# The Construction of a Local Lie Group from Its Lie Algebra

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by

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# Contents

## Abstract

In "Symmetry Groups and Their Applications", by W. Miller [1, p.152-206], Miller discusses local Lie group theory and certain resulting applications in special function theory. In the course of this discussion Miller considers local Lie transformation groups and Lie derivatives. Miller is able to prove that any Lie algebra of differential operators is the set of Lie derivatives for some local Lie transformation group (G, **Q**), where G = ($V$, φ) is the underlying local Lie group and **Q** is the action. Miller's proof shows that the action **Q** can be found by solving a system of ordinary differential equations. His proof does not explicitly give the underlying local Lie group G. It only shows that such an underlying local Lie group exists.

We show that if you restrict the Lie algebras of differential operators to ones with a basis of the form $\{L_k\}_{k=1}^n$, such that

$$L_k = \sum_{i=1}^{n} P_{ik}(\mathbf{x}) \frac{\partial}{\partial x_i},$$

where $P_{ik}(\mathbf{e}) = \delta_{ik}$, $1 \le i, k \le n$, then we can construct a local Lie group G = ($V$, φ) such that the local Lie transformation group (G, φ) has Lie derivatives $\mathrm{Span}(\{L_k\}_{k=1}^n)$. The Lie product φ of G is found by solving a system of ordinary differential equations. Our proof is an adaptation of the one Miller uses to find the action **Q** of a local Lie transformation group with Lie derivatives $\{L_k\}_{k=1}^n$. We also show that the Lie algebra of G is isomorphic to $\mathrm{Span}(\{L_k\}_{k=1}^n)$. Thus we have found a method of constructing a local Lie group from its Lie algebra when the Lie algebra is realized as differential operators having the above form.

The fact that any Lie algebra of differential operators is the set of Lie derivatives for some local Lie transformation group is important in applying local Lie theory to special function theory. By means of local Lie groups that are not sets of matrices, we verify known addition formulas for polynomials of binomial type, $_2F_0$ hypergeometric series, Eulerian polynomials and Hermite polynomials.

Although our results can be derived by various special function techniques, our examples are interesting in that they show that the various addition formulas can all be obtained by using the same local Lie group theory.

# Chapter 1

## Elementary Local Lie Group Theory

### Section 1.1 Introduction

In this thesis we consider the relationship between local Lie groups, Lie algebras and Lie derivatives. We are particularly interested in two things. One, we are interested in using a Lie algebra of differential operators, $\mathcal{A}$, to construct a local Lie group G with Lie algebra isomorphic to $\mathcal{A}$ and two, we are interested in applying local Lie theory to special function theory. In order for the thesis to be fairly self-contained, a review of elementary local Lie group theory is necessary. To avoid complex topological problems, we consider a simplified, algebraic development of parts of local Lie group theory as provided by Miller in [1] and [2].

We begin by defining a local Lie group. Let $F$ be the field of either real numbers, $\mathfrak{R}$, or complex numbers, $\mathfrak{C}$. Let $F_n$ be the vector space of all n-tuples $\mathbf{g} = (g_1, g_2, \ldots, g_n)$, $g_i \in F$. Let $\mathbf{e} = (0, 0, \ldots, 0)$ be the zero vector of $F_n$ and suppose $V$ is an open set in $F_n$ containing $\mathbf{e}$. We assume that $F_n$ has the usual topology.

**Definition 1.1.1** $(V, \varphi)$ is an n-dimensional **local Lie group** if $\varphi$ is a function, $\varphi: V X V \to F_n$ such that

(1) For all $\mathbf{g}, \mathbf{h} \in V$, $\varphi(\mathbf{g}, \mathbf{h})$ is an analytic function in each of its 2n arguments, $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n$. (1.1.1)

(2) If $\varphi(\mathbf{g}, \mathbf{h}) \in V$ and $\varphi(\mathbf{h}, \mathbf{k}) \in V$, then $\varphi(\varphi(\mathbf{g}, \mathbf{h}), \mathbf{k}) = \varphi(\mathbf{g}, \varphi(\mathbf{h}, \mathbf{k}))$. (1.1.2)

(3) $\varphi(\mathbf{e}, \mathbf{g}) = \varphi(\mathbf{g}, \mathbf{e}) = \mathbf{g}$ for all $\mathbf{g} \in V$. (1.1.3)

If $(V, \varphi)$ is a local Lie group we call $\varphi$ its **Lie product** and we define the **inverse** of the element $\mathbf{g}$, $\mathbf{g}^{-1}$, if it exists, to be that n-tuple in $F_n$ such that $\varphi(\mathbf{g}, \mathbf{g}^{-1}) = \varphi(\mathbf{g}^{-1}, \mathbf{g}) = \mathbf{e}$. Note that we do not require the existence of inverse elements in the definition of a local Lie group.

The reader should notice a similarity between the definition of a local Lie group and the definition of an ordinary group of elementary algebra. Local Lie groups are derived from Lie groups, which are themselves ordinary groups with special topological properties. We will deal only with local Lie groups.

Local Lie groups first appeared in the work of S. Lie and his colleagues [1], as Lie transformation groups . Lie groups, local Lie groups and their representations have important applications in the study of special functions (see Miller [2] and Vilenkin [1]), and in the theory of differential equations (see Pontryagin [1] and Pommaret [1]).

Since the most important tool for studying local Lie groups is the correspondence between the local Lie group and the structure known as its Lie algebra, we discuss some of the theory concerning local Lie groups and Lie algebras in Chapter One. Lie algebras are a field of study unto themselves, (see Jacobson [1] and Bourbaki [1]), but we are only interested in them with respect to their relationship to local Lie groups. The material presented in Chapter One serves as the background theory for the later chapters.

In Chapter Two we introduce local Lie transformation groups and the differential operators known as Lie derivatives. We prove that the set of all Lie derivatives of a local Lie transformation group forms a Lie algebra of differential operators and we show that any Lie algebra of differential operators is the set of all Lie derivatives of some local Lie transformation group. For the most part, the material of Chapters One and Two is covered in Miller [1] and [2].

In Chapter Three we consider a specific type of differential operators that generate a Lie algebra and use the techniques of Chapter Two to find a local Lie group $G = (V, \varphi)$ such that the Lie algebra of G is isomorphic to the algebra of differential operators. This result differs from the results of Chapter Two in that we not only prove the existence of a local Lie transformation group with a given set of Lie derivatives, we actually find one. Essentially it provides a means to construct a local Lie group from a Lie algebra satisfying certain properties.

In Chapter Four we utilize the material of Chapters One and Two and the additional concept of multiplier representations to prove some addition theorems for special functions of Mathematics. The goal here is to provide simple applications of the theory. Miller [2] and Vilenkin [1] do a more thorough examination of the application of the local Lie group theory to special function theory.

## Section 1.2 Local Lie Groups and Local Linear Lie Groups

In Section 1.1 we defined an n-dimensional local Lie group G = ($V$, $\varphi$) on the field $F$. If $F = \mathbb{C}$, then we have a **complex local Lie group**. If $F = \mathbb{R}$, then we have a **real local Lie group**.

Since $\varphi: V \times V \rightarrow F_n$, $V \subseteq F_n$, $\varphi(\mathbf{g}, \mathbf{h})$ is an n-tuple. Let $\varphi_j(\mathbf{g}, \mathbf{h})$ denote the $j^{th}$ component of $\varphi(\mathbf{g}, \mathbf{h})$, j = 1, 2, . . . , n. The following Lemma guarantees that $\mathbf{g}^{-1}$ exists for $\mathbf{g}$ in some open neighborhood of $\mathbf{e}$.

**Lemma 1.2.1** (See Miller, [1, p.163]). *Let* G = ($V$, $\varphi$) *be an n-dimensional local Lie group. Then there exists an open neighborhood* $V^{-1}$ *about* $\mathbf{e}$, $V^{-1} \subseteq V \subseteq F_n$, *such that for* $\mathbf{g} \in V^{-1}$, *there is a unique element* $\mathbf{g}^{-1} \in V$ *such that*

$$\varphi(\mathbf{g}, \mathbf{g}^{-1}) = \varphi(\mathbf{g}^{-1}, \mathbf{g}) = \mathbf{e}.$$

**Proof:** Let G = ($V$, $\varphi$) be an n-dimensional local Lie group. Fix $\mathbf{g} \in$ G and let $\mathbf{f}(\mathbf{h}) = \varphi(\mathbf{g}, \mathbf{h})$. Now,

$$\left.\frac{\partial \varphi_j(\mathbf{e}, \mathbf{h})}{\partial h_i}\right|_{\mathbf{h}=\mathbf{e}} = \delta_{ij},$$

where $\delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, is the **Kronecker delta function**. Thus, if $\mathbf{g}$ is close to $\mathbf{e}$, say $\mathbf{g}$ in some neighborhood $V_r$ of $\mathbf{e}$, the Jacobian

$$\det\left[\left(\left.\frac{\partial \varphi_i(\mathbf{g}, \mathbf{h})}{\partial h_k}\right|_{\mathbf{h}=\mathbf{e}}\right)_{1 \leq i,k \leq n}\right] \neq 0.$$

Thus, by the Inverse Function Theorem (see Apostol [1, p.144], there exists a neighborhood of $\mathbf{e}$, such that for all $\mathbf{h}$ close to $\mathbf{e}$, $\mathbf{f}^{-1}(\mathbf{h})$ exists, and is analytic in $\mathbf{g}$ and $\mathbf{h}$. In particular $f^{-1}(\mathbf{e})$ exists.

$$\mathbf{f} \circ \mathbf{f}^{-1}(\mathbf{e}) = \mathbf{e} = \varphi(\mathbf{g}, \mathbf{f}^{-1}(\mathbf{e})).$$

Thus, the right inverse, $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{e})$ exists for $\mathbf{g} \in V_r$. Similarly, there is an open set $V_l$ about $\mathbf{e}$, $V_l \subseteq V$, such that all $\mathbf{g}$ in $V_l$ have a unique left inverse, $\mathbf{y}$.

$V^{-1} = V_l \cap V_r$, is an open set about $\mathbf{e}$ such that all $\mathbf{g}$ in $V^{-1}$ have a unique left inverse, $\mathbf{y}$, and a unique right inverse, $\mathbf{x}$. Since $\varphi$ is a Lie product:

$$\mathbf{y} = \varphi(\mathbf{y}, \mathbf{e}) = \varphi(\mathbf{y}, \varphi(\mathbf{g}, \mathbf{x})) = \varphi(\varphi(\mathbf{y}, \mathbf{g}), \mathbf{x}) = \varphi(\mathbf{e}, \mathbf{x}) = \mathbf{x}.$$

Thus, every $\mathbf{g} \in V^{-1}$, has a unique inverse, $\mathbf{g}^{-1} = \mathbf{x} = \mathbf{y} \in V$.

Q.E.D

Unfortunately, the current definition of a local Lie group will not provide simple proofs of certain material covered in Chapter One. Thus, we introduce the concept of a local linear Lie group.

**Definition 1.2.1** **A local linear Lie group** is a set of mxm nonsingular matrices $A(\mathbf{g}) = A(g_1, g_2, \ldots, g_n)$ defined for each $\mathbf{g} \in W$ (where $W$ is an open sphere about $\mathbf{e} \in F_n$), such that

(1) $A(\mathbf{e}) = E_m$, the mxm identity matrix. (1.2.1)

(2) The matrix elements of $A(\mathbf{g})$ are analytic functions of $g_1, g_2, \ldots, g_n$ and the map $\mathbf{g} \to A(\mathbf{g})$ is one-to-one. (1.2.2)

(3) The n matrices $\dfrac{\partial A(\mathbf{g})}{\partial g_j}$, $j = 1, \ldots, n$, are linearly independent for each $\mathbf{g} \in W$. (1.2.3)

(4) There exists a neighborhood $W'$ of $\mathbf{e}$ in $F_n$, $W' \subseteq W$, with the property that, for $\mathbf{g}, \mathbf{h} \in W'$ there is a $\mathbf{k} \in W$ such that $A(\mathbf{g})A(\mathbf{h}) = A(\mathbf{k})$, where juxtaposition means matrix multiplication. (1.2.4)

Local linear Lie groups are essential to our development of Lie theory, particularly in Section 1.5. They also have important applications in representation theory. We justify calling local linear Lie groups a special type of local Lie group with the following Lemma.

**Lemma 1.2.2** (See Miller [1, p.164]). *Every local linear Lie group* G *defines a local Lie group,* $(W', \varphi)$, *where the Lie product,* $\varphi$, *is given by* $A(\mathbf{g})A(\mathbf{h}) = A(\varphi(\mathbf{g}, \mathbf{h}))$.

**Proof:** Let $W$ be an open sphere about $\mathbf{e}$ in $F_n$, and let the set of mxm matrices $A(\mathbf{g})$, $\mathbf{g} \in W$, be a local linear Lie group. Let $W'$ be the open neighborhood of $\mathbf{e}$ in $F_n$, $W' \subseteq W$, such that Eq.(1.2.4) holds. For $\mathbf{g}, \mathbf{h} \in W'$, let $\varphi(\mathbf{g}, \mathbf{h}) = \mathbf{k}$.

First, we show that $\varphi(\mathbf{g}, \mathbf{h})$ satisfies Property(1.1.1). From the implicit function theorem (see Apostol [1, p.147]) and Eq.'s (1.2.2) and (1.2.3) we have that the $g_i$,

$1 \le i \le n$, are analytic functions of the matrix elements of $A(\mathbf{g})$, $A_{ij}(\mathbf{g})$, $1 \le i, j \le n$. Therefore, $\mathbf{k} = \varphi(\mathbf{g}, \mathbf{h})$ is an analytic function of the $A_{ij}(\mathbf{k})$. Since $A(\mathbf{k}) = A(\mathbf{g})A(\mathbf{h})$, and the $A_{ij}(\mathbf{g})$ are analytic functions of $g_1, \ldots, g_n$, then $\varphi(\mathbf{g}, \mathbf{h})$ is an analytic, vector-valued function of $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n$.

Obviously, $\varphi: W' \times W' \to F_n$, and, $\varphi(\mathbf{g}, \mathbf{e}) = \varphi(\mathbf{e}, \mathbf{g}) = \mathbf{g}$ since

$$A(\mathbf{g})A(\mathbf{e}) = A(\mathbf{e})A(\mathbf{g}) = A(\mathbf{g}).$$

Furthermore, since matrix multiplication is associative,

$$\big(A(\mathbf{g})A(\mathbf{h})\big)A(\mathbf{k}) = A(\mathbf{g})\big(A(\mathbf{h})A(\mathbf{k})\big).$$

I.e.
$$A(\varphi(\mathbf{g}, \mathbf{h}))A(\mathbf{k}) = A(\mathbf{g})A(\varphi(\mathbf{h}, \mathbf{k})).$$

Thus, if $\varphi(\mathbf{g}, \mathbf{h})$ and $\varphi(\mathbf{h}, \mathbf{k}) \in W'$ then $\varphi(\varphi(\mathbf{g}, \mathbf{h}), \mathbf{k}) = \varphi(\mathbf{g}, \varphi(\mathbf{h}, \mathbf{k}))$, and $(W', \varphi)$ is a local Lie group, as required.

Q.E.D.

We now provide some examples of local Lie groups.

**Example 1.2.1** $G = (F_n, +)$, where '+' is ordinary vector addition, and n is a positive integer, is obviously an n-dimensional local Lie group. It is also **commutative**, since $\varphi(\mathbf{g}, \mathbf{h}) = \varphi(\mathbf{h}, \mathbf{g})$ for all $\mathbf{g}, \mathbf{h} \in F_n$. Note, G is also an ordinary group, where $\mathbf{g}^{-1} = -\mathbf{g}$.

**Example 1.2.2** The set of one-dimensional local Lie groups defined by $G_\gamma = (F, \varphi)$, where $\varphi(g, h) = g + h + \gamma gh$, $g, h \in F$, $\gamma$ is a constant from $F$, and juxtaposition is ordinary multiplication.

Let $g, h, \varphi(g, h), \varphi(h, k) \in F$. Then

$$\begin{aligned}\varphi(\varphi(g, h), k) &= \varphi(g + h + \gamma gh, k) = (g + h + \gamma gh) + k + \gamma(g + h + \gamma gh)k \\ &= g + (h + k + \gamma hk) + \gamma g(h + k + \gamma hk) = \varphi(g, \varphi(h, k)).\end{aligned}$$

Thus, Eq.(1.1.2) holds. Eq.'s (1.1.1) and (1.1.3) obviously hold, so $G_\gamma$ is a local Lie group, for all $\gamma \in F$. If $\gamma = 0$, then, for all $g \in F$, $g^{-1} = -g$. For $\gamma \neq 0$,

$$g^{-1} = \frac{-g}{1 + \gamma g} \quad \text{for } g \neq \frac{-1}{\gamma}\text{, and no inverse exists for } g = \frac{-1}{\gamma}.$$

**Example 1.2.3** The one-dimensional real local Lie group $G = (V, \varphi)$ where $V = \{g \in \mathbb{R} \mid -0.4 < g < 0.4\}$ and $\varphi(g, h) = \ln(e^g + e^h - 1)$.

$V$ is open and contains $\mathbf{e} = 0$, and if $g, h > -0.4$, then $e^g + e^h - 1 > 0$, so $\varphi(g, h)$ is defined for $g, h \in V$ and $\varphi: V \times V \to \mathbb{R}$. Also, $\varphi(g, e) = \ln(e^g + e^0 - 1) = g = \varphi(e, g)$, for $g \in V$. Furthermore, if $\varphi(g, h) \in V$ and $\varphi(h, k) \in V$ then

$$\varphi(\varphi(g, h), k) = \ln((e^g + e^h - 1) + e^k - 1) = \ln(e^g + (e^h + e^k - 1) - 1) = \varphi(g, \varphi(h, k)).$$

Thus, we need only satisfy ourselves that $\varphi(g, h)$ is analytic in g and h.

A function $f(x)$ is analytic in $x$ on an interval $(a - \delta, a + \delta)$, if $f$ is equal to the sum of a power series in $x$ throughout $(a - \delta, a + \delta)$. From elementary calculus we know

$$\ln(1 + x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}, \qquad -1 < x \leq 1.$$

Let $x = (e^g + e^h - 2)$. Then for $1 < e^g + e^h \leq 3$,

$$\ln(e^g + e^h - 1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (e^g + e^h - 2)^{k+1}.$$

If $g, h \in V$ then $1 < e^g + e^h < 3$. Hence, $\varphi$ is analytic in g and h, as required. Thus, G is indeed a one-dimensional local Lie group.

Now,

$$\ln(e^g + e^h - 1) = 0 \Leftrightarrow e^h = 2 - e^g \Leftrightarrow h = \ln(2 - e^g).$$

Thus, if $e^g < 2$, then $g^{-1} = \ln(2 - e^g)$, and if $e^g \geq 2$ then $g^{-1}$ does not exist. The open sphere $V^{-1} = V = \{g \in \mathbb{R} \mid -0.2 < g < 0.2\}$ is an open neighborhood about 0 such that if $g \in V^1$ then there exists a unique $g^{-1} \in V$ such that $\varphi(g, g^{-1}) = 0 = e$.

**Example 1.2.4** The 3-dimensional complex local Lie group $G = (\mathbb{C}^3, \varphi)$, where

$$\begin{aligned}\varphi(\mathbf{g}, \mathbf{h}) &= \varphi((g_1, g_2, g_3), (h_1, h_2, h_3)) \\ &= \big((1 + g_1)(1 + h_1) - 1,\ (1 + g_1)h_2 + g_2(1 + h_1)^2, \\ &\qquad (1 + g_1)h_3 + 2g_2h_2(1+h_1) + g_3(1+h_1)^3\big)\end{aligned}$$

Verification that G is a local Lie group is similar to Example 1.2.2, and is thus omitted. It follows that

$$\mathbf{g}^{-1} = \left(\frac{-g_1}{1 + g_1}, \frac{-g_2}{(1 + g_1)^3}, \frac{2g_2^2}{(1 + g_1)^5} - \frac{g_3}{(1 + g_1)^4}\right), \qquad \text{if } g_1 \neq -1,$$

and $\mathbf{g}$ does not have an inverse if $g_1 = -1$.

**Example 1.2.5** Let $W = \{(x, y, z) |\ |x| > -1, x, y, z \in \mathbb{R}\}$. The set of 3x3 matrices,

$$A(\mathbf{g}) = \begin{pmatrix} 1+g_1 & g_2 & g_3 \\ 0 & (1+g_1)^2 & 2g_2(1+g_1) \\ 0 & 0 & (1+g_1)^3 \end{pmatrix}, \qquad \mathbf{g} \in W,$$

forms a 3-dimensional complex local linear Lie group.

Since $g_1 \neq -1$, $\det(A(\mathbf{g})) \neq 0$ so $A(\mathbf{g})$ is nonsingular if $\mathbf{g} \in W$. $A(\mathbf{g}) = A(g_1, g_2, g_3)$ is defined for each $\mathbf{g} \in W$. Properties (1.2.1), (1.2.2) and (1.2.3) obviously hold. We now verify that property (1.2.4) is true.

Let $W'$ be the open sphere of radius 1/2 about $\mathbf{e}$. For $\mathbf{g}, \mathbf{h} \in W'$, $g_1 + h_1 + g_1h_1 > -1$. If $\mathbf{g}, \mathbf{h} \in W'$, let $\mathbf{k} = (k_1, k_2, k_3)$ where

$$k_1 = g_1 + h_1 + g_1h_1, \qquad k_2 = (1 + g_1)h_2 + g_2(1 + h_1)^2,$$

and

$$k_3 = (1 + g_1)h_3 + 2g_2h_2(1 + h_1) + g_3(1 + h_1)^2.$$

Then $\mathbf{k} \in W$, and $A(\mathbf{g})A(\mathbf{h}) = A(\mathbf{k})$, as required.

Note that, if we define the Lie product $\varphi$ by $A(\mathbf{g})A(\mathbf{h}) = A(\varphi(\mathbf{g}, \mathbf{h}))$, then we essentially have the local Lie group G of Example 1.2.4, defined on the reals instead of the complex field.

From the examples, it is clear that for a fixed open set $V$ containing $\mathbf{e}$, many Lie products $\varphi$, are possible such that $(V, \varphi)$ is a local Lie group. Example 1.2.2 provides an infinite number of Lie products on $V = F$. Furthermore, the choice of the open set $V$ is not unique. In fact, we have the following Lemma:

**Lemma 1.2.3** *Let* G = (*V*, φ) *be a local Lie group. If* $V' \subseteq V$, *and* $V'$ *is an open, connected set containing* **e**, *then* G' = ($V'$, φ) *is also a local Lie group.*

**Proof:** Obvious.

Thus we can shrink the open neighborhood *V*, about **e**, on which G is defined, without further consideration. Since *V* can vary, the Lie product, φ, shall be considered the determining factor of a local Lie group. Since there are an infinite number of local Lie groups, the concept of locally isomorphic local Lie groups is useful.

**Definition 1.2.2** Let G = (V, φ) and G' = (V', φ') be 2 local Lie groups. Let μ map an open neighborhood *W* of **e** ∈ G into an open neighborhood *W'* of **e'** ∈ G'. Then μ is **(local) analytic isomorphism** if μ is one-to-one and onto, and

$$\mu(\varphi(\mathbf{g}, \mathbf{h})) = \varphi'(\mu(\mathbf{g}), \mu(\mathbf{h})), \quad \text{where } \mathbf{g}, \mathbf{h} \text{ and } \varphi(\mathbf{g}, \mathbf{h}) \in G,$$

such that μ and its functional inverse, $\mu^{-1}: W' \to W$, are both analytic functions of the coordinates of G. If such a mapping exists, G is said to be **(locally) isomorphic** to G'.

In the remainder of the thesis any references to local Lie group isomorphisms are actually references to local isomorphisms. The local Lie groups of Example 1.2.2 can be used to provide an example of a local isomorphism. $G_0 = (F, \varphi)$ and $G_1 = (F, \varphi')$, are (locally) isomorphic with isomorphism $\mu: G_0 \to G_1$ defined by $\mu(g) = e^g - 1$. μ is an analytic function of g and since $e^g - 1 = e^h - 1 \Leftrightarrow g = h$, μ is one-to-one. Now if $h \in G_1$, then $\mu(\ln(h + 1)) = h$ for $|h| > -1$. Thus, μ is onto for the neighborhood of $G_1$ where $g \in G_1$, $|g| > -1$. Finally note that μ is a local isomorphism because

$$\begin{aligned}\mu(\varphi(g, h)) &= \mu(g + h) = e^{g+h} - 1\\ &= (e^g - 1) + (e^h - 1) + (e^g - 1)(e^h - 1) = \varphi'(\mu(g), \mu(h)).\end{aligned}$$

Obviously, local isomorphism is an equivalence relation that can be used to partition the set of all local Lie groups into equivalence classes.

Now, consider the Lie product φ of a local Lie group G. Since φ is an analytic function of its 2n arguments, we can expand $\varphi_j(\mathbf{g}, \mathbf{h})$ as a Taylor series about $\mathbf{g} = \mathbf{h} = \mathbf{e}$.

$$\varphi_j(\mathbf{g},\mathbf{h}) = \varphi_j(\mathbf{g},\mathbf{h})\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} + \sum_{r=1}^{n} \frac{\partial\varphi_j(\mathbf{g},\mathbf{h})}{\partial g_r}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} g_r + \sum_{s=1}^{n} \frac{\partial\varphi_j(\mathbf{g},\mathbf{h})}{\partial h_s}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} h_s$$

$$+ \sum_{r,s=1}^{n} \frac{\partial^2\varphi_j(\mathbf{g},\mathbf{h})}{\partial g_r\partial g_s}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} g_r g_s + \sum_{r,s=1}^{n} \frac{\partial^2\varphi_j(\mathbf{g},\mathbf{h})}{\partial h_r\partial h_s}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} h_r h_s$$

$$+ \sum_{r,s=1}^{n} \frac{\partial^2\varphi_j(\mathbf{g},\mathbf{h})}{\partial g_r\partial h_s}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} g_r h_s + \{\text{terms of order} > 2 \text{ in } g_r, h_s\}.$$

Since $\varphi_j(\mathbf{g}, \mathbf{e}) = \varphi_j(\mathbf{e}, \mathbf{g}) = g_j$, $\mathbf{g} \in F$, we can simplify this expansion considerably. We find

$$\varphi_j(\mathbf{g},\mathbf{h}) = g_j + h_j + \sum_{r,s=1}^{n} \frac{\partial^2\varphi_j(\mathbf{g},\mathbf{h})}{\partial g_r\partial h_s}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}} g_r h_s + \{\text{terms of order} > 2 \text{ in } g_r, h_s\}$$

We write this as

$$\varphi_j(\mathbf{g},\mathbf{h}) = g_j + h_j + \sum_{r,s=1}^{n} c_{j,rs} g_r h_s + \{\text{terms of order} > 2 \text{ in } g_r, h_s\}, \qquad (1.2.5)$$

where

$$c_{j,rs} = \frac{\partial^2\varphi_j(\mathbf{g},\mathbf{h})}{\partial g_r\partial h_s}\Big|_{\mathbf{g}=\mathbf{h}=\mathbf{e}}. \qquad (1.2.6)$$

**Lemma 1.2.4:** $\sum_{r=1}^{n} c_{j,rs}c_{r,tv} - c_{j,tr}c_{r,vs} = 0$

**Proof**: From Eq.'s (1.1.2) and (1.2.5), it follows that

$$(g_j + h_j + \sum_{r,s=1}^{n} c_{j,rs} g_r h_s) + k_j + \sum_{r,s=1}^{n} c_{j,rs}\left(g_r + h_r + \sum_{t,v=1}^{n} c_{r,tv} g_t h_v\right) k_s$$

$$= g_j + (h_j + k_j + \sum_{r,s=1}^{n} c_{j,rs} h_r k_s) + \sum_{t,r=1}^{n} c_{j,tr} g_t\left(h_r + k_r + \sum_{v,s=1}^{n} c_{r,vs} h_v k_s\right),$$

where terms of order 2 or more in $g_s$, $h_t$ or $k_u$ are omitted. Equate coefficients of $g_t h_v k_s$, to get the required result.

