that having sufficient pedagogical knowledge of the representations useful for his lessons and using various representations to connect students' knowledge with new situations, were important. This appears to provide further evidence to support Scott's claim that he varied his instructional methods using different representations. For instance, Scott believed that, although there are situations that required more than one representation in a lesson, it would not be wise "to give too many representations right away" as it might cause confusion for the students. This was reaffirmed in his survey response in which he agreed with the statement that suggested which including a lot of representations in a lesson could add confusion for students. Scott talked about using more than one representation in "situations where the numbers are increasing fairly quickly".

He commented that "it will be inefficient to represent a growing pattern that is growing very quickly using algebra tiles". Scott believed that he might get them started with the first few terms using algebra tiles before asking them to suggest other representations. According to the survey, Scott strongly agreed with the statement that encouraging students to use representations can improve their problem-solving skills.

Scott knowledgeably described his students' learning challenges. He mentioned his students' inability to work with multiplication fluently as another challenge facing their learning of patterning and algebra. "For instance, when we are building linear patterns, making those predictions further down the road, if they do not have good number sense with their multiplication they may find it challenging making those predictions". He commented that some of his students misperceived use of manipulatives to be "very elementary". Despite Scott highlighting some challenges associated with the teaching of this strand of math, he disagreed with the statement in the survey that suggested that it is one of the difficult strands to teach.

What came out strongly in the survey and interview was that, while Scott had several ways of encouraging students to use representations, he also used his knowledge of the students to predict their approach to the questions. This was apparent during his response to the scenario questions. When Scott was asked about the types of representations, he would expect his students to build, he said that “they may want to have conversations about it”. Scott predicted that some of the students would “right away say, that makes sense, x and p can equal the same number of bananas”. Scott further explained that “if there is any disagreement, I can see a student drawing a picture or using some objects like smart cubes or something to make a model of the situation”. For Scott, the students’ excellent approach to solving any given task meant they would be better prepared to use the appropriate representations in problem-solving.

Scott indicated that the students’ excellent approach to solving any given task also meant encouraging them to use their own representations. While responding further to the scenario questions, Scott predicted that, for his students to use representations as tools to solve the tasks, they might use a table and some concrete materials. He felt that he could “potentially create a T-chart form”. Scott felt that he would make use of a concrete balancing model and allow them to get “comfortable with variables representing different numbers”. He predicted that his students would use a table of values, graph and algebraic equations as tools to solve the second task (Table 20). Scott further predicted that in order to support his students in solving this task, he would make the students come up with other forms of representations in order to solve the task in a different way.

In summary, Scott believed that teaching with different representations needs to be connected to the curriculum material. He felt that teachers should vary their approaches to instruction using multiple representations and encourage students to generate their own representations but only after he has provided his own. Scott spoke specifically about transitioning progressively over different representations and how this approach would mean students are better prepared for algebra. Scott strongly believed that he needed to first choose representations for his students before they could use multiple representations in problem-solving. He also thought teachers need to use multiple representations as a responsive mechanism to unpack students' misconceptions.

6.3.1.4 Summary

Scott felt that teachers would need in-depth content knowledge to use different representations appropriately. He spoke knowledgeably about his students' learning challenges and how he varied his approach to teaching in order to meet individual needs.

He used representations such as manipulatives, pattern blocks, smart cubes and algebra tiles. Scott reported trying to work flexibly with different representations in order to uncover student misconceptions and respond to their individual challenges. He began by first providing and developing representations in a lesson and then encouraging students to generate their own.

6.3.2 Silva

6.3.2.1 Background

One of the teachers who was not sure, at the beginning, whether or not to serve as one of the five case studies from Ontario, was Silva. She later indicated her willingness to participate and was glad she did. Silva, a teacher in her 30s, has taught mathematics for more than four years. She teaches at the Grades 6, 7, and 8 level in a northern Ontario board. Silva has her Bachelor of Arts degree and a Bachelor of Education degree. Silva's background was not mathematics. She recently completed an additional qualification in Intermediate division (mathematics).

6.3.2.2 Teachers' Goals for and Use of Representations in the Classroom

Silva stated that her goals of providing representations for her students is to let them see mathematics problems in different ways, be able solve math tasks in multiple ways, and clarify misunderstandings. This was also apparent in her survey response in which she agreed with the statement that appropriate representations should be used to clarify misunderstandings during classroom discussions. Silva felt that the effective use of classroom discussions and questions would greatly impact the students' use of representations when solving a mathematics problem. Silva believed that appropriate use of representations improved the quality of students' understanding of the context and therefore the content. She said, "with representations they see the context of the content really". According to the survey, Silva strongly agreed that providing representations to support reasoning is something she does to explain difficult concepts to her students.

Another goal for using representations in Silva's classroom was to help her students prepare for the challenges they will be facing in Grade 9. She believed that the more opportunities one has to discuss the expectations in the next grade, the better the chances of the students feeling comfortable with learning mathematics. Silva said, "because I teach Grades 7 and 8 math, so a lot of the time my Grade 7's will hear the conversations with the Grade 8, so they are a little bit more comfortable the following year. They say, oh I remember that". It appeared Silva made efforts to ensure that her students do not lack the representational fluency needed in the next grade (grade 9). According to Silva, "the graphing and the equation seem to be the big problem for a lot of them". We do a lot of graphing and then going back to the algebraic model because they are going to need it when they get into Grade 9". Silva further explained that “I let them pick which representations that make the most sense to them. I tried to push them to the graphing, but I noticed for some of them, one of their biggest weaknesses is finding that linear relationship”. Silva reported that she focused more on graphing and equations in her teaching of patterning and algebra.

Speaking with Silva, I noticed that, while responding to the scenario questions, her overriding concern was ensuring that her students understood the key mathematical ideas. She believed that an ongoing challenge for students was the realization of the fact that the variable is a quantity and not an object. She felt that using the correct mathematical language and effective use of appropriate representations would clarify some of her students’ misconceptions. For instance, she explained how she would address the algebraic misconceptions of the letters as an alphabetic representation of a specific value. She said, “I would explain that the letters are different because they are representing different people. If Sam were given more bananas, it would no longer be equal to Cody’s bananas. We are comparing bananas and also comparing people”.

In summary, Silva’s goals for and use of representation in the classroom are to prepare students for future grades. She felt they needed to understand what they were doing in order to be ready for the following year. Silva thought this could be best achieved by first modelling representations and later asking children to generate their own when possible. A second goal was to have them use multiple representations and because she felt that they would learn more if they use multiple representations and made connections rather than just using one. Silva seemed to refer most to more traditional tables, graphs and equations, and less to pattern blocks, algebra tiles, and other visuals.

6.3.2.3 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.3.2.3.1 *Content*

Silva stated that she had shifted from more traditional teaching to reform-based teaching. She mentioned that when she first started teaching mathematics, she regretted that she "taught the formulas and everything; I taught very old school", but now she said, "I am more comfortable with using different representations". Although she also said that she nonetheless struggled making the shift. She felt that she would need more time learning how to use some representations before she would confidently be able to teach with them. This was evident in Silva's survey response in which she agreed with the statement that suggested that teachers would need to learn more about teaching-related math before using representations in their teaching. This appears to provide further evidence to support Silva's comment that she had a hard time with visual representations. According to Silva, "even now, I do different representations in different strands, I still sometimes go back and practice it myself so, I am confident when I teach my students about what those representations mean. Because if I make an error, I might just confuse them".

Silva feels that specialized knowledge is important for teachers to teach effectively with representations. This was evident in her survey response in which she disagreed with the statement that suggested that teachers do not necessarily need a specialized understanding of elementary math in order to use representations effectively. She also spoke with concern about her own content knowledge of mathematics—particularly the correct representation. Silva further explained, "I have a hard time with the visual too. So, I need more practice on that one". Silva believed that the teacher needs to also have knowledge of their student's background before introducing a new topic.

6.3.2.3.2 ***Students and Pedagogy***

Silva had specific beliefs about how one should most effectively teach algebra. When asked whether Silva found it more helpful to generate representations for her students, she responded that “At the beginning of the year, I do a lot of the generating of representations, but as they get more comfortable with realizing there is more than one way to solve it, they get more confident in showing them”. For example, she agreed with the statement on the survey that suggested that teachers should assist students in choosing appropriate representations. Silva believed that the teacher should use representations initially when new content is introduced, followed by students generating their own representations because the representations help students “to be able to communicate” their thinking. This suggests that Silva used representations in her classroom as a way to show her solutions, rather than as tools for students to think with. Silva believed that students needed to see her representations and that it was important to go through different ways to represent mathematical ideas or to present what makes sense to the student.

Silva reported that she felt with her students that the activities that occurred in their previous years had been superficial. Silva, for instance, explained that “in the previous year, they only look at the relationship but were not being pushed to really look at the input/output relationship. “They have a hard time with the multiplication model like what the relationships between input and output numbers are”. For Silva, good instruction meant her students would be better prepared whether coming into her class or going out of it.

Silva indicated that this also meant meeting students at their level. While responding further to the scenario questions, Silva predicted she would use different tools based on the level of her learners and the knowledge she has had of these learners. Silva agreed with the survey statement that suggested that representations help in moving students from using concrete models to abstract representations.

Silva believed that using different types of representations to teach the abstract concept of different patterns would make patterning and algebra more comprehensible. She explained that "it [the representation] does not necessarily have to be using a graph, it could mean using manipulatives or drawing it out. It could be doing an algebraic formula to show your understanding about what the problem is asking". However, she tended to provide more algebraic forms than visual representations in much of her discussions. Silva disagreed with the statement in the survey that suggested that representations are usually not physically visible. Hence, her comments about using physical pictures, and words to communicate one's understanding.

When asked about the challenges Silva usually encountered when teaching patterning and algebra, she mentioned the students' language issue and her struggle as a teacher. Silva reported that "I have a lot of students on IEPs, and when using different types of representations, they can get a little bit confused". It became very clear why Silva agreed with the statement in the survey that suggested that one specific representation of a pattern may not be enough in a patterning and algebra lesson. Silva commented that her students are not too bad with the patterns "until the patterns get a little more complex, they kind of struggle with the algebraic equations". Silva agreed with the statement in the survey that suggested that teachers should flexibly shift among different representations.

Silva reported that patterning and algebra could seem "extremely complicated". Not surprisingly, Silva agreed on the survey that patterning and algebra is one of the more difficult strands to teach. She mentioned that sometimes she asked students to solve a task for which they have no idea how to start. She said that "after they look at it, they will tell you, I have no idea what that means". She reported, however, that unless students have the relevant representational skills, the class would quickly become lost not knowing what to do.

She discussed her instructional methods of supporting her students' understanding of the material. For example, when Silva was asked to give an example of how she connects mathematics content to the real-world contexts, Silva explained that "We will start a word problem and do a lot of modelling. I will give them an equation, and they would be asked to come up with a word problem and give it a context". Silva appeared to connect real-world examples with algebra in order to bring her learners to understand content and context. She explained, "With algebra, we started looking at real-life situations like cell phone plans, a diabetes problem and how many needles a person needs, and how much they make in certain summer jobs. I try to make it more relevant rather than say here is an algebraic equation solve it, I tried to give them context for everything". Silva reported that she taught algebra using pattern blocks, graphs, toothpicks, algebra models and graphing.

Silva mentioned that the use of manipulatives played an important role in mathematical meaning-making and communicating concepts in general. Not surprisingly, she disagreed with the statement in the survey that suggested that the use of manipulatives is only good for teaching patterning but not for algebra.For Silva, ensuring that her students gained conceptual understanding when learning algebra was important in order for them to make a meaningful connection between the content and context. This was apparent in her survey response in which she agreed with the statement that suggested that representations could help students draw on their conceptual understandings, particularly with new and unfamiliar problems. Silva commented that if learners benefitted in her classroom, it was directly linked to what she has learned attending various math professional development sessions.

Silva reported that her knowledge of the students' ability would determine the extent of the content and the approach to start off a lesson. Silva, for instance, said that "I try to start with basics like a pictorial model". Students' knowledge of mathematical vocabulary was important to Silva, and that was why she had to start her lessons with the basics. Silva expressed how the use of representations was a source of focus and positive motivation for students in her class who were either hands-on or visual learners. Silva reported that she noticed meaningful learning with some of her lessons, while in other lessons there were confusions in which a change in tactic was needed. She said that "if they are looking at me like I am speaking gibberish, I will use something to help them see a connection or have them tell me what I should do".

Speaking with Silva, I noticed that, while responding to the first scenario question, she has a good knowledge of her students. Silva, for instance, predicted that half of her students would be confused because of the letters involved while the remaining half would attempt to solve the task. She felt that the misconceptions could occur because some of the students might process the letters alphabetically rather than seen as a quantity. Silva said that "it took a while for them to get an understanding about what the equal sign means this year. Some will probably ask, what do you mean x is equal to p?". When Silva was asked the types of representations, she would be expecting her students to build, Silva said that "some of them would probably draw out the bananas... a stick person with a random number of bananas and equal sign beside the stick person with a number of bananas that are the same as Sam's". She predicted that some of her students would use pictures, while others would use random numbers. For Silva to help the students in their efforts to use representations as tools, she answered "we spent the first little bit of time explaining what the equal sign means because every year they come in thinking they have to solve something if there is that equal sign. They seem to think like 5 minus 2 is equal to 3, an answer. So, we spent time talking about balancing and what the equal sign means, same as… like 5 - 2 is the same as 3 not solving". Silva was unsure about how students in her class would approach the second scenario problem. "I have never given out a question like this. I think some of them would jump into asking how many text messages is Olamide sending? Their biggest concern would be whether or not there is data included and how many gigabytes". As Silva continued to think about the approaches to solve this task, she mentioned table and graph but felt that some of her students might just give an explanation because "plan B has unlimited text messages". In her effort to support them, Silva predicted that "my first way to support them would be to give them a different amount of text messages so they could work out the deal. I always start with the conversation". **Summary**

In summary, Silva strongly believed that teaching algebra with multiple representation would better meet the needs of her students. She felt that teachers should begin with student's knowledge and build from there. In the beginning, this would mean the teacher modelling different representations and later asking students to generate their own. She also believed that students would learn by being challenged. Silva felt that teachers would need to know the mathematical content of what they were teaching in order to do it well. She had changed her instructional practice fairly dramatically from traditional direct instruction of procedures to reform-oriented instruction. She was still concerned about her own understanding of the concepts and appropriate representation at times, especially the more visual representations, such as manipulatives and models. Hence, supporting her comments about having clear concerns about her content knowledge. Silva reported that she taught algebra using pattern blocks, graphs, toothpicks, algebraic models, and graphing. She firmly believed that specialized content and pedagogical knowledge was necessary for teachers to teach with multiple representations effectively. Silva felt that this would mean students would be better prepared for future grades.

6.3.3 Susan

6.3.3.1 Background

Susan, a female teacher in her early 50s, was very enthusiastic about participating in the study. Susan has more than 15 years of mathematics teaching experience. Susan holds a degree in science and also additional qualifications in special education, mathematics education, and her principal's qualifications. Susan currently has been in a non-teaching role after many decades of teaching. She goes into the classroom to support ESL students. Susan taught mathematics to Grades 7 and 8 students previously.

6.3.3.2 Teachers' goals for and use of representations in the classroom

Susan stated that her goals for, and use of, representations was to make her teaching of algebra more comprehensible for her students. She felt that representations are "extremely helpful for kids" as they work through a classroom task or "something hard in an assignment". According to Susan, representations involved "giving students the opportunity to use certain models other than just numbers or digits". She believed that students should be allowed to use graphs, manipulatives, diagrams, and different types of representations that could help them communicate their understanding of the abstract concept.

Susan stated that her other goal of providing representations for her students was to show relationships between concepts. Susan felt that, if the students had not yet understood the relationships between different algebraic concepts, it would be difficult for them to follow up with the procedures as the teacher expects from them. During the interview, she said, "they see the math and the procedures very clearly, but they do not really understand the relationships to the algebra". She believed that it is necessary that teachers work on the expectations that the students should be able to unpack the math by showing the relationships in different patterning concepts.

In summary, Susan's goals for and use of representations in the classroom are to show relationships between concepts. She thought this would mean first understanding the relationships between the concepts and later starting to use the procedures. A second goal was to have her students explore multiple representations in order to make algebra understandable for them to learn. Susan felt they needed to use multiple representations in order to understand abstract ideas.

6.3.3.3 Beliefs and knowledge: Mathematics, Students and Pedagogy

6.3.3.3.1 ***Content***

Susan believed that, in order to teach curricular content with clarity, teachers need to use different representations in their lessons. Susan stated that she considered the curriculum expectations and resources that are available for access and the student's need before she decided on the type of representations to use in her classroom. She was especially worried about the various topics in algebra in order to be able to meet the curriculum expectations. Susan believed that the teacher ought to reflect on how different representations could be used to explore and investigate the math ideas as contained in the curriculum. This was needed to be done because the teacher would have to understand the curriculum expectations first in order to use the appropriate representations for the level of the students she is teaching. Not surprisingly, Susan agreed with the statement that suggested that the use of multiple representations is not clearly explained in the curriculum.

Susan stated that content knowledge helps teachers to make appropriate instructional decisions and enables them to choose the right representations. This was evident in her survey response as Susan disagreed with the statement that suggested that a teacher doesn't necessarily need a specialized understanding of elementary math in order to use representations. She spoke with concern about her own content knowledge of mathematics. Although Susan expressed confidence in her own mathematics knowledge, she acknowledged that there was more to learn about the use of representations, particularly when the teacher lacked deep content knowledge. She stated that whenever she lacked the content knowledge “I really have a hard time feeling comfortable letting the kids explore representations, and I am feeling a bit nervous even to just sort of using representations myself through my own guided instructions”. Susan expressed, however, that “If we (teachers) do not understand the math ourselves then, it will be difficult to go deep”.

6.3.3.3.2 ***Students and Pedagogy***

Susan expressed some specific beliefs that indicated that the use of representations was part of her classroom teaching tools and how she allowed her students to explore different representations. Susan believed that representations should be part of the classroom, and students should be engaged with “graphs, diagrams, using some sorts of concrete tools to help them”, particularly in patterning. She also believed that students should be encouraged to use different representations to unpack or figure out their understanding of different patterns. Susan believed that students are able to access different representations when they can restate a given task in their own words. As such, they are able to learn abstract ideas that are rooted in meaningful, concrete models. Susan noted that, through the appropriate use of symbolic notation, students could build algebra reasoning. For Susan, avoiding inappropriate use of representations that might disrupt students’ learning opportunities meant she needed to plan her lessons ahead of time.

Susan stated that “planning ahead for instruction goes hand in hand with the use of representations”. This was apparent in her survey response in which she agreed with the statement that suggested that effective use of representations requires a lot of planning. Susan stated that whenever she had the opportunity to be involved at the planning stage of her lesson with other teachers, she put together resources that were not procedurally based. She believed that learning from other teachers might be helpful “because sometimes knowing it is very different from teaching it”. Susan also suggested that as part of the planning, the teacher should engage in pre-teaching around representations before encouraging students to generate their own representations.

Susan indicated that planning meant that teachers are focusing on making sure that the students understand the material, meeting their needs and supporting them. While responding to the scenario questions, Susan demonstrated her knowledge of her students by predicting that some of the students would solve the first scenario question using counters (red and yellow counters), draw on scrap paper and numbers. She further predicted that there would be some conversations as well as back and forth arguments. When Susan was asked the type of representations she would expect her students to build, she said, “the first I can think of is a concrete sort of thing. A bit of table for this type of question. Yeah, I am not really sure, I cannot really think of any other representations”. She stated that in order to support them, she would want to see the type of representation they had started using. “I guess if they are using something concrete, we kind of transition them into numbers. I am not sure. Again, as I mentioned before, I might just put in a table to show the equality”. Despite the fact that Susan was not too sure of the type of representations that students would build, she strongly agreed with the statement that suggested that teachers should flexibly shift among different representations as they are generated by the students.

She noted that one representation was not enough in an algebra lesson. Susan, for instance, discussed solving an equation with the students and using different representations such as manipulatives to represent different variables in an equation, numbers, and graphs. She, however, disagreed with the statement in the survey that suggested that the use of manipulatives is only good for teaching patterning but not for algebra. Based on the interview and the second scenario interview question, Susan predicted that she might ask the students to give her the best guess, looking at cost and each of the plans. She predicted that some of the students would use money, “they will try to use actual cents as manipulatives, and hopefully moving to the math representation”.

Susan believed that, when different representations are used in the classroom, it would help students explain difficult concepts, solve math tasks that appeared difficult for them and clarify misunderstandings. Not surprisingly, Susan agreed with the statements in the survey that suggested that providing representations would support an explanation of difficult concepts to students and, appropriate representations should be used to clarify misunderstandings. She talked about a particular scenario with her students where an ESL (English as a Second Language) student misunderstood a physical table for the term “table” used in mathematics. Susan identified some of the challenges that might create misconceptions for students. She also felt that, when the teacher encouraged students to generate their own representations and “represent their thinking, it can be very frustrating”.

