Positron/Electron Annihilation via the Two-Photon Pathway

by

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Abstract

When a positron/electron pair annihilate via the two-photon pathway, the emitted photons are momentum correlated. This correlation ensures that they move along a straight line path in opposite directions. An experiment performed in 2004 by Dr. V.D. Irby measured the time interval between detection of the photons. He observed a decay in the number of counts with increasing detection time interval, which he described using a Lorentzian, the line width of which at full-width half-maximum is measured to be 120ps. The data collected by Irby is interesting because current theory predicts that because the source is so localized (the effective source width used by Irby is safely within 5mm) the photons should be detected within a time interval of $\Delta t=d/c$ where d is the thickness of the source. This time interval corresponds to 17ps. This thesis fits the results to an exponential, and shows that this exponentially decaying nature of the coincidence time interval is characteristic of the entanglement of the two photons. We find that the wavefunctions of the photons decoheres in space according to how long the particle pair took to decay (which is exponential), and that the probability of simultaneous detection depends on the exponential of the product of the lifetime of positronium and the detection time interval.

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Chapter 1

Introduction

Quantum mechanics was first discovered as a solution to the problems of blackbody radiation, the photo-electric effect and other phenomena such as how atoms emit radiation only at discrete energies. It provides a good explanation of atomic and photonic phenomena in vacuum and within the framework of interaction forces. This description of reality does come with conceptual difficulties, however. Arising out of the quantum mechanical description of the world are the ideas of action-at-a-distance, non-locality and entanglement, all of which seem to operate in counter-intuitive and mysterious ways. Many great scientists (and lots of mediocre ones, too) have had great difficulties in accepting that quantum mechanics may be complete in its description of reality and that it is reality that is confusing.

In their oft-cited paper "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [1] Einstein, Podolsky and Rosen contemplate the idea of non-commutating operations performed on an entangled system. They look at a system described by a two part wavefunction $\Psi(x_1, x_2) = \sum_n \psi_n(x_1) u_n(x_2)$ where $\psi_n(x_1)$ describes one component (which could be photons or electrons etc.) and $u_n(x_2)$ is the set of eigenfunctions of Q (the position operator) and describes the second component. Measuring the position of one of the particles completely determines the wavefunction of the second particle. Measuring the complementary property momentum also completely determines the wavefunction of the second particle, only in this case the determined wavefunction is now an eigenfuntion of

the momentum operator P. Since P and Q are non-commuting operators, position and momentum cannot both be described completely at the same time. Thus Einstein, Podolsky and Rosen conclude that this description of reality must be faulty since the second system has not been interacted with. In their words

> "This makes the reality of P and Q depend upon the process of measurement carried out on the first system which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this."

As it happens, the EPR paradox or the idea that interacting with one particle may alter the state of another distant particle instantaneously is now a familiar concept in quantum mechanics reflective of a property called "entanglement". When two or more things (like photons) are created from the same process (such as an annihilation event) and must conserve some property between them (such as momentum) the two objects may be entangled. A familiar example of entanglement are the Bell thought experiments [2]. In these experiments, two electrons which must conserve angular momentum between them are sent in different directions and the different spatial components of spin are measured as in the Stern-Gerlach experiments. The measurement of these spatial components corresponds to non-commuting operators, just as in the EPR example.

Once we measure the x-component of spin of one of the particles using a Stern-Gerlach magnet, we can infer the x-component of spin of the other particle. If we believed the EPR argument it would be reasonable to assume that we could measure the x-component of spin of one of the particles and the z-component of spin of the other particle and infer the corresponding spin components for each particle. Then we would have shown that the spacial components of spin can in fact be measured simultaneously and that quantum mechanics is not a valid theory.

However, application of Bell's theorem and non-locality to this EPR thought experiment shows that measuring a component of spin of one of the particles changes the wavefunction of the other instantaneously, and that subsequent measurement of the second particle's spin is reflective of this altered state not the original spin-conserved state. Many experiments have been performed to test Bell's inequality and non-locality [3][4][5]. No matter the distance between the entangled electrons, they are described by a single wavefunction and a change in that wavefunction caused by a measurement of the properties of either particle instantaneously affects the state of the other. This is the essence of entanglement. The wavefunction that describes the entangled states cannot be separated into the product of the individual component states. They are linked at the most basic level.

This thesis is concerned with the wavefunction that can be used to describe the entangled state of the photons emitted via the annihilation of the matter/anti-matter pair of an electron and a positron. When an electron and a positron are in close enough proximity they can annihilate each other. This annihilation process converts the energy associated with the mass and motion of the particles into radiation. There are two probable pathways for the emittance of this radiation. The first pathway is taken when the positron/electron pair has a total spin one, and sees the energy released in the form of three photons. The second pathway is taken when the positron/electron pair has spin zero, and sees the energy released in the form of two photons. We are interested in the two-photon process.

The two photons created from this annihilation process are described using the theory of wavefunction collapse[1]. Between them they must conserve the energy and momentum of the matter/anti-matter pair. The energy conservation property ensures that both photons are approximately .511MeV in energy. The momentum conservation property means that these two photons travel in opposite directions, and the angular momentum conservation property is what results in the two-photon

process where one is spin up and the other spin down. These three properties taken together make this positron/electron two-photon system an ideal candidate to study the nature of their entanglement, and to examine what phenomenon may arise from this entanglement. The measurement of one of the photon's momentum results in the collapse of the wavefunction of the second photon.

This theoretical work examines the entanglement discovered by an American scientist Victor D. Irby [6]. He performed an experiment that used very fast detection equipment to detect the two photons to confirm the hypothesis that these two photons could be detected simultaneously. The experiment allowed Irby to measure accurately the time difference between detection events. He observed what we claim is an exponential decay function characterizing the difference in detection times versus number of counts. This means that although some photon pairs are detected almost simultaneously, most are separated by a significant time interval. The number of decay events decreases exponentially with the time interval between detection of the correlated photons. Irby estimates a characteristic line width of around 100ps.

This exponential decay in the number of counts with increasing detection time interval is interesting because current theory predicts that because the source is so localized (the effective source width used by Irby is safely within 5mm) it was expected that the arrival time difference should be the time it takes to traverse the source $\Delta t = d/c$ where d is the thickness of the source. It takes light about 17ps to traverse 5mm. An uncertainty of this magnitude would account for the annihilation occurring at an extreme end of the source. However, the line width of the exponential decay at full-width half-maximum is ten times greater than this source location uncertainty, measured by Irby as about 120ps. Light takes this long to traverse 3.6cm. We claim that the exponentially decaying nature of the coincidence time interval is characteristic of the entanglement of the two photons.

For a practical application, time of flight positron emission technology (TOF PET) uses the time difference between detection events of photons produced through two-photon annihilation to localize the source of radiation along the line of response [7]. The Irby experiment begins with a highly localized source, and so should give us a good idea about how accurate coincident detection can be for applications such as TOF PET.

This thesis explains this time difference between detection events using the phenomenon of decoherence. Using the results pertaining to two-photon detection experiments published in 2010 (see reference [8]) we present a theory linking the time-dependent decoherence of the particle pair with the time detection difference. It begins by deriving the atomic "before" and radiative "after" states using a combination of solutions to the Dirac equation and theories of spontaneous emission. We then switch to centre of mass and relative coordinates as in other published work[8] and elucidate a possible explanation for the non-simultaneous detection phenomenon.

Chapter 2

The Irby Experiment

In 2004 a paper [6] was published in the Journal of Measurement Science and Technology that presented an experiment which concluded that there exists a fundamental uncertainty in the arrival times of two photons which are emitted from the same annihilation event. The author, V. Irby, was striving to the answer the question "what is the minimum quantum uncertainty in the time interval between detection of the two annihilation photons?". The conclusion reached was that the linewidth of the curve formed by plotting time difference between detection events against number of counts has a line width which agrees very well with the lifetime of positronium. The challenge for us is to explain why, and to present a formalism which describes the event accurately. The experiment of Irby will now be presented.

The heart of the experimental set up is a 3mm wide plastic disc containing ^{22}Na. This form of radioactive sodium decays via one of two pathways. The first pathway, accounting for 90% of the decay, is β_+ decay wherein a photon (later referred to as the "start" photon), a positron and a neutrino are released. The second pathway is electron capture, and does not include the release of a positron. Sodium 22 has a half-life of about 2.6 years [9]. This capsule of sodium serves as our positron source. It is sandwiched between two aluminum plates of 3mm thickness. At this thickness it is highly probable that all positrons emitted from ^{22}Na decay will find an electron to annihilate with within this sandwich. In fact, virtually all positrons are annihilated within one millimeter of aluminum, so we consider our positronium to be localized within 5mm.

Since the photons emitted via two-photon decay are highly momentum-correlated and the center of mass momentum of the electron/positron pair (positronium) is almost zero we expect the sum of momentum of the photons emitted from this process to be close to zero. This means that the photons fly off at almost exactly 180° in order to conserve the approximately zero momentum of the center of mass. So the next part of the set-up is the placement of two photon detectors on opposite sides of the sample. The photon detectors used in this experiment cannot distinguish between photon energies, although detection efficiency does vary depending on photon frequency. This means that coincidence rates of the detectors include three different rates. A coincidence count could be a result of a start photon followed by one of the singlet decay photons, a start photon followed by a triplet decay photon, two photons from the three-photon annihilation process or our desired correlated annihilation photons. The formula describing the coincident rate for annihilation/annihilation events is calculated by Irby [6] as

$$R_{AA} = \left[\epsilon_{start}\epsilon_{stop}\frac{\Omega_{stop}}{4\pi}\alpha f_1 R_\circ\right] \tag{2.1}$$

where ϵ denotes detection efficiency, Ω is the solid angle of the detector as viewed by the source, $R_\circ$ is the source activity, α takes into account how many photons emitted are annihilation photons and f_1 is the fraction of annihilation events that occur through the singlet channel. The subscripts 'start' and 'stop' label the detectors. The 'stop' detector electronically delays the signal so that it is always last to arrive at the picosecond timing analyzer. The role of these labels becomes clearer once the experiment has been fully explained. The formula describing the coincidence rate for decay/annihilation events (when the photon emitted at the beginning of the decay process is recorded within coincidence limit with an annihilation photon), R_{DA}, is

$$R_{DA} = 2\left[\epsilon_{start}\frac{\Omega_{start}}{4\pi}R_{\circ}\right]\epsilon_{stop}\frac{\Omega_{stop}}{4\pi}\left(\alpha f_1 + \beta f_3\right) \tag{2.2}$$

where β takes into account how many photons emitted are decay prompt photons and where f_3 is the fraction of annihilation events that occur through the triplet channel. If we plot these two formulas against each other inserting values reflective of Irby's experimental set-up we see that when the detectors are set 10cm apart with the source exactly between them, coincidence rates are dominated by annihilation/annihilation events. The placement of the detectors along a line bisecting the source assures us that the coincidence rates picked up by the detectors is highly dominated by annihilation/annihilation events (see Figure 2.1).

Now we have established that the experimental set-up filters out undesired photons. In order to have confidence in the results we must establish that errors present in the experiment are less than the accuracy claimed. The author summarizes all expected experimental uncertainties as

$$\left[\begin{array}{cc} \text{electronic jitter} & \Delta t_j \approx 38\ ps \\ \text{electronic walk} & \Delta t_w \approx 45\ ps \\ \text{transit time spread} & \Delta t_{tts} \approx 52\ ps \\ \text{source location} & \Delta t_s \approx 47\ ps \end{array}\right]$$ These errors are added in quadrature.

There is not much to be done about electronic jitter or electronic walk. These uncertainties are inherent to the instruments used. However, Irby uses an interesting method to reduce transit time spread while increasing accuracy.