Q.E.D.

In a similar manner, fix $\mathbf{h}$ and expand $\varphi_j(\mathbf{g}, \mathbf{h})$ in a Taylor Series about $\mathbf{g} = \mathbf{e}$ as

$$\varphi_j(\mathbf{g}, \mathbf{h}) = h_j + \sum_{k=1}^{n} F_{jk}(\mathbf{h}) g_k + \{\text{terms of order} \geq 2 \text{ in the } g_k\}, \qquad (1.2.7)$$

where

$$F_{jk}(\mathbf{h}) = \left.\frac{\partial \varphi_j(\mathbf{g}, \mathbf{h})}{\partial g_k}\right|_{\mathbf{g}=\mathbf{e}} . \qquad (1.2.8)$$

The $F_{jk}(\mathbf{h})$ will be used in Section 1.4. The associative law of the Lie product proves the following identity:

**Lemma 1.2.5** (See Miller [1, p.175], Eq.(5.4)).

$$F_{ij}(\varphi(\mathbf{h}, \mathbf{k})) = \sum_{r=1}^{n} \frac{\partial \varphi_i(\mathbf{h}, \mathbf{k})}{\partial h_r} F_{rj}(\mathbf{h}), \quad 1 \leq i, j \leq n.$$

**Proof**: By Eq.(1.1.2), for $\varphi(\mathbf{g}, \mathbf{h})$ and $\varphi(\mathbf{h}, \mathbf{k}) \in V$, $1 \leq i \leq n$

$$\varphi_i(\mathbf{g}, \varphi(\mathbf{h}, \mathbf{k})) = \varphi_i(\varphi(\mathbf{g}, \mathbf{h}), \mathbf{k}).$$

Expand both sides of this expression about $\mathbf{g} = \mathbf{e}$ using Eq.'s(1.2.7), (1.2.8) and the chain rule. Compare coefficients of $g_j$ to obtain the required result.

Q.E.D.

We will now consider Lie algebras.

## Section 1.3 Lie Algebras

Lie algebras are an important tool for studying local Lie groups. The Lie algebra of a local Lie group is created from the structure of local Lie groups. First, we define curves and tangent vectors on the local Lie groups as follows:

**Definition 1.3.1** Let $G = (V, \varphi)$ be an n-dimensional local Lie group. Let $t \to \mathbf{g}(t) = (g_1(t), \ldots, g_n(t))$, $t \in F$, be an analytic mapping of a neighborhood of $0 \in F$ into $V$ such that $\mathbf{g}(0) = \mathbf{e}$. Then $\mathbf{g}(t)$ is an **analytic curve through the identity** on G. The **tangent vector** to $\mathbf{g}(t)$ at $\mathbf{e}$ is the vector

$$\alpha = \left.\frac{d\mathbf{g}(t)}{dt}\right|_{t=0} = \left.\left(\frac{dg_1(t)}{dt}, \ldots, \frac{dg_n(t)}{dt}\right)\right|_{t=0} \in F_n. \qquad (1.3.1)$$

If $\alpha \in F_n$, then $\alpha t = (\alpha_1 t, \ldots, \alpha_n t)$, t sufficiently close to $0 \in F$, is one analytic curve in G with tangent vector $\alpha \in F_n$. Conversely, $\alpha \in F_n$ is the tangent vector of an infinite number of analytic curves in G. The following is also true:

**Lemma 1.3.1** (See Miller [1, p.166]). *If* $\mathbf{g}(t)$ *and* $\mathbf{h}(t)$ *are analytic curves through the identity in* G *with tangent vectors* $\alpha, \beta$, *respectively, then* $\varphi(\mathbf{g}(at), \mathbf{h}(bt))$ *is an analytic curve through* $\mathbf{e}$ *with tangent vector* $a\alpha + b\beta$.

**Proof:** Follows from Definition 1.3.1 and Eq.(1.2.5).

Q.E.D.

Thus, the vector space $F_n$ with ordinary vector addition and scalar multiplication, is the tangent space of the space of analytic curves in G.

**Definition 1.3.2** Let $\mathbf{g}(t)$ and $\mathbf{h}(t)$ be analytic curves through the identity on $G = (V, \varphi)$ with tangent vectors $\alpha, \beta$, respectively. The **commutator** $[\alpha, \beta]$ of $\alpha$ and $\beta$ is as follows:

$$[\alpha, \beta] = \left.\frac{d\mathbf{k}(t)}{dt}\right|_{t=0}, \tag{1.3.2}$$

where $\mathbf{k}(t) = \varphi\big(\mathbf{g}(\tau), \varphi(\mathbf{h}(\tau), \varphi(\mathbf{g}^{-1}(\tau), \mathbf{h}^{-1}(\tau)))\big)$, $t = \tau^2$. Since $\varphi$ is associative for $\tau$ close to $\mathbf{e}$, we write $\mathbf{k}(t) = \mathbf{g}(\tau)\mathbf{h}(\tau)\mathbf{g}^{-1}(\tau)\mathbf{h}^{-1}(\tau)$, where juxtaposition denotes the Lie product $\varphi$.

Eq.(1.3.2) is valid as long as the coefficient of $\tau$ in $\mathbf{k}(t)$ is 0. The validity of Eq.(1.3.2) is confirmed in the proof of the following theorem. Theorem 1.3.1 provides a simple way of calculating the commutator of two tangent vectors.

**Theorem 1.3.1** (See Miller [1, p.167], Theorem 5.6).

$$[\alpha, \beta]_j = \sum_{r,s=1}^{n} c_j^{rs} \alpha_r \beta_s, \tag{1.3.3}$$

*where* $c_j^{rs} = c_{j,rs} - c_{j,sr}$, *and* $c_{j,rs}$ *is given by* Eq.(1.2.6).

**Proof:** Let $\mathbf{g}(\tau)$ and $\mathbf{h}(\tau)$ be analytic curves in G, with tangent vectors $\alpha$ and $\beta$, respectively. By Lemma 1.2.1, $\mathbf{g}^{-1}(\tau)$ and $\mathbf{h}^{-1}(\tau)$ exist for $\tau$ close to 0, thus

$$\mathbf{k}(t) = \mathbf{g}(\tau)\mathbf{h}(\tau)\mathbf{g}^{-1}(\tau)\mathbf{h}^{-1}(\tau), \qquad t = \tau^2$$

is an analytic curve in G. Write $g_j(\tau)$, $k_j(t)$ and $h_j(\tau)$ as Taylor series, omitting terms of order > 2 in $\tau$, as

$$g_j(\tau) = \alpha_j\tau + b_j\tau^2 + \ldots \quad , \qquad h_j(\tau) = \beta_j\tau + c_j\tau^2 + \ldots \quad ,$$

and
$$k_j(t) = \rho_j\tau + a_j\tau^2 + \ldots$$

By Eq.(1.3.2), if $\rho_j = 0$, then $[\alpha, \beta]_j = a_j$. From our definition of $\mathbf{k}(t)$ it follows that $\varphi(\mathbf{k}(t), \varphi(\mathbf{h}(\tau), \mathbf{g}(\tau))) = \varphi(\mathbf{g}(\tau), \mathbf{h}(\tau))$. Then, by Eq.(1.2.5), omitting terms of order > 2 in $\tau$,

$$\varphi_j(\mathbf{g}(\tau), \mathbf{h}(\tau)) = (\alpha_j\tau + b_j\tau^2) + (\beta_j\tau + c_j\tau^2) + \sum_{r,s=1}^{n} c_{j,rs}\alpha_r\beta_s\tau^2,$$

and

$$\varphi_j(\mathbf{k}(\tau), \varphi(\mathbf{h}(\tau), \mathbf{g}(\tau))) = \rho_j\tau + a_j\tau^2 + (\beta_j\tau + c_j\tau^2) + (\alpha_j\tau + b_j\tau^2)$$

$$+ \sum_{r,s=1}^{n} c_{j,rs}\beta_r\alpha_s\tau^2 + \sum_{r,s=1}^{n} c_{j,rs}\rho_r\tau(\alpha_s\tau + \beta_s\tau).$$

Equating coefficients of $\tau$ and $\tau^2$ in $\varphi_j(\mathbf{k}(\tau), \varphi(\mathbf{h}(\tau), \mathbf{g}(\tau)))$ and $\varphi_j(\mathbf{g}(\tau), \mathbf{h}(\tau))$ to obtain:

$$\rho_j + \alpha_j + \beta_j = \beta_j + \alpha_j,$$

and

$$a_j + c_j + b_j + \sum_{r,s=1}^{n} c_{j,rs}(\beta_r\alpha_s + \rho_r\beta_s\tau + \rho_r\alpha_s) = (c_j + b_j) + \sum_{r,s=1}^{n} c_{j,rs}\alpha_r\beta_s .$$

Thus, $\rho_j = 0$, and $a_j + \sum_{r,s=1}^{n} c_{j,rs}\beta_r\alpha_s = \sum_{r,s=1}^{n} c_{j,rs}\alpha_r\beta_s$, $j = 1, 2, \ldots, n$. Rearrange to get the required result:

$$[\alpha, \beta]_j = a_j = \sum_{r,s=1}^{n}(c_{j,rs} - c_{j,sr})\alpha_r\beta_s.$$

Q.E.D.

The constants $c_j^{rs}$, $1 \le j, r, s \le n$, are called the **structure constants** of the local Lie group G.

**Corollary 1.3.1** (See Miller [1, p.168], Theorem 5.7). *For all* $\alpha, \beta, \gamma \in F_n$ *and for all* a, b $\in$ *F*:

(1) $[\alpha, \beta] = -[\beta, \alpha]$. (1.3.4)

(2) $[a\alpha + b\beta, \gamma] = a[\alpha, \gamma] + b[\beta, \gamma]$. (1.3.5)

(3) $[[\alpha, \beta], \gamma] + [[\gamma, \alpha], \beta] = -[[\beta, \gamma], \alpha]$. (1.3.6)

**Proof**: Eq.'s (1.3.4) and (1.3.5) follow directly from Eq.(1.3.3). Now, by Eq.(1.3.3), for $1 \le j \le n$

$$[[\alpha, \beta], \gamma]_j + [[\gamma, \alpha], \beta]_j + [[\beta, \gamma], \alpha]_j = \sum_{q,r,s,t=1}^{n} \left( c_j^{qr} c_q^{st} \alpha_s \beta_t \gamma_r + c_j^{qr} c_q^{st} \alpha_t \beta_r \gamma_s + c_j^{qr} c_q^{st} \alpha_r \beta_s \gamma_t \right)$$

$$= \sum_{r,s,t=1}^{n} \alpha_s \beta_t \gamma_r \left( \sum_{q=1}^{n} c_j^{qr} c_q^{st} + c_j^{qr} c_q^{st} + c_j^{qr} c_q^{st} \right)$$

However, by Lemma 1.2.4 and the fact that $c_j^{qr} = c_{j,qr} - c_{j,rq}$, it follows that

$$\left( \sum_{q=1}^{n} c_j^{qr} c_q^{st} + c_j^{qr} c_q^{st} + c_j^{qr} c_q^{st} \right) = 0.$$

Thus, Eq.(1.3.6) is satisfied, as required.

Q.E.D.

**Definition 1.3.3** The **Lie algebra** L(G), of a local Lie group G is the space of all tangent vectors at **e** equipped with the operations of scalar multiplication, vector addition, and commutator product.

To deal with the special case of local linear Lie groups we need to define tangent matrices and the matrix commutator.

**Definition 1.3.4** Let G be an n-dimensional local linear Lie group of mxm matrices. Let $A(t) = A(\mathbf{g}(t))$, $A(0) = E_m$, be an **analytic curve through the identity**. Then the **tangent matrix** to $A(\mathbf{g}(t))$ at **e** is the matrix

$$\mathbf{A} = \left.\frac{dA(\mathbf{g}(t))}{dt}\right|_{t=0} = \sum_{j=1}^{n} \left.\frac{\partial A(\mathbf{g})}{\partial g_j}\right|_{\mathbf{g}=\mathbf{e}} \left.\frac{dg_j(t)}{dt}\right|_{t=0} .$$

If A(t) and B(t) are two analytic curves in a local linear Lie group G, with tangent matrices **A** and **B**, respectively, then the **commutator of A and B, [A, B]**, is

$$[\mathbf{A}, \mathbf{B}] \equiv \frac{d}{dt}[A(\tau)B(\tau)A^{-1}(\tau)B^{-1}(\tau)]\Big|_{t=0},$$

where $t = \tau^2$ and $A^{-1}(\tau)$ is the matrix inverse of $A(\tau)$.

**Theorem 1.3.2** (See Miller [1, p.169], Theorem 5.8).

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}.$$

**Proof**: The proof is similar to the proof of Theorem 1.3.1. Let

$$C(t) \equiv A(\tau)B(\tau)A^{-1}(\tau)B^{-1}(\tau).$$

We can express $A(\tau)$, $B(\tau)$ and $C(t)$ as follows:

$$A(\tau) = E_m + \mathbf{A}\tau + \mathbf{A}'\tau^2 + \ldots,$$
$$B(\tau) = E_m + \mathbf{B}\tau + \mathbf{B}'\tau^2 + \ldots,$$
$$C(t) = E_m + \mathbf{C}\tau + \mathbf{C}'\tau^2 + \ldots,$$

where $E_m$ is the mxm identity matrix. From the definition of C(t), it follows that $C(t)B(\tau)A(\tau) = A(\tau)B(\tau)$. Replace $A(\tau)$, $B(\tau)$ and $C(t)$ by their power series expansions to conclude that

$$E_m + \mathbf{C}\tau + \mathbf{B}\tau + \mathbf{A}\tau + \mathbf{C}'\tau^2 + \mathbf{B}'\tau^2 + \mathbf{A}'\tau^2 + \mathbf{CB}\tau^2 + \mathbf{CA}\tau^2 + \mathbf{BA}\tau^2$$

$$= E_m + \mathbf{A}\tau + \mathbf{B}\tau + \mathbf{A}'\tau^2 + \mathbf{B}'\tau^2 + \mathbf{AB}\tau^2,$$

where terms of order > 2 in $\tau$ are omitted. Equate coefficients of $\tau$ to conclude that $\mathbf{C} = Z_m$. Equate coefficients of $\tau^2$ to conclude that $[\mathbf{A}, \mathbf{B}] = \mathbf{C}' = \mathbf{AB} - \mathbf{BA}$, as required.

Q.E.D.

The commutator of a local linear Lie group is known as the **matrix commutator.**

For examples of Lie algebras, we need only only take Examples 1.2.1 through 1.2.5 and apply Theorems 1.3.1 and 1.3.2. In particular, the Lie algebra of Example 1.2.4 is the vector space $\mathfrak{C}^3$, together with the commutator

$$[(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)] = \left(0, \alpha_2\beta_1 - \alpha_1\beta_2, 2(\alpha_3\beta_1 - \alpha_1\beta_3)\right).$$

The local linear Lie group, G, of Example 1.2.5 has, as a Lie algebra, the set of all matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 2\alpha_1 & 2\alpha_2 \\ 0 & 0 & 3\alpha_1 \end{pmatrix}, \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

If $A(\mathbf{g}(t))$ is an analytic curve through the identity in G, then

$$\mathbf{A} = \frac{dA(\mathbf{g}(t))}{dt}\bigg|_{\mathbf{g}=\mathbf{e}} = \begin{pmatrix} g_1'(0) & g_2'(0) & g_3'(0) \\ 0 & 2g_1'(0) & 2g_2'(0) \\ 0 & 0 & 3g_1'(0) \end{pmatrix},$$

But $g_i'(0)$, $i = 1, 2, 3$, can be any real number, depending on the choice of $\mathbf{g}(t)$. Hence L(G) has the prescribed form.

Apply Theorem 1.3.2 to obtain

$$[\mathbf{A}, \mathbf{B}] = \begin{pmatrix} 0 & \alpha_2\beta_1 - \alpha_1\beta_2 & 2(\alpha_3\beta_1 - \alpha_1\beta_3) \\ 0 & 0 & 2(\alpha_2\beta_1 - \alpha_1\beta_2) \\ 0 & 0 & 0 \end{pmatrix}.$$

Lie algebras can be defined without considering local Lie groups. These are known as abstract Lie algebras.

**Definition 1.3.5** An **abstract Lie Algebra** $\mathfrak{G}$ over $F$ is a vector space over $F$ together with a multiplication $[\alpha, \beta] \in \mathfrak{G}$ defined for all $\alpha, \beta \in \mathfrak{G}$ such that Eq.'s (1.3.4), (1.3.5) and (1.3.6) hold, for all $\alpha, \beta, \gamma \in \mathfrak{G}$ and for all $a, b \in F$.

Obviously, any Lie algebra is an abstract Lie algebra. Furthermore, any set of mxm matrices closed under matrix addition, scalar multiplication and the matrix commutator forms an abstract Lie algebra. Now, we define isomorphic Lie algebras.

**Definition 1.3.6** Let $\mathfrak{G}$ and $\mathfrak{G}'$ be abstract Lie Algebras, with commutators [-, -] and [-, -]', respectively. A **Lie algebra isomorphism** from $\mathfrak{G}$ to $\mathfrak{G}'$ is a one-to-one map, $\tau$, from $\mathfrak{G}$ onto $\mathfrak{G}'$ such that

(1) $\tau(a\alpha + b\beta) = a\tau(\alpha) + b\tau(\beta)$, $a, b \in F$, $\alpha, \beta \in \mathfrak{G}$.
(2) $\tau([\alpha, \beta]) = [\tau(\alpha), \tau(\beta)]'$.

For example, let $\mathfrak{G}$ be obtained by restricting the Lie algebra of Example 1.2.4 to the real numbers. We can construct an isomorphism, $\tau$, from $\mathfrak{G}$ to the Lie algebra of Example 1.2.5, by

$$\tau\big((\alpha_1,\alpha_2,\alpha_3)\big) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 2\alpha_1 & 2\alpha_2 \\ 0 & 0 & 3\alpha_1 \end{pmatrix}, \quad (\alpha_1, \alpha_2, \alpha_3) \in L(G).$$

It is not a coincidence that the local Lie group of Example 1.2.4 (defined over the real instead of the complex numbers) is locally isomorphic to the local Lie group of Example 1.2.5. We will show in Section 1.5, that G and G' are locally isomorphic Lie groups if and only if L(G) and L(G') are isomorphic Lie algebras.

Eq.(1.3.3) gives a straightforward method of finding the commutator for the Lie algebra, L(G), of a local Lie group $G = (V, \varphi)$. In chapter 3 we address the reverse problem. That is, given an abstract Lie algebra $\mathcal{A}$, construct a local Lie group $G = (V, \varphi)$ shuch that its Lie algebra, L(G) is isomorphic to $\mathcal{A}$. We completely solve this problem when $\mathcal{A}$ is realized as a set of linear differential operators having certain properties.

## Section 1.4 One-parameter Subgroups and the Exponential Mapping

The purpose of this section is to provide a means of expressing the elements of a local Lie group G in terms of the elements of the Lie algebra L(G). In order to do so, we need a special class of analytic curves from G.

**Definition 1.4.1** Let G be an n-dimensional local Lie group. The analytic curve $\mathbf{g}(t)$, defined for $t$ in some neighborhood $W$ of $0 \in F$ such that $\mathbf{g}(0) = \mathbf{e}$, is a **one-parameter subgroup** of G if

$$g_i(s+t) = \varphi_i(\mathbf{g}(s), \mathbf{g}(t)), \quad s, t, s+t \in W, \quad i = 1, 2, \ldots, n. \tag{1.4.1}$$

**Theorem 1.4.1** (See Miller [1, p.176], Theorem 5.16). *Let* G *be an* n-*dimensional local Lie group with Lie algebra* L(G), *and let* $\mathbf{g}(t)$ *be an analytic curve in* G *defined for* t *in some suitably small neighborhood of* 0. $\mathbf{g}(t)$ *is a one-parameter subgroup of* G *with tangent vector* $\alpha$ *at* $\mathbf{e}$ *if and only if* $\mathbf{g}(t)$ *is the unique solution of the system of differential equations*

$$\frac{dg_i(t)}{dt} = \sum_{k=1}^{n} \alpha_k F_{ik}(\mathbf{g}(t)), \quad g_i(0) = 0, \quad i = 1, 2, \ldots, n, \tag{1.4.2}$$

*where* $\alpha_k \in F$, *and* $F_{ik}(\mathbf{h})$ *is given by* Eq.(1.2.8).

**Proof:** Eq.(1.4.2) is a system of first order differential equations. Thus, by standard existence and uniqueness theorems for ordinary differential equations, (see Petrovski [1, p.96]), it has a unique solution, $\mathbf{g}(t)$, satisfying the initial condition $\mathbf{g}(0) = \mathbf{e}$. This solution is defined and analytic for all $|t| < \varepsilon$, where $\varepsilon$ is a positive number, depending on $F_{ik}$, but not on $\alpha$.

($\Rightarrow$) Let $\mathbf{g}(t)$ be a one-parameter subgroup of G with tangent vector $\alpha$ at $\mathbf{e}$. Then, $\mathbf{g}(0) = \mathbf{e}$, and Eq.(1.4.1) holds. Differentiate both sides of Eq.(1.4.1) with respect to s and evaluate at $s = 0$ to obtain:

$$\frac{dg_i(t)}{dt} = \left.\frac{d\varphi_i(\mathbf{g}(s), \mathbf{g}(t))}{ds}\right|_{s=0} = \sum_{k=1}^{n} \left.\frac{\partial\varphi_i(\mathbf{g}(s), \mathbf{g}(t))}{\partial g_k(s)}\right|_{s=0} \left.\frac{dg_k(s)}{ds}\right|_{s=0}$$

which, by Eq.(1.2.8), becomes Eq.(1.4.2), $\quad i = 1, 2, \ldots, n$.

($\Leftarrow$) For $\alpha \in L(G)$, let $\mathbf{g}(t)$ be the unique solution of Eq.(1.4.2), with $\mathbf{g}(0) = \mathbf{e}$. Since $F_{ik}(\mathbf{e}) = \delta_{ik}$, then by Eq.(1.4.2), $g_k'(0) = \alpha_k$, $k = 1, 2, \ldots, n$. To show that $\mathbf{g}(t)$ is a one-parameter subgroup with tangent vector $\alpha$ at $\mathbf{e}$, we need only show that it satisfies Eq.(1.4.1). Fix s close to zero and let $h_i(t) = g_i(t+s)$, and $k_i(t) = \varphi_i(\mathbf{g}(t), \mathbf{g}(s))$. Then

$$\frac{dh_i(t)}{dt} = \frac{dg_i(t+s)}{dt} = \sum_{a=1}^{n} F_{ia}(\mathbf{h}(t)), \qquad h_i(0) = g_i(s).$$

Similarly,

$$\frac{dk_i(t)}{dt} = \frac{d\varphi_i(\mathbf{g}(t), \mathbf{g}(s))}{dt} = \sum_{r=1}^{n} \frac{\partial \varphi_i(\mathbf{g}(t), \mathbf{g}(s))}{\partial g_r(t)} \frac{dg_r(t)}{dt},$$

which by Eq.(1.4.2) becomes

$$\frac{dk_i(t)}{dt} = \sum_{r=1}^{n} \frac{\partial \varphi_i(\mathbf{g}(t), \mathbf{g}(s))}{\partial g_r(t)} \sum_{a=1}^{n} \alpha_a F_{ra}(\mathbf{g}(t)).$$

Apply Lemma 1.2.5, to obtain

$$\frac{dk_i(t)}{dt} = \sum_{a=1}^{n} \alpha_a F_{ia}(\mathbf{k}(t))), \qquad k_i(0) = g_i(s), \quad i = 1, \ldots, n.$$

Since $\mathbf{h}(t)$ and $\mathbf{k}(t)$ satisfy the same first order differential system and initial conditions, then by the uniqueness of solution $\mathbf{h}(t) \equiv \mathbf{k}(t)$. Thus, $g_i(t + s) = \varphi_i(\mathbf{g}(t), \mathbf{g}(s))$, for suitably small $s$ and $t$. Hence, $\mathbf{g}(t)$ is a one-parameter subgroup, as required.

Q.E.D.

Denote the one-parameter curve with tangent vector $\alpha$ at $\mathbf{e}$ by $\mathbf{g}(t) = \mathbf{EXP}(\alpha, t)$.

**Corollary 1.4.1** *For each* $\alpha \in L(G)$, *there is a unique one-parameter subgroup,* $\mathbf{EXP}(\alpha, t)$, *with tangent vector* $\alpha$ *at* $\mathbf{e}$.

**Proof:** This is a direct consequence of Theorem 1.4.1.

**Corollary 1.4.2** $\mathbf{EXP}(a\alpha, t) = \mathbf{EXP}(\alpha, at)$, *for* $a \in F$, $\alpha \in L(G)$, $|t|$ *sufficiently close to* 0.

**Proof:** Obviously, $\mathbf{EXP}(a\alpha, t)$ and $\mathbf{EXP}(\alpha, at)$ are both one-parameter subgroups of $G$ and both have tangent vector $a\alpha$ at $\mathbf{e}$, thus by Corollary 1.4.1, $\mathbf{EXP}(a\alpha, t) = \mathbf{EXP}(\alpha, at)$.

