Robust H_∞ Model Reference Tracking Control of Singular Systems Using T-S Fuzzy Model and LMI

by

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Abstract

In this study, a Takagi-Sugeno fuzzy model based tracking controller is proposed for nonlinear high index singular systems with a guaranteed H_∞ model reference tracking performance. A coordinate transformation is introduced, which allows to represent the nonlinear high index singular systems using T-S fuzzy model. This transformation facilitates the specification of a dynamic reference system that satisfies the same constraints as the actual system. Moreover, the proposed transformation relaxes the existence condition of a full order observer for a nonlinear high index singular system. A modification of the existing linear matrix inequality optimization algorithm has been proposed to make the algorithm compatible for singular systems. The proposed approach is general and can be used for designing the tracking controller and defining the dynamic reference model for any singular systems with any index. Without using the exact feedback linearization technique and complicated adaptive schemes, this tracking controller design approach is simple and feasible for practical applications. Finally, two high index singular systems from two different fields are used as examples to demonstrate the proposed design effectiveness.

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List of Abbreviations

H_∞	Hilbert-Schmidt operator infinity-norm
λ	Lagrangian multiplier
LMI	Linear matrix inequality.
MIMO	Multi-input multi-output.
ODE	Ordinary differential equation.
DAE	Differential Algebraic equation.
CSTR	Continuously stirred tank reactor
PDC	Parallel distributed compensation
YALMIP	Yet Another LMI Parser
QSS	quasi-steady-state

1 Introduction

The main focus of this chapter is to give an overall idea of singular systems along with the available methods to analyze and design controllers for singular systems. The discussion will emphasize the state space representation of singular systems while giving priority towards the nonlinear singular systems involving nonlinear constraints, which is the primary direction of this study. A brief introduction of the state space representation of singular systems is given in the Section 1.1. Section 1.2 illustrates a physical example of a singular system, specifically, a constrained mechanical system. Section 1.3 covers the review of the studies on both linear and nonlinear singular systems. Moreover, this section defines some important terms regarding singular systems such as index and solvability. The differences between the ordinary differential equations (ODE) and the high index singular systems are explained in Section 1.3.4 Section 1.3.5 deals with the important term known as regularity of singular systems. Section 1.4 provides a motivation for this research, pointing out the existing problems related to design dynamic tracking controllers for high index singular systems. Sub-Section 1.5 explains the contribution of this study. Finally, the organization of the following chapters of this research note is given in the Section 1.6.

1.1 Singular Systems

Based on state space models, system analysis and synthesis are the core features in modern control theory, which were developed at the end of the 1950s and the beginning of the 1960s. To produce a state space model, some variables are chosen as state variables, such as speed, weight, temperature, and acceleration. These must be sufficient to characterize the system of interest. Then, according to the laws of physics, several equations are established to represent the relationships among the variables. Naturally, it is differential and algebraic equations that form the mathematical model of the system. Its general form is given as follows:

$$f(x^n(t), \ldots, x(t), u(t), t) = 0 \tag{1}$$

$$g(x^m(t), u(t), y(t), t) = 0 \tag{2}$$

where $x(t)$ is composed of state variables and called the state of the system , $u(t)$ is the control input, $y(t)$ is the measured output, f and g are vector-valued functions of $x^n(t)$,. . ., $x(t)$, $u(t)$, and t with appropriate dimensions. Generally Eq. (1) and Eq. (2) are called state and output equations, respectively.

Systems represented using differential algebraic equations are referred to as singular, implicit descriptor, semi-state, and generalized systems. This types of representation is very natural and commonly found in dynamic models of chemical [1],[2], electrical [19],[3], and mechanical engineering [4],[29] and their applications [6]. Singular systems have fundamentally different characteristics from ordinary differential equation (ODE) systems [7]. For example, unlike ODEs, arbitrary initial conditions or non-smooth inputs in singular systems may lead to impulsive solutions, which are a common cause of failure for standard ODE simulation methods. A approach commonly used to measure the differences between singular systems and ODE systems is that of the differential index. In simple words, the index of a singular system is the minimum number of differentiations required to obtained an equivalent ODE system. Singular systems with indices exceeding one are referred to as high-index singular systems. High-index singular systems have been a subject of vast research [8],[9]. A variety of numerical simulation methods have been developed for specific classes of high-index singular systems, ranging from those resulting from index reduction techniques [10], [11], [12], and [14] to those from nonlinear constrained optimization problems [71],[72]. The high-index singular systems is still an area of active research [15],[16].

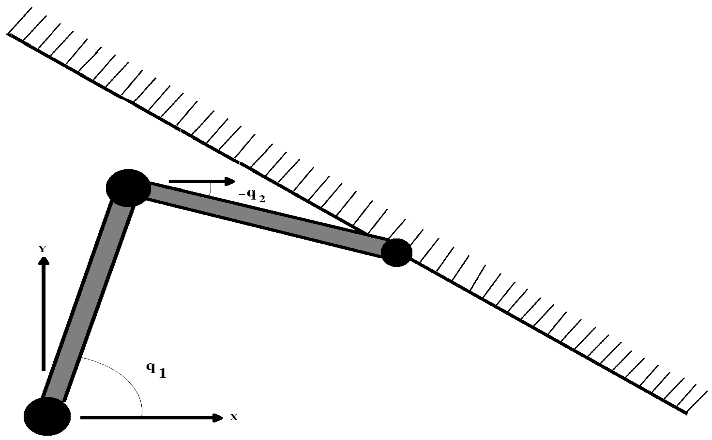

Figure 1: Constrained robot link

1.2 A Physical Example of Singular Systems

Consider a holonomic constrained robotic system shown in the Fig. 1 This robotic link has 2 joints and one end effector which is confined to a line, i.e., the movement of the end effector is holonomically constrained.

The dynamics of this constrained robotic link can be expressed as follows:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = T + J_\phi(q)^T\lambda \tag{3}$$

$$\phi(q) = 0 \tag{4}$$

where $q = [q_1\ q_2]^T$, q_1, q_2 are generalized coordinates, $M(q)$ denotes the inertial matrix, $C(q,\dot{q})$ is a matrix which characterizes the Coriolis or centrifugal terms, $G(q)$ represents gravitational effects, $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^T$ is the input torque vector at the joints, $J_\phi(q)$ is the Jacobian of $\phi(q)$ which defines the constraint, λ is Lagrangian multiplier, and

$$M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$$

$$C(q,\dot{q}) = m_2 l_1 l_2 (\cos q_1 \sin q_2 - \sin q_1 \cos q_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} -(m_1+m_2) l_1 g \sin q_1 \\ -m_2 l_2 g \sin q_2 \end{bmatrix}$$

The constraint in working space is given by

$$\phi\left(x\right) = -x + a(l_1+l_2) - y$$

and in joint space

$$\phi\left(q\right) = a\left(l_1+l_2\right) - l_1\left(\cos q_1 + \sin q_1\right) - l_2\left(\cos q_2 + \sin q_2\right)$$

$$J_\phi(q) = \begin{bmatrix} l_1 \sin q_1 - l_1 \cos q_1 & l_2 \sin q_2 - l_2 \cos q_2 \end{bmatrix}$$

with $m_{11} = (m_1+m_2)l_1^2$, $m_{12} = m_2 l_1 l_2 (\sin q_1 \sin q_2 + \cos q_1 \cos q_2)$, $m_{22} = m_2 l_2^2$, m_1 and m_2 being link masses, l_1 and l_2 being link lengths, and g denoting the acceleration due to gravity.

The dynamics can be expressed in the vector form by

$$\begin{bmatrix} M(q) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -C(q,\dot{q})\dot{q} - G(q) + T + J_\phi(q)^T \lambda \\ a\left(l_1+l_2\right) - l_1\left(\cos q_1 + \sin q_1\right) - l_2\left(\cos q_2 + \sin q_2\right) \end{bmatrix} \tag{5}$$

which consists of two differential equations and one algebraic equation where there are three unknowns, i.e., q_1, q_2 and λ. These differential equations and algebraic equations are intrinsically coupled. Since the matrix of coefficients of the derivative term is singular, 5 is a high-index singular system.

1.3 Literature Review

Early research on the control of singular systems focused on linear systems [17],[56] and [18]. For such systems, fundamental properties, such as solvability, stability, and solution characteristics [19],[20], and concepts related to control, such as controllability and observability [17],[31], [21], and [22], system equivalence [73],[75], [76], and [77], and minimal realizations [23],[24], [25], and [26], have been intensively studied. Based on these controller design methods can be broadly categorized into two classes. The first class of these methods is developed within the framework of smooth solutions corresponding to consistent initial conditions [17],[27] and [74], and the second one is for singular systems with arbitrary initial conditions [48],[29], [30],[49],[31],[32],[33] and [47].

The advancements in control of nonlinear ODE systems [78],[79],[80], and [81] have accelerated the growth of the research activities on the control of nonlinear singular systems over the several decades. In this regard, system-theoretic properties, such as existence and uniqueness of solutions [57],[82],[83],[59],[60], and [64], and stability analysis using Lyapunove techniques [61],[62], and [31], have been studied for different classes of nonlinear singular systems. A variety of controller design results have also been derived [34],[35],[36],[37],[38],[39],[40],[41],[42], and [43]. These results lie within the classical framework of smooth solutions and in general, rely on the derivation of an ODE representation of the original singular system or singular system modified through feedback. Finally, the design of optimal controllers through nonlinear programming has also been addressed for certain classes of nonlinear singular systems [44],[45].

1.3.1 Linear Singular Systems

Consider a linear system with the following form

$$E\dot{x} = Ax(t) + Bu(t) \tag{6}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, $u(t) \in \mathbb{R}^m$ is the vector of input variables, and E, A and B are constant matrices. It is assumed that E is singular, that is why the system of the form Eq. (6) is called singular system [85],[78]. The solution characteristics of the singular system in Eq. (6) are determined by the corresponding matrix pencil $P = (sE - A)$ [84], where $s \in \mathbb{C}$. More specifically, the system is solvable if and only if the pencil P is regular, that is $\det(P) \neq 0$. A practical procedure for verifying the regularity of the matrix pencil is provided by Luenberger's shuffle algorithm [86].

If $rank(E) = r < n$ and $\det(P)$ is a polynomial of degree $d(0 \leq d \leq r)$, then there exist non-singular matrices P and Q such that pre-multiplying Eq. (6) with P and employing a coordinate change.

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = Qx$$

yields the following standard canonical form representation [84]

$$\begin{aligned} \dot{x}_1 &= A_1\bar{x}_1 + B_1u(t) \\ N\dot{x}_2 &= \bar{x}_2 + B_2u(t) \end{aligned} \tag{7}$$

where $\bar{x}_1 \in \mathbb{R}^s$ and $\bar{x}_2 \in \mathbb{R}^{n-s}$. In the above description, A_1, B_1, and B_2 are constant matrices of appropriate dimensions, and N is a $(n-s) \times (n-s)$ matrix of nil-potency φ, i.e., all eigen values of N are zero and $N^{\varphi} = 0$ while $N^i \neq 0$ for $i \leq \varphi - 1$. When $d = r$, the matrix N is identically zero with an index of nil-potency $\varphi = 1$, and the x_2 subsystem in Eq. (7) is a purely algebraic system.

In the system of Eq. (7), the ODE subsystem in x_1 is decoupled from the x_2 subsystem and has a solution given by

$$x_1(t) = e^{At}x_1(0) + \int_0^t e^{A(t-\tau)}x_1 B_1 u(\tau)\, d\tau, \quad t \geq 0 \tag{8}$$

for any initial condition $x_2(0)$ and continuous inputs $u(t)$, which is smooth.

The singularity of the matrix E may arise from a sudden change in the dynamic system or from approximating some small parameters with zero. Therefore, the state x of the system at time $t = 0$ need not be constrained. For such an arbitrary initial condition $x(0_-) \in \mathbb{R}^n$, the system of Eq. (6) has a distributional solution under the same condition of regularity of the matrix pencil P. The system in Eq. (7) with the arbitrary initial condition $x_2(0_-)$, the x_2 subsystem has the following distributional solution [20],[87].

$$x_2(t) = -\sum_{i=0}^{\varphi-1} \delta^{(i-1)} N^i x_2(0_-) - \sum_{i=0}^{\varphi-1} N^i B_2 u^i(t), t \geq 0 \tag{9}$$

where δ denotes the unit impulse (Dirac delta) function and $\delta^{(i)}$ denotes the i-th distributional derivative. Starting from an arbitrary initial condition $x_2(0_-)$, the above solution for $x_2(t)$ exhibits an impulsive behavior at $t = 0$.

For the consistent initial conditions $x_2(0_-)$, $N^i x_2(0_-) = 0$. As a result, the solution for $x_2(t)$ is uniquely determined by the forcing inputs $u(t)$:

$$x_2(t) = -\sum_{i=0}^{\varphi-1} N^i B_2 u^i(t), t \geq 0 \tag{10}$$

where $u^{(i)}$ denotes the i-th derivative of the input. Thus, unlike ODE systems, singular systems of Eq. (6) do not have smooth solutions for arbitrary initial conditions. Only initial conditions $x(0)$ that are consistent, i.e., $x(0)$ such that $x_2(0)$ satisfies Eq. (10) at $t = 0$, yield smooth solutions. Furthermore, unlike ODE systems, if the index φ for singular systems of Eq. (6) exceeds one, then the solution of the singular system may depend on derivatives of the inputs $u(t)$.

The solvability of linear time-invariant systems of Eq. (6) is equivalent to the regularity of the matrix pencil P. However, for linear time-varying systems with the form

$$E(t)\dot{x}(t) = A(t)x\,(t) + B(t)u\,(t) \tag{11}$$

solvability and regularity of the pencil, $P = (sE(t) - A(t))$, are totally independent concepts [8]. A characterization of solvability for linear time-varying systems of Eq. (11) and a general canonical form representation for solvable systems were obtained in [46].

The impulsive behavior in the solution for x_2 essentially corresponds to the presence of poles at infinity in the system of Eq. (6). In a generalized state-space framework [31], the system of (6) has impulsive modes if and only if $\varphi > 1$, or equivalently $d < r$, i.e. the degree of polynomial $\det(sE - A)$ is strictly less than the rank of matrix E.

The systems that are controllable at infinity [48], and [31], or equivalently, systems for which the pencil $(sE - A - BK)$ has index $\varphi = 1$ for some feedback u, those systems can have arbitrary initial conditions. For such systems, the problem of feedback pole placement [30] and optimal control [48], [49], and [32] through state feedback have been addressed.

A key requirement in the above-mentioned results is that singular system of (6) must be controllable at infinity. However, there is a broad class of singular systems that are not controllable at infinity, for which it is not possible to eliminate the impulsive behavior arising from arbitrary initial conditions. Thus, another research direction has focused on addressing the control of singular systems within the conventional perspective of smooth solutions corresponding to consistent initial conditions [17],[27], and [28]. This study is also focusing on singular systems that are not controllable at infinity.

1.3.2 Nonlinear Singular Systems

The general fully-implicit form of nonlinear singular systems,

$$F(\dot{x}, x, u(t)) = 0 \tag{12}$$

where $x \in \chi \subset \mathbb{R}^n$ is the vector of state variable (χ is an open connected set), $u(t) \in \mathbb{R}^m$ is the vector of input variables and F : $\mathbb{R}^n \times \chi \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth function. In this subsection, the characteristics of singular systems with specified time-varying inputs $u(t)$ will be discussed. Clearly, the system in Eq. (12) is an implicit ODE system, if the Jacobian $\left(\frac{\partial F}{\partial \dot{x}}\right)$ is non-singular. On the other hand, if $\left(\frac{\partial F}{\partial \dot{x}}\right)$ is singular, then the system exhibits fundamentally different characteristics from ODE systems.

There has been a substantial amount of research on addressing basic system theoretic issues such as existence and uniqueness of solutions [57],[58],[59], and [60], stability [61],[62] and [63], for fully implicit singular systems of Eq. (6). However, the generality of the form of the system in Eq. (6) does not allow the development of explicit controller synthesis results. Furthermore, the majority of mechanical, electrical, and chemical process applications are modeled by singular systems in the so-called semi-explicit form where there is a distinct separation of differential and algebraic equations. In particular for mechanical systems, energy balances of mass dampers and springs yield explicit differential equations, while holonomic constraints on the motion give algebraic equations. On the other hand, for chemical processes, the standard dynamic balances of mass and energy yield explicit differential equations, while thermodynamic relations, empirical correlations, quasi-steady state relations, etc. comprise the algebraic equations.

Nonlinear singular systems can be represented with the following semi-explicit form for the modeling of mechanical, electrical, and chemical process applications:

$$\begin{aligned}\dot{x} &= f(x) + b(x)z_\alpha + g(x)u(t) \\ 0 &= k(x) + l(x)z_\alpha + c(x)u(t)\end{aligned} \tag{13}$$

where $x \in \chi \subset \mathbb{R}^n$ is the vector of differential variables for which we have explicit differential equations, $z \in \mathbb{Z} \subset \mathbb{R}^p$ is the vector of algebraic variables that vary according to the algebraic equations, χ and $\mathbb{Z}$ are open connected sets, $u(t) \in \mathbb{R}^n$ is vector of the input variables, $f(x)$ and $k(x)$ are smooth matrices of appropriate dimensions. The inputs u and the algebraic variables z_α appear in the system equations in a linear fashion, which is typical of most practical applications. Systems which are nonlinear in u and/or z_α can also be easily transformed into the above form [81].

1.3.3 Solvability and Index

Initial research on solvability of nonlinear singular systems focused on specialized classes of systems [64], the solvability of the class of systems with the form:

$$F(y,t) = 0, \qquad A(y,t)\frac{dy}{dt} = G(y,t) \tag{14}$$

was studied using the theory of differential equations on manifolds. In (14), F and G are vector valued mappings of dimensions n and m ($m < n$), respectively, and A is a matrix operator of dimension $m \times n$. Under sufficient smoothness of F, A, and G and the condition that the matrix:

$$\det\left(\begin{bmatrix} D_y F(y,t) \\ A(y,t) \end{bmatrix}\right) \neq 0 \tag{15}$$

on the manifold where $F(y,t) = 0$, a solution exists. In Eq. (15), $D_y F(y,t)$ denotes the Jacobian $\left(\frac{\partial F(y,t)}{\partial y}\right)$. The system of Eq. (14) corresponds to a locally unique vector field on this manifold. On the other hand, if the matrix in (15) is singular, then the system in Eq.

(14) permits solutions only on a lower dimensional manifold. This latter class of systems was termed as algebraically incomplete; they are essentially high index systems. The work of [64] was extended in [65] to a more general class of first and second order singular systems with indices not exceeding two and three, respectively.

This differential-geometric approach of viewing singular systems as appropriate vector fields on manifolds was used in [66] to address the solvability of singular systems with the form:

$$A(y,t)\frac{dy}{dt} = f(z) \tag{16}$$

where the matrix $A(z)$ is singular and then generalized in [60] to more general systems of the form:

$$F(t, x, \dot{x}) = 0 \tag{17}$$

The approach for the specification of sufficient conditions for solvability entailed a recursive identification of a family of constraint manifolds M_i, $i = 0, \ldots s$, where $M_{(i+1)} \subset M_i$ and s, called the degree of the singular system, is the largest integer such that $M_{(s-1)} \neq M_s$. The singular system was termed as regular if, under appropriate conditions, it corresponded to a unique vector field on the constrained manifold and thus, permitted a locally unique solution on the manifold. An analogous approach was followed in [58],[59] to derive sufficient conditions under which a general singular system of the form in Eq. (17) is equivalent to an ODE system on an appropriately constrained manifold.