Perhaps this was why Susan agreed with the statement in the survey that suggested that teachers should assist in choosing appropriate representations for students. During the scenario interview, Susan expressed her willingness to help her students uncover the misconception in the first question by encouraging them to prove their thinking. “Tell me how you know that? More so, I might suggest that Colin come and represent it with something. I think you need to explain to Codi what your thinking is”. This somewhat contrasted with her comments about students’ frustration when asked to generate their own representations and represent their thinking. However, Susan felt that teachers are expected to ask students to explain their responses to a mathematical problem or concept.

She indicated that her approaches to the use of representations to solve a mathematical problem depends on the student learning style. She shared her experiences about how she taught algebra in the last few years. However, she expressed concern about how teachers taught algebra. She talked about how teachers need to recognize that students all have different strengths. Susan expressed concerns over the challenges for students in math and reported that “we first need to know what their strengths are and what accommodation is needed”. Susan reported that “I think of different research studies that suggested that students should come up with their own representations, but I think in my experience, we need to guide a lot of the students”. This appears to provide further evidence to support Susan’s claim that students get frustrated when asked to represent their thinking. She explained that “I think it is sort of, listening to what they say and anticipating where they are going and coming from, then doing a bit of guided instruction”. She noted that some students could do well under guided instruction while some other students are more able to explore and discover things on their own. Susan believed that whether a teacher provides representations, or the students are left to provide their representations, depends on the student’s experience and his/her confidence with patterns. It became obvious why Susan agreed with the statement that suggested that knowing which representations to use is sometimes confusing to students.

Susan stated that knowledge of teaching requires teachers to have a repertoire of teaching techniques to help students with their needs. She stated that representations could either be beneficial or frustrating to students. During the interview, Susan mentioned the significant contributions of using representations in algebra lessons and she expressed a positive attitude towards helping students understand how to represent algebraic concepts in multiple ways. Susan felt that “providing representations helped us to determine which kid is really following and understanding the concept as supposed to just make them do the math”. However, it also appeared that Susan understood how the majority of her students struggled with the use of representations. For instance, Susan stated that understanding the math language is important because "we really recognize that language is a big barrier for them in math". She further stated that "I am not sure that they always understood why they were doing what they were doing". Susan however, was undecided with the statement in the survey that suggested that teachers would need to learn more about teaching-related math before using representations. Susan concluded that "I think using representations come back in a way of knowing your students very well". For Susan, good knowledge of students meant meeting them at their level.

Speaking with Susan, I noticed she was not sure how she would solve the task in the second scenario question and expressed that she was not comfortable with the question. She herself did not seem to be aware of some of the misunderstandings in the students' responses. Susan said that "I'm going to be honest with you, I haven't taught any of that in the last little while. And that's something I'm not too comfortable with right now". Susan believed the question was an EQAO kind of question, possibly it required interpretation. Further, she stated that "I haven't worked with students in developing this type of equation very much at all in the last years probably more than five years". This somewhat contrasted with her survey response in which she disagreed with the statement that suggested that patterning and algebra is one of the more difficult strands to teach.

6.3.3.4 Summary

Susan believed that representation in math is about using visuals such as math diagrams, pictures charts tables etc or concrete materials to explain or communicate an understanding of different mathematical concepts and how they relate to each other. Susan reported that she considered the curriculum expectations and resources that are available and the student's need before making decisions on the type of representations to use. She believed that students would learn more by being engaged with multiple representations. Susan felt that teachers should begin with the student's representations and build from there. She strongly believed that adequate planning with multiple representations would better meet the needs of her students. Susan also felt that representations would help students clarify misunderstandings, and because she felt that they would be discouraged if they were asked to generate their own representations, it was necessary for the teacher to assist them. She believed that students should be encouraged to use multiple representations to unpack mathematical ideas, but only after she has guided them. She spoke knowledgeably about her students varied learning styles and how she would begin teaching with representations in order to meet individual needs.

Susan felt that teachers would need to know the mathematical content of what they were teaching in order to increase student learning of algebra. She was still concerned about her own understanding of the concepts and expressed discomfort in allowing her students to use representations due to her superficial knowledge. Susan felt that teachers should use representations appropriately and because she felt that students would be able to understand why it was being used, it was necessary for the teacher to have good knowledge of the student. She spoke with limited knowledge about transition among different representations. Hence, she lacked the approach needed to support the students with the transition.

6.3.3.5 Sonia

6.3.3.6 Background

Sonia, in her 30s, was very passionate about teaching mathematics. She teaches mathematics at the elementary level in northern Ontario. Sonia has more than 7 years of mathematics teaching experience, but less than that in Grade 8. Sonia holds a degree in Arts and also additional qualifications in mathematics education, English and history.

6.3.3.7 Teachers' Goals for and Use of Representations in the Classroom

One of Sonia's goals for and use of representations in her classroom was to help students prepare for future grades. She stated that a teacher needs to "know where the students are mathematically and developmentally to get them ready for Grade 9". She believed that representations would aid the students' understanding of different concepts and how they approach mathematics problems. For instance, Sonia reported that "in Grade 6, they have been introduced to a variable and then, the conversations about the 'x', because, from Grade K or Grades 2 to 5, the 'x' represents multiplication and then, all of a sudden that 'x' could mean something else". Sonia noted that some of her students would understand that they "do not need a multiplication sign to represent multiplication" they understood how to "reidentify that as a variable". She felt that "some other students couldn't," and she would need to "represent the concepts concretely first before they are able to represent them as a variable".

Another goal for using representations in Sonia's classroom was to ensure that students transition from concrete representations to abstract. She reported that she wanted to make sure that "students are able to represent math ideas abstractly with mathematical equations". Although she felt that representations, particularly concrete and visual representations, were important, she stated that elementary school students need to move from using the concrete model to an abstract model for any strands in mathematics. Sonia believed that this approach also applied to patterning and algebra, including concrete models or contexts to show connections with abstract representations in number. She stated that the teacher should

> "have the student to choose which representation they need or where they need to start, with the idea or the knowledge that the end result or the end in mind is that they should be able to represent it abstractly with mathematical equations".

A third goal of using representations in Sonia's classroom was to clarify misunderstandings. For instance, based on her response to the first scenario question, Sonia noted that there was evidence that the students hold the misconception or misunderstanding that letters represented specific known values. According to the survey, Sonia agreed with the statement that suggested that appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings. Sonia believed that the teacher should be able to explain the misconceptions by coming up with another form of representation.

In summary, Sonia's goals for and use of representations in the classroom are to prepare students for future grades. She felt that they needed to understand certain concepts in order to be ready for the following year. Sonia thought this could be best achieved by first establishing the understanding of certain representations and later asking them to have a discussion when possible. A second goal was to have them transition from concrete representations to abstract because she felt that they would learn to show the connections with abstract ideas rather than just using concrete models. A third goal was for her to make students to use multiple representations to clarify misunderstandings. She felt that teachers needed to use another representation to further explain the ideas being presented.

6.3.3.8 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.3.3.8.1 ***Content***

Sonia expressed specific beliefs about teachers' pedagogical knowledge with respect to her use of representations in her classroom. She stated that a teacher has to "be an expert in understanding how to make the connection between the concrete models and the maths". Sonia explained that representations could take the form of concrete models, abstract representations, written words, or numbers and could be used to support patterning and algebra learning as well as make connections to mathematics concepts. She noted that the pillars of mathematics in the curriculum document are cross-curricular, and it was necessary for teachers to be knowledgeable about and understand the contents in order to make the connections to different mathematics strands. Sonia reported that despite having the curriculum expectations as guides to the teachers' teaching, it was not enough. Sonia reiterated that, as a teacher, you must "know what you are doing" because it takes time to build the understanding and "being able to really understand as a professional first before you take it to the classroom". She believed that teacher with a good understanding of the curriculum should be able "to really look at each strand of math and be able to represent the concepts that are in the curriculum expectations". She felt that "there is a lot more thinking in conceptual understanding that needs to happen than is stated in the curriculum document". Perhaps this was why Sonia was undecided about the survey statement that suggested that the use of multiple representations is not clearly explained in the curriculum.

Sonia believed that specialized content knowledge is "absolutely" necessary to teach mathematics and use representations. She stated that teachers require profound knowledge in order to develop or show patterns in a way that makes the mathematics ideas meaningful to the students. This was evident in her survey response in which she strongly disagreed with the statement that suggested that teachers do not necessarily require a specialized understanding of elementary mathematics in order to use representations effectively. Sonia reported that "when you look at the strand of maths—patterning and algebra pose a big challenge for the Grade 6 teachers. She explained, "I think that there needs to be specialized training and specialized resources". Sonia further explained that "as a math person, the way I taught maths, I'm realizing that I do need specialized knowledge and I do need to do a lot of learning behind the scene in order for me to be more effective in my teaching". Sonia commented that professional development training would help teachers in shaping their specialized knowledge.

Sonia stated that a teacher should acquire the knowledge of mathematics content first before he/she can effectively apply appropriate representations. She noted that "I do not think that teachers start with the representations and build on it with the content. I think it's the content, then representations". Sonia expressed that teachers with a good specialized knowledge would be able to show how algebraic equations can be derived from patterns. She mentioned different types of patterns she works with her students to include multiple patterns, such as growing patterns and shrinking patterns. However, while responding to the first scenario question, Sonia was unsure of the type of model (representation) she might use to help the students. Sonia stated that “I am really struggling with the mathematical model for the first scenario question”.

6.3.3.8.2 ***Students and Pedagogy***

Sonia expressed some beliefs that influence her decision making in the course of her instructional practice. During the interview, Sonia reported that the Ontario Ministry of Education’s emphasis on backward planning and the use of big ideas in order to foster conceptual understanding were the major ideas that informed her use of representations. She believed that some of the challenges confronting the teaching of mathematics, particularly in North-western Ontario, was “having time to be able to plan and become really good at teaching it”.

Sonia had a specific approach on how one should effectively plan towards using representations in a lesson rather than just relying on the traditional approach. She did seem confident with how she planned her lessons and gave a detailed explanation. Sonia reported that she set up her lessons to start with her unit plan and diagnostic assessment to determine the level of her student. Not surprisingly, Sonia strongly agreed with the statement that suggested that the effective use of representations requires a lot of planning. She stated that “before we start any unit of study, whether it is math or otherwise, the students write a diagnostic assessment specifically on the big ideas that I am looking at teaching them from my backward plan”. For Sonia, if she were working with her Grade 8 students, she would rather give them diagnostic test that focuses on algebra than on patterning. She stated that she would need to see where they are, based on the outcome of the diagnostic assessment. She believed she needed to know her students' level of understanding in order to plan for instruction. Sonia expressed that the backward plan and the diagnostic assessment "tell me where the students are,…using both of those combined allows me to develop or choose or select the models that I feel are appropriate and effective for students in my instruction." This perhaps explained why Sonia disagreed with the statement that suggested that patterning and algebra is one of the more difficult strands to teach. For Sonia, excellent planning meant she would be better prepared using relevant math resources.

Sonia stated that "you draw on tons of math resources such as The PRIME Kits". "You have got Marian Small, John Van de Walle; all these people have these Big Ideas books that show a variety of examples of representations". Sonia believed that, before you could use these resources, you need to be an expert because "whether it is patterning or algebra, you have to be an expert in what you are teaching". Although Sonia expressed that she had access to math resources, she stated that some of the resources were not applicable. Sonia reported that she had "a lot of resources and the ones that focus on the theoretical or the pedagogical understanding of patterning and algebra are helpful, but they are not applicable".

Speaking with Sonia, I noticed that while responding to the second scenario question, she has a good knowledge of the question. She felt that the question was similar to what was done with her students recently. She noted that a similar question could be found in the Math Makes Sense textbook. This appears to provide further evidence to support Sonia's claim that she has access to relevant math resources. Sonia explained that, she would ask her students to prove to her how they got the algebraic expressions. She believed that for her students to have generated the algebraic equations "probably, they do not need a table, they are very comfortable using variables and also the use of graph". She further explained that "I will probably have them prove it to me by plugging in numbers, especially the first equation".

Sonia felt that these students already grasped the understanding of variables. She explained, "these students are telling me they don't need anything but the numbers and that's where we are expecting them to be. Essentially, I think I need to prove them wrong or prove them right. Students learn a lot from their mistakes". While Sonia was explaining her approach to the second scenario question, she shared her concern that sometimes the approach she has put in place might fail. She, however, reported that "this is when you talk to the student individually or talk to colleagues that this isn't working".

What came out strongly in the survey and interview was that Sonia described knowledgeably how teachers had shifted from more traditional teaching to reform-based teaching. Sonia stated that, there had been a prime shift from how patterning and algebra have been taught in the past to new materials that are applicable. She reported that she taught patterning and algebra with different forms of representations such as simple blocks (unit squares), algebra tiles, input-output machine, and equations. This somewhat supported her survey response in which Sonia disagreed with the statement that suggested that graphical representations are the most important kind to illustrate algebra concepts. Furthermore, she stated that it was not sufficient to just teach a lesson, a teacher needs to be "responsive to the discussions, what the conversations are, what misunderstanding, and the gaps in misunderstanding". For instance, Sonia believed that, for a teacher to be able to teach mathematics effectively at Grade 6, he/she should look at both Grades 5 and 7 curricular expectations in order to understand the areas of emphasis. She believed that "teachers need something that they can take and implement in the classroom, flip and tweak, and figure out as they go through it". Narrating her experience about how she learned representations, Sonia noted that “I learned about representations just by trial and error” with the available resources. Perhaps this was why Sonia disagreed with the statement in the survey that suggested that representations are hard to use in teaching.

The discussion on Sonia’s instructional practice was further expanded as she shared how she supports her students’ understanding of the math content. Sonia believed that it is necessary for teachers to know how to sequence learning because it helps in building on different math concepts. She felt that as a teacher, “if you are not able to understand how that sequencing happens, then you’re going to be confusing the kids”. Sonia explained, “I will start with a simple representation and build into the second one and then, dig into the conceptual understanding that needs to happen”. It became clear talking with Sonia that students’ conceptual understanding was important to her in order to foster a student’s ability to handle unfamiliar situations. Not surprisingly, Sonia agreed with the statement on the survey that suggested that representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.

Sonia valued her experiences teaching with representations, ensuring that all her students were willingly working with different kinds of representations during mathematics classes. When Sonia was asked to describe how she uses representations in her lessons, she said, “basically, I model.”. This response was aligned with her survey response in which Sonia agreed with the statement that suggested that providing representations to support reasoning is something she would do. She further described how she used representations during her teaching. For instance, Sonia explained that she used algebra tiles to represent the algebraic equations before encouraging students to solve for x. She explained that she used simple blocks to build the patterns after which she describes the patterns in an equation form. However, Sonia expressed concern about how to explain the meaning of a variable to her students. She stated that “because variables are abstract ideas and I actually do not know how to teach the definition of a variable, what is the why behind the variable, I think it is a conversation”. Sonia believed that engaging the students in a discussion would be more helpful than using concrete models.

Sonia appeared to have a good knowledge of her students as she described two categories of students in her class to include those that “use concrete representations when they are struggling” and “then you also have students that do not do well with those concrete representations and just understand the math equations themselves”. Sonia believed that learning style was important because students learn in different ways. According to the survey, Sonia agreed with the statement that suggested that encouraging students to use representations could improve their problem-solving skills. She noted that if a student were “so into certain representations and wouldn’t want to move to another form of representation,” she would consider where she wanted the student to be before “pushing it”. Sonia discussed one particular lesson she did with her grade 8 students that one particular student could not model using the concrete algebra tiles. Instead, “he completely starts by drawing them himself”. She talked about giving her students more than one representation and encouraging them to choose which one they would need to start with when solving a mathematics problem. This was reaffirmed in her survey response in which Sonia agreed with the statement that suggested that knowing which representations to use is sometimes confusing for students.

The conversations with Sonia that included ways to support students’ understanding of both context and content was another area she felt was important. She believed that mathematics wasn’t a silo subject but was linked to real life. She explained that teachers should not just focus on the teaching of algebra concepts and variables, they should learn how to change the concept by putting topics in context for proper application in a different situation. According to the survey, Sonia agreed with the statement that suggested that selecting a worthwhile task determines what representations to use.

6.3.3.9 Summary

Sonia strongly believed that teachers need a comprehensive knowledge of both math content and representations. She felt that teachers should begin with studying the curriculum and an understanding of different math strands and how they are connected, drawing on relevant math resources. Sonia spoke confidently about her planning strategies and how these approaches would mean students' level of preparedness are better determined. Sonia had varied her instructional practice using sequencing in order to improve students' understandings. In the beginning, this would mean the teacher representing different simple representations and later transitioned to other representations. Sonia was also concerned that some of her strategies might fail.

Sonia felt that teachers would need specialized knowledge in order to use different representations effectively. She had shifted to using reformed-oriented instruction rather than using traditional direct instruction of procedures. Sonia was concerned about her own understanding of the appropriate representations at times. Hence, she tried to learn more about them. Sonia spoke knowledgeably about her students' learning styles, and how she would begin encouraging the use of other representations that makes sense to them. She also believed that students' understanding of multiple representations would mean they could solve any math task independently.

6.3.3.10 Sara

6.3.3.11 Background

Sara was one of the Ontario teachers who showed a definite interest in the current study, even before approval was received from the Lakehead University Research Ethics Board. She also indicated a willingness to serve as one of the five case studies. Sara, in her 50s, has more than 20 years of teaching experience, and more than 10 years teaching mathematics. She taught mathematics at the elementary grades in northern Ontario. Currently, Sara has a non-teaching role after many decades of teaching and only goes into the classroom to support math students with intellectual deficits or any aspects of learning deficits. She has a Bachelor of Arts degree and a Bachelor of Education degree. Her basic qualification was in visual arts (J/I division), and her additional qualifications were in mathematics education (P/J division and specialist).

6.3.3.12 Teachers' Goals for and Use of Representations in the Classroom

One of Sara's goals for using representations in her teaching was to support students to solve a math task in more than one way. She stated that she preferred expanding her students' knowledge and not just restricting them to solving a problem in one way. For instance, Sara noted that "even starting with something very simple leads to where they can see and say, Oh, there is more than one way…there's two more than N". Sara explained, "just because you understand one way, even if your one way is more sophisticated, I would hope you would go back in time, to see how they are connected". She felt the reason for that was to help students "to be able to show me you can solve this algebraic question". Sara believed that students' use of representations would improve their problem-solving performances, especially solving problems in multiple ways.

Another goal for using representations in Sara's class was that she used representations as a scaffold to help students clarify misunderstandings. Not surprisingly, Sara strongly agreed with the statement in the survey that suggested that appropriate representations should be used to clarify misunderstandings. Based on the interview and scenario interview questions, Sara expressed her enthusiasm to help students' reasoning and clarify their misunderstandings by asking more questions to prompt their approach to solving the questions. She explained, "I would ask more questions from the students. I might be inclined to give a slightly different scenario. S and T are not equal because S and T are different letters". Sara showed sufficient knowledge of the content and source of students' mistakes as well as effective teaching strategies to help students with their misconceptions. She understood that if they are both x, they have to be the same and if they are x + p they could be the same.

Sara believed that using different variables might be helpful to clarify students' misconception. As such, she felt that it is important to have students in a group and allow them to have a discussion. For Sara, she would ask students to "group bananas like each had 10 and 10". She further explained that "if we suddenly had 21 bananas would X and P still be the same? Would you want to say X and P are equal to each other in every case?". While responding to the second scenario question, Sara reported that she would ask more questions regarding this task. She explained that "for student one, I would say, could you tell me which one is plan A? Which one is plan B? And which one is plan C?". She felt that students should be able to communicate their thinking, especially "how the words connect to the expressions as well as the equations". Sara explained that her next question would be "how many text messages would you be able to send before you know you get up to the $90.20". She believed that since students often send messages from their phone, it would be easier to push them to explain their thinking and come up with the plans that work best.

In summary, Sara's goals for and use of representations in the classroom was to help students understand multiple ways to think about and solve math problems. She felt that they would learn to make connections to the math concepts rather than just using one approach. Another goal for the use of representations in her classroom was to help students clarify misunderstandings. She thought this could be best achieved by first asking more questions and later asking them to discuss their possible solutions.

6.3.3.13 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.3.3.13.1 ***Content***

Sara's beliefs about what constitutes effective use of representations differed considerably from the other teachers. For instance, Sara believed that representations are something so important that they should be treated seriously and used frequently in the classroom. She said, "it sounds facetious to say representations mean representations. I cannot think of another word other than represent". Sara believed that representations are different ways to show mathematical ideas. She listed different types of representations, such as graphic, pictorial, objects, symbols, numbers, and letters. Sara explained, "to me when I think of representations, it means all the different ways you can show an idea and show how it must exist, and how it is related, connected to the other parts of it". The way Sara described her view of representation was different from every other participant in the study. I believed that she was more experienced and familiar with the use of representation in terms of types used in teaching patterning and algebra. Sara was confident about her knowledge of algebra and representations. She expressed delight and satisfaction with her teaching and indicated that she used different representations to explain mathematical ideas.

Sara stated that how to use multiple representations efficiently and effectively comes with good knowledge of it and experience. She disagreed with the statement in the survey that suggested that teachers don't necessarily need a specialized understanding of elementary math in order to use representations effectively. She explained that teachers are often reliant on what they have used in the past. Sara appeared to be in support of teachers' use of representations related to their knowledge. She said that "I think it's really related to their knowledge and maybe their level of comfort". She commented that "once teachers find what works, what makes sense, I mean they won't drop them. Right?".