Transit Time Spread and Pulse Height Distribution

Photons are picked up by a multi-channel plate detector (MCPD), which is essentially a plate with many channels running diagonally through it. The channel pores are $10\mu m$ in diameter and are slanted 8° from the vertical. When an incident photon has been absorbed by the detector it triggers an electron avalanche through

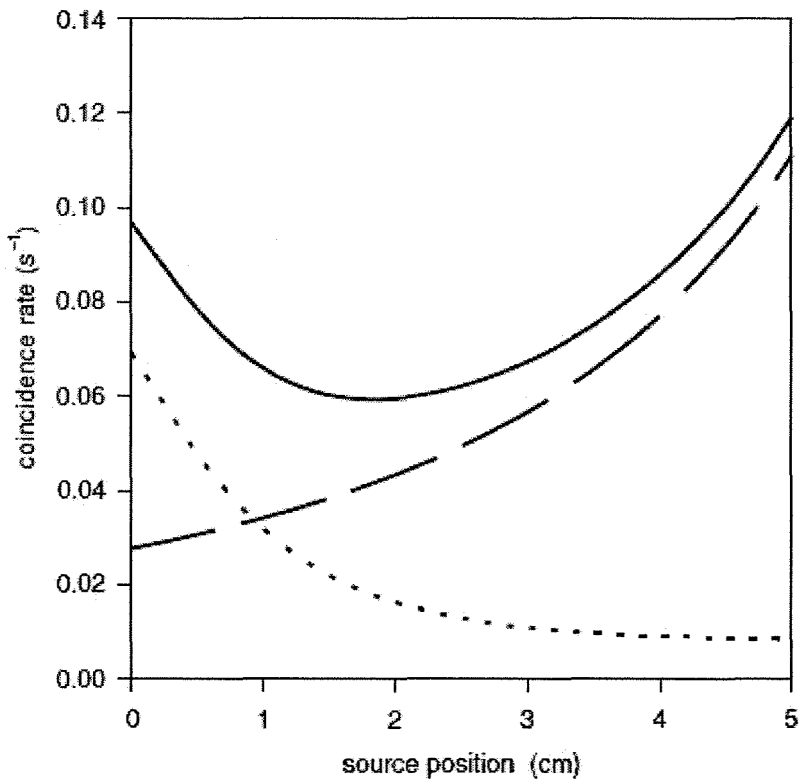

Figure 2.1: Coincidence rates vs Source position. The short-dashed line represents coincident detection of a decay photon with either a photon from the three-photon decay process or the two-photon decay process. The long-dashed line represents coincident detection of the photons created by the two-photon annihilation process. The solid line represents total coincindence detections. As the detectors are moved farther from the source, coincident detection events are dominated by two-photon annihilation [6].

the detector creating a pulse which is then picked up by an anode connected to the MCPD. It follows that photons of lower energy which cannot penetrate through matter create the largest pulses at the detector because they are absorbed close to the top of plate and an electron avalanche has the entirety of the plate in which to propagate. On the other hand, gamma photons released in two photon annihilation events have very high penetration depths through matter. They are likely to be absorbed close to the anode creating a pulse of relatively low height. Irby filters out pulses of high amplitude while maintaining the lower threshold. This is done by selecting pulses within the range ΔV which correspond to the incident radiation being absorbed at Δx according to the relation [6]

$$|\Delta x| = \frac{s_\circ}{\ln(n_\circ)} \ln\left(\frac{V + \Delta V}{V}\right) \tag{2.3}$$

Since Δx tends to zero as ΔV tends to zero, and since x is directly proportional to transit time spread, we can reduce Δt_{tts} by making ΔV as small as possible while keeping it close to the lower detection limit to filter out other radiation detection events.

Looking at pulse height distribution qualitatively as in Figure 2.2 we see that for lower energy electron pulses there is a peak around $17mV$. This peak corresponds to radiation hitting the MCP detector as deep into the channel as possible without penetrating matter. The Gaussian shape of the peak is explained by the rapid attenuation of incident low-energy radiation through matter. However the purely exponential decay of the high-energy γ-photon pulse amplitude is well supported by its high penetration through matter. As pulse height increases to above $20mV$, those originating from low-energy sources rapidly go to zero and pulse counts are dominated by the γ-photons. Irby also electronically tags all over-range events and does not use them when calculating coincidences, as these γ-photons likely originate as the start photon in positron emission or as triple-decay photons. Irby's results

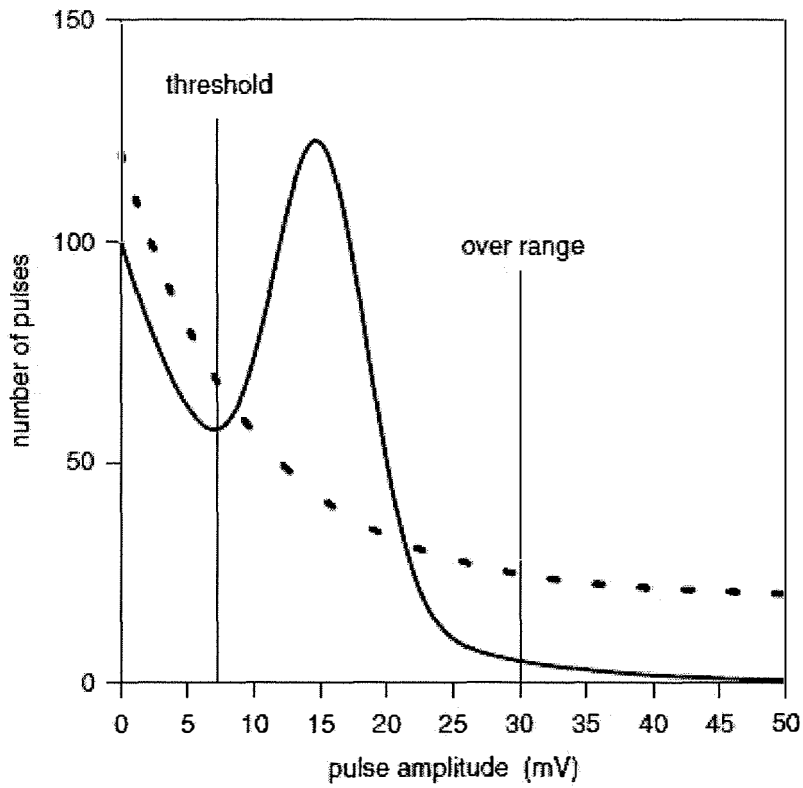

Figure 2.2: Pulse Height Distribution for a MicroChannel-Plate Detector. The solid line represents the pulse height distribution for mid-energy (1 keV) electrons. The dashed line represents the pulse height distribution for .511 MeV photons [6].

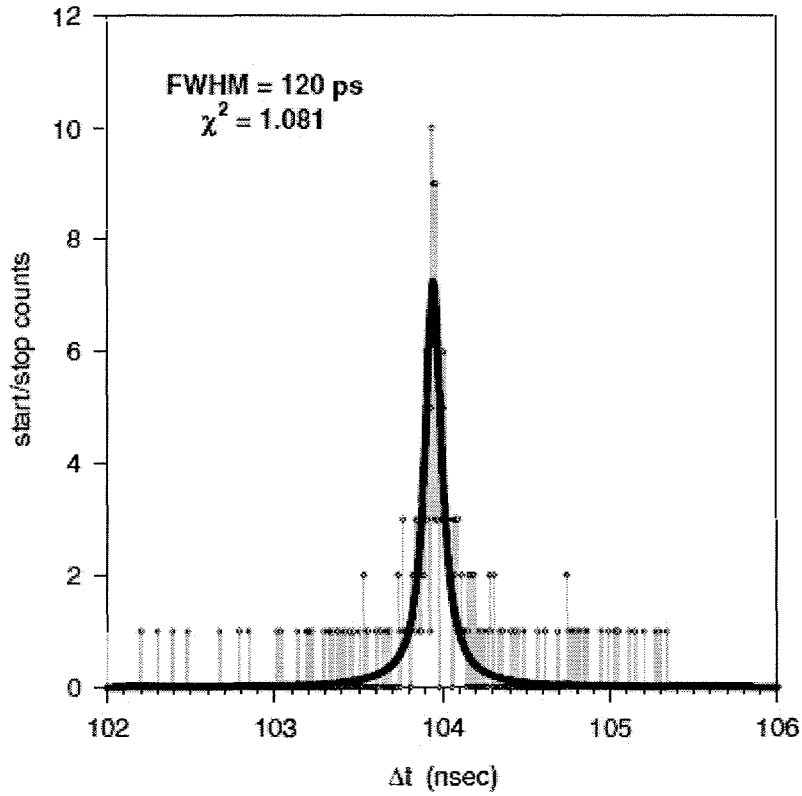

Figure 2.3: Final results by Irby (2004). This data is fit with a Lorenztian function, where a double exponential is required to describe positronium decay.

are plotted as photon counts vs. time interval between detection events in Figure 2.3.

The two photons that are detected by Irby are highly correlated, and there exists a single wave function that describes both of them together. As stated in the Introduction, it was expected that Irby would see simultaneous detection of the two momentum correlated photons produced in the two-photon positron/electron annihilation process when the source is midway between the detectors. Since this is not the case all of the time and the detection time difference at FWHM is very large

when compared to the time it takes light to traverse a 5mm sample, the correlated photons cannot generally be described using a delta function. There must be a wavefunction that can be used to find the probability of simultaneous detection. The rest of this thesis is concentrated on finding this wavefunction and using it to derive the probability of simultaneous detection.

Chapter 3

The Dirac Equation

The purpose of using the Dirac Equation is to calculate the probability that a positron will annihilate with an electron to form two photons, as in Sakurai [10]. Sakurai uses conservation of energy, relativity and creation and destruction operators in constructing his field operator Ψ. It is Lorentz covariant, conserves momentum as its basic premise and naturally describes the annihilation and creation of charged spin-1/2 particles. In the end, Sakurai calculates a lifetime for the positron/electron bound state. We continue by using his S- and M-matrix elements combined with the theory of spontaneous emission. But first...Dirac!

3.1 Derivation

We begin with a derivation of the Dirac equation. In relativity, conservation of energy is described by

$$\left(\frac{E^2}{c^2}\right) - \mathbf{p}^2 = (mc)^2 \tag{3.1}$$

In relativistic quantum mechanics the operator $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$ becomes $\boldsymbol{\sigma} \cdot (-i\hbar\boldsymbol{\nabla})$ where $\boldsymbol{\nabla}$ equals $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\boldsymbol{\sigma}$ is an operator introduced by Dirac to factor the p^2 term. The operator form of E is $\mathbf{E}^{(op)} = i\hbar\frac{\partial}{\partial t}$. The equation (3.1) becomes

$$\left(\frac{\mathbf{E}^{(op)}}{c} - \boldsymbol{\sigma} \cdot \mathbf{p}\right)\left(\frac{\mathbf{E}^{(op)}}{c} + \boldsymbol{\sigma} \cdot \mathbf{p}\right) = (mc)^2 \tag{3.2}$$

Making substitutions for E and $\mathbf{p}$ and using the parameterization $(x_1, x_2, x_{3,} ct = x_\circ)$, equation (3.1) becomes

$$\left(i\hbar\frac{\partial}{\partial x_\circ} + \boldsymbol{\sigma}\cdot \mathbf{i}\hbar\boldsymbol{\nabla}\right)\left(i\hbar\frac{\partial}{\partial x_\circ} - \boldsymbol{\sigma}\cdot i\hbar\boldsymbol{\nabla}\right)\phi = (mc)^2\phi. \tag{3.3}$$

Manipulation of this equation led Dirac to the famous Dirac Equation

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\Psi = 0 \tag{3.4}$$

where γ_μ with $\mu = 1, 2, 3, 4$ are 4×4 matrices given by

$$\gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu \\ i\sigma_\mu & 0 \end{pmatrix}. \tag{3.5}$$

The usual choice for the set of σ_μ is the Pauli matrices. The free-particle ($A_\mu = (\mathbf{0}, 0)$) solution of the Dirac equation with $p \neq 0$ is

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} u_A(\mathbf{p}) \\ u_B(\mathbf{p}) \end{pmatrix} \exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar} - i\frac{Et}{\hbar}\right) \tag{3.6}$$

The two-by-two matrices σ_μ must satisfy certain mathematical requirements (See appendix B). Ψ is explicitly defined as a *Dirac spinor.*

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{3.7}$$

Each ψ_i is either positive or negative in energy. We now use the Dirac equation to find plane wave solutions for Ψ. This set of plane wave-solutions can then be used as components in a Fourier series to describe an event.

