A special case of the range of invariant problem for AF type actions of $\mathbb{Z}_2$

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2025

Abstract

Elliott and Su gave a classification of inductive limit type $\mathbb{Z}_2$ actions on AF algebras using a K-theoretic invariant. In this paper, we consider the range of invariant problem, and give a sufficient condition for an object to be one of their invariants for such a system. In addition, we defines some structures that will be useful for further investigation of this problem.

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Chapter 1

Introduction

1.1 C*-Algebras

To introduce everything, we need some basic definitions about C*-algebras. To introduce what a C*-algebra is, first we introduce what a Banach algebra is.

Definition 1.1.1. *An algebra is a vector space A together with a bilinear map $A \times A \to A$, written as $(a, b) \to ab$, such that $\forall a, b, c \in A$, $a(bc) = (ab)c$.*
A normed algebra is an algebra A equiped with a submulitiplicative norm, that is, $\forall a, b \in A$, $||ab|| \leq ||a||||b||$.
A subalgebra of an algebra A is a vector subspace which is closed under multiplication.
An ideal of an algebra A is a vector subspace I such that $AI \subseteq I$ and $IA \subseteq I$.
A complete normed algebra is called a Banach algebra. A complete unital normed algebra is called a unital Banach algebra.[11]

Next, we introduce C*-Algebras.

Definition 1.1.2. *An involution on an algebra is a conjugate linear map $a \mapsto a^*$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$.*
*A *-algebra is an algebra with an involution on it.*
*A Banach *-algebra is a *-algebra A together with a complete submulitiplicative norm such that $||a^*|| = ||a||$. A unital Banach *-algebra is a Banach *-algebra with a unit 1 such that $||1|| = 1$.*
A C-algebra is a complex Banach *-algebra such that $||a^*a|| = ||a||^2$. A unitial C*-algebra is a C*-algebra with a unit.*

A C^*-subalgebra of a C^*-algebra A is a closed subalgebra B of A such that $B^* = B$.
A C^*-algebra A is simple when 0 and A are its only closed ideals.
If $\phi : A \to B$ is a homomorphism of *-algebras A and B and $\phi(a^*) = (\phi(a))^*$, then we say ϕ is a *-homomorphism. [11]

Here are some examples of C*-algebras.

Example 1.1.3. *Matrix algebras $M_n(\mathbb{C})$ are C^*-algebras with * the Hilbert adjoint. Commutative C^*-algebras are of the form $C_0(X)$ for a locally compact Hausdorff space X.*

The following theorem characterizes the finite-dimensional C*-algebras.

Theorem 1.1.4. *If A is a non-zero finite-dimensional C^*-algebra, it is $*$-isomorphic to $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ for some integers $n_1, n_2, \ldots, n_k$. [11]*

There are two special classes of C*-algebras built from finite-dimensional C*-algebras that we will discuss a lot.

Definition 1.1.5. *A uniformly hyperfinite algebra, or UHF algebra, is a unitial C^*-algebra A that has an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite-dimensional simple C^*-subalgebras each containing the unit of A such that $\overline{\bigcup_{n=1}^{\infty} A_n} = A$. An approximately finite-dimensional, or AF, algebra is a C^*-algebra A that has an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite-dimensional C^*-subalgebras such that $\overline{\bigcup_{n=1}^{\infty} A_n} = A$. [11]*

1.2 K_0 functor

K-Theory plays an important role in this subject. There are a lot of different way to define the K_0 functor. We choose one convenient way to define it for the algebras in this thesis. First, we introduce the projections in a C*-algebra.

Definition 1.2.1. *An element $p \in A$ for a C^* algebra A is a projection if $p = p^* = p^2$. [11]*

Next, we introduce stable equivalence of projections.

Definition 1.2.2. *If p, q are projections in a C^*-algebra A, they are called Murray-v. Neumann equivalent, which is written as $p \sim q$, if there exists $u \in A$ such that $u^*u = p$ and $uu^* = q$. Write 1_n for the unit of $M_n(\tilde{A})$ where $\tilde{A}$ is the unitisation of A. We view $M_n(\tilde{A})$ as a subalgebra of $M_{n+1}(\tilde{A})$ included as the upper left corner. If there exists $n, k \in \mathbb{N}$ such that $1_n \oplus p \sim 1_n \oplus q$ in $M_k(\tilde{A})$, then p, q are called stably equivalent. For the equivalence classes, we difine the sum $[p] + [q] = [p \oplus q]$. [11]*

Next, we introduce the Grothendieck group.

Definition 1.2.3. *For a cancellative abelian semigroup i.e. one where $x + y = z + y \implies x = z$, N with a zero element, we define an equivalence relation $\sim$ on $N \times N$ by setting $(a, b) \sim (c, d)$ if $a+d = b+c$. Let $[a, b]$ be the equivalence class of (a, b). The Grothendieck group of N is defined to be the collection of equivalence classes $[a, b]$ under the operation $[a, b] + [c, d] = [a + c, b + d]$. [11]*

For a *-algebra, we have this:

Definition 1.2.4. *For a unital *-algebra A, denote by $K_0(A)^+$ the semigroup of all the stable equivalence classes of projections of A. This is a cancellative abelian semigroup, so we may define $K_0(A)$ to be the Grothendieck group of $K_0(A)^+$. [11]*

Example 1.2.5. *For a C^*-algebra $A = M_n(\mathbb{C})$, $K_0(A)^+ \cong \mathbb{Z}^+$, $K_0(A) \cong \mathbb{Z}$, where the $\cong$ is given by the trace.*

There is a very useful property:

Theorem 1.2.6. *For two C^*-algebra, A, B, $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$.*

1.3 Partially Ordered Abelian Groups

The main objects of this thesis are partially ordered abelian groups. First, we introduce what these are:

Definition 1.3.1. *A partial order of an abelian group G is called translation-invariant if given any $x, y, z \in G$ with $x \leq y$ it follows that $x + z \leq y + z$. A partially ordered abelian group is an abelian group equipped with a specified translation-invariant partial order. We call an element $x \in G$ positive when $x \geq 0$. The positive cone of a partially ordered abelian group G is the set G^+ of all positive elements of G. A positive homomorphism is one that maps positive elements to positive elements. An order-unit in a partially ordered abelian group G is a positive element $u \in G^+$ such that for any $x \in G$, there is some positive integer n for which $x \leq nu$. For two partially ordered abelian group G and H with order units u and v, a normalized positive homomorphism from (G, u) to (H, v) is any positive homomorphism $f : G \to H$ such that $f(u) = v$. [8]*

For a partially ordered abelian group G, if the partial order on it is a lattice, we call *it a lattice order abelian group. If $a, b \in G^+$ are such that $a \wedge b = 0$, we write $a \perp b$. For a partially ordered abelian group G, for all $a \in G$, we write $x = x^+ + x^-$, in which $x^+ = x \vee 0$ and $x^- = x \wedge 0$.*

For AF algebras, we have:

Example 1.3.2. *For an AF algebra A, $(K_0(A), K_0(A)^+, [1_A])$ is a partially ordered abelian group $K_0(A)$ with positive cone $K_0(A)^+$ and order unit $[1_A]$.*

Theorem 1.3.3. *Every unitial $*$-homomorphism $\alpha : A \to B$ for unitial AF algebras A and B induces a normalized positive homomorphism of partially order abelian groups with order units $\alpha_* : (K_0(A), K_0(A)^+, [1_A]) \to (K_0(B), K_0(B)^+, [1_A])$. [11]*

Here is a special kind of order:

Definition 1.3.4. *Let $\{G_i \mid i \in I\}$ be a nonempty collection of partially ordered abelian groups. There is a natural partial order, called the product order, on the abelian group $G = \prod_{i \in I} G_i$, in which $\forall x, y \in G$, $x \leq y$ if and only if $\forall i \in I$, $x_i \leq y_i$. [8]*

Then we introduce some properties of some partially ordered abelian groups:

Definition 1.3.5. *A partially ordered abelian group is said to have the interpolation property, if given $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leq y_j$ for all i, j, there exists z in G such that $x_i \leq z \leq y_j$ for all i, j. A directed abelian group is any partially ordered abelian group G which is upward directed. That is, $\forall a, b \in G, \exists c \in G$ such that $c \geq a$ and $c \geq b$. An unperforated abelian group is a partially ordered abelian group G such that for all positive interger n, $nx \geq 0 \to x \geq 0$. [8]*

1.4 Direct Limits

Direct limit is an idea connecting a lot of subjects together. We can express it in category theory. First, we introduce some basic notation of category:

Definition 1.4.1. *A category is a quadruple $\mathbf{A} = (\mathcal{O}, hom, id, \circ)$ consisting of*
(1) a class $\mathcal{O}$, whose members are called A-objects. The class $\mathcal{O}$ of A-objects is often denoted by $Ob(\mathbf{A})$,
(2) for each pair (A, B) of $\mathbf{A}$-objects, a set $hom(A, B)$ whose members are called $\mathbf{A}$-morphisms from A to B,

(3) for each $\mathbf{A}$*-object* A*, a morphism* $id_A \in hom(A, A)$ *called the* $\mathbf{A}$*-identity on* A*,*
(4) a composition law associating with each $\mathbf{A}$*-morphism* $f : A \to B$ *and each* $\mathbf{A}$*-morphism* $g : B \to C$ *an morphism* $g \circ f : A \to C$*, called the composition of* f *and* g*, subject to the following conditions:*
(a) composition is associative; i.e. for morphisms $f : A \to B$*,* $g : B \to C$*, and* $h : C \to D$*, the equation* $h \circ (g \circ f) = (h \circ g) \circ f$ *holds,*
(b) $\mathbf{A}$*-identities act as identities with respect to composition; i.e. for* $\mathbf{A}$*-morphisms* $f : A \to B$ *the equations* $id_B \cdot f = f = f \cdot id_A$ *hold,*
(c) the sets $hom(A, B)$ *are pairwise disjoint.[1]*

Example 1.4.2. *There are some examples we will mention:*
(a) Let $\mathcal{O}$ *be the class of all partially ordered abelian groups,* hom *be the class of positive homomorphisms,* id *be the class of identity map, and* $\circ$ *denote the composition of group homomorphisms. Then,* $(\mathcal{O}, hom, id, \circ)$ *forms a category.*
(b) Let $\mathcal{O}$ *be the class of all partially ordered abelian groups with units,* hom *be the class of normalized positive homomorphisms,* id *be the class of identity map, and* $\circ$ *denote the composition of group homomorphisms. Then,* $(\mathcal{O}, hom, id, \circ)$ *forms a category.*
(c) Let $\mathcal{O}$ *be the class of all* C^**-algebras,* hom *be the class of all *-homomorphisms,* id *be the class of the identity maps, and* $\circ$ *denote the composition of *-homomorphisms. Then,* $(\mathcal{O}, hom, id, \circ)$ *forms a category.*

Next, we have the definition of functor:

Definition 1.4.3. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are categories, then a (covariant) functor* F *from* $\mathbf{A}$ *to* $\mathbf{B}$ *is a map that assigns to each* $A \in Ob(\mathbf{A})$ *an* $F(A) \in Ob(\mathbf{B})$ *and to each* $f \in hom(A, B)$ *an* $F(f) \in hom(F(A), F(B))$*, so that the following conditions hold:*
(1) $F(g \circ f) = F(g) \circ F(f)$ *whenever* $g \circ f$ *is defined, and*
(2) $F(id_A) = id_{F(A)}$ *for all* $A \in Ob(\mathbf{A})$*.[1]*

Next, we have the definition of direct system:

Definition 1.4.4. *A direct system in a category* $\mathbf{C}$ *consists of an ordered pair* $\{M_i, \phi_{ij}\}$*, where* $(M_i)_{i \in I}$ *is a family of objects in* $\mathbf{C}$ *indexed by a partially ordered set* $(I, \leq)$ *and* $(\phi_{ij} : M_i \to M_j)_{i \leq j}$ *is a family of morphisms, such that* $\forall i \leq j \leq k \in I$*,* $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$*.[13]*

Finally, we have the definition of direct limit:

Definition 1.4.5. *Let* I *be a partially ordered set, and let* $\{M_i, \phi_{ij}\}$ *be a direct system over* I *in category* $\mathbf{C}$*. The direct limit is an object* $\varinjlim M_i$ *and a family of morphisms* $(\alpha_i : M_i \to$

$\varinjlim M_i)_{i\in I}$ *such that*
(i) $\alpha_j\phi_{ij} = \alpha_i$ *whenever* $i \leq j$.
(ii)for every object $X \in Ob(\mathbf{C})$ *having maps* $f_i : M_i \to X$ *satisfying* $f_j\phi_{ij} = f_i$ *for all* $i \leq j$, *there exists a unique map* $\theta : \varinjlim M_i \to X$ *making* $\theta \circ \alpha_i = f_i$ *for all* $i \in I$.*[13]*

Example 1.4.6. *Consider the categories of partially ordered abelian groups, partially ordered abelian groups with units, C*-algebras, or Elliott-Su systems (to be introduced shortly). In these cases, the direct limit always exists[7][8][11].*

Remark 1.4.7. *AF-algebra can be defined in another way: if* A *is a direct limit of a direct sequence* $\{A_n, \phi_{ij}\}$ *of C*-algebras, where the* A_n *are finite-dimensional, then* A *is an AF-algebra.[11]*

1.5 Classification and Dimension Groups

The following theorem describes the K_0 functor for non-zero matrix C*-algebras:

Theorem 1.5.1. *If* $A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus ... \oplus M_{n_k}(\mathbb{C})$ *for some integers* $n_1, n_2, ..., n_k$, *then the map* $\tau : K_0(A) \to \mathbb{Z}^k$ *given by traces is an order isomorphism, in which* $\mathbb{Z}^k$ *is equipped with product ordering, and* $(n_1, n_2, ..., n_k) = \tau([1]_A)$.*[11]*

Below, we will introduce several classification theorems. These theorems share a common point: they were all proved using a similar method, which we call Elliott's intertwining argument. The pattern of this argument is as follows. Given two objects A and B, which is given by a sequence of $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ and a kind of functor F that maps every $\cdot \mapsto \mathrm{Inv}(\cdot)$, if we have $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$ like the diagram below.

$$\begin{array}{ccccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & ... & \longrightarrow & A & \quad & \mathrm{Inv}(A) \\
 & & & & & & & & & & \Updownarrow\cong \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & ... & \longrightarrow & B & \quad & \mathrm{Inv}(B)
\end{array}$$

We can get:

$$\begin{array}{ccccccccc}
\mathrm{Inv}(A_1) & \longrightarrow & \mathrm{Inv}(A_2) & \longrightarrow & \mathrm{Inv}(A_3) & \longrightarrow & ... & \longrightarrow & \mathrm{Inv}(A) \\
 & & & & & & & & \Updownarrow\cong \\
\mathrm{Inv}(B_1) & \longrightarrow & \mathrm{Inv}(B_2) & \longrightarrow & \mathrm{Inv}(B_3) & \longrightarrow & ... & \longrightarrow & \mathrm{Inv}(B)
\end{array}$$

Then, we pull back the map from Inv(A) $\cong$ Inv(B) to a commuting diagram:

$$\begin{tikzcd}
\mathrm{Inv}(A_1) \arrow[r] \arrow[d, "\nu_1"] & \mathrm{Inv}(A_2) \arrow[r] \arrow[d, "\nu_2"] & \mathrm{Inv}(A_3) \arrow[r] \arrow[d, "\nu_3"] & \ldots \arrow[r] & \mathrm{Inv}(A) \arrow[d, shift right, "\nu"'] \\
\mathrm{Inv}(B_1) \arrow[r] \arrow[ur, "\mu_1"] & \mathrm{Inv}(B_2) \arrow[r] \arrow[ur, "\mu_2"] & \mathrm{Inv}(B_3) \arrow[r] \arrow[ur] & \ldots \arrow[r] & \mathrm{Inv}(B) \arrow[u, shift right, "\mu"']
\end{tikzcd}$$

Next, we prove two lemmas:

(1) (Existence lemma) For $C, D \in \{A_i\}_{i\in\mathbb{N}} \bigcup \{B_i\}_{i\in\mathbb{N}}$, if there is a morphism ϕ : Inv(C) $\mapsto$ Inv(D), we will have a morphism $\hat{\phi} : C \mapsto D$ such that $F(\hat{\phi}) = \phi$.

(2) (Uniqueness lemma) For $C, D \in \{A_i\}_{i\in\mathbb{N}} \bigcup \{B_i\}_{i\in\mathbb{N}}$, if there are two morphisms $\hat{\phi}_1, \hat{\phi}_2$: $C \mapsto D$ such that $F(\hat{\phi}_1) = F(\hat{\phi}_2)$, then $\exists u \in D$ such that $\hat{\phi}_2 = Ad(u) \circ \hat{\phi}_1$.

Then, we use the existence lemma to show there exists this diagram:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\hat{\nu}_1"] & A_2 \arrow[r] \arrow[d, "\hat{\nu}_2"] & A_3 \arrow[r] \arrow[d, "\hat{\nu}_3"] & \ldots \arrow[r] & A \arrow[d, shift right, "\hat{\nu}"'] \\
B_1 \arrow[r] \arrow[ur, "\hat{\mu}_1"] & B_2 \arrow[r] \arrow[ur, "\hat{\mu}_2"] & B_3 \arrow[r] \arrow[ur] & \ldots \arrow[r] & B \arrow[u, shift right, "\hat{\mu}"']
\end{tikzcd}$$

Finally, we use uniqueness lemma to modify this diagram above to make it commute. Then we can prove that these objects can be classified by the functor F.

In 1976, Elliott classified unitial AF-algebras by their K_0 groups, by expressing AF-algebras as direct limits of sequences of finite-dimensional C*-algebra.[6]:

Theorem 1.5.2. *Two unital AF-algebra A and B are isomorphic if and only if the triples $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$ are isomorphic.[12]*

In [6], Elliott called $(K_0(A), K_0(A)^+)$ for an AF-algebra A a dimension group. These are inductive limits of sequences of simplicial groups $\mathbb{Z}^n$. After some years, Effros, Handelman and Shen gave an axiomatisation of a more general class of groups, which are now called dimension groups[5]:

Theorem 1.5.3. *An ordered group G is a dimension group if and only if it is a Riesz group.[5]*

Then, we can define:

Definition 1.5.4. *A dimension group is a directed, unperforated, interpolation partially ordered abelian group.[8]*

What Elliott called dimension groups are now countable dimension groups. Effros, Handelman and Shen found that the range of the Elliott classification is the countable dimension groups.[5] After this, Handelman and Rossmann showed that for a UHF algebra $\bar{A}$, $(K_0(A^G), [1])$, as an ordered $K_0(G)$-module, classifies the product type actions up to stable conjugacy [9]:

Theorem 1.5.5. *Suppose that $(K_0(A^{G,\alpha}), [1]) \simeq (K_0(A^{G,\beta}), [1])$ as ordered $K_0(G)$-modules, where α and β are outer product type actions of the finite group G on the UHF algebra $\bar{A}$. Then α is stably conjugate to β.[9]*

Just one year later, Handelman and Rossmann showed that for an AF algebra A, locally representable actions α of a finite group G can be classified by $K_0(A \times_\alpha G)$ with a condition[10]:

Theorem 1.5.6. *For the AF algebra A, let $\alpha, \beta : G \to Aut(A)$ be two locally representable actions of the finite group G. Then α and β are stably conjugate if and only if there is an order isomorphism $\rho : K_0(A \times_\alpha G) \to K_0(A \times_\beta G)$ such that $\rho([A_\alpha]\chi_{reg}) = [A_\beta]\chi_{reg}$. [10]*

Chapter 2

The Elliott-Su Classification of AF type $\mathbb{Z}_2$ actions

After Handelman and Rossmann's works, Elliott and Su dropped the locally representable condition but restricted the group to $\mathbb{Z}_2$. They found that AF-type inductive limit actions of $\mathbb{Z}_2$ can be classified by an invariant consisting of: $(K_0(A), a_*)$, $(K_0(A \times_\alpha \mathbb{Z}_2), \hat{\alpha}_*)$, the map between them and a pair of special elements: the class of the unit and the class of the projection that comes from averaging the unitaries of $\mathbb{Z}_2$ in the cross product. We call this the Elliott-Su invariant:

Theorem 2.0.1. *Let $(A, \alpha, \mathbb{Z}_2) = \lim_\rightarrow(A_n, \alpha_n, \mathbb{Z}_2)$ and $(B, \beta, \mathbb{Z}_2) = \lim_\rightarrow(B_n, \beta_n, \mathbb{Z}_2)$ be two inductive limit C^*-dynamical systems, let F be an order-preserving group isomorphism from $(K_0(A), \alpha_*)$ to $(K_0(B), \beta_*)$ mapping $[1_A]$ to $[1_B]$, and let ϕ be an order-preserving group isomorphism from $(K_0(A \times_\alpha \mathbb{Z}_2), \hat{\alpha}_*)$ to $(K_0(B \times_\beta \mathbb{Z}_2), \hat{\beta}_*)$ mapping the special element to the special element. Suppose that the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \longrightarrow & K_0(A \times_\alpha \mathbb{Z}_2) \\ F \downarrow & & \downarrow \phi \\ K_0(B) & \longrightarrow & K_0(B \times_\beta \mathbb{Z}_2). \end{array}$$

Then there is an isomorphism ψ from (A, α) to (B, β) such that $\psi_* = F$ and such that the extension of a map from $A \times_\alpha \mathbb{Z}_2$ to $B \times_\beta \mathbb{Z}_2$ gives rise to the map ϕ.[7]