[57] gives a method of repetitive differentiation of Eq. (17) to obtain sufficient conditions for solvability in terms of an extended set of equations, called the derivative array equations. More specifically, differentiating Eq. (17) k times with respect to t yields the following

extended set of equations

$$F_k(t,x,\dot{x},w) = \begin{bmatrix} F(t,x,\dot{x}) \\ F_l(t,x,\dot{x}) + F_x(t,x,\dot{x})\dot{x} + F_{\dot{x}}(t,x,\dot{x})\ddot{x} \\ \vdots \\ \frac{d^k}{dt^k}F(t,x,\dot{x}) \end{bmatrix} = 0 \tag{18}$$

which involves higher-order time derivatives of $x, w = [x^{(2)}, \ldots, x^{(k+1)}]$. While the above set of equations is still singular with respect to $x^{(k+1)}$ owing to the singularity of $F_{\dot{x}}(t,x,\dot{x})$, it is possible that they uniquely determine $\dot{x} = \phi(x,t)$ for some $k \geq 1$. Indeed, under certain rank conditions for suitable Jacobians of F_k. [57] gives a detailed description of these conditions and their comparisons with other results of solvability conditions.

For the general fully implicit singular system of Eq. (17), the index φ is defined as the smallest integer such that the derivative array equation $F_{vd}(t,x,\dot{x},w) = 0$ in Eq. (18) uniquely determines $\dot{x}$ as a function of t, x [8]. In practice, depending on the structure of the system in Eq. (17), it may not be necessary to differentiate all equations. In particular, for the semi-explicit singular system of (13), one needs to differentiate only the algebraic equations, and the index of such systems has the following definition [8].

Definition 1.1:[90] The index φ of the singular system in Eq. (13), with specified smooth input $u(t)$, is the minimum number of times the algebraic equations or their subsets have to be differentiated to obtain a set of differential equations for z_α, i.e.

$$\dot{z}_\alpha = F_\varphi(x, z_\alpha, t).$$

The index φ provides a measure of the "singularity" of the algebraic equations and the resulting differences from ODE systems. More specifically, consider the singular system of Eq. (13) with a non-singular matrix $l(x)$. Clearly, the algebraic equations can be solved for

z_α:

$$z_\alpha = -l(x)^{-1}\left[k(x) + c(x)u(t)\right] \tag{19}$$

and one differentiation of the algebraic equations in Eq. (13) or equivalently the solution for z_α, would yield the differential equations for z_α, i.e., the singular system has an index $\varphi = 1$. For such systems, a direct substitution of the solution for z_α in the differential equations for x, yields an equivalent ODE representation:

$$\dot{x} = \bar{f}(x) + \bar{g}(x)\bar{u}(x) \tag{20}$$

where

$$\begin{aligned} \bar{f}(x) &= f(x) - b(x)l(x)^{-1}k(x) \\ \bar{g}(x) &= g(x) - b(x)l(x)^{-1}c(x) \end{aligned}$$

Thus singular systems of Eq. (13) with an index of 1 are essentially the same as ODE systems, and the simulation and control of such systems can be easily addressed on the basis of their ODE representations. Note that, due to linear appearances of the algebraic variables z_α in Eq. (13), the linearity with respect to the inputs u is preserved in Eq. (20).

In contrast, systems with singular algebraic equations, more specifically a singular matrix $l(x)$, cannot be readily reduced into ODE systems and they may have a high index $\varphi > 1$. It is worth noting that some singular systems may not have any index. This research note will focus on only singular systems which have an index greater than one. Moreover, a finite index for some smooth input $u(t)$ is assumed, since this is necessary for the existence of a locally unique smooth solution $x(t)$ and $z_\alpha(t)$.

1.3.4 Difference Between High Index Singular Systems and ODE Systems

Considering an ODE system with the following form is considered

$$\dot{x} = f(x) + g(x)u(t) \tag{21}$$

where $x \in \chi \subset \mathbb{R}^n$ is the vector of state variables, $u(t) \in \mathbb{R}^n$ is the vector of input variables, $f(x)$ is smooth vector field, and $g(x)$ is a smooth matrix of dimension $(n \times m)$. The ODE system can be viewed as a special class of singular systems of the form in (13) with an index $\varphi = 0$. As mentioned before, index-1 singular systems of Eq. (13) with non-singular matrices $l(x)$ are also the same as ODE systems of Eq. (21).

High index singular systems with a singular matrix $l(x)$ are different from ODE systems. The singular algebraic equations in Eq. (13) imply the presence of algebraic constraints in the differential variables x. If the index φ exceeds 2, then the algebraic equations have to be differentiated several times in order to obtain a solution for z_α, and the differentiated equations impose additional constraints in x. These constraints restrict the solution for x in a state space of dimensions less than n. Consequently, for arbitrary initial conditions $x(0)$, the singular system exhibits an impulsive behavior at the initial time $t = 0$. This is a common cause of failure in the numerical simulation of singular systems. Only the initial conditions $x(0)$ that satisfy the underlying algebraic constraints in the x, allow smooth solutions $(x(t), z_\alpha(t))$ and are referred to as consistent initial conditions [68],[69] and [70]. This is clearly in contrast with ODE systems of Eq. (21) for which a well defined solution exists for any initial condition $x(0)$. Furthermore, unlike ODE systems in Eq. (21), the solution $(x(t), z_\alpha(t))$ of high index singular systems in Eq. (13) may also depend on the time derivatives of the inputs $u(t)$, in which case these inputs must be sufficiently smooth. These fundamental differences between singular systems and ODE systems can be illustrated more clearly through the following simple example.

Example 1.1

Consider the following singular system [90]

$$\begin{aligned} x_1 &= x_2 + z \\ x_2 &= x_1 + x_3 \\ x_3 &= x_1 + x_2 + u(t) \\ 0 &= x_2 + x_3 \end{aligned} \tag{22}$$

with one algebraic variable z and one input $u(t)$ specified as a function of time. Note that the algebraic equation

$$0 = x_2 + x_3 \tag{23}$$

does not involve z_α, i.e. $l(x) = 0$, and thus the system has a high index. Moreover, the algebraic equation already denotes a constraint in the differential variables x. Differentiating this constraint once and using the relation for $\dot{x}_2$ and $\dot{x}_3$ yields the following new algebraic equation:

$$0 = 2x_1 + x_2 + x_3 + u(t) \tag{24}$$

which denotes another constraint in x. Note however that this constraint explicitly involves the input $u(t)$. Another differentiation of this constraint yields the following equation:

$$0 = 2x_1 + 3x_2 + x_3 + u(t) + \dot{u}(t) + 2z_\alpha \tag{25}$$

which can be solved

$$z_\alpha = -0.5(2x_1 + 3x_2 + x_3 + u(t) + \dot{u}(t)$$

Thus the singular system in Eq. (22) has an index $v_d = 3$. Another differentiation of Eq. (25) would yield the differential equation for z. The algebraic equations in Eq. (23) and Eq. (24) denote two constraints in x, one of which involves $u(t)$. These two constraints restrict the evolution of $x(t)$ in a one dimensional state and a smooth solution $(x(t),\ z(t))$ exists only for consistent initial conditions $x(0)$ which satisfy the consistencies $x_2 + x_3 = 0$ and $2x_1 + x_2 + x_3 + u(t) = 0$. The solution for z depends on the derivative of the input, $u(t)$, implying that the input $u(t)$ must be continuously differentiable at least once.

1.3.5 Regularity

In the context of numerical simulations of singular systems with specified inputs $u(t)$, this implies that the inputs $u(t)$ must vary smoothly with time. However, in the context of feedback control, $u\,(t)$ is the vector of manipulated inputs that are not specified a priori as a function of time. As such, a feedback control law is designed for these inputs. The presence of these manipulated inputs in the underlying algebraic constraints in x, has important ramifications on controller designs due to the fact that the constrained state space region for x depends on the feedback control law for u. Regularity can be defined simply as follows:

Definition 1.2: [90] a singular system is called regular if the following conditions are satisfied.

1. There exists an index φ.

2. The set where the differential variable x are constrained to evolve is invariant under any control law for u.

Remark 1.1: Condition 2 in the above definition of regular systems essentially states that the underlying algebraically constrained state space region does not involve the inputs u. Thus, the requirement for a finite index in condition (1) is independent of the time-varying nature of these inputs, since the index does not depend on u under condition(2).

The notions of regularity is different from existing notions of regular systems used in the literature in the context of solvability; in [8], a linear singular system of Eq. (6) is referred to as regular if the associated matrix pencil P is regular, while in [66] and [60], a nonlinear singular system is said to be regular if it corresponds to a locally unique vector field on a manifold. This study is limited to regular singular systems that are not controllable in infinity. In the future, this study will be extended to the non-regular singular system.

1.4 Research Motivation

Singular systems describe a wider class of systems more naturally than regular systems [94]-[100]. Like regular systems, the stabilization problem of singular systems is comparatively easier than the tracking problem. A number of studies have been done on the tracking control problem of nonlinear singular systems using different controller design methods [101]-[105], [101]. Independently of this, nonlinear control systems based on Takagi-Sugeno (T-S) fuzzy model have also received a great deal of attention over the last several decades because of the model's simplicity and feasibility of industrial applications [106]-[110]. T-S fuzzy approach provides a simple and straightforward way to decompose the task of modeling and control design into a group of local tasks, which tend to be easier to handle. It also provides mechanism to blend these local tasks together to deliver the overall model and control design. It allows to use all the available advanced linear controller design tools as well. These tools for linear systems range from elegant state space optimal control to robust control paradigms. T-S fuzzy approach is more structure based and systematic to design controller than other fuzzy approaches. Moreover, advancement in convex optimization techniques makes complex Linear Matrix Inequality (LMI) problems easier to solve, which results in a simpler and more organized controller design technique based on the Takagi-Sugeno (T-S) fuzzy model and the Lyapunov method. There are many papers on the stability of index-1 singular systems using T-S fuzzy model based controllers [111]-[115], however very few studies have examined the tracking control of singular systems using T-S fuzzy model based methods [111], [115] and [116]. All of these studies related to the tracking problem use the same procedure proposed by Tadanari and Tanaka [115]. The approach followed in those papers takes advantage of the redundancy of singular systems to reduce the number of LMI conditions. This approach is only valid for regular systems, not for singular systems. From this author's best knowledge, until now, the robust tracking problem of high index singular systems using the T-S fuzzy model based controller design approach with LMI has not been discussed. Moreover, the reference model can not be arbitrarily chosen for singular systems. The reference model must satisfy the same algebraic constraints as the actual system to get an impulse-free solution. This condition makes the tracking controller design of nonlinear high index singular systems more difficult. Studies have been done to generate reference models for nonholonomic constraint (programmed constraint) systems [117]. Until now, for holonomic constraint systems, no attempts have been made to define a dynamic reference system that satisfies the same holonomic constraints of the plant. Using identification techniques, generating a stable nonlinear dynamic reference model for high index singular systems is not always feasible. The level of difficulty of the generation of the reference model increases with the increment of the index of the singular system. That is why a general and organized way is needed to solve this problem. Moreover, working with nonlinear constraints complicates the controller design technique. Therefore, a simple approach is necessary to design tracking controllers for nonlinear high index singular systems.

On the other hand, very few studies have been done on designing full order observers for nonlinear singular systems using T-S fuzzy logic based model. There are a number of research works on the observer design for linear singular systems [91],[92]. From the author's best knowledge, there is no work on the observer design for high index singular system using T-S fuzzy model. Singular systems are needed to be I-observable and R-observable to ensure the existence of a full order Luenberger observer. To satisfy this condition, the level of difficulty increases with the increment of the index of singular systems. Therefore, a convenient method is needed to design a full order robust optimal observer for the nonlinear singular systems, relaxing some of the conditions such as I-observability.

1.5 Contributions

In this thesis, the tracking control problem is discussed for a class of nonlinear high-index singular systems relying on T-S fuzzy modeling approach. First, a coordinate transformation is introduced to change a nonlinear high index singular system into a singular system composed of a linear fast subsystem and a nonlinear slow subsystem with strict-feedback form. With consistent initial conditions, the transformed singular system is equivalent to a nonlinear index-1 singular system with affine constraints. A stable linear reference system is proposed for the tracking control problem based on the nonlinear index-1 singular system. This reference system shares the same linear fast subsystem with the transformed singular system so that its response to the consistent initial conditions satisfies both the algebraic equation and hidden constraints. As a result, the dynamics of both the original system and the reference system are on the same constrained manifold. This transformed system satisfies the existence condition of a full order Luenberger observer, relaxing the I-observability condition.

Second, a T-S fuzzy system is used to model a nonlinear index-1 singular system. Based on this T-S fuzzy model, initially a state feedback controller and later on a observer-based state feedback controller are designed. In both cases, error dynamics are derived. From the derived error dynamics H_∞ tracking problems are formulated.

Third, a Lyapunov function method is used to develop sufficient conditions which guarantee the stability of the closed-loop systems and a prescribed H_∞ tracking performance. An LMI-based optimization problem is formulated from the sufficient conditions to obtain feedback gains for a prescribed attenuation level ρ. The simplicity of the proposed tracking controller design approach makes it suitable for practical applications.

Finally, two high index nonlinear singular systems from different fields are taken as examples to illustrate the entire design procedure and to verify the robustness of the proposed controllers for all bounded disturbances.

1.6 Thesis Outline

This thesis is organized as follows: In Chapter 2, a coordinate transformation is proposed to transform a high index singular system into a singular system with index-1 with hidden and explicit constraints in affine form. A dynamic reference model following the tracking control problem for the high index singular system is discussed. A state feedback controller is designed based on the T-S fuzzy model. The gains of the feedback controller are calculated by solving the optimization problem which is formulated by converting the H_∞ tracking problem into a LMI-based optimization problem. Sufficient conditions are derived to guarantee the solvability of the H_∞ tracking problem. In Chapter 3, the same transformation is derived but in a more general form. An observer-based reference model following the tracking control problem for the high index singular system is proposed, following the same procedure introduced in Chapter 2. Moreover, in this chapter, sufficient conditions are derived to ensure the existence of the T-S fuzzy based observer for high index singular systems. Chapter 4 provides two comprehensive examples to demonstrate the effectiveness of the designs introduced in Chapters 2 and 3. In Chapter 5, the positive aspects, along with the limitations of the proposed designs are discussed along with indications of future research directions of this study.

2 State Feedback Controller Design

Modeling is probably the most basic topic in any system theory. Fuzzy T-S model of any nonlinear system is actually based on the combination of linearized models along the system's trajectory. On the other hand, linearization methods of the nonlinear singular systems with nonlinear constraints need to satisfy several strong assumptions [89] and [88]. Therefore, a simple way is necessary to linearize the nonlinear singular systems having non-linear constraints. Section 2.1 provides a short discussion about the difficulties involved in linearizing the nonlinear singular systems with nonlinear constraints. A coordinate transformation is proposed to transform a high index singular system into a singular system with index-1 and to convert the system into a special form, suitable for linearization, which is given in Section 2.2. This transformation is slightly biased by the dynamics of the constrained mechanical systems. In Section 2.3, a fuzzy T-S model is derived for this transformed system. A dynamic reference model for the nonlinearly constrained singular system is introduced in Section 2.4. A state feedback tracking controller is designed based on T-S fuzzy model in the Section 2.5. The gains of the feedback controller are calculated by solving the optimization problem which is formulated, converting the H_∞ tracking problem into a LMI-based optimization problem. The H_∞ tracking problem is formulated in Section 2.6 and the conversion of the H_∞ problem into an LMI-based optimization problem is derived in Section 2.7, satisfying sufficient conditions to guarantee the solvability of the H_∞ tracking problem. Section 2.8 deals with the solution of the formulated LMI problem.

2.1 The Necessity of the Transformation

Special classes of singular systems can be linearized considering a number of assumptions [89]. It is easy to linearize a non-linear singular system with linear constraints, but when the constraints become nonlinear, the linearization process fails to approximate the linear version of the non-linear constraints. After linearization the stability becomes different from the stability of the actual nonlinear system excepts some special cases [89],[88]. The solution of the constrained state variables solved from the linearized constraints are not the same as those solved from the nonlinear constraints. The deviation of the error increases with the increase of the index of singular systems and the non-linearity of the constraints. After linearization, the linear constraints become different from the actual constraints. The problem regarding the linearization process of nonlinear singular systems can be overcome if nonlinear algebraic constraints can be linearized exactly. This is possible, by changing the basis of the actual state space to a new basis, where the non-linear constraints become linear. Moreover, it is easier to analyze the system and design the controller if the singular system is decoupled into two subsystems. One is a dynamic system which is called the slow system, and the other is an algebraic system which is known as the fast system. Therefore, a transformation is necessary to change the basis of the state space along with decoupling the system into a fast and a slow sub-systems..

2.2 Transformation for Nonlinear High Index Singular Systems

Consider a general singular system (or implicit system) described by the following differential-algebraic equations

$$f\left(\frac{d^n x(t)}{dt^n}, \frac{d^{n-1} x(t)}{dt^n}, ..., x(t), u(t), \lambda\right) = 0 \quad (26)$$

$$f_c(x(t)) = 0 \quad (27)$$

where $x = [x_d^T, x_c^T]^T \in R^m$, $x_d \in R^{m-\beta}$ is a vector of dynamic variables, $x_c \in R^\beta$ is a vector of constraint variables, $u \in R^m$ is a vector of control variables, $\lambda \in R^\beta$ is a vector of algebraic variables, and $f(\cdot) \in R^m$ and $f_c(\cdot) \in R^\beta$ are sufficiently smooth functions. It is known that most of constrained mechanical systems can be modelled by Eq. (26)-Eq. (27) [96], [131]. It is assumed that $\frac{\partial f}{\partial\left(\frac{d^n x}{dt^n}\right)}$ is non-singular, which means that, by the implicit function theorem [132], there exists a function $F\left(\frac{d^{n-1}x(t)}{dt},\ldots,x(t),u(t),\lambda\right)$ so that

$$f\left(F\left(\frac{d^{n-1}x}{dt^n},\ldots,x,u,\lambda\right),\frac{d^{n-1}x}{dt^{n-1}},\ldots,x,u,\lambda\right)=0$$

that is, $\frac{d^n x(t)}{dt^n}=F\left(\frac{d^{n-1}x(t)}{dt^{n-1}},\ldots,x(t),u(t),\lambda\right)$. It is further assumed that $\left(\frac{df_c}{dx}\frac{\partial F}{\partial\lambda}\right)$ is non-singular so that the index of the system is $\eta=n+1$. As a matter of fact, λ can be uniquely determined by differentiating (27) n times,

$$\begin{aligned}
\frac{d}{dt}f_c(x(t) &= h_{1}\left(x\left(t\right)\right)+\frac{d}{dx}f_c(x(t))\frac{d}{dt}x(t)\\
\frac{d^2}{dt^2}f_c(x(t) &= h_{2}\left(\frac{d}{dt}x(t),x\left(t\right)\right)+\frac{d}{dx}f_c(x(t))\frac{d^2}{dt^2}x(t)\\
\vdots &= \vdots\\
\frac{d^{n-1}}{dt^{n-1}}f_c(x(t)) &= h_{n-1}\left(\frac{d^{n-2}}{dt^{n-2}}x\left(t\right),\ldots,x\left(t\right)\right)+\frac{d}{dx}f_c(x(t))\frac{d^{n-1}}{dt^{n-1}}x(t)\\
\frac{d^n}{dt^n}f_c(x(t)) &= h_{n}\left(\frac{d^{n-1}}{dt^{n-1}}x\left(t\right),\ldots,x\left(t\right)\right)+\frac{d}{dx}f_c(x(t))\frac{d^n}{dt^n}x(t)\\
&= h_n\left(\frac{d^{n-1}}{dt^{n-1}}x(t),\ldots,x(t)\right)+\frac{df_c(x(t))}{dx}\times\\
&\quad\times F\left(\frac{d^{n-1}}{dt^{n-1}}x,\ldots,x(t),u(t),\lambda\right)\\
&= H\left(\frac{d^{n-1}}{dt^{n-1}}x\left(t\right),......x(t),u(t),\lambda\right)
\end{aligned}$$

It is convenient to introduce the variables

$$
\begin{aligned}
z_1 &= x_c,\ \ z_2 = \frac{d}{dt}x_c(t), \ldots, z_n = \frac{d^{n-1}}{dt^{n-1}}x_c(t) \\
z_{n+1} &= \lambda \\
z_{n+2} &= x_d, z_{n+3} = \frac{d}{dt}x_d(t), \ldots, z_{2n+1} = \frac{d^{n-1}}{dt^{n-1}}x_d(t)
\end{aligned}
$$