3.2 Plane Wave Solutions

The plane wave solutions used to construct appropriate Fourier series must be orthonormal and comprise a complete set. The solutions derived below satisfy these conditions and will form the basis of the quantized Dirac field. This description of the quantized Dirac field will lead us towards a method of describing the physical event of positron/electron pair annihilation. In order to describe interaction with photons, we must re-write $(\boldsymbol{\sigma} \cdot \mathbf{p})$ in the Hamiltonian with $\mathbf{p} \longrightarrow \mathbf{p} - eA_\mu/c$ where A_μ is the time-independent electromagnetic interaction potential. It is then assumed that $\Psi(x,t)$ is an eigenfuction of $i\hbar\frac{\partial}{\partial t}$ with eigenvalue E. In solving the Dirac equation for this potential, $\Psi_i(x,0) = \psi_i e^{(-i\mathbf{p}\cdot\mathbf{x}/\hbar)}$ are four component spinors with the ψ_i being the time-independent portion, a four vector. To avoid confusion we point out that the ψ_i are functions of a four vector in the sense that they are defined by space-time coordinates, and $\Psi(x,t)$ is a four vector in the sense that it is composed of 4 vectors. Once the Dirac equation has been evaluated for our potential A_μ (see Appendix A), our $u(\mathbf{p})$ functions like in equation (3.6) are as follows:

$$u^{(1)}(\mathbf{p})=\sqrt{\left(|E|+mc^2\right)/2mc^2}\begin{pmatrix}1\\0\\p_3c/\left(E+mc^2\right)\\\left(p_1+ip_2\right)c/\left(E+mc^2\right)\end{pmatrix} \tag{3.8}$$

$$u^{(2)}(\mathbf{p})=\sqrt{\left(|E|+mc^2\right)/2mc^2}\begin{pmatrix}0\\1\\\left(p_1-ip_2\right)c/\left(E+mc^2\right)\\-p_3c/\left(E+mc^2\right)\end{pmatrix}$$

$$u^{(3)}(\mathbf{p})=\sqrt{\left(|E|+mc^2\right)/2mc^2}\begin{pmatrix}-p_3c/\left(E+mc^2\right)\\-\left(p_1+ip_2\right)c/\left(E+mc^2\right)\\1\\0\end{pmatrix} \tag{3.9}$$

$$u^{(4)}(\mathbf{p})=\sqrt{\left(|E|+mc^2\right)/2mc^2}\begin{pmatrix}-\left(p_1+ip_2\right)c/\left(E+mc^2\right)\\p_3c/\left(E+mc^2\right)\\0\\1\end{pmatrix}$$

If we note that the time evolution of a positron in the vector potential $+eA_\mu/c$ is the same as that of an electron in a field described by $-eA_\mu/c$ we see that the $u^{(n)}(\mathbf{p})$ solutions presented above describe, in order, a spin-up electron, a spin-down electron, a spin-up positron and a spin-down positron. These together form an orthonormal set and the law of conservation of charge holds true. These solutions taken at $t=0$ can be expanded in a Fourier series so that one may describe all possible states for an electron and a positron of a certain spin and momentum. They cannot, however, describe a system in which positrons and electrons are created and/or destroyed in finite number until we replace the Fourier coefficients with creation and destruction

operators. After adding the operators and completing the solution, we are left with the operator

$$\Psi(\mathbf{x},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_{r=1}^{4}\sqrt{\frac{mc^2}{|E|}}b_{\mathbf{p}}^{(r)}(t)u^{(r)}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} \tag{3.10}$$

where r denotes collectively the sign of the energy (the sign of the charge on the particle) and the spin state of that particle. In this way, the four possibilities for r are a positron spin up or down or an electron spin up or down. It will be noted that after we took the time dependence out of the free field solutions (3.8), the time dependence of $\Psi(\mathbf{x},t)$ arises out of the time dependent nature of the creation and destruction operators $b_{\mathbf{p}}^{\dagger(r)}(t)$ and $b_{\mathbf{p}}^{(r)}(t)$. By the Heisenberg equation of motion

$$\begin{aligned}
\dot{b_{\mathbf{p}}^{(r)}}&=\frac{i}{\hbar}\left[H,b_{\mathbf{p}}^{(r)}\right]=\mp\frac{i}{\hbar}b_{\mathbf{p}}^{(r)}\,|E|\\
\dot{b_{\mathbf{p}}^{\dagger(r)}}&=\frac{i}{\hbar}\left[H,b_{\mathbf{p}}^{\dagger(r)}\right]=\pm\frac{i}{\hbar}b_{\mathbf{p}}^{(r)}\,|E| \qquad\qquad \text{for } r=\{1,2,3,4\}
\end{aligned}$$

we see that

$$\begin{aligned}
b_{\mathbf{p}}^{(r)}(t)&=b_{\mathbf{p}}^{(r)}(0)e^{\mp i|E|t/\hbar}\\
b_{\mathbf{p}}^{\dagger(r)}(t)&=b_{\mathbf{p}}^{\dagger(r)}(0)e^{\pm i|E|t/\hbar}
\end{aligned}$$

which enables us to write $\Psi(\mathbf{x},t)$ as

$$\begin{aligned}
\Psi(\mathbf{x},t) \;=\; &\frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sqrt{\frac{mc^2}{|E|}}\left(\sum_{r=1,2}b_{\mathbf{p}}^{(r)}(0)u^{(r)}(\mathbf{p})\exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}-\frac{i\,|E|\,t}{\hbar}\right]\right.\\
&\left.+\sum_{r=3,4}b_{\mathbf{p}}^{\dagger(r)}(0)u^{(r)}(\mathbf{p})\exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}+\frac{i\,|E|\,t}{\hbar}\right]\right).
\end{aligned} \tag{3.11}$$

The elements of (3.11) can be interpreted so that the sign of the energy denotes the species of particle, whether electron or positron. By doing this we further associate the first and second parts of (3.11) as Hermitian conjugates and make the following substitutions in the creation and destruction operators

$$\begin{aligned} &\text{for } r = 1,2 \quad b_{\mathbf{p}}^{(r)} = b_{\mathbf{p}}^{(s)} \quad u^{(r)}(\mathbf{p}) = u^{(s)}(\mathbf{p}) \\ &\text{for } r = 3,4 \quad b_{-\mathbf{p}}^{(r)} = d_{\mathbf{p}}^{(s)} \quad u^{(r)}(-\mathbf{p}) = v^{(s)}(\mathbf{p}) \end{aligned}$$

where $d_{\mathbf{p}}^{(s)}$ and $d_{\mathbf{p}}^{\dagger(s)}$ satisfy the same anticommutation relations as $b_{\mathbf{p}}^{(s)}$ and $b_{\mathbf{p}}^{\dagger(s)}$ but are defined as for electrons instead of positrons.

Our expressions for Ψ and its conjugate $\overline{\Psi}$ are finally

$$\Psi(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sqrt{\frac{mc^2}{|E|}} \left(b_{\mathbf{p}}^{(s)} u^{(s)}(\mathbf{p}) \exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{i\,|E|\,t}{\hbar} \right] + d_{\mathbf{p}}^{\dagger(s)} v^{(s)}(\mathbf{p}) \exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i\,|E|\,t}{\hbar} \right] \right) \tag{3.12}$$

$$\overline{\Psi}(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sqrt{\frac{mc^2}{|E|}} \left(d_{\mathbf{p}}^{(s)} \overline{v}^{(s)}(\mathbf{p}) \exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{i\,|E|\,t}{\hbar} \right] + b_{\mathbf{p}}^{\dagger(s)} \overline{u}^{(s)}(\mathbf{p}) \exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i\,|E|\,t}{\hbar} \right] \right). \tag{3.13}$$

The above expression for $\Psi(\mathbf{x},t)$ is an evolution of the plane wave function characterized by $(\mathbf{p},r)$, and is now a field operator, operating in the number space of electrons and positrons of momentum $\mathbf{p}$ and spin s existing at points (x,t). The shift from the plane wave function to field operator is known as the second quantization.

We will break down Ψ and $\overline{\Psi}$ into expressions which will prove to be convenient later. Separating the positive and negative frequency parts of Ψ and $\overline{\Psi}$ we can write them as

$$\psi^{(+)} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{s=1,2} \sqrt{\frac{mc^2}{|E|}} b_{\mathbf{p}}^{(s)} u^{(s)}(\mathbf{p}) \exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{i\,|E|\,t}{\hbar} \right] \tag{3.14}$$

which annihilates electrons,

$$\psi^{(-)} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{s=1,2} \sqrt{\frac{mc^2}{|E|}} d_{\mathbf{p}}^{\dagger(s)} v^{(s)}(\mathbf{p}) \exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i\,|E|\,t}{\hbar}\right] \tag{3.15}$$

which creates positrons

$$\overline{\psi}^{(+)} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{s=1,2} \sqrt{\frac{mc^2}{|E|}} d_{\mathbf{p}}^{(s)} \overline{v}^{(s)}(\mathbf{p}) \exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{i\,|E|\,t}{\hbar}\right] \tag{3.16}$$

which annihilates positrons and

$$\overline{\psi}^{(-)} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{s=1,2} \sqrt{\frac{mc^2}{|E|}} b_{\mathbf{p}}^{\dagger(s)} \overline{u}^{(s)}(\mathbf{p}) \exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i\,|E|\,t}{\hbar}\right] \tag{3.17}$$

which creates electrons.

3.2.1 Interaction Hamiltonian

Thus far we have been using SI units, but from this point on we will switch to the natural units where $\hbar = c = 1$. We assume that our problem is solvable through perturbation theory, so that the evolution of our state vector Φ in the interaction picture is equal to the interaction Hamiltonian acting on that state, i.e.

$$i\frac{\partial}{\partial t}\Phi = H_I\Phi. \tag{3.18}$$

We begin finding a solution by defining an operator $U(t, t_\circ)$ such that $\Phi(t) = U(t, t_\circ)\Phi(t_\circ)$. It is apparent then that there exists the boundary condition $U(t_\circ, t_\circ) = 1$. Equation (3.18) can then be written $i\frac{\partial}{\partial t}U(t, t_\circ) = H_I U(t, t_\circ)$. If we combine this differential equation with the boundary condition, we get

$$\begin{aligned} U(t,t_\circ) &= 1 - i\int_{t_\circ}^{t} H_I U(t,t_\circ)dt \\ &= 1 - i\int_{t_\circ}^{t} H_I(t_1)dt_1 + (-i)^2 \int_{t_\circ}^{t} dt_1 \int_{t_\circ}^{t_1} H_I\,(t_1)H_I(t_2)\,dt_2 \\ &\quad +(-i)^n \int_{t_\circ}^{t} dt_1 \int_{t_\circ}^{t_1} dt_2 ... \int_{t_\circ}^{t_{n-1}} H_I\,(t_1)\,H_I(t_2)...H_I(t_n)dt_n + ... \end{aligned} \tag{3.19}$$

when solved iteratively. What is exciting about this solution is that each expression pertains to an outcome of higher order than the last, so that "1" represents no change in the system (i.e., when the initial state $\Phi(t_\circ)$ is identical to the final state $\Phi(t)$), the next expression represents a first-order change in the system etc. What is meant by 'first order change' cannot be defined until both the Hamiltonian and the state vectors (initial and final) are defined, which will be done for positron annihilation shortly. It will be noted, however, that this iterative solution proves to be the key to describing physical interactions that may result in many different outcomes. For example, it is known that positron-electron annihilation may result in the release of two photons (a second-order solution) or three photons (a third order solution). There is also more than one possible second or third-order outcome, although the probabilities of these alternate solutions may be very small. In fact, the most useful thing about (3.19) is that by bracketing it with the initial and final states of interest and then squaring one obtains the probability of that transition.

$$P_{if} = \left|\langle \Phi_f | \, U(t, t_\circ) \, |\Phi_i\rangle\right|^2 \tag{3.20}$$

Although $U(t, t_\circ)$ has proven to be a very useful function, it is possible to encounter difficulties with its dependence on the time interval $\triangle t = t - t_\circ$. When this interval is small enough (depending on the process) we might choose t to come before the process has finished and not get any useful information. To avoid this scenario we create a new operator S such that $S = U(\infty, -\infty)$ as in (3.21) below.

$$\begin{aligned} S(\infty, -\infty) \;=\; & 1 - i \int_{-\infty}^{\infty} H_I(t_1) dt_1 + \\ & (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} H_I\,(t_1) H_I(t_2)\, dt_2 + \\ & (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 ... \int_{-\infty}^{t_{n-1}} H_I\,(t_1)\, H_I(t_2) ... H_I(t_n) dt_n + ... \end{aligned} \tag{3.21}$$

Although it is impossible for us to know what has happened and will happen to particles from the beginning of time to the end of it (if there is such a thing as a beginning and end of time), our choice of infinities is intended to represent a time interval significant enough to encapsulate the relevant process in time so that the history of the initial state and the fate of the final state are irrelevant.