In theorem above, we can find the most important object that we research:

Definition 2.0.2. *For a C^*-dynamical systems $(A, \alpha, \mathbb{Z}_2)$,*

we call $\begin{tikzcd} K_0(A) \arrow[loop above, "\alpha_*"] \arrow[r, "\phi"] & K_0(A \times_\alpha \mathbb{Z}_2) \arrow[loop above, "\hat{\alpha}_*"] \end{tikzcd}$ *, along with* $[1]$ *and* $[(1+g)/2]_{A\times_\alpha \mathbb{Z}_2}$ *where* g *is the non-identity element of* $\mathbb{Z}_2$*, an Elliott-Su invariant.*

We call $\begin{tikzcd} K_0(A) \arrow[loop above, "\alpha_*"] \arrow[r, "\phi"] & K_0(A \times_\alpha \mathbb{Z}_2) \arrow[loop above, "\hat{\alpha}_*"] \end{tikzcd}$ *an Elliott-Su system.*

Given two dimension groups G *and* H*,* $\mathbb{Z}_2$ *actions* α *and* β *and equivariant homomorphisim* ϕ*, we call* $\begin{tikzcd} G \arrow[loop above, "\alpha"] \arrow[r, "\phi"] & H \arrow[loop above, "\beta"] \end{tikzcd}$ *a* $\mathbb{Z}_2$ *dimension group system.*

We call $\begin{tikzcd} \mathbb{Z} \arrow[loop above, "identify"] \arrow[r, "x\to(x,x)"] & \mathbb{Z}^2 \arrow[loop above, "flip"] \end{tikzcd}$ *or* $\begin{tikzcd} \mathbb{Z}^2 \arrow[loop above, "flip"] \arrow[r, "(x,y)\to x+y"] & \mathbb{Z} \arrow[loop above, "identify"] \end{tikzcd}$ *a basic Elliott-Su system.*
We call a direct sum of finitely many basic Elliott-Su systems a simplicial Elliott-Su system.

Remark 2.0.3. *We can see that if* $A \cong M_n \oplus M_n$ *for some* n *with a* $\mathbb{Z}_2$ *action* $\alpha : (x, y) \to (y, x)$ *on it,* $A \times_\alpha \mathbb{Z}_2 \cong M_{2n}$*. We find* $\begin{tikzcd} K_0(A) \arrow[loop above, "\alpha_*"] \arrow[r, "\phi"] & K_0(A \times_\alpha \mathbb{Z}_2) \arrow[loop above, "\hat{\alpha}_*"] \end{tikzcd}$ *is isomorphic to*

$\begin{tikzcd} \mathbb{Z}^2 \arrow[loop above, "flip"] \arrow[r, "(x,y)\to x+y"] & \mathbb{Z} \arrow[loop above, "identify"] \end{tikzcd}$.

Remark 2.0.4. *We can see that if* $A \cong M_n$ *for some* n *with a* $\mathbb{Z}_2$ *action* α *inner on it,* $A \times_\alpha \mathbb{Z}_2 \cong M_n \oplus M_n$*. We can find* $\begin{tikzcd} K_0(A) \arrow[loop above, "\alpha_*"] \arrow[r, "\phi"] & K_0(A \times_\alpha \mathbb{Z}_2) \arrow[loop above, "\hat{\alpha}_*"] \end{tikzcd}$ *is isomorphic to* $\begin{tikzcd} \mathbb{Z} \arrow[loop above, "identify"] \arrow[r, "x\to(x,x)"] & \mathbb{Z}^2 \arrow[loop above, "flip"] \end{tikzcd}$.

Remark 2.0.5. *Similarly, we can see that a simplicial Elliott-Su system is isomorphic to*

$\begin{tikzcd} K_0(A) \arrow[loop above, "\alpha_*"] \arrow[r, "\phi"] & K_0(A \times_\alpha \mathbb{Z}_2) \arrow[loop above, "\hat{\alpha}_*"] \end{tikzcd}$ *for some finite dimensional* C^**-algebra* A *with a* $\mathbb{Z}_2$ *action* α*.*

After Elliott and Su gave their classification, Choi and Dean modified the Effros, Handelman and Shen theorem[4]:

Theorem 2.0.6. *If a $\mathbb{Z}^2$ action is given on a countable lattice-ordered dimension group, then it can be expressed as an inductive limit of $\mathbb{Z}^2$ actions in simplicial groups.[4]*

Theorem 2.0.7. *Let G be a countable lattice-ordered dimension group, and let α be a $\mathbb{Z}_2$ action on G. Then $E = \{a \in G^+ | \forall b \in G^+ \ s.t. \ b \leq a, b = \alpha(b)\}$ is a convex submonoid of G, and $E + (-E)$ is an ideal of G.*

Proof. Suppose $a, b \in E$, and let $c = a + b$. If $d \in G^+$, $d \leq c = a + b$, by interpolation properties of G, there exist $x_1, x_2 \in G^+$ such that $x_1 \leq a$, $x_2 \leq b$ with $d = x_1 + x_2$. By E's property, $\alpha(x_1) = x_1, \alpha(x_2) = x_2$. So $\alpha(d) = \alpha(x_1 + x_2) = x_1 + x_2 = d$, which means $c \in E$. Thus E is a submonoid.
We see that E is convex because $\forall a \in E$, $\forall b \in G$ such that $0 \leq b \leq a$, $\forall c$ such that $0 \leq c \leq b$, $0 \leq c \leq b \leq a$ so $\alpha(c) = c$. Thus $E + (-E)$ is convex subgroup, and because $E = (E + (-E))^+$, by proposition 1.3. of [8], $E + (-E)$ is directed.
Thus $E + (-E)$ is an ideal of H. □

Definition 2.0.8. *Let G be a countable lattice-ordered dimension group, and let α be a $\mathbb{Z}_2$ action on G. We call the monoid E defined in Theorem 2.0.7 the super fixed submonoid. We call the subgroup $E + (-E)$ defined in Theorem 2.0.7 the super fixed subgroup, denoted by $G^{\alpha\alpha}$. Each element which is in the super fixed subgroup is called a super fixed element.*

Definition 2.0.9. *We say that an $\mathbb{Z}_2$ dimension group system $(G, \alpha) \xrightarrow{\phi} (H, \beta)$ satisfies the magic conditions if the following hold.*
(1)

$$\phi(G^+) = (H^+)^\beta$$

$$\phi(G^{\alpha+}) = \{x + \beta(x) | x \in H^+\}$$

$$ker(\phi) = G(-1) = \{x \in G | x + \alpha(x) = 0\}$$

and there exists a map η from H to G which satisfies

$$\eta(H^+) = (G^+)^\alpha$$

$$\eta(H^{\beta^+}) = \{x + \alpha(x) | x \in G^+\}$$

$$ker(\eta) = H(-1) = \{x \in H | x + \beta(x) = 0\}$$

and:

$$\eta \circ \phi = \sigma$$

$$\phi \circ \eta = \xi$$

where $\sigma : G \to G$ is given by $\sigma(x) = x + \alpha(x)$, and $\xi : H \to H$ is given by $\xi(x) = x + \beta(x)$.

$$\begin{tikzcd} G \arrow[loop above, "\alpha"] \arrow[r, bend left, "\phi"] & H \arrow[loop above, "\beta"] \arrow[l, dashed, bend left, "\eta"] \end{tikzcd}$$

(2) For all $a \in G^+$ such that $a \wedge \alpha(a) = 0$, $\phi(a) \in H^+$ will be in the super fixed submonoid of H.
(3) For all $y \in H^+$ such that $\beta(y) \perp y$, $\exists x \in G^{\alpha\alpha+}$, such that $\phi(x) = y + \beta(y)$.

Remark 2.0.10. *It is easy to see that a inductive limit of simplicial Elliott-Su systems satisfies (1) of magic conditions.*

Theorem 2.0.11. *Suppose $(G, \alpha) \to^{\varphi} (H, \beta)$ is an inductive limit of simplical Elliott-Su systems, and that $x \in G^+$ satisfies $x \perp \alpha(x)$. Then $\varphi(x)$ is a superfixed element of H^+.*

Proof. We have a commutative diagram expressing $(G, \alpha) \to^{\varphi} (H, \beta)$ as an inductive limit

$$\begin{tikzcd} (G_1, \alpha_1) \arrow[r, "\mu_{1,2}"] \arrow[d, "\varphi_1"] & (G_2, \alpha_2) \arrow[r, "\mu_{2,3}"] \arrow[d, "\varphi_2"] & (G_3, \alpha_3) \arrow[r] \arrow[d, "\varphi_3"] & \dots \arrow[r] & (G, \alpha) \arrow[d, "\varphi"] \\ (H_1, \beta_1) \arrow[r, "\nu_{1,2}"] & (H_2, \beta_2) \arrow[r, "\nu_{2,3}"] & (H_3, \alpha_3) \arrow[r] & \dots \arrow[r] & (H, \beta) \end{tikzcd}$$

in which each column is a simplicial Elliott-Su system. By passing to subsequences and renumbering, in any such system we may assume that $ker(\mu_{n\infty}) = ker(\mu_{n,n+1})$ for all n, and similarly for the νs.

Let J_m be the direct summand of G_m generated by those simplicial basis elements e such that $\mu_{m,m+1}(e) = 0$, and let I_m be the ideal of H_m generated by $\varphi_m(J)$. Since $\mu_{m,m+1}$ is equivariant, J_m is an α invariant ideal of G_m, so G_m/J_m is a simplicial group with an action $\tilde{\alpha}$ given by $\tilde{\alpha}(x + J_m) = \alpha(x) + J_m$. By commutativity of the squares, $I_m \subseteq ker(\nu_{m,m+1})$, and I_m is β invariant. Thus H_m/I_m is a simplicial group with action $\tilde{\beta}$ given by $\tilde{\beta}(x + I_m) = \beta(x) + I_m$. Define $\tilde{\varphi}_m : (G_m/J_m) \to (H_m/I_m)$ by $\tilde{\varphi}(x + J_m) = \varphi_m(x) + I_m$. This makes $((G_m/J_m), \tilde{\alpha}) \to^{\tilde{\varphi}} ((H_m/I_m), \tilde{\beta})$ into a simplicial Elliott-Su system. We get a commutative diagram

$$\begin{tikzcd}
(G_m, \alpha_m) \arrow[r] \arrow[d, "\varphi_m"] & (G_m/J_m, \tilde{\alpha}_m) \arrow[r] \arrow[d, "\tilde{\varphi}_m"] & (G_{m+1}, \alpha_{m+1}) \arrow[d, "\varphi_{m+1}"] \\
(H_m, \beta_m) \arrow[r] & (H_m/I_m, \tilde{\beta}_m) \arrow[r] & (H_{m+1}, \beta_{m+1})
\end{tikzcd}$$

In which the compositions along the top and bottom rows are the maps $\mu_{m,m+1}$ and $v_{m.m+1}$ respectively. Proceeding left to right through our diagram, we can insert such an intermediate step at every stage. Then passing to subsequences, we can express our original limit with a sequence in which if $g \in G_k^+$ and $g \neq 0$, then $\mu_{k,\infty(g)} \neq 0$.