Q.E.D.

At this point, $\mathbf{EXP}(\alpha, t)$ is defined only for $|t| < \varepsilon$, where $\varepsilon$ is a constant for the local Lie group. We extend the domain of the $\mathbf{EXP}$ function to $|t| \geq \varepsilon$, while still satisfying Eq.(1.4.1), by using Corollary 1.4.2. Let

$$\overline{\mathbf{EXP}}(\alpha, t) = \begin{cases} \mathbf{EXP}(\alpha, t) & \text{for } |t| < \varepsilon \\ \mathbf{EXP}\left(a\alpha, \left(\frac{t}{a}\right)\right) & \text{for } |t| \geq \varepsilon, \text{ where } a \in F, \text{ such that } \left|\frac{t}{a}\right| < \varepsilon. \end{cases}$$

This is a valid definition since, by Corollary 1.4.2, it does not depend on a. Furthermore, $\overline{\mathbf{EXP}}(\alpha, t)$ is an analytic curve which satisfies Eq.(1.4.1). Thus, we shall not differentiate between $\overline{\mathbf{EXP}}(\alpha, t)$ and **EXP**($\alpha$, t), and we shall simply refer to the one-parameter subgroups **EXP**($\alpha$, t).

**Lemma 1.4.1** (See Miller [1, p.177]). *For fixed* $t \in F$, **EXP**($\alpha$, t) *is an analytic function of* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

**Proof**: **g**(t) = **EXP**($\alpha$, t) is an analytic curve in G. Thus, we can expand it in a Taylor series about t = 0:

$$g_k(t) = \sum_{m=1}^{\infty} \left.\frac{d^m g_k(s)}{ds^m}\right|_{s=0} \frac{t^m}{m!}, \qquad 1 \leq k \leq n.$$

From Eq.(1.4.2), the chain rule and math induction we find that $\left.\frac{d^m g_k(s)}{ds^m}\right|_{s=0}$ is a homogeneous polynomial of degree m in the $\alpha_i$, i.e. it consists only of terms of the form $c\alpha_1^{a_1}\alpha_2^{a_2}\ldots\alpha_n^{a_n}$, where $a_1 + a_2 + \ldots + a_n = m$, $c \in F$. For fixed t close to 0, the Taylor's series converges for all $\alpha$, thus **g**(t) is analytic in $\alpha_1, \alpha_2, \ldots, \alpha_n$, as required.

Q.E.D.

The proof of Lemma 1.4.1 confirms the fact that **EXP**($\alpha$, t) is actually a function of $\alpha t = (\alpha_1 t, \alpha_2 t, \ldots, \alpha_n t)$. Thus for the remainder of the thesis we denote the one-parameter subgroups by **EXP**($\alpha t$).

**Definition 1.4.2** Let G be an n-dimensional local Lie group with Lie algebra L(G) and one-parameter subgroups **EXP**($\alpha t$), $\alpha \in$ L(G). The **exponential map**, **EXP**, is the mapping from L(G) into G such that **EXP**($\alpha$) = **EXP**($\alpha$(1)).

Obviously, the exponential map takes a neighborhood of $(0, 0, \ldots, 0) \in$ L(G) into G. {If $\alpha$ is not close enough to $(0, 0, \ldots, 0)$ then **EXP**($\alpha$(1)) might not be in G}.

**Lemma 1.4.2** (See Miller [1, p.177]). *The exponential map defines an analytic coordinate transformation on some neighborhood of* **e** = (0, 0, . . . , 0) ∈ L(G). *Thus, the* n *coordinates* $\alpha_j$ *of* $\alpha \in$ L(G) *can be used to parameterize the local Lie group* G.

**Proof:** By Lemma 1.4.1 and its proof it follows that **EXP**($\alpha$) is an analytic function of $\alpha_i$, $1 \le i \le n$, and

$$\left.\frac{\partial \mathrm{EXP}_i(\alpha)}{\partial \alpha_k}\right|_{\alpha=\mathbf{e}} = \delta_{ik}, \quad 1 \le i, k \le n.$$

Thus, the Jacobian of the exponential map is non-zero for some neighborhood of the zero vector in L(G). Thus, by the inverse function theorem (Apostol [1, p. 144] there is an open neighborhood *X* of the zero vector in L(G) and an open neighborhood *Y* of **e** in *V*, such that **EXP**:*X* → *Y* is one-to-one and onto and the inverse function, $\mathbf{EXP}^{-1}$, exists and is analytic. Thus, $\alpha \in X \Leftrightarrow$ **EXP**($\alpha$) ∈ *Y*, and we can use the elements of the Lie algebra to parameterize the local Lie group, as required.

Q.E.D.

If G is a local linear Lie group the following adjustments must be made. A **one-parameter subgroup** of G is an analytic curve, A(t) = A(**g**(t)), such that

$$A(s)A(t) = A(s+t), \qquad |s|, |t| \text{ sufficiently small.}$$

For tangent matrix **A** ∈ L(G), **EXP**(**A**t) is the unique solution of

$$\frac{d}{dt}A(t) = \mathbf{A}(A(t)), \qquad A(0) = E_m. \tag{1.4.3}$$

By Theorem 1.4.1, the one-parameter subgroups of G are exactly the analytic curves **EXP**(**A**t), **A** ∈ L(G), |t| suitably small. Since G is a local linear Lie group, it follows that,

$$\mathbf{EXP}(\mathbf{A}t) = \sum_{j=0}^{\infty} \frac{\mathbf{A}^j t^j}{j!}, \tag{1.4.4}$$

where $\mathbf{A}^0 = E_m$.

This comes from the fact that Eq.(1.4.3) has the unique solution **EXP**(**A**t) and the fact that

$$\frac{d}{dt}\left(\sum_{j=0}^{\infty}\frac{A^j t^j}{j!}\right)=\sum_{j=1}^{\infty}\frac{A^j t^{j-1}}{(j-1)!}=A\sum_{j=0}^{\infty}\frac{A^j t^j}{j!},$$

satisfies Eq.(1.4.3) with $\sum_{j=0}^{\infty}\frac{A^j 0^j}{j!}=E_m$.

Thus, in order to obtain the one-parameter subgroups of a local linear Lie group, one uses Eq.(1.4.4) whereas to find the one-parameter subgroups of local Lie groups one must solve the differential system given by Eq.(1.4.2).

## Section 1.5 Isomorphism Theorems

Local Lie group isomorphisms and Lie algebra isomorphisms were defined in Sections 1.2. and 1.3, respectively. The final section of this Chapter deals with the relationship between isomorphic local Lie groups and Lie algebras as well as the relationship between abstract Lie algebras and the Lie algebras of local Lie groups. For brevity, the following theorems are stated without proof, but the proofs can be found in the given references. Our intention here is to state results important to the thesis and give an idea of how they are derived.

**Theorem 1.5.1** (Ado's Theorem). *Every abstract Lie algebra is isomorphic to some matrix Lie algebra.*

**Proof:** (See Jacobson [1, p.202], Miller [1, p.170], Theorem 5.9, and Ado [1, p.309-327]).

**Theorem 1.5.2** *If* G, G' *are isomorphic* n-*dimensional local Lie groups with Lie algebras* L(G) *and* L(G'), *respectively then* L(G) *and* L(G') *are isomorphic Lie algebras.*

**Proof**: (See Miller [1, p.179], Theorem 5.17).

**Theorem 1.5.3** *Let* G, G' *be local linear Lie groups and let* L(G) *and* L(G') *be isomorphic Lie algebras. Then* G *and* G' *are (locally) isomorphic local linear Lie groups.*

**Proof:** (See Miller [1, p.180], Theorem 5.18).

**Corollary 1.5.1** *Two local linear Lie groups,* G *and* G', *are (locally) isomorphic if and only if* L(G) *and* L(G') *are isomorphic.*

**Proof:** Obvious conclusion of Theorems 1.5.2 and 1.5.3.

Note that Theorem 1.5.3 can be extended to all local Lie groups (see Miller [1, p.181]). Thus we have the following Corollary.

**Corollary 1.5.2** *Two local Lie groups,* G *and* G', *are (locally) isomorphic if and only if* L(G) *and* L(G') *are isomorphic.*

**Theorem 1.5.4** *If* $\mathfrak{G}$ *is a matrix Lie algebra, then there exists a local linear Lie group* G *such that* $L(G) = \mathfrak{G}$.

**Proof:** (See Miller [1, p.181], Theorem 5.19).

**Theorem 1.5.5** *Every abstract Lie algebra is the Lie algebra of some local Lie group.*

**Proof**: (See Kostrikin and Shafarevich [1, p.196], Lie's Theorem or deduce Theorem 1.5.5 from Theorems 1.5.1, 1.5.4 and Corollary 1.5.2).

The most important results of this section are Corollary 1.5.2 and Theorem 1.5.5. From these two results it follows that, given any abstract Lie algebra, $\mathfrak{G}$, there exists a local Lie group with Lie algebra isomorphic to $\mathfrak{G}$. The ability to go from Lie algebra to local Lie group will be important in the next Chapter, where we deal with local Lie transformation groups, Lie derivatives, and Lie algebras of Differential operators.

We end this chapter on Elementary Local Lie Group Theory with an examination of the most elementary of local Lie groups, the one-dimensional local Lie groups. An immediate conclusion from Corollary 1.5.2 is that all one-dimensional local Lie groups are (locally) isomorphic and if G is a one-dimensional local Lie group then it is commutative.

However, the fact that all one-dimensional local Lie groups are isomorphic does not help in the generation of all one-dimensional Lie products. Consider Examples 1.2.2 and 1.2.3. The one-dimensional Lie product $\varphi(g, h) = g + h + \gamma gh$,

$\gamma \in F$ generates an infinite number of local Lie products that are polynomials in g and h. $\varphi(g, h) = \ln(e^g + e^h - 1)$ is an example of a one-dimensional Lie product that is a power series of g and h.

The associative property restricts the form of a local Lie product, even in the one-dimensional case. Suppose $G = (V, \varphi)$ is a one-dimensional Lie group such that $\varphi(g, h)$ can be expressed as a polynomial of g and h. The following argument shows that $\varphi(g, h) = g + h + \gamma gh, \gamma \in F$ is the only possibility.

Assume that there exists a one-dimensional Lie product $\varphi$ such that $\varphi(g, h)$ is a polynomial of order 3 or more in g and h, say $\varphi(g, h) = \sum_{i,k=0}^{n} f_{ik} g^i h^k$, $n > 2$. By Eq.(1.1.3),

$$f_{0k} = f_{k0} = \delta_{1k}, \quad k = 0, 1, 2, 3, \ldots, n .$$

Thus,

$$\varphi(g, h) = g + h + \sum_{i,k=1}^{n} f_{ik} g^i h^k .$$

Since $\varphi$ is a Lie product, $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$. By direct computation,

$$\varphi(\varphi(x, y), z) = x + y + z + \sum_{i,k=1}^{n} f_{ik} x^i y^k + \sum_{i,k=1}^{n} f_{ik} \left( x + y + \sum_{r,s=1}^{n} f_{rs} x^r y^s \right)^i z^k,$$

and

$$\varphi(x, \varphi(y, z)) = x + y + z + \sum_{i,k=1}^{n} f_{ik} y^i z^k + \sum_{i,k=1}^{n} f_{ik} x^i \left( y + z + \sum_{r,s=1}^{n} f_{rs} y^r z^s \right)^k.$$

Compare coefficients of $z^{n^2}$, to conclude that the polynomial

$$\sum_{i=1}^{n} f_{in} x^i [\sum_{r=1}^{n} f_{rn} y^r ]^n = 0.$$

Thus, the coefficient of $x^n y^{n^2}$ is $f_{nn}{}^{n+1} = 0$, i.e., $f_{nn} = 0$.
Thus,

$$\sum_{i,k=1}^{n} f_{in} x^i \left( \sum_{r=1}^{n} f_{rn} y^r \right)^n = \sum_{i=1}^{n-1} f_{in} x^i \left( \sum_{r=1}^{n-1} f_{rn} y^r \right)^n = 0.$$

Thus, the coefficient of $x^{n-1}y^{n(n-1)}$ is zero, i.e., $(f_{n-1n})^{n+1} = 0$, i.e., $f_{n-1n} = 0$.

Repeat the same argument to conclude that $f_{r,n} = 0$, $r = 1, 2, \ldots, n$, $n > 1$. Since all one-dimensional local Lie groups are commutative, $f_{ik} = f_{ki}$, $1 \le i, k \le n$. Thus, $\varphi(g, h)$ is not a polynomial of degree $n > 2$ in g and h. Thus, $\varphi(g, h)$ can be expressed as a polynomial in g and h only if $\varphi(g, h) = g + h + \gamma gh$, $\gamma \in F$.

In Chapter 3 we characterize all one-dimensional Lie products in terms of solutions of ordinary differential equations of the form

$$\frac{dx}{dt} = \alpha P(x), \quad x(0) = g,$$

where $P(0) = 1$. This characterization is useful in constructing examples of one-dimensional local Lie groups.

# Chapter 2

# The Lie Derivative

## Section 2.1 Local Lie Transformation Groups

We now have sufficient background to discuss local Lie transformation groups and Lie derivatives. We begin with the definition of a local Lie transformation group.

**Definition 2.1.1** Let G = ($V$, $\varphi$) be an n-dimensional local Lie group and let $U$ be an open connected set in $F_m$. Let **Q** be a mapping, **Q**:$U$x$G$ $\rightarrow F_m$. Then (G, **Q**) acts on the manifold $U$ as a **local Lie transformation group** if **Q** satisfies the following properties:

1. **Q[x, g]** is analytic in the m + n coordinates of **x** and **g**. (2.1.1)
2. **Q[x, e]** = **x**, for all **x** $\in$ $U$. (2.1.2)
3. If **Q[x, g]** $\in U$ then **Q[Q[x, g], h]** = **Q[x, $\varphi$(g, h)]**, (2.1.3)
for **g**, **h**, $\varphi$(**g**, **h**) $\in$ G.

If (G, **Q**) is a local Lie transformation group, then **Q** is called the **group action** and G is called the **underlying local Lie group**.

In order to distinguish the elements of G and L(G), which are n-tuples, from the m-tuples of $U$, we will use lightface type to represent the non-identity elements of G and L(G) for the remainder of Chapter 2. Thus, **Q[x, g]** becomes **Q[x**, g**]**. We can consider the m components, $Q_i$**[x**, g**]**, of **Q[x**, g**]**.

For fixed g, the map **x** $\rightarrow$**Q[x**, g**]** is locally analytic and one-to-one. The map is analytic because Property(2.1.1) guarantees that **Q[x**, g**]** is an analytic function of **x**. The map is one-to-one because, if **x** $\in U$ and g is sufficiently close to **e**, then $g^{-1}$ exists. Thus, by Properties (2.1.2) and (2.1.3),

$$\begin{aligned}\mathbf{Q}[\mathbf{x}, g] = \mathbf{Q}[\mathbf{y}, g] &\Leftrightarrow \mathbf{Q}[\mathbf{Q}[\mathbf{x}, g], g^{-1}] = \mathbf{Q}[\mathbf{Q}[\mathbf{y}, g], g^{-1}] \\ &\Leftrightarrow \mathbf{Q}[\mathbf{x}, e] = \mathbf{Q}[\mathbf{y}, e] \\ &\Leftrightarrow \mathbf{x} = \mathbf{y}.\end{aligned}$$

Given any local Lie group, $G = (V, \varphi)$, a simple example of a local Lie transformation group is obtained by letting the group action be the Lie product, $\varphi$. I.e., $\mathbf{Q}: V \times G \rightarrow F_n$ such that $\mathbf{Q}[\mathbf{g}^0, \mathbf{g}] = \varphi(\mathbf{g}^0, \mathbf{g})$.

Let $\text{EXP}(\alpha t)$ be a one-parameter subgroup of G. By Property(2.1.1), $\mathbf{Q}[\mathbf{x}, \text{EXP}(\alpha t)]$ is analytic in $\text{EXP}(\alpha t)$, thus $Q_i[\mathbf{x}, \text{EXP}(\alpha t)]$ can be expanded in a Taylor series in t about $t = 0$, as

$$Q_i[\mathbf{x}, \text{EXP}(\alpha t)] = x_i + t\sum_{r=1}^{n} P_{ir}(\mathbf{x})\alpha_r + \ldots, \quad 1 \le i \le m, \tag{2.1.4}$$

where

$$P_{ir}(\mathbf{x}) = \left.\frac{\partial Q_i[\mathbf{x}, g]}{\partial g_r}\right|_{g=e}, \quad 1 \le i \le m, \quad 1 \le r \le n. \tag{2.1.5}$$

**Definition 2.1.2** Let $\mathcal{A}_\mathbf{x}$ be the space of all functions $f$, analytic in some open neighborhood of $\mathbf{x}$. The **Lie derivative of the function $f \in \mathcal{A}_\mathbf{x}$**, $L_\alpha f$, is

$$L_\alpha f(\mathbf{x}) = \left.\frac{d}{dt}\Big(f(\mathbf{Q}[\mathbf{x}, g(t)])\Big)\right|_{t=0}, \quad \alpha \in L(G), \tag{2.1.6}$$

where $g(t)$ is any analytic curve in G with tangent vector $\alpha$ at $\mathbf{e}$. Note that $g(t) = \text{EXP}(\alpha t)$ is one such curve. We now give an alternate form of the Lie derivative which shows that Eq.(2.1.6) is independent of the choice of curve $g(t)$.

**Lemma 2.1.1**
$$L_\alpha = \sum_{i=1}^{m}\sum_{r=1}^{n} \alpha_r P_{ir}(\mathbf{x})\frac{\partial}{\partial x_i}, \tag{2.1.7}$$

*where* $P_{ir}(\mathbf{x})$ *is given by* Eq.(2.1.5).

**Proof:** Apply the chain rule to Eq.(2.1.6) and then evaluate it at $t = 0$. Then use Eq.(2.1.5) and Eq.(2.1.2), and the fact that $g(0) = \mathbf{e}$ and $g(t)$ has tangent vector $\alpha$ at $\mathbf{e}$ to obtain Eq.(2.1.7).

Q.E.D.

Note that, for Lemma 2.1.1, we only need $\frac{dg(t)}{dt}$ to exist at $t = 0$. Thus, Definition 2.1.2 is stronger than necessary, since it limits the choice of the curve $g(t)$ to analytic functions. Eq.(2.1.6) is valid for any function g(t) defined on G, such that g(t) is once differentiable, g(0) = **e** and g(t) has tangent vector $\alpha$ at **e**.

Thus, Lie derivatives are linear differential operators mapping $\mathcal{A}_{\mathbf{x}} \to \mathcal{A}_{\mathbf{x}}$. The purpose of Chapter 2 is to show that the set of all Lie derivatives of a local Lie transformation group form a Lie algebra of differential operators and to show that every Lie algebra of differential operators is the set of all Lie derivatives of some local Lie transformation group.

In the next section we examine Lie derivatives and local Lie transformation groups in more detail, and show that the set of Lie derivatives of a local Lie transformation group is a Lie algebra of differential operators.

## Section 2.2 Theorems Concerning Lie Derivatives

Throughout Section 2.2, assume that G = ($V$, $\varphi$) is an n-dimensional local Lie group with one-parameter subgroups EXP($\alpha$t), and that (G, **Q**) is a local Lie transformation group. By Lemma 2.1.1, (G, **Q**) has a unique set of Lie derivatives, $L_\alpha$, $\alpha \in$ L(G). The following is also true.

**Theorem 2.2.1** (See Miller [1, p.191], Theorem 5.24). *Let* (G, **Q**) *be a local Lie transformation group with Lie derivatives* $L_\alpha$. *Then the unique solution of*

$$\frac{d}{dt}\mathbf{x}(t) = L_\alpha \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}^0, \tag{2.2.1}$$

*is* $\mathbf{x}(t) = \mathbf{Q}[\mathbf{x}^0, \mathrm{EXP}(\alpha t)]$.

**Proof:** Due to Eq.(2.1.7), Eq.(2.2.1) can be written

$$\frac{d\mathbf{x}(t)}{dt} = \sum_{i=1}^{m} \sum_{k=1}^{n} \alpha_k P_{ik}(\mathbf{x}(t)) \frac{\partial \mathbf{x}(t)}{\partial x_i(t)} ,$$

where $P_{ik}(\mathbf{x})$ is given by Eq.(2.1.5). Separate the components to conclude that vector ordinary differential equation, (V.O.D.E.), (2.2.1) is equivalent to the following system of differential equations,

$$\frac{dx_i(t)}{dt} = \sum_{k=1}^{n} \alpha_k P_{ik}(\mathbf{x}(t)), \quad i = 1, 2, \ldots, m, \qquad \mathbf{x}(0) = \mathbf{x}^0. \tag{2.2.2}$$

Now, let $(G, \mathbf{Q})$ be a local Lie transformation group with Lie derivatives $L_\alpha$ and let $\mathbf{x}^0 \in U$ and $\alpha \in L(G)$. From the existence and uniqueness theorems (see Petrovski [1, p.96] ) for a V.O.D.E., we know that the V.O.D.E.(2.2.1) has a unique solution $\mathbf{x}(t)$. However, $\mathbf{Q}[\mathbf{x}^0, EXP(\alpha(0))] = \mathbf{Q}[\mathbf{x}^0, \mathbf{e}] = \mathbf{x}^0$ and

$$\frac{d}{dt} Q_i[\mathbf{x}^0, EXP(\alpha t)] = \left.\frac{dQ_i[\mathbf{x}^0, EXP(\alpha(t+s))]}{ds}\right|_{s=0}$$

$$= \left.\frac{dQ_i[\mathbf{x}^0, \varphi(EXP(\alpha t), EXP(\alpha s))]}{ds}\right|_{s=0}$$

$$= \left.\frac{dQ_i[\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)], EXP(\alpha s)]}{ds}\right|_{s=0}.$$

By Eq.(2.1.4) it follows that

$$\frac{d}{dt} Q_i[\mathbf{x}^0, EXP(\alpha t)] = \sum_{k=1}^{n} P_{ik}(\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)])\alpha_k.$$

Thus, $\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)]$ is the unique solution of the V.O.D.E.(2.2.1), as required.

Q.E.D.

Thus, given a local Lie group G and the Lie derivatives $L_\alpha$, $\mathbf{Q}$ is completely determined. However, the following simple example illustrates that the Lie derivatives alone do not <u>uniquely</u> determine a local Lie transformation group.

**Example 2.2.1** Consider the one-dimensional local Lie groups $G_0 = (F, \varphi)$, and $G_1 = (F, \varphi')$ in Example 1.2.2. As noted in Section 1.2, $G_0$ and $G_1$ are locally isomorphic, with isomorphism $\mu: G_0 \to G_1$ such that $\mu(g) = e^g - 1, g \in G_0$. Let $U$ be an an open set in $F_m$, and suppose that $\mathbf{Q}: U x\, G_0 \to F_m$ is a group action on $G_0$. Let $\mathbf{Q}'[\mathbf{x}, \mu(g)] \equiv \mathbf{Q}[\mathbf{x}, g]$ and restrict $G_1$ to the open set around $\mathbf{e}$ for which $\mu(g) \in G_1$, for $g \in G_0$. Then $\mathbf{Q}': U x\, G_1 \to F_m$ is an action on $G_1$ because

1. $\mathbf{Q}'[\mathbf{x}, 0] = \mathbf{Q}'[\mathbf{x}, \mu(0)] = \mathbf{Q}[\mathbf{x}, 0] = \mathbf{x}$.
2. $\mathbf{Q}'[\mathbf{x}, \mu(g)]$ is an analytic function of the m + 1 coordinates of $\mathbf{x}$ and

$\mu(g)$. This follows from the fact that **Q**[**x**, g] is an analytic function of the m + 1 coordinates of **x** and g, and the fact that $g = \mu^{-1}(\mu(g))$ is an analytic function of $\mu(g)$.

3. If **Q'**[**x**, $\mu(g)$] $\in U$, then

$$\mathbf{Q}'[\mathbf{Q}'[\mathbf{x}, \mu(g)], \mu(h)] = \mathbf{Q}[\mathbf{Q}[\mathbf{x}, g], h] = \mathbf{Q}[\mathbf{x}, \varphi(g, h)] = \mathbf{Q}'[\mathbf{x}, \mu(\varphi(g, h))]$$
$$= \mathbf{Q}'[\mathbf{x}, \varphi'(\mu(g), \mu(h))].$$

Clearly, ($G_0$, **Q**) is a local Lie transformation group if and only if ($G_1$, **Q'**) is one as well.

Now let

$$P_{i1}(\mathbf{x}) \equiv \left.\frac{\partial Q_i'[\mathbf{x}, \mu(g)]}{\partial \mu(g)}\right|_{\mu(g)=0}, \qquad i = 1, 2, \ldots, m.$$

By Eq.(2.1.7) the Lie derivatives of ($G_1$, **Q'**) are

$$L_\alpha = \sum_{i=1}^{m} \alpha P_{i1}(\mathbf{x})\frac{\partial}{\partial x_i}, \alpha \in F.$$

However,

$$\left.\frac{\partial Q_i[\mathbf{x}, g]}{\partial g}\right|_{g=0} = \left.\frac{\partial Q_i'[\mathbf{x}, \mu(g)]}{\partial g}\right|_{g=0} = \left.\frac{\partial Q_i'[\mathbf{x}, \mu(g)]}{\partial \mu(g)}\frac{\partial \mu(g)}{\partial g}\right|_{g=0}.$$

Thus,

$$\left.\frac{\partial Q_i[\mathbf{x}, g]}{\partial g}\right|_{g=0} = P_{i1}(\mathbf{x})\left.\frac{d(e^g - 1)}{dg}\right|_{g=0} = P_{i1}(\mathbf{x}).$$

Therefore, by Eq.(2.1.7) the Lie derivatives of ($G_0$, **Q**) are the same as the Lie derivatives of ($G_1$, **Q'**), but the underlying local Lie groups $G_1$ and $G_0$ are different. Thus, the unique solution of Eq.(2.2.1) is not sufficient by itself to uniquely determine a local Lie transformation group. Without knowledge of the one-parameter subgroups EXP($\alpha$t), we do not know how to interpret the solution of the V.O.D.E.(2.2.1).