6.3.3.13.2 ***Students and Pedagogy***

Sara's beliefs further suggested that the use of representations would promote constructive learning among students and impact her instructional decisions. At the heart of her instructional decisions was the way she planned her lessons. Sara reported that, as she planned her lessons, she found she was "trying to think about what I'm really going to let kids learn". Not surprisingly, Sara agreed with the statement in the survey that suggested that the effective use of representations requires a lot of planning. She stated that it is important to introduce a lesson with a simple explanation of mathematical language and the use of familiar words in questions. Sara said, "sometimes we will start with the simpler problem. Right?". She believed that starting a lesson with a simple problem was a great way to help the student understand new topics. Sara explained, "I want to be able to work with students so that if they get part of the relationship, you can use inverse operations to get to where you want to go.., other parts like generalization". She further explained, "the second part has been patterning, being able to take a pattern, like linear patterning and be able to think about it algebraically". Sara strongly agreed with the survey statement that suggested that teachers should flexibly shift among different representations as they are generated by students.

Sara stated that teachers need to ensure that students are flexible in their thinking as they approach any given task. She believed that they needed to be flexible in their reasoning in order to improve their problem-solving skills. Her sentiments are reflected in her explanation during the interview. Sara said that "in order to make mathematics students flexible in their thinking, if I only show one representation, how am I, in any way, helping them to be flexible in their thinking?".

Sara's notion of instructional strategies also rests on the beliefs that teachers may need to first generate representations for students because she sees this strategy as what each student builds on as they start to generate their representations. Sara stated that it is better for a teacher to use representations initially when new content is introduced, followed by students generating their own representations, as they become comfortable "to be able to communicate what your thinking is". This response was aligned with her survey response in which Sara agreed with the statement that suggested that it is necessary that teachers assist in choosing appropriate representations for students. Sara however, suggested that students should be encouraged and supported to generate their own representations. She explained, "sometimes when we are initially working together I will, sometimes, simply by the question you ask, the pattern you describe, you ask them to spend time working on the T-chart". Although she noted that she always pushed them to come up with the chart, the format of the questions would help them to work on how to generalize. She also noted that "it is about what makes sense" when it comes to generating representations for the children or they (students) generating it themselves.

Sara's teaching methods and beliefs reflected on the way she responded to the scenario questions. For example, when Sara was asked how she would support the students in their efforts to use representations as tools to solve the task on the first scenario question she said, "I would have to have some manipulatives handy. They would be out. They would not be hidden". Sara expressed that if she already knew where she was going with the task, she said that she would "even honestly have bananas ready close at hand".

She also expected the students to come up and show their thinking. This appears to provide further evidence to support Sara's claim that she used representations to ensure that students are flexible in their thinking. She explained, "I want them to come and show me what so and so means. I would have that as part of the discussion, honestly, graphics and bananas". Sara predicted that students in her class would approach the task just the same way it was in the question. She expressed that "I would teach that X and P are exactly the same, call them both X and some kids would say No! Different letters; they cannot be the same". As mentioned previously, she did understand that if they are both x, they have to be the same and if they are x + p they could be the same. Sara indicated that she knows not only what students could do but also what the students are thinking while they are producing the answers. Sara further predicted that the type of representations her students would be using to approach this task would include numerical and centicubes.

While responding further to the second scenario question, Sara explained that she would support the students using an excel program to graph the solution. She further explained, "because you have three different graphs, we could figure out where they meet and where they intersected. It is about choosing the representation that matches where the students' thinking is". For Sara, although being able to help them solve the task was important, she would not impose any kind representations on them. She felt it was necessary to encourage them to have a conversation about the task. Sara reported that "ideally, being able to lead the conversation to see where the students see the connection" would be her first approach.

Sara stated that the relationship between her teaching knowledge and the knowledge of the students are very connected. When she was asked to explain her experience with the use of representation during her teaching, she said, "well, I think, one of my favourite areas to teach is algebra because I really love how you can go from the use of objects right into the algebraic expressions. Students can feel their fingers and manipulate objects, and then you can connect it so beautifully to the abstract 3n + 2 as a way of representing it". She further elaborated on the range of her students' answers for the algebraic expressions. Sara mentioned that students would often prefer to use a graph to show how they are related rather than just drawing out the solution in order to express what the multiplier and the constant are. She reported that "students can find it their own way and then, another student gives another way, you can then bring it together". To her, it was “beautiful and super exciting”. Sara based her decision to use representations in her teaching on “where students are”. She believed in “finding things that I believed have worked to help students move forward with their thinking so that they can go from the object to the N chart table of values”.Speaking with Sara, I noticed that she had a strong knowledge of her students. She stated that she was able to understand students’ misconceptions as she worked together with them. Sara mentioned that she used more than one representation to handle situations of students’ misconceptions. Talking about using more than one representation in an algebraic lesson, She said, “it depends on where they (students) are”.

Sara predicted that some of her students might find drawing the graph for a linear equation painful, so they would prefer working with the table of values to figure out the multiplier and the constant. She explained, “some would say just give me the table of values. I don’t need to draw it. I can find it in the numbers”. Sara felt that “so, it really depends on where they are, and sometimes I’m almost going to say, the level of confidence and belief”. She reported that using this approach enables them to “internalize it” and make them use different representations. Sara explained that when learners are being made to show the connections and relationships, they pick up both implicit and explicit knowledge. She said that she allowed her students to use some blocks in order to establish the relationships between the mathematics concepts they are learning.

Sara identified some challenges she has had and how she helped students in order to overcome these challenges. She said that “my challenge is when I get to inverse operations, we usually get there. The challenges within that are collecting like terms when we have N on both sides of the equation. Subtracting N from both sides of the equation is tricky”. For Sara, she purposefully uses integers and fractions to push them to think about the inverse operation, “to me that’s the algebra”. She felt that she also needed to teach the patterning aspect, and throughout the interview, she referred to patterning as the "second part". According to Sara, "when we get to the second part describing patterning, I have students who perform below provincial average just because they failed to connect and represent the negative numbers". She further explained that even though "of course they can solve problems lots of other ways besides algebraically," the challenge for them was "different signs to be an x". Sara reported that students depend on how teachers represent negative numbers. She appeared to have devised a way to address this problem with her students, "zombie and zombie is a fun way I do it with my students". She stated that ensuring that the contents were taught in the context of how students would make sense of it was important. This appears to provide further evidence to support that Sara has a great deal of pedagogical knowledge.

When Sara was asked the impact of representations on the success of her lessons, she said, "to me, it is inconceivable". She explained, "to me, representations really show our thinking; they absolutely do". This further supports her claim of using representation to ensure that students are flexible in their thinking by solving the task in multiple ways. She expressed that it would increase confidence to help her students solve mathematical problems in multiple ways. Not surprisingly, Sara strongly agreed with the statement in the survey that suggested that encouraging students' use of representations could improve their problem-solving skills. Additionally, she said, "I really enjoy thinking about how to represent something that is abstract that makes it more tangible or looking at a tangible thing and then say how can I represent it in another way".

6.3.3.14 Summary

Sara reported that she taught patterning and algebra with pictures, objects, symbols, numbers, and letters. She reported that representation is something so important that it should be a mainstay in the classroom. Sara strongly believed that teaching algebra with multiple representations is most in the classroom as it would better meet the needs of her students. She felt that teachers should plan their lessons with simple math language, concrete situations and build from there. Sara thought this would mean the teacher is encouraging students to generate their own representations after she has provided her own. She also felt that teachers need to choose appropriate task in order to allow transition between different representations and because students would learn more if they transition between representations and understood the relationships rather than only using one. Sara thought teachers need to allow students to have a discussion about their approach to any given task.

Sara felt that teachers would need specialized content knowledge to use different representations effectively. She also felt very confident in her knowledge of representations and how to use them. Sara spoke knowledgeably about her students varied understanding of inverse operations and how she would begin teaching with multiple representations in order to meet individual needs. She had changed her instructional practice dramatically based on the level of her students, and what worked for each student. Sara felt that students' understanding of multiple representations would mean they have more confidence in solving math problems in multiple ways.

6.3.4 Summary of the Ontario Participants' Goals, Beliefs and Knowledge

Table 21 provides a snapshot of the five Ontario participants' goals, knowledge and beliefs regarding the use of representations in patterning and algebra.

Table 21:

Summary of Ontario teachers' perspectives and instructional practice.

Participants	Scott	Silva	Susan	Sonia	Sara
Qualifications and experience	HBORBEd in Mathematics Education (Primary/Junior) with six years' experience of teaching mathematics.	BABEd in Mathematics Education (Intermediate). She had five years' experience of teaching mathematics.	BEd in Mathematics Education (Primary/Junior) with more than 10 years' experience of teaching mathematics	BEd in Mathematics Education (Primary/Junior) with six years' experience of teaching mathematics.	BEd in Mathematics Education (Primary/Junior) with more than 10 years' experience of teaching mathematics
Modes of representation	Used representations such as manipulatives, pattern blocks, smart cubes and algebra tiles.	Used representations such as pattern blocks, graphs, toothpicks, algebraic models, real-world, manipulatives and equations	Used representations such as graphs, manipulatives, diagrams, numbers, models, and concrete tools	Used representations such as simple blocks, algebra tiles, input-output machine, equations, concrete models, abstract representations, written words, graphs, manipulatives, equations or numbers.	Used representations such as graphic, pictorial, objects, symbols, numbers and letters, pentacubes, manipulatives, pattern blocks, algebra tiles, chart, Excel software.
Goals	The various representations allowed him to highlight important mathematical ideas,	The various representations are used to prepare students for future grades.	The various representations allowed her to highlight important mathematical ideas,	The various representations allowed her to support students' reasoning and move from using	The various representations allowed her to push students to be

	clarify misunderstandings and build appropriate mathematical language. Scott's description of his approach was to allow students to clarify misconceptions.	She believed she initially had to generate representations before students are comfortable generating theirs. She believed students should be able to solve any given task in multiple ways.	clarify misunderstandings and show relationships.	concrete to abstract representations.	flexible in their thinking. She believed students should be able to solve any given task in multiple ways. She predicted correctly what students would and would not understand.
Beliefs and Knowledge: Content	Scott reported that a teacher needs in-depth content knowledge to use different representations. It seemed as if she had sufficient SCK regarding the specific content in the scenario questions.	She believed specialized knowledge is a prerequisite to use representations. It appeared as if her SCK is insufficient regarding the specific content in the scenario questions.	She believed specialized knowledge is a prerequisite to use representations. It appeared as if her SCK is insufficient regarding the specific content in the scenario questions.	She believed specialized knowledge is a prerequisite to use representations. It seemed as if she had sufficient SCK regarding the specific content in the scenario questions.	She believed specialized knowledge is a prerequisite to use representations. No misconceptions were expressed, and it seemed as if she had sufficient SCK regarding the specific content in

	Scott had knowledge of how the curriculum integrates the use of representations.	She had no knowledge of how the curriculum integrates the use of representations.	She had no knowledge of how the curriculum integrates the use of representations.	She had knowledge of how the curriculum integrates the use of representations.	the scenario questions. She had knowledge of how the curriculum integrates the use of representations.
Beliefs and Knowledge: Students and Pedagogy	Scott used his knowledge of representations and math content to explain his scaffolding approach for his students' learning. He recognized and clarified students' misconceptions.	She provided scaffolding to supports students' understanding. She was unaware of students' misconceptions and showed little understanding of how to address their misunderstanding.	She provided scaffolding to supports students' understanding. Susan was unaware of students' misconceptions and showed little understanding of how to address their misunderstanding.	She provided scaffolding to supports students' understanding. She was aware of students' misconceptions and how to address their misunderstanding.	She provided scaffolding to supports students' understanding. She was aware of students' misconceptions and how to address their misunderstanding.
	Scott believed in teachers building on students' existing knowledge.	She believed in connecting students' prior knowledge with new situations.	She believed in connecting students' prior knowledge with new situations.	She believed in connecting students' prior knowledge with new situations	She believed in connecting students' prior knowledge with new situations.
	He encouraged his students to generate their own representations after	She encouraged his students to generate their own representations after	She encouraged his students to generate their own representations after	She encouraged his students to generate their own representations after	She believed she initially had to generate representations before students are

	first providing for them.	first providing for them.	first providing for them	first providing for them.	comfortable generating theirs and, that students had to give explanations and justifications for their thinking.
	Scott believed in transitioning among different representations.	Not much evidence of transitioning among different representations.	Some evidence of transitioning among different representations.	Some evidence of transitioning among different representations.	Sara believed in transitioning among different representations.
	Not much evidence of adequate planning was expressed.	Some evidence of planning before using representation was mentioned.	Some evidence of planning before using representation was mentioned.	Strong evidence of planning before using representation was mentioned.	Strong evidence of planning before using representation was mentioned.
	His teaching style varied from traditional to reform-oriented in order to build his students' confidence.	Her teaching style varied from traditional to reform-oriented in order to support her students' understanding.	She believed in a reform-oriented teaching style but showed no evidence of her knowledge of it.	Her teaching style varied from traditional to reform-oriented in order to support her students' understanding.	She believed in a reform-oriented teaching style and showed her vast understanding of it.
	He believed content should be taught in context.	She believed tasks should be context related.	She believed tasks should be context related.	She believed tasks should be context related.	She believed tasks should be context related.
	Discussions and students working in groups were appropriate strategies.	Discussions and students working in groups were appropriate strategies.	Discussions and students working in groups were appropriate strategies.	Discussions and students working in groups were appropriate strategies.	Discussions and students working in groups were

			appropriate strategies.

6.4 Presenting Lagos Teachers

In the next section, biographical information regarding the five Lagos case study participants Bola, Beth, Ben, Baker and Bryce is provided as well as a brief description of the qualification and experience of each of the five teachers. I present and discuss the findings from the survey and interviews. Discussion on the sub-themes of goals, beliefs, and knowledge follows the specific order of the different survey dimension descriptors. A summary is provided at the end of this section in Table 22.

It should be noted that the topic of factorisation, when the coefficient of x is greater than one, and simultaneous equations, are noteworthy topics that were mentioned in the Lagos Grade 8 curriculum but were not in the Ontario Grade 8 curriculum expectations. The difference in curriculum distinguishes the two locations.

6.4.1.1 Bola

6.4.1.2 Background

Bola, in his late 30s, holds a bachelor's degree in physics and electronics, and a postgraduate diploma in education degree. He had been teaching science for between six and ten years but had fewer than four years of Grade 8 mathematics teaching experience at the time of the interview. He taught a group of learners ages 11 – 13 years in a public school located within an urban area of the Ojo, Lagos educational zone.

Prior to the interview, it seemed he had sufficient knowledge of the topic, but his interview responses were incoherent, repetitive, and showed he had limited knowledge of the topic. The interview also revealed that his insufficient knowledge of representations limited him from responding thoroughly to the interview questions, particularly the scenario interview questions.

6.4.1.3 Teachers' Goals for and Use of Representations in the Classroom

Bola stated that one of his goals of teaching with representations was for him to support students' use of different physical objects. He felt that teachers should avoid focusing on numbers, signs, or formal language that made students "get bored easily in the mathematics classroom". For example, Bola reported that "when you are teaching a concept that is abstract or teaching a new topic, you don't want to start with all the grammar and the terminologies of the concepts" but rather you use "things that they can connect with". He believed that, for students to have a grasp of a new topic at the beginning of a lesson, it must be related to something they can see and make sense of. According to the survey, Bola agreed with the statement that suggested that appropriate representations should be used to highlight important mathematical ideas.

Another goal of using representations in Bola's classroom was "only to pass information to students". For example, he stated that, rather than asking students to memorise their multiplication facts by completing the table, teachers should "for instance, create an image of 3 x 5 using boxes, put 3 groups of 5 together, by the time you add up it gives 15". Bola expressed that using representations to pass the information would help students gain the necessary knowledge. He felt that forcing students to memorise multiplication facts would not be helpful as they moved to their next grade particularly if they didn't understand the concept. Bola noted that this approach would "put students in the mood instantly" as well as "bring the students closer to learning".

In summary, one of Bola's goals for and use of representations in the classroom was for students to use something they are familiar with and relate it to any math ideas being taught. He felt that they needed to make sense of what they were learning in order to understand the math ideas. A second goal of using representations was to disseminate information to the students.

Bola felt that they needed to be interested in whatever they were learning, as such, he would use representations to encourage them to acquire the required knowledge that may be helpful in future grades.

6.4.1.4 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.4.1.4.1 *Content*

Bola believed that representation "is not rigid" rather, it is a way of making use of physical materials in order to represent abstract ideas. According to the survey, he disagreed with the statement that suggested that representations were usually not physically visible. He felt that representations have helped his students' understanding, particularly "imagery, a physical thing to learn faster." Furthermore, Bola believed that representations are "certain concepts that require you to create an image for them (students) to be able to develop an idea of a concept or something they can connect with". For Bola, representations is "creating imagery of reality of what it's really like in a real-life situation." It might be that he was only thinking about how students can be given different tools to think with, which in itself is a mental representation through external representation. While Bola believed in using creating physical images for students he had difficulty giving any examples of what he believed representations meant to him. He repeatedly use the term "connect" for almost every explanation he gave. He stated generally that, "representations can be created from an image, scenario, or even grouping the students". Not surprisingly, he strongly agreed with the statement in the survey that suggested that representations can be mental images.

Bola stated that knowledge of mathematics is important for teachers to be able to use representation effectively. This aligned with his survey response in which Bola disagreed with the statement that suggested that a teacher does not necessarily need a specialized understanding of elementary mathematics in order to use representations effectively. Bola reported that a teacher with specialized knowledge would easily notice when his students are understanding the concept he is teaching or not. He believed that a teacher needs to have a grasp of the topics in the mathematics curriculum. Bola mentioned that most teachers go to the classroom and write out equations they found in the textbook without necessarily following the curriculum expectations. He commented that “if you are not good at the content, you cannot create the representations”. Bola concluded that “representations is directly proportional to the knowledge of the topic the teacher is presenting”.

He believed that having a specialized knowledge was critically important for a teacher to understand what the curriculum describes and to use representation effectively but as stated earlier his specialized knowledge wasn’t strong.

6.4.1.4.2 ***Students and Pedagogy***

Bola had specific beliefs about how one should effectively teach with representations and whether there are other types of representations. For example, based on the survey response and interview with Bola, it appeared that he was only thinking mathematically about the scenario questions presented and how the problem could be addressed in a middle-school classroom. He did not see the problem as a misconception. Essentially, he explained how he would address the letters as an alphabet. Bola said, “based on my experience as a teacher, I will let the students understand that this is like a container”. He further explained, “X and P are not really the content, they are the container carrying the content”. It was revealing to know that Bola wanted his students to see X and P as objects that contain the same item. He worried about making sure he could address everyone’s needs in his class, from the student who is struggling to those that need little or no guidance, in each lesson. He had thoughtful pedagogical intentions but lacked the math content to be able to implement them with this question.

Bola felt that having a good knowledge of students and how they learn with representations are critical. He stated that he had not learned anything new from using representations but spoke with deep concern about his students' experiences of its use in the classroom. Bola talked about teaching the topic of *sets,* and his experience with the students. He discussed one particular lesson he did with his Grade 8 class in which he had used mangoes to illustrate a topic on *sets* that was not successful due to how the "slow learners" misunderstood the representations. Bola said, "next time I asked a student to describe a *set*, and he was mentioning 3 mangoes, 2 oranges". He stated that sometimes, the "slow learners" in his classroom focused on the types of representations he presented to them and lost the context and purpose in which it was being used.

He explained that he only used the "fruits to explain the concept," but the "slow learners" did not realize that the representations were meant to ease their understanding. As such, he felt that teachers should not place too much emphasis on a particular representation. Bola, however, noted that for the average learners and the fast learners the use of representations "increases their understanding". He strongly agreed with the statement on the survey that suggested that representations can help students draw on their conceptual understandings to solve new and unfamiliar problems. He seemed to lack the pedagogical knowledge necessary to make these representations useful to his struggling students.

Bola agreed with the survey statement that suggested that one specific representation of a pattern might not be enough in a patterning and algebra lesson. However, during the interview, Bola reported that there is no such thing as modes or types of representations. He said, "I do not believe in anything like types of representations". There was some discrepancy of how Bola views and relates to different representations; Bola reported that he does not believe in types of representations, and yet he claimed that "representations have really helped me" cover key mathematics concepts in algebra. Although it was not clear why Bola agreed with the survey statement that one specific representations of a pattern might not be enough, he did not recognize that the same idea can be presented in different forms of representations. He also did not believe that it was possible to switch from one mode of representation to the other, in which one is able to move back and forth between representations and understanding the correspondence between the representations. This was evident based on his comment that he did not believe in types of representations.