Now suppose that $x \in G^+$ is such that $x \perp \alpha(x)$. Assume that we have our system expressed as an inductive limit with the above property that if $g \in G_m^+$ and $\mu_{m,\infty}(g) = 0$, then $g = 0$. We may choose an m and $g \in G_m^+$ with $\mu_{m,\infty}(g) = x$. If $k \geq m$ and $h \in G_k^+$ satisfies $h \leq \mu_{m,k}(g)$ and $h \leq \mu_{m,k}(\alpha_m(g))$, then $0 \leq \mu_{k,\infty}(h) \leq x$ and $0 \leq \mu_{k,\infty}(h) \leq \alpha(x)$, so $\mu_{k,\infty}(h) = 0$, and $h = 0$. Thus $\mu_{m,k}(g) \perp \alpha_k(\mu_{m,k}(g))$ for all $k \geq m$. It follows that for all $k \geq m$, we have $\varphi_k(\mu_{m,k}(g)) \in H_k^{+\beta\beta}$, in other words, $\varphi_k(\mu_{m,k}(g))$ is superfixed in H_k^+. Now consider $\varphi(x) \in H^+$. Suppose we have $y \in H^+$ with $0 \leq y \leq \varphi(x)$. We can choose a $k > m$ and a $v \in H_k$ with $v_{k\infty}(v) = y$, $v \leq \varphi_k(\mu_{m,k}(g))$. It follows that $\beta_k(v) = v$, so $\beta(y) = y$. Since y was arbitrary, $\varphi(x)$ is superfixed. □

Corollary 2.0.11.1. *Suppose $(G, \alpha) \to^{\phi} (H, \beta)$ is an inductive limit of simplicial Elliott-Su systems, and that $y \in H^+$ satisfies $\beta(y) \perp y$. Then there exists $x \in G^{\alpha\alpha+}$ such that $\phi(x) = y + \beta(y)$.*

Proof. Suppose $y \in H^+$ and $\beta(y) \perp y$. Let $x = \eta(y)$. Then $y + \beta(y) = \phi(\eta(y)) = \phi(x)$. Suppose $0 \leq z \leq x$. Let $w = z - z \wedge \alpha(z)$. Then $0 \leq w \leq x$. Also, $w \perp \alpha(w)$, so $\phi(w) \in H^{\beta\beta+}$ from theorem 2.0.11 above. We have $0 \leq \phi(w) \leq y + \beta(y)$, so by interpolation there exist y_1, y_2 with $0 \leq y_1 \leq y$, $0 \leq y_2 \leq \beta(y)$, and $\phi(w) = y_1 + y^2$. We have $y_1 \perp \beta(y_1)$ and $0 \leq y_1 \leq \phi(w) \in H^{\beta\beta+}$, so $y_1 = 0$. Similarly $y_2 = 0$, so $\phi(w) = 0$, and $w = 0$. Thus $z = z \wedge \alpha(z)$, so $z \in G^{\alpha}$. Since z was arbirtary, $x \in G^{\alpha\alpha+}$.

□

Remark 2.0.12. *The above proof only used magic condition (1) and (2).*

Combining the results above, we have shown:

Theorem 2.0.13. *A inductive limit of simplicial Elliott-Su system satisfies magic conditions.*

Corollary 2.0.13.1. *If* $(G, \alpha) \xrightarrow{\phi} (H, \beta)$ *is an* $\mathbb{Z}_2$ *dimension group system that satisfies the magic conditions,* $\phi(G^\alpha) = \{x + \beta(x) | x \in H\}$

Proof. For all $\forall x \in G^\alpha$, write $x = x^+ + x^-$, $\phi(x) = \phi(x^+ + x^-) = \phi(x^+) + \phi(x^-)$. Since $x^+ \in G^{\alpha+}$, there exists $y^+ \in H^+$ such that $\phi(x^+) = y^+ + \beta(y^+)$ by magic condition (1). Applying map $- : G \to G, x \to -x$ to second condition, we can show $\phi(G^{\alpha-}) = \{x + \beta(x) | x \in H^-\}$, so there exists $y^- \in H^-$ such that $\phi(x^-) = y^- + \beta(y^-)$. Thus $y = y^+ + y^-$, $y + \beta(y) = \phi(x)$. $\square$

Corollary 2.0.13.2. *If* $(G, \alpha) \xrightarrow{\phi} (H, \beta)$ *is an* $\mathbb{Z}_2$ *dimension group system that satisfies the magic conditions,* $\phi(G) = H^\beta$

Proof. For all $g \in G$, $g = g^+ + g^-$ $\phi(g^+) \in (H^+)^\beta$, dually, $\phi(g^-) \in (H^-)^\beta$, so $\phi(g^+ + g^-) \in H^\beta$ $\square$

Chapter 3

Main Theorem

The range of invariant problem of the Elliott-Su Classifation has not been solved. In this section, we give a sufficient condition under which a $\mathbb{Z}_2$ dimension group system will be an inductive limit of simplicial Elliott-Su systems. First, we introduce a structure which we call a flank monoid of a countable lattice-ordered dimension group with $\mathbb{Z}_2$ action.

Theorem 3.0.1. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. There exists a maximal subset $F \subseteq H^+$ that satisfies:*

$$\forall a, b \in F, a \wedge \beta(b) = 0$$

.

Proof. Let P be the collection of all the subsets $G \subseteq H$ satisfying $\forall a, b \in G, a \wedge \beta(b) = 0$, and order P by $\subseteq$. (Note, this implies $G \subseteq H^+$.) To apply Zorn's Lemma, take a chain $T = \{G_i\}_{i \in I}$ in P. If T is empty, then $\{0\}$ is an upper bound for T in P. Assume then that T is non-empty. Let $F' = \bigcup \{G_i\}_{i \in I}$ be the union of all subsets in T. We just need to prove $\forall a, b \in F', a \wedge \beta(b) = 0$. For all $a \in F'$ if $\exists b \in F'$ such that $a \wedge \beta(b) \neq 0$, then there exist $i, j \in I$ such that $a \in G_i$, $b \in G_j$, which means $a, b \in G_{max\{i,j\}}$, so $a \wedge \beta(b) = 0$ by $G_{max\{i,j\}} \in P$, which makes a contradiction. Thus $\forall a, b \in F', a \wedge \beta(b) = 0$. Thus by Zorn's Lemma, there exists a maximal element F in P. $\square$

Theorem 3.0.2. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Every subset F satisfying the conclusion of Theorem 3.0.1 must be a submonoid of H.*

Proof. By Lattice-ordered groups: an introduction Prop 1.1.5.(page 3)[2], since H is a dimension group, it must have the interpolation property[8]. Thus $\forall a, b, c \in F$ such that $a \wedge \beta(c) = 0, b \wedge \beta(c) = 0$, using the property, $(a+b) \wedge \beta(c) \leq (a \wedge \beta(c)) + (b \wedge \beta(c)) = 0+0 = 0$. From $a \wedge \beta(a) = 0, b \wedge \beta(b) = 0$ and $c \wedge \beta(c) = 0$, $a, b, c \geq 0$, we have $a + b \geq 0$, $(a+b) \wedge \beta(c) \geq 0$. Thus $(a+b) \wedge \beta(c) = 0$ and $c \wedge \beta(a+b) = 0$. Also, $0 \leq (a+b) \wedge \beta(a+b) \leq a \wedge \beta(a+b) + b \wedge \beta(a+b) \leq a \wedge \beta(a) + a \wedge \beta(b) + b \wedge \beta(a) + b \wedge \beta(b) = 0+0+0+0 = 0$, so $(a+b) \wedge \beta(a+b) = 0$. By the maximality of F, if $a + b \notin F$, $F \cup \{a+b\}$ satisfies $\forall c, d \in F \cup \{a+b\}, c \wedge \beta(d) = 0$, which makes a contradiction, so $a + b \in F$. □

Definition 3.0.3. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Every subset F satisfying the conclusion of Theorem 3.0.1 is called a flank monoid of H.*

Corollary 3.0.3.1. *If $(G, \alpha) \xrightarrow{\phi} (H, \beta)$ is an $\mathbb{Z}_2$ dimension group system that satisfies the magic conditions, and F is a flank monoid of H, then $\xi(F) \subseteq \phi(G^{\alpha+})$.*

Theorem 3.0.4. *[2](page 3)For a latticed ordered group G:*
(a) $\forall a, b \in G, a + b = a \vee b + a \wedge b$,
(b) the lattice $(G, \vee, \wedge)$ of the lattice-ordered group G is always distributive,
(c) $(a+c) \wedge (b+c) = a \wedge b + c$ and $(a+c) \vee (b+c) = a \vee b + c$.

Theorem 3.0.5. *For a latticed ordered group G, let $g \in G$. Then for $g^+ = g \vee 0$ and $g^- = g \wedge 0$, we have $g^+ \wedge (-g^-) = 0$.*

Proof. We can see $g^+ \wedge (-g^-) = (g^+ + g^- - g^-) \wedge (-g^-) = (g^+ + g^-) \wedge 0 - g^- = g \wedge 0 - g^- = g^- - g^- = 0$. □

Theorem 3.0.6. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Let F be a flank monoid of H. For all $a, b \in F$ satisfying $a + \beta(a) = b + \beta(b)$, $a = b$.*

Proof. Because F is a flank monoid of H, we have $a \wedge \beta(a) = 0$, $a \wedge \beta(b) = 0$, $b \wedge \beta(a) = 0$, $b \wedge \beta(b) = 0$. Let $a + \beta(a) = b + \beta(b) = c$. By theorem 3.0.4(a), $a + \beta(a) = a \wedge \beta(a) + a \vee \beta(a) = 0 + a \vee \beta(a) = a \vee \beta(a) = c$. Similarly $b \vee \beta(b) = c$. We have $c \geq a$, $c \geq b$.
By theorem 3.0.4(b), this lattice is distributive. Therefore: $\beta(b) \wedge (a \vee b) = (\beta(b) \wedge a) \vee (\beta(b) \wedge b) = 0 \vee 0 = 0$ and $\beta(b) \vee (a \vee b) = \beta(b) \vee b \vee a = c \vee a = c$.
Thus by theorem 3.0.4(a), $\beta(b) \wedge (a \vee b) + \beta(b) \vee (a \vee b) = \beta(b) + (a \vee b) = c + 0 = \beta(b) + b$, so $b = a \vee b$. By similar a process, $a = a \vee b$, so $a = b = a \vee b$. □

Example 3.0.7. *These are some examples of flank monoids:*
(a) For $H = \mathbb{Z} \oplus \mathbb{Z}$ in product ordering, $\beta : (a,b) \to (b,a)$ an action on H, $F = \{(a,0) \in H | a \in \mathbb{Z}^+\}$ is a flank monoid of H.
(b) For $H = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ in product ordering, $\beta : (a,b) \to (b,a)$ an action on H, $F = \{(a,0) \in H | a \in \mathbb{Z}[1/2]^+\}$ is a flank monoid of H.
(c) For $H = \mathbb{Z}[1/2] \oplus \mathbb{Z} \oplus \mathbb{Z}$ in product ordering, $\beta : (a,b,c) \to (a,c,b)$ an action on H, $F = \{(0,a,0) \in H | a \in \mathbb{Z}^+\}$ is a flank monoid of H.