**Lemma 2.2.1** (See Miller [1, p.191-192]). *If* $f: U \to F$ *is an analytic function in some neighborhood of* $\mathbf{x}^0$, *then*

$$\frac{d^k f(\mathbf{x}(t))}{dt^k} = L_\alpha{}^k f(\mathbf{x}(t)), \qquad k = 0, 1, 2, \ldots \qquad (2.2.3)$$

*where* $\mathbf{x}(t) = \mathbf{Q}[\mathbf{x}^0, \text{EXP}(\alpha t)]$, and $L_\alpha{}^k f(\mathbf{x}(t)) = L_\alpha[L_\alpha{}^{k-1} f(\mathbf{x}(t))]$, $k = 1, 2, 3, \ldots$

**Proof:** $L_\alpha^0 f(\mathbf{x}(t)) = f(\mathbf{x}(t))$. Now consider $\frac{df(\mathbf{x}(t))}{dt}$. Apply the chain rule to obtain

$$\frac{df(\mathbf{x}(t))}{dt} = \sum_{i=1}^{m} \frac{\partial f(\mathbf{x})}{\partial x_i} \frac{dx_i(t)}{dt} .$$

By Eq.'s (2.2.2) and (2.1.7), $f(\mathbf{x}(t))$ satisfies the differential equation

$$\frac{df(\mathbf{x}(t))}{dt} = L_\alpha f(\mathbf{x}(t)). \tag{2.2.4}$$

Since $L_\alpha f(\mathbf{x}(t))$ is itself an analytic function of $\mathbf{x}(t)$, then by Eq.(2.2.4)

$$\frac{d}{dt}\left(\frac{df(\mathbf{x}(t))}{dt}\right) = L_\alpha\left(\frac{df(\mathbf{x}(t))}{dt}\right) = L_\alpha\left(L_\alpha f(\mathbf{x}(t))\right) = L_\alpha^2 f(\mathbf{x}(t)).$$

Eq.(2.2.3) follows from repetition of the above argument k times.

Q.E.D.

Since $f(\mathbf{x}(t))$ is an analytic function of t, we can expand it in a Taylor series about $t = 0$. We obtain the following theorem:

**Theorem 2.2.2** (See Miller [1, p.192], Theorem 5.25). *If f is a function analytic in some neighborhood of* $\mathbf{x}^0 \in U$ *then*

$$f(\mathbf{x}(t)) = \left(\sum_{j=0}^{\infty} \frac{t^j}{j!} L_\alpha^j\right) f(\mathbf{x}^0) \equiv (\exp(tL_\alpha)) f(\mathbf{x}^0),$$

*where* $\mathbf{x}(t) = \mathbf{Q}[\mathbf{x}^0, \mathrm{EXP}(\alpha t)]$.

**Proof:** Expand $f(\mathbf{x}(t))$ in a Taylors series about $t = 0$. Then the result follows immediately from Eq.'s (2.2.3) and (2.1.2).

Q.E.D.

We are now in a position to prove the main result in this section, namely that the set of Lie derivatives of a local Lie transformation group form a Lie algebra of differential operators.

**Theorem 2.2.3** (See Miller [1, p.192], Theorem 5.26). *The set of all Lie derivatives of a local Lie transformation group* (G, **Q**) *form a Lie algebra. In fact, for all* $\alpha, \beta \in L(G)$, *and all* $a, b \in F$

$$L_{(a\alpha+b\beta)} = aL_\alpha + bL_\beta, \tag{2.2.5}$$

*and*

$$L_{[\alpha,\beta]} = L_\alpha L_\beta - L_\beta L_\alpha \equiv [L_\alpha, L_\beta]. \tag{2.2.6}$$

**Proof:** Clearly, the commutator $[L_\alpha, L_\beta] \equiv L_\alpha L_\beta - L_\beta L_\alpha$ satisfies Eq.'s (1.3.4), (1.3.5) and (1.3.6). Thus, to show that the Lie derivatives form a Lie algebra we only need to show that the set of Lie derivatives constitutes a vector space and is closed under the commutator. Thus it is sufficient to show that the Lie derivatives of (G, **Q**) satisfy Eq.'s (2.2.5) and (2.2.6).

Eq.(2.2.5) follows immediately from Lemma 2.1.1. Now consider Eq.(2.2.6). By the definition of a Lie derivative,

$$L_{[\alpha,\beta]} f(\mathbf{x}) = \frac{d}{dt} f(\mathbf{Q}[\mathbf{x}^0, g(t)])\bigg|_{t=0},$$

where $f$ is a function analytic in some neighborhood of $\mathbf{x}^0$ and $g(t)$ is a curve in G with tangent vector $[\alpha, \beta]$. In particular,

$$g(t) = \varphi\big(\varphi(EXP(\alpha\tau), EXP(\beta\tau)), \varphi(EXP(-\alpha\tau), EXP(-\beta\tau))\big),$$

is a curve in G, with tangent vector $[\alpha, \beta]$. Since $\varphi$ is associative, we can write

$$g(t) = EXP(\alpha\tau)EXP(\beta\tau)EXP(-\alpha\tau)EXP(-\beta\tau),$$

where juxtaposition denotes Lie product. Then, by Eq.(2.1.3), with $\tau$ suitably small,

$$f(\mathbf{Q}[\mathbf{x}^0, g(t)]) = f\Big(\mathbf{Q}\big[\mathbf{x}^0, \big(EXP(\alpha\tau)EXP(\beta\tau)EXP(-\alpha\tau)EXP(-\beta\tau)\big)\big]\Big),$$

$$= f\Big(\mathbf{Q}\big[\,\mathbf{Q}[\mathbf{x}^0, EXP(\alpha\tau)EXP(\beta\tau)EXP(-\alpha\tau)], EXP(-\beta\tau)\big]\Big),$$

which by Theorem 2.2.2 becomes

$$f(\mathbf{Q}[\mathbf{x}^0, g(t)]) = (\exp(-\tau L_\beta)) f\Big(\mathbf{Q}\big[\mathbf{x}^0, \big(EXP(\alpha\tau)EXP(\beta\tau)\big)EXP(-\alpha\tau)\big]\Big).$$

Repeat this process until we obtain:

$$f(\mathbf{Q}[\mathbf{x}^0, g(t)]) = (\exp(\tau L_\alpha))(\exp(\tau L_\beta))(\exp(-\tau L_\alpha))(\exp(-\tau L_\beta))f(\mathbf{x}^0).$$

Expand in $\tau$ to get :

$$f(\mathbf{Q}[\mathbf{x}^0, g(t)]) = (1 + \tau^2(L_\alpha L_\beta - L_\beta L_\alpha) + \ldots \text{ terms of order} \geq 3 \text{ in } \tau \ldots)f(\mathbf{x}^0), \quad t = \tau^2.$$

Differentiate with respect to $t$ and evaluate at $t = 0$ to obtain Eq.(2.2.6).

Q.E.D

Thus, the set of Lie derivatives of a local Lie transformation group forms a Lie algebra of differential operators, which we will call $\mathfrak{G}$.

**Definition 2.2.1** Let $(G, \mathbf{Q})$ be a local Lie transformation group with Lie algebra of Lie derivatives $\mathfrak{G}$. If the map $\alpha \to L_\alpha$ is a Lie algebra isomorphism from L(G) onto $\mathfrak{G}$, then $(G, \mathbf{Q})$ acts **effectively** as a local Lie transformation group. (By the nature of the mapping, this is equivalent to saying that $(G, \mathbf{Q})$ acts effectively as a local Lie transformation group if $\dim(L(G)) = \dim(\mathfrak{G})$).

In the next section we discuss differential operators in general. We are working toward proving that any Lie algebra of differential operators is actually the algebra of Lie derivatives of some local Lie transformation group. We conclude this section with the following Lemma, which shows that every local Lie group acts effectively as a local Lie transformation group on itself.

**Lemma 2.2.2** *Let* $G = (V, \varphi)$ *be an* n*-dimensional, local Lie group and let* $\mathbf{Q}$:VxG $\to F_n$ *be defined by* $\mathbf{Q}[\mathbf{g}, \mathbf{h}] = \varphi(\mathbf{g}, \mathbf{h})$. *Then* $(G, \mathbf{Q})$ *is a local Lie transformation group acting effectively on* $V$ *with Lie derivatives*

$$L_\alpha = \sum_{i,k=1}^{n} \alpha_k R_{ik}(\mathbf{g}) \frac{\partial}{\partial g_i}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in L(G), \tag{2.2.7}$$

*where*

$$R_{ik}(\mathbf{g}) = \left.\frac{\partial \varphi_i(\mathbf{g}, \mathbf{h})}{\partial h_k}\right|_{\mathbf{h}=\mathbf{e}}. \tag{2.2.8}$$

**Proof:** $\mathbf{Q} = \varphi$ satisfies Eq.'s (2.1.1), (2.1.2) and (2.1.3), thus $(G, \mathbf{Q})$ is a local Lie transformation group. By Lemma 2.1.1, the Lie derivatives of $(G, \mathbf{Q})$ are

$$L_\alpha = \sum_{i,k=1}^{n} \alpha_k P_{ik}(\mathbf{g}) \frac{\partial}{\partial g_i}, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in L(G),$$

where
$$P_{ik}(\mathbf{g}) = \left.\frac{\partial \varphi_i(\mathbf{g}, \mathbf{h})}{\partial h_k}\right|_{\mathbf{h}=\mathbf{e}} = R_{ik}(\mathbf{g}), \; 1 \le i, k \le n.$$

Let $\mathfrak{G}$ be the algebra of Lie derivatives of ( G, **Q**). Clearly, the map $\alpha \to L_\alpha$ is onto and $[\alpha, \beta] \to L_{[\alpha, \beta]} = [L_\alpha, L_\beta]$.

Suppose $L_\alpha = L_\beta$. Then by Eq.(2.2.7),

$$\sum_{i,k=1}^{n} \alpha_k R_{ik}(\mathbf{g}) \frac{\partial}{\partial g_i} = \sum_{i,k=1}^{n} \beta_k R_{ik}(\mathbf{g}) \frac{\partial}{\partial g_i}.$$

Fix i to consider the coefficient of $\frac{\partial}{\partial g_i}$, and let $\mathbf{g} = \mathbf{e}$. By Eq.'s (2.2.8) and (1.1.3), $R_{ik}(\mathbf{e}) = \delta_{ik}$, thus $\alpha_i = \beta_i$, $1 \le i \le m$, as required. Thus L(G) is isomorphic to $\mathfrak{G}$. Thus, (G, **Q**) acts effectively as a local Lie transformation group.

Q.E.D.

The type of local Lie transformation groups considered in Lemma 2.2.2 will have importance later on it this Chapter as well as in Chapter 3.

## Section 2.3 Lie Algebras of Differential Operators

In Section 2.1, Lemma 2.1.1, we found that a Lie derivative of a local Lie group is a sum of differential operators. We now define precisely what is meant by a differential operator. Let $U$ be an open set in $F_m$ and let $f$ be a function analytic in some neighborhood of $\mathbf{x}^0 \in U$. We let $\mathcal{A}_{\mathbf{x}0}$ be the space of all functions analytic in some neighborhood of $\mathbf{x}^0$, where the neighborhood varies with the function, $f$.

**Definition 2.3.1** We will say that L is a **linear differential operator on** ***U*****,** if

$$L = \sum_{i=1}^{m} P_i(\mathbf{x}) \frac{\partial}{\partial x_i},$$

where $P_i(\mathbf{x})$, $1 \le i \le m$, is a function analytic on $U$ and $Lf(\mathbf{x})$ is defined by

$$Lf(\mathbf{x}) = \sum_{i=1}^{m} P_i(\mathbf{x})\frac{\partial f(\mathbf{x})}{\partial x_i} . \qquad (2.3.1)$$

We say two differential operators, A and B are **equal** on $U$ if $Af(\mathbf{x}) = Bf(\mathbf{x})$ for all $f \in \mathcal{A}_{\mathbf{x}0}$, $\mathbf{x}^0 \in U$ and all $\mathbf{x}$ where $f(\mathbf{x})$ is defined. The zero differential operator, $\theta$, is the differential operator that takes all analytic functions on $U$ to the null function. A set of differential operators, $L_1, L_2, \ldots, L_n$, is **linearly dependent** on $U$ if there exists constants $\alpha_k$, $1 \le k \le n$, not all zero, such that

$$\sum_{k=1}^{n} \alpha_k L_k = \theta$$

on $U$, otherwise the set of differential operators is **linearly independent**.

The product, AB, of two differential operators A and B is defined in the usual manner by $AB(f(x)) = A(B(f(x)))$. The **commutator** of two linear differential operators A and B is denoted by [A, B], and is defined by

$$[A, B] \equiv AB - BA. \qquad (2.3.2)$$

**Lemma 2.3.1** *The commutator of 2 linear differential operators is a linear differential operator.*

**Proof:** Let $A = \sum_{i=1}^{m} P_i(\mathbf{x})\frac{\partial}{\partial x_i}$ and $B = \sum_{k=1}^{m} R_k(\mathbf{x})\frac{\partial}{\partial x_k}$ be any two linear differential operators. By Eq.(2.3.2)

$$[A, B] = \left(\sum_{i=1}^{m} P_i(\mathbf{x})\frac{\partial}{\partial x_i}\right)\left(\sum_{k=1}^{m} R_k(\mathbf{x})\frac{\partial}{\partial x_k}\right) - \left(\sum_{k=1}^{m} R_k(\mathbf{x})\frac{\partial}{\partial x_k}\right)\left(\sum_{i=1}^{m} P_i(\mathbf{x})\frac{\partial}{\partial x_i}\right).$$

Expand it and simplify to obtain:

$$[A, B] = \sum_{k=1}^{m}\left(\sum_{i=1}^{m}\left(P_i(\mathbf{x})\frac{\partial R_k(\mathbf{x})}{\partial x_i} - R_i(\mathbf{x})\frac{\partial P_k(\mathbf{x})}{\partial x_i}\right)\right)\frac{\partial}{\partial x_k} , \qquad (2.3.3)$$

which is a linear differential operator, as required.

Q.E.D.

We now outline conditions under which a set of linear differential operators form a Lie algebra.

**Lemma 2.3.2** *Let* $L_1, L_2, \ldots, L_n$ *be* n *linear differential operators. If there exists constants* $c_{sk}^r$ *such that for* $1 \le s, k, r \le n$,

$$[L_s, L_k] = L_s L_k - L_k L_s = \sum_{r=1}^{n} c_{sk}^r L_r, \tag{2.3.4}$$

*then* $\mathrm{Span}(\{L_k\}_{k=1}^n)$ *is Lie algebra under the linear differential operator commutator.*

**Proof:** $\mathrm{Span}(\{L_k\}_{k=1}^n)$ is the vector space generated by $L_1, L_2, \ldots, L_n$. It is trivial to show that the linear differential operator commutator satisfies Eq.'s (1.3.4), (1.3.5) and (1.3.6). By Eq.(2.3.4) the vector space $\mathrm{Span}(\{L_k\}_{k=1}^n)$ is closed under the linear differential operator commutator. Thus $\mathrm{Span}(\{L_k\}_{k=1}^n)$ is a Lie algebra.

Q.E.D.

Define the linear differential operators

$$L_s = \sum_{i=1}^{m} P_{is}(\mathbf{x}) \frac{\partial}{\partial x_i}, \qquad s = 1, 2, \ldots, n, \tag{2.3.5}$$

acting on functions that are analytic on $U \subset F_m$. If there exists $c_{jk}^r$ such that Eq.(2.3.4) holds then by Lemma 2.3.2,

$$\mathrm{Span}(\{L_k\}_{k=1}^n) = \left\{ \sum_{s=1}^{n} \sum_{i=1}^{m} \alpha_s P_{is}(\mathbf{x}) \frac{\partial}{\partial x_i} \,\middle|\, \alpha_s \in F \right\}$$

forms a Lie algebra over $F$. An important relationship between the $P_{is}(\mathbf{x})$'s and the $c_{jk}^r$'s is given in the following Lemma.

**Lemma 2.3.3** (See Miller [1, p.195-196]). *If the linear differential operators* $L_k$, $1 \le k \le n$, *are defined by* Eq.(2.3.5) *and if there exists constants* $c_{jk}^r$, $1 \le r, j, k \le n$ *such that* Eq.(2.3.4) *is satisfied, then for all* $\mathbf{x} \in U$, *and for* $1 \le q \le m$, $1 \le k, s \le n$,

$$\sum_{i=1}^{m} \left( P_{is}(\mathbf{x}) \frac{\partial P_{qk}(\mathbf{x})}{\partial x_i} - P_{ik}(\mathbf{x}) \frac{\partial P_{qs}(\mathbf{x})}{\partial x_i} \right) = \sum_{r=1}^{n} c_{sk}^r P_{qr}(\mathbf{x}). \tag{2.3.6}$$

**Proof:** Substitute the $L_s$'s as given by Eq.(2.3.5) into Eq.(2.3.4) to obtain:

$$\sum_{q,i=1}^{m} \left( P_{is}(\mathbf{x}) \frac{\partial P_{qk}(\mathbf{x})}{\partial x_i} - P_{ik}(\mathbf{x}) \frac{\partial P_{qs}(\mathbf{x})}{\partial x_i} \right) \frac{\partial}{\partial x_q} = \sum_{q=1}^{m} \left( \sum_{r=1}^{n} c_{sk}^r P_{qr}(\mathbf{x}) \right) \frac{\partial}{\partial x_q},$$

and then obtain Eq.(2.3.6) by equating coefficients of $\frac{\partial}{\partial x_q}$, for $1 \le q \le m$.

Q.E.D

## Section 2.4 Preliminaries Concerning Systems of Differential Equations

We now prove some preliminary results which are used in Section 2.6 and in Chapter 3. Let $V$ be an open set in $F_n$ such that $\mathbf{e} \in V$, and let $U$ be an open, connected subset of $F_m$. Further, let $g \in V$, and $\mathbf{x} \in U$ and let $S_{kj}(g)$ and $P_{ik}(\mathbf{x})$ be analytic functions of $g$ and $\mathbf{x}$, respectively, $1 \le i \le m$, $1 \le j, k \le n$. We are concerned with two different systems of differential equations. Firstly, we are concerned with the V.O.D.E.(2.2.1), where $\mathbf{x}(t)$ is an analytic function of $t$, and $L_\alpha$ is given by Eq.(2.1.7). Secondly, we are concerned with the system of partial differential equations (P.D.E.) of the form

$$\left.\begin{aligned} &\frac{\partial T_q(g)}{\partial g_k} = \sum_{a=1}^{n} P_{qa}(\mathbf{T}) S_{ak}(g), \quad 1 \le k \le n,\ 1 \le q \le m, \\ &T_q(\mathbf{e}) = \mathbf{x}^0, \end{aligned}\right\} \tag{2.4.1}$$

where $P_{qa}(\mathbf{T})$ and $S_{ak}(g)$ are analytic functions of $\mathbf{T} \in U$ and $g \in V$, respectively.

We have already seen that the V.O.D.E.(2.2.1) has a unique solution. The following theorem provides necessary and sufficient conditions for the system of P.D.E.(2.4.1) to have a unique solution.

**Theorem 2.4.1** (See Pontryagin [1, p.398], Theorem 85). *Let* $\mathbf{T} \in U$, $g \in V$ *and let* $\Psi_k^q(\mathbf{T}, g)$ *be an analytic function of* $\mathbf{T}$ *and* $g$. *The system of* P.D.E.

$$\left.\begin{aligned} &\frac{\partial T_q(g)}{\partial g_k} = \Psi_k^q(\mathbf{T}, g), \quad 1 \le k \le n,\ 1 \le q \le m, \\ &T_q(\mathbf{e}) = \mathbf{x}^0,\ \mathbf{x}^0 \in U, \end{aligned}\right\} \tag{2.4.2}$$

*has a unique solution for* $g$ *in a neighborhood of* $\mathbf{e}$, *if and only if, for all* $\mathbf{T} \in U$, $g \in V$, *the following equation is satisfied identically,*

$$\sum_{q=1}^{m} \frac{\partial \Psi^{i}_{k}(\mathbf{T}, g)}{\partial T_q} \Psi^{q}_{r}(\mathbf{T}, g) + \frac{\partial \Psi^{i}_{k}(\mathbf{T}, g)}{\partial g_r} \equiv \sum_{q=1}^{m} \frac{\partial \Psi^{i}_{r}(\mathbf{T}, g)}{\partial T_q} \Psi^{q}_{k}(\mathbf{T}, g) + \frac{\partial \Psi^{i}_{r}(\mathbf{T}, g)}{\partial g_k}, \qquad (2.4.3)$$

*where* $1 \le i \le m$, $1 \le r, k \le n$.

**Proof**: See Pontryagin.

Eq.(2.4.3) is known as the **integrability conditions** for the system of P.D.E.(2.4.2). Since $P_{ir}(\mathbf{T})$ and $S_{rk}(g)$ are analytic functions of $\mathbf{T}$ and $g$, respectively,

$$\sum_{r=1}^{n} P_{ir}(\mathbf{T}) S_{rk}(g), \quad 1 \le i \le m,\ 1 \le k \le n,$$

is analytic in both $\mathbf{T}$ and g. Thus we can apply Theorem 2.4.1 to the system of P.D.E.(2.4.1), with

$$\Psi^{i}_{k}(\mathbf{T}, g) = \sum_{r=1}^{n} P_{ir}(\mathbf{T}) S_{rk}(g), \qquad 1 \le i \le m,\ 1 \le k \le n.$$

We obtain the following result.

**Corollary 2.4.1** (See Miller [1, p.195], Eq.(9.25)). *The system of* P.D.E.(2.4.1) *has a unique solution if, for* $1 \le i \le m$, $1 \le k, r \le n$, *the following integrability conditions hold for all* $\mathbf{T} \in U$, $g \in V$:

$$\sum_{a,s=1}^{n} \left( \sum_{q=1}^{m} P_{qa}(\mathbf{T}) \frac{\partial P_{is}(\mathbf{T})}{\partial T_q} - P_{qs}(\mathbf{T}) \frac{\partial P_{ia}(\mathbf{T})}{\partial T_q} \right) S_{sr}(g) S_{ak}(g)$$
$$= \sum_{a=1}^{n} P_{ia}(\mathbf{T}) \left( \frac{\partial S_{ak}(g)}{\partial g_r} - \frac{\partial S_{ar}(g)}{\partial g_k} \right) \qquad (2.4.4)$$

**Proof:** By Theorem 2.4.1, the system of P.D.E.(2.4.1) has a unique solution if Eq.(2.4.3) holds for

$$\Psi^{i}_{k}(\mathbf{T}, g) = \sum_{r=1}^{n} P_{ir}(\mathbf{T}) S_{rk}(g).$$

Substitute for $\Psi^{i}_{k}(\mathbf{T}, g)$ in Eq.(2.4.3) to obtain the equivalent expression,

$$\sum_{a,s=1}^{n} \sum_{q=1}^{m} \frac{\partial P_{ia}(\mathbf{T})}{\partial T_q} S_{ak}(g) P_{qs}(\mathbf{T}) S_{sr}(g) + \sum_{a=1}^{n} P_{ia}(\mathbf{T}) \frac{\partial S_{ak}(g)}{\partial g_r}$$
$$= \sum_{a,s=1}^{n} \sum_{q=1}^{m} \frac{\partial P_{is}(\mathbf{T})}{\partial T_q} S_{ak}(g) P_{qa}(\mathbf{T}) S_{sr}(g) + \sum_{a=1}^{n} P_{ia}(\mathbf{T}) \frac{\partial S_{ak}(g)}{\partial g_r}.$$

Rearrange to obtain Eq.(2.4.4).

Q.E.D.

Our choice of $P_{is}(\mathbf{x})$ and $S_{sk}(g)$ will guarantee that the V.O.D.E.(2.2.1) and the system of P.D.E.(2.4.1) have the same unique solution. We begin by defining the $S_{sk}(g)$ in terms of invertible matrices as follows. Let $R_{sk}(g)$ be an analytic function for $g \in V$, $1 \le s, k \le n$. For $g \in V$, define the nxn matrix $\mathbf{R}(g)$ by

$$\mathbf{R}(g) = \left(R_{sk}(g)\right)_{s,k=1}^{n} . \qquad (2.4.5)$$

Assume that if $g \in V$, $\mathbf{R}(g)$ has a matrix inverse, $\mathbf{S}(g) = \left(S_{sk}(g)\right)_{s,k=1}^{n}$. That is, for all $g \in V$,

$$\mathbf{S}(g)\mathbf{R}(g) = E_n, \qquad (2.4.6)$$

where $E_n$ is the nxn identity matrix. Now suppose that there exists constants $c_{sk}^{r}$ such that the $R_{sk}(g)$ satisfy Eq.(2.3.6) with $m = n$. That is, assume

$$\sum_{i=1}^{n} \left( R_{is}(g) \frac{\partial R_{qk}(g)}{\partial g_i} - R_{ik}(g) \frac{\partial R_{qs}(g)}{\partial g_i} \right) = \sum_{r=1}^{n} c_{sk}^{r} R_{qr}(g), \quad 1 \le k, q, s \le n. \qquad (2.4.7)$$

We are interested in the form that Eq.(2.4.7) takes when written in terms of the $S_{ik}(g)$'s.

**Lemma 2.4.1** (See Miller [1, p.195], Equation 9.26). *For* $g \in V$, *let* $R_{ik}(g)$ *and* $S_{ik}(g)$ *satisfy* Eq.(2.4.6). $R_{ik}(g)$ *satisfies* Eq.(2.4.7) *if and only if* $S_{ik}(g)$ *satisfies*

$$\left.\begin{aligned} &\frac{\partial S_{iq}(g)}{\partial g_k} - \frac{\partial S_{ik}(g)}{\partial g_q} = \sum_{r,s=1}^{n} c_{rs}^{i} S_{rq}(g) S_{sk}(g), \quad 1 \le i, k, q \le n, \\ &\textit{with } c_{rs}^{i} = -c_{sr}^{i}. \end{aligned}\right\} \qquad (2.4.8)$$

**Proof**: Eq.(2.4.6) implies that, for $g \in V$,

$$\sum_{a=1}^{n} S_{ia}(g) R_{ak}(g) = \sum_{a=1}^{n} R_{ia}(g) S_{ak}(g) = \delta_{ik}, \qquad 1 \le i, k \le n, \qquad (2.4.9)$$

and, differentiating both sides of Eq.(2.4.6) with respect to $g_i$ yields the identity:

$$\frac{\partial R_{bk}(g)}{\partial g_i} = -\sum_{a,r=1}^{n} R_{ba}(g)\frac{\partial S_{ar}(g)}{\partial g_i} R_{rk}(g), \quad 1 \le i, b, k \le n. \tag{2.4.10}$$

(⇐) First we will show that Eq.(2.4.8) implies Eq.(2.4.7). By using Eq.(2.4.10), it is easy to show that

$$\sum_{i=1}^{n} \left(R_{is}(g) \frac{\partial R_{qk}(g)}{\partial g_i} - R_{ik}(g)\frac{\partial R_{qs}(g)}{\partial g_i}\right) = \sum_{i,a,r=1}^{n} -R_{rs}(g)\, R_{qa}(g)\, R_{ik}(g) \left(\frac{\partial S_{ai}(g)}{\partial g_r} - \frac{\partial S_{ar}(g)}{\partial g_i}\right)$$

Substitute for $\left(\frac{\partial S_{ai}(g)}{\partial g_r} - \frac{\partial S_{ar}(g)}{\partial g_i}\right)$ from Eq.(2.4.8), and simplify using Eq.(2.4.9) to conclude,

$$\sum_{i=1}^{n} \left(R_{is}(g) \frac{\partial R_{qk}(g)}{\partial g_i} - R_{ik}(g)\frac{\partial R_{qs}(g)}{\partial g_i}\right) = \sum_{i,a,r=1}^{n} -R_{rs}(g)R_{qa}(g)R_{ik}(g)\left(\sum_{b,t=1}^{n} c_{bt}^{a} S_{bi}(g)S_{tr}(g)\right)$$

$$= -\sum_{a,b,t=1}^{n} \left(\delta_{ts}R_{qa}(g)\delta_{kb}c_{bt}^{a}\right),$$

$$= -\sum_{a=1}^{n} \left(R_{qa}(g)c_{ks}^{a}\right)$$

Thus,

$$\sum_{i=1}^{n} \left(R_{is}(g) \frac{\partial R_{qk}(g)}{\partial g_i} - R_{ik}(g)\frac{\partial R_{qs}(g)}{\partial g_i}\right) = \sum_{a=1}^{n} c_{sk}^{a}R_{qa}(g).$$

Hence, Eq.(2.4.8) implies Eq.(2.4.7).