What came out strongly in the interview scenario questions was that while Bola claimed to use representations in his math classroom, he had a superficial knowledge of representations. This was apparent during his response to the scenario questions. When Bola was asked to predict the types of representations his students would use to solve this task, he said, “I do not really have a grasp of what you mean by types of representations”. “Sincerely, I have not done any course that discusses the types of representation”. Not surprisingly, for Bola, given that he had difficulty recognizing that it was possible to translate a concept from one representation to the other, he had difficulty predicting appropriate representations that students were likely to use. Overall, this indicated he has a very weak understanding of multiple representations.

During the interview, Bola described the impact of using too many representations in a lesson on the “slower learners” as a challenge to his teaching of patterning and algebra. He believed that “too much representation kills the learning of new concepts”. Bola noted that, for the slow learners, the use of representations “draw or hold them back” and they have “a problem transitioning among different representations”. Not surprisingly, Bola strongly agreed with the survey statement that suggested that including a lot of representations within a lesson could add confusion for students. However, he noted that using more than one representation depends on the topic and the concept he intended to teach. Perhaps this was why Bola was undecided about the survey statement that suggested that teachers should flexibly shift among different representations.

Bola spoke briefly with deep concern about inadequate technology in the classroom and how that affected his teaching of patterning and algebra. He said, “we don’t have facilities for the technology to create visuals and diagrams to aid teaching”. Bola further noted that “visualization helps a lot”.

In summary, Bola reported that teachers’ understanding of the curriculum expectations would mean they are able to use representations effectively. He felt that teachers would need specialized knowledge in order to create multiple representations. However, Bola did not have sufficient content knowledge and he has a limited understanding of representations. He is not strong in specialized knowledge himself and also could not respond to some of the math tasks.

Bola spoke about his students’ learning experiences with representations and how using more than one representation would negatively affect their learning, especially the students with learning disabilities. Bola believed that representations are physical materials that are needed to support how students relate with new math ideas. He felt that teachers should relate images to math ideas for students’ understanding. He wanted to help his students learn the material by using representations but it would seem lacked the mathematical content knowledge to do so.

6.4.1.5 Summary

Bola believed that representations are physical materials that support the way students relate with math ideas. Bola’s goal for and use of representations in his classroom was for students to relate math ideas with something they are familiar with and to disseminate information to the students. He reported that a teacher with specialized knowledge would notice when students are understanding the concept he is teaching. Bola reported that there is no such thing as types of representations. He has limited content knowledge and lacked understanding of representations.

6.4.2 Beth

6.4.2.1 Background

Beth, in her early 40s, was very enthusiastic about participating in the study. She teaches at the same school as another study participant Baker (before he moved to a senior secondary school where he now teaches chemistry). Beth has more than ten years of mathematics teaching experience, but less of that in the intermediate class. She teaches a group of learners ages 11 – 14 years in a public school located in Lagos. Beth has a first degree in mathematics education in addition to the NCE, which is a Nigerian teaching qualification obtained after three years of post-secondary training at a college of education.

6.4.2.2 Teachers' Goals for and Use of Representations in the Classroom

Beth stated that one of her goals for using representations in her classroom was to improve students' understanding. She reported that she feels teachers need to assist students in using an appropriate representation for proper understanding, in particular, conceptual understanding. Not surprisingly, Beth strongly agreed with the statement that suggested that representations can help students draw on their conceptual understanding to solve unfamiliar problems. Beth explained that a teacher needs to use representations in their teaching especially at the primary (elementary) school level. She stated that even at the secondary school level representations are required to improve the students' understanding.

Another goal of using representations in Beth's classroom was to help in building students' confidence in problem-solving in particular, "when they are left with some exercises to work with". Beth believed that teachers' constant use of representations can help students' confidence in solving unfamiliar mathematics problems once they are comfortable. She stated that, with adequate exposure to the use of representations, students could confidently solve different math problems on their own with or without assistance from the teacher. She mentioned that representations “assist students to learn the basic concepts particularly in building the necessary math foundation” to be used in the future.

In summary, Beth’s goal for and use of representations in the classroom are to improve students’ understanding. She felt that representations are useful to students at both elementary and secondary levels. A second goal was to build students’ confidence in problem solving and because she felt that they would be more able to draw on their conceptual understandings when they are exposed to the use of multiple representations.

6.4.2.3 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.4.2.3.1 ***Content***

Beth believed that algebra is a branch of mathematics that requires appropriate use of representations such as symbols or graphs to make it easier to learn. Beth, for instance, talked about showing her students how to properly use symbolic and graphical representations and had given them a procedure for using these representations in algebra lessons. She further stated that “most of them [students] need symbols, that is how they can learn math”. Beth mentioned that “representation is actually referring to mathematical instructional material”. For Beth, representations are “symbolic and graphical”, rather than referring to a broader range of visuals. Her beliefs about what representations mean are limited to symbolic and graphical representations as she could not use any other example to describe what she meant.

Beth believed that “the teachers’ knowledge of representation is important”. She stated that “when you do not know what to teach you can’t know the representations to use”. Beth disagreed with the survey statement that suggested that teachers do not necessarily need a specialized understanding of elementary math in order to use representations effectively.

Although Beth expressed comfort with her own mathematical knowledge, she admitted that “I really do not understand what you mean by patterning but the basic algebra I understood”. For her, representations mean using symbols and graphs to solve any math problem. Beth noted that representations are not only useful in algebra but can be used in statistics, probability and other strands for “actual understanding of math”. She reported that only a few topics such as simultaneous equations, matrices, sets, and factorization need numerical representations while other areas of math need graphical representations.

6.4.2.3.2 ***Students and Pedagogy***

Beth had specific beliefs about how teachers’ instructional practice serves as the driving force behind the way students are appropriately guided to learn algebra. She stated that planning was an important part of her teaching, in particular selecting the appropriate task that would keep the students engaged. Despite the importance Beth claimed to have attached to planning, it was not clear why she was undecided with the survey statement that suggested that effective use of representations requires a lot of planning. However, Beth agreed with the survey statement that suggested that selecting a worthwhile task determines what representations to use.

Beth believed that using more than one representation in a lesson “might be beyond the scope of the class” and because she felt there were no alternative approach to solve a math problem rather than just one way. As such, she felt providing more than one representation in a lesson was beyond the level of her students. She said, “having more than one representation will create a little problem with the level of the students I am taking”. Another reason why Beth felt using more than one representation was beyond the level of her students was because she felt the students would find it difficult to understand and would cause confusion for them. Beth did talk about using one representation at a time without making links between different representations.

Beth described her experience with representations using a topic on simultaneous equations, one of her past lessons. She mentioned three different approaches (graphical, elimination and substitution), which she regarded as “representations of different forms”. It was revealing to know that Beth thought these different procedural methods were representations. She further stated that the three different procedural approaches she regarded as representations would enable students to see the beauty of mathematics rather than viewing it as an “abstract subject”. As such, Beth commented that teachers should teach in a way that makes sense to the students.

While Beth was explaining her approach to the first scenario question, it appeared that she was thinking about letting the students know that there could be a value for x. She believed that in order to solve the task, “we need to actually get some other complete information that will direct us”. It was not clear from Beth’s explanation whether she was trying to find values for x and p. Beth said, “we have x banana and p banana as unknown, if we get the total number of bananas that all of them are going to share from there, we can say x + p = certain amount”. Beth did not see that if they are both x, they have to be the same and, she also did not know that if they are x + p they could be the same. She was only thinking mathematically about the task and how the problem could be addressed given that there was a total. She did not see the problem as a misconception. Further, in her response to the second scenario question, Beth reported that she would be responsive to her students’ misunderstanding as she would encourage them to use representations they are familiar with to explain the plan. Beth, for instance, discussed students using a prepaid plan to figure out the best plan. She predicted that graphical representations, linear equations, and concrete materials would be the representations her students will use. Not surprisingly, Beth strongly agreed with the statement in the survey that suggested that representations help in moving students from using concrete models to abstract representations.

Beth believed that for teachers to effectively use representations in teaching a concept, they need an adequate level of mathematical knowledge of the concept in order to provide instruction. In particular, teachers' knowledge of representation is an important aspect of their pedagogical knowledge that is required to make their teachings understandable to students. Beth stated that representations are important in every one of her lessons. During the interview, she reported that "without representations, they (the students) cannot understand how to solve abstract mathematics tasks". She believed that there are times when a teacher needs to figure out and learn how to approach a lesson using different representations in order to bring the students to understand the concept being taught. This was contrary to her survey response as Beth disagreed with the statement that suggested that a teacher would need to learn more about teaching-related mathematics before using representations in her teaching. While Beth had high expectations of the central role of teacher's content knowledge in good instruction, she was herself unsure of many basic concepts in early algebra.

Speaking with Beth I noticed that, while responding to one of the scenario questions, she worried about how to simplify the idea of a variable for the students. She reported that she would mention to the students that "the x is a symbol representing an unknown for which you have to find the number of bananas the person has". She further explained that "If I want to teach the student, I will let them know that the x does not actually mean anything, it is just representation, a variable, unknown symbol we have to look for in solving it only when we know the total". Beth's response to the scenario questions further provide evidence that she lacked a good understanding of representation. When Beth was asked they type of representations she would expect her students to use, she said, "I would expect them to build from the first one, since it is x banana I can say they should write the representation they are getting, for it is symbolic representation because x and p are variables".

When Beth was asked how she would bring her students to understand both context and therefore content as she planned her lessons, she talked about teaching quadratic equations, one of her past lessons. She said, “you have to let the students know that they cannot find the values for y except when they have values for x”. Beth stated that she tried to use a table of values as a tool in order to teach them how they can manipulate the value of x to find a corresponding value for y. Also, Beth discussed one particular lesson on quadratic equations she did with her Grade 8 class where she had to use the factoring technique before approaching the same task using the graphing technique. Basically, she tended to refer most to more traditional tables and graphs and hardly mentioned any visuals. Beth did explain the mathematics content, but there was no explanation of what context she would use to relate to the content.

Beth felt that “maybe complex algebraic expression will have more than one algebraic expression, I am not really sure; maybe undecided or so”. She stated that she spent a lot of time understanding “where the students are, and then continue from the place of their prior knowledge”. Furthermore, she reported on how she used representations with her students as they explore and investigate topics on one-variable equations and quadratic equations. Although the Ontario Grade 8 mathematics curriculum expectations required students to solve and verify linear equations involving a one-variable term, quadratic equations are not included in the curriculum expectations at that level. She talked about teaching one-variable equations such as $2a + 3 = 5$ using a two-step process. Beth did not describe the solution to this problem using a balance scale, rather she said, “when trying to explain that to the student you can tell them that 3 can cross over to the other side”.

Beth’s approach to this task was direct instruction of procedures and this shows that she lacks familiarity with other of visual representations. She further explained that “I mean without using the variable when you transfer from the other side the student can get that missing variable". She reported that if representations are used, "a student will be able to understand what you are teaching". It was not clear the kind of representation Beth meant. Overall, Beth wasn't clear on how to apply representations in teaching the aforementioned topics. She tends to rely on traditional teaching methods. According to the survey, Beth strongly agreed with the statement that suggested that the use of multiple representations is not clearly explained in the curriculum.

6.4.2.4 Summary

Beth strongly believed that teaching with representations can only be done using a narrow list of traditional mathematical tools and misunderstood different procedural methods to mean different representations. She reported that planning was an important part of her teaching, in particular selecting the appropriate task that would keep the students engaged. Beth believed that students' understanding of different representations would mean they are not able to deal with more than one representation at a time.

Beth felt that for teachers to effectively use representations in their teaching they would need to first understand the math content themselves, this was an area she struggled with. She was concerned about her own understanding of patterning and use of appropriate representations as she uses more of traditional tables and graphs. Beth believed that representations are very helpful tools in solving algebraic problems as well as other strands of mathematics. She felt that teachers should understand students' prior knowledge and build from there. Her instructional practice was fairly focused on traditional direct instruction of procedures.

6.4.3 Ben

6.4.3.1 Background

Ben was very willing and happy to participate in the study. In his early 40's, he has taught mathematics for more than ten years. He had fewer than seven years of intermediate class teaching experience. Ben teaches a group of learners ages 5 – 12 years in a private co-educational school located within an urban area in Ojo, Lagos educational zone. He has a first degree in mathematics education in addition to the NCE, which is a Nigerian teaching qualification obtained after three years of post-secondary training at a College of Education. Ben recently completed a Master of Science Education in Mathematics, Grades K – 6 at Walden University, Minnesota.

6.4.3.2 Teachers' Goals for and Use of Representations in the Classroom

Ben stated that one of his goals for using representations in his classroom was to facilitate learning. While he was explaining his approach to the first scenario question, he mentioned that he will use "pictures, diagrams" to facilitate the learning of the concept to his students. Ben believed that teachers should use representations to facilitate students' understanding of any given concepts in the classroom. He expressed that representations assist students in remembering the different concepts. Ben stated that students were able to take responsibility for their own learning as he introduced different representations to them. While responding further to the first scenario question, Ben predicted that the children in his class would approach the task with "some students agreeing with what Cody said and some with what Sam said".

A second goal of using representations in Ben's classroom was to clarify misunderstandings students experience particularly when they are learning difficult concepts. This response was aligned with his survey response in which Ben agreed with the statement that suggested that appropriate representations should be used to highlight important math ideas in order to clarify misunderstandings. While responding to the first scenario question, Ben explained how he would be responsive to his students' challenges as he addressed the algebraic misconceptions of the letters as an alphabet and as a specific value. Ben said, "looking at this question because the children would say, Cody would be right if he says x and p are different letters. That is algebra. But let me put myself in place of Collin, ok? Count the number of bananas each of them has and finds they are the same. I will start with numbers".

Ben gave an example in order to uncover the misconception by focusing on the mathematical idea. He said, "let's assume somehow Sam has two bananas and Cody has two bananas. What does that mean?" He explained that he would give an additional two bananas to each of the students. "So, for Sam to have written that $x = p$, I will tell the students that, in algebra that is an aspect that has to do with letters. First, I will need to explain with numbers that are equal that two bananas equal to two bananas and four bananas equal to four bananas before I can arrive at $x = p$. So that's what I will tell my students". Ben's explanation suggested that he understood the related concept and would guide the students to find the misconceptions. He would explain how this could be the case rather than having students use inquiry to determine whether or not this could be true.

Another goal for and use of representations in Ben's classroom was to support students' reasoning. Based on the interview and the scenario interview questions, Ben reportedly linked the underlying ideas involved to different representations that would support students' reasoning. This was evident in his survey response in which he strongly agreed with the statement that suggested that providing representations during classroom discussions could be used to support reasoning and explain difficult concepts. Ben was able to think mathematically about the scenario questions presented as well as demonstrate a good understanding of how the problem could be addressed in a middle-school classroom.

In summary, Ben's goals for and use of representations in the classroom are to facilitate learning and support reasoning. He felt students needed to understand different representations in order to take responsibility for their own learning. A second goal was to clarify misunderstandings and because he felt they would learn difficult concepts if they use multiple representations. Ben thought this could be achieved by giving similar examples.

6.4.3.3 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.4.3.3.1 *Content*

Ben believed that using multiple representations to teach any math concepts would make it more understandable. He stated that he taught algebra in ways he termed as graphically, numerically and verbally. Ben believed that "the use of representations will really help them to understand the concept not just the procedural but the concept." He mentioned that he decided to use representations to teach his entire algebra unit for the school year through presenting different examples and activities.

Ben was very confident of his knowledge of the curriculum and mentioned that "there is nothing like representations", in the Nigeria mathematics curriculum. This was contrary to his survey response as Ben indicated "don't know" to the statement that suggested that the use of representations is not clearly explained in the curriculum. He noted that "we have algebra, we have patterning in our curriculum, but there is no specific word like representations". Ben stated that he came across the word "representation" at the various professional development sessions he had attended.

Ben believed that knowledge of mathematics is a fundamental component of what is needed by all teachers. He said he was an algebra lover, and he believed that teachers' use of representations is related to their knowledge. Ben commented that "as a math teacher, you are not just a mathematics teacher, you must have good knowledge of what you are teaching". He noted that "teachers' understanding of representations will go a long way to really help in the proper teaching of a strand such as patterning and algebra". According to the survey, Ben therefore disagreed with the statement that suggested that teachers do not necessarily need a specialized understanding of elementary math in order to use representations effectively.

Based on the interview it appeared that Ben has knowledge of what is involved in using different types of representations. He reported that his criteria for the choice of representations depended on “readiness of the pupils” and “the difficulty of the concept of algebra” he intended teaching. When Ben was asked to give an example of an area that students find difficult, he noted that factorization, when the coefficient of x is greater than one, is an aspect that is “most frustrating” to his students.

6.4.3.3.2 ***Students and Pedagogy***

Ben described specific beliefs about his use of representations and what representations should be emphasized and how the representations would be understood by the students. He firmly believed that “getting to use representations, you need to prepare with some other stuff around your work” and that “it is not all the time we prepare for that; that is the bad side of it”. During the interview, Ben said, “just teaching alone without representations makes the topics difficult for them”. He stated that he was committed to a high level of planning and uses various representations to supplement his lessons. Ben mentioned that when planning his lessons, “concepts or topics are broken into bits, and for each of these topics/concepts for each day, I look at what works well and what’s my objective and what to make up”. Ben discussed his use of appropriate representations as a way of highlighting the relevant mathematical ideas that would follow the students through their lives as mathematicians. He said, “the appropriate representations that would help the learners be great mathematicians in the future”.

Overall, Ben was not clear on the benefits of using representations in his teaching. He said he would rather have his students memorize the procedures than take time to generate representations to simplify difficult concepts. Although Ben was not clear on the benefits of using representations in his classroom, he mentioned the importance of teaching with representations, and stated that representations make his “work easier”, and that his students get to “understand easily”. It became very clear through talking with Ben that he believed representations could ease teaching of algebra. This was because Ben felt that, if teachers allow students to explore math ideas on their own using different representations, they discover things on their own and could also make meaning from the ideas being presented.

Ben preferred for his students to come up with their own representations than for him to provide representations for them. Ben strongly agreed with the statement in the survey that suggested that allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra. He commented that students should be expected to generate representations by themselves but would only help them only when there was a need, “by that they learn more and it becomes part of them.” Not surprisingly, Ben disagreed with the statement in the survey that suggested that it is necessary that teachers should assist in choosing representations for students. He explained, “before I give them my own idea, I will ask them what method, what representations could be used? And, if they could not get the right terminology, I would then start with my method.” Ben stated that he encouraged his students to use symbols, “kind of representations that would really help them to understand word problems.” He reported that, during his math lesson, he encouraged his students to continue with the same form of representations if they were not ready to move beyond the concrete representations into abstract representations; particularly the students that “might still be struggling to understand”. According to the survey, Ben agreed with the statement that suggested that representations help in moving students from concrete models to abstract representations. Ben further explained, “when students come up with their representations it becomes easier for them, and they understand the concept easily and faster, so it makes learning very fast, and we go at a faster pace than it used to be when we could not use representations”.

During the interview, Ben was asked to explain how he uses some of the representations to explore and investigate ideas, and he discussed two of his past lessons. He talked about teaching addition and subtraction of fractions with fraction strips. He said, "most students do not understand addition and subtraction of fractions in my class." He explained, "A child comes into my class, I asked him if you add a half and a half together what do you get? And the child was trying to think, oh let me find the LCM, let me do this…" Ben felt that was not "too good." He reported that he helped the child think through the task using fraction strips manipulatives. He reported to have said to the student, "Why not think about this… I have to use the manipulatives, get fraction strips of half and another half combined together. The child was like wow. If I combine two halves together it gives me one." Ben believed that his students were able to create a mental image from that experience of using the fraction strips to establish that adding a half and a half will give one and also use it with other fractions. He also discussed another lesson where he mentioned that "for a graphical method I use pictures." For example, Ben reported that: "take ten students in a test; let's give all different grades. Two of them had grade A, three got grade B and one got grade C. And let's say another got grade D and nobody had grade E. I can use pictures to represent, to really let them understand."

Ben stated that one of the challenges for not using representations in his lessons sometimes was because of the workload. For example, he explained that, "as it was in my own school, you have 10 topics to be taught in 10 weeks, and you need to cover these topics, and there is no way you can. You need to look for a way to cover the topics, as a result, you just have to rush things over quickly to stay on track—do not let me waste time on the use of representations, just let me move on to the next topic." "I would rather teach my topics abstractly, rather than come up with representations." As he talked about his experience, he felt that "you really need more time to develop these representations, to use for these children." "I think that one of the challenges is that when you ask students to come up with their own representations to try to interpret the question in their own way, it takes a while.” He also felt that it took students time to be able to use representations “because algebra is a strand of mathematics that poses challenges to them.”

Ben described his confidence in both pedagogical and knowledge of student. He mentioned the significant role of using representations to achieve success in his lessons. During the interview, Ben shared the impact of using representations in his lessons in a particular school year. He said, “there was a session (a school year), like that we didn’t really have many extra curricular activities in the school. I was able to extend the topics I couldn’t cover in a week into another week. I was able to use multiple representations, manipulatives and technology. I used everything to introduce the topics to the children. At the end of the year, they really, I mean the children really came out very well. They really did well in their examinations, so it really helped”. Further, Ben reported that the availability of manipulatives usually helps his students to develop conceptual understandings. Not surprisingly, he disagreed with the statement in the survey that suggested that the use of manipulatives is only good for teaching patterning but not algebraic concepts. Ben explained, however, that there are occasions when he does not have access to manipulatives. He said, “if the manipulatives are not available or not reachable at the time I need to use them, I just resolve to teach it abstractly or just let the students learn it like that without any form of representations”.