Theorem 3.0.8. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Let F be a flank monoid of H. For all $a \in F, b \in H$ satisfying $0 \leq b \leq a$, $b \in F$. That is, F is convex.*

Proof. Let $a \in F, b \in H$ satisfy $0 \leq b \leq a$. Then for all $c \in F$, $\beta(c) \wedge a = 0$, therefore, $\beta(c) \wedge b \leq \beta(c) \wedge a = 0$, but $b \geq 0$, and $c \wedge \beta(c) = 0$, so $\beta(c) \geq 0$, so $\beta(c) \wedge b \geq 0$, so $\beta(c) \wedge b = 0$ and $c \wedge \beta(b) = 0$. Also, $0 \leq \beta(b) \wedge b \leq \beta(a) \wedge a = 0$, so $\beta(b) \wedge b = 0$. Because of the maximality of F, if $b \notin F$, $F \cup \{b\}$ satisfies $\forall d, e \in F \cup \{b\}, d \wedge \beta(e) = 0$, which makes a contradiction, so $b \in F$. □

Theorem 3.0.9. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Let F be a flank monoid of H. Then $F + (-F)$ is an ideal of H.*

Proof. Since F is an submonoid, $\forall a, b \in F, c, d \in -F$, $(a + c) - (b + d) = (a + (-d)) + (b + (-c)) \in F + (-F)$, so it is a subgroup of H. To show it is convex, we need to show $(F + (-F))^+$ is convex. Suppose $a \in F, b \in -F$ and $a + b \geq 0$. We have $b \leq 0$, so $0 \leq a + b \leq a \in F$, and by theorem 3.0.8, $a + b \in F$, and $F \subseteq (F + (-F))^+$, so $F = (F + (-F))^+$. By theorem 3.0.8, $(F + (-F))^+$ is convex. Thus $F + (-F)$ is convex subgroup. Because $F = (F + (-F))^+$, by proposition 1.3. of [8], $F + (-F)$ is directed, so $F + (-F)$is an ideal of H. □

Using the action β, $\beta(F) + (-\beta(F))$ is an ideal of H.

Corollary 3.0.9.1. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Let F be a flank monoid of H. Then $\tilde{F} = F + \beta(F) + (-F) + (-\beta(F))$ is an ideal of H.*

Proof. The sum of two ideals in an interpolation group is an ideal. ([8] prop 2.4) □

Theorem 3.0.10. *Let H be a countable lattice-ordered dimension group, and let β be a $\mathbb{Z}_2$ action on H. Let F be a flank monoid of H. Then $(F + (-F)) \cap (\beta(F) + (-\beta(F))) = \{0\}$*

Proof. Suppose $a \in (F+(-F)) \cap (\beta(F)+(-\beta(F)))$. Since $(F+(-F))$ and $(\beta(F)+(-\beta(F)))$ are directed groups, $a^+ = a \wedge 0, a^- = a \vee 0$ exist in both of them. We have $a \in F+(-F)$, $a^+ = a \wedge 0 \in (F+(-F))^+ = F$, $a^+ = a \wedge 0 \in (\beta(F)+(-\beta(F)))^+ = \beta(F)$ (by the proof of theorem 3.0.9). Thus there exists $b \in F$ such that $\beta(b) = a^+$. By the definition of flank monoid, $a^+ \wedge \beta(b) = a^+ \wedge a^+ = 0$, so $a^+ = 0$. In addition, a^- is dually equal to 0. So $a^+ + a^- = a = 0$. $\square$

Next, we need a special theorem:

Theorem 3.0.11. *Suppose H is an countable latticed ordered-dimension group with $\mathbb{Z}_2$ action β with a flank monoid F. If $a \in F+(-F), b \in \beta(F)+\beta(-F)$ are such that $a+b \geq 0$, then $a, b \geq 0$.*

Proof. By the proof of theorem 3.0.10, $a^+, (-a^-) \in F, b^+, (-b^-) \in \beta(F)$ and by theorem 3.0.5, $a^+ \wedge (-a^-) = 0$. Similarly, $b^+ \wedge (-b^-) = 0$. We also have $a^+ \wedge (-b^-) = 0$, $a^+ \wedge b^+ = 0$, $(-a^-) \wedge b^+$ and $(-a^-) \wedge (-b^-) = 0$. By $a^+, b^+, (-a^-)$ and $(-b^-)$ are positive, $0 \leq (a^+ + b^+) \wedge ((-a^-) + (-b^-)) \leq a^+ \wedge (-a^-) + a^+ \wedge (-b^-)) + b^+ \wedge (-a^-) + b^+ \wedge (-b^-)) = 0$. Since $(a^+ + b^+) \wedge ((-a^-) + (-b^-)) = 0$, $(a^+ + b^+) \vee ((-a^-) + (-b^-)) = (a^+ + b^+) + ((-a^-) + (-b^-))$. It follows that $(a+b) \vee 0 = ((a^+ + b^+) - ((-a^-) + (-b^-))) \vee 0 = (a^+ + b^+) \vee ((-a^-) + (-b^-)) - ((-a^-) + (-b^-)) = (a^+ + b^+) + ((-a^-) + (-b^-)) - ((-a^-) + (-b^-)) = a^+ + b^+ = (a+b)^+ = a + b = a^+ + b^+ + a^- + b^-$. Thus $a^- + b^- = 0$. By $a^- \vee b^- = 0$ and $a^- \wedge b^- = 0$, $a^- = b^- = 0$. We have $a, b \geq 0$. $\square$

Definition 3.0.12. *Suppose H is a countable latticed-ordered dimension group, and β is a $\mathbb{Z}_2$ action on H. If there exists a flank monoid F and $\tilde{F} = F + \beta(F) + (-F) + (-\beta(F))$, and there exists a subgroup $E \subseteq H$ such that*

$$\forall x \in E, x = \beta(x)$$

$$E \cap \tilde{F} = \{0\}$$

$$E + \tilde{F} = H$$

Then we call E a fixed subgroup for the flank monoid F in H. We call H an easy countable latticed-ordered dimension group with action β.

Definition 3.0.13. *For H a countable latticed-ordered dimension group, β a $\mathbb{Z}_2$ action on H, and a flank monoid F such that $\{a \in H | a \wedge \beta(a) = 0\} \subseteq F + \beta(F)$, we call F a major flank monoid. If there exists a fixed subgroup for the flank monoid F in H, we call it a major fixed subgroup.*

Corollary 3.0.13.1. *Suppose H is an easy countable latticed-ordered dimension group with action β with a flank monoid F and a fixed subgroup E for F. Then if $a \in H$, $\exists$ unique $b \in E, e \in F + (-F) + \beta(F) + \beta(-F)$ such that $a = b + e$, and $\exists$ unique $c \in F + (-F), d \in \beta(F) + \beta(-F)$ such that $e = c + d$.*

Proof. First, We show there exist unique $b \in E, e \in \tilde{F}$ such that $b + e = a$. If $\exists b_1, b_2 \in E$, $e_1, e_2 \in \tilde{F}$, such that $b_1 + e_1 = b_2 + e_2 = a$, we have $b_1 - b_2 \in E, e_1 - e_2 \in \tilde{F}$, $(b_1 - b_2) + (e_1 - e_2) = 0$, $0 \in E \cap \tilde{F}$ so $b_1 - b_2, e_1 - e_2 \in E \cap \tilde{F}$. But $E \cap \tilde{F} = 0$, so $0 = b_1 - b_2 = e_1 - e_2$. By the same reason(theorem 3.0.10), $\exists$ unique $c \in F + (-F), d \in \beta(F) + \beta(-F)$ such that $c + d = e$. Thus $a = b + e = b + c + d$. □

Corollary 3.0.13.2. *Suppose H is an easy countable latticed-ordered dimension group with action β with a flank monoid F and a fixed subgroup E for F. Then if $a \in H^{\beta}$, $\exists$ unique $b \in E, c \in F + (-F)$ such that $a = b + c + \beta(c)$.*

Proof. Using corollary 3.0.13.1, $\exists$ unique $b \in E, c \in F + (-F), d \in \beta(F) + \beta(-F)$ such that $a = b + c + d$. Since $a \in H^{\beta}$, $a = \beta(a)$. $\beta(b) = b$, so $\beta(c + d) = \beta(c) + \beta(d) = c + d$. $\beta(c) \in \beta(F) + \beta(-F)$, $\beta(d) \in F + (-F)$, and by uniqueness of corollary 3.0.13.1, $a = b + \beta(d) + \beta(c)$ with $b \in E, \beta(d) \in F + (-F), \beta(c) \in \beta(F) + \beta(-F)$, so $\beta(c) = d, \beta(d) = c$. □

Corollary 3.0.13.3. *Suppose H is an easy countable latticed-ordered dimension group with action β with a flank monoid F and a fixed subgroup E for F. Then F must be a major flank monoid.*

Proof. Suppose $a \in H$ with $a \wedge \beta(a) = 0$. Since $a \in H = E + \tilde{F}$, there exists unique $b \in E, c \in F + (-F), d \in \beta(F) + \beta(-F)$ such that $a = b + c + d$. We have $a \wedge \beta(a) = 0$, and $b = \beta(b)$ by definition. Then $(b + c + d) \wedge \beta(b + c + d) = (b + c + d) \wedge (b + \beta(c) + \beta(d)) = b + (c + d) \wedge (\beta(c) + \beta(d)) = 0$. By $0 \in E \cap \tilde{F}$, $b \in E$, so $(c + d) \wedge (\beta(c) + \beta(d)) \in E$. But $(c + d) \wedge (\beta(c) + \beta(d)) \in \tilde{F}$, so $(c + d) \wedge (\beta(c) + \beta(d)) = 0$, and so $b = 0$. By theorem 3.0.11, $a \in F + \beta(F)$. □

Now, we are going to state our main theorem.

Theorem 3.0.14. *Suppose G is a countable latticed-ordered dimension group with a $\mathbb{Z}_2$ action α, $H = E \oplus (F + (-F)) \oplus \beta(F + (-F))$ is an easy countable latticed-ordered dimension group with action β, and $(G, \alpha) \xrightarrow{\phi} (H, \beta)$ satisfies the magic conditions. Suppose further it satisfies:*
(i) $\phi(G^{\alpha\alpha+}) = \{a + \beta(a) | a \in H, a \wedge \beta(a) = 0\}$

(ii) For every $a \in G^+$, $\exists b \in E^+, c \in F + \beta(F)$ such that $\phi(a) = b + c$.
(iii) For all $a \in E^+$, there exists $b \in G^+$ such that $\phi(b) = a$ and $b \wedge \alpha(b) = 0$.
Then this $\mathbb{Z}_2$ dimension group system will be an inductive limit of a sequences of simplicial Elliott-Su systems.