Now define a change of coordinates $v = \varphi\left(z\right)$, which is given by

$$
\begin{aligned}
v_1 &= -f_c(x(t)) = -f_c(z_1, z_{n+2}) \\
v_2 &= -\frac{dv_1}{dt} = \frac{d}{dt}f_c(x(t)) = h_1\left(x\left(t\right)\right) + \frac{d}{dx}f_c(x(t))\frac{d}{dt}x(t) \\
&= h_1\left(z_1, z_{n+2}\right) + \frac{d}{dx}f_c(z_1, z_{n+2})\begin{bmatrix} z_{n+3} \\ z_2 \end{bmatrix} \\
v_3 &= -\frac{dv_2}{dt} = -\frac{d^2}{dt^2}f_c(x(t)) = h_2\left(\frac{d}{dt}x(t), x\left(t\right)\right) + \frac{d}{dx}f_c(x(t))\frac{d^2}{dt^2}x(t) \\
&= h_2\left(z_1, z_2, z_{n+2}, z_{n+3}\right) + \frac{d}{dx}f_c(z_1, z_{n+2})\begin{bmatrix} z_{n+4} \\ z_3 \end{bmatrix} \\
\vdots &= \vdots \\
v_n &= -\frac{dv_{n-1}}{dt} = \left(-1\right)^n \frac{d^{n-1}}{dt^{n-1}}f_c(x(t)) \\
&= h_{n-1}\left(\frac{d^{n-2}}{dt^{n-2}}x\left(t\right), \ldots, x\left(t\right)\right) + \frac{d}{dx}f_c(x(t))\frac{d^{n-1}}{dt^{n-1}}x(t) \\
&= h_{n-1}\left(z_1, \ldots, z_{n-1}, z_{n+2}, \ldots, z_{2n}\right) + \frac{d}{dx}f_c(z_1, z_{n+2})\begin{bmatrix} z_{2n+1} \\ z_n \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}
v_{n+1} &= H\left(\frac{d^{n-1}}{dt}x\left(t\right), ...x(t), u(t), \lambda\right) \\
&= H\left(z_1, \ldots, z_n, z_{n+2}, \ldots, z_{2n+1}, u(t), z_{n+1}\right) \\
v_{n+2} &= x_d = z_{n+2} \\
v_{n+3} &= \frac{dv_{n+2}}{dt} = \frac{d}{dt}x_d(t) = z_{n+3} \\
\vdots &= \vdots \\
v_{2n+1} &= \frac{dv_{2n}}{dt} = \frac{d^{n-1}}{dt^{n-1}}x_d(t) = z_{2n+1}
\end{aligned}$$

and its inverse $z = \varphi^{-1}\left(v\right)$ exists if $\frac{df_c(x(t))}{dx_c}$ and $\frac{\partial}{\partial \lambda}H\left(\frac{d^{n-1}}{dt^{n-1}}x\left(t\right), ..., x(t), u(t), \lambda\right) = \left(\frac{df_c}{dx}\frac{\partial F}{\partial \lambda}\right)$ are non-singular. It can be verified that, in the new coordinates v, Eq. (26)-Eq. (27) takes the form of Eq. (28)-Eq. (29)

$$\begin{aligned}
0 &= -v_1 \\
\frac{dv_1}{dt} &= -v_2 \\
\frac{dv_2}{dt} &= -v_3 \\
\vdots &= \vdots \\
\frac{dv_{n-1}}{dt} &= -v_n \\
\frac{dv_n}{dt} &= -v_{n+1} \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
\frac{dv_{n+2}}{dt} &= v_{n+3} \\
\vdots &= \vdots \\
\frac{dv_{2n}}{dt} &= v_{2n+1} \\
\frac{dv_{2n+1}}{dt} &= g\left(v, u\right) \qquad (29)
\end{aligned}$$

with $g(v,u) = \frac{d^n}{dt^n} x_d = F_1\left(\frac{d^{n-1}}{dt} x(t), ...x(t), u(t), \lambda\right) = F_1(z, u(t)) = F_1\left(\varphi^{-1}(v), u(t)\right)$. It is obvious that the fast subsystem Eq. (28) is linear and the slow subsystem Eq. (29) is in a strict feedback form.

In order to guarantee that solutions to Eq. (28) and Eq. (29) are impulse-free, initial conditions are required to be consistent, which means that $v_i(0) = 0$ for $i = 1, \ldots, n+1$. It can be verified that, for the consistent initial conditions, solutions to (28) are $v_i(t) = 0$ for $i = 1, \ldots, n+1$. As a result, (28) and Eq. (29) are equivalent to

$$\begin{aligned} 0 &= -v_1 \\ 0 &= -v_2 \end{aligned}$$

$$\begin{aligned} 0 &= -v_3 \\ \vdots &= \vdots \\ 0 &= -v_n \\ 0 &= -v_{n+1} \end{aligned} \tag{30}$$

and

$$\begin{aligned} \frac{dv_{n+2}}{dt} &= v_{n+3} \\ \vdots &= \vdots \\ \frac{dv_{2n}}{dt} &= v_{2n+1} \\ \frac{dv_{2n+1}}{dt} &= g(v,u) \end{aligned} \tag{31}$$

respectively. It is obvious that the system described by Eq. (30) and (31) is regular and impulse-free for any initial consistent conditions.

2.3 T-S Fuzzy Model of The Transformed Nonlinear High Index Singular System

In order to represent a Takagi-Sugeno fuzzy model for Eq. (30) and (31), the function $g(v_1, \ldots, v_{2n+1}, u)$ is linearized at different operating points (v_i, u_i) using Taylor series expansion, that is,

$$\begin{aligned} a_i &= \left.\frac{\partial}{\partial v} g(v,u)\right|_{(v_i,u_i)} \\ b_i &= \left.\frac{\partial}{\partial u} g(v,u)\right|_{(v_i,u_i)} \end{aligned}$$

Then, the linearized singular system in v space becomes

$$E\dot{v}(t) = A_i v + B_i u \tag{32}$$

where $E = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}$, $B_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & b_i^T \end{bmatrix}^T$,

$$A_i = \begin{bmatrix} -I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ c_1^i & c_2^i & c_3^i & c_4^i & \cdots & c_{n+1}^i \end{bmatrix} \text{ with } \begin{bmatrix} c_1^i & c_2^i & c_3^i & c_4^i & \cdots & c_{n+1}^i \end{bmatrix} = a_i.$$

2.4 Dynamic Reference Model for Transformed Nonlinear High Index Singular System

Consider a reference system in v space as

$$
\begin{aligned}
0 &= -v_{r1}(t) \\
0 &= -v_{r2}(t) \\
0 &= -v_{r3}(t) \\
\vdots &= \vdots \\
0 &= -v_{r(n+1)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dv_{n+2}}{dt} &= v_{r(n+3)} \\
\vdots &= \vdots \\
\frac{dv_{2n}}{dt} &= v_{r(2n+1)} \\
\frac{dv_{2n+1}^{j}}{dt} &= \sum_{i=1}^{n+1} a_{cri}^{j} v_{ri} + \sum_{i=1}^{n} a_{dr(n+1+i)}^{j} v_{r(n+1+i)}^{j} + r_I^j(t), \\
\text{where } j &= 1, \ldots, m-\beta
\end{aligned}
$$

where v_{ri} has the same dimension as v_i for $i = 1, \ldots, 2n+1$, $v_{r(n+1+i)}^{j}$ is the j-th element of $v_{r(n+1+i)}$, $a_{cri}^{j} \in R^{1\times(n+1)\beta}$ can be chosen arbitrarily for $j = 1, \ldots, m-\beta$, $a_{dr(n+1+i)}^{j} \in R$, $i = 1, \ldots, n$, are the coefficients of a stable polynomial for $j = 1, \ldots, m-\beta$, $r_I = \begin{bmatrix} r_I^1 & \cdots & r_I^{m-\beta} \end{bmatrix}^T \epsilon R^{(m-\beta)\times 1}$ is the vector of reference inputs. This reference system automatically satisfies the constraints of the original system in v space, which can be written as

$$E\dot{v}_r(t) = A_r v_r(t) + B_r r_I(t) \tag{33}$$

where $B_r = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}^T$,

$$A_r = \begin{bmatrix} -I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ a_{r1} & a_{r(n+2)} & a_{r(n+3)} & a_{r(n+4)} & \cdots & a_{r(2n+1)} \end{bmatrix} \text{with for } , i = 1, \ldots, n$$

$$a_{r1} = \begin{bmatrix} a^1_{cr1} & \cdots & a^1_{cr(n+1)} \\ \vdots & \ddots & \vdots \\ a^{m-\beta}_{cr1} & \cdots & a^{m-\beta}_{cr(n+1)} \end{bmatrix}, \text{ and } a_{r(n+1+i)} = \begin{bmatrix} a^1_{dr(n+1+i)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^{m-\beta}_{dr(n+1+i)} \end{bmatrix}.$$

2.5 T-S Fuzzy Model and State Feedback Controller

A nonlinear singular system can be approximated by a Takagi-Sugeno fuzzy model with linear rule consequences [112]-[114]. The i-th rule of the continuous-time fuzzy model for the nonlinear singular system is of the following form,

If $z_1(t)$ is M_{i1} and $\cdots$ and $z_p(t)$ is M_{ip}

Then $E\dot{x}(t) = A_i x(t) + B_i u(t) + d_S(t)$

output

$$y(t) = C_i x(t) + d_O(t)$$

Here, M_{ip} is the fuzzy set, L is the number of rules, $x(t) \in R^{(2n+1)}$, $u(t) \in R^m$, and $d_S(t) \in R^{(2n+1)}$ are the bounded external disturbance, $E \in R^{(2n+1)\times(2n+1)}$, $A_i \in R^{(2n+1)\times(2n+1)}$, and $B_i \in R^{(2n+1)\times m}$ are defined in Section 2.2. $z(t) = [z_1(t), z_2(t), \ldots, z_p(t)]$ is the vector of premise variables. It is assumed that the premise variables are not functions of the input vector $u(t)$. $d_S(t)$ and $d_O(t)$ are the bounded disturbance in the states and in the outputs. Given a pair of $(x(t), u(t))$, the outputs of the fuzzy systems are inferred to as

$$\begin{aligned} E\dot{x}(t) &= \frac{\sum_{i=1}^{L} \mu_i(z(t)) [A_i x(t) + B_i u(t)] + d_S(t)}{\sum_{i=1}^{L} \mu_i(z(t))} \\ &= \sum_{i=1}^{L} h_i(z(t)) [A_i x(t) + B_i u(t)] + d_S(t) \end{aligned} \tag{34}$$

$$\begin{aligned} y(t) &= \frac{\sum_{i=1}^{L} \mu_i(z(t)) C_i x(t) + d_O(t)}{\sum_{i=1}^{L} \mu_i(z(t))} \\ &= \sum_{i=1}^{L} h_i(z(t)) C_i x(t) + d_O(t) \end{aligned} \tag{35}$$

where

$$\begin{aligned} \mu_i(z(t)) &= \prod_{j=1}^{p} M_{ij}(z_j(t)) \\ h_i(z(t)) &= \frac{\mu_i(z(t))}{\sum_{i=1}^{L} \mu_i(z(t))} \end{aligned}$$

for all t. The term $M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in M_{ij}.It is assumed that for all t, $\sum_{i=1}^{L} \mu_i(z(t)) > 0$ and $\mu_i(z(t)) \geq 0$ for $i = 1, 2, \ldots, L$. It follows from [118]-[120] that for all t, $\sum_{i=1}^{L} h_i(z(t)) = 1$ and $h_i(z(t)) \geq 0$ for $i = 1, 2, \ldots, L$.

Consider a reference model as follows:

$$E_r \dot{x}_r(t) = A_r x_r(t) + B_r r_I(t) \tag{36}$$

where $x_r(t) \in R^n$ is the vector of reference state variables, $A_r \in R^{n\times n}$ is an asymptotically stable matrix, $B_r \in R^{n\times m}$, and $r_I(t) \in R^m$ represents the reference input.

Now, parallel distributed compensation (PDC) [121]-[123] is used to design the fuzzy controller with a T-S fuzzy model.

Control rule i:

If $z_1(t)$ is M_{i1} and $\cdots$ and $z_p(t)$ is M_{ip}

Then $u(t) = K_i[x(t) - x_r(t)]$

The overall fuzzy PDC controller is given by

$$\begin{aligned} u(t) &= \frac{\sum_{i=1}^{L} \mu_i(z(t)) \left[K_j(x(t) - x_r(t))\right]}{\sum_{i=1}^{L} \mu_i(z(t))} \\ &= \sum_{i=1}^{L} h_i(z(t)) \left[K_i(x(t) - x_r(t))\right] \end{aligned} \tag{37}$$

The error dynamics are formulated by combining the actual system Eq. (34) and the reference system Eq. (36). After simplification, the augmented system can be expressed as:

$$\tilde{E}\dot{\tilde{x}}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \tilde{A}_{ij}^T \tilde{x}(t) + \tilde{W}(t) \tag{38}$$

where

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & E_r \end{bmatrix}, \tilde{x}(t) = \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}$$
$$\tilde{A}_{ij} = \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_r \end{bmatrix}, \tilde{W}(t) = \begin{bmatrix} d_S(t) \\ B_r r_I(t) \end{bmatrix}$$

Define the following H_∞ tracking performance [124]-[125], related to tracking error $[x(t) - x_r(t)]$, considering initial conditions $\tilde{x}(0)$,

$$\begin{aligned} & \int_0^{t_f} [x(t) - x_r(t)]^T Q [x(t) - x_r(t)] dt \\ \leq\ & \tilde{x}^T(0) \tilde{P} \tilde{x}(0) + \rho^2 \int_0^{t_f} \tilde{W}(t)^T \tilde{Q} \tilde{W}(t) dt \end{aligned} \tag{39}$$

where $\tilde{P}$ is a symmetric positive definite weighting matrix, t_f is the final time of control, ρ is the prescribed attenuation level, Q is a positive definite weighting matrix, and $\tilde{Q}$ is defined by

$$\tilde{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}$$

In Eq. (37) the objective is to bound the infinity norm of the systems operator smaller than the ρ, where L_2 norms are considered for the input and the output signals.

Definition 2.1: The problem of robust H_∞ fuzzy tracking control is solvable if there exist a fuzzy tracking controller in Eq. (37) for the augmented system in Eq. (38) so that an H_∞ tracking performance in Eq. (39) is achieved for all $\tilde{W}(t)$ with an attenuation level ρ and the closed loop system in Eq. (40) is quadratically stable.

$$\tilde{E}\dot{\tilde{x}}(t) = \sum_{i=1}^{r} h_i(z(t)) \sum_{j=1}^{r} h_j(z(t)) \tilde{A}_{ij} \tilde{x}(t) \tag{40}$$

2.6 Design for H_∞ Tracking Controller

It can be easily verified that the transformed system Eq. (32), $i = 1, \ldots, L$, and the reference system Eq. (33) are regular and impulse-free. The transformed open loop system Eq. (34) is regular, due to the fact that $\sum_{i=1}^{L} h_i(z(t)) = 1$,

$$
\begin{aligned}
&\sum_{i=1}^{L} h_i(z(t))\, A_i \\
&= \sum_{i=1}^{L} h_i(z(t)) \left(\begin{bmatrix} -I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ c_1^i & c_2^i & c_3^i & c_4^i & \cdots & c_{n+1}^i \end{bmatrix} \right) = \left(\begin{bmatrix} -I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ \bar{c}_1^i & \bar{c}_2^i & \bar{c}_3^i & \bar{c}_4^i & \cdots & \bar{c}_{n+1}^i \end{bmatrix} \right)
\end{aligned}
$$

which results in $\det\left(sE - \sum_{i=1}^{L} h_i(z(t))\, A_i\right) \neq 0$, where $\bar{c}_j^i = \sum_{i=1}^{L} h_i(z(t))\, c_j^i$. Again the transformed open loop system Eq. (34) is impulse-free because,

the fast system

$$
\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -I & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{n+1} \end{bmatrix}
$$

where $A_c = \begin{bmatrix} -I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -I \end{bmatrix}$, is always invertible therefore, the initial values of the constrained states variables can be uniquely determined from the independent state variable, resulting in a impulse free open loop system.

The following theorem provides sufficient conditions for the problem of robust H_∞ fuzzy tracking control to be solvable.

Theorem 2.1: If there exists a common matrix $\tilde{P} \epsilon R^{2n\times 2n} > 0$ such that

$$\begin{aligned} \tilde{E}^T\tilde{P} = \tilde{P}^T\tilde{E} &\geq 0 && (41)\\ \tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij} + \frac{1}{\rho^2}\tilde{P}\tilde{P} + \tilde{Q} &< 0 && (42) \end{aligned}$$

for $h_i(z(t))\,h_j(z(t)) \neq 0$, $\forall t$, $i,j = 1,2,\cdots,L$, for a prescribed attenuation level ρ, then the problem of robust H_∞ fuzzy tracking control is solvable.

Proof:

Consider the Lyapunov like function candidate $V(\tilde{x}(t)) = \tilde{x}^T(t)\,E^T\tilde{P}\tilde{x}(t)$. Then, the time derivative of $V(\tilde{x}(t))$ is

$$\begin{aligned}
&\dot{V}(\tilde{x}(t))\\
=\;& \dot{\tilde{x}}^T(t)\,\tilde{E}^T\tilde{P}x(t) + \tilde{x}^T(t)\,\tilde{E}^T\tilde{P}\dot{\tilde{x}}(t) = \left(\tilde{E}\dot{\tilde{x}}(t)\right)^T\tilde{P}\tilde{x}(t) + \tilde{x}^T(t)\,\tilde{P}^T\left(\tilde{E}\dot{\tilde{x}}(t)\right)\\
=\;& \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))\,h_j(z(t))\,\tilde{x}^T(t)\left[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}^T\tilde{A}_{ij}\right]\tilde{x}(t) + \tilde{W}^T(t)\,\tilde{P}\tilde{x}(t) + \tilde{x}^T(t)\,\tilde{P}\tilde{W}(t)\\
=\;& \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))\,h_j(z(t))\,\tilde{x}^T(t)\left[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}^T\tilde{A}_{ij}\right]\tilde{x}(t) - \frac{1}{\rho^2}\tilde{x}^T(t)\,\tilde{P}\tilde{P}\tilde{x}(t) + \tilde{x}^T(t)\,\tilde{P}\tilde{W}(t)\\
&+\tilde{W}^T(t)\,\tilde{P}\tilde{x}(t) - \rho^2\tilde{W}^T(t)\,\tilde{W}(t) + \frac{1}{\rho^2}\tilde{x}^T(\,t)\,\tilde{P}\tilde{P}\tilde{x}(\,t) + \rho^2\tilde{W}^T(t)\,\tilde{W}(t)\\
=\;& \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))\,h_j(z(t))\,\tilde{x}^T(t)\left[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}^T\tilde{A}_{ij}\right]\tilde{x}(t)\\
&-\left(\frac{1}{\rho}\tilde{P}\tilde{x}(t) - \rho\tilde{W}(t)\right)^T\left(\frac{1}{\rho}\tilde{P}\tilde{x}(t) - \rho\tilde{W}(t)\right) + \frac{1}{\rho^2}\tilde{x}^T(\,t)\,\tilde{P}\tilde{P}\tilde{x}(\,t) + \rho^2\tilde{W}^T(t)\,\tilde{W}(t)\\
\leq\;& \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))\,h_j(z(t))\,\tilde{x}^T(t)\left[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}^T\tilde{A}_{ij}\right]\tilde{x}(t)\\
&+\frac{1}{\rho^2}\tilde{x}^T(\,t)\,\tilde{P}\tilde{P}\tilde{x}(\,t) + \rho^2\tilde{W}^T(t)\,\tilde{W}(t)
\end{aligned}$$

$$\leq \sum_{i=1}^{L}\sum_{j=1}^{L} h_i\left(z\left(t\right)\right) h_j\left(z\left(t\right)\right) \tilde{x}^T\left(t\right)\left[\tilde{A}_{ij}^T\tilde{P} + \tilde{P}^T\tilde{A}_{ij} + \frac{1}{\rho^2}\tilde{P}\tilde{P}\right]\tilde{x}\left(t\right) + \rho^2\tilde{W}^T\left(t\right)\tilde{W}\left(t\right)$$

It follows from Eq. (42) that

$$\begin{aligned}
\dot{V}\left(\tilde{x}\left(t\right)\right) &\leq -\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z\left(t\right)\right) h_j\left(z\left(t\right)\right)\tilde{x}^T\left(t\right)\tilde{Q}\tilde{x}\left(t\right) + \rho^2\tilde{W}^T\left(t\right)\tilde{W}\left(t\right) \\
&\leq -\tilde{x}^T\left(t\right)\tilde{Q}\tilde{x}\left(t\right)\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z\left(t\right)\right) h_j\left(z\left(t\right)\right) + \rho^2\tilde{W}^T\left(t\right)\tilde{W}\left(t\right) \\
&\leq -\tilde{x}^T\left(t\right)\tilde{Q}\tilde{x}\left(t\right) + \rho^2\tilde{W}^T\left(t\right)\tilde{W}\left(t\right)
\end{aligned} \tag{43}$$

Integrating Eq. (43) from 0 to t_f yields

$$V\left(t_f\right) - V\left(0\right) \leq -\int_0^{t_f} \tilde{x}^T\left(t\right)\tilde{Q}\tilde{x}\left(t\right)dt + \rho^2\int_0^{t_f} \tilde{W}^T\left(t\right)\tilde{W}\left(t\right)$$

which is equivalent to

$$\begin{aligned}
& \int_0^{t_f} \left(x\left(t\right) - x_r\left(t\right)\right)^T\tilde{Q}\left(x\left(t\right) - x_r\left(t\right)\right)dt \\
= & \int_0^{t_f} \tilde{x}^T\left(t\right)\tilde{Q}\tilde{x}\left(t\right)dt \\
\leq & \ \tilde{x}^T\left(0\right)\tilde{P}\tilde{x}\left(0\right) - \tilde{x}^T\left(t_f\right)\tilde{P}\tilde{x}\left(t_f\right) + \rho^2\int_0^{t_f} \tilde{W}^T\left(t\right)\tilde{W}\left(t\right) \\
\leq & \ \tilde{x}^T\left(0\right)\tilde{P}\tilde{x}\left(0\right) + \rho^2\int_0^{t_f} \tilde{W}^T\left(t\right)\tilde{W}\left(t\right)
\end{aligned}$$

Therefore, the augmented system Eq. (38) is stable for $\tilde{W}^T\left(t\right) = 0$ and the H_∞ tracking performance (39) for a prescribed attenuation level ρ is achieved if Eq. (41) and Eq. (42) are satisfied.