We have been referring to the two-photon annihilation process as second order. The proof of this lies in the interaction Hamiltonian. The interaction Hamiltonian density can be derived from the Lagrangian density to be $\mathcal{H}_{int} = -ie\overline{\Psi}\gamma_\mu\Psi A_\mu$ with A_μ defined as the well-known quantized radiation field operator

$$\mathbf{A}_\mu(\mathbf{x},t) = \left(\frac{1}{\sqrt{V}}\right)\sum_k\sum_\alpha c\sqrt{\hbar/2\omega}\left[\mathbf{a}_{k,\alpha}(t)\boldsymbol{\epsilon}_\mu^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{a}_{k,\alpha}^\dagger(t)\boldsymbol{\epsilon}_\mu^{(\alpha)}e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \qquad (3.22)$$

in the Coulomb gauge where $\frac{1}{\sqrt{V}}$ vanishes under integration, $\mathbf{k}$ is the wave number, and $\epsilon^{(\alpha)}$ indicates the polarization. The derivation of this operator can be found in almost any quantum mechanics textbook [**?**]. It is worthwhile to note that while $\mathbf{x}$ and t parameterize $\mathbf{A}$, they do not define the operator space. Rather than operating in space and time, the quantized radiation field operator operates in number space via the creation and annihilation operators $\mathbf{a}^\dagger$ and $\mathbf{a}$. Thus this operator acts as a mechanism whereby we can create and/or destroy a photon of energy ω, wave vector $\mathbf{k}$ and polarization α at position $\mathbf{x}$ and time t.

For later purposes, we will write $\mathbf{A}_\mu(\mathbf{x},t)$ as the sum of the positive and negative frequency parts.

$$\mathbf{A}_\mu(\mathbf{x},t) = A_\mu^{(+)} + A_\mu^{(-)}$$

where

$$A_\mu^{(+)} = \left(\frac{1}{\sqrt{V}}\right)\sum_k\sum_\alpha c\sqrt{\hbar/2\omega}\mathbf{a}_{k,\alpha}(t)\boldsymbol{\epsilon}_\mu^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}} \text{ annihilates a photon}$$

$$A_\mu^{(-)} = \left(\frac{1}{\sqrt{V}}\right)\sum_k\sum_\alpha c\sqrt{\hbar/2\omega}\mathbf{a}_{k,\alpha}^\dagger(t)\boldsymbol{\epsilon}_\mu^{(\alpha)}e^{-i\mathbf{k}\cdot\mathbf{x}} \text{ creates a photon}$$

The interaction Hamiltonian is then

$$H_I = -ie\int \overline{\Psi}\gamma_\mu\Psi A_\mu dx^3 \tag{3.23}$$

Equation (3.23) appears attractively simple, but it is deceivingly complex. Now that Ψ, Ψ, and A_u have been defined, their product in the interaction Hamiltonian results in eight possible combinations, and this is only when they appear in the first order expression in (3.21)! To see explicitly what outcomes are possible, we write down the arguments of the integrations of the first order of the Hamiltonian

$$\begin{aligned}
&H_I(t_1)\\
&= -ie\int \overline{\Psi}\gamma_\mu\Psi A_\mu dx^3\\
&= -ie\int \left[\left(\overline{\psi}^{(+)} + \overline{\psi}^{(-)}\right)\gamma_\mu\left(\psi^{(+)} + \psi^{(-)}\right)\left(A_\mu^{(+)} + A_\mu^{(-)}\right)\right]dx^3\\
&= -ie\int \left[\left(\overline{\psi}^{(+)}\gamma_\mu\psi^{(+)} + \overline{\psi}^{(+)}\gamma_\mu\psi^{(-)} + \overline{\psi}^{(-)}\gamma_\mu\psi^{(+)} + \overline{\psi}^{(-)}\gamma_\mu\psi^{(-)}\right)\left(A_\mu^{(+)} + A_\mu^{(-)}\right)\right]dx^3
\end{aligned}$$

It is clear, after a little bit of trial and error, that for our choice of A_μ the below transitions are possible and indeed are the only ones possible. Any other choices for initial or final states will result in either an invalid operation or the creation of the null vector. The arguments of bras and kets are as follows: $e^-, e^+, photons$. For the sake of clarity, only creation and annihilation operators will be written down, and other factors omitted.

$Transition$	$bra, operators, ket$
$\overline{\psi}^{(+)}\gamma_\mu\psi^{(+)}A^{(+)}$	$\langle 0,0,n-1\vert\, dba\,\vert 1,1,n\rangle$
$\overline{\psi}^{(+)}\gamma_\mu\psi^{(+)}A^{(-)}$	$\langle 0,0,n+1\vert\, dba^\dagger\,\vert 1,1,n\rangle$
$\overline{\psi}^{(+)}\gamma_\mu\psi^{(-)}A^{(+)}$	$\langle 0,0,n-1\vert\, dd^\dagger a\,\vert 0,0,n\rangle$
$\overline{\psi}^{(+)}\gamma_\mu\psi^{(-)}A^{(-)}$	$\langle 0,0,n+1\vert\, dd^\dagger a^\dagger\,\vert 0,0,n\rangle$
$\overline{\psi}^{(-)}\gamma_\mu\psi^{(+)}A^{(+)}$	$\langle 1,0,n-1\vert\, b^\dagger ba\,\vert 1,0,n\rangle$
$\overline{\psi}^{(-)}\gamma_\mu\psi^{(+)}A^{(-)}$	$\langle 1',0,n+1,\vert\, b^{\dagger'}ba^\dagger\,\vert 1,0,n\rangle$
$\overline{\psi}^{(-)}\gamma_\mu\psi^{(-)}A^{(+)}$	$\langle 1,1,n-1\vert\, b^\dagger d^\dagger a\,\vert 0,0,n\rangle$
$\overline{\psi}^{(-)}\gamma_\mu\psi^{(-)}A^{(-)}$	$\langle 1,1,n+1\vert\, d^\dagger ba^\dagger\,\vert 0,0,n\rangle$

The first order equation as written above can, depending on which initial and final states are of interest, give the probabilities for eight possible transitions. It should be noted that notation for photon occupation number is different than that for particles, for there may exist two photons of exactly the same state in a system whereas two identical electrons (or positrons) are forbidden. This difference has its roots in the commutation relations that define the relationship between creation and annihilation operators. The commutation relations are as follows, with O_r representing annihilation operators and $O_r^\dagger$ representing creation operators in momentum-spin state r:

$$\left\{O_r, O_{r'}^\dagger\right\} = \delta_{rr'}$$
$$\{O_r, O_{r'}\} = 0$$
$$\left\{O_r^\dagger, O_{r'}^\dagger\right\} = 0$$

For photons the term in brackets is defined as $\{A,B\} = AB - BA$ (the commutation relation), and for the particles is $\{A,B\} = AB + BA$ (the anti-commutation relation). These differences reflect the physical reality of the Fermionic nature of electrons and positrons, and the Bosonic nature of photons.

The second order element $(-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} H_I(t_1) H_I(t_2)\, dt_2$ which we will call $S^{(2)}$ has sixty-four elements which in different combination predict specific outcomes. If we substitute (3.23) for H_I we get

$$
\begin{aligned}
& S^{(2)} \\
& = (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} H_I(t_1) H_I(t_2)\, dt_2 \\
& = (-i)^2 \int d^4x_1 \int_{t_2>t_1} d^4x_2 \mathcal{H}_{int}(x_2)\, \mathcal{H}_{int}(x_1) \\
& = e^2 \int d^4x_1 \int_{t_2>t_1} d^4x_2 \overline{\Psi}(x_2)\, \gamma_\mu \Psi(x_2)\, A_\mu(x_1)\, \overline{\Psi}(x_1)\, \gamma_\nu \Psi(x_1)\, A_v(x_2) \\
& = e^2 \int d^4x_1 \int_{t_2>t_1} d^4x_2 \left\{ \overline{\psi}^{(+)}(x_2) + \overline{\psi}^{(-)}(x_2) \right\} \gamma_\mu \left\{ \psi^{(+)}(x_2) + \psi^{(-)}(x_2) \right\} A_\mu(x_1) \\
& \quad \times \left\{ \overline{\psi}^{(+)}(x_1) + \overline{\psi}^{(-)}(x_1) \right\} \gamma_\nu \left\{ \psi^{(+)}(x_1) + \psi^{(-)}(x_1) \right\} A_v(x_2)
\end{aligned}
$$

There are sixteen terms in this equation, and therefore sixteen possible transitions before you take into account the different variations possible after A_u and A_v are expanded (then the possibilities swell to sixty-four), so I will not list them all. But once again, the probability of transition from one state to another may be found by bracketing this expression with appropriate initial and final states. In our case the final state is $\langle \Phi_f | = \langle 1_{\mathbf{k}}, 1_{-\mathbf{k}} |$, the initial $|\Phi_i\rangle = |e^+, e^-\rangle$. Upon inserting these states, we see that only two of the sixty-four expressions survive. $S^{(2)}$ becomes S_{fi}, and the initial and final photon states may be separated from the particle states because their respective operators do not interact. We can also interchange summation indices (which have been omitted here) in order that the two particle terms may look as similar as possible, but this must be accompanied by the addition of the step functions $\Theta(t_x - t_y)$ and the recognition that our integration parameter now depends on whether t_x follows t_y or vice versa. After these changes are made, $S^{(2)}$ becomes

$$S_{fi} = (-e)^2 \int d^4x_1 \int d^4x_2 \, \langle 1_{\mathbf{k}}, 1_{-\mathbf{k}}| \, A_\mu(x_1) \, A_v(x_2) \, |0\rangle \, (\gamma_\mu)_{\alpha\beta} \, (\gamma_v)_{\gamma\delta} \tag{3.24}$$
$$\times \Big[\langle 0| \, \psi_\beta^{(+)}(x_1) \, \overline{\psi}_\gamma^{(-)}(x_2) \, \overline{\psi}_\alpha^{(+)}(x_1) \, \psi_\delta^{(+)}(x_2) \, \left|e^+e^-\right\rangle \Theta(t_1 - t_2)$$
$$- \langle 0| \, \overline{\psi}_\gamma^{(+)}(x_2) \, \psi_\beta^{(-)}(x_1) \, \overline{\psi}_\alpha^{(+)}(x_1) \, \psi_\delta^{(+)}(x_2) \, \left|e^+e^-\right\rangle \Theta(t_2 - t_1) \Big] .$$