Proof. We are going to construct a diagram like this:

$$\begin{tikzcd}
(G_1,\alpha_1) \arrow[r, "\mu_{1,2}"] \arrow[d, shift left, "\phi_1"] & (G_2,\alpha_2) \arrow[r, "\mu_{2,3}"] \arrow[d, shift left, "\phi_2"] & (G_3,\alpha_3) \arrow[r, "\mu_{3,4}"] \arrow[d, shift left, "\phi_3"] & \dots & \dots \arrow[r] & (G,\alpha) \arrow[dl, "\phi_\infty"'] \arrow[dd, shift left, "\phi"] \\
(H_1,\beta_1) \arrow[r, "\nu_{1,2}"] \arrow[u, shift left, "\rho_1"] \arrow[drrrrr, "\tau_1"'] & (H_2,\beta_2) \arrow[r, "\nu_{2,3}"] \arrow[u, shift left, "\rho_2"] \arrow[drrrr, "\tau_2"] & (H_3,\beta_3) \arrow[r, "\nu_{3,4}"] \arrow[u, shift left, "\rho_3"] \arrow[drrr, "\tau_3"] & \dots \arrow[r] & (H_\infty,\beta_\infty) \arrow[dr, "\tau"] & \\
 & & & & & (H,\beta) \arrow[uu, shift left, "\eta"]
\end{tikzcd}$$

Given (G, α), by the proof in [4], we may write

$$\begin{tikzcd}
G_1 = \mathbb{Z}^{m_1} \arrow[loop above, "\alpha_1"] \arrow[r, "\mu_{12}"] & G_2 = \mathbb{Z}^{m_2} \arrow[loop above, "\alpha_2"] \arrow[r, "\mu_{23}"] & \dots \arrow[r] & G \arrow[loop above, "\alpha"]
\end{tikzcd}$$

where for each simplicial basis element $e \in G_i$, we have $\mu_{i\infty}(e) = \alpha(\mu_{i\infty}(e))$ or $\mu_{i\infty}(e) \wedge \alpha(\mu_{i\infty}(e)) = 0$. Because each α_i is a $\mathbb{Z}_2$ action, we can write each $G_i = E_i \oplus F_i \oplus F_i'$, in which $\alpha_i : E_i \to E_i$ is the identity, and $\alpha_i : F_i \oplus F_i' \to F_i \oplus F_i'$ where $\forall x \in F_i, y \in F_i'$ we have $\alpha_i(x, y) = (y, x)$ and F_i is isomorphic to F_i'. Let each $I_{dim(E_i)}$ be the identity matrix of dimension $dim(E_i)$, $I_{dim(F_i)}$ be the identity matrix of dimension $dim(F_i)$, let $\phi_i = \begin{bmatrix} I_{dim(E_i)} & 0 & 0 \\ I_{dim(E_i)} & 0 & 0 \\ 0 & I_{dim(F_i)} & I_{dim(F_i)} \end{bmatrix}$

and let $H_i = \mathbb{Z}^{n_i} = \phi_i(G_i)$ with action $\beta_i = \begin{bmatrix} 0 & I_{dim(E_i)} & 0 \\ I_{dim(E_i)} & 0 & 0 \\ 0 & 0 & I_{dim(F_i)} \end{bmatrix}$. It can be checked directly that $\phi_i : G_i \to H_i = E_i \oplus E_i' \oplus F_i$ is a equivariant map.

Let $\mu_{ii+1} = \begin{bmatrix} a & c & c \\ b & d & e \\ b & e & d \end{bmatrix}$, in which a is a $dim(E_i) \times dim(E_{i+1})$ matrix, d and e are $dim(F_i) \times dim(F_{i+1})$ matrices, b is a $dim(E_i) \times dim(F_{i+1})$ matrix, and c is a $dim(F_i) \times dim(E_{i+1})$ matrix.

Let $v_{ii+1} = \begin{bmatrix} a & 0 & c \\ 0 & a & c \\ b & b & d+e \end{bmatrix}$, where v_{ii+1} maps H_i to H_{i+1}. It can be checked directly that the following diagram commutes and each map is an equivariant positive map:

$$\begin{tikzcd} G_i \arrow[loop above, "\alpha_i"] \arrow[r, "\mu_{ii+1}"] \arrow[d, "\phi_i"] & G_{i+1} \arrow[loop above, "\alpha_{i+1}"] \arrow[d, "\phi_{i+1}"] \\ H_i \arrow[loop below, "\beta_i"] \arrow[r, "v_{ii+1}"] & H_{i+1} \arrow[loop below, "\beta_{i+1}"] \end{tikzcd}$$

Now we are going to define a pair of maps: ϕ^l and ϕ^r. First we need to show some decomposition properties. By condition(ii), for each $m \in G^+$, $\exists n \in E^+, w \in F + \beta(F)$ such that $\phi(m) = n + w$. By Magic condition (1), $\phi(m) \in H^{\beta^+}$, so by Corollary 3.0.13.2 there exists unique $n' \in E^+$ and $p \in F + (-F)$ such that $\phi(m) = n'+p+\beta(p) = n'+(p+\beta(p)) = n+w$. By Corollary 3.0.13.1, $n = n'$ and $p + \beta(p) = w \geq 0$. By theorem 3.0.11, $p \geq 0$, $p \in F$. Assume that $m \in G^{\alpha+}$. By magic condition (1), $\exists r \in H^+$ such that $\phi(m) = n + p + \beta(p) = r+\beta(r)$. Apply Corollary 3.0.13.1 to r, let $s \in E, h \in F+\beta(F)+(-F)+\beta(-F)$, $r = s+h$ and $(s+h)+\beta(s+h) = n+(p+\beta(p)) = (2s)+(h+\beta(h))$. By the uniqueness in Corollary3.0.13.1, $2s = n$, and because it is a dimension group, it is unperforated, so $s \geq 0$. Therefore, for every $m \in G^{\alpha+}$, there are unique $s \in E^+$ and $p \in F$ such that $\phi(m) = s + p + \beta(s + p)$ and $s + p \geq 0$. Similarly, from Corollaries 2.0.13.1, 2.0.13.2, for every $m \in G^\alpha$, there are unique $s \in E$ and $p \in F + (-F)$ such that $\phi(m) = s + p + \beta(s + p)$. For all $m \in G^\alpha$, let $\phi^l(m) = s + p$. For $u, v \in G^\alpha$, let $s_u, s_v \in E$, $p_u, p_v \in F + (-F)$ be the elements in the decompositions above. We have $s_u + s_v \in E$ and $p_u + p_v \in F + (-F)$. Consider $((s_u + s_v) + (p_u + p_v)) + \beta(((s_u + s_v) + (p_u + p_v))) = \phi(u) + \phi(v) = \phi(u + v)$. From the proof above, the choice of these two element is unique, so $s_u + s_v$ and $p_u + p_v$ are the elements in the decompositions for $\phi(u + v)$. Thus $\phi^l(u + v) = \phi^l(u) + \phi^l(v)$, so ϕ^l is a homomorphism. Because for all $m \in G^{\alpha+}$, $\phi^l(m) \geq 0$ it is also a positive group homomorphism. We define $\phi^r(m) = \beta(s + p) = \beta(\phi^l(m))$ for $m \in G^\alpha$, which is a positive homomorphism.

Now we are going to define a sequence of maps $\tau_i : H_i \to H$.
Let f' be a simplicial basis element for G_i in F_i. Because $\mu_{i\infty}(f') \wedge \alpha(\mu_{i\infty}(f')) = 0$, by the magic condition (1)(2): $\phi(\mu_{i\infty}(f')) = \phi(\mu_{i\infty}(\alpha_i(f')))$ is in $H^{\beta\beta}$, so $\phi(\mu_{i\infty}(f')) + \phi(\mu_{i\infty}(\alpha_i(f'))) = \phi(\mu_{i\infty}(f')) + \phi(\mu_{i\infty}(f')) = \phi(\mu_{i\infty}(f' + \alpha_i(f'))) \in H^{\beta\beta}$. Because $f' \geq 0$, so $\phi(\mu_{i\infty}(f')) \geq 0$.

Because $f' + \alpha_i(f') \in G_i^{\alpha_i}$, $\mu_{i\infty}(f' + \alpha_i(f')) \in G^{\alpha}$. If $s \in E^+$ and $p \in F$ are such that $\phi(\mu_{i\infty}(f' + \alpha_i(f'))) = s + p + \beta(s + p)$, because $0 \leq p \leq \phi(\mu_{i\infty}(f' + \alpha_i(f')))$, p is also in super fixed submonoid of H. Thus $p \wedge \beta(p) = 0$ and $p = \beta(p)$, so $p = 0$ in this situation. Then $\phi^l(\mu_{i\infty}(f' + \alpha_i(f'))) = \phi^r(\mu_{i\infty}(f' + \alpha_i(f'))) = s = \beta(s)$. Because $\phi(\mu_{i\infty}(f')) = \phi(\mu_{i\infty}(\alpha_i(f')))$ by magic conditions and $\phi(\mu_{i\infty}(f' + \alpha_i(f'))) = 2s$, $\phi(\mu_{i\infty}(f')) = \phi(\mu_{i\infty}(\alpha_i(f'))) = s = \phi^l(\mu_{i\infty}(f' + \alpha_i(f'))) = \phi^r(\mu_{i\infty}(f' + \alpha_i(f')))$ in this situation.

Let $\rho_i : H_i \to G_i$ be the map $\rho_i = \begin{bmatrix} I_{dim(E_i)} & I_{dim(E_i)} & 0 \\ 0 & 0 & I_{dim(F_i)} \\ 0 & 0 & I_{dim(F_i)} \end{bmatrix}$.

Let each $\tau_i : H_i \to H$ be the linear map defined as follows:

If f is a simplicial basis element for H_i in E_i, $\tau_i(f) = \phi^l(\mu_{i\infty}(\rho_i(f)))$.

If f is a simplicial basis element for H_i in E_i', $\tau_i(f) = \phi^r(\mu_{i\infty}(\rho_i(f)))$.

If f is a simplicial basis element for H_i in F_i, $\tau_i(f) = \phi^l(\mu_{i\infty}(\rho_i(f))) = \phi^r(\mu_{i\infty}(\rho_i(f)))$.

Because $\phi^l, \phi^r, \mu_{i\infty}$ and ρ_i are positive maps, τ_i is positive. Because it is defined on basis elements, it is a homomorphism.