2.7 Conversion of $\mathbf{H}_\infty$ Optimal Problem into LMI Problem

It is known that the problem of robust H_∞ tracking control for regular systems can be formulated as a minimization problem [126]. The minimization problem consists in finding a common matrix $\tilde{P}$ such that both Eq. (41) and Eq. (42) are satisfied for a minimum attenuation level ρ. This minimization problem can be transformed into a minimization problem subject to some LMIs. The LMI problem can be solved in a computationally efficient manner using a convex optimization technique such as interior point method [127]-[128]. The following theorem provides sufficient conditions for the solvability of the problem of robust H_∞ tracking control based on LMIs.

Theorem 2.2: If there exist symmetric positive definite matrices S_{11}, S_{22}, C_{11}, C_{22} such that $P_{11} = diag(S_{11}, S_{22})$ and $P_{22} = diag(C_{11}, C_{22})$ satisfying the following LMIs for a prescribed attenuation level ρ and a positive definite weighting matrix Q

$$S_{11} = S_{11}^T > 0 \tag{44a}$$

$$S_{22} = S_{22}^T > 0 \tag{44b}$$

$$C_{11} = C_{11}^T > 0 \tag{44c}$$

$$C_{22} = C_{22}^T > 0 \tag{44d}$$

$$\begin{bmatrix} M_{11} & -P_{11}B_iK_j - Q & 0 \\ (-P_{11}B_iK_j - Q)^T & M_{22} & P_{22} \\ 0 & P_{22} & -\rho^2 I \end{bmatrix} < 0 \tag{45}$$

then the problem of robust H_∞ tracking control is solvable, where

$$\begin{aligned} M_{11} &= (A_i + B_iK_j)^T P_{11} + P_{11}(A_i + B_iK_j) + \frac{1}{\rho^2}P_{11}P_{11} + Q \\ M_{22} &= A_r^T P_{22} + P_{22}A_r + Q \end{aligned}$$

Proof: Define $\tilde{P} = diag(P_{11}, P_{22})$. First, let us show that $\tilde{P}$ satisfies Eq. (41). Due to Eq.

(44a) to Eq. (44d)

$$
\begin{aligned}
E^T P_{11} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \\
P_{11}^T E &= \begin{bmatrix} S_{11}^T & 0 \\ 0 & S_{22}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_{11}^T & 0 \\ 0 & 0 \end{bmatrix} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
E_r^T P_{22} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & 0 \\ 0 & S_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \\
P_{11}^T E_r &= \begin{bmatrix} C_{11}^T & 0 \\ 0 & C_{22}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_{11}^T & 0 \\ 0 & 0 \end{bmatrix} \geq 0
\end{aligned}
$$

It follows that $E^T P_{11} = P_{11}^T E \geq 0$ and $E_r^T P_{22} = P_{11}^T E_r \geq 0$. Therefore,

$$
\begin{aligned}
\tilde{E}^T \tilde{P} &= \begin{bmatrix} E^T & 0 \\ 0 & E_r^T \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} = \begin{bmatrix} E^T P_{11} & 0 \\ 0 & E_r^T P_{22} \end{bmatrix} \\
&= \begin{bmatrix} P_{11}^T E & 0 \\ 0 & P_{22}^T E_r \end{bmatrix} = \begin{bmatrix} P_{11}^T & 0 \\ 0 & P_{22}^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E_r \end{bmatrix} \\
&= \tilde{P}^T \tilde{E} \geq 0.
\end{aligned}
$$

To show that Eq. (42) is satisfied with $\tilde{P}$, replacing $\tilde{P}$ with $diag(P_{11}, P_{22})$ in Eq. (42) gives

$$
\begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_r \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}^T \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_r \end{bmatrix}
$$

$$
+\frac{1}{\rho^2} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} + \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} < 0 \tag{46}
$$

After simplification and taking the Schur complements, it can be shown that (46) can be expressed as Eq. (45) [126].

2.8 Solution of the LMI Problem

Note that Eq. (45) can not be solved in a single step using LMI toolbox. However, it can be solved by the following two-step procedure [126].

Step 1: Find P_{11} and K_j

$$\begin{aligned} S_{11} &= S_{11}^T > 0 & (47) \\ S_{22} &= S_{22}^T > 0 & (48) \end{aligned}$$

$$(A_i + B_i K_j)^T P_{11} + P_{11} (A_i + B_i K_j) + \frac{1}{\rho^2} P_{11} P_{11} + Q < 0 \tag{49}$$

for P_{11}. Set $\bar{Z}_{11} = P_{11}^{-1}$, $\bar{Z}_{11} = diag(z_{11}, z_{22})$, $z_{11} = S_{11}^{-1}$, $z_{22} = S_{22}^{-1}$ and $Y_j = K_j \bar{Z}_{11}$. Eq. (47), Eq. (48) and Eq. (49) can be rewritten as

$$\begin{aligned} z_{11} &= z_{11}^T > 0 & (50) \\ z_{22} &= z_{22}^T > 0 & (51) \end{aligned}$$

$$\begin{bmatrix} \bar{Z}_{11} A_i^T + A_i \bar{Z}_{11} + B_i Y_j + (B_i Y_j)^T + \frac{1}{\rho^2} I & \bar{Z}_{11} \\ \bar{Z}_{11} & -Q^{-1} \end{bmatrix} < 0 \tag{52}$$

For a prescribed attenuation level ρ, the matrices $\bar{Z}_{11}$ and Y_j (thus $P_{11} = \bar{Z}_{11}^{-1}$ and $K_j = Y_j W_{11}^{-1}$) can be obtained by solving LMIs Eq. (50), Eq. (51) and Eq. (52).

Step-2: Find P_{22}

P_{11} and K_j are substituted into Eq. (45). Then the LMIs (44c), Eq. (44d) and Eq. (45) are solved for P_{22}.

The attenuation level ρ can be minimized by searching for $P_{11} > 0, P_{22} > 0$ so that $\min_{P_{11},P_{22}} \rho^2$. This optimization problem can be solved by reducing ρ and solving the LMIs with the above two-step procedure until just before no feasible solutions $P_{11} = P_{11}^T > 0$, $P_{22}^T > 0$ of the LMI problem can be found.

3 Observer-Based State Feedback Controller Design

The transformation introduced in the previous chapter is biased by the mechanical system dynamics but a more general form of the transformation is given in the Section 3.1, dealing with semi-explicit representation of the singular system, as most of the electrical, mechanical, and chemical systems are modeled in that form. A full-order fuzzy T-S model based observer for the singular system is designed in Section 3.2. Necessary conditions to ensure the existence of an observer are derived in Section 3.3. In the same section, an H_∞ problem is formulated by augmenting the tracking error and observer error dynamics. All the gains of the controller as well as the observer are obtained by solving this H_∞ optimal problem. Sufficient conditions are derived to guarantee the solvability of the H_∞ tracking problem. The H_∞ problem is converted into an LMI-based optimization problem to solve the problem using the convex optimization technique. These conversions are discussed in Section 3.4.The solution of the convex optimization problem is given in Section 3.5.

3.1 Transformation for Nonlinear High Index Singular System in Semi-explicit Form

Consider nonlinear multi-input multi-output (MIMO) singular systems with the semi-explicit form:

$$\frac{d^n x}{dt^n} = f(x(t)) + b\,(x(t))\,z + g\,(x(t))\,u(t) \tag{53}$$

$$0 = k\,(x(t)) + l\,(x(t))\,z + c\,(x(t))\,u \tag{54}$$

$$y_i = h_i(x), i = 1, \cdots, m \tag{55}$$

where $x = [x_d^T, x_c^T]^T \epsilon R^m$, $x_d \epsilon R^{m-\beta}$ is a vector of dynamic variables, $x_c \epsilon R^\beta$ is a vector of constraint variables, $z \epsilon R^p$ is a vector of algebraic variables, $u \epsilon R^m$ is a vector of control variables, and y_i is the i-th output. $f(x(t)) \epsilon R^m$ and $k(x(t)) \epsilon R^p$ are sufficiently smooth functions. $b(x(t))$, $l(x(t))$, $c(x(t))$, and $g(x(t))$ are smooth matrices of appropriate dimensions. In the above description, the input $u(t)$ and the algebraic variable z appear in the affine form in the system equations, which is typical of most practical applications. singular systems have a high index only if $l(x(t)) = 0$, i.e.

$$0 = k(x(t)) + c(x(t))\, u \tag{56}$$

As the focus of this study is only regular singular system therefore $c(x(t)) = 0$, then the algebraic equation becomes,

$$0 = k(x(t)) \tag{57}$$

Let consider the index of this system is $\eta = n + 1$.Then, z can be uniquely determined from the equation obtained by differentiating equation (57) n times with respect to time, that is

$$\begin{aligned}
\frac{d}{dt} k(x(t)) &= \frac{d}{dx} k(x(t)) \frac{dx(t)}{dt} \\
\frac{d^2}{dt^2} k(x(t)) &= \frac{d^2}{dx} k(x(t)) \frac{d^2 x(t)}{dt} \\
\vdots &= \vdots \\
\frac{d^{n-1}}{dt^{n-1}} k(x(t)) &= \frac{d^{n-1}}{dx} k(x(t)) \frac{d^{n-1} x(t)}{dt} \\
\frac{d^n}{dt^n} k(x(t)) &= \frac{d^n}{dx} k(x(t)) \frac{d^n x(t)}{dt} = \frac{d^n}{dx} k(x(t)) [f(x(t)) + b(x(t))\, z + g(x(t))\, u(t)] \\
&= H\left(\frac{d^{n-1}}{dt^{n-1}} x(t), ...x(t), u(t), z\right)
\end{aligned}$$

For simplification of the representation a new variable p, is introduced

$$\begin{aligned} p_1 &= x_c,\ p_2 = \frac{d}{dt}x_c(t), \ldots, p_n = \frac{d^{n-1}}{dt^{n-1}}x_c(t) \\ p_{n+1} &= x_d, p_{n+2} = \frac{d}{dt}x_d(t), \ldots, p_{2n} = \frac{d^{n-1}}{dt^{n-1}}x_d(t) \end{aligned}$$

Now define a change of coordinates $v = \varphi(p)$, which is given by

$$\begin{aligned} v_1 &= -k(x(t)) = -k(p_1, p_{n+1}) \\ v_2 &= -\frac{dv_1}{dt} = \frac{d^1}{dx}k(x(t))\frac{d^1 x(t)}{dt} = \frac{d}{dx}k(p_1, p_{n+1})\begin{bmatrix} p_{n+2} \\ p_2 \end{bmatrix} \\ \vdots &= \vdots \\ v_n &= -\frac{dv_{n-1}}{dt} = (-1)^n \frac{d^{n-2}}{dx}k(x(t))\frac{d^{n-2}x(t)}{dt} \\ &= \frac{d}{dx}k(p_1, p_{n+1})\begin{bmatrix} p_{2n-1} \\ p_{n-1} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v_{n+1} &= -\frac{dv_n}{dt} = H\left(\frac{d^{n-1}}{dt^{n-1}}x(t), ...x(t), u(t), z\right) \\ &= H(p_1, \ldots, p_n, \ldots, p_{2n}, u(t), z) \end{aligned}$$

$$\begin{aligned} v_{n+2} &= x_d = p_{n+1} \\ v_{n+3} &= \frac{dv_{n+2}}{dt} = \frac{d^1}{dt}x_d(t) = p_{n+2} \\ \vdots &= \vdots \\ v_{2n} &= \frac{dv_{2n-1}}{dt} = \frac{d^{n-2}}{dt^{n-2}}x_d(t) = p_{2n-1} \\ v_{2n+1} &= \frac{dv_{2n}}{dt} = \frac{d^{n-1}}{dt^{n-1}}x_d(t) = p_{2n} \end{aligned}$$

and its inverse $p = \varphi^{-1}(v)$ exists if $\frac{d}{dx_c}k(x(t))$ is non-singular. It can be verified that, in the new coordinates v, (53)-(54) takes the same form as equations (28)-Eq. (29). The rest of the part will be same as shown, in the Chapter 2. The difference between the previous transformation and the transformation derived in this chapter is, in the constrained mechanical system, the objective was to keep the algebraic variable (i.e., contact force) $\lambda \approx 0$ or any constant value, where as in other singular systems the algebraic variables have a value which changes with time depending on the nature of the constraint. That is why, in the transformation derived in this chapter, the n-th derivative of the constrained function is constrained to zero not the algebraic variable.

3.2 T-S Fuzzy Model Based Observer and Observer Based State Feedback Controller

Based on T-S fuzzy model Eq. (34) and Eq. (35), the i-th rule of the continuous-time fuzzy model based observer for the nonlinear singular system is,

Observer Rule i

If $z_1(t)$ is M_{i1} and $\cdots$ and $z_p(t)$ is M_{ip}

Then $E\dot{\hat{x}}(t) = A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t)),$

$\hat{y}(t) = C_i\hat{x}(t)$

Remark 3.1: The premise variables $Z(t)$ can be measurable state variables, outputs or combination of measurable state variables. For T-S type fuzzy model, using state variables as premise variables are common, but not always [101]-[103], [103]-[109]. The limitation of this approach is that some state variables must be measurable to construct the fuzzy observer and fuzzy controller. This is a common limitation for control system design of T–S fuzzy approach [107]-[108], If the premise variables of the fuzzy observer depend on the estimated state variables, i.e., $Z(t)$ instead of $\hat{Z}(t)$ in the fuzzy observer, the situation becomes more complicated. In this case, it is difficult to directly find control gains K_j and observer gains L_i. The problem has been discussed in [108]. For simplification, it is considered that premise variables do not depend on estimated state variables.

The overall fuzzy observer is represented as follows:

$$\begin{aligned} E\dot{\hat{x}}(t) &= \frac{\sum_{i=1}^{L} \mu_i(z(t))\left[A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\right]}{\sum_{i=1}^{L} \mu_i(z(t))} \\ &= \sum_{i=1}^{L} h_i(z(t))\left[A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\right] \end{aligned} \tag{58}$$

$$\begin{aligned} \hat{y}(t) &= \frac{\sum_{i=1}^{L} \mu_i(z(t))\, C_i\hat{x}(t)}{\sum_{i=1}^{L} \mu_i(z(t))} \\ &= \sum_{i=1}^{L} h_i(z(t))\, C_i\hat{x}(t) \end{aligned} \tag{59}$$

Consider the estimation error as

$$\hat{e} = x(t) - \hat{x}(t) \tag{60}$$

By differentiating (60) with respect to time, we get

$$\begin{aligned} \dot{\hat{e}} &= \dot{x}(t) - \dot{\hat{x}}(t) \\ E\dot{\hat{e}} &= E\dot{x}(t) - E\dot{\hat{x}}(t) \\ E\dot{\hat{e}} &= \sum_{i=1}^{L} h_i(z(t))\left[A_i x(t) + B_i u(t)\right] + d_S(t) \\ &\quad -\sum_{i=1}^{L} h_i(z(t))\left[A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t)) + L_i d_O(t)\right] \end{aligned}$$

$$E\dot{\hat{e}} = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i h_j (z(t)) [\{A_i x(t) + B_i u(t)\} - \{A_i \hat{x}(t) + B_i u(t) + L_i C_j (x(t) - \hat{x}(t))\} + L_i d_O(t)] + d_S(t)$$

$$E\dot{\hat{e}} = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i h_j (z(t)) \{[A_i (x(t) - \hat{x}(t)) - L_i C_j (x(t) - \hat{x}(t))] + L_i d_O(t)\} + d_S(t)$$

$$E\dot{\hat{e}} = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i h_j (z(t)) \{[(A_i - L_i C_j)\hat{e}] + L_i d_O(t)\} + d_S(t) \tag{61}$$

For this case, the parallel distributed compensation (PDC) [121]-[123] controller takes the following form

Control rule i:

If $z_1(t)$ is M_{i1} and $\cdots$ and $z_p(t)$ is M_{ip}

Then $u(t) = K_i[\hat{x}(t) - x_r(t)]$

The overall fuzzy PDC controller is given by

$$u(t) = \frac{\sum_{i=1}^{L} \mu_i(z(t)) [K_j(\hat{x}(t) - x_r(t))]}{\sum_{i=1}^{L} \mu_i(z(t))} = \sum_{i=1}^{L} h_i(z(t)) [K_i(\hat{x}(t) - x_r(t))] \tag{62}$$

where $i = 1, 2, \ldots, L$.