Ben appeared to understand how the majority of his students felt about algebra. He mentioned that “Children find patterning and algebra very difficult.” and “are always thrown off balance”. As such, he tried to look for ways around it in order to make it simpler for the students to understand as he reported that he does not “have any choice but to teach it.” Ben stated that “students’ lack of understanding” and “when the algebra is very difficult for both the teacher and the learners to interpret sometimes" are some of the challenges associated with the teaching and learning of algebra. He did reiterate that "factorization is a bit of a challenge for most students." Ben reported that he tried to act appropriately to facilitate learning among his students whenever there was a difficult concept to learn. He explained, "when I enter the class and see their mood, and I see these children are really ready to learn, that is the time I ask them to come up with their own representation or ask them to get into groups or work individually". He did note that "My job is to make them understand difficult concepts in algebra" by pushing them to look for appropriate representations or multiple representations to make the work easier for them. The desire for students to build off their own understandings through using representations was a common emphasis throughout his discussions.

Speaking with Ben I noticed that, while responding to the second scenario question, he was both thinking of his own concerns and the concerns of his students simultaneously. He assumed that if he was struggling to understand something that his students are likely to experience these same issues, and therefore anticipated students' thinking based upon his own concerns. Ben stated that "for my own level of students that I am teaching, I would rather ask them to come up with their own table. To use a table to really help them to know what plan would be best for Olamide rather than go by these expressions. They might not understand it. They will not understand these expressions. In conclusion, Ben explained how he would support his students and mentioned that he would put them in groups and allow them to have discussions before suggesting representations such as numerical, graphical or tables. He said, "this is how we do tasks and check for what they are doing and just support them on how to come up with a good and appropriate answer".

6.4.3.4 Summary

Ben felt that teaching with representations should focus most importantly on students' understanding of the concept. He believed that teachers should begin with thorough planning of their lessons however, he was concerned about his own time for preparation, as a result of heavy workloads. Ben reported however that the time investment was not always worth it, and that his teaching workload affects his use of representations. Ben felt that students would learn more if they generate their own representations, especially the use of visuals such as manipulatives but he does not have the time to do so. Ben believed that the use of manipulatives can help students develop the ability to create a mental image. He was confident of his knowledge of the Nigeria Grade 8 math curriculum document and he believed that representations are not contained in the curriculum document.

Ben felt that specialized knowledge is critical for teachers to be able to use representations in any strand of math. He believed this would mean the teacher understanding the nature of the concept to teach and how prepared the students are to learn. Ben was concerned about his own understanding of the content and expressed similar concern for his students. He reported that when his students are left to explore different representations they become seemingly engaged.

6.4.3.5 Baker

6.4.3.6 Background

Baker, in his mid 30s, teaches mathematics at the middle-school level in Lagos. He has more than 10 years of science teaching experience, but less than that in grade 8 mathematics. He teaches in a public school that has a large student population. Baker holds a bachelor's degree in chemistry and an education degree.

6.4.3.7 Teachers' Goals for and Use of Representations in the Classroom

Baker stated that one of his goals for using representations is to help students construct meaning from any given task. Baker felt that his students need to construct their own meaning from any math problem before he would be able to support their reasoning, he taught by "trying a problem that makes sense to the students so that they are able to construct meaning from the problem". Baker felt that the algebra his students were learning should apply to their lives. He, for instance, talked about relating to his students how their classroom building can be used to learn *perimeter*. Baker reported that, with this approach, students would generate the formula 2Length + 2Breadth on their own and as such make the task more realistic. He said, "you are actually doing a process that they won't forget".

Baker shared that he tried to set up effective lessons for his students. He believed that when teachers make use of concrete objects to describe a new concept, students don't easily forget. As such, "they have self-confidence", and the concept you are teaching is "imbued in them". He felt that "most teachers do not see the importance of [using] representations to make students be more confident in mastering different concepts". According to the survey, Baker agreed with the statement that providing representations during classroom discussions could be used to support students' reasoning.

Another goal of using representations in Baker's classroom was to show relationships between different concepts. He stated that making connections and showing relationships was important in enhancing mathematics learning. Baker, for instance, felt that representation is "a relationship that expresses or explains the similarity between objects". He believed that teachers should be able to put students through a teacher-supported transition for which the students are developing conceptual understanding as they learn a new concept. It became clear why Baker disagreed with the statement that suggested that it is necessary that teachers should assist in choosing appropriate representations for students.

In summary, Baker's goal for and use of representations was to support students to construct meaning from any math problem. He felt that students needed to make sense of what they were learning in order to be ready to establish meaning from it. Baker thought this could be achieved by relating to concrete objects. A second goal was to have students use multiple representations because he felt that they would be able to show relationships and connections.

6.4.3.8 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.4.3.8.1 *Content*

Baker believed that "algebra appeared to students as a threat". He was concerned that the use of representations is not clearly explained in the Nigeria mathematics curriculum, which made the teaching and learning of mathematics, particularly algebra, stressful for both the teacher and the students. This somewhat contrasted with his survey response as Baker was undecided about the statement that suggested that the use of representations is not clearly explained in the curriculum. He stated that "a good teacher must find a way of putting students through how to effectively use representations, that is the only way you can achieve the curriculum objectives".

Baker believed that teachers' use of representations is related to their knowledge. He appeared to be confident of his knowledge of mathematics and representations. Baker did comment that he found "teachers' mastery of the content to be helpful particularly in applying the appropriate representations". According to the survey, Baker disagreed with the statement that suggested that teachers don't necessarily need a specialized understanding of elementary math in order to use representations effectively. During the interview, he said, "teachers should understand the type of representations to use when teaching a particular math concept". He noted that "a fraction is a fraction but can be explained in different ways".

6.4.3.8.2 ***Students and Pedagogy***

Baker shared some of his beliefs about how one should use different representations and teaching strategies that would impact on students' understanding of math concepts. In order to teach algebra in the method, Baker felt was best for his students, he mentioned pictures, objects, symbols, diagrams, charts and graphs as some of the representations that students should be able to use comfortably when solving any given algebra task. Baker, for instance, while responding to the second scenario question predicted that he would use graphs and letters to support his students in their effort to use representations. According to the survey, he strongly agreed with the statement that suggested that appropriate representations should be used to highlight important mathematical ideas during classroom discussions. He believed that, when teachers failed to use representations during their teaching they are very likely to go through the stress of explaining the new ideas to their students. On the other hand, they do less work when they make use of different representations.

Baker believed that, in order to bring his students to understand both the context and the content, he would use some concrete materials to explain difficult areas. He reported that students should be able to represent through an investigation with concrete materials a problem such as 3 + 4. Baker, for instance, explained that task such as this could be solved by asking the students to make use of objects or blocks for which "the students use 3 blocks and 4 blocks, adding up to get 7 blocks". Baker gave this example as a simple way of describing the importance of using concrete materials. He further explained that, whenever "a student finds a similar problem in the future, he/she could choose to use another form of representations different from what the teacher had used after he/she has had a grasp of the concept". Baker stated that math will be more meaningful to students if teachers use contextual tasks. This response aligned with his survey response in which Baker agreed with the statement that suggested that selecting a worthwhile task determines what representations to use.

Baker did not consider moving students from concrete representations to abstract representations in his explanation as to why he should use more than one representation. This somewhat contrasted with his survey response in which Baker agreed with the statement that suggested that representations help in moving students from using concrete models to abstract representations. He shared that sometimes, he used one mode of representation with a set of students in a lesson but another mode of representation with another set of students at the same level/grade. Baker reiterated that for a mathematical problem to make sense to the student, “they must be able to construct meaning from the problem”. It became clear through talking with Baker about algebra that he was determined to use more than one representation in every one of his lessons so that the students could be successful.

Baker discussed using real-life examples in his algebra class to ease students’ learning because “if you do not do that it will look very strange to the students”. By recognising the difficulties students were having with algebra at all levels, Baker used his own classroom situations to identify the letters involved and how to manipulate these letters as variables as being the reason why it is viewed as a difficult strand. Further, Baker shared his experiences and challenges associated with the teaching of this strand. Baker, for instance, felt that some of the representations he needed in his teaching of algebra are not always available. Overcrowded classrooms are another challenge that limits Baker’s use of representations during his teaching of patterning and algebra. Most classrooms in Lagos are overcrowded with the class-size ranging between 35 and 40 students causing thin spread of resources and thereby affecting the quality of teaching. The large class population can be linked to one of the reasons that teachers tend to adopt traditional transmissive strategies. He also mentioned that, since the government is the sole supplier of the different representation materials, there are times when “those things are not always handy”.

Baker stated that one of his instructional strategies was to allow students to generate their own representations. Baker stated he encouraged his students to come up with their own representations after he has shown them. This somewhat contrasted with his survey response as he was undecided with the statement that suggested that allowing students to generate their own representations is an excellent way to develop student understanding of this strand.

Baker stated that teachers’ pedagogical knowledge would help in facilitating instructions that involve the use of representations. He talked about making sure all of his lessons integrated some common representations or a few representations so that students do not get confused as a result of including a lot of representations. Not surprisingly, Baker agreed with the survey statement that suggested that knowing which representation to use is sometimes confusing to students. Baker, for instance, felt there was no need for him to switch between representations if “there was no hiccup anywhere and everything was flowing”. Perhaps this was why he agreed with the statement that suggested that representations are hard to use in teaching.

Baker felt it was important for teachers to integrate representations into their lessons, he shared his concerns over how “most teachers just allow students to learn the formula without showing the relationships”. He reported that this approach was one of the reasons “why equations look threatening to students”. It became obvious why Baker agreed with the survey statement that suggested that patterning and algebra is one of the more difficult strands to teach.

According to the survey, Baker disagreed with the statement that suggested that the use of manipulatives is only good for teaching patterning but not for algebra. During the interview, Baker shared a lesson he used with his students to help them gain an understanding of symbols and manipulatives. He shared that he used this lesson to give his students time to learn what they needed to explore, as well as investigate tasks with representations in order to improve on their problem-solving skills. Baker talked about using manipulatives and symbols and then described ideas where students are able to use representations that are relevant. “Once students are able to get the concepts”, Baker believed that “they are more likely to solve problems on their own using different symbols to represent the key concepts”.

6.4.3.9 Summary

Baker believed that representations are pictures, charts, objects, symbols, diagrams and graphs that help students to learn. He believed that for teachers to achieve the curriculum objectives, the use of representations would need to be clearly explained in the Nigeria mathematics curriculum. He felt that teachers should engage in teaching strategies that use different representations and encourage students to generate their own but only after he has provided his own. Baker reported that pictures, objects, symbols, diagrams, charts and graphs are helpful representations for students. Baker strongly believed that teachers should focus more on contextual tasks in their teaching and because he felt that the use of letters makes algebra a difficult strand of math.

Baker reported that teachers would need specialized knowledge in order to use appropriate representations. He felt that teachers should be mindful about switching between representations because he felt it might confuse some of the students. Baker was still concerned about how teachers should integrate some common representations especially manipulatives and symbols into their lessons because using it effectively would help students’ understanding. He believed that students would improve their mathematical skills if they are able to understand different representations.

6.4.3.10 Bryce

6.4.3.11 Background

Bryce was happy to participate in the study. In his early 40s, he teaches mathematics at the Grade 8 level in Lagos state. He has more than six years of mathematics teaching experience, but less than that in Grade 8. Bryce has a first degree in mathematics education in addition to the NCE, which is a Nigeria teaching qualification obtained after three years of post-secondary training at a College of Education.

6.4.3.12 Teachers' Goals for and Use of Representations in the Classroom

Bryce stated that one of his goals for using representations in his classroom was to enable him to cover the scheme of work. The scheme of work is usually a document that summarizes the content of a course of instruction and divides the content into manageable portions for logical and organized teaching and assessment. The scheme of work is a plan that shows work to be done in the classroom. Apparently, one of Bryce's goals for using representations was to help him cover the algebra contents he was meant to teach in each of his lesson. Bryce explained that providing representations has helped him to cover the key math concepts in patterning and algebra. He further explained that using representations "helped me to cover my scheme of work on time, saves time, saves energy, and make me feel relaxed while the students are engaged in math activities".

Another goal for using representations in Bryce's classroom was to help students' understanding of the mathematical language. He commented that, as a teacher, "I am a guide". He stated that accessing and understanding the mathematical language was important for the students to engage in generating their own representations. According to the survey, Bryce strongly agreed with the statement that suggested that providing representations to support reasoning is something he will do to explain difficult concepts. Bryce explained that he would need to get the students acquainted with the language, the meanings of difficult words "by looking at the meaning in the dictionary and later in real-life examples".

In summary, Bryce's goal for and use of representations in the classroom was to help students develop the math language and because he felt they would generate their own representations if they are able to access and understand the math language. A second goal was to assist Bryce in covering his scheme of work. He felt when multiple representations are used students would be more engaged.

6.4.3.13 Beliefs and Knowledge: Mathematics, Students and Pedagogy

6.4.3.13.1 ***Content***

Bryce stated that he found using real-life examples, such as a balance scale as well as the students themselves, as an excellent way to generate representations in his teaching. When he was asked if this was the representation he uses as tools to solve mathematics problems, he said, "those were the representations I use, they are not tools". Bryce had to see representations as teaching aids before he could acknowledge that they are tools that help in solving mathematics problems. He described himself as using more real-life problem-solving in his classroom as a result of the professional development sessions he attended some years ago, although his description of what representations mean was vague. According to Bryce, "representations mean using real-life method to teach".

According to the survey, Bryce disagreed with the statement that suggested that teachers do not necessarily need a specialized understanding of elementary math in order to use representations. During the interview, Bryce was not able to explain how teachers' use of representations is related to their knowledge. Bryce, for instance, shared a time when the language teachers in his school had a cultural day event, and he and a few other teachers who had the opportunity to attend a training session trained other teachers on how they could approach the event. He said, "when we went for the training program, we shared some of the knowledge we acquire during the training with other teachers, and we train the other teachers". Bryce's explanation was not related to math in any way. This appears to provide further evidence to support Bryce's limited knowledge of both representations and math content. In addition, Bryce had a hard time with the scenario question about students' misconceptions where a student said X = P, while the other said that X and P are different letters and so cannot be the same.

6.4.3.13.2 ***Students and Pedagogy***

Bryce had specific beliefs about how teachers should effectively use representations to improve the learning of algebra. He talked about experiential learning as part of his efforts to effectively use representations with his students. When asked whether Bryce found it more helpful to generate representations for his students, he responded verbally and on the survey that allowing students to generate their own representations makes it much easier for them to understand. He said, "when they generate their own representations, they understand better". This somewhat contrasted with his comments in which Bryce felt that teachers need to generate representations for the students before they are comfortable generating theirs. Bryce, for instance, shared that "I think the teacher has to generate one as a sample before students can be able to generate their own". It became clear why Bryce agreed with the survey statement that suggested that it is necessary that teachers should assist in choosing appropriate representations for students. For Bryce, good understanding of representations meant his students would be better exposed to using more than one representation. He stated that, for algebraic lessons, he would need more than one representation but did not give a particular example. This was affirmed in his survey response in which he agreed with the statement that suggested that one specific representation of a pattern may not be enough in a patterning and algebra lesson. Bryce felt that "for better understanding, you need more than one. So that the information would go deep down in their mind”. Bryce stated that while some students would easily understand with one representation, others would expect the teacher to generate more.

Based on the interview and scenario interview with Bryce, he predicted that the students would look at the questions from a different perspective. While responding to the first scenario question, Bryce predicted that some of the students would agree while others will disagree that X and P are equal. He further predicted that there would be a discussion among the students. Bryce said, “for better understanding, they can use real life, or they can use two sets of a bag containing the same content”. He agreed with the students as stated in the question that, “they are both correct”. Bryce explained that because the contents of X and P are bananas, Sam is correct “if you say $X = P$, it is correct and if you say $X \neq P$ because they are different letters is also correct”. However, it seemed Bryce was translating the question statements in order to gain a correct meaning for the mathematical context. First, Bryce felt because they both have equal number of bananas (same number of the same item), then the student is correct to have said $X = P$. Second, Bryce felt that this student is also correct to have said that $X \neq P$ because he gave fixed positions to the algebraic letter as it was understood as one of the 26 alphabetical letters in English language and not as a quantity. He further explained that the students looked at the question from a different perspective. Bryce’s lack of confidence and superficial knowledge of the concepts came out during the conversations on the scenario questions. He was unable to explain how he would assist the students in using representations as tools to solve the task.

Although Bryce stated that he taught abstractly and traditionally in the past, he said, his teaching changed from being abstract to using real-life representations after attending intervention training. Bryce reported that “before the training, I do teach in the abstract. I had so many problems with my students understanding me”. He said he was confident in his use of representations after attending various professional development sessions. Bryce discussed the advantages of using representations in his lessons and how that has resulted in a sharp increase in students' performance in his school. He stated that representations have been "helping a lot" in his teaching. Bryce was no longer worried that his lessons would be boring to his students as he said, "representations made my class very interesting". He stated that he drew on different resources he felt would be needed to teach his lessons.

Bryce commented that he now has access to "tons of resources". For example, Bryce cited www.mathisfun.com as one of the websites he uses during his teaching. Another type of instructional material Bryce used to teach his lessons on algebra was videos. He explained that during the course of his teaching, some of the students would understand better after they linked his explanation to that of the video. Bryce noted that some of the students sometimes understood his explanation and language better than what they had watched in the video. He advised he would approach an algebraic lesson by arousing the interest of his students.

Although Bryce disagreed with the survey statement that suggested that teachers need to learn more about teaching-related math before using representations, he suggested that teachers need to be supported in using modern technologies such as smartboard to support their use of representations. He said, "I suggest we should go for modern representations, and also we need smartboard in all classrooms in Lagos state". Bryce noted that intervention training had had an impact on the way he teaches as he now uses audio-visuals in his classroom. This appears to provide further evidence to support Bryce's claim that he uses video in his lessons. He described how he would use audio-visuals to start his lesson, after which he gave his explanation and allowed students to ask questions before giving the students a task to solve. He thought using this approach indicated that he had changed his instructional practice from abstract teaching of procedures to reformed-oriented instruction, whereas he was still using traditional style of teaching. Bryce explained that while they were watching a video, he would ask the students to "note the questions they intended to ask". He also mentioned that he could construct or improvise materials that are not readily available to support his teaching.

Bryce reported that his choice of representations was informed by factors such as the topic he intended to teach, the age of the students, readiness of the students, the grade level of the students, and his time limit. He commented that these factors were some of the reasons why he struggled to find the appropriate approach to cover all the materials demanded by the curriculum. As such, Bryce reported that he felt the need to use representations in order to make his teaching effective. While responding to the first scenario question, Bryce explained that he would look for a similar question to simplify the question in order to support their thinking. He did not mention any mode of representations the students would be using. Bryce was not able to suggest any form of support he would offer his students in their effort to use representations to solve the task.

Bryce described his teaching challenges. When asked about the challenges he usually encountered when teaching patterning and algebra, he appeared to focus on the challenges facing his daily teaching rather than talking about the challenges facing the teaching and learning of algebra. It was revealing to find that these challenges include: students being underage; parents' attitudes; lack of learning materials for some of the students; teachers' lackadaisical attitude towards teaching as a result of government abolishing corporal punishment; and inadequate funding from the government.

6.4.3.14 Summary

Bryce's goal for and use of representations in the classroom was to help students develop the math language and to help him cover his scheme of work as required in the curriculum expectations. He reported that he taught algebra using videos and real-life examples such as balance scale, which he regarded as reformed-based approach and because he felt there would be meaningful learning and improve in students' performance. Bryce strongly believed that teachers should allow students to generate their own representations. He felt that, when students are left to generate and explore different representations, they would understand better.

Although Bryce regards representations as teaching aids, he did not see them as tools to solve math problems. He did not elicit any discussions regarding the meaning of representations in real-life situations. He strongly believed that teachers should use real-life context and relates it with the math content. Bryce felt that teaching algebra with real-life context would better improve students' understandings.

Bryce felt that teaching with videos and other technology would support students' understanding. He believed this was possible as a result of attending various professional development training sessions. It would appear that Bryce lacked the skills and knowledge of reformed-based teaching as he was still using transmission (video and explanation) to teach his lessons. He was still concerned about his own understanding of representations especially using representations as tools versus as final answers. He spoke specifically about his teaching challenges and how the challenges limit his choice of representations. He also reported that he struggled with how he would vary his approaches to instruction using multiple representations in order to meet individual needs.

6.4.4 Summary of the Lagos Participants' Goals, Knowledge and Beliefs

Table 22 provides a snapshot of the five Lagos participants' goals, knowledge and beliefs regarding the use of representations in patterning and algebra.

Table 22:

Summary of Lagos teachers' perspectives and instructional practice.