(⇒) To prove the converse, assume Eq.(2.4.7). It follows immediately that $c_{rs}^{i} = -c_{sr}^{i}$.

Substitute Eq.(2.4.10) into Eq.(2.4.7) to obtain

$$\sum_{a,r,i=1}^{n} R_{ik}(g)R_{qa}(g)R_{rs}(g)\left(\frac{\partial S_{ar}(g)}{\partial g_i} - \frac{\partial S_{ai}(g)}{\partial g_r}\right) = \sum_{r=1}^{n} c_{sk}^{r}R_{qr}(g). \tag{2.4.11}$$

Multiply both sides of Eq.(2.4.11) by $S_{sx}(g)S_{yq}(g)S_{kz}(g)$ and take the sum as q, s and k go from 1 to n. Simplify using Eq.(2.4.9) to obtain

$$\frac{\partial S_{yx}(g)}{\partial g_z} - \frac{\partial S_{yz}(g)}{\partial g_x} = \sum_{k,s=1}^{n} c_{sk}^{y} S_{sx}(g) S_{kz}(g).$$

This is Eq.(2.4.8) under a change of variables.

Q.E.D.

## Section 2.5 Preliminaries Concerning Local Lie Groups and Lie Derivatives

We shall now deal with some results involving local Lie groups, and local Lie transformation groups. We apply the general material of Section 2.4, as needed. We use Lemma 2.2.2 in the following Lemma.

**Lemma 2.5.1** (See Miller [1, p.195-196]). *Let* $G = (V, \varphi)$ *be an* $n$*-dimensional local Lie group. Let* $R_{jk}(g)$ *be obtained from* Eq.(2.2.8) *and let* $c_r^{jk}$ *be the structure constants of* G, $1 \le j, k, r \le n$. *Then the* $R_{jk}(g)$ *satisfy* Eq.(2.4.7) *with* $c_r^{jk} = c_{jk}^{r}$.

**Proof**: By Lemma 2.2.2, (G, $\varphi$), is a local Lie transformation group with Lie derivatives

$$L_\alpha = \sum_{a,k=1}^{n} \alpha_k R_{ak}(g) \frac{\partial}{\partial g_a}, \quad \alpha \in F_n.$$

Let $B_k = (\delta_{1k}, \delta_{2k}, \ldots, \delta_{nk})$, $1 \le k \le n$. Then $L_k \equiv L_{B_k} = \sum_{a=1}^{n} R_{ak}(g)\frac{\partial}{\partial g_a}$. Thus,

$$[L_j, L_k] = L_jL_k - L_kL_j = \left(\sum_{a=1}^{n} R_{aj}(g)\frac{\partial}{\partial g_a}\right)\left(\sum_{s=1}^{n} R_{sk}(g)\frac{\partial}{\partial g_s}\right) - \left(\sum_{a=1}^{n} R_{ak}(g)\frac{\partial}{\partial g_a}\right)\left(\sum_{s=1}^{n} R_{sj}(g)\frac{\partial}{\partial g_s}\right)$$

$$= \sum_{s,a=1}^{n} \left(R_{aj}(g)\frac{\partial R_{sk}(g)}{\partial g_a} - R_{ak}(g)\frac{\partial R_{sj}(g)}{\partial g_a}\right)\frac{\partial}{\partial g_s}. \tag{2.5.1}$$

However, by Eq.(2.2.6),

$$[L_j, L_k] = L_{[B_j, B_k]} = \sum_{r,s=1}^{n} [B_j, B_k]_r R_{sr}(g) \frac{\partial}{\partial g_s}. \tag{2.5.2}$$

Now, from Theorem 1.3.1, $[\beta_j, \beta_k]_r = c_r^{jk}$, $1 \le j, k, r \le n$. Substitute $[\beta_j, \beta_k]_r = c_r^{jk}$ into

Eq.(2.5.2) and compare the resulting equation with Eq.(2.5.1). Fix s and consider coefficients of $\frac{\partial}{\partial g_s}$ to obtain Eq.(2.4.7).

Q.E.D.

For $g \in V$, define the nxn matrix $\mathbf{R}(g)$ by

$$\mathbf{R}(g) = \left(R_{jk}(g)\right)_{j,k=1}^{n} . \tag{2.5.3}$$

Now, it follows from Eq.'s (2.2.8) and (1.1.3) that $R_{jk}(\mathbf{e}) = \delta_{jk}$, so $\mathbf{R}(\mathbf{e}) = E_n$, the nxn identity matrix. Thus, the determinant of $\mathbf{R}(\mathbf{e})$, $\det|\mathbf{R}(\mathbf{e})| = 1$. Since the $R_{jk}(g)$, $1 \le j, k \le n$ are analytic functions of $g_1, g_2, \ldots, g_n$, there exists a neighborhood, *W*, around $\mathbf{e}$ in which the determinant of $\mathbf{R}(g)$ is non-zero. Thus, the matrix inverse of $\mathbf{R}(g)$, $\mathbf{R}^{-1}(g)$ exists for $g \in W$. We will denote the matrix inverse of $\mathbf{R}(g)$ by $\mathbf{S}(g) = \left(S_{jk}(g)\right)_{j,k=1}^{n}$. Then for $g \in W$, Eq.(2.4.6) holds.

**Corollary 2.5.1** *The* $S_{jk}(g)$'s *satisfy* Eq.(2.4.8), *for* $g \in W$, $1 \le j, k \le n$.

**Proof:** The proof follows immediately from Lemmas 2.4.1 and 2.5.1, and the definition of $S_{jk}(g)$.

Q.E.D.

Unlike the V.O.D.E.(2.2.1) which yields the action of a local Lie transformation group, the P.D.E.(2.4.1) has thus far had no relation to our discussion of Lie derivatives. This next Lemma illustrates the importance of the solution of the P.D.E.(2.4.1).

**Lemma 2.5.2** (See Miller [1, p.194], Lemma 5.5). *Let* (G, **Q**) *be an* n-*dimensional local Lie transformation group acting on* $U \subset F_m$, *where the action* **Q** *is found by solving the V.O.D.E.* (2.2.1). *Then* $\mathbf{Q}[\mathbf{x}, g]$ *satisfies the following system of partial differential equations,*

$$\frac{\partial Q_i[\mathbf{x}, g]}{\partial g_k} = \sum_{r=1}^{n} P_{ir}(\mathbf{Q}[\mathbf{x}, g]) S_{rk}(g), \quad 1 \le i \le m,\ 1 \le k \le n, \tag{2.5.4}$$

*where* $P_{ir}(\mathbf{x})$ *is given by* Eq.(2.1.5), $R_{jk}(g)$ *is given by* Eq.(2.2.8) *and*

$$\mathbf{S}(g) = \left(S_{jk}(g)\right)_{j,k=1}^{n} = \mathbf{R}^{-1}(g),$$

*for g in some open neighborhood, W, of* $\mathbf{e} \in G$.

**Proof:** By Eq.(2.1.3), for $t$ close to 0, $g$ close to $\mathbf{e}$,

$$Q_i[\mathbf{x}, \varphi(g, EXP(\alpha t))] = Q_i[Q[\mathbf{x}, g], EXP(\alpha t)].$$

Differentiate this expression with respect to $t$, evaluate at $t = 0$ and compare coefficients of $\alpha_s$, $1 \le s \le n$, to obtain the identity:

$$\sum_{j=1}^{n} \frac{\partial Q_i[\mathbf{x}, g]}{\partial g_j} R_{js}(g) = P_{is}(Q[\mathbf{x}, g]), \quad 1 \le i \le m, 1 \le s \le n, \tag{2.5.5}$$

which holds for all $Q[\mathbf{x}, g] \in U$. Use the fact that $\mathbf{S}(g) = \mathbf{R}^{-1}(g)$, for $g \in W$, to transform Eq.(2.5.5) into Eq.(2.5.4), as required.

Q.E.D.

**Lemma 2.5.3** *Let* $G = (V, \varphi)$ *be an* $n$*-dimensional local Lie group with structure constants ,* $c_r^{jk}$, $1 \le r, j, k \le n$ *and let* $\{L_k\}_{k=1}^{n}$ *be* $n$ *linearly independent differential operators defined on* $U \subset F_m$ *satisfying* Eq.(2.3.5). *If the* $\{L_k\}_{k=1}^{n}$ *satisfy* Eq.(2.3.4) *with* $c_{jk}^{r} = c_r^{jk}$, *then the system of* P.D.E.(2.4.1) *has a unique solution, where* $S(g) = \mathbf{R}^{-1}(g)$ *and* $R_{jk}(g)$ *is obtained from* Eq.(2.2.8).

**Proof:** Let $G$ be a local Lie group of the required form, let $W$ be an open set about $\mathbf{e}$ where $\mathbf{S}(g)$ is defined and let $c_{jk}^{r} = c_r^{jk}$, $1 \le j, k, r \le n$. By Lemma 2.3.3, Eq.(2.3.6) holds and by Lemma 2.5.1, Eq.(2.4.7) holds. Thus by Lemma 2.4.1, Eq.(2.4.8) holds as well.

Now, from Eq.'s(2.3.6) and (2.4.8), it follows that

$$\sum_{a,s,q=1}^{n} \left(P_{qa}(\mathbf{T})\frac{\partial P_{is}(\mathbf{T})}{\partial T_q} - P_{qs}(\mathbf{T})\frac{\partial P_{ia}(\mathbf{T})}{\partial T_q}\right) S_{sr}(g)S_{ak}(g) = \sum_{a,s,q=1}^{n} c_q^{as} P_{iq}(\mathbf{T}) S_{sr}(g) S_{ak}(g)$$

$$= \sum_{q=1}^{n} P_{iq}(\mathbf{T}) \left( \sum_{a,s=1}^{n} c_q^{as} S_{sr}(g) S_{ak}(g) \right)$$

$$= \sum_{q=1}^{n} P_{iq}(\mathbf{T}) \left( \frac{\partial S_{qk}(g)}{\partial g_r} - \frac{\partial S_{qr}(g)}{\partial g_k} \right).$$

Thus, Eq.(2.4.4) is satisfied for $g \in W$, $\mathbf{T} \in U$. Thus, by Corollary 2.4.1, the system of P.D.E.(2.4.1) has a unique solution.

Q.E.D.

Lemma 2.5.2 illustrates that a group action $\mathbf{Q}$ satisfies both the V.O.D.E.(2.2.1) and the system of P.D.E.(2.4.1), with $\mathbf{T}(g) = \mathbf{T}(\mathbf{x}^0, g)$. We wish to verify this relationship between solutions in general. Consider the unique solution of the system of P.D.E.(2.4.1). Obviously, it is a function of $g$ and $\mathbf{x}^0$, let us denote it by $\mathbf{W}(\mathbf{x}^0, g)$.

**Lemma 2.5.4** *Let* $P_{ik}(\mathbf{x})$, $R_{jk}(g)$ *and* $S_{kj}(g)$ *be as given in* Lemma 2.5.2. $\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t))$ *is the unique solution of the* V.O.D.E.(2.2.1) *if and only if* $\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t))$ *is the unique solution of the system of* P.D.E.(2.4.1).

**Proof:** ($\Leftarrow$) We have already seen that for $\mathbf{x}^0 \in U$ and $\mathrm{EXP}(\alpha t) \in V$, both the V.O.D.E.(2.2.1) and the system of P.D.E.(2.4.1) have a unique solution. Assume $\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t))$ is the unique solution of the system of P.D.E.(2.4.1). By the chain rule,

$$\frac{dW_i(\mathbf{x}^0,\mathrm{EXP}(\alpha t))}{dt} = \sum_{k=1}^{n} \frac{\partial W_i(\mathbf{x}^0, \mathrm{EXP}(\alpha t))}{\partial \mathrm{EXP}_k(\alpha t)} \frac{d(\mathrm{EXP}_k(\alpha t))}{dt}.$$

However, $\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t))$ is a solution of the system of P.D.E.(2.4.1) and

$$\frac{d\mathrm{EXP}_k(\alpha t)}{dt} = \left.\frac{d\mathrm{EXP}_k(\alpha(t+s))}{ds}\right|_{s=0}$$

$$= \sum_{r=1}^{n} \frac{\partial \varphi_k(\mathrm{EXP}(\alpha t), \mathrm{EXP}(\alpha s))}{\partial \mathrm{EXP}_r(\alpha s)} \left.\frac{d\mathrm{EXP}_r(\alpha s)}{ds}\right|_{s=0} = \sum_{r=1}^{n} R_{kr}(\mathrm{EXP}(\alpha t))\alpha_r.$$

It follows that:

$$\frac{dW_i(\mathbf{x}^0, \mathrm{EXP}(\alpha t))}{dt} = \sum_{k,j,r=1}^{n} P_{ij}(\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t)))S_{jk}(\mathrm{EXP}(\alpha t))\alpha_r R_{kr}(\mathrm{EXP}(\alpha t)).$$

By Eq.(2.4.9) this simplifies to:

$$\frac{dW_i(\mathbf{x}^0, \mathrm{EXP}(\alpha t))}{dt} = \sum_{j=1}^{n} \alpha_j P_{ij}(\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t))).$$

Since $\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha 0)) = \mathbf{x}^0$, $\mathbf{W}(\mathbf{x}^0, \mathrm{EXP}(\alpha t))$ is a solution of the V.O.D.E.(2.2.1).

($\Rightarrow$) Fix $\mathbf{x}^0 \in U$ and $\text{EXP}(\alpha t) \in V$. Let $\overline{\mathbf{W}}(\mathbf{x}^0, \text{EXP}(\alpha t))$ be the solution of the V.O.D.E.(2.2.1), which we know is unique, (see Petrovski [1, p.96]). But from the above argument, $\mathbf{W}(\mathbf{x}^0, \text{EXP}(\alpha t))$ is a solution of the P.D.E.(2.4.1) as well as the V.O.D.E.(2.2.1). Consequently,

$$\overline{\mathbf{W}}(\mathbf{x}^0, \text{EXP}(\alpha t)) = \mathbf{W}(\mathbf{x}^0, \text{EXP}(\alpha t))$$

is the solution as that of the system of P.D.E.(2.4.1), as required.

Q.E.D.

The system of P.D.E.(2.4.1) is more powerful than the V.O.D.E.(2.2.1) in the sense that its solution is defined for all elements of G and not simply for one-parameter subgroups. We will be using this fact in the next section to prove that a Lie algebra of differential operators is an algebra of Lie derivatives.

## Section 2.6 Lie Algebras of Differential Operators and Lie Derivatives

We are now ready to show that every Lie algebra of differential operators is the algebra of Lie derivatives of some local Lie transformation group. Let $U$ be an open, connected set such that $U \subset F_m$.

**Theorem 2.6.1** (See Miller [1, p.194], Theorem 5.27). *Let* $\{L_s\}_{s=1}^{n}$ *be* n *linearly independent differential operators defined by* Eq.(2.3.5), *and analytic on some open, connected set* $U$ *of* $F_m$. *Suppose there exists* $c_{sk}^{r}$ *such that* Eq.(2.3.4) *holds,* $1 \le s, k, r \le n$. *Then the* n*-dimensional Lie algebra,* $\mathfrak{G}$, *generated by the* $L_s$ *is the algebra of Lie derivatives of a local Lie transformation group* (G, $\mathbf{Q}$), *acting effectively on* $U$. $\mathbf{Q}[\mathbf{x}^0, \text{EXP}(\alpha t)]$ *is the unique solution of the* V.O.D.E.(2.2.1), *for* $\mathbf{x}^0 \in U$.

**Proof:** Since $L_s$, $1 \le s \le n$, are linearly independent, it follows that the $\{L_s\}_{s=1}^{n}$ form the basis of a Lie algebra $\mathfrak{G}$. Express $L_\alpha \in \mathfrak{G}$ by

$$L_\alpha = \sum_{s=1}^{n} \alpha_s L_s . \tag{2.6.1}$$

Clearly $\mathfrak{G}$ is isomorphic to a second Lie algebra, $\mathfrak{G}'$, with isomorphism given by

$L_\alpha \to \alpha$. By Theorem 1.5.5 and Corollary 1.5.2, $\mathfrak{G}'$ is the Lie algebra of some n-dimensional local Lie group G = (*V*, φ), unique up to isomorphism. It follows that the structure constants of G are the $c^r_{sk}$ of Eq.(2.3.4).

By Lemma 1.4.2, we can label the elements of G by means of the exponential mapping. Once we have shown that the solution of the V.O.D.E.(2.2.1), $\mathbf{Q}[\mathbf{x}^0, \mathrm{EXP}(\alpha t)]$, does indeed produce an action on G, then Theorem 2.2.1 guarantees that $\mathfrak{G}$ is the algebra of Lie derivatives of (G, **Q**). Thus, we need only show that **Q** is an action.

Let $R_{ks}(g)$, $1 \le s, k \le n$, be obtained from Eq.(2.2.8), limit g to an open neighborhood of **e** for which the inverse element exists and let $S_{ks}(g)$ be the elements of the matrix inverse of $\left(R_{ks}(g)\right)^n_{k,s=1}$. Let $P_{is}(\mathbf{x})$, $1 \le i \le m$, be obtained from the differential operator $L_s$, $1 \le s \le n$, using Eq.(2.3.5). Consider the system of P.D.E.(2.4.1).

By Lemma 2.5.3, the system of P.D.E.(2.4.1) has a unique solution, which we now define to be $\mathbf{Q}[\mathbf{x}^0, g]$. Let g = EXP(αt). Then by Lemma 2.5.4, $\mathbf{Q}[\mathbf{x}^0, \mathrm{EXP}(\alpha t)]$ is the unique solution of both the V.O.D.E.(2.2.1) and the system of P.D.E.(2.4.1). By definition, $\mathbf{Q}[\mathbf{x}^0, \mathbf{e}] = \mathbf{x}^0$. We now show **Q** is associative.

Let $\mathbf{y}(t) = \mathbf{Q}[\mathbf{Q}[\mathbf{x}^0, \mathrm{EXP}(\alpha)], \mathrm{EXP}(\beta t)]$, and let $\mathbf{z}(t) = \mathbf{Q}[\mathbf{x}^0, \varphi(\mathrm{EXP}(\alpha), \mathrm{EXP}(\beta t))]$. Clearly, $\mathbf{y}(0) = \mathbf{z}(0) = \mathbf{Q}[\mathbf{x}^0, \mathrm{EXP}(\alpha)]$. By the definition of **Q**,

$$\frac{dy_i(t)}{dt} = \sum_{s=1}^{n} \beta_s P_{is}(\mathbf{y}(t)), \qquad 1 \le i \le m.$$

Now, by the chain rule of differentiation,

$$\frac{dz_i(t)}{dt} = \sum_{q=1}^{n} \left(\frac{\partial Q_i[\mathbf{x}^0, \varphi(\mathrm{EXP}(\alpha), \mathrm{EXP}(\beta t))]}{\partial \varphi_q(\mathrm{EXP}(\alpha), \mathrm{EXP}(\beta t))}\right) \frac{d\varphi_q(\mathrm{EXP}(\alpha), \mathrm{EXP}(\beta t))}{dt}.$$

But

$$\frac{d\varphi_q(\mathrm{EXP}(\alpha), \mathrm{EXP}(\beta t))}{dt} = \left.\frac{d\varphi_q(\mathrm{EXP}(\alpha), \mathrm{EXP}(\beta(t+s)))}{ds}\right|_{s=0}$$

$$= \left. \frac{d\varphi_q(\varphi(EXP(\alpha), EXP(\beta t)), EXP(\beta s))}{ds} \right|_{s=0}$$

$$= \sum_{r=1}^{n} \left. \frac{\partial\varphi_q(\varphi(EXP(\alpha), EXP(\beta t)), EXP(\beta s))}{\partial EXP_r(\beta s)} \frac{dEXP_r(\beta s)}{ds} \right|_{s=0} .$$

Thus,

$$\frac{d\varphi_q(EXP(\alpha), EXP(\beta t))}{dt} = \sum_{r=1}^{n} R_{qr}(\varphi(EXP(\alpha), EXP(\beta t)))\beta_r. \tag{2.6.2}$$

By Eq.(2.6.2) and the fact that $\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)]$ satisfies the P.D.E.(2.4.1),

$$\frac{dz_i(t)}{dt} = \sum_{q,r,s=1}^{n} P_{is}(z(t))S_{sq}(\varphi(EXP(\alpha), EXP(\beta t)))R_{qr}(\varphi(EXP(\alpha), EXP(\beta t)))\beta_r .$$

By Eq.(2.4.9) this simplifies to the following,

$$\frac{dz_i(t)}{dt} = \sum_{r=1}^{n} P_{ir}(\mathbf{z}(t))\beta_r , \quad 1 \le i \le m.$$

Since $\mathbf{y}(t)$ and $\mathbf{z}(t)$ satisfy the same differential equation and initial conditions, and by the uniqueness of solution of a V.O.D.E., (Petrovski [1, p.96]), $\mathbf{y}(t) \equiv \mathbf{z}(t)$. Although our argument only works for $t$ close to 0, analytic function theory, (see Hille, [1, p.3 ], Law of Permanence of Functional Equations) allows us to extend it to $t = 1$. Thus, $\mathbf{Q}$ is associative, as required.

Finally, we need to verify that $\mathbf{Q}$ is an analytic function of the $m + n$ components of $\mathbf{x}$ and $g$. $\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)]$ is a solution of the V.O.D.E.(2.2.1),

$$\frac{d\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)]}{dt} = L_\alpha \mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)],$$

where $L_\alpha$ is given by Eq.(2.6.1). Thus, by an argument similar to that employed for the proof of Lemma 2.2.1, it follows that

$$\frac{d^k\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)]}{dt^k} = L_\alpha{}^k\mathbf{Q}[\mathbf{x}^0, EXP(\alpha t)], \quad k = 0, 1, \ldots$$

Thus,

$$Q[\mathbf{x}, EXP(\alpha t)] = \exp tL_\alpha \mathbf{x} = \left(\sum_{j=0}^{\infty} \frac{t^j}{j!} L_\alpha^j\right)\mathbf{x}.$$

Now,

$$L_\alpha^0 \mathbf{x} = \mathbf{x}$$

$$L_\alpha \mathbf{x} = \sum_{i=1}^{m} \sum_{k=1}^{n} \alpha_k P_{ik}(\mathbf{x}) \frac{\partial \mathbf{x}}{\partial x_i},$$

$$L_\alpha(L_\alpha \mathbf{x}) = \sum_{i,q=1}^{m} \sum_{k,l=1}^{n} \alpha_k \alpha_l P_{ql}(\mathbf{x}) \frac{\partial P_{ik}(\mathbf{x})}{\partial x_q} \frac{\partial \mathbf{x}}{\partial x_i},$$

and

$$L_\alpha(L_\alpha(L_\alpha \mathbf{x})) = \sum_{i,q,r=1}^{m} \sum_{j,k,l=1}^{n} \alpha_j \alpha_k \alpha_l P_{rj}(\mathbf{x}) \left[\frac{\partial P_{ql}(\mathbf{x})}{\partial x_r} \frac{\partial P_{ik}(\mathbf{x})}{\partial x_q} \frac{\partial \mathbf{x}}{\partial x_i} + P_{ql}(\mathbf{x}) \frac{\partial^2 P_{ik}(\mathbf{x})}{\partial x_r \partial x_q} \frac{\partial \mathbf{x}}{\partial x_i}\right],$$

etc.

Obviously, $L_\alpha^m \mathbf{x}$ is both a homogeneous polynomial of order $m$ in $\alpha_1, \alpha_2, \ldots, \alpha_n$, and an analytic function of $\mathbf{x}$. Thus, for $EXP(\alpha t) \in V$, $Q[\mathbf{x}^0, EXP(\alpha t)]$ is an analytic function of the $m$ components of $\mathbf{x}^0$ and $\alpha_1 t, \alpha_2 t, \ldots, \alpha_n t$. Therefore, $Q[\mathbf{x}^0, EXP(\alpha)]$ is an analytic function of $\mathbf{x}^0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since the exponential map is locally one-to-one and analytic we conclude that $Q[\mathbf{x}^0, EXP(\alpha)]$ is an analytic function of the components of $\mathbf{x}$ and $EXP(\alpha)$, as required.

Q.E.D.

Thus, every Lie algebra of differential operators is the set of all Lie derivatives of some local Lie transformation group (G, **Q**) acting effectively on *U*, and the set of all Lie derivatives of a local Lie transformation group (G, **Q**) is a Lie algebra. However, Theorem 2.6.1 only shows that the action $Q[\mathbf{x}^0, EXP(\alpha t)]$ can be found by solving the V.O.D.E.(2.2.1). It does not say anything about how one would use the Lie algebra of differential operators $Span(\{L_k\}_{k=1}^n)$ to construct the underlying local Lie group G. We only know by Theorem 1.5.5 and Corollary 1.5.3 that G exists. In Chapter 3, we consider a special type of Lie algebra of differential operators and adapt the proof of Theorem 2.6.1 to actually construct a local Lie group from a Lie algebra of differential operators, without using the material of Section 1.5.