The augmented system is formulated by combining the error dynamics of estimation error and tracking error. After simplification, the augmented error dynamics can be expressed as:

$$\tilde{E}\dot{\tilde{x}}\left(t\right) = \sum_{i=1}^{L} h_i\left(z\left(t\right)\right) \sum_{j=1}^{L} h_j\left(z\left(t\right)\right) \tilde{A}_{ij}\tilde{x}\left(t\right) + \tilde{E}_i\tilde{W}(t) \tag{63}$$

where

$$\begin{aligned}
\tilde{E} &= \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}, \ \tilde{x}\left(t\right) = \begin{bmatrix} \hat{e}(t) \\ x(t) \\ x_r(t) \end{bmatrix}, \ \tilde{N}_i = \begin{bmatrix} L_i & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
\tilde{A}_{ij} &= \begin{bmatrix} A_i + C_i L_i & 0 & 0 \\ -B_i K_j & A_i + B_i K_j & -B_i K_j \\ 0 & 0 & A_r \end{bmatrix}, \ \tilde{W}(t) = \begin{bmatrix} d_O(t) \\ d_S(t) \\ r(t) \end{bmatrix}
\end{aligned}$$

Define the following H_∞ tracking performance [124]-[125], related to tracking error $\left(x\left(t\right) - x_r\left(t\right)\right)$, considering initial conditions $\left(x\left(0\right) - x_r\left(0\right)\right)$,

$$\begin{aligned}
&\int_0^{t_f T} \left(x\left(t\right) - x_r\left(t\right)\right)^T \left(t\right) \tilde{Q} \left(x\left(t\right) - x_r\left(t\right)\right) dt \\
\leq \ & \tilde{x}^T(0)\tilde{P}\tilde{x}(0) + \rho^2 \int_0^{t_f} \tilde{W}(t)/^T \tilde{W}(t) dt
\end{aligned} \tag{64}$$

where $\tilde{P}$ is a symmetric positive definite weighting matrix, t_f is the final time of control, ρ is the prescribed attenuation level, Q is a positive definite weighting matrix, and $\tilde{Q}$ is defined by

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & -Q \\ 0 & -Q & Q \end{bmatrix}$$

Definition 3.1: The problem of robust H_∞ fuzzy tracking control is solvable if there exists a fuzzy tracking controller in (62) for the augmented system in (63) so that an H_∞ tracking performance in (64) is achieved for all $\tilde{W}(t)$ with an attenuation level ρ and the closed loop system in (65) is quadratically stable.

$$\tilde{E}\dot{\tilde{x}}(t) = \sum_{i=1}^{r} h_i(z(t)) \sum_{j=1}^{r} h_j(z(t)) \tilde{A}_{ij}\tilde{x}(t) \tag{65}$$

3.3 Design for H_∞ Observer Based Tracking Controller

First, let us check if the system Eq. (61) with zero inputs is regular and impulse-free. To this end, suppose the output matrix C_i can be decomposed as

$$C_i = \begin{bmatrix} \bar{C}_{1i} & \bar{C}_{2i} & \bar{C}_{3i} & \cdots & \bar{C}_n & \bar{C}_{(n+1)i} \end{bmatrix}$$

Similarly, L_i can be denoted by

$$L_i = \begin{bmatrix} L_{1i} \\ L_{2i} \\ L_{3i} \\ \vdots \\ L_{ni} \\ L_{(n+1)i} \end{bmatrix}$$

Then, a straightforward derivation gives

$$
\begin{aligned}
& \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_i C_i\right) \\
= \ & \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(\begin{bmatrix} -I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ c_1^i & c_2^i & c_3^i & c_4^i & \cdots & c_{n+1}^i \end{bmatrix}\right. \\
& \left. - \begin{bmatrix} L_{1i} \\ L_{2i} \\ L_{3i} \\ \vdots \\ L_{ni} \\ L_{(n+1)i} \end{bmatrix} \begin{bmatrix} \bar{C}_{1i} & \bar{C}_{2i} & \bar{C}_{3i} & \bar{C}_{4i} & \cdots & \bar{C}_{(n+1)i} \end{bmatrix} \right)
\end{aligned}
$$

$$
= \sum_{i=1}^{L} h_i\left(z\left(t\right)\right) \left(\begin{bmatrix} -I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ c_1^i & c_2^i & c_3^i & c_4^i & \cdots & c_{n+1}^i \end{bmatrix} \right.
$$

$$
\left. - \begin{bmatrix} L_{1i}C_{1i} & L_{1i}C_{2i} & L_{1i}C_{3i} & L_{1i}C_{4i} & \cdots & L_{1i}C_{(n+1)i} \\ L_{2i}C_{1i} & L_{2i}C_{2i} & L_{2i}C_{3i} & L_{2i}C_{4i} & \cdots & L_{2i}C_{(n+1)i} \\ L_{3i}C_{1i} & L_{3i}C_{2i} & L_{3i}C_{3i} & L_{3i}C_{4i} & \cdots & L_{3i}C_{(n+1)i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{ni}C_{1i} & L_{ni}C_{2i} & L_{ni}C_{3i} & L_{ni}C_{4i} & \cdots & L_{in}C_{(n+1)i} \\ L_{(n+1)i}C_{1i} & L_{(n+1)i}C_{2i} & L_{(n+1)i}C_{3i} & L_{(n+1)i}C_{4i} & \cdots & L_{(n+1)i}C_{(n+1)i} \end{bmatrix} \right)
$$

$$
= \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)
$$

$$
\left(\begin{bmatrix} -I - L_{1i}C_{1i} & -L_{1i}C_{2i} & -L_{1i}C_3 & -L_{1i}C_4 & \cdots & -L_{1i}C_{(n+1)i} \\ -L_{2i}C_{1i} & -L_{2i}C_{2i} & I - C_3L_{2i} & -C_4L_{2i} & \cdots & -L_{2i}C_{(n+1)i} \\ -L_{3i}C_1 & -L_{3i}C_{2i} & -C_3L_{3i} & I - C_4L_{3i} & \cdots & -L_{3i}C_{(n+1)i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -L_{ni}C_1 & -L_{ni}C_{2i} & -L_{ni}C_3 & -L_{ni}C_4 & \cdots & I - L_{ni}C_{(n+1)i} \\ c_1^i - L_{(n+1)i}C_1 & c_2^i - L_{(n+1)i}C_2 & c_3^i - L_{(n+1)i}C_3 & c_4^i - L_{(n+1)i}C_4 & \cdots & c_{n+1}^i - L_{(n+1)i}C_{(n+1)i} \end{bmatrix} \right) \tag{66}
$$

Note that the solution to Eq. (30) is zero, thus L_1 can be set to 0. As a result, Eq. (66) becomes

$$
\begin{aligned}
& \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_i C_i\right) \\
= & \sum_{i=1}^{L} h_i\left(z\left(t\right)\right) \\
& \begin{bmatrix}
-I & 0 & 0 & 0 & \cdots & 0 \\
-L_{2i}C_{1i} & -L_{2i}C_{2i} & I - C_3 L_{2i} & -C_4 L_{2i} & \cdots & -L_{2i}C_{(n+1)i} \\
-L_{3i}C_1 & -L_{3i}C_{2i} & -C_3 L_{3i} & I - C_4 L_{3i} & \cdots & -L_{3i}C_{(n+1)i} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-L_{ni}C_1 & -L_{ni}C_{2i} & -L_{ni}C_3 & -L_{ni}C_4 & \cdots & I - L_{ni}C_{(n+1)i} \\
c_1^i - L_{(n+1)i}C_1 & c_2^i - L_{(n+1)i}C_2 & c_3^i - L_{(n+1)i}C_3 & c_4^i - L_{(n+1)i}C_4 & \cdots & c_{n+1}^i - L_{(n+1)i}C_{(n+1)i}
\end{bmatrix}
\end{aligned}
$$

which is equivalent to

$$
\det\left(sE - \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_i C_i\right)\right) \neq 0 \tag{67}
$$

Therefore, the system Eq. (61) is regular.

From [91], the singular system Eq. (34) and Eq. (35) has a full-order impuse-free singular state observer in the form of Eq. (58) if and only if it is R-observrable and I-observable.

After the transformation, the condition can be relaxed for the existence of a full-order impulse-free singular state observer for the system.

Theorem 3.1: The singular system Eq. (34) and Eq. (35) has a full-order impulse-free singular state observer in the form of Eq. (58) if it is R-observrable.

Proof:

The fast system is impulse free because $v\left(t\right) = 0$, $t \geq 0$, which has been proved in Section

2.5. Therefore, The singular system Eq. (34) and Eq. (35) is I-observable. As a result, only R-observability is sufficient for the existence of a full-order singular state observer.

Condition 3.1: [91] The slow subsystem is R-observable if and only if

$$rank \begin{bmatrix} sI - A_{22i} \\ C_{2i} \end{bmatrix} = n_1 \tag{68}$$

where A_{22i} is the i-th system matrix of the slow system. Eq. (68) will satisfy the existence condition of a full order state observer given below,

$$eig\left(E,\ \left(A_i - L_iC_i\right)\right) \subset \mathbb{C}^- \tag{69}$$

As the pair $(E,\ (A_i - L_iC_i))$ is regular according to Eq. (67), there exists gain L_i so that

$$eig\left(\lambda E - \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_iC_i\right)\right) \subset \mathbb{C}^- \tag{70}$$

Moreover, setting $C_{1i} = 0$ and $L_{1i} = 0$ for, $i = 1, \ldots, L$, in the equation Eq. (66), it can be shown that, $\det\left(sE - \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_iC_i\right)\right) \neq 0$, In that case, the condition $eig\left(\lambda E - \sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_iC_i\right)\right) \subset \mathbb{C}^-$ is still satisfied.

For a certain L_i, $\sum_{i=1}^{L} h_i\left(z\left(t\right)\right)\left(A_i - L_iC_i\right) \neq 0$,therefore, the matrix pair $(E,\ (A_i - L_iC_i))$ is regular.

The following theorem provides sufficient conditions for the problem of robust H_∞ fuzzy observer based tracking control to be solvable.

Theorem 3.2: If there exists a common matrix $\tilde{P}\ \epsilon\ R^{2n\times 2n} > 0$ such that

$$\tilde{E}^T\tilde{P} = \tilde{P}^T\tilde{E} \quad \geq \quad 0 \tag{71}$$

$$\tilde{A}_{ij}^T\tilde{P} + \tilde{P}^T\tilde{A}_{ij} + \frac{1}{\rho^2}\tilde{P}\tilde{E}_i\tilde{E}_i^T\tilde{P} + \tilde{Q} \quad < \quad 0 \tag{72}$$

for $h_i(z(t))\, h_j(z(t)) \neq 0$, $\forall t$, $i, j = 1, 2, \cdots, L$, for a prescribed attenuation level ρ, then the problem of robust H_∞ fuzzy tracking control is solvable.

Proof:

Considering the Lyapunov like function $V(\tilde{x}(t)) = \tilde{x}^T(t)\, E^T \tilde{P} \tilde{x}(t)$, theorem 2 can be proved following the same proof shown in chapter 2.

A optimization problem is derived to find a common matrix $\tilde{P}$ such that both Eq. (71) and Eq. (72) are satisfied for a minimum attenuation level ρ. This optimization can be transferred into a optimization problem subject to some LMIs. The LMI problem can be solved in a computationally efficient manner using a convex optimization technique such as interior point method [127]-[128]. The following theorem provides sufficient conditions for the solvability of the problem of robust H_∞ observer based tracking control based on LMIs.

3.4 Conversion of $\mathbf{H_\infty}$ Optimal Problem into LMI Problem

Theorem 3.3: If there exist positive definite matrices Z_{11}, Z_{22}, H_{11}, H_{22}, N_{11}, N_{22} such that, for a prescribed attenuation level ρ and a positive definite weighting matrix Q, $P_{11} = diag(Z_{11}, Z_{22})$, $P_{22} = diag(H_{11}, H_{22})$, and $P_{33} = diag(N_{11}, N_{22})$ satisfy the following LMIs

$$Z_{11} = Z_{11}^T > 0 \tag{73a}$$

$$Z_{22} = Z_{22}^T > 0 \tag{73b}$$

$$H_{11} = H_{11}^T > 0 \tag{73c}$$

$$H_{22} = H_{22}^T > 0 \tag{73d}$$

$$N_{11} = N_{11}^T > 0 \tag{73e}$$

$$N_{22} = N_{22}^T > 0 \tag{73f}$$

$$
\begin{bmatrix}
M_{11} & P_{11} & X_i & M_{41}^T & 0 & 0 \\
P11 & -p^2 I & 0 & 0 & 0 & 0 \\
X_i^T & 0 & -p^2 I & 0 & 0 & 0 \\
M_{41} & 0 & 0 & M_{44} & M_{45} & 0 \\
0 & 0 & 0 & M_{45}^T & M_{55} & P_{33} \\
0 & 0 & 0 & 0 & P_{33} & -p^2 I
\end{bmatrix} < 0 \tag{74}
$$

then the problem of robust H_∞ tracking control is solvable, where

$$
\begin{aligned}
M_{11} &= (A_i - L_i C_i)^T P_{11} + P_{11}(A_i - L_i C_i) \\
M_{41} &= -(B_i K_j)^T P_{22} + \frac{1}{\rho^2} P_{22} P_{11} \\
M_{44} &= (A_i + B_i K_j)^T P_{22} + P_{22}(A_i + B_i K_j) + \frac{1}{\rho^2} P_{22} P_{22} + Q \\
M_{45} &= -P_{22} B_i K_j - Q \\
M_{55} &= A_i^T P_{22} + P_{22} A_i + Q
\end{aligned}
$$

Proof:

Define $\tilde{P} = diag(P_{11}, P_{22})$. First, let us show that $\tilde{P}$ satisfies (71). Due to (73a) to (73f).

$$
\begin{aligned}
E^T P_{11} &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Z_{22} \end{bmatrix} \geq 0 \\
P_{11}^T E &= \begin{bmatrix} Z_{11}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Z_{22}^T \end{bmatrix} \geq 0 \\
E^T P_{22} &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Z_{22} \end{bmatrix} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
P_{22}^T E &= \begin{bmatrix} H_{11}^T & 0 \\ 0 & H_{22}^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & H_{22}^T \end{bmatrix} \geq 0 \\
E^T P_{33} &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & X_{22} \end{bmatrix} \geq 0 \\
P_{33}^T E &= \begin{bmatrix} X_{11}^T & 0 \\ 0 & X_{22}^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & X_{22}^T \end{bmatrix} \geq 0
\end{aligned}
$$

It follows that $E^T P_{11} = P_{11}^T E \geq 0$ and $E_r^T P_{22} = P_{11}^T E_r \geq 0$. Therefore,

$$
\begin{aligned}
\tilde{E}^T \tilde{P} &= \begin{bmatrix} E^T & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & E^T \end{bmatrix} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} = \begin{bmatrix} E^T P_{11} & 0 & 0 \\ 0 & E^T P_{22} & 0 \\ 0 & 0 & E^T P_{33} \end{bmatrix} \\
&= \begin{bmatrix} P_{11}^T E & 0 & 0 \\ 0 & P_{22}^T E & 0 \\ 0 & 0 & P_{33}^T E \end{bmatrix} = \begin{bmatrix} P_{11}^T & 0 & 0 \\ 0 & P_{22}^T & 0 \\ 0 & 0 & P_{33}^T \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix} \\
&= \tilde{P}^T \tilde{E} \geq 0.
\end{aligned}
$$

To prove that (72) is satisfied with $\tilde{P}$, replacing $\tilde{P}$ with $diag(P_{11}, P_{22}, P_{33})$ in (72) gives

$$
\begin{aligned}
&\begin{bmatrix} A_i + C_i L_i & 0 & 0 \\ -B_i K_j & A_i + B_i K_j & -B_i K_j \\ 0 & 0 & A_r \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \\
&+ \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}^T \begin{bmatrix} A_i + C_i L_i & 0 & 0 \\ -B_i K_j & A_i + B_i K_j & -B_i K_j \\ 0 & 0 & A_r \end{bmatrix} \\
&+ \frac{1}{\rho^2} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \begin{bmatrix} -L_i & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -L_i & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}
\end{aligned}
$$

$$
+\begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & -Q \\ 0 & -Q & Q \end{bmatrix} < 0
$$

$\Rightarrow$

$$
\begin{bmatrix} (A_i + C_i L_i)^T & -(B_i K_j)^T & 0 \\ 0 & (A_i + B_i K_j)^T & 0 \\ 0 & -(B_i K_j)^T & A_r^T \end{bmatrix} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}
+\begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \begin{bmatrix} A_i + C_i L_i & 0 & 0 \\ -B_i K_j & A_i + B_i K_j & -B_i K_j \\ 0 & 0 & A_r \end{bmatrix}
$$

$$
+\frac{1}{\rho^2} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \begin{bmatrix} -L_i & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -L_i^T & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}
$$

$$
+\begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & -Q \\ 0 & -Q & Q \end{bmatrix} < 0
$$

$\Rightarrow$

$$
\begin{bmatrix} (A_i + C_i L_i)^T P_{11} & -(B_i K_j)^T P_{22} & 0 \\ 0 & (A_i + B_i K_j)^T P_{22} & 0 \\ 0 & -(B_i K_j)^T P_{22} & A_r^T P_{33} \end{bmatrix}
+\begin{bmatrix} P_{11}(A_i + C_i L_i) & 0 & 0 \\ -P_{22} B_i K_j & P_{22}(A_i + B_i K_j) & -P_{22} B_i K_j \\ 0 & 0 & P_{33} A_r \end{bmatrix}
$$

$$
+\frac{1}{\rho^2} \begin{bmatrix} -P_{11} L_i & P_{11} & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \begin{bmatrix} -L_i^T & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}
$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & -Q \\ 0 & -Q & Q \end{bmatrix} < 0$$

$\Rightarrow$

$$\begin{bmatrix} (A_i + C_iL_i)^T P_{11} & -(B_iK_j)^T P_{22} & 0 \\ 0 & (A_i + B_iK_j)^T P_{22} & 0 \\ 0 & -(B_iK_j)^T P_{22} & A_r^T P_{33} \end{bmatrix}$$

$$+\begin{bmatrix} P_{11}(A_i + C_iL_i) & 0 & 0 \\ -P_{22}B_iK_j & P_{22}(A_i + B_iK_j) & -P_{22}B_iK_j \\ 0 & 0 & P_{33}A_r \end{bmatrix}$$

$$+\frac{1}{\rho^2}\begin{bmatrix} P_{11}L_iL_i^T + P_{11} & P_{11} & 0 \\ P_{22} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}\begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & -Q \\ 0 & -Q & Q \end{bmatrix} < 0$$

$\Rightarrow$

$$\begin{bmatrix} (A_i + C_iL_i)^T P_{11} & -(B_iK_j)^T P_{22} & 0 \\ 0 & (A_i + B_iK_j)^T P_{22} & 0 \\ 0 & -(B_iK_j)^T P_{22} & A_r^T P_{33} \end{bmatrix}$$

$$+\begin{bmatrix} P_{11}(A_i + C_iL_i) & 0 & 0 \\ -P_{22}B_iK_j & P_{22}(A_i + B_iK_j) & -P_{22}B_iK_j \\ 0 & 0 & P_{33}A_r \end{bmatrix}$$

$$+\frac{1}{\rho^2}\begin{bmatrix} P_{11}L_iL_i^T P_{11} + P_{11}P_{11} & P_{11}P_{22} & 0 \\ P_{22}P_{11} & P_{22}P_{22} & 0 \\ 0 & 0 & P_{33}P_{33} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & -Q \\ 0 & -Q & Q \end{bmatrix} < 0$$

$\Rightarrow$

$$\begin{bmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{bmatrix} < 0 \tag{75}$$

where

$$\begin{aligned}
f_{11} &= (A_i + C_i L_i)^T P_{11} + P_{11} (A_i + C_i L_i) + \frac{1}{\rho^2} \left(P_{11} L_i L_i^T P_{11} + P_{11} P_{11} \right) \\
f_{12} &= -(B_i K_j)^T P_{22} + \frac{1}{\rho^2} P_{11} P_{22} \\
f_{21} &= -P_{22} B_i K_j + \frac{1}{\rho^2} P_{22} P_{11}
\end{aligned}$$

$$\begin{aligned}
f_{22} &= (A_i + B_i K_j)^T P_{22} + P_{22} (A_i + B_i K_j) + \frac{1}{\rho^2} P_{22} P_{22} + Q \\
f_{23} &= -P_{22} B_i K_j - Q \\
f_{32} &= -(B_i K_j)^T P_{22} - Q \\
f_{33} &= A_r^T P_{33} + P_{33} A_r + \frac{1}{\rho^2} P_{33} P_{33} + Q
\end{aligned}$$

After simplification and taking the Schur complements, it can be shown that (75) can be expressed as (74) [126]. where, $X_i = P_{11} L_i$,

$$\begin{bmatrix} M_{11} & P_{11} & X_i & M_{41}^T & 0 & 0 \\ P11 & -p^2 I & 0 & 0 & 0 & 0 \\ X_i^T & 0 & -p^2 I & 0 & 0 & 0 \\ M_{41} & 0 & 0 & M_{44} & M_{45} & 0 \\ 0 & 0 & 0 & M_{45}^T & M_{55} & P_{33} \\ 0 & 0 & 0 & 0 & P_{33} & -p^2 I \end{bmatrix} < 0$$

3.5 Solution of the LMI Problem

Note that (74) is not a linear matrix inequality problem. Therefore it can not be solved in a single step using LMI toolbox. However, it can be solved by the following two-step procedure [126].