This expression for S_{fi} can be further simplified by inserting a complete set of states $|n\rangle \langle n|$ between the second and third operators in the electron field expressions as follows, where $\langle 0| \, \psi_\beta^{(+)}(x_1) \, \overline{\psi}_\gamma^{(-)}(x_2) \, \overline{\psi}_\alpha^{(+)}(x_1) \, \psi_\delta^{(+)}(x_2) \, |e^+e^-\rangle = \langle 0| \, \psi_1^{(+)} \overline{\psi}_2^{(-)} \, |n\rangle \, \langle n| \, \overline{\psi}_1^{(+)} \psi_2^{(+)} \, |e^+e^-\rangle$. We see that only the vacuum term contributes, leaving us with $\langle 0| \, \overline{\psi}_2^{(+)} \psi_1^{(-)} \, |0\rangle \, \langle 0| \, \overline{\psi}_1^{(+)} \psi_2^{(+)} \, |e^+e^-\rangle$. Similarly, $\langle 0| \, \overline{\psi}_2^{(+)} \psi_1^{(-)} \overline{\psi}_1^{(+)} \psi_2^{(+)} \, |e^+e^-\rangle = \langle 0| \, \overline{\psi}_2^{(+)} \psi_1^{(-)} \, |0\rangle \, \langle 0| \, \overline{\psi}_1^{(+)} \psi_2^{(+)} \, |e^+e^-\rangle$. S_{fi} has now become

$$S_{fi} = (-e)^2 \int d^4x_1 \int d^4x_2 \, \langle 1_{\mathbf{k}}, 1_{-\mathbf{k}}| \, A_\mu(x_1) \, A_v(x_2) \, |0\rangle \, \gamma_u \gamma_v \tag{3.25}$$
$$\times \Big[\langle 0| \, \psi_1^{(+)} \overline{\psi}_2^{(-)} \, |0\rangle \, \langle 0| \, \overline{\psi}_1^{(+)} \psi_2^{(+)} \, \left|e^+e^-\right\rangle \Theta(t_1 - t_2)$$
$$- \langle 0| \, \overline{\psi}_2^{(+)} \psi_1^{(-)} \, |0\rangle \, \langle 0| \, \overline{\psi}_1^{(+)} \psi_2^{(+)} \, \left|e^+e^-\right\rangle \Theta(t_2 - t_1) \Big] .$$

This turns out to be a very interesting and revealing expression. The first part states that two photons will be created, each at two different places and times x_1 and x_2 (remembering of course that these are four-vectors), and that the photons will have wave vectors $\mathbf{k}$ and $-\mathbf{k}$. The second part reads that if $t_1 > t_2$, an electron is annihilated and another created at x_2 (the earlier time). These events are followed by the annihilation of both a positron and an electron at x_1. The third part reads exactly the same as the second, but with the exchange of $t_1 \leftrightarrow t_2$, and $x_1 \leftrightarrow x_2$. The S-matrix element may be further simplified by substituting the proper expressions for $\overline{\psi}^{(+)} \, \psi^{(+)}$ into the term that annihilates the positron and original electron After

substitution of (3.16) and (3.14) into (3.25) we are left with

$$S_{fi} = (-e)^2 \int d^4x_1 \int d^4x_2 \, \langle 1_{\mathbf{k}}, 1_{-\mathbf{k}}| \, A_u(x_1) \, A_v(x_2) \, |0\rangle \, (\gamma_u)_{\alpha\beta} \, (\gamma_v)_{\gamma\delta} \tag{3.26}$$
$$\times \left[\langle 0| \, \psi_\beta^{(+)}(x_1) \, \overline{\psi}_\gamma^{(-)}(x_2) \, |0\rangle \, \langle 0| \, \overline{\psi}_\alpha^{(+)}(x_1) \, \psi_\delta^{(+)}(x_2) \, \left|e^+e^-\right\rangle \phi(t_1 - t_2) \right.$$
$$\left. - \langle 0| \, \overline{\psi}_\gamma^{(+)}(x_2) \, \psi_\beta^{(-)}(x_1) \, |0\rangle \, \langle 0| \, \overline{\psi}_\alpha^{(+)}(x_1) \, \psi_\delta^{(+)}(x_2) \, \left|e^+e^-\right\rangle \phi(t_2 - t_1) \right].$$

The expression $\langle 0| \, \overline{\psi}_1^{(+)} \psi_2^{(+)} \, |e^+e^-\rangle$ can be expanded to read $\left[\sqrt{\frac{m}{E_+ V}} \overline{\nu}_\alpha^{(s_+)}(\mathbf{p}_+) \, e^{ip_+ \cdot x_1} \right]$ $\left[\sqrt{\frac{m}{E_- V}} u_\delta^{(s_-)}(\mathbf{p}_-) \, e^{ip_- \cdot x_2} \right]$ which defines the annihilation of an electron at x_1 and a positron at x_2. We may also solve for $\langle 1_{\mathbf{k}}, 1_{-\mathbf{k}}| \, A_u(x_1) \, A_v(x_2) \, |0\rangle$. Since both A_u and A_ν can create a photon with momenta k_1 or k_2 at x_1 or x_2, the term $\langle 1_{\mathbf{k}}, 1_{-\mathbf{k}}| \, A_u(x_1) \, A_v(x_2) \, |0\rangle$ is defined as

Equation (3.26) now looks like

$$S_{fi} = (-e)^2 \int d^4x_1 \int d^4x_2 \left[\left(\frac{1}{\sqrt{2\omega_1 V}} \epsilon_\mu^{(\alpha_1)} e^{ik_1 \cdot x_1} \right) \left(\frac{1}{\sqrt{2\omega_2 V}} \epsilon_\nu^{(\alpha_2)} e^{-ik_2 \cdot x_2} \right) \right. \tag{3.27}$$
$$\left. + \left(\frac{1}{\sqrt{2\omega_2 V}} \epsilon_\mu^{(\alpha_2)} e^{ik_2 \cdot x_1} \right) \left(\frac{1}{\sqrt{2\omega_1 V}} \epsilon_\nu^{(\alpha_1)} e^{-ik_1 \cdot x_2} \right) \right] \left[\sqrt{\frac{m}{E_+ V}} \overline{\nu}_\alpha^{(s_+)}(\mathbf{p}_+) \, e^{ip_+ \cdot x_1} \right] \times$$
$$(\gamma_u)_{\alpha\beta} \left[\begin{array}{c} \langle 0| \, \psi_\beta^{(+)}(x_1) \, \overline{\psi}_\gamma^{(-)}(x_2) \, |0\rangle \, \phi(t_1 - t_2) - \\ \langle 0| \, \overline{\psi}_\gamma^{(+)}(x_2) \, \psi_\beta^{(-)}(x_1) \, |0\rangle \, \phi(t_2 - t_1) \end{array} \right] \times \tag{3.28}$$
$$(\gamma_v)_{\gamma\delta} \left[\sqrt{\frac{m}{E_- V}} u_\delta^{(s_-)}(\mathbf{p}_-) \, e^{ip_- \cdot x_2} \right]. \tag{3.29}$$

The probability of the annihilation of an electron-positron pair via the creation of two momentum-correlated photons is expressed as the product of the plane wave solutions for the correlated photons (3.27) with an intermediate time ordered state (3.28) and the plane wave particle expressions for an electron and a positron (3.29). The intermediate states sandwiched between the vacuum states in (3.28) are new and describe what is happening in between x_1 and x_2. The operator ϕ is one when its argument is greater than one and zero otherwise. The expression then reads that

if $t_1 > t_2$ a virtual electron propagates from x_2 to x_1, and if $t_2 > t_1$ a virtual positron propagates from x_1 to x_2.

We can consider both scenarios (the propagation of a virtual electron from x_2 to x_1 and the propagation of a virtual positron from x_1 to x_2) simultaneously using the time ordering product $T\left(\psi_\alpha(x)\,\overline{\psi}_\beta(x')\right)$. In evaluating this product it is shown that regardless of whether t_1 is later or earlier than t_2,

$$\begin{aligned}\left\langle 0\right|\psi_\beta^{(+)}(x_1)\,\overline{\psi}_\gamma^{(-)}(x_2)\left|0\right\rangle &= \left\langle 0\right|\overline{\psi}_\gamma^{(+)}(x_2)\,\psi_\beta^{(-)}(x_1)\left|0\right\rangle \\ &= \frac{-i}{(2\pi)^3}\int d^4p\frac{(-i\gamma\cdot p+m)_{\alpha\beta}}{p^2+m^2-i\epsilon}\exp\left[ip\cdot(x-x')\right].\end{aligned} \tag{3.30}$$

The meaning of this product is that we can consider what is happening between x_1 and x_2 as either a virtual electron going forward in time from x_2 to x_1 or an electron going backward in time from x_1 to x_2. Whether the reader chooses to interpret virtual positrons and electrons or solely one kind of virtual particle going backwards or forwards in time is a matter of preference. What is important is that considering both situations together leaves us with (3.30), which is Lorentz invariant and covariant in $(\mathbf{x}, x_\circ)$.

Substituting (3.30) into (3.26) and rearranging we get

$$\begin{aligned}S_{fi} = {} & (-e)^2\frac{m}{2V^2}\sqrt{\frac{1}{E_-E_+\omega_1\omega_2}}\int d^4x_1\int d^4x_2 e^{-ik_1\cdot x_1-ik_2\cdot x_2}\epsilon_\mu^{(\alpha_1)}\epsilon_\nu^{(\alpha_2)}\times \\ & e^{-ip_+\cdot x_1+ip_-\cdot x_2}\overline{v}\gamma_\mu\left(-\frac{i}{(2\pi)^3}\int d^4q\frac{e^{iq\cdot(x_1-x_2)}(-i\gamma\cdot q+m)}{q^2+m^2-ie}\right)\gamma_\nu u + \\ & \left\{\begin{matrix}k_1\longleftrightarrow k_2\\ \alpha_1\longleftrightarrow\alpha_2\end{matrix}\right\}\end{aligned} \tag{3.31}$$

So, for example, S_{fi} may describe the following scenario: an electron of positive spin is annihilated at x_1 creating a photon and a virtual electron in the same instant. The photon flies off with momentum k_1, the virtual electron propagates to x_2 where

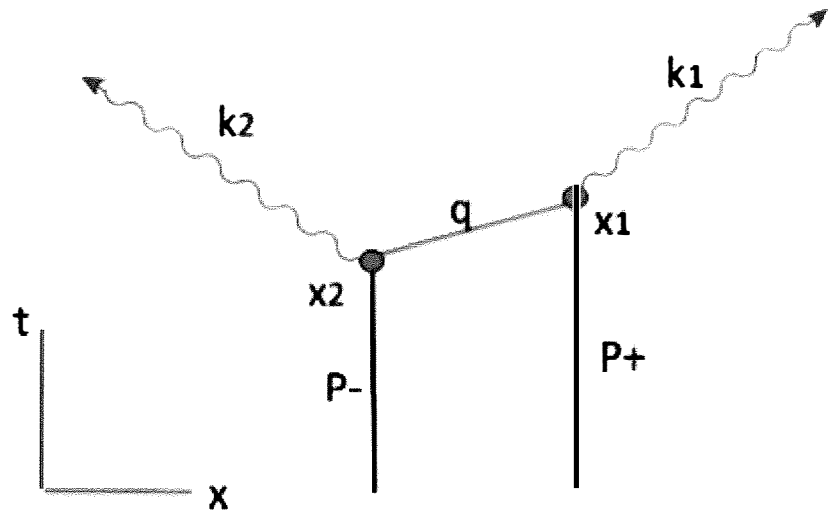

Figure 3.1: A positron (p+) and an electron (p-) moving at low speed are annihilated at x1 and x2. A virtual particle moves from one annihilation location to the other, represented by q. Two photons of k1 and k2 fly off along a line.

it is annihilated at t_2 along with a positron of negative spin. In that instant, a photon of momentum $-k$ is created at x_2. The ordering of events is easily pictured using a Feynman diagram 3.1 Defining a covariant matrix element $M_{fi}(q)$ as

$$M_{fi}(q) = ie^2\overline{v}\gamma\epsilon^{(\alpha_1)}\frac{[-i\gamma q + m]}{q^2 + m^2 - i\varepsilon}\gamma\epsilon^{(\alpha_2)}u + \{q \leftrightarrow -q\}. \tag{3.32}$$

Then the covariant matrix element describes the exchange of a virtual particle from points q to $-q$, used as dummy variables for x_1 and x_2 the space-time coordinates of the annihilation events.