# Chapter 3

# The Construction of a Local Lie Group From Its Lie Algebra

## Section 3.1 Introduction and Examples

We continue with the notation of Chapters One and Two. That is, we shall let $F$ be either the field of complex or real numbers, $F_n$ be the space of n-tuples with coordinates from $F$, and let $V$ be an open set in $F_n$ containing $\mathbf{e} = (0, 0, \ldots, 0)$.

In Section 2.6, Theorem 2.6.1, we presented Miller's proof that any Lie algebra of linear differential operators is the set of all Lie derivatives of some local Lie transformation group, (G, $\mathbf{Q}$). Theorem 2.6.1 only shows that the action $\mathbf{Q}$ of a local Lie transformation group (G, $\mathbf{Q}$) can be found by solving a system of differential equations. It tells us nothing about the Lie product, $\varphi$, that makes G into a local Lie group in some neighborhood of $\mathbf{e} \in F_n$. Instead, Theorem 2.6.1 relies on Theorem 1.5.5 to guarantee the existence of an underlying local Lie group G, such that (G, $\mathbf{Q}$) is a local Lie transformation group with the prescribed Lie derivatives.

In this Chapter we restrict the Lie algebra of differential operators to Lie algebras having a basis $\{L_k\}_{k=1}^{n}$, of the form

$$L_k = \sum_{i=1}^{n} P_{ik}(\mathbf{x})\frac{\partial}{\partial x_i}, \qquad 1 \le k \le n, \tag{3.1.1}$$

where $P_{ik}(\mathbf{g})$ is a function defined and analytic for all $\mathbf{g}$ in some open, connected neighborhood, $V$ of $\mathbf{e}$, such that

$$P_{ik}(\mathbf{e}) = \delta_{ik} \quad 1 \le i, k \le n. \tag{3.1.2}$$

We show that we can construct a Lie product $\varphi$ from the solution of a system of ordinary differential equations such that $G = (V, \varphi)$ is a local Lie group and $(G, \varphi)$ is a local Lie transformation group whose Lie algebra of Lie derivatives has basis $\{L_k\}_{k=1}^{n}$.

Our proof is an adaptation of Miller's proof of Theorem 2.6.1. This result differs from Theorem 2.6.1 in that we do not require Theorem 1.5.5 and we actually find both an underlying local Lie group, G, and a local Lie transformation group, (G, $\mathbf{Q}$). Let us call the Lie algebra generated by $\{L_k\}_{k=1}^{n}$, $\mathcal{L} = \text{Span}\left(\{L_k\}_{k=1}^{n}\right)$, and as in Section 2.6, we let

$$L_\alpha = \sum_{k=1}^{n} \alpha_k L_k, \text{ for all } \alpha \in F_n.$$

We also show that L(G), the Lie algebra of G, is isomorphic to $\mathcal{L}$. This provides a constructive proof that every Lie algebra is the Lie algebra of some local Lie group, up to isomorphism in the case when the Lie algebra is realized by $\mathcal{L}$.

If we were to apply Theorem 2.6.1 to $\mathcal{L}$, we could conclude that there exists a local Lie transformation group (G, $\overline{\mathbf{Q}}$ ) with $\mathcal{L}$ as its set of Lie derivatives. It follows from Definition 2.1.1 that $\overline{\mathbf{Q}} : V \times V \to V$, such that

$$\overline{\mathbf{Q}}[\mathbf{x}, \mathbf{e}] = \mathbf{x}, \quad \mathbf{x} \in V;$$

$\overline{\mathbf{Q}}[\mathbf{x}, \mathbf{g}]$ is analytic in $\mathbf{x}$ and $\mathbf{g}$, for $\mathbf{x}$ and $\mathbf{g} \in V$,

and $\overline{\mathbf{Q}}[\mathbf{x}, \varphi(\mathbf{g}, \mathbf{h})] = \overline{\mathbf{Q}}[\overline{\mathbf{Q}}[\mathbf{x}, \mathbf{g}], \mathbf{h}]$, if both $\varphi(\mathbf{g}, \mathbf{h})$ and $\overline{\mathbf{Q}}[\mathbf{x}, \mathbf{g}] \in V$.

This is not enough to conclude that $\overline{\mathbf{Q}}$ is a Lie product. However, the Proof of Theorem 2.6.1 can be modified to reach this conclusion.

Thus, we provide a method of constructing a local Lie group whose Lie algebra is realized by $\mathcal{L}$. Pontryagin [1, p.401] starts with coordinate increments to construct a local Lie group having prescribed structure constants. Our method is simpler but limited. The main result of the thesis is the following Theorem.

**Theorem 3.1.1** *Let* $L_k$ *be defined by* Eq.(3.1.1), $1 \le k \le n$ , *be* n *linearly independent differential operators defined and analytic in a connected, open set* $U \subset F_n$, *such that* $\mathbf{e} \in U$ *and the* $P_{ik}(\mathbf{g})$ *satisfy* Eq.(3.1.2). *Also, assume there exists constants* $c_{jk}^{r}$ *such that* Eq.(2.3.4) holds, $1 \le j, k, r \le n$. *If* $\mathbf{Q}[\mathbf{g}, \alpha t]$ *is the solution of the* V.O.D.E.(2.2.1), *then there exists a connected, open neighborhood* $V$ *of* $\mathbf{e}$, *contained in* $U$ *such that the function* $\varphi: V \times V \to F_n$ *defined by*

$$\varphi(\mathbf{g}, \mathbf{Q}[\mathbf{e}, \alpha t]) = \mathbf{Q}[\mathbf{g}, \alpha t] \tag{3.1.3}$$

*is the Lie product for a local Lie group* $(V, \varphi) = G$.

In fact we show that the structure constants $c_r^{jk}$ of G are the $c_{jk}^r$'s used in

Eq.(2.3.4). This suggests an obvious isomorphism between L(G) and $\mathcal{L}$. We also show that $\mathcal{L}$ is the set of all Lie derivatives of the local Lie transformation group (G, φ).

Note that Eq.(3.1.1) is actually Eq.(2.3.5) with m = n. The method we use to prove Theorem 3.1.1 is similar to the one used to prove Theorem 2.6.1. However, we need not make any assumptions about the existence of a local Lie group with the given structure constants. Many of the Lemmas in Section 2.4 which were used to prove Theorem 2.6.1 will be used again here.

Since $\mathbf{Q}[\mathbf{e}, \alpha]$ plays a critical role in the definition of φ, and in order to simplify notation, we introduce the notation

$$\mathbf{EX}(\alpha t) \equiv \mathbf{Q}[\mathbf{e}, \alpha t], \qquad \alpha \in F_n,.$$

The importance of $\mathbf{EX}(\alpha t)$ is further illustrated by the following Lemma.

**Lemma 3.1.1** $\mathbf{EX}(\alpha)$ *defines an analytic coordinate transformation from an open set* $U$ *to an open set* $V$, both in $F_n$, *such that:*

i) $\mathbf{e} \in U$ *and* $\mathbf{e} \in V$,
ii) *for all* $\mathbf{h} \in V$, *there exists a unique* $\alpha \in U$ *such that* $\mathbf{EX}(\alpha) = \mathbf{h}$. *That is,* $\mathbf{EX}^{-1}(\mathbf{h})$ *exists for* $\mathbf{h} \in V$.

**Proof:** By an argument similar to the one used in the proof of Theorem 2.6.1 to show that the group action is analytic in its m + n arguments, we conclude that $\mathbf{Q}[\mathbf{e}, \alpha t]$ is analytic in $\alpha t$. Since

$$\left.\frac{d\mathbf{EX}_i(\alpha)}{d\alpha_k}\right|_{\alpha=0} = \left. P_{ik}(\mathbf{EX}(\alpha))\right|_{\alpha=0} = \delta_{ik}, \quad 1 \le i, k \le n,$$

then the determinant of the matrix $\left(P_{ik}(\mathbf{g})\right)_{i,k=1}^{n}$ is non-zero in some neighborhood of $\mathbf{e}$. Thus, by an argument similar to that used in Section 1.4, Lemma 1.4.2, for the exponential map, $\mathbf{EX}$ maps a neighborhood of $\mathbf{e} \in U$ onto a neighborhood of $\mathbf{e} \in V$ such that, for all $\mathbf{h} \in V$, there exists a unique $\alpha \in U$ such that $\mathbf{EX}(\alpha) = \mathbf{h}$, and $\mathbf{EX}^{-1}$ exists and is analytic on $V$.

Q.E.D.

Note that Lemma 3.1.1 does not actually require $P_{kk}(\mathbf{e}) = \delta_{kk} = 1$, $1 \le k \le n$, but only that $P_{kk}(\mathbf{e}) \neq 0$. Further explanation of why we require Eq.(3.1.2) to hold will be given in the next two sections. Given a Lie algebra of linear differential operators, $\mathcal{L}$, with basis $\{L_k\}_{k=1}^{n}$ satisfying Eq.(3.1.1), such that

$$P_{ik}(\mathbf{e}) = \begin{cases} 0 & \text{if } i \neq k \\ c_i \neq 0, & \text{if } i = k \end{cases}, \tag{3.1.4}$$

it is a simple matter to create a second basis of linear differential operators for $\mathcal{L}$ satisfying Eq.(3.1.2). Thus, Eq.(3.1.4) and Eq.(3.1.2) are equivalent conditions.

We wish to motivate Theorem 3.1.1 before proving it. To this end, we give a simple application of Theorem 3.1.1 and an example concerning why the condition Eq.(3.1.2) is necessary. In Section 3.2.2 we give a simple proof of Theorem 3.1.1 in the one-dimensional case and show why the technique used cannot be extended to higher dimensions. Consider the following one-dimensional example.

**Example 3.1.1** From Chapter One, Example 1.2.2, we know that the function $\varphi: F X F \rightarrow F$, defined by

$$\varphi(g, h) = g + h + \gamma gh, \tag{3.1.5}$$

such that $\gamma \in F$, is a one-dimensional Lie product on $F$. By Theorem 1.4.1 the one-parameter subgroup $\text{EXP}(\alpha t)$ of $(F, \varphi)$ is found by solving the ordinary differential equation (O.D.E.)

$$\frac{dy(t)}{dt} = \alpha(1 + \gamma y), \qquad y(0) = 0.$$

Thus,

$$\text{EXP}(\alpha t) = \begin{cases} \dfrac{e^{\gamma\alpha t} - 1}{\gamma}, & \text{if } \gamma \neq 0 \\ \alpha t, & \text{if } \gamma = 0 \end{cases}. \tag{3.1.6}$$

By definition, $\text{EXP}(\alpha t)$ satisfies Eq.(1.4.1) and has tangent vector $\alpha$ at $t = 0$.

Now we wish to go in the other direction. That is, start with the O.D.E.

$$\frac{dy(t)}{dt} = \alpha(1 + \gamma y), \qquad y(0) = g, \tag{3.1.7}$$

and obtain the Lie product (3.1.5). $Q[g, \alpha t]$, the solution of the O.D.E.(3.1.7), is given by

$$Q[g, \alpha t] = \begin{cases} \dfrac{(1 + \gamma g)e^{\gamma \alpha t} - 1}{\gamma}, & \text{if } \gamma \neq 0 \\ \alpha t + g, & \text{if } \gamma = 0 \end{cases},$$

which is analytic in $F X F$. From this we make the following two important observations:

$$Q[0, \alpha t] = EXP(\alpha t)$$

and

$$\begin{aligned} Q[g, \alpha t] &= g + EXP(\alpha t) + \gamma g EXP(\alpha t) \\ &= \varphi(g, EXP(\alpha t)). \end{aligned}$$

By Lemma 1.4.2, the exponential map defines an analytic coordinate transformation on some neighborhood of 0. Thus, there exists two neighborhoods of 0, call them $U$ and $V$, such that for all $h \in V$, there exists a unique $\alpha \in U$ such that $EXP(\alpha) = h$. Thus the Lie product $\varphi$ defined by Eq.(3.1.3) can also be defined locally on $V$ in terms of $Q[g, \alpha]$, by

$$\varphi(g, h) = \varphi(g, EXP(\alpha)) = \varphi(g, Q[0, \alpha]) = Q[g, \alpha],$$

and $\varphi: V X V \to F$. Thus, we have taken the one-dimensional Lie algebra of linear differential operators with basis $(1 + \gamma g)\frac{d}{dg}$, applied Eq.(3.1.3) and obtained the local Lie group with Lie product given by Eq.(3.1.5).

We will show that this is also true for the n-dimensional case, $n \geq 1$. To justify the initial condition $P(0) \neq 0$ we need only consider the following example.

**Example 3.1.2** Consider the Lie algebra of differential operators with basis $L_1 = g\frac{d}{dg}$. Then $P(0) = 0$ and the V.O.D.E.(2.2.1) becomes the O.D.E.

$$\frac{dy(t)}{dt} = \alpha y, \quad y(0) = g.$$

It has solution $y(t) = Q[g, \alpha t] = ge^{\alpha t}$. If we define $\varphi$ by means of Eq.(3.1.3), then $\varphi$ is not a Lie product. It follows from the fact that $Q[0, \alpha t] = 0$, that 0 is not the identity element since

$$\varphi(g, Q[0, \alpha t]) = \varphi(g, 0) = Q[g, \alpha t] = ge^{\alpha t}.$$

Thus, there are restrictions on P(g).

## Section 3.2 The One-Dimensional Case

In this section, we give a simple proof of the one-dimensional case of Theorem 3.1.1. By Corollary 1.5.2, all one-dimensional local Lie groups are locally isomorphic and all one-dimensional Lie algebras are isomorphic. Consequently, using Theorem 3.1.1 to construct a one-dimensional local Lie group with Lie algebra isomorphic to a given one-dimensional Lie algebra provides a complex solution to a simple problem. Any one-dimensional local Lie group would do. Nonetheless, the simplicity of the one dimensional version of Theorem 3.1.1 does justify its consideration. We begin by stating Theorem 3.1.1 for the one-dimensional case.

**Theorem 3.2.1** *Let $W$ be an open neighborhood of 0 in $F$ and let $P{:}W \to F$ be an analytic function on $W$ such that $P(0) = 1$. Let $Q[g, \alpha t]$ be the solution of the O.D.E.*

$$\frac{dy(t)}{dt} = \alpha P(y(t)), \quad y(0) = g, \tag{3.2.1}$$

*and let $EX(\alpha) \equiv Q[0, \alpha]$, for $\alpha \in U$. Then $\varphi$ defined by*

$$\varphi(g, EX(\alpha)) = Q[g, \alpha], \tag{3.2.2}$$

*is a Lie product on some open, connected set, $V$, containing 0.*

We shall prove Theorem 3.2.1 with the aid of the following Lemma. From the theory of O.D.E., (see Petrovski [1, p.96]), we know there exists an open neighborhood $U$ of 0 in $F$, such that $Q[g, \alpha t]$ is analytic for $(g, \alpha t) \in U X U$. From Lemma 3.1.1 we know $\varphi$ is well-defined. For the remainder of Section 3.2 we will always use $U$ and $V$ as they were used in Lemma 3.1.1.

**Lemma 3.2.1:** *Let* $\alpha, \beta, \alpha + \beta$ *all belong to* $U$. *Then*

$$EX(\alpha + \beta) = Q[EX(\alpha), \beta] = \varphi(EX(\alpha), EX(\beta)). \tag{3.2.3}$$

**Proof:** If $\beta = 0$, Eq.(3.2.3) is obviously true. Thus, assume $\beta \neq 0$ and let

$$y(t) = Q[0, \alpha + \beta t] = EX(\alpha + \beta t)$$

and

$$z(t) = Q[EX(\alpha), \beta t] = \varphi(EX(\alpha), EX(\beta t)).$$

Then by the definition of Q, $y(0) = z(0) = EX(\alpha)$. Now, since $\alpha$ and $\beta$ are scalars,

$$\frac{dy(t)}{dt} = \frac{dQ[0, \alpha + \beta t]}{dt}$$

$$= \frac{dQ[0, \beta(\frac{\alpha}{\beta} + t)]}{dt}$$

$$= \frac{dQ[0, \beta(\frac{\alpha}{\beta} + t)]}{d(\frac{\alpha}{\beta} + t)} \frac{d(\frac{\alpha}{\beta} + t)}{dt}$$

$$= \beta P(Q[0, \alpha + \beta t])$$

$$= \beta P(y(t)).$$

Furthermore,

$$\frac{dz(t)}{dt} = \frac{dQ[EX(\alpha), \beta t]}{dt}$$

$$= \beta P(z(t)).$$

It follows from the uniqueness of solution of an O.D.E. that $y(t) \equiv z(t)$, for $t$ in some neighborhood of $0$. It is then a simple matter to use analytic continuation, (Hille[1, p.3]), to extend the equality to $t = 1$ and conclude that Eq.(3.2.3) holds, as required.

Q.E.D.

The fact that $P(0) \neq 0$ was required in Lemma 3.1.1. To this point we have not needed $P(0) = 1$. However,

$$\left.\frac{dEX(\alpha t)}{dt}\right|_{t=0} = \alpha P(0). \tag{3.2.4}$$

Thus in order for the analytic curve $EX(\alpha t)$ to have tangent vector $\alpha$ at 0, we require $P(0) = 1$. We want $EX(\alpha t)$ to be the one-parameter subgroup of $(V, \varphi)$ with tangent vector $\alpha$ at **e**, thus we will always require that $P(0) = 1$. We are now prepared to prove Theorem 3.2.1.

**Proof of Theorem 3.2.1:** Define $\varphi(g, EX(\alpha))$ by Eq.(3.2.2), for g and $EX(\alpha) \in V$. Obviously, $\varphi: V X V \rightarrow F$, and by the definition of Q, $\varphi$ is analytic in $VXV$. Furthermore,

$$\varphi(0, EX(\alpha)) = Q[0, \alpha] = EX(\alpha)$$

and because Q is the solution of the O.D.E.(3.2.1),

$$\varphi(g, 0) = Q[g, 0] = g.$$

Thus, 0 is the identity element for $\varphi$.

Thus we need only show that $\varphi$ is associative on $V$. Let g, h, k, $\varphi(g, h)$, $\varphi(h, k)$ all belong to $V$. By Lemma 3.1.1, there exists $\alpha$, $\beta$, and $\gamma$ belonging to $U$ such that $g = EX(\alpha)$, $h = EX(\beta)$ and $k = EX(\gamma)$. Now use Eq.(3.2.3) as follows,

$$\begin{aligned}
\varphi(g, \varphi(h, k)) &= \varphi\big(EX(\alpha), \varphi(EX(\beta), EX(\gamma))\big) \\
&= \varphi\big(EX(\alpha), EX(\beta + \gamma)\big) \\
&= EX(\alpha + (\beta + \gamma)) \\
&= EX((\alpha + \beta) + \gamma) \\
&= \varphi\big(EX(\alpha + \beta), EX(\gamma)\big) \\
&= \varphi\big(\varphi(EX(\alpha), EX(\beta)), EX(\gamma)\big) \\
&= \varphi(\varphi(g, h), k).
\end{aligned}$$

Thus, $\varphi$ is a Lie product on $V$, as required.

Q.E.D.

The Lie algebra of $G = (V, \varphi)$ is the field $F$ together with the commutator $[\alpha, \beta] = 0$, for $\alpha, \beta \in F$. L(G) is trivially isomorphic to the Lie algebra generated by $L_1 = P(g)\frac{d}{dg}$, because all one-dimensional Lie algebras are isomorphic .

**Lemma 3.2.2** *The one-parameter subgroup of* $(V, \varphi)$ *with tangent vector* $\alpha$ *at* 0 *is* EX($\alpha$t), *for* $\alpha \in F$.

**Proof:** Obviously, EX($\alpha$(0)) = Q[0, 0] = 0. By Eq.(3.2.4) and the fact that P(0) = 1, EX($\alpha$t) has tangent vector $\alpha$ at 0. Finally, Eq.(1.4.1) follows immediately from Lemma 3.2.1. Thus, by Definition 1.4.1, EX($\alpha$t) is the one-parameter subgroup of $(V, \varphi)$ with tangent vector $\alpha$ at 0, as required.

Q.E.D.

Lemma 3.2.1 plays an important role in the proof of Theorem 3.2.1. Lemma 3.2.1 depends on the fact that any element of $U$ can be expressed as a scalar multiple of any other non-zero element of $U$. This is only true if $U$ is one-dimensional. In one dimension, all elements of G are in the same one-parameter subgroup. Lemma 3.2.1 does not hold for higher dimensions. This is the main reason why the proof of the one-dimensional case of Theorem 3.1.1 is so easy and why this technique can't be extended to the higher dimensional cases. (The same idea works in Theorem 2.6.1 for a proof of the one-dimensional case, but we shall not go over it here). In the general case of Theorem 3.1.1, one has to use the idea of a Lie algebra. The set of differential operators defined by Eq.(3.1.1) must form a basis for the Lie algebra, $\mathcal{A}$.

We can use Theorem 3.2.1 to generate examples of one-dimensional local Lie groups. If P(x) is a function analytic at x = 0, such that P(0) = 1, then by Theorem 3.2.1 we can obtain a one-dimensional Lie product if we can solve the differential equation

$$\frac{dx(t)}{dt} = \alpha P(x(t)), \qquad x(0) = x^0. \tag{3.2.5}$$

For the open neighborhood where a solution exists, EXP($\alpha$t) is the solution of Eq.(3.2.5) with $x^0 = 0$ and $\varphi$(g, EXP($\alpha$t)) is the solution of Eq.(3.2.5) with $x^0 = g$.

For example, P(x) = 1 yields the one-parameter subgroups EXP($\alpha$t) = $\alpha$t together with the Lie product $\varphi(g, h) = g + h$; $P(x) = \gamma x + 1, \gamma \neq 0$, yields the the one-parameter subgroups EXP($\alpha$t) = $\frac{1}{\gamma}(e^{\alpha\gamma t} - 1)$ together with the Lie product $\varphi(g, h) = g + h + \gamma gh$ and $P(x) = e^{\gamma x}, \gamma \neq 0$ yields the one-parameter subgroups EXP($\alpha$t) = $\frac{-1}{\gamma}\ln(1 - \gamma\alpha t)$ together with the Lie product $\varphi(g, h) = \frac{-1}{\gamma}\ln(e^{-\gamma g} + e^{-\gamma h} - 1)$.

In fact, we have the following characterization of Lie products of one-dimensional local Lie groups.

**Theorem 3.2.2** $\varphi(\mathbf{g}, \mathbf{h})$ *is a Lie product for a one-dimensional local Lie group with one-parameter subgroups* $\mathrm{EXP}(\alpha t)$ *if and only if*

$$\mathrm{EXP}(\alpha t) = Q[0, \alpha t]$$
$$\varphi(g, \mathrm{EXP}(\alpha t)) = Q[g, \alpha t]$$

*where* $Q[g, \alpha t]$ *is the solution of the* O.D.E.

$$\frac{dx(t)}{dt} = \alpha P(x), \quad x(0) = g,$$

*and* $P(x)$ *is analytic at* $x = 0$ *and* $P(0) = 1$.

**Proof**: The proof follows from Theorem 3.2.1 and Lemma 3.2.2.

Q.E.D.

## Section 3.3 The n-Dimensional Case

We now prove Theorem 3.1.1 in general. Let $U$ be an open connected set in $F_n$ such that $\mathbf{e} \in U$. Throughout this section we will assume that $P_{ik}(\mathbf{g})$, $1 \le i, k \le n$, is an analytic function for $\mathbf{g} \in U$ such that Eq.(3.1.2) is satisfied. Define the linear differential operators $L_k$ by Eq.(3.1.1), $1 \le k \le n$, acting on functions that are analytic on $U$ and assume that there exists constants $c_{jk}^{r}$, $1 \le j, k, r \le n$, such that Eq.(2.3.4) is satisfied. We wish to show that $\varphi$ defined by Eq.(3.1.3) is a Lie product, where $\mathbf{Q}[\mathbf{g}, \alpha t]$ is the solution of the V.O.D.E.(2.2.1). Again we use the analytic coordinate transformation

$$\mathbf{EX}(\alpha t) = \mathbf{Q}[\mathbf{e}, \alpha t], \quad \alpha \in F_n.$$

Now, for $\mathbf{g} \in U$, define the nxn matrix $\mathbf{P}(\mathbf{g})$ by

$$\mathbf{P}(\mathbf{g}) = \left(P_{ik}(\mathbf{g})\right)_{i,k=1}^{n}.$$

By Eq.(3.1.2), $\mathbf{P}(\mathbf{e}) = E_n$. Since $P_{ik}(\mathbf{g})$ is analytic, $1 \le i, k \le n$, then for $\mathbf{g}$ in some neighborhood of $\mathbf{e}$, $\mathbf{P}(\mathbf{g})$ has a matrix inverse, that we will denote by

$$\mathbf{S}(\mathbf{g}) = \left(S_{ik}(\mathbf{g})\right)_{i,k=1}^{n}.$$

For convenience, we will begin by limiting the set $U$ to an open, connected neighborhood of $\mathbf{e}$ where $\mathbf{P}^{-1}(\mathbf{g})$ exists. We noted in Section 3.1 that, by Theorem 2.6.1, the n-dimensional Lie algebra, $\mathcal{L}$, generated by the $\{L_k\}_{k=1}^{n}$ is the algebra of Lie derivatives of some local Lie transformation group (G, $\overline{\mathbf{Q}}$) acting effectively on $U$, where $\mathbf{EXP}(\alpha t)$, $\alpha \in F_n$, are the one-parameter subgroups of the unknown group G, for t suitably close to 0, and $\overline{\mathbf{Q}}[\mathbf{g}, \mathbf{EXP}(\alpha t)]$ is the unique solution of the V.O.D.E.(2.2.1).

Thus,

$$\varphi(\mathbf{g}, \mathbf{EX}(\alpha t)) = \mathbf{Q}[\mathbf{g}, \alpha t] = \overline{\mathbf{Q}}[\mathbf{g}, \mathbf{EXP}(\alpha t)]. \tag{3.3.1}$$

Now, since $\overline{\mathbf{Q}}$ is an action, it follows that $\mathbf{e}$ is the identity element of $\varphi$ since

$$\varphi(\mathbf{g}, \mathbf{e}) = \overline{\mathbf{Q}}[\mathbf{g}, \mathbf{e}] = \mathbf{g},$$

and Eq.(3.1.3) together with the definition of $\mathbf{EX}(\alpha)$ imply that

$$\varphi(\mathbf{e}, \mathbf{EX}(\alpha)) = \mathbf{Q}[\mathbf{e}, \alpha] = \mathbf{EX}(\alpha).$$

Furthermore, $\overline{\mathbf{Q}}[\mathbf{g}, \mathbf{EXP}(\alpha)] = \varphi(\mathbf{g}, \mathbf{EX}(\alpha))$ is an analytic function of the 2n components of $\mathbf{g}$ and $\mathbf{EXP}(\alpha)$. However, since $\mathbf{EXP}(\alpha)$ and $\mathbf{EX}(\alpha)$ are both analytic coordinate transformations on some neighborhood of $\mathbf{e} \in F_n$, it follows that in some neighborhood of $\mathbf{e}$ we can express $\mathbf{EXP}(\alpha)$ as an analytic function of $\mathbf{EX}(\alpha)$. Thus, $\varphi(\mathbf{g}, \mathbf{EX}(\alpha))$ is an analytic function of the 2n components of $\mathbf{g}$ and $\mathbf{EX}(\alpha)$, when $\mathbf{g}$ and $\mathbf{EX}(\alpha)$ are close to $\mathbf{e}$, as required.