Participants	Bola	Beth	Ben	Baker	Bryce
Qualifications and experience	BSc in Physics and electronics with no mathematics Education training. He had four years' experience of teaching mathematics.	BScBEd in Mathematics Education (Intermediate/Senior). She had ten years' experience of teaching mathematics.	BScEd in Mathematics Education (Intermediate/Senior) with more than 10 years' experience of teaching mathematics	BSc in Chemistry with no Mathematics Education training. He had five years' experience of teaching mathematics.	BEd in Mathematics Education (Primary/Junior) with more than seven years' experience of teaching mathematics
Modes of representation	Used representations such as numbers, signs, terminologies image, scenario, concrete materials.	Used representations such as symbols, graphs and equations.	Used representations such as graphical, numerical, verbal and manipulatives.	Used representations such as pictures, objects, symbols, diagrams, charts and graphs.	Used representations such as real-life and computer software.
Goals	The various representations allowed him to put students in the mood. Bola's description of his approach was to allow students to make connections with math ideas.	The various representations are used to improve students' understanding. She believed the use of representations is an excellent approach to help students build their confidence in problem-solving.	The various representations allowed him facilitate learning, support reasoning and clarify misunderstandings.	The various representations allowed him to make students construct meaning from any given task.	The various representations allowed his students to understand much better.

Beliefs and Knowledge: Content	He believed specialized knowledge is a prerequisite to use representations. It appeared as if his SCK is insufficient regarding the specific content in the scenario questions. Bola had no knowledge of how the curriculum integrates the use of representations.	She believed specialized knowledge is a prerequisite to use representations. It appeared as if her SCK is insufficient regarding the specific content in the scenario questions. Beth had no knowledge of how the curriculum integrates the use of representations and, no knowledge of patterning aspect of patterning and algebra strand.	He believed specialized knowledge is a prerequisite to use representations. It seemed as if his SCK is sufficient regarding the specific content in the scenario questions. Ben had knowledge of how the curriculum integrates the use of representations	He believed specialized knowledge is a prerequisite to use representations. It appeared as if his SCK is insufficient regarding the specific content in the scenario questions. Baker had no knowledge of how the curriculum integrates the use of representations	Bryce believed specialized knowledge is a prerequisite to use representations. It seemed as if his SCK is insufficient regarding the specific content in the scenario questions. He had no knowledge of how the curriculum integrates the use of representations.
Beliefs and Knowledge: Students and Pedagogy	He could not recognize and clarified students' misconceptions.	She was unaware of students' misconceptions and showed no understanding on how to address their misunderstanding.	Ben was aware of students' misconceptions and showed understanding on how to address their misunderstanding.	He was unaware of students' misconceptions and showed little understanding on how to address their misunderstanding.	He was unaware of students' misconceptions and showed no understanding on how to address their misunderstanding.

	He believed in connecting students' prior knowledge with new situations.	She believed in connecting students' prior knowledge with new situations.	Ben believed in teachers building on students' existing knowledge.	He believed in connecting students' prior knowledge with new situations.	He believed in connecting students' prior knowledge with new situations.
	He did not mention how he encouraged his students to generate their own representations.	No evidence of encouraging her students to generate their own representations.	He believed he initially had to generate representations before students are comfortable generating theirs.	He encouraged his students to generate their own representations after first providing for them.	He encouraged his students to generate their own representations after first providing for them.
	Some evidence of transitioning among different representations but believed such affect learning negatively.	Some evidence of transitioning among different representations.	Some evidence of transitioning among different representations.	Some evidence of transitioning among different representations.	Some evidence of transitioning among different representations.
	His teaching style were traditional.	Her teaching style were traditional.	He believed in reform-oriented teaching style and showed evidence of his knowledge of it.	He believed in reform-oriented teaching style but showed no evidence of his knowledge of it.	He believed in reform-oriented teaching style but showed no evidence of his knowledge of it.
	Not much evidence of adequate planning was expressed.	Not much evidence of planning before using representation was mentioned.	Some evidence of planning before using representation was mentioned.	Not much evidence of planning before using representation was mentioned.	No evidence of planning before using representation was mentioned.

	He believed tasks should be context related.	She believed tasks should be context related.	He believed tasks should be context related.	He believed tasks should be context related.	He believed tasks should be context related.

CHAPTER SEVEN: CROSS-CASE ANALYSIS AND DISCUSSION

7.1 Introduction

This chapter presents a cross-case analysis and discussion of the key findings drawn from the two jurisdictions. The cross-case analysis enables me to set out and explain similarities and differences among the case studies, to consider and make sense of their relationships. The analysis of the teachers' use of representations was carried out based on the underlying structure of their goals, beliefs and knowledge, discussed earlier in the thesis. Themes emerging from the cross-case analysis will be discussed in relation to the existing research literature.

The Participants

The ten teachers in the study shared their time, knowledge and personal experience with how they use representations when teaching patterning and algebra and the beliefs that underscore their actions. The teachers were from two geographical locations: Ontario, Canada and Lagos, Nigeria. The ten teachers in the study had varied amounts of experience in teaching patterning and algebra to Grade 8 students.

7.2 Comparison of the Teachers' Goals for and Use of Representations

Previous research has indicated that teachers' goals for and the use of representations are a driving force behind how students obtain knowledge in algebra and other strands of mathematics. There were some distinguishable differences in the teachers' use of representations between the two jurisdictions. For example, the Ontario teachers' goals for and use of representations were directed towards supporting students to make connections, show relationships, develop reasoning and solve math problems in multiple ways. This is consistent with what was recommended by NCTM (2000), namely that representations are needed to enhance students' understanding, and ability to make connections in mathematics. Similarly, Rittle-Johnson, Loehr and Durkin (2017) suggest that the use of different representations is needed to facilitate the learning of algebra and improve understanding. In contrast, the Lagos teachers' goals for and use of representations were to improve understanding, facilitate learning, show relationships, disseminate information and cover the scheme of work.

Analysis of the key findings from all ten teachers' interviews revealed that their goals for and use of representations relate to three themes:

- opportunities for students to show connections, relationships and reasoning;
- supporting students' confidence in problem-solving; and
- facilitation and opportunities for questioning and discussion.

7.2.1 Opportunities for Students to Show Connections, Relationships and Reasoning

My findings in this regard revealed that the five Ontario participants in my study had certain goals in mind, one of which was to ensure that representations are effectively used during their teaching. My findings differ from those of Moyer (2001) who found that teachers' goals for using representations included using them for the pedagogically questionable reasons. For example, some of the teachers in Moyer's study stated that the main purpose for using representations such as manipulatives was "fun math". Moyer revealed that, by "fun math", teachers artificially set up a classroom situation in which materials may not be used effectively.

In Ontario, however, both Scott and Susan focus on giving students the opportunity to show relationships between mathematical concepts and different representations. I found that Silva, Susan and Sara seemed to believe that students learn through establishing connections between mathematical ideas. Both Sonia and Sara further believed that using representations can support students in making sense of and reasoning about mathematical tasks and concepts, as is desirable (Mitchell, et al. 2013).

Conversely, for the Lagos teachers, the goals for and use of representations differed, first among themselves, and also from the Ontario teachers. Only Baker reported that he used representation to support students in showing relationships and making connections. Ben stated during the interview that he used representations to support students' reasoning. Bola wanted students to construct meaning from mathematical ideas by using a context they could connect with. This finding is consistent with that of Beswick (2010), who highlights the importance of contexts in assisting learners to make connections and engage with challenging mathematics, rather than obscuring it.

7.2.2 Supporting Students' Confidence in Problem-Solving

Four of the five Ontario teachers in my study focused their goals for and use of representations on communicating mathematical ideas in order to support students' confidence in problem solving. Scott for example, mentioned that he ensures students have access to representations that give them the confidence to attempt abstract mathematics concepts. Scott and Silva felt that adequate communication of mathematical ideas through multiple representations would prepare students to be capable of solving tasks in the future. Scott ensured that students develop the right mathematical language whereas Silva used traditional graphs. Four of the Ontario teachers claimed to communicate mathematical concepts through multiple representations in order to support student confidence in problem-solving, which is consistent with the finding of Stylianou (2010). Mitchell et al. (2013) also suggest that teachers must be able to communicate mathematical ideas to students in a comprehensive manner.

Silva, Susan and Sara spoke about using multiple representations in order to expose students to multiple approaches to solving a task. Sara commented that teachers may need to communicate mathematics concepts through multiple representations that match how the students are thinking in order to help them move forward. As Star and Rittle-Johnson (2009) reminded us: it is not a good idea to teach students one and only one way to approach a type of mathematics problem. Research indicates that expert mathematics teachers recognize the importance of comparing multiple strategies (Ball, 1993; Stigler & Hiebert, 1999).

In contrast, only Beth of the five Lagos teachers, explained she provided representations to support her students in solving problems and building students' confidence instead of merely teaching them how to solve problems or solving the problem for them. This is somewhat consistent with the finding of Moyer (2001), who found that only three of the ten teachers in her study used representations such as manipulatives for problem-solving and enrichment. Interestingly, whereas Bola mentioned that he used representation simply for disseminating information, in Stylianou's (2010) study, ten of the 18 teachers used representation for the purpose of helping students understand the given information. Stylianou stated that the teachers' purpose of using representation was to understand information and plan problem-solving. Bryce for example, was more into using representations to support the development of students' mathematical language.

7.2.3 Facilitation and Opportunities for Questioning and Discussion

To encourage the participation of students in algebra class, a few of the teachers believed it was essential to create the opportunity for students to question and discuss their ideas. Except for Ontario teachers Susan and Sara (whose goals were to give students the opportunity to question and discuss their thinking), the other three Ontario teachers did not focus on questioning and discussion. Only Sara reported that she asked her students to explain their thinking, a strategy that would prompt further discussion between her and the students in classroom. Three (Scott, Sonia and Sara) of the five teachers also used representations for the purpose of clarifying misunderstandings.

In contrast, only Ben, a Lagos teacher, reported that he uses representations to facilitate learning in his classroom. This is similar with that of Stylianou (2010) who reported teachers need to be made aware of explicit ways in which representations can be enacted in order to help them facilitate their students' learning.

Bola reported that his use of representations was to arouse student interest whereas Bryce reported that it was for him to cover the scheme of work meant for the school year.

7.3 Goals Discussion

When comparing teacher practice across and within countries such as in my study, it is important to consider the teachers' goals for and use of representations (Chappuis & Stiggins, 2002). "Learning is easier when learners understand what goal they are trying to achieve, the purpose of achieving the goal, and the specific attributes of success" (Chappuis & Stiggins, 2002, p. 42). Each of the case teachers had a unique set of instructional goals and ways of supporting student learning. Teachers' goals such as clarifying misunderstandings, making connections, and showing relationships were similar in the ten case studies. What set the ten teachers apart was: language development, using multiple approaches, building confidence, problem-solving, supporting reasoning, and facilitating learning. Some, but not all, of the teachers used different representations as tools to emphasise important concepts.

One of the goals for and use of representations identified by all ten teachers was to facilitate, question and engage students in meaningful discussions. However, only one of the five Lagos teachers suggested that the teacher should act as a facilitator to the construction of knowledge for students (Green, Flowers, & Piel, 2008). Mitchell, Charalambous, and Hill (2013) remind us that representations can facilitate student learning. As in the case of Sara, teachers need to guide and support students through relevant prompts by giving clues (Wildani, 2014). Explanation and discussion are important goals of using representations to promote mathematics learning, according to the literature. Research has shown that teachers should facilitate discussion of different students' explanations, helping them build upon each other's reasoning (Stein, Engle, Smith & Hughes, 2008).

Representation can foster connection-making between concepts and procedures or between various strategies (NCTM, 2000). Ontario case study participants claimed they believed in setting appropriate goals for and use of representations in the classrooms. Specifically, Ontario teachers expressed that teachers must be able to make connections and show relationships among mathematical ideas in order to devise appropriate problem-solving strategies (Kajander & Boland, 2014). Moyer (2001) found that some of the teachers in her study believed that using representations such as manipulatives is "playing and not working" (p. 188). This was not the case in my research, where at least some of the teachers used representations for the purposes of connecting math ideas and showing relationships, and all claimed to believe in their value.

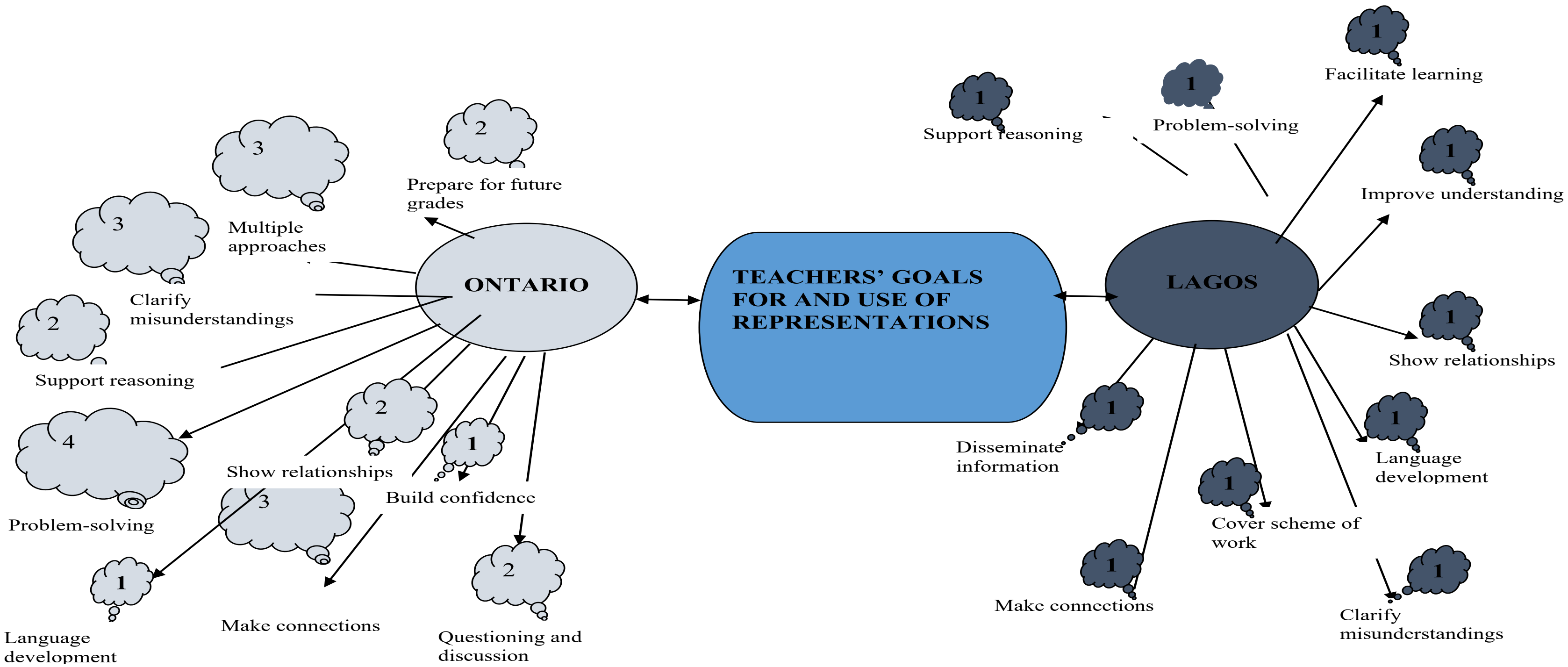
My research supported the idea that some teachers believed that visual representations enable students to make connections between their own experiences and mathematical concepts (Post & Cramer, 1989). However, in the case of Lagos teachers, a lack of well articulated goals relating to making connections, showing relationships and supporting students' reasoning inhibited their goal setting towards more helpful use of representations.

In summary, across the two jurisdictions, all ten teachers claimed they attempted to use representations to support students' learning. All of the teachers felt their goals for and use of representation were to support learning opportunities; however, the teachers would need to be observed in the classroom in order to ascertain if the goals they discussed during the interviews actually match with their practices. This is acknowledged as a limitation of the study. All of the teachers who participated in my study showed evidence that they may use representations to do at least some of: solve a mathematics problem in multiple ways, make connections, show relationships, facilitate learning, support students' language development and reasoning and clarify misunderstandings in their teaching (Figure 8).

Teachers from the Ontario subsample tended to describe more reform-oriented (NCTM, 2000) goals for and uses of representation when compared to their colleagues from Lagos. As mentioned previously, this finding may be related to the depth of teachers' specialized content knowledge. In the next section, participants' mathematical content knowledge is further compared and discussed.

Figure 8:

Comparison of key teachers' goals for and use of representations in Ontario and Lagos



7.4 Comparison of the Teachers' Beliefs and Knowledge: Mathematics, Students, and Pedagogy

In this section, I will discuss the findings from this study, particularly the results illustrated in Table 21 and Table 22, and compare them with other research.

7.4.1 Teachers' Beliefs and Knowledge: Mathematics

I wanted to know from the teachers how important they felt it was for a teacher to have sufficient specialized content knowledge in order to effectively use representations in teaching patterning and algebra. The majority of the teachers believed that specialized content knowledge is a prerequisite to teach patterning and algebra using representations.

Cross-case analysis revealed three common themes in the interviews, which will be discussed next:

- the need for specialized content knowledge in teaching;
- teachers' level of specialized content knowledge; and
- representations in the curriculum.

7.4.1.1 Specialized Content Knowledge in Teaching

In the interviews, the five Ontario teachers maintained that having SCK was critical, especially as it helps teachers to link representation to the underlying ideas and other representations (Ball, Thames, & Phelps 2008; Silva & Thompson, 2008). For example, Sara, Scott and Sonia each argued that a teacher needs "profound", "in-depth" or "comprehensive" content. This finding is consistent with that of Ball et al. (2008), who suggested that SCK is important knowledge for all teachers to have and, that SCK plays an important role in knowing "how to choose, make and use mathematical representations effectively" (p. 400).

Similarly, perhaps more surprisingly were the Lagos teachers, who all agreed that SCK was necessary despite not showing evidence of it. On the basis of Ball et al.'s (2008) spectrum of

SCK, ranging from how to explain, to justifying one's mathematical ideas, to making and using mathematical representations effectively, the majority of the Lagos teachers recognized the need for SCK. For example, Ben felt that SCK is critical while Beth believed that content knowledge is very important in order to effectively use representations in the classrooms. Overall, despite the majority of the Lagos teachers demonstrating weak mathematics content knowledge, they all appeared to be promoting the need for a teacher to have a strong SCK in order to teach with representations. As Silverman and Thompson (2008) remind us, the mathematical demands of teaching do require specialized content knowledge.

7.4.1.2 Teachers' Level of Specialized Content Knowledge.

Due to the differences in the teachers' beliefs and knowledge, it was not surprising that there were some distinguishable differences in the level of SCK between the two jurisdictions, but evidence of strong SCK was not generally found even in Ontario. In fact, of the ten case study teachers, only Sara from Ontario demonstrated strong SCK in her response to the second scenario question. The other four Ontario teachers (Scott, Silva, Susan, and Sonia) stated that they understood how representation might be used for problem-solving, and also claimed to have sufficient mathematical knowledge of patterning and algebra but did not demonstrate this during the interview. For example, Susan and Silva were not sure how to approach the second scenario task, while Scott mentioned traditional tables and graphs as the only representations that could be used to solve the task. Sonia claimed a comfort level but had no comment about the student solutions presented. This finding is consistent with Baumert et al. (2010) and Silva and Thompson (2008) who found evidence of teachers' struggling with the specialized knowledge of mathematics.

None of the five Lagos teachers, despite being subject specialists, were able to respond proficiently to the second scenario question or demonstrate specialized content knowledge to make connections between multiple representations. For instance, Bola, Bryce, and Baker did not have a grasp of the second scenario question at all. These three teachers demonstrated weak overall specialized content knowledge (SCK). Bola's SCK was inconsistent, and he made several mistakes in his explanations to the scenario questions. Although Beth mentioned the use of concrete materials and traditional graphs, she could not explain how these representations may help students to unpack the mathematical ideas. The finding that the five Lagos teachers were unable to unpack the mathematical ideas and explore different representations contrasts with Mitchell et al. (2014) who reported that one of the two teachers in their study did not understand how to unpack the use of different representations.

7.4.1.3 Representations in the Curriculum

Studies (e.g., Handal & Herrington, 2003) have suggested that the gap between the goal of the curriculum and teachers' beliefs may cause the failure of curricular change in their classrooms. Apart from all of the necessary detail of the content standard requirements, the teacher may also need to take into consideration how they can integrate different representations in their teaching (NCTM, 2000). Consistent with the Ontario curriculum: Grades 1-8, Mathematics, three of the Ontario teachers (Scott, Sonia, and Sara) understood that teachers are supposed to facilitate students' use of algebraic representations to model and interpret mathematical ideas. Scott, Sonia, and Sara expressed that teachers should represent the concepts that are stated in the curriculum expectations with different representations.

Although Scott and Sonia appeared to have insufficient knowledge of SCK, they understood the curriculum and how mathematical ideas are linked so that students' understanding can be deepened. For example, Scott felt that students should be able to apply their knowledge of patterns to do practical calculations. Sonia felt that there was a need for more emphasis on conceptual understandings in the curriculum document. Susan believed that the teacher should use multiple representations to explore mathematical ideas as contained in the curriculum, but she was concerned about how to meet the curriculum expectations.