Step 1: Solving $M_{44} < 0$, find P_{11} and K_j

$$H_{11} = H_{11}^T > 0 \tag{76}$$

$$H_{22} = H_{22}^T > 0 \tag{77}$$

$$(A_i + B_i K_j)^T P_{22} + P_{22} (A_i + B_i K_j) + \frac{1}{\rho^2} P_{22} P_{22} + Q \tag{78}$$

for P_{22}. Set $R_{22} = P_{22}^{-1}$, $R_{22} = diag(r_{11}, r_{22})$, $r_{11} = H_{11}^{-1}$, $r_{22} = H_{22}^{-1}$ and $Y_j = K_j R_{22}$. (76), (77), and (78) can be rewritten as

$$r_{11} = r_{11}^T > 0 \tag{79}$$

$$r_{22} = r_{22}^T > 0 \tag{80}$$

$$\begin{bmatrix} R_{22} A_i^T + A_i R_{22} + B_i Y_j + (B_i Y_j)^T + \frac{1}{\rho^2} I & R_{22} \\ R_{22} & -Q^{-1} \end{bmatrix} < 0 \tag{81}$$

For a prescribed attenuation level ρ, the matrices R_{22} and Y_j (thus $P_{22} = R_{22}^{-1}$ and $K_j = Y_j R_{22}^{-1}$) can be obtained by solving LMIs (79), (80). and (81).

Step 2: Find P_{11}, P_{33} and L_i

The values of P_{11} and K_j (where $j = 1, \ldots, L$) obtained in the Step 1 are substituted into (74). Then the LMIs (74) are solved for P_{11}, P_{33} and L_i, $i = 1, \ldots, L$.

The attenuation level ρ can be minimized by searching for $P_{11} > 0, P_{22} > 0$ so that $\min\limits_{P_{11}, P_{22}} \rho^2$.

This optimization problem can be solved by reducing ρ and solving the LMIs with the above two-step procedure until just before no feasible solutions $P_{11} = P_{11}^T > 0$, $P_{22} = P_{22}^T > 0$ and $P_{33} = P_{33}^T > 0$ of the LMI problem can be found.

4 Simulations

4.1 Simulations for State Feedback Controller

Consider a two link planar elbow manipulator, as shown in Fig. 1, where joint-1 is driven by a motor mounted at the base and joint 2 is passive. The dynamic equation of the 2-link planar elbow manipulator, with a given constraint as shown in Fig. 1, is given below [96], [129], [131].

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = T + J_{\phi}(q)^T \lambda \tag{82}$$

$$\phi(q) = 0 \tag{83}$$

where $q = [q_1\ q_2]^T$, q_1, q_2 are generalized coordinates, $M(q)$ denotes the inertial matrix, $C(q,\dot{q})$ is a matrix which characterizes the Coriolis or centrifugal terms, $G(q)$ represents gravitational effects, $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^T$ is the input torque vector at the joints, $J_{\phi}(q)$ is the Jacobian of $\phi(q)$, λ is Lagrangian multiplier, and

$$M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$$

$$C(q,\dot{q}) = m_2 l_1 l_2 (\cos q_1 \sin q_2 - \sin q_1 \cos q_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} -(m_1 + m_2) l_1 g \sin q_1 \\ -m_2 l_2 g \sin q_2 \end{bmatrix}$$

$$\begin{aligned}\phi(q) &= a(l_1+l_2) - l_1(\cos q_1 + \sin q_1) - l_2(\cos q_2 + \sin q_2) \\ &= -x + a(l_1+l_2) - y\end{aligned}$$

$$J_\phi(q) = \begin{bmatrix} l_1 \sin q_1 - l_1 \cos q_1 & l_2 \sin q_2 - l_2 \cos q_2 \end{bmatrix}$$

with $m_{11} = (m_1+m_2)l_1^2$, $m_{12} = m_2 l_1 l_2(\sin q_1 \sin q_2 + \cos q_1 \cos q_2)$, $m_{22} = m_2 l_2^2$, m_1 and m_2 being link masses, l_1 and l_2 being link lengths, and $g = 9.8\,(m/s^2)$ is the acceleration due to gravity.

The constraint Eq. (83) is a straight line which intersects the x-axis and y-axis at $(a(l_1+l_2), 0)$ and $(0, a(l_1+l_2))$, respectively. It is assumed that $a < 1$ to avoid singularity of the Jacobian. Set $q_1 = q_1$, $q_2 = q_2$, $q_3 = \dot{q}_1$ and $q_4 = \dot{q}_2$. Then, Eq. (82) and Eq. (83) can be expressed as the following state space form including external disturbances d_{Si}, $i = 1, 2$:

$$\begin{aligned}\begin{bmatrix}\dot{q}_1\end{bmatrix} &= q_3 \\ \begin{bmatrix}\dot{q}_2\end{bmatrix} &= q_4 \\ \begin{bmatrix}\dot{q}_3 \\ \dot{q}_4\end{bmatrix} &= M(q_1,q_2)^{-1}\left[-C(q_1,q_2,q_3,q_4)\dot{q} - G(q_1,q_2)\right. \\ &\quad \left. + T + J_\phi(q_1,q_2)^T\lambda\right] + \begin{bmatrix} d_{S1} \\ d_{S2}\end{bmatrix} \\ 0 &= a(l_1+l_2) - l_1(\cos q_1 + \sin q_1) \\ &\quad - l_2(\cos q_2 + \sin q_2)\end{aligned}$$

Because q_1 is an independent joint angle, q_2 has to be determined as a function of q_1 from the constraint equation Eq. (83). As a result, by setting $x_d = q_1$ and $x_c = q_2$, the model Eq.

(82)-(83) is equivalent to Eq. (30) and Eq. (31) with $n = 2$, which can be expressed as

$$\begin{aligned} 0 &= v_1 \\ 0 &= v_2 \\ 0 &= v_3 \\ \dot{v}_4 &= v_5 \\ \dot{v}_5 &= f_1 + g_{11}T_1 + g_{12}T_2 \end{aligned} \tag{84}$$

where $T_2 = 0$,

$$\begin{aligned} &\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ = \ & -M^{-1}C\dot{q} - M^{-1}G + M^{-1}J^T \left(JM^{-1}J^T\right)^{-1} \\ & \left(v_3 + JM^{-1}C\dot{q} + JM^{-1}G - JM^{-1}\dot{J} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} \right) \end{aligned}$$

and

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = M^{-1} - M^{-1}J^T \left(JM^{-1}J^T\right)^{-1} JM^{-1}$$

The parameters are $l_1 = 1m$, $l_2 = 1m$, $m_1 = 1kg$, $m_2 = 1kg$, $a = 0.8$. q_1 is constrained within $\left[0, \frac{\pi}{2}\right]$. The T-S fuzzy model for the system in Eq. (84) is given by the following 3 fuzzy rules.

Rule i:

If v_4 is about θ_i

then $Ev = A_i v + B_i u + d_S$

where $i = 1, 2, 3$, $\theta_1 = \frac{1}{36}\pi$, $\theta_2 = \frac{1}{4}\pi$, $\theta_3 = \frac{9}{4}\pi$,

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -0.0777 & 0 & 0.0926 & 2.5347 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2.4495 & 0 & 0.0469 & -3.8893 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.2058 & 0 & 0.0973 & -3.7057 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.7352 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5044 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4476 \end{bmatrix}.$$

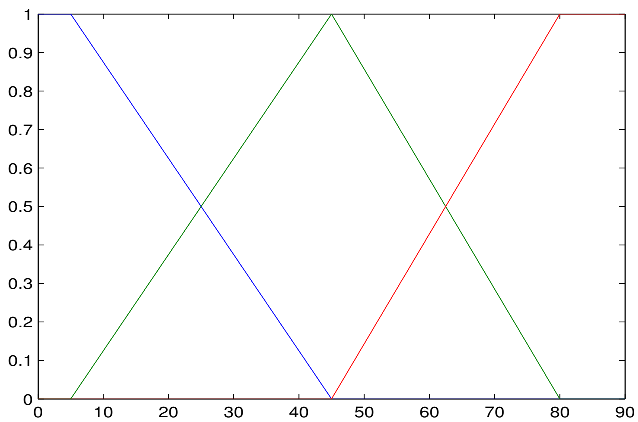

Figure 2: Membership function for joint angle q_1

The dynamic reference Eq. (33) is used with

$E_r = E$, $r_I = 8(\frac{5\pi}{18} + \frac{5\pi}{18}\sin(t))$,

$$A_r = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 & -5 \end{bmatrix} \text{ and } B_r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the convenience of design, S-shape, Z-shape and Triangle membership functions, as shown in Fig. 2, are considered. External disturbances in Newton meters are assumed to be $d_{S1} = 5\sin(2t)$ and $d_{S2} = 0$.

The LMI optimization is done using YLMIP toolbox incorporating with Matlab. The solutions of the LMI problems are

$$
\begin{aligned}
P_{11} &= \begin{bmatrix} 0.9453 & 0 & 0.0000 & 0 & 0 \\ 0 & 0.0095 & 0 & 0 & 0 \\ 0.0000 & 0 & 0.0001 & 0 & 0 \\ 0 & 0 & 0 & 11.840 & 0.0273 \\ 0 & 0 & 0 & 0.0273 & 0.0023 \end{bmatrix}, \\
P_{22} &= \begin{bmatrix} 176.68 & 0 & 0.0000 & 0 & 0 \\ 0 & 176.77 & 0 & 0 & 0 \\ 0.0000 & 0 & 176.77 & 0 & 0 \\ 0 & 0 & 0 & 569.02 & 325.89 \\ 0 & 0 & 0 & 325.89 & 248.68 \end{bmatrix}, \\
K_1 &= \begin{bmatrix} 3.859 & 0 & -0.1405 & -20964.4 & -1733.7 \end{bmatrix}, \\
K_2 &= \begin{bmatrix} 3.859 & 0 & -0.1405 & -20964.4 & -1733.7 \end{bmatrix}, \\
K_3 &= \begin{bmatrix} 3.859 & 0 & -0.1405 & -20964.4 & -1733.7 \end{bmatrix}.
\end{aligned}
$$

The trajectories of the state variables q_1 to q_4, along with q_{r1} to q_{r4}, are given in the figures from Fig. 3 to Fig. 10 respectively, including disturbances.

The initial condition is assumed to be $(q_1(0), q_3(0), q_{r1}(0), q_{r3}(0)) = (\frac{5\pi}{18}, 0, \frac{5\pi}{18}, 0)^T$ in the simulation. The states q_2, q_4 and q_{r2}, q_{r4} are calculated from the algebraic constraint. In the presence of disturbances, the state q_3 follows reference trajectory with a very small error. From the simulation results, the performance of the proposed controller is obviously satisfactory in the presence of disturbances.

Imposing 10% parameters uncertainties on the lengths l_1, l_2 and the masses m_1, m_2 and keeping the disturbance same as before the simulation is run. The tracking responses obtained are given in the figures from Fig. 11 to Fig. 18,

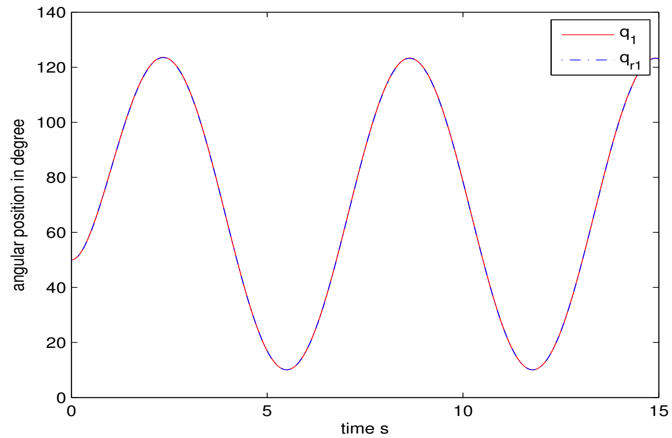

Figure 3: The trajectories of q_1 and q_{r1}

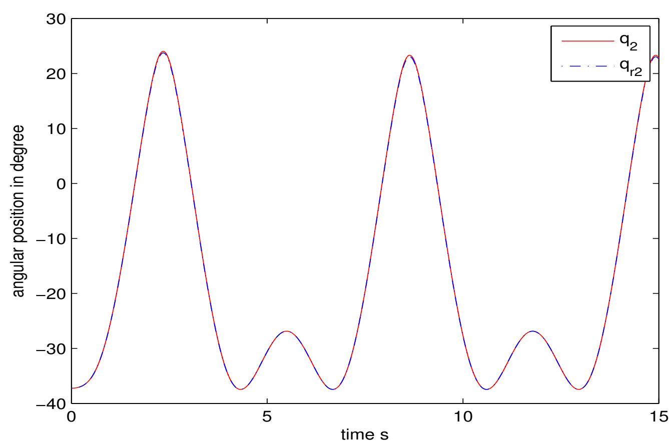

Figure 4: The trajectories of q_2 and q_{r2}

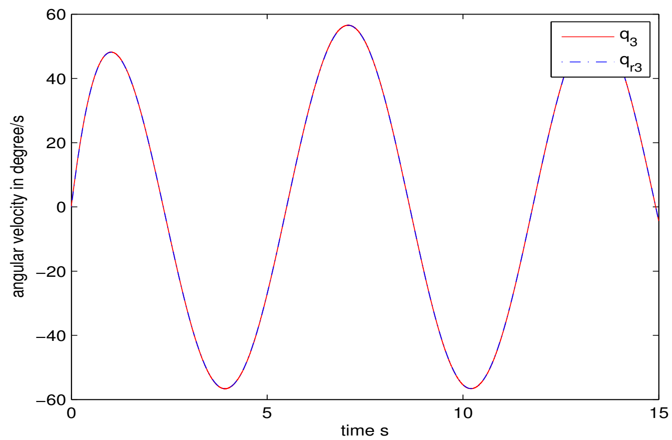

Figure 5: The trajectories of q_3 and q_{r3}

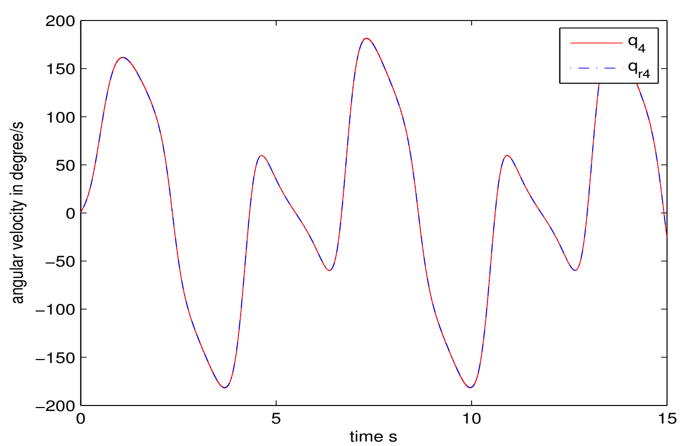

Figure 6: The trajectories of q_4 and q_{r4}

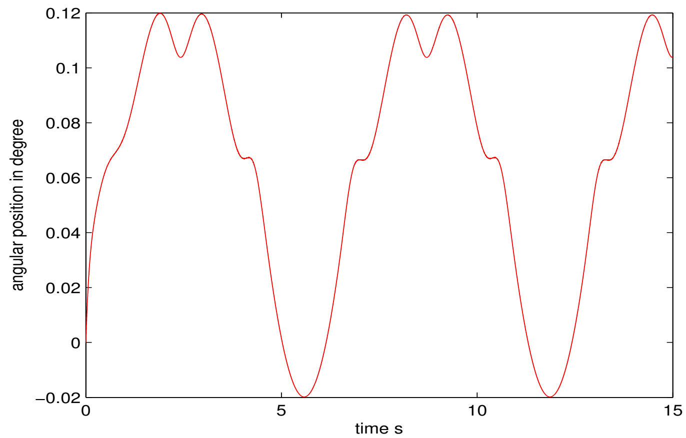

Figure 7: Tracking error (q_1-q_{r1})

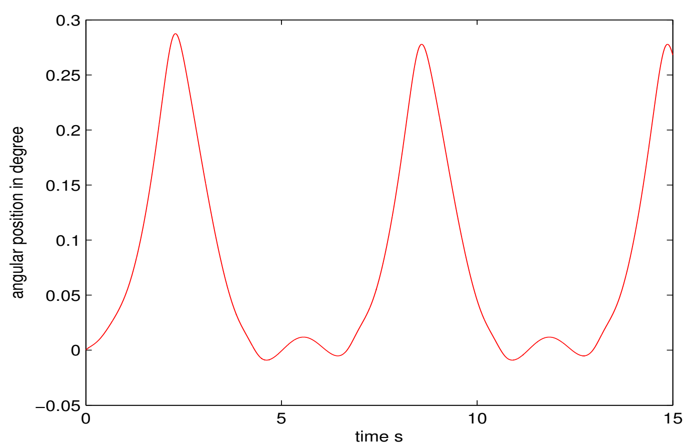

Figure 8: Tracking error (q_2-q_{r2})

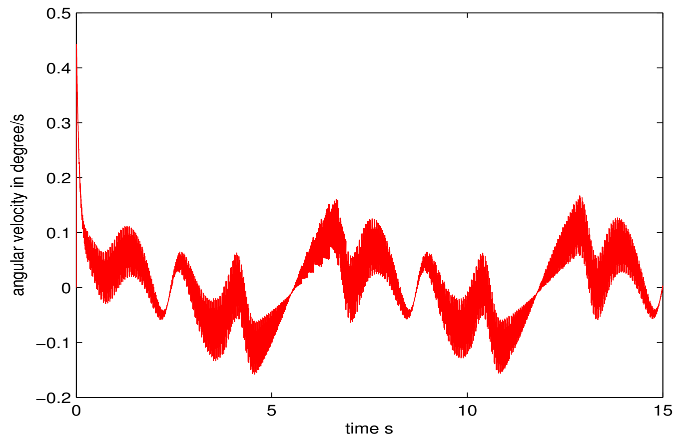

Figure 9: Tracking error (q_3-q_{r3})

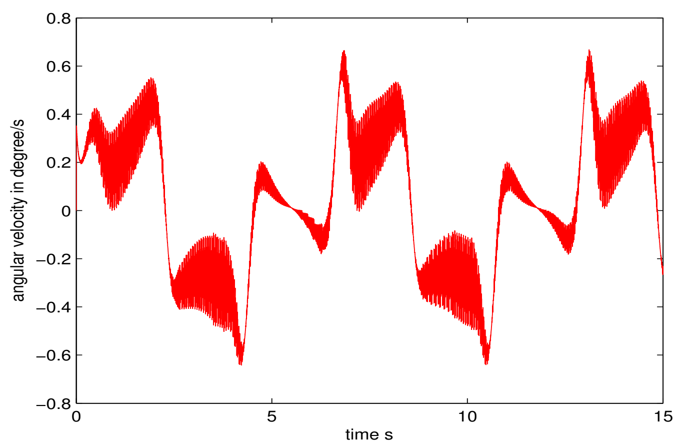

Figure 10: Tracking error (q_4-q_{r4})