Thus, to conclude that $\varphi$ is a Lie product, we need only show that for $\varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta))$ and $\varphi(\mathbf{g}, \mathbf{EX}(\alpha))$ both belonging to $V$,

$$\varphi(\varphi(\mathbf{g}, \mathbf{EX}(\alpha)), \mathbf{EX}(\beta)) = \varphi(\mathbf{g}, \varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta))). \tag{3.3.2}$$

Unfortunately, Eq.(3.3.1) does not provide sufficient information to obtain Eq.(3.3.2), in terms of the group action $\overline{\mathbf{Q}}$. In particular, we do not know the meaning of $\varphi(\mathbf{g}, \varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta)))$. In any event, we wish to construct the Lie product $\varphi$ without using Theorem 1.5.5, i.e., without prior knowledge of the existence of a local Lie group G with Lie algebra isomorphic to the Lie algebra of differential operators. We show Eq.(3.3.2) holds by using a technique similar to the one use to prove Theorem 2.6.1. Most of the necessary Lemmas were proved in Chapter 2.

In the proof of Theorem 2.6.1, we needed $P_{ik}(\mathbf{g})$ and $S_{kj}(\mathbf{g})$ to have certain properties. Also, we needed $\mathbf{Q}[\mathbf{g}, \mathbf{EXP}(\alpha t)]$ to satisfy a second system of differential equations. We find that the following properties still hold:

From Lemma 2.3.3 it follows immediately that, for $1 \le q, k, s \le n$,

$$\sum_{i=1}^{n} \left( P_{is}(\mathbf{g}) \frac{\partial P_{qk}(\mathbf{g})}{\partial g_i} - P_{ik}(\mathbf{g}) \frac{\partial P_{qs}(\mathbf{g})}{\partial g_i} \right) = \sum_{r=1}^{n} c_{sk}^{r} P_{qr}(\mathbf{g}). \tag{3.3.3}$$

From Lemma 2.4.1, with $R_{ik}(\mathbf{g}) = P_{ik}(\mathbf{g})$ it follows that $c_{rs}^{i} = -c_{sr}^{i}$ and

$$\frac{\partial S_{iq}(\mathbf{g})}{\partial g_k} - \frac{\partial S_{ik}(\mathbf{g})}{\partial g_q} = \sum_{r,s=1}^{n} c_{rs}^{i} S_{rq}(\mathbf{g}) S_{sk}(\mathbf{g}), \quad 1 \le i, k, q \le n. \tag{3.3.4}$$

Thus, by Eq.'s (3.3.3) and (3.3.4) and Corollary 2.4.1 and Lemma 3.1.1, it follows that the system of partial differential equations

$$\left.\begin{aligned} \frac{\partial T_i(\mathbf{g}, \mathbf{EX}(\alpha))}{\partial EX_k(\alpha)} &= \sum_{r=1}^{n} P_{ir}(\mathbf{T}(\mathbf{g}, \mathbf{EX}(\alpha))) S_{rk}(\mathbf{EX}(\alpha)), \quad 1 \le i, k \le n \\ \mathbf{T}(\mathbf{g}, \mathbf{e}) &= \mathbf{g}, \end{aligned}\right\} \tag{3.3.5}$$

has a unique solution for $\mathbf{g}$ and $\mathbf{EX}(\alpha)$ in some open neighborhood of $V$ of $\mathbf{e} \in F_n$.

The following is also true:

**Lemma 3.3.1** Let $\mathbf{g}, \mathbf{EX}(\alpha t) \in V$. $\mathbf{T}(\mathbf{g}, \mathbf{EX}(\alpha t))$ *is the unique solution of the system of* P.D.E.(3.3.5) *if and only if it is also the unique solution of the* V.O.D.E.(2.2.1).

**Proof:** The proof concerns the uniqueness of solution of systems of differential equations and is similar to the proof of Lemma 2.5.4 and is thus omitted.

Q.E.D.

We now relate Lemma 3.3.1 to the the given situation with the following Lemma.

**Lemma 3.3.2** $\varphi(\mathbf{g}, \mathbf{EX}(\alpha t))$ *defined by* Eq.(3.1.3) *satisfies the system of* P.D.E.(3.3.5).

**Proof:** By Eq.(3.1.3), $\varphi(\mathbf{g}, \mathbf{EX}(\alpha t))$ satisfies the V.O.D.E.(2.2.1). It follows from Lemma 3.3.1 that $\varphi(\mathbf{g}, \mathbf{EX}(\alpha t))$ is also a solution to the system of P.D.E.(3.3.5), as required.

Q.E.D.

We now have sufficient background to complete the proof of Theorem 3.1.1.

**Proof of Theorem 3.1.1:** It follows immediately from Eq.(3.1.3) that **e** is the identity element because

$$\varphi(\mathbf{g}, \mathbf{e}) = \mathbf{Q}[\mathbf{g}, \mathbf{e}] = \mathbf{g} \text{ and } \varphi(\mathbf{e}, \mathbf{EX}(\alpha t)) = \mathbf{Q}[\mathbf{e}, \alpha t] = \mathbf{EX}(\alpha t).$$

By an argument similar to the one used in Theorem 2.6.1 to show that a group action is an analytic function of its m + n arguments, we conclude that $\varphi(\mathbf{g}, \mathbf{EX}(\alpha))$ is analytic in the 2n coordinates of **g** and $\mathbf{EX}(\alpha)$. In order to complete the proof, we need only show that $\varphi$ is associative.

Let $\mathbf{y}(t) = \varphi(\varphi(\mathbf{g}, \mathbf{EX}(\alpha)), \mathbf{EX}(\beta t))$ and $\mathbf{z}(t) = \varphi(\mathbf{g}, \varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t)))$. Clearly, $\mathbf{y}(0) = \mathbf{z}(0) = \varphi(\mathbf{g}, \mathbf{EX}(\alpha))$. Now, by Eq.(3.1.3)

$$\frac{dy_i(t)}{dt} = \sum_{k=1}^{n} \beta_k P_{ik}(\mathbf{y}(t)).$$

By the chain rule, Lemma 3.3.2 and Eq.(3.1.3) we obtain

$$\frac{dz_i(t)}{dt} = \sum_{k=1}^{n} \frac{\partial\varphi_i(\mathbf{g}, \varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t)))}{\partial\varphi_k(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t))} \frac{d\varphi_k(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t))}{dt}$$

$$= \sum_{k,r,s=1}^{n} P_{ir}(\mathbf{z}(t)) S_{rk}(\varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t))) \beta_s P_{ks}(\varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t))).$$

Since $\mathbf{S}(\varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t))) = \left(\mathbf{P}(\varphi(\mathbf{EX}(\alpha), \mathbf{EX}(\beta t)))\right)^{-1}$, the expression simplifies to

$$\frac{dz_i(t)}{dt} = \sum_{s=1}^{n} \beta_s P_{is}(\mathbf{z}(t)).$$

Thus, since $\mathbf{y}(t)$ and $\mathbf{z}(t)$ satisfy the same differential equation and have the same initial conditions, we conclude by uniqueness, (Petrovski [1, p.96]), that $\mathbf{y}(t) \equiv \mathbf{z}(t)$, t suitably close to 0. Use analytic continuation (Hille [1, p.3]) to extend the proof to $t = 1$ and conclude that Eq.(3.3.2) is true under the necessary conditions. Thus, $\varphi$ is a Lie product on $V$, as required.

Q.E.D

Theorem 2.6.1 differs from Theorem 3.1.1 in that Theorem 2.6.1 only concludes that there exists a local Lie group G such that the local Lie transformation group (G, **Q**) had the prescribed Lie derivatives, whereas Theorem 3.1.1 shows that the Lie product of G can be found by solving a system of differential equations. In both Theorems the group action is found by solving a system of ordinary differential equations, however Theorem 2.6.1 is applicable to a much broader range of Lie algebras of differential operators than Theorem 3.1.1.

We complete this section by examining the constructed local Lie group $(V, \varphi)$ in more detail. We verify that the local Lie transformation group $((V, \varphi), \varphi)$ has, as its set of Lie derivatives, the Lie algebra of differential operators $\mathcal{L} = \text{Span}(\{L_k\}_{k=1}^{n})$.

**Lemma 3.3.3** *The one-parameter subgroups of* $(V, \varphi)$ *are the* $\mathbf{EX}(\alpha t)$, $\alpha \in F_n$.

**Proof:** Clearly $\mathbf{EX}(\alpha(0)) = \mathbf{e}$, and by Eq.(3.1.2),

$$\left.\frac{dEX_k(\alpha t)}{dt}\right|_{t=0} = \left.\frac{dQ_k[\mathbf{e}, \alpha t]}{dt}\right|_{t=0} = \left.\sum_{r=1}^{n} \alpha_r P_{kr}(\mathbf{Q}[\mathbf{e}, \alpha t])\right|_{t=0} = \alpha_k.$$

Furthermore, by Eq.(3.1.3), if $\mathbf{y}(t) = \mathbf{EX}(\alpha(t+s))$ and $\mathbf{z}(t) = \varphi(\mathbf{EX}(\alpha s), \mathbf{EX}(\beta t))$, then it is a simple exercise to show that

$$\frac{dy_i(t)}{dt} = \frac{dz_i(t)}{dt}, \qquad 1 \le i \le n,$$

and $y(0) = z(0)$. Thus by the uniqueness of a solution for a V.O.D.E. (Petrovski [1, p.96]), it follows that

$$\mathbf{EX}(\alpha(t+s)) = \varphi(\mathbf{EX}(\alpha s), \mathbf{EX}(\beta t)).$$

Thus, by Definition 1.4.1, $\mathbf{EX}(\alpha t)$ is the one-parameter subgroup of $(V, \varphi)$ with tangent vector $\alpha$ at $\mathbf{e}$.

Q.E.D.

**Lemma 3.3.4** *The structure constants* $c_r^{sk}$ *for the local Lie group* $(V, \varphi)$ *are the* $c_{sk}^r$ *that satisfy* Eq.(2.3.4).

**Proof:** From our construction of the local Lie group $(V, \varphi)$, we know that

$$[L_s, L_k] = \sum_{r=1}^{n} c_{sk}^r L_r.$$

By Eq.(1.2.6), it follows that

$$c_r^{sk} = \left(\frac{\partial^2 \varphi_r(\mathbf{g}, \mathbf{EX}(\alpha))}{\partial g_s \partial EX_k(\alpha)} - \frac{\partial^2 \varphi_r(\mathbf{g}, \mathbf{EX}(\alpha))}{\partial g_k \partial EX_s(\alpha)}\right)\Bigg|_{\mathbf{g}=\mathbf{EX}(\alpha)=\mathbf{e}} .$$

Now, by Lemma 3.3.2,

$$\frac{\partial^2 \varphi_r(\mathbf{g}, \mathbf{EX}(\alpha))}{\partial g_s \partial EX_k(\alpha)} = \frac{\partial}{\partial g_s} \sum_{q=1}^{n} P_{rq}(\varphi(\mathbf{g}, \mathbf{EX}(\alpha))) S_{qk}(\mathbf{EX}(\alpha))\Bigg|_{\mathbf{g}=\mathbf{EX}(\alpha)=\mathbf{e}}$$

$$= \frac{\partial P_{rk}(\mathbf{g})}{\partial g_s}\Bigg|_{\mathbf{g}=\mathbf{e}} .$$

Thus,

$$c_r^{sk} = \frac{\partial P_{rk}(\mathbf{g})}{\partial g_s}\Bigg|_{\mathbf{g}=\mathbf{e}} - \frac{\partial P_{rs}(\mathbf{g})}{\partial g_k}\Bigg|_{\mathbf{g}=\mathbf{e}} .$$

Now, evaluate Eq.(3.3.3) at $\mathbf{g} = \mathbf{e}$ to find that,

$$\frac{\partial P_{rk}(\mathbf{g})}{\partial g_s}\Bigg|_{\mathbf{g}=\mathbf{e}} - \frac{\partial P_{rs}(\mathbf{g})}{\partial g_k}\Bigg|_{\mathbf{g}=\mathbf{e}} = c_{sk}^r .$$

Thus, for $1 \le r, s, k \le n$, $c_r^{sk} = c_{sk}^r$, as required.

Q.E.D.

Thus, the Lie algebra of $(V, \varphi)$ is isomorphic to the given algebra of differential operators, $\mathcal{L}$, with isomorphism defined by $\alpha \to L_\alpha$.

**Lemma 3.3.5** *The local Lie group* $G = (V, \varphi)$, *where* $\varphi$ *is given by* Eq.(3.1.3), *together with* $\varphi$ *as the group action form a local Lie transformation group with Lie derivatives*

$$L_\alpha = \sum_{k=1}^{n} \alpha_k L_k, \qquad \alpha \in L(G),$$

*where* $L_k$ *is given by* Eq.(3.1.1).

**Proof:** From Theorem 3.1.1 we know that $(V, \varphi)$ is a local Lie group, and from Lemma 2.2.2 we know that $((V, \varphi), \varphi)$ is a local Lie transformation group acting on $V$, with Lie derivatives

$$L_\alpha = \sum_{i,k=1}^{n} \alpha_k R_{ik}(g)\frac{\partial}{\partial g_i}, \quad \alpha \in F_n,$$

where $R_{ik}(g)$ is given by Eq.(2.2.8). Thus, we need only show that $R_{ik}(g) = P_{ik}(g)$, $g \in V$, $1 \le i, k \le n$.

By Eq.'s (2.2.8), (3.3.9) and the fact that $\mathbf{P}(\mathbf{e}) = \mathbf{S}(\mathbf{e}) = \mathbf{E}_n$,

$$\begin{aligned} R_{ik}(\mathbf{g}) &= \left.\frac{\partial \varphi_i(\mathbf{g}, \mathbf{EX}(\alpha))}{\partial EX_k(\alpha)}\right|_{\mathbf{EX}(\alpha)=\mathbf{e}} \\ &= \left.\sum_{r=1}^{n} P_{ir}(\varphi(\mathbf{g}, \mathbf{EX}(\alpha)))S_{rk}(\mathbf{EX}(\alpha))\right|_{\mathbf{EX}(\alpha)=\mathbf{e}} \\ &= \sum_{r=1}^{n} P_{ir}(\mathbf{g})\delta_{rk} \\ &= P_{ik}(\mathbf{g}). \end{aligned}$$

Thus, given an n-dimensional Lie algebra of differential operators with a basis $\{L_k\}_{k=1}^n$, defined by Eq.(3.1.1), such that Eq.(3.1.2) holds, we have that these differential operators are the Lie algebra of Lie derivatives of the local Lie group transformation group $((V, \varphi), \varphi))$ where $\varphi$ is given by Eq.(3.1.3).

Q.E.D.

**Corollary 3.3.1** *$\mathfrak{L}$ is the set of all Lie derivatives of a local Lie transformation group $((V, \varphi), \varphi)$ if and only if $\mathfrak{L}$ is a Lie algebra of differential operators with basis $\{L_k\}_{k=1}^{n}$ acting on $V$, satisfying* Eq.'s(3.1.1) and (3.1.2).

**Proof**: It follows immediately from Lemmas 3.3.5 and 2.2.2.

Q.E.D.

Thus, the proof of Theorem 2.6.1 can be adapted to construct a local Lie group from a Lie algebra. In the next Chapter we use a generalization of Theorem 2.6.1 to prove addition theorems for special functions.

# Chapter 4

## Multiplier Representations and Special Functions

### Section 4.1 Generalized Lie Derivatives and Multiplier Representations

In this Chapter we shall use local Lie transformation groups to prove addition theorems for certain special functions of mathematics. 'Special functions' are classes of functions that appear frequently in mathematical discussion and have some noted importance. The classification of special functions is objective, but usually includes such functions as Legendre, Jacobi and Hermite polynomials and Beta, Gamma and Theta functions to name a few.

Before we can use local Lie transformation groups to obtain addition theorems for special functions, we need the following additional theory. See Miller [1, p.196-199] for the derivation of the following material. We shall use the notation of Chapter 2. That is, plainface type will be used for the nonidentity n-tuples from G and L(G) and boldface type will be used for the m-tuples of $U \subset F_m$.

**Definition 4.1.1** Let $(G, \mathbf{Q})$ be a local Lie transformation group acting on a neighborhood of $U \subset F_m$ with Lie derivatives $L_\alpha$, $\alpha \in L(G)$, and let $\mathcal{A}_{\mathbf{x}0}$ be the set of all functions analytic in some neighborhood of $\mathbf{x}^0$. Let $\nu: U \times G \rightarrow F$, such that $\nu(\mathbf{x}, g)$ is a scalar-valued, analytic function of $\mathbf{x}$ and g and

1. $\nu(\mathbf{x}, \mathbf{e}) = 1.$ (4.1.1)
2. $\nu(\mathbf{x}, \varphi(g, h)) = \nu(\mathbf{x}, g)\nu(\mathbf{Q}[\mathbf{x}, g], h),$ (4.1.2)

for $\mathbf{Q}[\mathbf{x}, g] \in U$, $\varphi(g, h) \in G$. A **local multiplier representation T** of $(G, \mathbf{Q})$ on $\mathcal{A}_{\mathbf{x}0}$ **with multiplier** $\nu$, consists of a mapping $\mathbf{T}(\mathbf{g})$ of $\mathcal{A}_{\mathbf{x}0}$ onto $\mathcal{A}_{\mathbf{x}0}$ defined for all $\mathbf{g} \in G$, $f \in \mathcal{A}_{\mathbf{x}0}$ such that

$$[\mathbf{T}(\mathbf{g})f\,](\mathbf{x}) = \nu(\mathbf{x}, g) f(\mathbf{Q}[\mathbf{x}, g]), \tag{4.1.3}$$

for $\mathbf{x}$ suitably close to $\mathbf{x}^0$.

If $\nu(\mathbf{x}, g) = 1$, then **T** is known as an **ordinary representation**, (which we used, without naming, in the discussion of ordinary Lie derivatives). There is an important relationship between ordinary representations and multiplier representations that the following Lemma explains.

**Lemma 4.1.1** *Let* **T** *be a multiplier representation of the local Lie transformation group* (G, **Q**) *on* $U \subset F_m$ *and let* $U' = U x F$. *Then the mapping* $\mathbf{Q}': U' \times G \rightarrow F_{m+1}$ *such that*

$$\mathbf{Q}'[(\mathbf{x}, x_{m+1}), g] = \big(\mathbf{Q}[\mathbf{x}, g], x_{m+1} + \ln v(\mathbf{x}, g)\big), \tag{4.1.4}$$

*is the group action of the local Lie transformation group* (G, **Q'**) *acting on* $U'$.

**Proof**: By Eq.(2.1.1) and Definition 4.1.1, **Q** and ν are both analytic in **x** and g, thus it follows from Eq.(4.1.4), that $\mathbf{Q}'[(\mathbf{x}, x_{m+1}), g]$ is an analytic function of $\mathbf{x}, x_{m+1}$ and g. It follows from Eq.'s(4.1.4), (4.1.1) and (2.1.2) that $\mathbf{Q}'[(\mathbf{x}, x_{m+1}), \mathbf{e}] = (\mathbf{x}, x_{m+1})$. Now, by Eq.'s(4.1.4), (2.1.3) and (4.1.2)

$$\begin{aligned}\mathbf{Q}'[(\mathbf{x}, x_{m+1}), \varphi(g, h)] &= \big(\mathbf{Q}[\mathbf{x}, \varphi(g, h)], x_{m+1} + \ln v(\mathbf{x}, \varphi(g, h))\big)\\ &= \big(\mathbf{Q}[\mathbf{Q}[\mathbf{x}, g], h], x_{m+1} + \ln\big(v(\mathbf{x}, g)v(\mathbf{Q}[\mathbf{x}, g], h)\big)\big)\\ &= \big(\mathbf{Q}[\mathbf{Q}[\mathbf{x}, g], h], x_{m+1} + \ln v(\mathbf{x}, g) + \ln v(\mathbf{Q}[\mathbf{x}, g], h)\big)\\ &= \mathbf{Q}'[\mathbf{Q}'[(\mathbf{x}, x_{m+1}), g], h],\end{aligned}$$

for $\mathbf{Q}[\mathbf{x}, g] \in U$ ( or $\mathbf{Q}'[(\mathbf{x}, x_{m+1}), g] \in U'$ ) and $\varphi(g, h) \in G$, as required.

Q.E.D.

Since (G, **Q'**) is a local Lie transformation group it has Lie derivatives $L'_\alpha$, $\alpha \in L(G)$. By Lemma 2.1.1

$$L'_\alpha = \sum_{i=1}^{m+1} \sum_{r=1}^{n} \alpha_r P'_{ir}((\mathbf{x}, x_{m+1}))\frac{\partial}{\partial x_i}, \tag{4.1.5}$$

where $P'_{ir}((\mathbf{x}, x_{m+1})) = \left.\dfrac{\partial Q'_i[(\mathbf{x}, x_{m+1}), g]}{\partial g_r}\right|_{g=e}$. However, it follows from Eq.(4.1.4) that

$$P'_{ir}((\mathbf{x}, x_{m+1})) = P_{ir}(\mathbf{x}), \quad 1 \le i \le m, \quad 1 \le r \le n, \tag{4.1.6}$$

where $P_{ir}(\mathbf{x})$ is derived from $\mathbf{Q}[\mathbf{x}, \mathrm{EXP}(\alpha t)]$ by Eq.(2.1.5), and

$$P'_{m+1r}((\mathbf{x}, x_{m+1})) = \left.\frac{\partial v(\mathbf{x}, g)}{\partial g_r}\right|_{g=e} \equiv P_r(\mathbf{x})), \quad 1 \le r \le n. \tag{4.1.7}$$

Obviously $P'_{ir}((\mathbf{x}, x_{m+1}))$ does not depend on $x_{m+1}$, $1 \le i \le m+1$, $1 \le r \le n$.

Now, given the local multiplier representation, **T**, we define the generalized Lie derivatives $D_\alpha$, $\alpha \in L(G)$, as follows.

**Definition 4.1.2** Let **T** be a multiplier representation and let $f \in \mathcal{A}_{\mathbf{x}^0}$. The **generalized Lie derivative $D_\alpha f$** of $f$ under the one-parameter subgroup EXP($\alpha$t) is the analytic function

$$D_\alpha f(\mathbf{x}) = \left.\frac{d[\mathbf{T}(\mathrm{EXP}(\alpha t))f](\mathbf{x})}{dt}\right|_{t=0}. \qquad (4.1.8)$$

It follows from the product rule of differentiation and Eq.'s (4.1.8), (4.1.3), (2.1.7), (4.1.1) and (2.1.2) that

$$D_\alpha = L_\alpha + \sum_{r=1}^{n} \alpha_r P_r(\mathbf{x}), \qquad (4.1.9)$$

where $P_r(\mathbf{x})$ is given by Eq.(4.1.7). (See Miller [1, p.198], Eq.(9.32)). It is clear from Lemma 4.1.1 and Eq.(4.1.9) that multiplier representations are really just a special type of ordinary representations. Thus, we have the following theorems corresponding to Theorems 2.2.3 and 2.6.1.

**Theorem 4.1.1** (See Miller [1, p.198], Theorem 5.28). *The generalized Lie derivatives of a local multiplier representation form a Lie algebra under the operations of addition of derivatives and Lie bracket*

$$[D_\alpha, D_\beta] \equiv D_\alpha D_\beta - D_\beta D_\alpha. \qquad (4.1.10)$$

*In fact,*

$$1.\ D_{a\alpha+b\beta} = aD_\alpha + bD_\beta, \qquad (4.1.11)$$

*and*

$$2.\ D_{[\alpha,\beta]} = [D_\alpha, D_\beta], \qquad (4.1.12)$$

*for* $\alpha, \beta \in L(G)$, $a, b \in F$ *and* $[D_\alpha, D_\beta]$ *defined by* Eq.(4.1.10).

**Proof**: Eq.(4.1.11) follows from Eq.(4.1.9) and Theorem 2.2.3. Thus the generalized Lie derivatives, $D_\alpha$, form a vector space under addition and scalar multiplication. Eq.(4.1.12) follows from the fact that

$$L'_{[\alpha,\beta]} = [L'_\alpha, L'_\beta],$$

where $L'_\alpha$ is given by Eq.(4.1.5), and the relationship between $D_\alpha$ and $L'_\alpha$.

Q.E.D.

Thus, the set of all generalized Lie derivatives of a local Lie transformation group forms an abstract Lie algebra under the commutator of Eq.(4.1.10). **T** is an **effective** multiplier representation if the map $\alpha \rightarrow D_\alpha$ is a Lie algebra isomorphism. We now adapt Theorem 2.6.1 to generalized Lie derivatives.

**Theorem 4.1.2** (See Miller [1, p.199], Theorem 5.31). *Let*

$$D_k = \sum_{i=1}^{m} P_{ik}(\mathbf{x})\frac{\partial}{\partial x_i} + P_k(\mathbf{x}), \quad 1 \le k \le n,$$

*be* n *linearly independent differential operators defined and analytic in an open set* $U \subset F_m$. *If there exists constants* $c^r_{jk}$ *such that*

$$[D_j, D_k] = \sum_{r=1}^{n} c^r_{jk} D_r, \quad 1 \le j, k \le n,$$

*then the* $D_k$ *form the basis for a Lie algebra which is the algebra of generalized Lie derivatives of an effective local multiplier representation* **T**. $\mathbf{x}(t) = \mathbf{Q}[\mathbf{x}, \mathrm{EXP}(\alpha t)]$ *and* $\nu(\mathbf{x}, \mathrm{EXP}(\alpha t))$ *are obtained by integration of the equations*

$$\frac{dx_i(t)}{dt} = \sum_{k=1}^{n} P_{ik}(\mathbf{x}(t))\alpha_k, \quad 1 \le i \le m, \quad \mathbf{x}(0) = \mathbf{x}, \tag{4.1.13}$$

*and*

$$\frac{d}{dt}\ln\nu(\mathbf{x}, \mathrm{EXP}(\alpha t)) = \sum_{j=1}^{n} \alpha_j P_j(\mathbf{x}(t)), \quad \nu(\mathbf{x}, \mathbf{e}) = 1. \tag{4.1.14}$$

**Proof**: Due to the restrictions imposed on $D_k$, $1 \le k \le n$, it follows from Theorem 2.6.1 that the unique solution of the differential system (4.1.13), $\mathbf{Q}[\mathbf{x}, \mathrm{EXP}(\alpha t)]$, is indeed a group action. Thus, we need only verify that the solution of Eq.(4.1.14) produces a multiplier.