Conversely, except Ben, the other four Lagos teachers did not demonstrate sufficient knowledge of the curriculum, the mathematics itself, or how to integrate different representations in their teaching. Consistent with the Lagos mathematics curriculum, Ben however appeared to have some understanding that teachers were supposed to facilitate students developing in-depth conceptual understanding of mathematical relationships and procedures. However, Ben felt there could be time issues for covering the curriculum if they were always using representations. Ben also felt that the use of representation is not clearly explained in the curriculum.

Overall, the findings indicated the need for teachers' specialized content knowledge to be much better developed. This finding may also have impacted how teachers interpreted other questions in the study, in that it is difficult to speak in depth about something one doesn't fully understand. A discussion of the ten teachers' beliefs and knowledge concerning students and pedagogy follows.

7.4.2 Teachers' Beliefs and Knowledge: Students and Pedagogy

Studies (e.g., Beswick, 2012; Bruce & Ross, 2008; Hackling, Ramseger, & Chen, 2017) have indicated that appropriate teachers' beliefs and depth of knowledge are required in order to teach mathematics well (Ma, 1999). The cross-case analysis revealed that there were five categories of beliefs and knowledge relating to students and pedagogy talked about by the teachers in both jurisdictions, which were.

- Planning and sequencing instruction,
- Use of contextual learning tasks,
- Opportunities for students to generate their own representations,
- Linking students' prior knowledge to new situations, and

- Translation among multiple representations.

7.4.2.1 Planning and Sequencing Instruction

Four (Silva, Susan, Sonia, and Sara) of the five Ontario teachers believed that teachers need to plan their lessons around multiple representations before using them. My finding is consistent with Pajares (1992) who found a strong relationship between teachers' education beliefs and their planning. Sara had a passion for algebra and mathematics teaching and was committed to her planning time. Silva reported that she researched on her own or consulted colleagues about how to use certain representations. In the interview, Susan explained that she planned her lessons based on her own creativity and by collaborating with other teachers. Only Sonia said she based her planning on the Ontario Ministry of Education's backward planning, diagnostic assessment, and the use of big ideas in order to use representations effectively.

In contrast, only Ben of the five Lagos teachers appeared to explicitly plan and employed a range of strategies that work for him in order to scaffold, support and create opportunities for students to use representations. My finding that only one Lagos teacher believed planning should be considered is consistent with the finding of Ladele (2013) who found that one of the four teachers in her study believed that teachers teaching with representations should do some form of planning. Only Ben stated that he was committed to a high level of planning and used various representations to supplement his lessons.

Research suggests that teachers should have the ability to sequence mathematics content to facilitate student learning (Ball, 1990; Hill et al. 2008). It is difficult to comment on the teachers' ability to sequence their use of various representations as the teachers were not observed in the classroom. However, the majority of the Ontario teachers explained that their sequencing approach involved supporting students' developing conceptual understandings when finding the solution to any given task. In the interviews, four of the five teachers claimed that teachers should sequence tasks efficiently to enable the students to progress in their cumulative understanding of a particular idea, and said they introduced their students to a variety of situations for them to practice any newly acquired skills. Sonia said that teachers should understand how sequencing learning works in order not to confuse the students.

In contrast, as far as the logical flow of the information is concerned, only Ben appeared to know how he could sequence the content in a lesson. Ben stated that he purposely planned his lessons sequentially building upon learning from one lesson to the next, and that he also sequenced activities within lessons.

There appears to be a gap in the literature in regard to the need to and how to sequence activities presented with representations, as no comparative studies were identified.

7.4.2.2 Use of Contextual Learning Tasks

The majority of the Ontario teachers believed that using appropriate contextual learning tasks engage students in applying representations to solve any given task. My finding is consistent with the research results obtained by Venkat (2010) who found that the teacher in her study used contextual tasks and discussions not just for the students to solve any given task, but also to reflect on their answers, explain, and justify their arguments. Sara and Sonia believed that, when students are given a complex learning task, they are required to apply their conceptual knowledge and deeper thinking. Sara mentioned that when students are struggling with a contextual task, she modelled the processes and skills that students lacked when handling such tasks, while also being careful not to give them all the steps involved. Sonia believed that teachers should put concepts in context for proper application in different situations. Sara and Scott noted that teachers should support students' thinking and reasoning through the type of learning tasks they choose.

Similarly, the majority of the Lagos teachers believed that teachers should use contextual learning tasks to inspire the students, and to engage them in the use of representations, and explain their answers. Moyer (2001), for example, found evidence of students being engaged in lessons because they enjoyed contextual tasks that are of interest to them. Bola stated that he used contextual learning tasks to motivate the students to reflect on their solutions and to explain their thinking. Similarly, Bryce said that using real-life contextual tasks that students can relate to was very useful in his teaching as he believed these forms of tasks would help to explore different representations as well as improve students' understanding. Baker, however, found that real-life problems were the most difficult tasks for students.

7.4.2.3 Opportunities for Student to Generate their own Representations

All five teachers in Ontario believed that teachers should allow students to generate their own representations. Sara also mentioned that students should be guided with the right prompts and questions in order for them to do so, which is consistent with the research results obtained by Terwel, Van Oers, Van Dijk, and Van Eeden (2009). Only Scott said that students should first be comfortable accessing the language before they are able to generate their own representations. This finding is consistent with Yackel, Cobb, and Wood (1991) who found that using the appropriate language helps students to develop their own mathematical understandings.

In contrast, Ben, Baker, and Bryce of the Lagos teachers said that students should be encouraged to generate their own representations but did not know how to go about it. Bryce mentioned this would help with understanding. However, Ben said that, due to time issues for using representations, he would rather provide representations for his students or use procedures. This finding is consistent with Stylianou (2010) where six of the 18 teachers said that the curriculum workload would make teaching students about representation an added burden.

7.4.2.4 Linking Students' Prior Knowledge to New Situations

In my study, the majority of the Ontario teachers believed that teachers should use scaffolding to make connections between students' prior knowledge and new situations. Their beliefs aligned with Myhill and Brackley's (2004) finding that teachers should consider students' prior knowledge to facilitate new content. Scott believed that teachers should build on students' existing knowledge in order to avoid a conflict between prior student learning and the teacher's intention. Sonia and Sara also mentioned that they believed learning should start with prior knowledge for students to gain understanding. Sonia added that she tested students' prior knowledge before introducing them to new content.

Similarly, all five Lagos teachers believed that teachers ought to assess their students' prior knowledge before starting to introduce new content, as advocated by Ferguson (2012). The majority of the Lagos teachers did not explain further; however, Ben did so saying he believed that this is necessary because new content can only be understood if it is linked to students' prior knowledge. Ben reported that he considered students' prior knowledge in order to determine what or what not to teach. As such, he got to know what students had done the previous year to gauge his expectations.

7.4.2.5 Translations Among Different Representations

The majority of the teachers believed that translations among multiple representations is important, but not all of the teachers were able to support their claim with examples. Only Sara, one of the Ontario teachers mentioned that translations among multiple representations around the same concept allowed her to present mathematical concepts in multiple ways, which, in turn, improve students' conceptual understanding. Sara's view strongly conforms with NCTM's (2000) position in their national standards document that stated that teachers should "translate among mathematical representations to solve problems" (p. 67). Sonia believed that, if students prefer certain representations, they should be allowed to work with such representations until they are ready to move into the next. Scott said that translations would help students to make connections between mathematical representations by encouraging them to move from concrete to abstract.

In contrast, Beth, Bola, and Bryce said that there were difficulties with translating among different representations. Beth said that translations within or between representations would cause confusion for her students. Bola said that translations among multiple representations could even hinder student learning particularly for the students with learning disabilities. Bola's view aligns with the views of the teachers in the van Garderen, Scheuermann, and Poch (2019) study, who felt that students with learning disabilities having difficulty in using multiple representations in mathematical problem solving. It should, however, be mentioned that Bola did not believe in multiple representations, and also saw representation as a topic of study rather than a means of understanding mathematics. This finding is similar to Stylianou (2010) who found that for about six of the 18 teachers in her study, representation seemed to be a topic of study.

7.4.3 Discussion of Beliefs and Knowledge: Mathematics, Students and Pedagogy

Researchers argue that teacher beliefs and knowledge are both driving forces behind their pedagogical approach, and strongly determine the way a teacher uses representation to support student understanding (Philip, 2007; Silvermann & Thompson, 2008). The majority of the teachers described here held similar beliefs, and that provided insights for their pedagogies, which appeared to support their claim about using multiple representations. However, most of the Lagos teachers were very weak in mathematics content knowledge compared with the Ontario teachers and this was reflected in their constraints about making connections amongst multiple representations.

A key finding of this study is that the beliefs of all ten teachers aligned with the tenets of mathematical knowledge for teaching: personal understanding, capacity to reflect on students' thinking, an understanding of multiple instructional sequences, and teachers' ability to think outside their own initial cognition (Silverman & Thompson, 2008). For example, Sonia, one of the Ontario teachers, said teachers must "understand as a professional first before you take it to the classroom". Silverman and Thompson (2008) state that teachers with a strong SCK would be able to model in a variety of ways students may understand content. In the same way, Kajander et al. (2010) state that there is need for teachers to have "other" mathematical understandings, which require "facility with appropriate mathematical models, alternate approaches to concepts and ways of thinking and reasoning conducive to students" (p. 50). Teachers require strategic knowledge and skills to choose the most appropriate representations for each situation (Uesaka & Manalo, 2006).

Strategic knowledge helps a teacher to determine when and why certain representations are more appropriate than others in particular occasions (Uesaka & Manalo, 2006). During the cross-case analysis it also became evident that only one teacher from Ontario demonstrated strong SCK while there was none among the Lagos teachers. That the quality of many teachers' SCK in my study is significantly low (Ball, Hill & Bass, 2005; Ma, 1999) should not be a surprise. Researchers (e.g., Ball, Hill & Bass, 2005) believe that most teachers are graduates of a system that needs improvement.

Most Lagos case study teachers could not talk much about the curriculum, partly because they have no access to the curriculum document (Ladele, 2013), and where they have access to it, the teachers hardly followed the content. Teachers either rely on the scheme of work as a guide or sometimes use the syllabus. A scheme of work is the interpretation and breakdown of the syllabus, which teachers believed would help them in preparing students for examinations.

That is a common problem in the implementation of the Lagos curriculum reform. This is not surprising, as prior research reported the impact of this problem, namely that Grade 8 students had not mastered up to a third of the curriculum content (Arisekola, 2010). Therefore, it would be difficult for the Lagos school systems to maximize the curriculum reform intended, due to a lack of access to the curriculum, and the weak SCK found among the teachers.

As mentioned previously, SCK is required for teachers to articulate and arrive at the mathematics as intended by the curriculum, but most of the teachers appeared weak in SCK. In contrast, the Ontario curriculum document is available online for teachers to access. However, Susan and Sonia believed that there were hindrances to overcome in the mathematics curriculum when using representations, even though the use of representations might improve students' understandings. A curriculum is more than a collection of activities but rather must be coherent, focused on important mathematical ideas, and well articulated across the grade levels (NCTM, 2000). Charalambous and Phillippou (2010) pointed out that teachers play a vital role in implementing reform in the curriculum.

Another strong belief held by the majority of the teachers is the need for selecting an appropriate contextual learning task. As Beswick (2010) reminded us, facilitating mathematical ideas in contexts would help students' understanding of mathematical procedures and abstract ideas. For example, Sara mentioned that, if content were taught in a context student could relate to, their conceptual understanding will be improved. When students are given contextual learning tasks, it "provides opportunities for learners to connect their knowledge to new information and to build on their knowledge and interest through active engagement in meaningful problem solving" (Artzt et al., 2008, p. 10).

Bola, one of the Lagos teachers, believed tasks should be relevant to what students can relate to in the environment. Boaler and Humphreys (2005) argue that contextual learning tasks support multiple voices, disagreements, challenges and support mathematical reasoning, when used appropriately. However, Brodie (2010) argues that choosing appropriate contextual tasks is necessary but not sufficient to support a learner to develop reasoning. Boaler (2014) opined that tasks need to give students the space to learn. Boaler said, "when students are just there to answer questions that are right or wrong it is very difficult to develop a learning orientation towards math" (video). Ferguson (2012) suggested that tasks should be adjusted in some way, to meet the needs of students struggling unproductively. She further stated that teachers should use appropriate prompts and consider students' prior knowledge before adjusting tasks. Nistal et al. (2009) said that contextual tasks can encourage students to not only switch strategies but also representations.

Some of the teachers also emphasized the need to consider students' prior knowledge in order to facilitate the assimilation of new content knowledge. Accessing prior knowledge is fundamental stage in the learning process (Christen & Murphy, 1991). Bransford, Brown and Cocking (2000) stated that linking students' prior knowledge creates an opportunity for "organizing information into a conceptual framework allows for greater transfer" (p. 17). Ontario case study teachers claimed to approach new concepts and content by building on students' foundation. Research suggests that the comprehension of new information can only be understood in relation to prior knowledge (Myhill & Brackley, 2004) as any attempt of teachers to ignore students' prior knowledge may result in the student learning information in conflict with the teachers' intention (Ambrose & Lovett, 2014). Research even suggests that students with low prior knowledge have problems with effective coordination and integration of multiple representations when solving problems (Ainsworth, 1999).

The teachers also discussed why students must be given the opportunity to generate their own representations. Teachers' ways of generating representations may either hinder or supports students' conceptual understanding of abstract algebra (Kaenders & Weiss, 2018). Ontario case study teachers believed that students' personal experience with representations would often influence their choice of representations (Uesaka & Manalo, 2006). Sonia mentioned that, if a teacher must suggest any representations in class, s/he should have tried it before attempting to use it with the students.

Ontario case study teachers believed that, if teachers employ plain language, unambiguous representations, as well as use precise and unbiased mathematical vocabulary, students' understanding would be improved (Ketterlin-Geller, Shivraj, Basaraba & Schielack, 2019). However, research also shows that, when students are given a choice among representations, they find it difficult to select the most appropriate representation for each occasion (Uesaka & Manalo, 2006). Bola, one of the Lagos teachers, believed that students' inability to choose among representations is due to teachers' inappropriate use of representations and that might hinder student's mathematical performance. As such, it may be that teachers need to be skilled in guiding students at making choices among representations. However, Boaler (2014) argued that students should be encouraged to think and develop mathematical models themselves.

Teachers from the Ontario subsample believed that translation between representations may help in improving students' conceptual understanding. Sara stated that it was necessary for students to understand a mathematical concept in multiple ways in order to enhance their level of thinking. Representation is a tool for gaining conceptual understanding in mathematics (Tackie, Sheppard, & Flint, 2019). Each representation can transform into another model of representation (Cathcart et al., 2006). However, one of the Lagos case study teachers said that using translation between representations was beyond her students' level of understanding. As such, she often chooses to use a traditional algorithm, which would develop only what Skemp (1978) calls instrumental understanding.

This was why Kaput (2000) lamented that, in some classrooms, teachers encouraged the use of rules and procedures instead of students' thinking and understanding. However, Stylianou (2010) pointed out that, when teachers have only a little understanding of representation, their vision of the use of representation can be little as well, and that was noticeable among the Lagos teachers. It has been argued that translations between representations in a flexible manner has the potential of making learning of mathematics more meaningful and effective (NCTM, 2000), but this was not evidenced in the responses of all the teachers, particularly the Lagos ones.

As far as the approach required for teachers to use representations was concerned, the teachers believed that all mathematical concepts must be thoroughly presented to sequence learning as it helps in building on different math concepts. Ontario case study teachers stated that adequate planning and effective teaching strategies were needed to deliver and communicate mathematical concepts effectively.

In summary, student success in learning algebra rests on what and how teachers teach the concepts (Kaput, 2008). Overall, all of the teachers agreed that SCK is a prerequisite in teaching with representations. SCK appeared generally weak for most teachers from both the Ontario and the Lagos subsamples. None of the Lagos case study teachers could successfully handle the second scenario question. It is important for teachers to not only have SCK of individual topics, but also connect their understanding of those topics (Wieman & Arbaugh, 2014).

Mitchell et al. (2014) argue that "strong and deep understanding is required to help students understand mathematical procedures and their underlying mathematical ideas, but also to gradually steer students to an abstract mathematical generalization" (p. 53). Teachers needed to prepare in advance and sequence their lessons. All ten teachers claimed that using contextual learning tasks should be related to students' experience (Boaler, 1998).

As a final word in this chapter, the findings from the interviews indicate that Ontario case study teachers' goals, beliefs, and knowledge may offer greater learning opportunities for students in algebra than those described by the Lagos case study teachers. These findings could explain why teachers from Ontario tended to describe more reform oriented (NCTM, 2000) approaches about beliefs and knowledge of representations when compared with their counterparts from Lagos in mathematics content, students, and pedagogy.

Given the findings of this study, it would seem that students whose teachers' instructional practices align with the NCTM's (2000) *Standards* are more likely to demonstrate a better conceptual understanding of patterning and algebra than their counterparts whose classroom instructional experiences follow a traditional approach. It would be worthwhile for both prospective and practising teachers who seek to improve their instructional practices and enhance their students' conceptual understanding of algebra to integrate representations in their teaching.

CHAPTER EIGHT: SUMMARY, IMPLICATIONS, AND CONCLUSIONS

8.1 Introduction

This study utilized a concurrent mixed methods design to explore how Grade 8 teachers in Ontario, Canada and Lagos, Nigeria generate and provide representations when teaching patterning and algebra. In this study, representation is defined as a variety of forms, including pictures (e.g., drawings, charts, graphs), written symbols (e.g., numbers, equations, words), manipulative models, oral language, real-world situations (Van de Walle, 2004), and images on computers or calculators. It is also the process of generating these forms. In this chapter, a summary of the findings is presented. Then implications of these findings are presented followed by an outline of limitations. Finally, possible areas for future research are identified.

8.2 Research Question: What are teachers' goals for and perspectives of using representations in Ontario and Lagos?

The purpose of this question was to address the goals and perspectives of teachers, and the MTMRI was specially designed to realize this goal. However, the findings from the close-ended responses of the MTMRI alone could only provide a part of the general picture of teachers' perspectives of representations. Hence, the open-ended question complements the results. The findings concerning this research question were presented and discussed in Chapter 5, and are briefly summarized here.

The findings from the online survey revealed that the participants in the study, all middle school teachers, had a generally positive perception of using representations when teaching algebra. The majority of the participants perceived that representations could help students draw on conceptual understandings and improve problem solving skills. Some of the teachers, particularly those from Ontario, were open to the use of representations in learning and believed in their value while, some of the Lagos teachers viewed representations more as a topic of study rather than as a general learning process, which is consistent with the findings of Stylianou (2010).

The findings further suggested that teachers generally believed that using representations was critical to highlight important mathematical ideas. However, there were mixed responses as to whether the use of representations makes patterning and algebra a difficult strand to teach. About two-thirds of the teachers believed that representations are not hard to use in teaching while the rest said they found them difficult. Nonetheless, the majority of the teachers believed that representations could help in moving students from using concrete to abstract understandings. Some teachers believed that understanding enough about teaching-related mathematics is important to use representations in their teaching.

The majority of the Ontario teachers perceived SCK (specialized content knowledge) as a very important factor compared with the Lagos teachers who were less sure. Most Ontario teachers strongly believed that representations can be used to build understanding, improve problem solving, solve a problem in multiple ways, communicate, and make connections. While Lagos teachers were more comfortable with the use of symbols than other types of representations, Ontario teachers tended to refer more to models, including concrete models.

8.3 Research Question: How do teachers' goals for and perspectives of using representations differ by region?

In order to answer this question, the data collected from both the interviews and survey were analyzed. The majority of all of the teachers reported using representations as a means of achieving their instructional goals, which were focused on opportunities for students to show connections, relationships, and reasoning, supporting students' confidence in problem-solving and facilitation and opportunities for questioning and discussion. However, it should be mentioned that, while the teachers claimed they were focussed on using strategies and representations that would have a positive impact on their students' learning, the majority appeared to lack the knowledge to do so effectively.

During the interviews, most of the Ontario teachers often talked about using classroom manipulatives such as algebra tiles, pattern blocks, smart cubes, and fraction strips, among others, in their teaching. However, it was only Ben, who talked about using manipulatives such as fraction strips in his teaching. Not only did his knowledge influence his approach, but his experiences also influenced his knowledge regarding the use of representations. In other instances, however, teachers argued that teaching mathematics more abstractly helped them lighten their workload, cover the curriculum, and make up for time spent outside of the classrooms due to professional development.

The teachers from both jurisdictions in this study stated they explore a variety of representations in their classrooms. The choice of representations reflected each teacher's personal preference, knowledge, beliefs, and experiences. The traditional graphs and symbols appeared to be the commonly used representations by some of the teachers particularly the Lagos teachers. Other teachers referred to their knowledge and experiences and talked about how they implement various reform-based tools to achieve success in their classrooms. Each teacher reported using representations that were easily accessible to him/her while teaching algebra.