Sakurai uses the S-matrix element to obtain the cross-section for pair annihilation in the center-of-mass coordinate system [10]

$$\sigma = \frac{\pi r_\circ^2}{\beta_+} \text{ for } \beta_+ << 1$$

with $r_\circ = \alpha/m$ where $\alpha = \frac{1}{137}$ is the fine structure constant and $\beta_+ = |\mathbf{p}_+|/m$. This is used to calculate the lifetime of positronium (a bound e^+e^- state) using the formula

$$\tau = (\sigma v_+ \rho)^{-1} \frac{1}{2}$$

where $\rho = 1/\left[\pi (2a_\circ)^3\right]$ is the square of the bound-state wave function at the origin, v_+ is the electron/positron velocity (low compared to that of light). Taking the limit as $v_+ \rightarrow 0$ Sakurai calculates

$$\tau_{para-positronium} = \frac{2}{\alpha^5 m} \simeq 1.25 \times 10^{-10}\text{sec.} \tag{3.33}$$

Chapter 4

Changing Bases

Our wavefunction is a linear combination of two states. Sakurai describes them in terms of the physical objects involved: the first state is that of the positron/electron pair and the second state is the two photons. This description is useful when describing the various annihilation and creation processes, but in order to describe the entanglement of the two photons and the results obtained by Irby we will effect a change of basis. Instead of using the positions of annihilation we switch to center and relative coordinates.

$$\mathbf{x}_c = \frac{1}{2}\left(\mathbf{x}_1 + \mathbf{x}_2\right), \quad \mathbf{x}_r = \mathbf{x}_1 - \mathbf{x}_2 \tag{4.1}$$

Likewise instead of writing momentum separated as that of the electron (p_-) and that of the positron (p_+) we describe momentum in terms of center and relative coordinates. This is very convenient because it allows us to easily describe a net momentum zero event (one where the positron and electron are moving at low and opposite velocities $\beta_+ = \beta_-$, which is an excellent approximation.

$$\mathbf{p}_c = \mathbf{p}_+ + \mathbf{p}_-, \quad \mathbf{p}_r = \frac{1}{2}\left(\mathbf{p}_1 - \mathbf{p}_2\right) \tag{4.2}$$

The third component to be affected by the change is the wave-vector $\mathbf{k}$.

$$\mathbf{k}_c = \mathbf{k}_+ + \mathbf{k}_-, \quad \mathbf{k}_r = \frac{1}{2}\left(\mathbf{k}_1 - \mathbf{k}_2\right)$$

Under these transformations the S-matrix (3.31) element of Sakurai is

$$S_{fi}^{(2)} = -i\frac{1}{(2\pi)^4}\sqrt{\frac{m}{E_-V}\frac{m}{E_+V}\frac{1}{2\omega_1 V}\frac{1}{2\omega_2 V}} \tag{4.3}$$
$$\times \int d^4x_c \int d^4x_r \int d^4q M_{fi}(q)$$
$$\times \exp\left(ip_c \cdot x_c + ip_r \cdot x_r - ik_c \cdot x_c - ik_r \cdot x_r + iq \cdot x_r\right)$$

Sakurai uses an infinite volume approximation in which integration of (4.3) gives

$$S_{fi}^{(2)} = -\frac{i(2\pi)^4}{2mV^2}\delta^4(k_c - p_c)\, M_{fi}(q)|_{q=k_r-p_r} \tag{4.4}$$

Since $\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_2 = \mathbf{k}_c \cdot \mathbf{x}_c + \mathbf{k}_r \cdot \mathbf{x}_r$ for the photons and $\mathbf{p}_+ \cdot \mathbf{x}_1 + \mathbf{p}_- \cdot \mathbf{x}_2 = \mathbf{p}_c \cdot \mathbf{x}_c + \mathbf{p}_r \cdot \mathbf{x}_r$, for the particles, our operators and their Fourier transform properties remain unchanged. These vector additions may be visualized using Figure 4.1. The linear superposition of states $\psi(\mathbf{x},t)$ now looks like

$$\psi(\mathbf{x},t) = c_{\mathbf{k}_c}(t)\,|1s, \mathbf{k}_c\rangle + c_{\mathbf{k}_r}(t)\,|\mathbf{k}_c, \mathbf{k}_r\rangle\,. \tag{4.5}$$

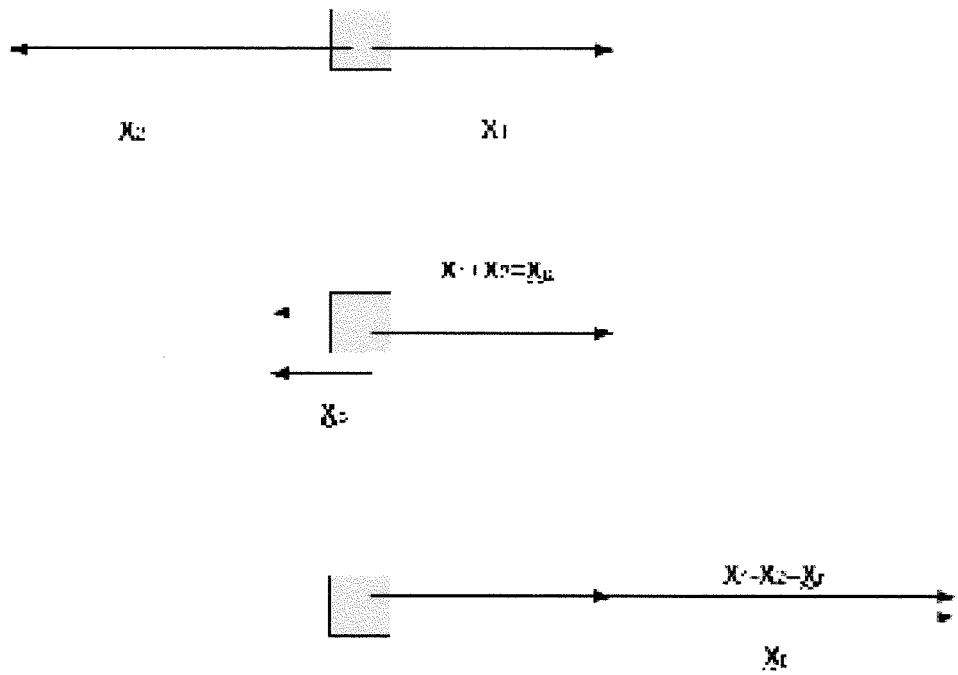

Figure 4.1: A visualization of the composition of the center of energy (x_c) and relative (x_r) coordinates. The blue box represents the source. The k_r, k_c, p_r, and p_c are constructed similarly.

Chapter 5

Theory of Spontaneous Emission

For a system decaying from one state to another such as ours the state vector is a combination of the two states. Each state $|n\rangle$ is preceded by a time-dependant coefficient $c_n(t)$.

$$|\psi(t)\rangle = c_a(t)|a\rangle + c_b(t)|b\rangle \tag{5.1}$$

$$\left|\dot{\psi}(t)\right\rangle = \dot{c}_a(t)|a\rangle + \dot{c}_b(t)|b\rangle \tag{5.2}$$

Solving the Schrodinger equation of motion $\left|\dot{\psi}\right\rangle = -\frac{i}{\hbar}V|\psi(t)\rangle$ where V is the interaction potential will yield a pair of coupled differential equations for $c_a(t)$ and $c_b(t)$, the time dependent coefficients. Adapting the Weisskopf-Wigner theory of spontaneous emission [11] will allow us to solve for these time-dependent coefficients, and the coefficient of the state describing the correlated photons will contain the decay constant describing the double-exponential shape of Irby's results (2.3).

Theories of spontaneous emission describe what happens when an atom in an excited state releases energy in the form of radiation and does so without the prompting of an external field. This absence of a prompting field means that the emitted photons are not subject to an harmonic field operating to drive the frequency of the atom's radiance. We use spontaneous emission to describe the decay of a para-positronium "atom" into two momentum-correlated photons. The methods used in [12] and [11] use the time-dependant Schrodinger equation to solve for the coefficients of the components of the wavefunction. We do not, but instead use the

function $U(t,t')$ derived by Sakurai to describe the time evolution of the function ψ.

At the time of annihilation it is the combination of properties of the particle pair that is of interest. We describe their pairing (positronium) using a single state vector ψ based on the observables momentum and energy as well as position. The photons produced by para-positronium decay are highly momentum and energy correlated so we describe them with a single ket.

The time evolution of our wavefunction is

$$|\psi\rangle = -iU(t,t')\,|\psi(t')\rangle$$

Our system begins in the state $|1s,\mathbf{k}_c\rangle$ and evolves into $|\mathbf{k}_c,\mathbf{k}_r\rangle$, satisfying the initial and final conditions $c_{\mathbf{k}_c}(t_\circ) = 1$ and $c_{\mathbf{k}_r}(t >> t_\circ) = 1$ and of course $|c_{\mathbf{k}_c}(t_\circ)|^2 + |c_{\mathbf{k}_r}(t)|^2 = 1$. We use the theory of spontaneous emission to obtain expressions for the coefficients. With $\dot{U}^{(2)} = U_r^{(2)}\ \ \delta(\mathbf{k}_c - \mathbf{p}_c)$, and $\dot{U}^{(0)} = U_r^{(0)}\ \ \delta(\mathbf{k}_c - \mathbf{p}_c)$

$$|\psi(t)\rangle = -U(t,t')\,|\psi(t')\rangle$$

$$|\psi(t+\Delta t)\rangle = -U(t+\Delta t,t')\,|\psi(t')\rangle \text{ let } t' = t$$

$$|\psi(t+\Delta t)\rangle = -\left[U_r^{(0)}(t+\Delta t,t) + U_r^{(1)}(t+\Delta,t) + U_r^{(2)}(t+\Delta,t)\right]|\psi(t)\rangle$$

$$\left[|\psi(t+\Delta t)\rangle - U_r^{(0)}(t+\Delta t,t)\,|\psi(t)\rangle\right]/\Delta t = -\left[U_r^{(1)}(t+\Delta t,t) + U_r^{(2)}(t+\Delta,t)\right]/\Delta t\,|\psi(t)\rangle$$

$$\left|\dot{\psi}(t)\right\rangle = -\left[\dot{U}_r^{(1)}(t,t') + \dot{U}_r^{(2)}(t,t')\right]|\psi(t)\rangle$$

$$\dot{c}_a(t)\,|1s,\mathbf{k}_c\rangle + \Sigma_{\mathbf{k}}\dot{c}_b(t)\,|\mathbf{k}_c,\mathbf{k}_r\rangle = -\left[\dot{U}_r^{(1)}(t,t') + \dot{U}_r^{(2)}(t,t')\right]c_a(t)\,|1s,\mathbf{k}_c\rangle + \Sigma_{\mathbf{k}}c_b(t)\,|\mathbf{k}_c,\mathbf{k}_r\rangle$$

To get the pair of coupled differential equations we multiply the above equation by bras from the left. Multiplying by $\langle 1s,\mathbf{k}_c|$, we get

$$\begin{aligned}
\dot{c}_a(t) &= -ic_a(t)\,\langle 1s,\mathbf{k}_c|\,\dot{U}_r^{(1)}(t,t')\,|1s,\mathbf{k}_c\rangle - \Sigma_{\mathbf{k}}c_b(t)\,\langle 1s,\mathbf{k}_c|\,\dot{U}_r^{(2)}(t,t')\,|\mathbf{k}_c,\mathbf{k}_r\rangle \\
\dot{c}_a(t) &= \Sigma_{\mathbf{k}}c_b(t)\,\langle 1s,\mathbf{k}_c|\,\dot{U}_r^{(2)}(t,t')\,|\mathbf{k}_c,\mathbf{k}_r\rangle \qquad (5.3)
\end{aligned}$$

Multiplying by $\langle \mathbf{k}_c, \mathbf{k}_r |$, we get the other equation

$$\dot{c}_b(t) = -c_a(t) \langle \mathbf{k}_c, \mathbf{k}_r | \dot{U}_r^{(2)}(t, t') | 1s, \mathbf{k}_c \rangle \tag{5.4}$$