Now, we are going to show τ_i is a equivariant map. If e_1 is a simplicial basis element for H_i in E_i and e_2 is a simplicial basis element for H_i in E_i' such that $e_1 = \beta_i(e_2)$, then $\beta(\tau(e_1)) = \beta(\phi^l(\mu_{i\infty}(\rho_i(e_1)))) = \phi^r(\mu_{i\infty}(\rho_i(e_1))) = \phi^r(\mu_{i\infty}(\rho_i(e_2))) = \tau_i(e_2)$, and if f is a simplicial basis element for H_i in F_i, then $f = \beta_i(f)$, so τ_i is a equivariant map.

Now we should check this diagram commutes:

$$\begin{tikzcd}
H_i \arrow[loop above, "\beta_i"] \arrow[r, "\nu_{ii+1}"] \arrow[dr, "\tau_i"'] & H_{i+1} \arrow[loop above, "\beta_{i+1}"] \arrow[d, "\tau_{i+1}"] \\
 & H \arrow[loop below, "\beta"]
\end{tikzcd}$$

Suppose f is the jth simplicial basis element for H_i in E_i, let a_j and b_j be the jth columns of the matrices a and b. We have $\tau_i(f) = \phi^l(\mu_{i\infty}(\rho_i(f))) = \phi^l(\mu_{i+1\infty}(\mu_{ii+1}(\rho_i(f)))) = \phi^l(\mu_{i+1\infty}(a_j^T, b_j^T, b_j^T))$, $\nu_{ii+1}(f) = (a_j^T, \underbrace{0, 0, \ldots, 0}_{\dim(E_{i+1})}, b_j^T)$, and $\tau_{i+1}(\nu_{ii+1}(f))$

$$= \phi^l(\mu_{2\infty}(a_j^T, \underbrace{0, 0, \ldots, 0}_{\dim(F_{i+1})}, \underbrace{0, 0, \ldots, 0}_{\dim(F_{i+1})})) + \phi^l(\mu_{i+1\infty}(\underbrace{0, 0, \ldots, 0}_{\dim(E_{i+1})}, b_j^T, b_j^T)) = \phi^l(\mu_{i+1\infty}(a_j^T, b_j^T, b_j^T)),$$

so $\tau_i(f) = \tau_{i+1}(\nu_{ii+1}(f))$. If f is a simplicial basis element for H_i in E_i', the proof is similar. Suppose f is the jth simplicial basis element for H_i in F_i. Let c_j, d_j and e_j be the jth columns of the matrices c, d and e. We have $\tau_i(f) = \phi^l(\mu_{i\infty}(\rho_i(f))) = \phi^l(\mu_{i+1\infty}(2c_j^T, (d+e)_j^T, (d+e)_j^T))$. Because $0 \leq \phi(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})})) \leq \phi(\mu_{i+1\infty}(2c_j^T, (d+e)_j^T, (d+e)_j^T))$, and $\phi(\mu_{i+1\infty}(2c_j^T, (d+e)_j^T, (d+e)_j^T)) = \phi(\mu_{i\infty}(\rho_i(f)))$, which is in $H^{\beta\beta}$, by the defination of super fixed submonoid, $\phi(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}))$ is also in $H^{\beta\beta}$ and $\phi(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})})) \in G^\alpha$, so $\phi^l(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})})) = \phi^r(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}))$. Because $\nu_{ii+1}(f) = (c_j^T, c_j^T, (d+e)_j^T)$, $\tau_{i+1}(\nu_{ii+1}(f)) =$

$$\phi^l(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})})) + \phi^r(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}))$$
$$+ \phi^l(\mu_{i+1\infty}(\underbrace{0,0,\dots,0}_{\dim(E_{i+1})}, (d+e)_j^T, (d+e)_j^T)) = \phi^l(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}))$$
$$+ \phi^l(\mu_{i+1\infty}(c_j^T, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})}, \underbrace{0,0,\dots,0}_{\dim(F_{i+1})})) + \phi^l(\mu_{i+1\infty}(\underbrace{0,0,\dots,0}_{\dim(E_{i+1})}, (d+e)_j^T, (d+e)_j^T)) =$$

$\phi^l(\mu_{i+1\infty}(2c_j^T, (d+e)_j^T, (d+e)_j^T))$. It follows that $\tau_i(f) = \tau_{i+1}(\nu_{ii+1}(f))$, so the diagram commutes.

Consider:

$$\begin{tikzcd}
H_1 \arrow[loop above, "\beta_1"] \arrow[r, "\nu_{12}"] \arrow[drrr, "\tau_1"] & H_2 \arrow[loop above, "\beta_2"] \arrow[r, "\nu_{23}"] \arrow[drr, "\tau_2"] & \dots \arrow[r] & H_\infty \arrow[loop above, "\beta_\infty"] \arrow[d, "\tau"] \\
& & & H \arrow[loop below, "\beta"]
\end{tikzcd}$$

We are going to show that τ is an equivariant isomorphism.

First, we prove τ is surjective, and $\tau(H_\infty^+) = H^+$.
Let $\phi_i^l = \begin{bmatrix} I_{dim(E_i)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{dim(F_i)} & 0 \end{bmatrix}$.

Recall that we have maps $\eta : H \to G$ and $\sigma : G \to G$ in magic condition (1).
For each $x \in F$, $\eta(x) \in G^{\alpha+}$, so $\phi(\eta(x)) = x + \beta(x)$ and $\phi^l(\eta(x)) = x$.
By condition(i) and F being a flank monoid, $x \wedge \beta(x) = 0$, so $x + \beta(x) \in \phi(G^{\alpha\alpha+})$. There exists $y \in G^{\alpha\alpha}$ such that $\phi(y) = x + \beta(x)$, so $\phi(y) = \phi(\eta(x))$. Then $y - \eta(x) \in ker(\phi)$. By magic condition (1), $y - \eta(x) + \alpha(y - \eta(x)) = 0 = 2y - \eta(x) - \alpha(\eta(x))$, therefore $2y = \eta(x) + \alpha(\eta(x))$. Because $\eta(x) + \alpha(\eta(x)) \in G^{\alpha\alpha}$ and $\eta(x) \geq 0$, $\eta(x) = \alpha(\eta(x))$, so $2\eta(x) = 2\alpha(\eta(x)) = 2y$. Because G a dimension group, it is unperforated, so $\eta(x) = y$ and $\eta(x) \in G^{\alpha\alpha}$.
Assume $z \in G_i{}^+$ is such that $\mu_{i\infty}(z) = \eta(x)$. Because $\eta(x)$ is a simplicial basis element in $G^{\alpha\alpha}$, if $w \in G_i$ is such that $0 \leq w \leq z$ and $w \in F_i$ or $w \in F_i'$, because $0 \leq \mu_{i\infty}(w) \leq \eta(x)$ we have $\mu_{i\infty}(w) \in G^{\alpha\alpha}$. Because $\mu_{i\infty}(w) \wedge \alpha(\mu_{i\infty}(w)) = 0$, $\mu_{i\infty}(w) = 0$, so $\mu_{i\infty}(z - w) = \eta(x)$. If $z = z_1 + z_2 + z_3$ such that $z_1 \in E_i, z_2 \in F_i$ and $z_3 \in F_i'$, then $\mu_{i\infty}(z_1) = \eta(x)$. Thus $\phi_i^l(z_1) \in H_i{}^+$ and $\tau_i(\phi_i^l(z_1)) = x$, so $x \in \tau(H_\infty{}^+)$. Similarly, for each $x \in \beta(F)$, $x \in \tau(H_\infty^+)$.

Consider $a \in H^+$. By Corollary 3.0.13.1, there exist $b \in E, c \in F + (-F)$ and $d \in \beta(F) + \beta(-F)$ such that $a = b + c + d$. We can see $a \geq 0$ and $\beta(a) \geq 0$, therefore $a \wedge \beta(a) = (b + c + d) \wedge (b + \beta(c) + \beta(d)) = b + (c + d) \wedge (\beta(c) + \beta(d)) \geq 0$. We can see $a \wedge \beta(a) = \beta(a \wedge \beta(a))$, so $a \wedge \beta(a) \in H^{\beta+}$. By magic conditions (1), there exists $t \in G^+$ such that $\phi(t) = a \wedge \beta(a)$. Thus we can find $i \in \mathbb{N}$ such that we can lift t to $\bar{t} \in G_i^+$, such that $\tau_i(\phi_i(\bar{t})) = \phi(t) = \phi(\mu_{i\infty}(\bar{t})) \in H$ and $\phi_i : G_i \to H_i$ is a positive map, so $\tau_i(\phi_i(\bar{t})) = a \wedge \beta(a) \in \tau(H_\infty^+)$. Because $(c+d) \wedge (\beta(c)+\beta(d)) \in F+(-F)+\beta(F)+\beta(-F)$ and $a \geq a \wedge \beta(a)$, $0 \leq a - a \wedge \beta(a) = c + d - (c+d) \wedge (\beta(c)+\beta(d)) \in F+(-F)+\beta(F)+\beta(-F)$. By Corollary 3.0.13.1, there exists $e \in F+(-F), f \in \beta(F)+\beta(-F)$ such that $e+f = a - a \wedge \beta(a)$. By theorem 3.0.11, $e, f \geq 0$, therefore, $e \in F$ and $f \in \beta(F)$. By what we have proved above, $e, f \in \tau(H_\infty^+)$. Therefore, $a = e + f + a \wedge \beta(a) \in \tau(H_\infty^+)$. Then we have $H^+ \subseteq \tau(H_\infty^+)$ and because τ is a positive map, $\tau(H_\infty^+) \subseteq H^+$, so we have $H^+ = \tau(H_\infty^+)$. We can use the map $-$ on H to see $H^- = \tau(H_\infty^-)$. Because each element $x \in H$ can be written as $x = x^+ + x^-$, $x^- \in H^-$, $x^+ \in H^+$, $H = \tau(H_\infty)$. Now we have τ is surjective.

Next, we prove τ is injective.

First, we prove $E = H^{\beta\beta}$. From condition (iii) and magic condition (2), $E \subseteq H^{\beta\beta}$. Suppose $z \in H^{\beta\beta}$. Assume $u \in E, v \in F + (-F)$ and $w \in \beta(F) + \beta(-F)$ are the elements given by Corollary 3.0.13.1 such that $z = u + v + w$. Since $u \in E \subseteq H^{\beta\beta}$, and $H^{\beta\beta}$ is an ideal, $v + w = z - u \in H^{\beta\beta}$, so $(w+v)^+ \in H^{\beta\beta+}$. By $(w+v)^+ \in F+(-F)+\beta(F)+\beta(-F)$, we have $v' \in F + (-F)$ and $w' \in \beta(F) + \beta(-F)$ such that $w' + v' = (w + v)^+ \geq 0$, and by Theorem 3.0.11, $w', v' \geq 0$. Therefore, we have $w' \in \beta(F)$, $v' \in F$ and $w' + v' = (w+v)^+ \in H^{\beta\beta+}$. By definition of $H^{\beta\beta+}$ and flank monoid, we have $w' = \beta(w')$ which forces $w' = 0$ and $v' = 0$, which makes $(w + v)^+ = 0$. Similarly, $(w + v)^- = 0$. Therefore, $w + v = 0$, then we have

$H^{\beta\beta} \subseteq E$. Thus, $H^{\beta\beta} = E$.