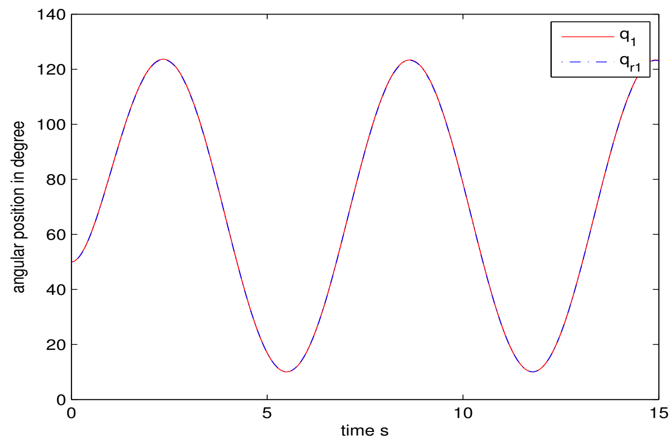

Figure 11: The trajectories of q_1 and q_{r1}

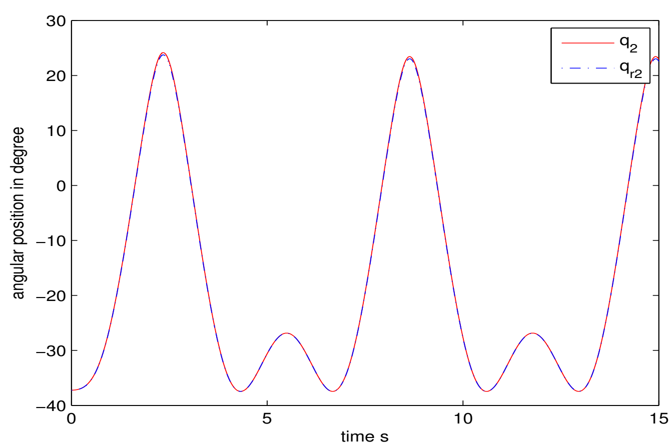

Figure 12: The trajectories of q_2 and q_{r2}

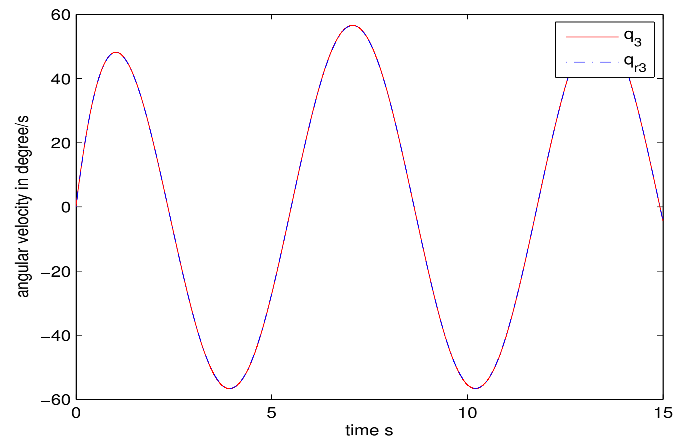

Figure 13: The trajectories of q_3 and q_{r3}

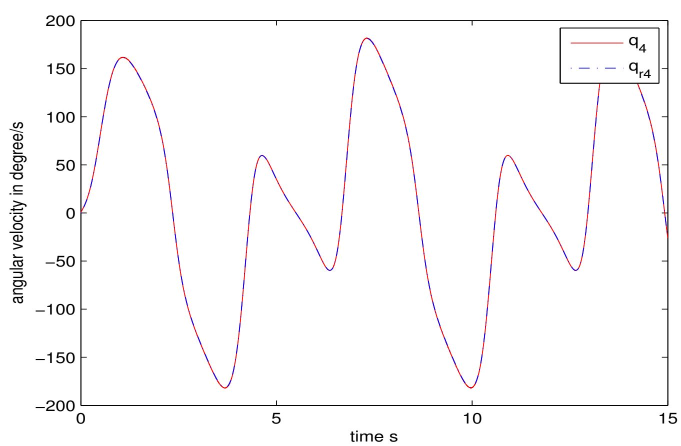

Figure 14: The trajectories of q_4 and q_{r4}

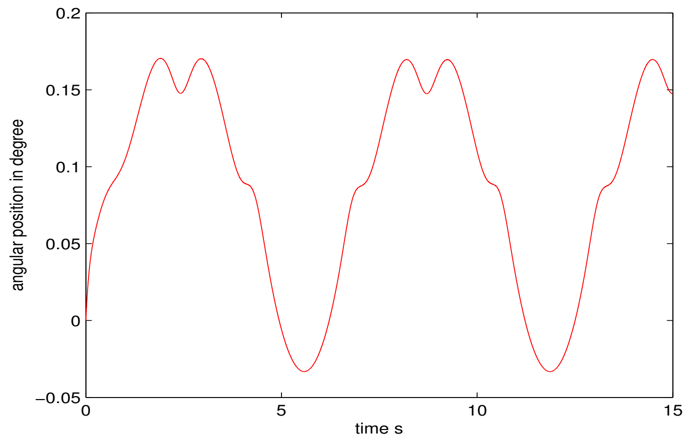

Figure 15: Tracking error (q$_1$-q$_{r1}$)

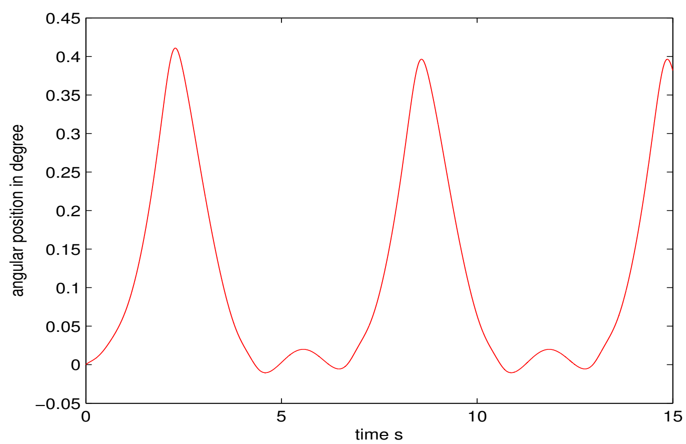

Figure 16: Tracking error (q$_2$-q$_{r2}$)

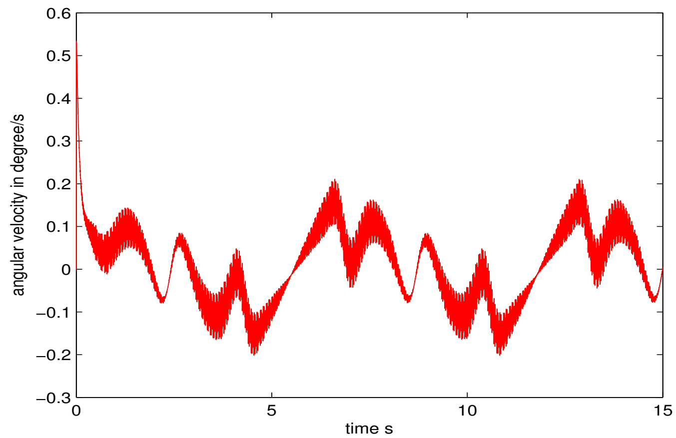

Figure 17: Tracking error (q_3-q_{r3})

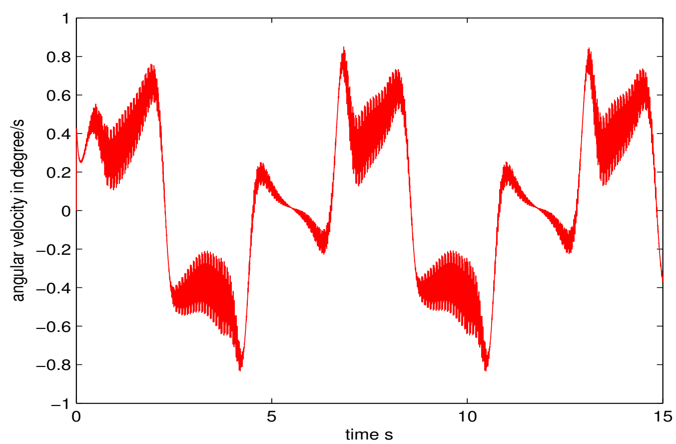

Figure 18: Tracking error (q_4-q_{r4})

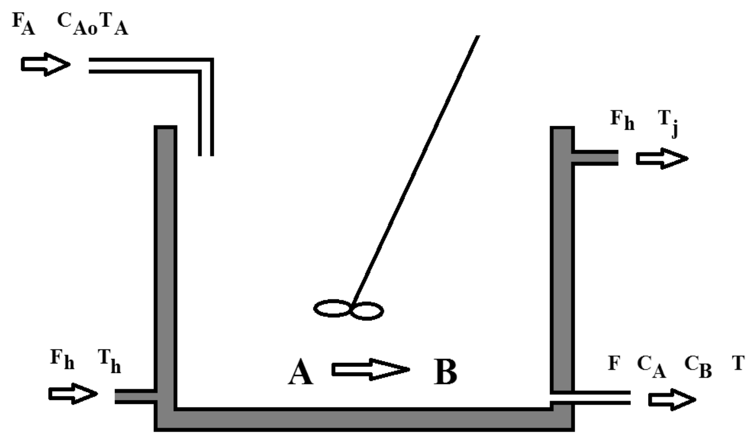

Figure 19: Continuously stirred tank reactor

4.2 Simulations for Observer Based State Feedback Controller

Consider a continuously stirred tank reactor (CSTR) with heating jacket [90], as shown in Fig. 19. Reactant A is fed at a flow rate F_A, molar concentration C_{A0}, and temperature T_A to the reactor, where the irreversible endothermic reaction $A \rightarrow B$ occurs. The rate of reaction is given by the following relation:

$$R_A = k_0 \exp\left(-\frac{E_a}{RT}\right) C_A \tag{85}$$

where k_0 and E are the reaction rare coefficient and activation energy, respectively. C_A is the molar concentration of A in the reactor holdup, and T is the reactor temperature. The product stream is withdrawn at a flow rate F and heat is provided to the reactor through the heating jacket, where the heating fluid is fed at a flow rate F_h. Consider the case when the heat transfer rate $Q = UA\,(T_j - T)$ is fast, i.e. the product of overall heat transfer U and heat transfer area A is large. The detailed rate-based model of the process, where the heat transfer rate expression is explicitly included, is given by the following ODE system:

$$\begin{aligned}
\dot{v} &= F_A - F \\
\dot{C}_A &= \frac{F_A}{V}(C_{A0} - C_A) - k_0 \exp(\frac{-E_a}{RT})C_A \\
\dot{C}_B &= -\frac{F_A}{V}C_B + k_0 \exp(\frac{-E_a}{RT})C_A \\
\dot{T} &= \frac{F_A}{V}(T_A - T) - k_0 \exp(\frac{-E_a}{RT})C_A \frac{dHr}{pc_p} + \frac{Q}{pVc_p} \\
\dot{T}_j &= \frac{F_h}{V_h}(T_h - T_j) - \frac{Q}{p_h V_h c_{p_h}}
\end{aligned} \tag{86}$$

$$0 = Q - UA\,(T_j - T) \tag{87}$$

Owing to the presence of the large parameter UA in the rate expression for the fast heat transfer, the ODE model in (86) exhibits stiffness. Equivalently, the fast heat transfer implies a time-scale multiplicity in the process dynamics where after an initial fast transience, the reactor and jacket are essentially at thermal equilibrium, i.e. $T_j \approx T$. Thus, under the quasi-steady-state (QSS) assumption of thermal equilibrium, the explicit rate expression for the fast heat transfer is replaced by the relation $T_j = T$, to obtain the following singular system model:

$$\dot{v} = F_A - F \tag{88}$$

$$\dot{C}_A = \frac{F_A}{V}(C_{A0} - C_A) - k_0 \exp(\frac{-E_a}{RT})C_A \tag{89}$$

$$\dot{C}_B = -\frac{F_A}{V}C_B + k_0 \exp(\frac{-E_a}{RT})C_A \tag{90}$$

$$\dot{T} = \frac{F_A}{V}(T_A - T) - k_0 \exp(\frac{-E_a}{RT})C_A \frac{dHr}{pc_p} + \frac{Q}{pVc_p} \tag{91}$$

$$\dot{T}_j = \frac{F_h}{V_h}(T_h - T_j) - \frac{Q}{p_h V_h c_{p_h}} \tag{92}$$

$$0 = T_j - T \tag{93}$$

Clearly due to the assumption of thermal equilibrium, the algebraic equation is singular and can not be solved for the algebraic variable Q. It can be verified that above singular system

Table 1: Nominal value of the process parameters [90]

Variables	Description	Nominal value
C_{A0}	feed reactant concentration $(mole/l)$	5.0
C_A	reactant concentration in reactor (mol/l)	1.596
C_B	product concentration in reactor (mol/l)	3.404
c_p	specific heat capacity $(J/g\ K)$	6.0
E	activation energy $(J/mol\ K)$	50000
F_A	outlet flow rate from reactor (l/min)	3.0
F_h	heating fluid flow rate (l/min)	.1
k_0	pre-exponential factor in reaction rate $(l/mol\ min)$	1.0×10^9
T_A	feed reactant temperature (K)	300
T	reactor temperature (K)	284.08
T_h	heating fluid temperature (K)	375
T_j	jacket temperature (K)	285.37
V	reactor hold up volume (l)	10.0
V_h	jacket volume (l)	1.0
p	liquid density (g/l)	600
ΔH_r	heat of reaction (J/mol)	20000
R	universal gas constant $(J/mole\ K)$	8.134

has an index two.

Now consider the case of $F_0 = F_A$, which implies that the reactor holdup volume V is constant. For simplicity, it is assumed that the density and specific heat capacities of the two liquids are the same, i.e., $p_h = p$ and $c_{ph} = c_p$.

In this process, it is desired to keep reactant concentration C_A, the product concentration C_B, and reactor temperature T at a desired level using reactant flow rate F_A and the heating fluid flow rate F_h as the manipulated inputs.

As the reactor holdup volume is constant, the dynamics of the CSTR becomes

$$\dot{C}_A = \frac{F_A}{V}(C_{A0} - C_A) - k_0 \exp(\frac{-E_a}{RT})C_A \tag{94}$$

$$\dot{C}_B = -\frac{F_A}{V}C_B + k_0 \exp(\frac{-E_a}{RT})C_A \tag{95}$$

$$\dot{T} = \frac{F_A}{V}(T_A - T) - k_0 \exp(\frac{-E_a}{RT})C_A \frac{dHr}{pc_p} + \frac{Q}{pVc_p} \tag{96}$$

with an algebraic constraint on the dynamics

$$0 = T_j - T \tag{97}$$

By setting $x_d = \begin{bmatrix} C_A & C_B & T \end{bmatrix}$ and $x_c = T_j$, the model (94)-(97) is equivalent to Eq. (30) and Eq. (31) with $n = 2$, which can be expressed as

$$0 = v_1 \tag{98}$$

$$0 = v_2 \tag{99}$$

$$\dot{v}_3 = f_1 + p_1 z + g_1 u_1 \tag{100}$$

where

$$f_1 = \begin{bmatrix} -k_0 \exp(\frac{-E_a}{RT})C_A \\ k_0 \exp(\frac{-E_a}{RT})C_A \\ -k_0 \exp(\frac{-E_a}{RT})C_A \frac{dHr}{pc_p} \end{bmatrix}, \; p_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{pVc_p} \end{bmatrix},$$

$$z = \frac{k_0 \exp(\frac{-E_a}{RT})C_A \frac{dHr}{pc_p} + \frac{1}{V}(T - T_A) F_A + \frac{1}{V_h}(T_h - T_j) F_h}{\frac{V_h + V}{pVV_hc_p}}$$

$$g_1 = \begin{bmatrix} \frac{F_A}{V}(C_{A0} - C_A) & 0 \\ -\frac{F_A}{V}C_B & 0 \\ \frac{F_A}{V}(T_A - T) & 0 \end{bmatrix}, \; u_1 = \begin{bmatrix} F_A \\ F_h \end{bmatrix}$$

The nominal values of the parameters are given in a tabular form in Table 1. To shorten the simulation time, the reactor is considered pre-heated. The T-S fuzzy model for CSTR system is given by the following 9 fuzzy rules.

Rule 1:

If v_4 is about C_{B1} and v_5 is about T_1

then $Ev = A_1 v + B_1 u + d_S$

Rule 2:

If v_4 is about C_{B1} and v_5 is about T_2

then $Ev = A_2 v + B_2 u + d_S$

Rule 3:

If v_4 is about C_{B1} and v_5 is about T_3

then $Ev = A_3 v + B_3 u + d_S$

Rule 4:

If v_4 is about C_{B2} and v_5 is about T_1

then $Ev = A_4 v + B_4 u + d_S$

Rule 5:

If v_4 is about C_{B2} and v_5 is about T_2

then $Ev = A_5 v + B_5 u + d_S$

Rule 6:

If v_4 is about C_{B2} and v_5 is about T_3

then $Ev = A_6 v + B_6 u + d_S$

Rule 7:

If v_4 is about C_{B3} and v_5 is about T_1

then $Ev = A_7 v + B_7 u + d_S$

Rule 8:

If v_4 is about C_{B3} and v_5 is about T_2

then $Ev = A_8 v + B_8 u + d_S$

Rule 9:

If v_4 is about C_{B3} and v_5 is about T_3

then $Ev = A_9 v + B_9 u + d_S$

where $C_{B1} = 3.4$, $C_{B3} = 3.8$, $C_{B3} = 4.2$, $T_1 = 284.08$,$T_2 = 284.3$, $T_3 = 284.5$, $C_{Ai} = C_{A0} - C_{Bi}$ $(i = 1, 2, 3)$, and

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.9408 & 0 & -0.0763 \\ 0 & 0 & 0.6398 & -0.3011 & 0.0763 \\ 0.0089 & 0.0909 & -3.2311 & 0 & -0.6679 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.9564 & 0 & -0.0774 \\ 0 & 0 & 0.6503 & -0.3060 & 0.0774 \\ 0.0098 & 0.0909 & -3.2845 & 0 & -0.6790 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.9707 & 0 & -0.0785 \\ 0 & 0 & 0.6601 & -0.3106 & 0.0785 \\ 0.0106 & 0.0909 & -3.3337 & -3.8893 & -0.6893 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.8418 & 0 & -0.0572 \\ 0 & 0 & 0.6398 & -0.2020 & 0.0572 \\ 0.0105 & 0.0909 & -3.2311 & 0 & -0.4831 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.8557 & 0 & -0.0581 \\ 0 & 0 & 0.6503 & -0.2054 & 0.0581 \\ 0.0111 & 0.0909 & -3.2845 & 0 & -0.4911 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.8685 & 0 & -0.0589 \\ 0 & 0 & 0.6601 & -0.2084 & 0.0589 \\ 0.0117 & 0.0909 & -3.3337 & 0 & -0.4985 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.7616 & 0 & -0.0381 \\ 0 & 0 & 0.6398 & -0.1219 & 0.0381 \\ 0.0090 & 0.0909 & -3.2311 & 0 & -0.3124 \end{bmatrix},$$

$$A_8 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.7742 & 0 & -0.0387 \\ 0 & 0 & 0.6503 & -0.1239 & 0.0387 \\ 0.0095 & 0.0909 & -3.2845 & 0 & -0.3176 \end{bmatrix},$$

$$A_9 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.7858 & 0 & -0.0392 \\ 0 & 0 & 0.6601 & -0.1257 & 0.0392 \\ 0.0099 & 0.0909 & -3.3337 & 0 & -0.3223 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3400 & 0 \\ -0.3400 & 0 \\ 1.5920 & 0.0025 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3400 & 0 \\ -0.3400 & 0 \\ 1.5700 & 0.0025 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3400 & 0 \\ -0.3400 & 0 \\ 1.5500 & 0.0025 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3800 & 0 \\ -0.3800 & 0 \\ 1.5920 & 0.0025 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3800 & 0 \\ -0.3800 & 0 \\ 1.5700 & 0.0025 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3800 & 0 \\ -0.3800 & 0 \\ 1.5500 & 0.0025 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.4200 & 0 \\ -0.4200 & 0 \\ 1.5920 & 0.0025 \end{bmatrix}, B_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.4200 & 0 \\ -0.4200 & 0 \\ 1.5700 & 0.0025 \end{bmatrix}, B_9 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.4200 & 0 \\ -0.4200 & 0 \\ 1.5500 & 0.0025 \end{bmatrix}.$$

The dynamic reference Eq. (33) is used with

$$E_r = E,\ r_I = \begin{bmatrix} 1.5800 & 3.42 & 284.3 \end{bmatrix},$$

$$A_r = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } B_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For the convenience of design, S-shape, Z-shape, and Triangle membership functions, as shown in Figs. 20-21, are considered. External disturbances in measuring the concentration and temperature are assumed to be $0.2\sin(t)$ and $0.2\sin(t)$. The LMI optimization is done using YLMIP toolbox incorporating with Matlab. The solutions of the LMI problems are