The differential equation for the first coefficient is

$$\dot{c_a}(t) = \Sigma_{\mathbf{k}} c_b(t) \langle 1_s, \mathbf{k}_c | \dot{U}_r^{(2)}(t, t_\circ) | \mathbf{k}_c, \mathbf{k}_r \rangle \tag{5.5}$$

where $\langle f | U^{(2)}(t, t_\circ) | i \rangle = U_{fi}^{(2)}$.Writing out $\dot{U}_{fi}^{(2)}$

$$\dot{U}_{fi}^{(2)}(t_0, t) = \frac{\partial U_{fi}^{(2)}(t_0, t)}{\partial t}. \tag{5.6}$$

For times $(t - t_\circ) >> (t_c - t_r)$ we can replace $U_{fi}^{(2)}$ with $S_{fi}^{(2)}$ and get

$$\dot{U}_{fi}^{(2)}(t_0, t) = \frac{\partial S_{fi}^{(2)}(t_0, t)}{\partial t}. \tag{5.7}$$

Substituting (4.4) in the above equation

$$\dot{U}_{fi}^{(2)}(t_0, t) = \frac{\delta}{\delta t} \left[-\frac{i(2\pi)^4}{2mV^2} \delta^4(k_c - p_c) M_{fi}(q)|_{q=k_r - p_r} \right] \tag{5.8}$$

we can see that the variable t is buried in the four-component delta function. We can exchange $2\pi\delta(\omega_c - E_c)$ (the energy part) with $T = t - t_0$ (time derivative unity because the energy will not change over time) and $\delta^3(\mathbf{k}_c - \mathbf{p}_c)(2\pi)^3/V$ with $\delta_{\mathbf{k}_c, \mathbf{p}_c}$, appropriate for discrete and well-defined momentum, which our photons certainly have. Our result for (5.8) is then

$$\dot{U}_{fi}^{(2)}(t_0, t) = -\frac{i}{mV} \delta_{\mathbf{k}_c, \mathbf{p}_c} M_{fi}(q)|_{q=k_r - p_r}$$

Substituting this into (5.4) and (5.5) we get

$$c_a(t) = C\exp\left[\left(-iE_c - \Gamma\right)(t - t_0)\right] \tag{5.9}$$

$$\begin{aligned} c_{\mathbf{b}}(t) = -C\overset{.}{U}{}^{(2)}_{fi} \{&\exp\left[\left(-iE_c - \Gamma\right)(t - t_0)\right] \\ &- \exp\left[-i\omega_c(t - t_0)\right]\} \left(\omega_c - E_c + i\Gamma\right)^{-1} \end{aligned} \tag{5.10}$$

The time $t_\circ$ is the time of positron ejection from the radioactive decay. For times much greater than Γ^{-1} the first term of (5.10) can be neglected, leaving $c_{\mathbf{b}}(t) = -\exp\left[-i\omega_c(t - t_0)\right]\left(\omega_c - E_c + i\Gamma\right)^{-1}$, where $S^{(2)}$ is the S-matrix element describing two-photon annihilation. Then following the theory of Weisskopf-Wigner [11], with $\omega = 2\left|\mathbf{k}_r\right|$, and $E = 2m$ (to first order in E) we get for the coefficient $c_b(t) = c_{\mathbf{k}_r}(t)$

$$c_{\mathbf{k}_r}(t) = A\frac{\exp\left[-i\omega(t - t_\circ)\right]}{\omega - E + i\Gamma} \tag{5.11}$$

Where A is the normalization constant. We evaluate for A as in [8] and get

$$c_{\mathbf{k}_r}(t) = \sqrt{\frac{8\pi\Gamma}{VE^2}}\frac{\exp\left[-i\omega(t - t_\circ)\right]}{\omega - E + i\Gamma}$$

This is the coefficient needed to describe the evolution of the two-photon radiation. The coefficient $c_a(t) = c_{\mathbf{k}_c}(t)$ is essentially zero after a few decay time constants.

Chapter 6

Discussion

Now that we have found the time dependent coefficient describing the evolution of the wavefunction of the correlated photons, we will use this coefficient to find an expression describing the time difference in detection of the two photons. The mixture of states describing the relative dynamics is constructed using $c_{\mathbf{k}_r}(t)$ and $|\mathbf{k}_r\rangle$ as

$$|\psi_r\rangle = \sqrt{\frac{8\pi\Gamma}{VE^2}} \sum_{\mathbf{k}_r} \frac{\exp\left[-i\omega\left(t-t_\circ\right)\right]}{\omega - E + i\Gamma} |\mathbf{k}_r\rangle . \tag{6.1}$$

Strictly speaking this mixture should be weighted as in the $1s$ state, but since $\Gamma >> a_\circ^{-1}$ the weights can be neglected, and we can treat (6.1) as a pure state. Now we have a wave vector in the momentum basis. However the detection device of Irby measures position and arrival time. In order to describe these, we switch to space time coordinates to get the space-time wavefunction $\psi(\mathbf{x}_r, t)$.

$$\psi(\mathbf{x}_r, t) = \langle \mathbf{x}_r | \psi_r \rangle$$

where

$$|\psi_r\rangle = \int d^3x \psi(\mathbf{x}_r, t) |\mathbf{x}_r\rangle .$$

To switch from position to momentum coordinates we use $|\mathbf{x}_r\rangle = e^{i\mathbf{k}_r\cdot\mathbf{x}_r} |\mathbf{k}_r\rangle / (2\pi)^{3/2}$ and then integrate over x_r and $\mathbf{k}_r$ as in [8] to find

$$\psi(|\mathbf{x}_r|, t) = \sqrt{\frac{\Gamma}{4\pi}} \frac{1}{|\mathbf{x}_r|} \exp\left[-(iE+\Gamma)\left(t - t_\circ - \frac{1}{2}|\mathbf{x}_r|\right)\right] \tag{6.2a}$$

It is important that we specify that $t > t_\circ$, and we need to include in this statement that the times implicit in $\mathbf{x}_r$ are also greater than $t_\circ$.

Irby tried to analyze his results in a paper published in 2003 (See Reference [13]) using the theory presented in the Einstein Podolsky Rosenthal paper of 1935 [1]. This theory relies on wavefunction collapse to describe the entanglement of two particles. Once the property in which the particles are entangled (in our case, momentum) has been measured for one particle, the wavefunction of the other particle collapses into an eigenfunction of the density matrix. Irby's attempt was unsuccessful largely because the wavefunctions he constructed could not describe the time difference in detection of the two photons. Because we have switched to center of energy and relative coordinates, the relative detection time is fairly easy to pull out, and we shall continue using the wavefunction collapse method. Like EPR and Irby we write (6.2a) as

$$\psi\left(\left|\mathbf{x}_r\right|, t\right) = \Theta\left(\tau_1 - t_\circ\right)\Theta\left(\tau_2 - t_\circ\right)\int_0^\infty dx\delta\left(\left|x_<\right| - x\right)\psi\left(\left|x_<\right| + x, t\right)$$

where $|x_r| = (|x_<| - x)$ and $\delta\left(|x_<| - x\right)$ is a position eigenvector with eigenvalue x. The step functions ensure that the photons are not released before the positron has been released at time $t_\circ$. We define τ_1 as the emission time of the first photon and τ_2 as the emission time of the second photon. The time $t_<$ is the time of the first photodetection, $t_<$ of the second. We integrate over x, which is still a four-dimensional space-time vector.

The probability for coincident detection is, within the center of energy and relative coordinates,

$$P = \frac{1}{V}\left|\psi\left(\left|\mathbf{x}_r\right|, \frac{t_1 + t_2}{2}\right)\right|^2 \tag{6.3}$$

Substituting (6.2a) into (6.3) and averaging over positron emission times we get

$$P = \frac{r}{8\pi\left|\mathbf{x}_r\right|^2 V}\exp\left[-\Gamma\left|\tau_1 - \tau_2\right|\right]. \tag{6.4}$$

where r is the rate of positron injection and $|\tau_1 - \tau_2|$ is time interval separating detection events. The formula (6.4) gives the probability of coincident detection for the entangled photons released via the two-photon pathway of positron-electron annihilation. The exponential term is a product of a characteristic decay time Γ (positronium lifetime) and the time interval between detection events. Figure (6.1) is the data collected by Irby (and generously sent to us), fitted to a comparison of curves (6.1). The curve that best fits both the peak and the tails is the exponential. Then the most probable result is $P = \frac{r}{8\pi|\mathbf{x}_r|^2 V}$, when $\tau_1 - \tau_2 = 0$, which is the peak, and as the time difference increases the probability of that result decreases exponentially.

We stated at the beginning of this thesis that our goal was to create an expression that accounts for the exponentially decaying count rates with respect to the size of the time difference between detection times. The expression (6.4) does so very neatly. The physics of it is this: averaging over the positron emission times indicates that we do not know the instant at which the positron is created. Also it is true that we do not know the instant of decay; it follows an exponential decay curve. This decay probability can be summed up as "The electron and positron most likely annihilate right away (corresponding to the sharp peak in detection of two photons arriving simultaneously). If the pair do not annihilate instantaneously, they are then most likely to decay in the next instant (corresponding to the two photons arriving a short time apart)" and so on such that the probability of the pair still existing decreases exponentially and the probability of detecting the two photons arriving a significant time apart has increased. The reason the time difference in photon detection reflects the probability curve of the annihilation event is because detection of the photon position is a consequence of causality and dependent on the occurrence of the annihilation. This is what we have built into equation (6.4) by using center of energy and relative coordinates, photon emission times that are dependent on

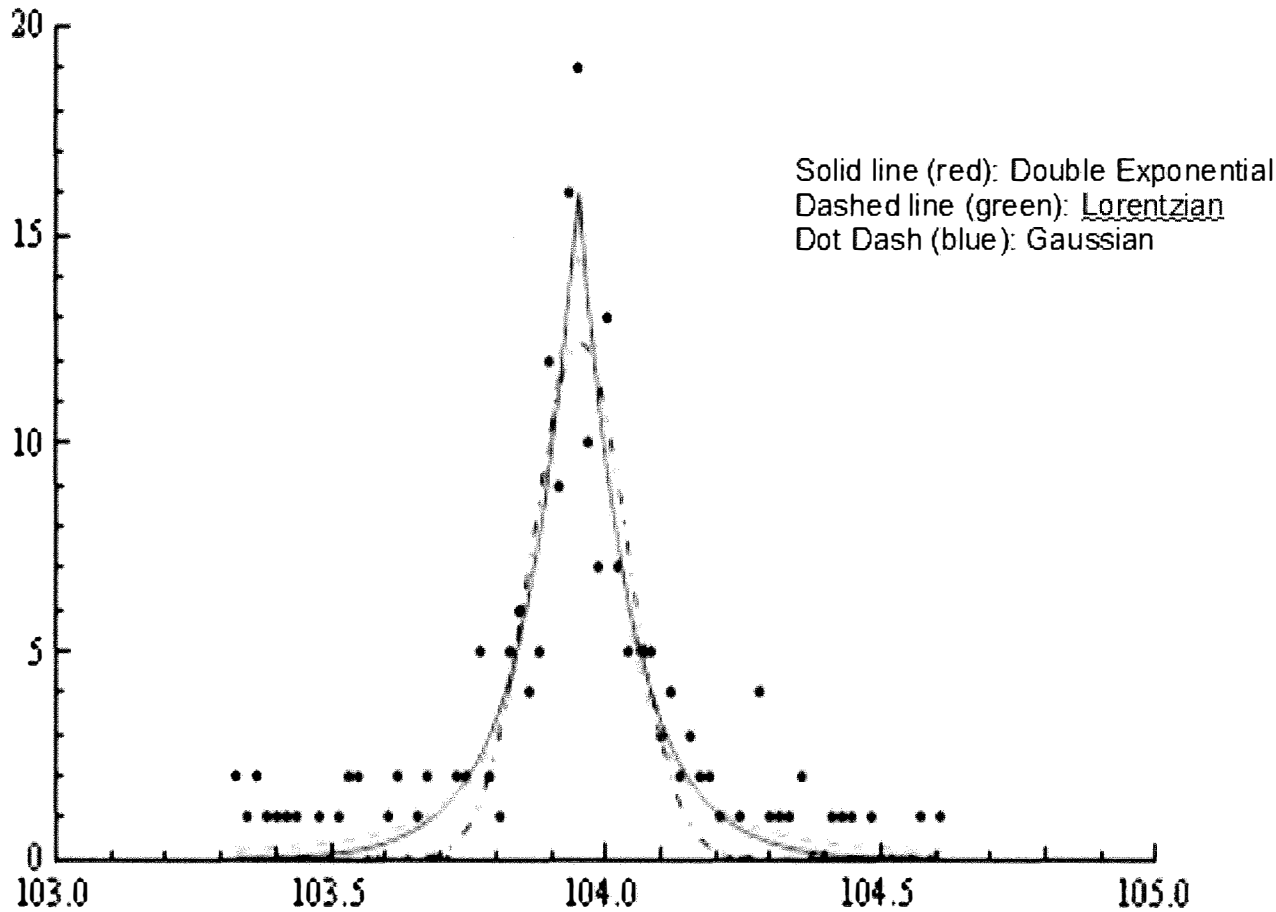

Figure 6.1: Irby's raw data, showing various function fittings. The double exponential is a better fit to the peak, and the tails follow the Lorentzian fitting. The standard deviation of the double exponential is aproximately 0.45, with FWHM of about 103ps.

positron emission timing and distance traveled to the detector, spontaneous emission theory and momentum correlation leading to wavefunction collapse.