The majority of the Lagos teachers believed that representations could be useful in their teaching, but they do not have the necessary tools or knowledge to fully utilize them in their classrooms. As mentioned by some of the teachers during the interviews, most of the Ontario teachers have access to resources, but Lagos teachers have a history of poor resource availability. However, one of the five Ontario teachers believed that teachers did not have enough time to explore the available resources for their lessons. The Lagos teachers also pointed to a lack of preparation time as negatively affecting their ability to utilize multiple representations in the classrooms. Most of the Lagos teachers did not construct their own manipulatives, and rather preferred to use symbols, traditional graphs, diagrams, and pictures.

8.4 Implications

8.4.1 Implications for Practice

Previous studies (e.g., Sloan, 2010) suggest that the use of representations such as manipulatives gives a better understanding of mathematical concepts, improves students' confidence, and encourages positive attitudes toward mathematics. Many teachers in my study were only able to recognize representations in the form of symbols, graphs, and diagrams. The implication here is that these teachers need to gain a deeper understanding of what representation is and what it looks like in the classroom, also need to further develop the specialised content knowledge (SCK) related to patterning and algebra. In particular, the majority of the Lagos participants lacked an understanding of representation and exhibited a weak understanding of content.

As revealed in this study, finding time in an already overloaded curriculum can make it difficult for teachers to create an opportunity for students to explore different representations and this is particularly true in Lagos where there are even more high-level algebra requirements in Grade 8 such as factoring. Yet, teachers need to provide learners with activities for exploring representations and making connections to real-life situations. The National Council of Teachers of Mathematics (2000) strongly advocates for mathematics instruction in which the teacher at all levels should support students in becoming fluent users of representations, and support their problem solving through the use of a variety of representations. This is particularly the case in Lagos, where the curricular expectations require students to develop mathematical proficiency and understanding of the mathematical processes, yet the formal and overloaded Grade 8 algebra content, with its focus on a procedural approach, may make teachers feel they are forced to teach abstractly. Hence, there is a need, particularly in Lagos to reduce the Grade 8 algebra content and encourage pedagogy to be more supportive of conceptual understanding than procedural skill, memorization, and regurgitation.

Following the launch of a new curriculum in Ontario (summer 2020), attention should be paid to building capacity among classroom teachers to implement this curriculum by ensuring that appropriate teaching and learning resources and strategies, as well as relevant professional development are in place. It is hoped that my research findings will open up conversations among stakeholders to give adequate support to the in-service teachers.

8.4.2 Implications for Research

Given that this was a small exploratory study into teachers' perspectives of representations, the generalisability of the findings is limited. A possibility for future research would be to observe student learning and work and use the data to further examine and compare teachers' beliefs, knowledge, and practice.

This study was inspired by and utilizes representations, one of the mathematical learning processes (NCTM, 2000), currently acclaimed as one of the reform-based instructional approaches to teaching and learning algebra. I believe it would be beneficial to compare my findings to further research about teachers' perspectives of representations in order to create a more generalizable description of the situation.

Researchers (e.g., Izsak & Sherin, 2003; Stylianou, 2010) have suggested that it is not only students who experience difficulties with representations, but also teachers may have gaps in their understanding of representations as revealed in this study. More research is needed in the area of algebra teaching using representations. As well, the MTMRI instrument should be tested on samples similar to the one used in this study to further verify its validity and reliability.

A cross-cultural study on mathematical learning processes involving middle school teachers from more countries may offer more insights on factors responsible for how teachers use representations in their teaching, and how well they understand them, as well as to explore student learning and support.

8.4.3 Implications for Teacher Professional Learning

One participant shared her concern that there was limited classroom time available to implement what was being learned at the various trainings. Opportunities should be given for teachers to reflect on their beliefs, knowledge, and practice about the teaching and learning of patterning and algebra using representations. The finding suggests that elementary in-service teachers in both locations need to be more informed about the importance and use of the mathematical learning processes, particularly representations. School boards should provide targeted and effective workshops that would focus on patterning and algebra, as well as other mathematics strands, for elementary in-service teachers particularly the beginning teachers, while allowing for this extra time away from the classroom.

Data from this study need to be taken into account when planning for teacher development programs in Ontario, since targeted professional learning may have an impact on how teachers implement *The Ontario Curriculum, Grades 1 – 8: Mathematics—Curriculum Context, 2020*.

It would be helpful to raise awareness about the importance of using multiple representations, especially among Lagos teachers. The superficial knowledge about multiple representations and specialized content knowledge that all of the Lagos teachers seem to have themselves, needs to be addressed during professional development. In particular, this study agrees with Hill's (2010) suggestion that, providing in-service teachers with early professional learning that includes content knowledge and importance of representations benefits the teachers' practice and deepens their content knowledge.

8.4.4 Implications for Teacher Education

The study revealed that there were limitations to some of the teachers' knowledge and practice. This suggests also that their teacher preparation may have been inadequate. Although the study was conducted with in-service teachers, pre-service teachers are products of the existing school systems, and thus some of them may likely have the same superficial knowledge. Therefore, this study has implications for teacher education programs. Previous studies (e.g., Sloan, 2010) reported that the use of representations among pre-service teachers gives better understanding of concepts, improves confidence levels of students and produces positive attitudes towards mathematics. Therefore, teacher educators, through the use of many different teaching strategies, should support pre-service teachers to develop a deeper understanding of mathematics concepts.

With respect to Ontario, the study confirmed other research (Kajander & Holm, 2013) that suggests that enhanced elementary mathematics education is needed to deepen prospective teachers' conceptual mathematical understanding.

In Lagos, the quality of the mathematics teacher education may improve if pre-service teachers are taught more about algebra content and multiple representations. The study revealed that there were limitations to all of the teachers' specialized content knowledge and use of reform-based approaches despite being content specialists. Teacher education that provides learning opportunities about multiple representations and content knowledge has the potential of not only improving the pre-service teachers' understanding, but of also ensuring the correct conceptual knowledge being passed on to their future students. In particular, teacher education programmes across Lagos institutions need to re-design their content for more effective preparation. As well, teaching approaches in the schools need to be re-examined and better supported.

8.5 Limitations of the Study

The study represents perspectives of a group of middle school teachers from two jurisdictions, and as such is limited in terms of generalizability. The number of teachers used was small and sampling was limited to one or two school boards each, so the findings may not be generalizable beyond the school boards used in the study. However, there may exist a transferability of the methods to produce results for other strands of the mathematics curriculum, other contexts, or even other mathematical learning processes.

In addition, although many attempts were made to collect data from Grade 8 in-service teachers, particularly Ontario teachers, only 20 of the Ontario teachers completed the online survey. However, the focused questions and the in-depth data collected may somewhat mitigate this weakness.

Interviews and surveys involving teachers, perhaps with an equal number of participants from each jurisdiction, would provide more opportunity for comparison and a better understanding of teachers' use of representations and mathematics teaching practices.

Another limitation of the study was that the ten teachers that were recruited for the interview were very different in terms of preparation, with teachers from Ontario being generalists while the Lagos sample teachers were subject specialists. Be that as it may, the responses from the Lagos teachers did not put them at an advantage over the Ontario teachers regarding their level of content knowledge. In addition, I observed differences in the curriculum, with more patterning in the Ontario curriculum and much more formal algebra in the Lagos curriculum, so direct comparison of what the teachers were trying to teach was also not fully possible.

Getting to know more about the beliefs, knowledge and practices of in-service teachers who do have experience with reform-based curriculum may help in identifying interventions that could help teachers integrate mathematical learning processes, in particular the use of representations. Specifically, the beliefs that some of the teachers (mostly Ontario) developed through having experience with reform-based curriculum may have helped these teachers align their beliefs with the use of representations.

8.6 Conclusion

In summary, exploring different representations and creating an opportunity for students to use representations are essential for developing conceptual understanding (Hiebert & Carpenter, 1992). Mitchell et al. (2014) reported that teachers' effective use of representations is needed for this to happen. Given that a deep understanding of mathematical ideas reflects how representations are integrated into every lesson (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003), the superficial understanding of representations among all the Lagos teachers' was notable and should be addressed. Ben was the only Lagos teacher in my study who had a relatively strong knowledge of representations, but it should be noted that Ben had his postgraduate education from North America.

Teachers should be afforded the opportunity to enhance their SCK as it is believed that this specialized content knowledge would strongly influence the effectiveness of teachers' instructional practices. Examples of knowledge that might be emphasized during professional training include experiences with many different types of models and representations, and the associated reasoning. As well, it might include how to monitor learners' misconceptions, how to logically sequence tasks, and acknowledging learners ideas and ways of thinking in order to help them move from concrete to abstract representations.

Although initially, I set out to conduct research that would involve classroom observations, I realized that this would be a challenging task because of the nature of observing both teachers and students at two different locations. However, the scenario questions provided a bit of insight into the teachers' content knowledge. In future studies, a more explicit emphasis on teachers' understanding of representation would be helpful.

Ideally, teachers' instructional practices should be predominantly learner-centred including encouraging students to generate their own representations. Not just the Lagos teachers but some of the Ontario teachers too need to understand more about the use and importance of representations in order to create the opportunity for students to effectively and appropriately use them. For this to happen, teachers should be taught how to actually teach using constructivist perspectives, rather than simply claiming to be teaching in a constructivist manner. Teachers should be sensitized to the importance of ensuring that their instructional practices are consistent with their beliefs.

Teachers need to be made aware of the role of representations, and recognize the potential of a student generating appropriate representation during problem solving and making appropriate connections related to a topic in the curriculum. An awareness of the different roles of representation can ultimately improve teachers' and students' understandings. As the results of the current study have shown, teachers need to consider how their students encounter connections with the real world and the mathematics they experience in the classroom. In particular, it was found during the interviews that some of the Lagos teachers were not aware of some aspects of the curriculum.

The importance of mathematics education preparation, including professional development in response to curriculum reforms such as that advocating for the use of representations in the teaching and learning of patterning and algebra, cannot be understated

(Stylianou, 2011). Such programs need to provide in-service teachers with tools to recognize the importance of using representations in problem solving and developing conceptual understanding (Hiebert & Carpenter, 1992). The most significant overall finding and recommendation of my study is that mathematics teacher education is required to enhance teachers' understanding of representation, particularly among Lagos teachers. Only then will the full potential of representations be realised in teaching and learning.

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Appendix A:

Survey Instrument:
MATHEMATICS TEACHERS MULTIPLE REPRESENTATIONS INVENTORY (MTMRI)

Thank you for considering to participate in this study.

This survey is meant to collect information on your opinion about the use of multiple representations during the teaching of patterning and algebra at the intermediate level.

The survey will take approximately 10 minutes to complete.

Kindly tick (√) the appropriate box to answer the questions using a six-point scale as defined.

SA-strongly agree
A-agree
N-undecided
D-disagree
SD- strongly disagree
Don't know
Your responses will be treated with utmost confidentiality.

S/N		SA	A	U	D	SD	Don't know
1	The use of multiple representations is not clearly explained in the curriculum.						
2	Providing representations to support reasoning is something I will often do to explain difficult concepts to students.						
3	The use of representations is not particularly useful in teaching and learning patterning and algebra.						
4	Allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.						
5	Representations can be mental images.						
6	I would need to learn more about teaching-related mathematics before using representations in my teaching.						
7	Representations are usually not physically visible.						
8	One specific representation of a pattern may not be enough in a patterning and algebra lesson.						
9	Representations help in moving students from using concrete models to abstract representations.						
10	Representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.						
11	Representations are less effective when suggested to students by teachers, rather than being generated by students.						
12	Graphical representations are the most important kind to illustrate algebraic concepts.						
13	Knowing which representation to use is sometimes confusing to students.						
14	Encouraging students to use representations can improve their problem solving skills.						
15	Representations are hard to use in teaching.						
16	A specialized understanding of elementary mathematics is not necessarily needed on the part of the teacher in order to use representations effectively in teaching patterning and algebra.						
17	The use of manipulatives is only good for teaching patterning but not for algebra.						
18	Patterning and algebra is one of the more difficult strands to teach as a lot of representation is involved.						
19	Selecting a worthwhile task determines what representation to use.						

20	Appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings.						
21	Including a lot of representations within a lesson could add confusion for students.						
22	Teachers should flexibly shift among different representations as they are generated by students.						
23	It is necessary that teachers should assist in choosing appropriate representations for students.						
24	The effective use of representations requires a lot of planning.						

What do you really think representations mean?

Part B

Please write your name (or nickname) ______________________________
Please provide a personal (non-professional) email address______________________
How long have you been teaching elementary school students (that is, Grades 1 to 8)?
1 - 3 years 4 – 6years 7 - 10 years
More than 10 years

Appendix B:

Interview Questions:

Semi-structured interview schedule Research questions 2 & 32. What representations do teachers in Ontario, Canada and Lagos, Nigeria use when teaching patterning and algebra?

3. How do teachers use these representations during instruction?

These are likely questions to be asked as a follow up after the classroom observations

Teacher identity____
School____
Date____
Time____

Interview questions

1. How can you explain your experience with the use of representations during teaching?
2. What informs your use of representations during mathematics teaching?
3. How did you plan to approach algebraic lesson in order to bring the learners to understand the content and context? Will you give examples of how you generate representation for your students?
4. In what situations do you think you will need more than one representation in algebra lesson or problem necessary?
5. In what ways has providing representations to your learners help you to cover the key mathematics concepts in algebra and patterning?
6. How are teachers' use of representations related to their knowledge? Have you and your learners learnt anything new from using representations to teach and learn mathematics?
7. How do you think intervention training (e.g., professional development learning) might be helpful to teachers' use of representations in the mathematics classroom?
8. Do you think you need any further support to use representations? If so, what kind of support do you need?

Scenarios interview questions

1. Sam has x bananas and Codi has p bananas. Collin counts the number of bananas each of them have and finds they are the same. Sam said you write as x= p, but Codi said that x and p are different letters and so cannot be the same. What would you say to these students?

2. Olamide just arrived in Canada and needed a phone in order to communicate. Olamide met Tyler who visually display three plans and point out the advantages of each plan to Olamide.

- Plan A costs a basic fee of $29.95 per month and 10 cents per text message
- Plan B costs a basic fee of $90.20 per month and has unlimited text messages
- Plan C costs a basic fee of $49.95 per month and 5 cents per text message

- All plans offer unlimited calling
- Calling on nights and weekends are free
- Long distance calls are included

Olamide wants to know how to decide which plan will save him most money. Your students were to determine which plan has the lowest cost given the number of text messages Olamide is likely to send. Present to Olamide by defining variables, writing equations, making tables, constructing graphs, finding slopes and intercepts and finding points of intersection. Two students came with initial thoughts below

S1: $f(x) = 29.95x + 0.10y$, $f(x) = y + 90.20x$, $f(x) = 49.95x + 0.05y$ adding extra variable each to the equations and incorrect graph

S2: $x = 29.85 + 10y$, $x = 90.20$, $x = 49.95 + 5y$, inverting the x and y axes.

What would you say to these students?

Appendix C:

Results of Original Version of the Survey

	Percent of teachers						
	Strongly agree	Agree	Undecided	Disagree	Strongly disagree	Don't know	Mean
Goals of teaching with representations							
Providing representations to support reasoning is something I will often do to explain difficult concepts to students.	56.0	41.8	1.1	0.0	1.1	0.	4.52
Appropriate representations should be used to highlight important mathematical ideas during classroom discussions in order to clarify misunderstandings.	36.3	62.6	1.1	0.0	0.0	0.0	4.35
It is necessary that teachers should assist in choosing appropriate representations for students. (R)	29.7	59.3	3.3	7.7	0.0	0.0	4.11
Knowledge regarding content							
A specialized understanding of elementary mathematics is not necessarily needed on the part of the teacher in order to use representations effectively in teaching patterning and algebra. (R)	16.5	60.4	3.3	14.3	5.5	0.0	2.32
Knowledge regarding learners							
Representations can help students draw on their conceptual understandings to solve new and unfamiliar problems.	50.5	45.1	3.3	0.0	0.0	0.0	4.48
Representations are less effective when suggested to students by teachers, rather than being generated by students.	12.1	30.8	19.8	30.8	4.4	1.1	3.12
Knowing which representation to use is sometimes confusing to students. (R)	1.1	4.4	5.5	65.9	23.1	0.0	4.06
Encouraging students to use representations can improve their problem-solving skills.	63.7	33.0	2.2	0.0	0.0	1.1	4.57
Including a lot of representations within a lesson could add confusion for students. (R)	1.1	6.6	3.3	62.6	26.4	0.0	4.07
Knowledge regarding teaching							
I would need to learn more about teaching-related mathematics before using representations in my teaching. (R)	2.2	16.5	5.5	61.5	14.3	0.0	3.69
Graphical representations are the most important kind to illustrate algebraic concepts. (R)	3.3	28.6	18.7	42.9	3.3	2.2	3.10
Representations are hard to use in teaching. (R)	7.7	54.9	15.4	22.0	-	0.0	2.52
The use of manipulatives is only good for teaching patterning but not for algebra. (R)	7.7	25.3	15.4	40.7	11.0	0.0	2.78
Patterning and Algebra is one of the more difficult strands to teach as a lot of representation is involved. (R)	9.9	32.5	6.6	47.3		1.1	2.89
Beliefs regarding content							
The use of multiple representations is not clearly explained in the curriculum. (R)	20.9	69.2	6.6		1.1	-	1.89
Representations can be mental images.	22.0	65.9	3.3	7.7	0.0	1.1	3.99
Representations are usually not physically visible. (R)	14.3	44.0	3.3	29.7	7.7	1.1	3.24
Beliefs about learners							
Allowing students to generate their own representations is an excellent way to develop student understanding of patterning and algebra.	45.1	48.4	6.6	0.0	0.0	0.0	4.39
Representations help in moving students from using concrete models to abstract representations.	54.9	39.6	4.4	0.0	1.1	0.0	4.47
Beliefs about teaching							
The use of representations is not particularly useful in teaching and learning patterning and algebra. (R)	28.6	42.9	5.5	17.6	5.5	0.0	2.29
One specific representation of a pattern may not be enough in a patterning and algebra lesson.	18.7	67.0	7.7	5.5	1.1	0.0	3.97
Selecting a worthwhile task determines what representation to use.	20.9	70.3	4.4	3.3	0.0	0.0	4.10
Teachers should flexibly shift among different representations as they are generated by students.	22.0	68.1	6.6	3.3	0.0	0.0	4.09
The effective use of representations requires a lot of planning.	34.1	60.4	3.3	2.2	0.0	0.0	4.26

Weighted average 3.65

Note: Confidence was scored on a 6 point scale Strongly agree=5; Agree=4; Undecided=3; Disagree=2; Strongly disagree=1; Don't know=0, R = reversed coded

Appendix D:

Research Study Ethics Approval

Lakehead UNIVERSITY

Research Ethics Board
t: (807) 343-8283
research@lakeheadu.ca

April 18, 2018

Principal Investigator: Dr. Elizabeth Kajander
Student: Mr. Jimmy Avoseh
Faculty of Education
Lakehead University
955 Oliver Road
Thunder Bay, ON P7B 5E1

Dear Dr. Kajander and Mr. Avoseh:

Re: REB Project #: 132 17-18 / Romeo File No: 1466286
Granting Agency: N/A
Agency Reference #:N/A

On behalf of the Research Ethics Board, I am pleased to grant ethical approval to your research project titled, "Engaging Multiple Representations and Modalities in Grade Eight: Exploring Mathematics Teachers' Perspectives and Instructional Practices in Algebra and Patterning".

Ethics approval is valid until April 18, 2019. Please submit a Request for Renewal to the Office of Research Services via the Romeo Research Portal by March 18, 2019 if your research involving human participants will continue for longer than one year. A Final Report must be submitted promptly upon completion of the project. Access the Romeo Research Portal by logging into myInfo at:

https://erpwp.lakeheadu.ca/

During the course of the study, any modifications to the protocol or forms must not be initiated without prior written approval from the REB. You must promptly notify the REB of any adverse events that may occur.

Best wishes for a successful research project.

Sincerely,

Dr. Kristin Burnett
A/Chair, Research Ethics Board

/sm

955 Oliver Road, Thunder Bay, ON, Canada, P7B 5E1 | lakeheadu.ca

Appendix E:

School Board Ethics Approval

SUPERIOR-GREENSTONE DISTRICT SCHOOL BOARD

P.O. Bag 'A', 12 Hemlo Drive
Marathon, Ontario P0T 2E0
Telephone: 807-229-0436 Fax: 807-229-1471
E-Mail: boardoffice@sgdsb.on.ca

May 8th, 2018

Ann Kajander
Associate Professor
Faculty of Education
955 Oliver Road
Lakehead University, Thunder Bay, ON
P7B 5E1

Dear Dr. Kajander,

On behalf of Superior-Greenstone District School Board I am pleased to grant ethical approval to your research project titled, "Engaging Multiple Representations and Modalities in Grade Eight: Exploring Mathematics Teachers' Perspectives and Instructional Practices in Algebra and Patterning".

Ethics approval is valid until May 8, 2019. If necessary please submit a request for renewal to Nicole Morden-Cormier nmorden-cormier@sgdsb.on.ca by April 8, 2019 if your research will continue for longer than one year.

We look forward to participating in this exciting research.

Best regards,

David Tamblyn

David Tamblyn
Director of Education

cc. Nicole Morden-Cormier
Kathleen Schram