For all $g_1, g_2 \in G^\alpha$, if $\phi(g_1) = \phi(g_2)$, $g_1 - g_2 \in ker(\phi)$, so $g_1 - g_2 + \alpha(g_1 - g_2) = 0$. Therefore $g_1 + \alpha(g_1) = g_2 + \alpha(g_2) = 2g_1 = 2g_2$. Because G is unperforated, $g_1 = g_2$. We knew that $0 \in G^\alpha$ and $\phi(0) = 0$, so $\forall g \in G^\alpha$ such that $\phi(g) = 0$, $g = 0$.

For all $h \in H_i$ such that $\tau_i(h) = 0$, $\rho_i(h) \in G_i^{\alpha_i}$, so $\phi(\mu_{i\infty}(\rho_i(h))) = \tau_i(\phi_i(\rho_i(h))) = \tau_i(h + \beta_i(h)) = 0 + \beta(0) = 0$ and $\mu_{i\infty}(\rho_i(h)) \in G^\alpha$, so $\mu_{i\infty}(\rho_i(h)) = 0$, so $\mu_{ii+1}(\rho_i(h)) = 0$ by Choi and Dean paper. Thus $\phi_{i+1}(\mu_{ii+1}(\rho_i(h))) = \nu_{ii+1}(\phi_i(\rho_i(h))) = \nu_{ii+1}(h + \beta_i(h)) = 0$, so $\forall N \geq i+1$, $\nu_{iN}(h + \beta_i(h)) = 0$. If $\nu_{iN}(h) = x + y$ such that $x \in F_N, y \in E_N \oplus \ E'_N$, $x + y + \beta_N(x+y) = y + \beta_N(y) + (2x) = 0$, and because x is orthogonal to y and $\beta_N(y)$, $x = 0$. Thus $\exists t \in E_{i+1} \subseteq G_{i+1}$ such that $\nu_{ii+1}(h) = \phi^l_{i+1}(t) - \beta_{i+1}(\phi^l_{i+1}(t))$. It can be checked directly by matrix product that $\forall s \in E_{i+1} \subseteq G_{i+1}$, $\nu_{i+1N}(\phi^l_{i+1}(s)) = \phi^l_N(\mu_{i+1N}(s))$.

Consider $t_\infty = \mu_{i+1\infty}(t) \in G^\alpha \subseteq G$. By magic condition (1) and Corollary 2.0.13.2, $\phi(t_\infty) \in H^\beta$. By Corollary 3.0.13.2, there exist $e \in E, f \in F + (-F)$ such that $\phi(t_\infty) = e + f + \beta(f)$. By condition (i), $f \wedge \beta(f) = 0$, and the map $-$ is dual, so $\exists c \in G^{\alpha\alpha}$ such that $\phi(c) = f + \beta(f)$. Since $(t_\infty - c) \in G^\alpha$, $(t_\infty - c)^+ \in G^{\alpha+}$. By magic condition (1), $\exists e'^+ \in H^+$ such that $\phi((t_\infty - c)^+) = e'^+ + \beta(e'^+)$. Similarly, $\exists e'^- \in H^-$ such that $\phi((t_\infty - c)^-) = e'^- + \beta(e'^-)$. Setting $e' = e'^+ + e'^-$, we get $\phi(t_\infty - c) = e = e' + \beta(e')$. Since E^+ is a super fixed submonoid, $0 \leq e'^+ \leq e^+$ and $0 \geq e'^- \geq e^-$, we have $e' = e'^+ + e'^- \in E$. By $e' \in E$ and condition (iii), there exists $\hat{e}^+ \in G^+$ such that $\hat{e}^+ \wedge \alpha(\hat{e}^+) = 0$ and $\phi(\hat{e}^+) = e'^+$. Dually, $\exists \hat{e}^- \in G^-$ with $\phi(\hat{e}^-) = e'^-$ and $\hat{e}^- \vee \alpha(\hat{e}^-) = 0$. Let $\hat{e}^+ + \hat{e}^- = \hat{e}$. Because $\phi(c + \hat{e} + \alpha(\hat{e})) = \phi(t_\infty)$, $t_\infty \in G^\alpha$, and $c + \hat{e} + \alpha(\hat{e}) \in G^\alpha$, we have $t_\infty = c + \hat{e} + \alpha(\hat{e})$.

We can lift $\hat{e}^+, \hat{e}^-, c^+, c^-$ to elements $\bar{e}^+, \bar{e}^-, \bar{c}^+, \bar{c}^- \in G_j$ for some $j \geq i+1$ such that $t_j = \bar{c}^+ + \bar{c}^- + \bar{e}^+ + \bar{e}^- + \beta(\bar{e})^+ + \beta(\bar{e})^-$. Thus $\nu_{ij}(h) = \nu_{i+1j}(\phi^l_{i+1}(t) - \beta_{i+1}(\phi^l_{i+1}(t))) = \phi^l_j(t_j) - \beta_j(\phi^l_j(t_j)) = \phi^l_j(\bar{c}^+ + \bar{c}^- + \bar{e}^+ + \bar{e}^- + \beta(\bar{e})^+ + \beta(\bar{e})^-) - \beta_j(\phi^l_j(\bar{c}^+ + \bar{c}^- + \bar{e}^+ + \bar{e}^- + \beta(\bar{e})^+ + \beta(\bar{e})^-))$. Since $\hat{e}^+ \wedge \alpha(\hat{e}^+) = 0$ in G, if b is a simplicial basis element in E_j in G_j such that $0 \leq b \leq \bar{e}^+$, $\mu_{j\infty}(b) = 0$. The case of $\bar{e}^-$ is similar. Every basis elements in E_j which is used to express $\bar{e}^+$ and $\bar{e}^-$ must map to 0 finally, and $0 \in G^{\alpha\alpha}$. Because $c^+ \in G^{\alpha\alpha}$, if b is a simplicial basis element in E_j in G_j such that $0 \leq b \leq \bar{c}^+$, since the super fixed monoid is convex, $\mu_{j\infty}(b) \in G^{\alpha\alpha}$. The case of $\bar{c}^-$ is similar. Similarly if b is a simplicial basis element in E_j in G_j such that $0 \leq b \leq -\bar{c}^-$, since the super fixed monoid is convex, $\mu_{j\infty}(b) \in G^{\alpha\alpha}$.

Suppose $\{b_r\}_{r=1,2,3,\ldots,s}$ is a list of simplicial basis elements for G_j such that $t_j = \bar{c}^+ + \bar{c}^- + \bar{e}^+ + \bar{e}^- + \beta(\bar{e})^+ + \beta(\bar{e})^- = \sum_{r=1}^{s} a_r b_r$. If $b_r \in F_j$, $\phi^l_j(b_r) - \beta_j(\phi^l_j(b_r)) = 0$. Let $\{b_r\}_{r=1,2,3\ldots,l} \subseteq$

$\{b_r\}_{r=1,2,3,\ldots,s}$ be the simplicial basis for $E_j \subseteq G_j$. We have $h_j = \sum_{r=1}^{l} a_r\phi_j^l(b_r) - \sum_{r=1}^{l} a_r\beta_j(\phi_j^l(b_r)) = \sum_{r=1}^{s} a_r\phi_j^l(b_r) - \sum_{r=1}^{s} a_r\beta_j(\phi_j^l(b_r))$. Since each $\mu_{i\infty}(b_r) \in G^{\alpha\alpha}$, by Condition(i), F being a major flank monoid, and Corollary 3.0.13.1, we have $\tau_j(\phi_j^l(b_r)) \in F$. It follows that $\tau_j(\sum_{r=1}^{l} a_r\phi_j^l(b_r) - \sum_{r=1}^{l} a_r\beta_j(\phi_j^l(b_r))) = \sum_{r=1}^{l} a_r\tau_j(\phi_j^l(b_r)) - \sum_{r=1}^{l} a_r\beta(\tau_j(\phi_j^l(b_r))) = 0$ and $\sum_{r=1}^{l} a_r\tau_j(\phi_j^l(b_r)) \in F + (-F)$, $\sum_{r=1}^{l} a_r\beta(\tau_j(\phi_j^l(b_r))) \in \beta(F + (-F))$. By Corollary 3.0.13.1, $\sum_{r=1}^{l} a_r\tau_j(\phi_j^l(b_r)) = 0$. Since $\beta(0) = 0$, $\tau_j(\sum_{r=1}^{l} a_r\phi_j^l(b_r) + \sum_{r=1}^{l} a_r\beta_j(\phi_j^l(b_r))) = \tau_j(\phi_j(\sum_{r=1}^{l} a_r b_r)) = \phi(\mu_{j\infty}(\sum_{r=1}^{l} a_r b_r)) = 0$. Because $\mu_{j\infty}(\sum_{r=1}^{l} a_r b_r) \in G^{\alpha}$, so $\mu_{jj+1}(\sum_{r=1}^{l} a_r b_r) = 0$ by passing to subsequences, as we have done. Because $\phi_{j+1}^l(\mu_{jj+1}(\sum_{r=1}^{l} a_r b_r)) - \beta_{j+1}(\phi_{j+1}^l(\mu_{jj+1}(\sum_{r=1}^{l} a_r b_r))) = h_{j+1}$ and $\phi_{j+1}^l(\mu_{jj+1}(\sum_{r=1}^{l} a_r b_r)) = 0$, $h_{j+1} = 0$ in H_{j+1}. If $h \in H^{\infty}$ and $\tau(h) = 0$, then $\exists h_i \in H_i$ for some i with $h = \nu_{i\infty}(h_i)$ and $\tau_i(h_i) = 0$. Thus τ is injective.

We have that this diagram

$$\begin{tikzcd} G \arrow[r, "id"] \arrow[d, "\phi_\infty"] \arrow[loop above, "\alpha"] & G \arrow[d, "\phi"] \arrow[loop above, "\alpha"] \\ H_\infty \arrow[r, "\tau"] \arrow[loop below, "\beta_\infty"] & H \arrow[loop below, "\beta"] \end{tikzcd}$$

commutes by construction and (id, τ) is an isomorphism of $\mathbb{Z}_2$ dimension group systems.

Now we have that $(G, \alpha) \xrightarrow{\phi} (H, \beta)$ is an inductive limit of a sequences of simplicial Elliott-Su systems. $\square$

Remark 3.0.15. *For main theorem, given any random order unit $u \in G$ that satisfies $u \in G^{\alpha}$ and $v \in H$ that satisfies $v + \beta(v) \in H^{\beta}$, this $\mathbb{Z}_2$ dimension group system can be identified as an Elliott-Su invariant with these two special elements.*

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