The time difference between detection of the two simultaneously created photons is the direct result of the decoherence of the wavepacket describing them. If the center of energy were not spread out over space, the fact that the photons are detected 120ps apart would have to mean that one photon was emitted 3cm closer to its detector than the other photon. However, Irby's source is so localized (to within 3-7mm) that the time interval between photon arrival times must be a result of some physical process that treats the photon wave packet as being spread out over an area. My theory is that if the energy of the photons is spread out over an area then it's arrival at the detector will not register at the instant that the leading edge of the pulse arrives. The absorption of the photon's energy does not necessarily happen at the instant of the wavefront's arrival, but rather according to a probability curve described by $P(t) = \frac{r}{8\pi|\mathbf{x}_r|^2 V} \exp\left[-\Gamma\left|\tau_1 - \tau_2\right|\right]$. I would put forward that the decoherence (the spreading over space) out of the wavefunction occurs over the time it takes for the matter/antimatter pair to annihilate. Thus over a series of measurements the detection time difference at FWHM is the lifetime of the positron/electron pair.

6.1 Application

These results can be immediately applied to existing technologies, most notably time-of-flight positron emission tomography (TOF PET). PET is used to image tumors in a person's body by having the patient ingest a substance tagged with a radioactive element that releases positrons in its decay process. The substance, glucose for example, is taken into the tumor along with the radioactive element and the tumor becomes the source of gamma ray emissions[15].

The PET detector is built as a long hollow tube of concentric detector rings into which the patient is inserted. When two detectors log near-simultaneous events (within the order of 10ns) the event is attributed to two-photon annihilation and computer software constructs a line between the two detectors called the line of response (LOR). Many of these LOR are logged, and wherever they intersect is a likely source of radiation. Time-of-flight PET is a refinement of PET based on the time interval between detection events. The time interval between detection events is used to calculate where along the LOR the photons originated[16] (see Figure 6.2).

Irby's experiment and our research is significant to TOF PET because it shows that there is a significant uncertainty in the location of the radiative source, or the tumor. An uncertainty of at least 120ps (or 3.6cm in distance) would be a reasonable margin of error, although there are still a significant number of photon pairs that are detected at greater time intervals than this.

6.2 Future Experimentation

The theoretical results obtained in this thesis paper could be corroborated by further experimentation. In order to test the theory that the detection time interval is a result of decoherence stemming from the decay process, other experiments must be devised to rule out other possibilities. For example, Irby's experiment occurs within high vacuum and are thus limited to the physical constraints of a high vacuum apparatus. The detectors are never more than 10cm apart. If, for example, the decoherence of the photon wavepacket occurs as a consequence of propagation over distance and not as proposed in this thesis, an experiment where the detectors are a larger distance apart would yield an exaggerated curve and larger FWHM. It must

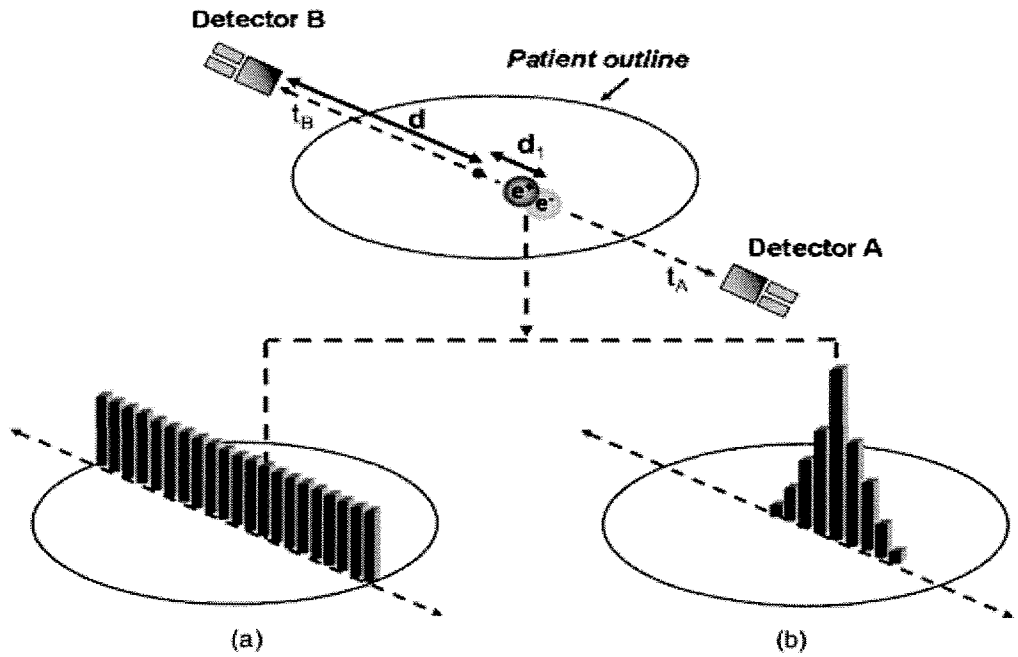

Figure 6.2: A diagram illustrating the detection of one pair of momentum correlated photons originating from the two-photon positron/electron annihilation process. Image (a) illustrates that in PET, location information is distributed equally along the LOR. Image (b) illustrates that in TOF PET, source location information is localized to a segment of the LOR [16].

be noted, however, that if this is the case Irby's results of 120ps (the lifetime of positronium) would be an enormous coincidence.

Another aspect of the theory that could be tested with a larger experimental set-up are the effects spatial localization perpendicular to the direction of propagation. The further apart the detectors are, the lesser the angle their surfaces subtend from the source, thus increasing our knowledge of their location. According to the Heisenberg uncertainty principle as it applies to the relationship between location and momentum $\Delta x \Delta p \geqslant \hbar/2$. Since the photons are so strongly momentum correlated, it would be interesting to see an analysis of the results of Irbys experiment and identical set-ups at further distances from the viewpoint of the relationship between highly localized position and correlated momentum.

Chapter 7

Appendices

7.1 A. Plane wave solution

In order to find plane wave solutions to the Dirac equation, we must re-write the Hamiltonian $(\sigma \cdot \mathbf{p})$ as $\mathbf{p} \longrightarrow \mathbf{p} - \frac{eA_\mu}{c}$ where A_μ is the time-independent electromagnetic interaction potential. It is then assumed that $\Psi(x,t)$ is an eigenfuction of $i\hbar\frac{\partial}{\partial t}$with eigenvalue E. Thus

$$\mathbf{A}(x,t) = (\mathbf{A}, iA_\circ) \tag{7.1}$$

and

$$\Psi(x,t) = \Psi(x,0)e^{-iEt/\hbar}. \tag{7.2}$$

After making these substitutions in (**??**), the equation becomes

$$\begin{pmatrix} -i\hbar\frac{\partial}{\partial x_\circ} + \frac{eA_\circ}{c} & -\boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla}+\frac{eA_\mu}{c}) \\ \boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla}+\frac{eA_\mu}{c}) & i\hbar\frac{\partial}{\partial x_\circ} + \frac{eA_\circ}{c} \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = -mc \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \tag{7.3}$$

This matrix reduces to the following two equations:

$$(-i\hbar\frac{\partial}{\partial x_\circ} - \frac{eA_\circ}{c})\psi_A - \boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla} - \frac{eA_\mu}{c})\psi_B = -mc\psi_A \tag{7.4}$$

$$-\boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla}-\frac{eA_{\mu}}{c})\psi_A+(i\hbar\frac{\partial}{\partial x_{\circ}}-\frac{eA_{\circ}}{c})\psi_B=-mc\psi_B \tag{7.5}$$

Equations (7.4) and (7.5) then simplify to

$$\boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla}-\frac{eA_{\mu}}{c})\psi_B=\frac{1}{c}(-mc^2+E-eA_{\circ})\psi_A \tag{7.6}$$

$$-\boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla}-\frac{eA_{\mu}}{c})\psi_A=-\frac{1}{c}(mc^2+E-eA_{\circ})\psi_B \tag{7.7}$$

The free-particle ($A=(0,0)$) solution of the Dirac equation with $p\neq 0$ is

$$\psi=\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}=\begin{pmatrix}u_A(\mathbf{p})\\ u_B(\mathbf{p})\end{pmatrix}\exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar}-i\frac{Et}{\hbar}\right) \tag{7.8}$$

Where $u_A(\mathbf{p})$ and $u_B(\mathbf{p})$ are four component spinors independent of x and t. If we substitute (7.8) into equations (7.6) and (7.7) we get

$$\boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla})u_B(\mathbf{p})\exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar}-i\frac{Et}{\hbar}\right)=\frac{1}{c}(-mc^2+E)u_A(\mathbf{p})\exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar}-i\frac{Et}{\hbar}\right) \tag{7.9}$$

$$\Downarrow$$

$$u_A(\mathbf{p})=\frac{c}{E-mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})u_B(\mathbf{p}) \tag{7.10}$$

and

$$\boldsymbol{\sigma}\cdot(i\hbar\boldsymbol{\nabla})u_B(\mathbf{p})\exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar}-i\frac{Et}{\hbar}\right)=\frac{1}{c}(-mc^2+E)u_A(\mathbf{p})\exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar}-i\frac{Et}{\hbar}\right)$$

$$\Downarrow$$

$$u_B(\mathbf{p}) = \frac{c}{E + mc^2}(\boldsymbol{\sigma} \cdot \mathbf{p})u_A(\mathbf{p}). \tag{7.11}$$

After some investigation, we find that there exist two appropriate solutions for both $u_A(\mathbf{p})$ and $u_B(\mathbf{p})$. Solutions $u^{(1)}(\mathbf{p})$ and $u^{(2)}(\mathbf{p})$ have $E = \sqrt{|\mathbf{p}|^2 c^2 + m^2c^4} > 0$, and are associated with$u_A(\mathbf{p})$. Solutions $u^{(3)}(\mathbf{p})$ and $u^{(4)}(\mathbf{p})$ have $E = \sqrt{|\mathbf{p}|^2 c^2 + m^2c^4} < 0$, and are associated with$u_B(\mathbf{p})$.

7.2 B. Sigma Matrices

There are many possibilities for our choices of σ_μ. In order to satisfy the requirements in using a Dirac spinor, the gamma matrices need only satisfy the following four requirements: that they be traceless, of even dimension greater than 2 and have eigenvalues of ± 1. By virtue of these characteristics of the σ_μ, the gamma matrices satisfy the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$, and are each Hermitian. These relationships are important because with them one may derrive the differential law of current conservation, and show that the Dirac equation is valid independent of the choice of representation. The choice of the placement and sign of the imaginary numbers in the γ_μ matrices determines the form of the 4-vector spinor. We make the easy choice for the σ_μ of (3.5) and choose the standard form of the Pauli matrices, which satisfy the four requirements. This choice of γ_μ corresponds to a four-vector of the form $\Psi = [\boldsymbol{\Psi}, i\Psi_\circ]$.

The 2×2 matrices σ_μ:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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