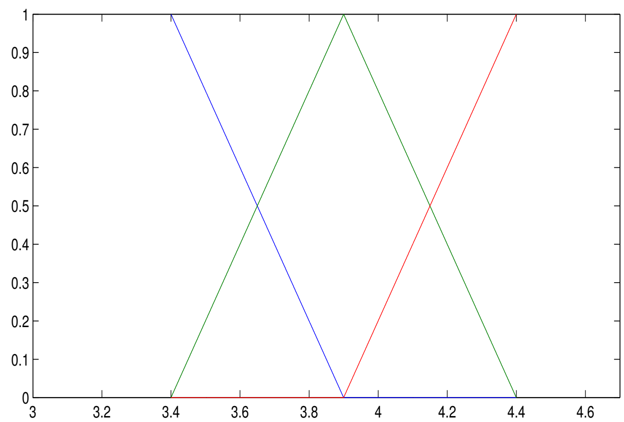

Figure 20: Membership function for concentration

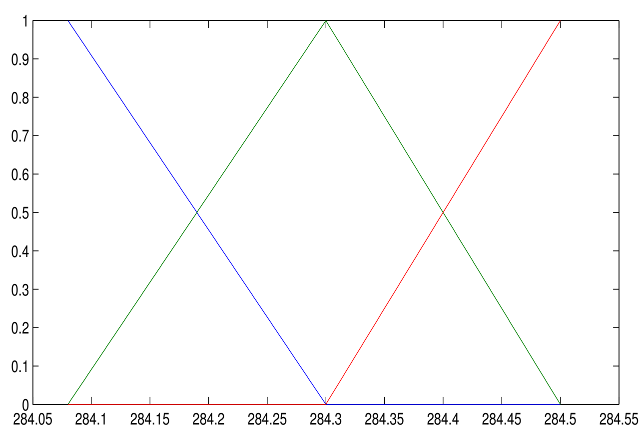

Figure 21: Membership function for temperature

$$P_{11} = \begin{bmatrix} 0.8858 & -0.0000 & 0 & 0 & 0 \\ -0.0000 & 0.8858 & 0 & 0 & 0 \\ 0 & 0 & 0.0167 & -0.0145 & 0.0000 \\ 0 & 0 & -0.0145 & 0.0178 & -0.0000 \\ 0 & 0 & 0.0000 & -0.0000 & 0.2208 \end{bmatrix},$$

$$P_{22} = \begin{bmatrix} 1.1289 & 0.0000 & 0 & 0 & 0 \\ 0.0000 & 1.1289 & 0 & 0 & 0 \\ 0 & 0 & 203.8549 & 165.5484 & -0.0050 \\ 0 & 0 & 165.5484 & 190.5492 & 0.0037 \\ 0 & 0 & -0.0050 & 0.0037 & 4.5285 \end{bmatrix},$$

$$P_{33} = \begin{bmatrix} 0.8858 & -0.0000 & 0 & 0 & 0 \\ -0.0000 & 0.8858 & 0 & 0 & 0 \\ 0 & 0 & 0.0167 & -0.0145 & 0.0000 \\ 0 & 0 & -0.0145 & 0.0178 & -0.0000 \\ 0 & 0 & 0.0000 & -0.0000 & 0.2208 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_2 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_3 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_4 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_5 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_6 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_7 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_8 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

$$K_9 = \begin{bmatrix} 0.0000 & 0.0000 & -0.0021 & 0.0014 & -0.0000 \\ -0.0000 & -0.0004 & 1.2998 & -0.8458 & -0.0705 \end{bmatrix} * 10^5$$

The trajectories of the state variables C_A, C_B, and T, along with C_{Ad}, C_{Bd}, and T_d. are given from Fig. 22 to Fig. 27, respectively, including sensor noise as disturbances.

Imposing 5% uncertainties in the process parameters ΔH_r, ρ and c_ρ and keeping sensor noise as disturbances as before the simulation is run. The tracking responses obtained are given from Fig. 28 to Fig. 33

4.3 Result Discussion

4.3.1 Holonomic Constrained Mechanical Link

The figures from Fig. 3 to Fig. 10 show the tracking responses of the constrained mechanical link described in Section 4.1. This simulation is run in the presence of 5 N disturbance force.

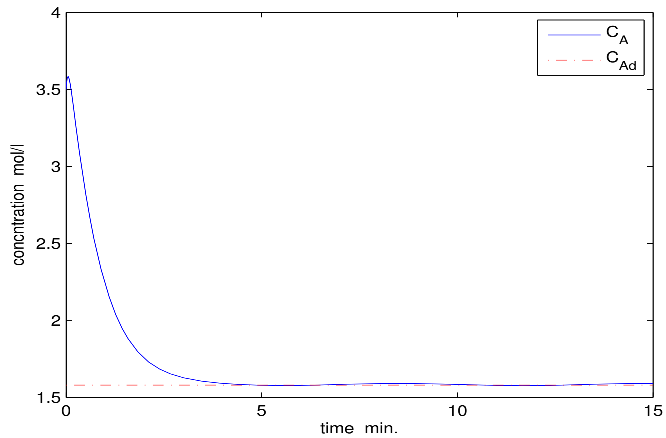

Figure 22: The trajectories of C_A and C_{Ad}

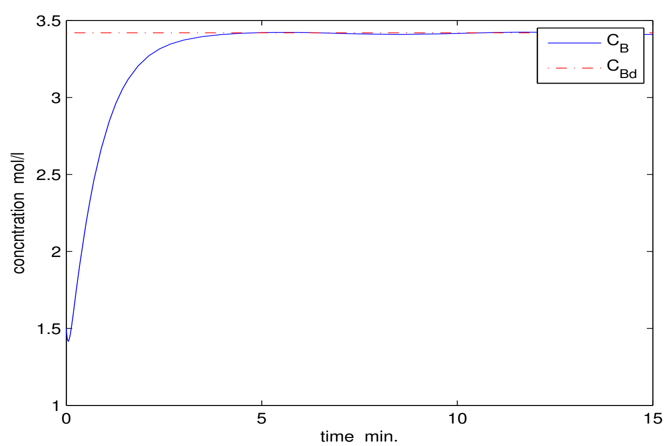

Figure 23: The trajectories of C_B and C_{Bd}

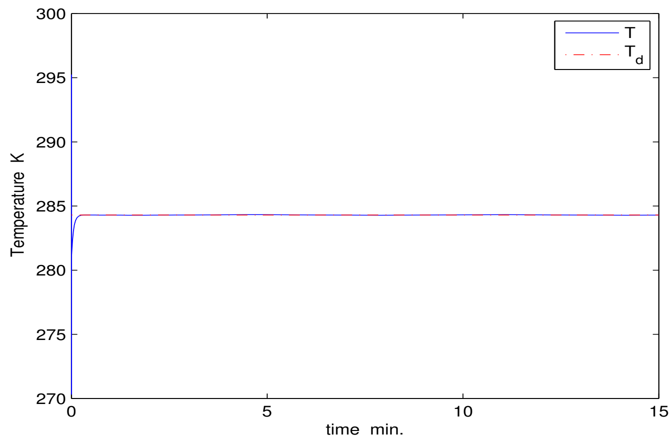

Figure 24: The trajectories of T and T$_d$

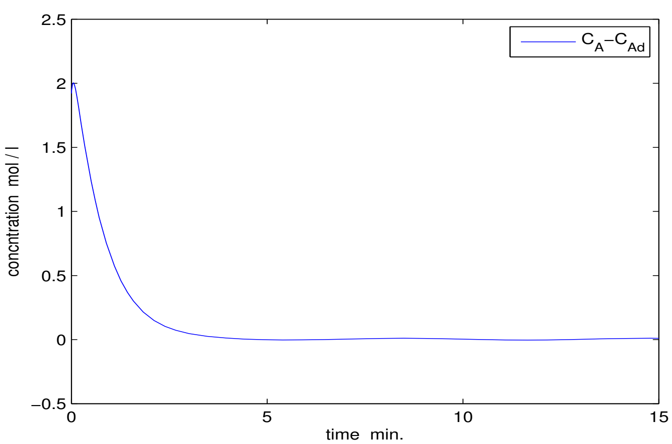

Figure 25: Tracking error (C$_A$-C$_{Ad}$)

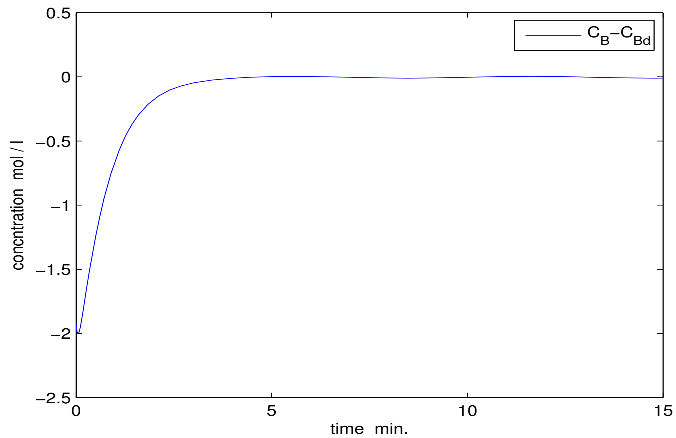

Figure 26: Tracking error (C_B-C_{Bd})

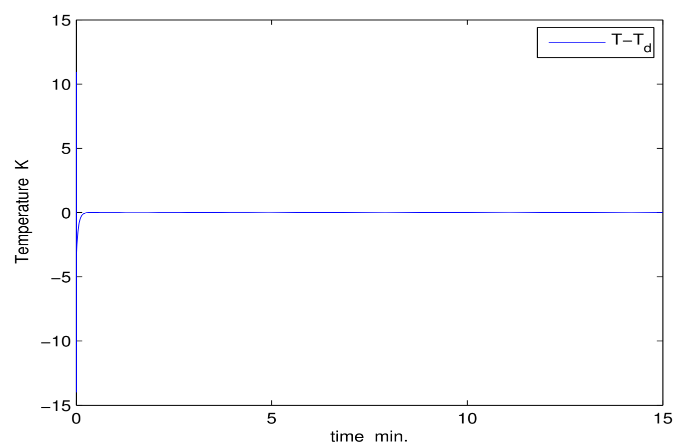

Figure 27: Tracking error (T-T_d)

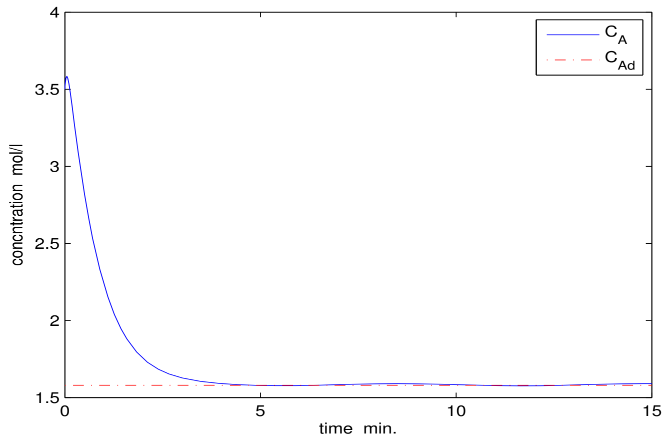

Figure 28: The trajectories of C_A and C_{Ad}

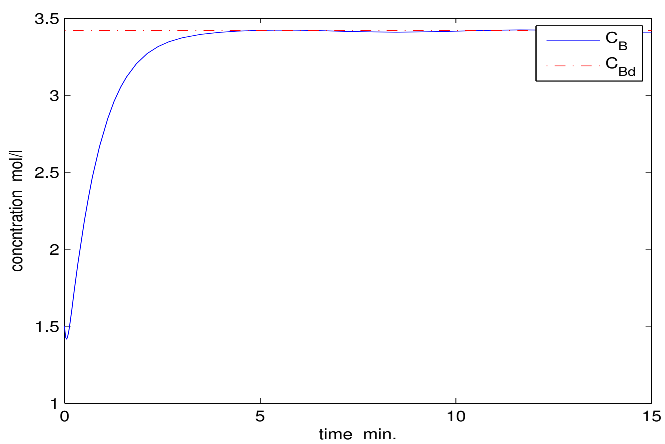

Figure 29: The trajectories of C_B and C_{Bd}

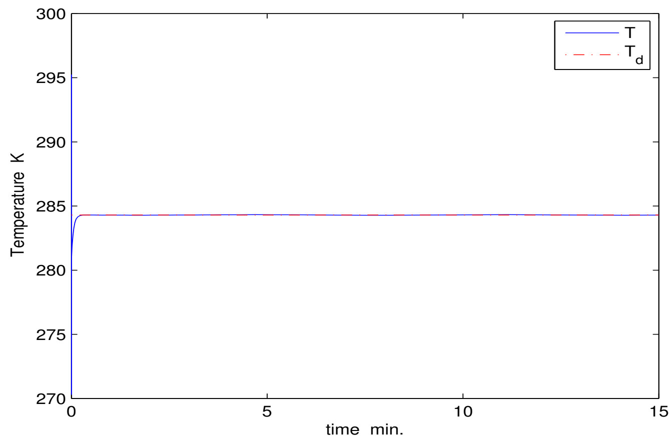

Figure 30: The trajectories of T and T_d

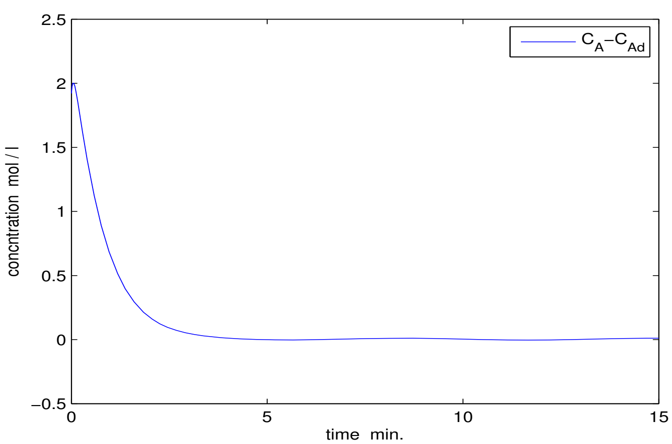

Figure 31: Tracking error (C_A-C_{Ad})

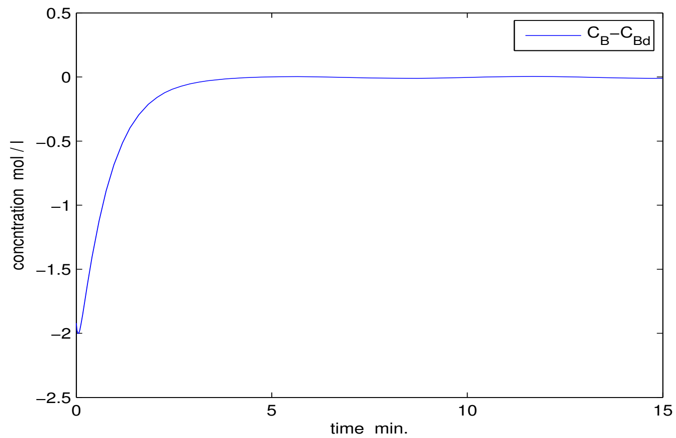

Figure 32: Tracking error (C_B-C_{Bd})

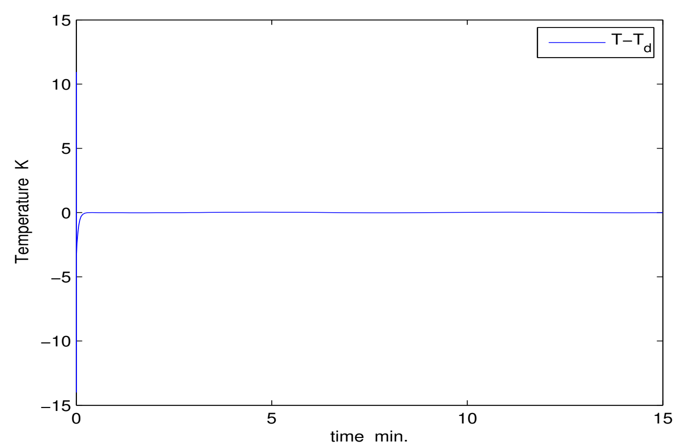

Figure 33: Tracking error (T-T_d)

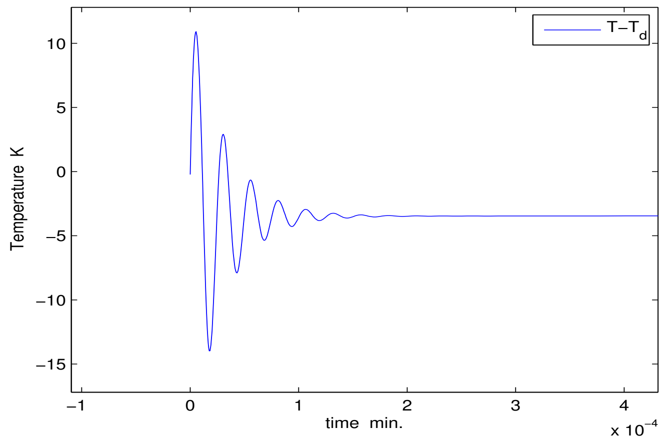

Figure 34: Magnified view of Fig. 33 to ensure the impulse freeness

In the exposure of such high magnitude disturbance, the error convergences to a bounded value which is quite satisfactory. On other hand, the figures from Fig. 11 to Fig. 18 show the responses, considering both 5 N disturbance and 10% uncertainty in the modeling parameters. The proposed controller still yields excellent performace. These simulations verify the robustness of the proposed controller.

4.3.2 Continuously Stirred Tank Reactor

The figures from Fig. 19 to Fig. 27 display the set point tracking responses of the continuously stirred tank reactor described in Section 4.2. This simulation is run with the measurement noise as disturbance. The controller makes the error converge to a bounded value which is small enough. The figures from Fig. 28 to Fig. 33 provide the responses in the presence of both measurement noise and 5% uncertainty in the modeling parameters. Also in this case the proposed controller keeps excellent performance. Fig. 34 ensures that the response to the fast system is impulse-free.

5 Conclusion and Future Work

5.1 Conclusion

In this work, the H_∞ tracking control problem has been discussed for high index singular systems using T-S fuzzy approach. The main achievements are summarized as follows.

1. The coordinate transformation has been introduced to decouple a nonlinear high index singular system into a linear fast subsystem and a nonlinear slow subsystem with strict-feedback form. A linear reference system has been proposed to share the same linear fast subsystem with the transformed singular system so that its response to the consistent initial conditions satisfies both the algebraic equation and hidden constraints. The proposed transformation also relaxes the I-observability condition of the existence of a full order Luenberger observer.

2. Based on this T-S fuzzy model, a state feedback controller and an observer-based state feedback controller have been designed for H_∞ tracking problems of the high index singular systems. Using Lyapunov function method, sufficient conditions for the stability of the closed-loop systems with H_∞ tracking performance have been derived.

3. An LMI-based optimization problem is formulated from the sufficient conditions to obtain feedback gains for a prescribed attenuation level ρ. YALMIP toolbox has been used to solve the LMI optimization problem. CVX toolbox can also be used to solve this optimization problem.

4. Two high index nonlinear singular systems from different fields are taken as examples to verify the robustness of the proposed controllers for both bounded disturbances and parameters uncertainties. Through the simulation, the performance of the proposed controllers are

verified.

The advantage of the proposed tracking control design approach is that a simple T-S fuzzy model based PDC controller is used for the controller design without the exact feedback linearization technique and complicated adaptive schemes. The design simplicity makes the proposed approach suitable to implement in practical applications where constraint systems are involved. The robustness of the proposed tracking controller is verified from the simulations of the illustrated physical problems.

5.2 Future Research Direction

The following problems are still open to be solved.

(1) The linearization of the non-regular high index singular system and model this high-index singular system using fuzzy T-S model.

(2) Observer design for the non-regular high index singular systems using the T-S model ensuring the existence of the observers.

(3) A simple way is needed to solve the LMI problems in a single step method instead of two step methods specifically when dealing with H_∞ tracking problem for nonlinear systems.

(4) A servo H_∞ controller for high index singular systems without using dynamic reference model, rather using constant reference only. For this case the deviation error will be estimated using observer not the states.

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