Construct the differential operators $L'_\alpha$ according to Eq.'s(4.1.5), (4.1.6) and (4.1.7) where the $P_{ik}(\mathbf{x})$'s, $1 \le i \le m$ and $P_k(\mathbf{x})$ are obtained from $D_k$, $1 \le k \le n$. Clearly these differential operators are linearly independent and satisfy Eq.(2.3.4). Thus, by Theorem 2.6.1 there exists a local Lie transformation group (G, $\mathbf{Q}'$) such that

$Q_i'[(\mathbf{x}, x_{m+1}), EXP(\alpha t)]$ is obtained by solving the system of P.D.E.(4.1.13), $1 \leq i \leq m$ and $Q'_{m+1}[(\mathbf{x}, x_{m+1}), EXP(\alpha t)]$ is obtained by solving the O.D.E(4.1.14), where

$$Q'_{m+1}[(\mathbf{x}, x_{m+1}), EXP(\alpha t)] = Inv(\mathbf{x}, EXP(\alpha t)) + x_{m+1}.$$

Since the group action **Q'** is associative,

$$Q'_{m+1}[(\mathbf{x}, x_{m+1}), \varphi(g, h)] = Q'_{m+1}[\mathbf{Q}'[(\mathbf{x}, x_{m+1}), g], h].$$

That is,

$$\begin{aligned} Inv(\mathbf{x}, \varphi(g, h)) + x_{m+1} &= Inv(\mathbf{Q}[\mathbf{x}, g], h) + Q'_{m+1}[(\mathbf{x}, x_{m+1}), g] \\ &= Inv(\mathbf{Q}[\mathbf{x}, g], h) + Inv(\mathbf{x}, g) + x_{m+1}. \end{aligned}$$

Thus, $$Inv(\mathbf{x}, \varphi(g, h)) = Inv(\mathbf{Q}[\mathbf{x}, g], h) + Inv(\mathbf{x}, g).$$

It follows that $\nu(\mathbf{x}, g)$ satisfies Eq.(4.1.2). Since $\nu(\mathbf{x}, g)$ also satisfies Eq.(4.1.1) and is a scalar-valued, analytic function of $\mathbf{x}$ and g, **T** is a multiplier representation, as required.

Q.E.D.

In the next section we show by example how multiplier representations can be used to prove addition theorems for special functions. The key to applying the local Lie theory of this section to prove addition theorems of special functions is the fact that, for a multiplier representation **T** of $G = (V, \varphi)$ on $\mathcal{A}_{\mathbf{x}0}$ with multiplier $\nu$,

$$[\mathbf{T}(\varphi(g, h))f](\mathbf{x}^0) = \left(\mathbf{T}(g)[\mathbf{T}(h)f]\right)(\mathbf{x}^0), \qquad (4.1.15)$$

where $f \in \mathcal{A}_{\mathbf{x}0}$ and $\varphi(g, h) \in G$. Eq.(4.1.15) is a direct consequence of Definition 4.1.1.

## Section 4.2 Proofs of Addition Theorems Using Multiplier Representations

We now show how local Lie theory can be used to find addition theorems for some special functions. We do not claim that the addition theorems we shall find are new or that our proofs of them are necessarily simpler than the usual proofs, using special function techniques. We give these proofs to show that the application of Eq.(4.1.15) gives addition theorems even in some of the simple cases. For a more rigorous and comprehensive study of the application of local Lie groups to special function theory see Miller [2], or Vilenkin [1]. Miller and Vilenkin use local linear Lie groups for their studies, we shall use the ordinary local Lie groups.

The hypergeometric function, $_2F_1$, is one well known special function. It is defined by

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \tag{4.2.1}$$

where $(a)_n = a(a+1)\ldots(a+n-1),\ n \geq 1,\ (a)_0 = 1, a \neq 0$. In Miller [1, p.199-204] or [2, p.20-24], Miller outlines a method of proving addition theorems for the hypergeometric polynomials using multiplier representations and a specified 3-dimensional local linear Lie group.

For the examples of this section we use the one-dimensional local Lie group $G = (F, \varphi)$, where $\varphi(g, h) = g + h$, $g, h \in F$. The Lie algebra of G, L(G) consists of the field $F$ together with the commutator $[\alpha, \beta] = 0$, $\alpha, \beta \in F$ and the one-parameter subgroups of G are $\mathrm{EXP}(\alpha t) = \alpha t$, $\alpha \in F$. Our first example deals with the generalized hypergeometric function.

The hypergeometric function is generalized from Eq.(4.2.1) as follows:

$$_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{k=1}^{q}(\beta_k)_n} \frac{z^n}{n!}, \tag{4.2.2}$$

where $\beta_k \neq 0$ and $\beta_k$ is not a negative integer, $k = 1, 2, \ldots, q$. (See Rainville [1, p.73]).

With the aid of Eq.(4.2.2) and the Ratio test of a power series, Rainville is able to show where $_pF_q$ is defined, that is, for what $z$ the power series of Eq.(4.2.2) converges.

If any $\alpha_i$ in Eq.(4.2.2) is zero or a negative integer, then the power series terminates and $_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z)$ is a polynomial in $z$. Otherwise we must consider the non-negative integers $p$ and $q$ of $_pF_q$ and apply the following guidelines.

1. If $p \le q$, then the series converges for all finite z.

2. If $p = q + 1$, then the series converges for $|z| < 1$, diverges for $|z| > 1$ and ${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z)$ is absolutely convergent on the circle $|z| = 1$ if

$$\mathrm{Re}\left(\sum_{k=1}^{q}\beta_k - \sum_{i=1}^{p}\alpha_i\right) > 0.$$

3. If $p > q + 1$, then the power series of Eq.(4.2.2) diverges for all $z \neq 0$.

Thus, unless a or b are zero or negative integers the ${}_2F_1(a, b; c; z)$ hypergeometric function is only defined for $|z| < 1$ and perhaps $|z| = 1$. The addition theorems Miller proves for the hypergeometric function take into account where ${}_2F_1$ is analytic. We shall provide an addition theorem for one of the generalized hypergeometric functions.

In the process of dealing with special functions, Rainville uses the following elementary series manipulation, (see Rainville [1, p.56]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k), \tag{4.2.3}$$

where A(k, n) is a function of k and n. This will be a useful technique in our examples of multiplier representations.

**Example 4.2.1** Let $\mathcal{A}_0$ be the set of all functions analytic in a neighborhood of zero. Let

$$D = x + \frac{d}{dx}$$

be a general differential operator and let G be the one-dimensional local Lie group with Lie product $\varphi(g, h) = g + h$, discussed at the beginning of this section. It follows from Theorem 4.1.2 that

$$Q[z, \mathrm{EXP}(\alpha t)] = Q[z, \alpha t] = z + \alpha t,$$

and

$$v(z, \mathrm{EXP}(\alpha t)) = v(z, \alpha t) = \exp\left(\frac{\alpha^2 t^2}{2} + \alpha t z\right),$$

for $z \in F$ and $\alpha \in G$, where $\exp(x) = e^x$ is the ordinary exponential function. Thus, for $\alpha \in G$, $f \in \mathcal{A}_0$, the multiplier representation $T(\alpha): \mathcal{A}_0 \to \mathcal{A}_0$ is defined by,

$$[T(\alpha)f](z) = \exp\left(\frac{\alpha^2}{2} + \alpha z\right)f(z + \alpha), \tag{4.2.4}$$

Now, it is simple to verify that T is a multiplier representation since $Q = \varphi$, thus (G, Q) is a local Lie transformation group. Furthermore, $\nu(z, EXP(\alpha t))$ is a scalar-valued function analytic in z and $\alpha t$, $\nu(z, 0) = 1$ and

$$\nu(z, \varphi(\alpha, \beta)) = \nu(z, \alpha)\,\nu(Q[z, \alpha], \beta) = \exp\left(\frac{\alpha^2}{2} + \frac{\beta^2}{2} + \alpha z + \beta z + \alpha\beta\right),$$

thus, $\nu(z, EXP(\alpha t))$ is a multiplier.

From Eq.(4.2.4) we can verify that $T(\alpha)$ maps a function analytic in a neighborhood of zero to another function analytic in a neighborhood of zero. We can consider $\mathcal{A}_0$ to be the infinite dimensional vector space with basis $\{h_k(z)\}_{k=0}^{\infty}$, $h_k(z) = z^k$. From Eq.(4.2.4), we conclude that there is no subspace S of $\mathcal{A}_0$ with a finite basis of polynomials such that $T(\alpha):S \to S$. For a non-negative integer,n, $\alpha \in G$ and $z \in F$,

$$[T(\alpha)h_n](z) = \exp\left(\frac{\alpha^2}{2} + \alpha z\right)(z + \alpha)^n. \tag{4.2.5}$$

It is trivial to check that Eq.(4.1.15) holds. However, in order to obtain an addition theorem from the multiplier representation given by Eq.(4.2.4), we use the fact that $[T(\alpha)h_n](z)$ is analytic about $t = 0$ to expand Eq.(4.2.5) as an infinite power series. Using the binomial theorem and the power series expansion for $e^x$ we find:

$$[T(\alpha)h_n](z) = \exp\left(\frac{\alpha^2}{2} + \alpha z\right)(z + \alpha)^n = \exp\left(\frac{\alpha^2}{2}\right)\sum_{k=0}^{\infty}\frac{\alpha^k z^k}{k!}\sum_{j=0}^{\infty}\binom{n}{j}\alpha^{n-j}z^j,$$

where $\binom{n}{j} = \begin{cases}\dfrac{n!}{(n-j)!j!} & \text{if } j \le n \\ 0 & \text{if } j > n\end{cases}$ .

If $\alpha \neq 0$, then by Eq.'s (4.2.2) and (4.2.3) we conclude that,

$$[T(\alpha)h_n](z) = \exp\left(\frac{\alpha^2}{2}\right)\sum_{k=0}^{\infty}\frac{\alpha^{n+k}}{k!}z^k\,{}_2F_0(-n, -k; \_\,; \frac{1}{\alpha^2}),\quad \alpha \neq 0. \tag{4.2.6}$$

By Eq.(4.2.6),

$$[T(\alpha+\beta)h_n](z) = \exp\left(\frac{(\alpha+\beta)^2}{2}\right)\sum_{k=0}^{\infty}\frac{(\alpha+\beta)^{n+k}}{k!}z^k\,{}_2F_0(-n,-k;\_\,;\frac{1}{(\alpha+\beta)^2}),$$

where $(\alpha+\beta)\neq 0$. However, by Eq.'s (4.1.15) and (4.2.6), if $\alpha$ and $\beta$ are both non-zero, then

$$[T(\alpha+\beta)h_n](z) = T(\alpha)\left[\exp\left(\frac{\beta^2}{2}\right)\sum_{r=0}^{\infty}\frac{\beta^{n+r}}{r!}\,{}_2F_0(-n,-r;\_\,;\frac{1}{\beta^2})z^r\right]$$

$$= \exp\left(\frac{\alpha^2+\beta^2}{2}\right)\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{\alpha^{r+k}}{k!}\frac{\beta^{n+r}}{r!}\,{}_2F_0(-n,-r;\_\,;\frac{1}{\beta^2})\,{}_2F_0(-r,-k;\_\,;\frac{1}{\alpha^2})z^k.$$

Equate coefficients of $z^k$ in the two expressions for $[T(\alpha+\beta)h_n](z)$ to obtain the following addition theorem for the ${}_2F_0$ function,

$$e^{\alpha\beta}(\alpha+\beta)^{n+k}\,{}_2F_0(-n,-k;\_\,;\frac{1}{(\alpha+\beta)^2})$$

$$= \sum_{r=0}^{\infty}\frac{\beta^{n+r}\alpha^{r+k}}{r!}\,{}_2F_0(-n,-r;\_\,;\frac{1}{\beta^2})\,{}_2F_0(-r,-k;\_\,;\frac{1}{\alpha^2}), \qquad (4.2.7)$$

where $\alpha\neq 0$, $\beta\neq 0$, $\alpha+\beta\neq 0$ and $k$ and $n$ are nonnegative integers. By Eq.(4.2.2), the summation on the right hand side of Eq.(4.2.7) actually terminates after $r = n$.

**Example 4.2.2** Now we shall deal with polynomial sequences. Let $p_i(x)$ be a polynomial of exactly degree i, for $i = 0, 1, 2, \ldots$ . Then

$$p_0(x), p_1(x), p_2(x), \ldots ,$$

is known as a polynomial sequence. The polynomial sequence, $\{p_n(x)\}_{n=0}^{\infty}$, is a **sequence of binomial type** if it satisfies the infinite set of identities:

$$p_n(x+y) = \sum_{k=0}^{\infty}\binom{n}{k}p_k(x)p_{n-k}(y), \quad n = 0, 1, 2, \ldots \qquad (4.2.8)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ if $k\leq n$, and $\binom{n}{k} = 0$ if $k > n$.

A necessary and sufficient condition that $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of binomial type is the existence of a unique infinite series H(x) such that H(0) = 0, H'(0) ≠ 0 and

$$\sum_{n=0}^{\infty} \frac{p_n(\alpha)}{n!} z^n = e^{\alpha H(z)}. \tag{4.2.9}$$

(See Rota [1, p.8-20] for a detailed discussion of sequences of binomial type).

Eq.(4.2.9) implies Eq.(4.2.8) due to the fact that

$$e^{\alpha H(z)} e^{\beta H(z)} = e^{(\alpha+\beta)H(z)}. \tag{4.2.10}$$

To see this, use Eq.(4.2.9) to substitute for $e^{\alpha H(z)}$, $e^{\beta H(z)}$ and $e^{(\alpha+\beta)H(z)}$ in Eq.(4.2.10) and then use Eq.(4.2.3) to obtain

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{p_k(\alpha)}{k!}\frac{p_j(\beta)}{j!} z^{k+j} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{p_{k-j}(\alpha)}{(k-j)!}\frac{p_j(\beta)}{j!} z^k = \sum_{k=0}^{\infty} \frac{p_k(\alpha+\beta)}{k!} z^k.$$

Equate the coefficients of $z^k$ and rearrange to obtain Eq.(4.2.8). Thus, given that Eq.(4.2.9) is the generating function of a sequence of binomial polynomials then there exists a simple and obvious proof that the polynomials satisfy Eq.(4.2.8).

We can also utilize multiplier representations to obtain the same result. Our point in doing so is simply to verify that the group representation technique can be used in this simple case.

Let

$$D = H(z) + 0\frac{d}{dz}$$

be a linear differential operator, where H(z) has unique infinite power series expansion such that H(0) = 0, H'(0) ≠ 0 and there exists a polynomial sequence such that Eq.(4.2.9) holds. By Theorem 4.1.2 we know that D generates the Lie algebra of generalized Lie derivatives of a local Lie transformation group with group action and multiplier obtained by solving the differential equations

$$\frac{dx(t)}{dt} = 0,\quad x(0) = x^0 \quad\text{and}\quad \frac{d\ln(v(x^0, EXP(\alpha t)))}{dt} = \alpha H(x(t)),\ v(x^0, 0) = 1.$$

Thus, using the one-dimensional local Lie group used in Example 4.2.1, we obtain the multiplier

$$[T(EXP(\alpha))f](z) = [T(\alpha)f](z) = e^{\alpha H(z)}f(z). \qquad (4.2.11)$$

In the case of polynomial sequences, we are more interested in form than in analyticity. Thus we shall not concerned ourselves with ensuring that $e^{\alpha H(z)}f(z)$ is a function analytic in some neighborhood of $z = 0$. We will manipulate the infinite polynomial expansions without worrying about where the power series converges. Consider how $T(EXP(\alpha))$ acts on $h_k(z) = z^k$, $k = 0, 1, 2, \ldots$

Use Eq.'s (4.2.11), (4.1.15), (4.2.9) and Eq.(4.2.3) to conclude that

$$[T(\alpha + \beta)h_k](z) = e^{(\alpha+\beta)H(z)}z^k = \sum_{m=0}^{\infty} \frac{p_m(\alpha + \beta)}{m!} z^{m+k}$$

and

$$\begin{aligned}
[T(\alpha + \beta)h_k](z) &= \{T(\alpha)[T(\beta)h_k]\}(z) \\
&= T(\alpha)[\sum_{m=0}^{\infty} \frac{p_m(\beta)}{m!} z^{m+k}] \\
&= \sum_{m=0}^{\infty} \frac{p_m(\beta)}{m!} [T(\alpha)h_{m+k}](z) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{p_m(\beta)}{m!} \frac{p_n(\alpha)}{n!} z^{m+k+n} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{p_{m-n}(\beta)}{(m-n)!} \frac{p_n(\alpha)}{n!} z^{m+k}.
\end{aligned}$$

Fix m and equate the coefficients of $z^{m+k}$ in the two power series expressions for $[T(\alpha + \beta)h_k](z)$ to derive Eq.(4.2.8).

We shall now give another example of using multiplier representations to prove an addition theorem for a sequence of polynomials.

**Example 4.2.3** The polynomial sequence, $\{p_n(x)\}_{n=0}^{\infty}$, is a **Eulerian family of polynomials**, (see Andrews [1]), if

i) $p_0(x) = 1$

ii) for each n,

$$p_n(xy) = \sum_{k=0}^{n} \binom{n}{k}_q p_k(x)p_{n-k}(y)y^k, \tag{4.2.12}$$

where $\binom{n}{k}_q = \dfrac{[q]_n}{[q]_{n-k}[q]_k}$ , $[q]_0 = 1$ and $[q]_n = (1 - q)(1 - q^2) \ldots (1 - q^n)$, $n > 0$, $q \in F$.

Andrews, [1, p.356], Corollary to Theorem 6, shows that a necessary and sufficient condition for $\{p_n(x)\}_{n=0}^{\infty}$ to be an Eulerian family of polynomials is that its generating function is of the form

$$\sum_{n=0}^{\infty} \frac{p_n(x)}{[q]_n} t^n = \frac{H(xt)}{H(t)}, \tag{4.2.13}$$

where H(t) has the formal power series expansion of the form $\sum_{k=0}^{\infty} c_k \frac{t^k}{k!}$ where $c_k$ is the coefficient of $x^k$ in $p_k(x)$, $k = 0, 1, 2, \ldots$, (by the definition of a polynomial sequence, $c_k \neq 0$, $c_0 = 1$).

Let

$$P_k(x) = \frac{1}{[q]_k} p_k(x), \quad k = 0, 1, 2, \ldots$$

Then Eq.(4.2.12) becomes

$$P_n(xy) = \sum_{k=0}^{n} P_k(x)P_{n-k}(y)y^k \tag{4.2.14}$$

and Eq.(4.2.13) becomes

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{H(xt)}{H(t)}. \tag{4.2.15}$$

Thus, we can make the Eulerian polynomials q-free. We shall still refer to them as Eulerian polynomials. It is not obvious from the generating function Eq.(4.2.15) that the family of Eulerian polynomials, $\{P_n(x)\}_{n=0}^{\infty}$, satisfies the addition theorems given by Eq.(4.2.12). However, we can use multiplier representations to prove it and thus obtain an alternate proof to the one Andrews gave, that Eq.(4.2.15) is sufficient to guarantee Eq.(4.2.14).

Let

$$D = x\frac{H'(x)}{H(x)} + x\frac{d}{dx}, \tag{4.2.16}$$

where $H(t) = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!}$, $c_k \neq 0$, $0 \leq k < \infty$, $c_0 = 1$, be a generalized differential operator. Then by Theorem 4.1.2 we obtain the multiplier representation

$$[T(\alpha)f](z) = \frac{H(ze^{\alpha})}{H(z)} f(ze^{\alpha}). \tag{4.2.17}$$

T is indeed a multiplier representation with multiplier $v(z, \alpha) = \frac{H(ze^{\alpha})}{H(z)}$, because $Q[z, \alpha] = ze^{\alpha}$ is an action for the one-dimensional local Lie group G, $v(z, 0) = 1$ and $v(z, \alpha)v(ze^{\alpha}, \beta) = v(z, \alpha+\beta) = \frac{H(ze^{\alpha+\beta})}{H(z)}$.

Now, let T act on the polynomials $h_n(z) = z^n$, $n = 0, 1, 2, \ldots$ . If there exists a polynomial sequence $\{P_k(x)\}_{k=0}^{\infty}$ such that Eq.(4.2.15) holds, then by Eq.(4.2.17)

$$[T(\alpha)h_n](z) = \frac{H(ze^{\alpha})}{H(z)} z^n e^{n\alpha} = \sum_{k=0}^{\infty} P_k(e^{\alpha}) z^{n+k} e^{n\alpha}. \tag{4.2.18}$$

For simplicity, let $x = e^{\alpha}$ and $y = e^{\beta}$. Then Eq.(4.2.18) becomes

$$[T(\alpha)h_n](z) = \sum_{k=0}^{\infty} P_k(x) z^{n+k} x^n.$$

Thus,

$$[T(\alpha+\beta)h_n](z) = \sum_{k=0}^{\infty} P_k(xy) z^{n+k} x^n y^n .$$

However,

$$[T(\alpha)\{T(\beta)h_n\}](z) = T(\alpha)\{\sum_{k=0}^{\infty} P_k(y)z^{n+k}y^n\}$$
$$= \sum_{k=0}^{\infty} y^n P_k(y)[T(\alpha)h_{n+k}](z)$$
$$= \sum_{k=0}^{\infty} y^n P_k(y) \sum_{j=0}^{\infty} P_j(x)z^{n+k+j}x^{n+k}.$$

Use Eq.(4.2.3) to obtain

$$[T(\alpha)\{T(\beta)h_n\}](z) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} P_{k-j}(y)P_j(x)x^{n+k-j}y^n z^{n+k}.$$

From Eq.(4.1.15) it follows that

$$\sum_{k=0}^{\infty} P_k(xy)z^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} P_{k-j}(y)P_j(x)x^{k-j}z^k.$$

Compare coefficients of $z^k$ to obtain

$$P_k(xy) = \sum_{j=0}^{k} P_{k-j}(y)P_j(x)x^{k-j}, \quad k = 0, 1, 2, \ldots \tag{4.2.19}$$

which is equivalent to Eq.(4.2.14).

We have only shown that Eq.(4.2.19) holds for $x = e^{\alpha}$ and $y = e^{\beta}$. If $x \in \mathfrak{R}$, $x \neq 0$, then

$$x = \begin{cases} e^{\ln x} & \text{if } x > 0 \\ e^{\ln|x| + i\pi} & \text{if } x < 0 \end{cases}.$$

Thus, Eq.(4.2.19) holds for all $x \in \mathfrak{R}/\{0\}$. It is simple to check that Eq.(4.2.19) holds for $x = 0$. Thus given the generalized differential operator of Eq.(4.2.16) and a sequence of polynomials satisfying the generating function Eq.(4.2.15) we can derive the addition formula for the Eulerian family of polynomials defined on the field of real numbers. In Section 4.3 we give a final example of the use of multiplier representations in special function theory using a local Lie group of a higher dimension.

**Section 4.3 Multiplier Representations and the Hermite Polynomials.**

The special functions known as the **Hermite Polynomials**, $H_n(x)$, are defined by the relation

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}, \qquad (4.3.1)$$

valid for all finite $x$ and $t$. For a further discussion of the Hermite Polynomials see Rainville [1, p.187-199]. Our final examples involves these polynomials. We shall use the 3-dimensional local Lie group introduced in Section 1.2, Example 1.2.4 to prove the addition theorem

$$\left(\sqrt{a_1^2 + a_2^2}\right)^n H_n\left(\frac{a_1x_1 + a_2x_2}{\sqrt{a_1^2 + a_2^2}}\right) = \sum_{k=0}^{n} \binom{n}{k} a_2^k a_1^{n-k} H_k(x_2)H_{n-k}(x_1). \qquad (4.3.2)$$

We can prove Eq.(4.3.2) in a simple but not obvious manner from the generating function Eq.(4.3.1). Substitute

$$x = \frac{a_1x_1 + a_2x_2}{\sqrt{a_1^2 + a_2^2}} \quad \text{and} \quad t = z\sqrt{a_1^2 + a_2^2}$$

into Eq.(4.3.1). Then

$$\exp(2z\,(a_1x_1 + a_2x_2) - z^2(a_1^2 + a_2^2)) = \sum_{n=0}^{\infty} \frac{H_n\left(\dfrac{a_1x_1 + a_2x_2}{\sqrt{a_1^2 + a_2^2}}\right)}{n!} (a_1^2 + a_2^2)^{n/2} z^n. \qquad (4.3.3)$$

But, by Eq.(4.3.1),

$$\exp(2z(a_1x_1 + a_2x_2) - z^2(a_1^2 + a_2^2)) = \exp(2za_1x_1 - z^2a_1^2)\exp(2za_2x_2 - z^2a_2^2)$$

$$= \left(\sum_{n=0}^{\infty} \frac{H_n(x_1)a_1^n}{n!} z^n\right)\left(\sum_{k=0}^{\infty} \frac{H_k(x_2)a_2^k}{k!} z^k\right)$$

Use Eq.(4.2.3), to conclude that

$$\exp\left(2z(a_1x_1 + a_2x_2) - z^2(a_1^2 + a_2^2)\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{H_{n-k}(x_1)a_1^{n-k}}{(n-k)!} \frac{H_k(x_2)a_2^k}{k!} z^n. \qquad (4.3.4)$$

Compare the coefficient of $z^n$ in Eq.'s(4.3.3) and (4.3.4) to obtain Eq.(4.3.2), as required. We shall now prove Eq.(4.3.2) using multiplier representations.

From Example 1.2.4 we have a 3-dimensional complex local Lie group G, with Lie product defined by

$$\varphi(\mathbf{g}, \mathbf{h}) = \big((1+g_1)(1+h_1) - 1,\ (1+g_1)h_2 + g_2(1+h_1)^2,\ (1+g_1)h_3 + 2g_2h_2(1+h_1) + g_3(1+h_1)^3\big).$$

$$= \big(g_1 + h_1 + g_1h_1,\ g_2 + h_2 + g_1h_2 + 2g_2h_1 + g_2h_1^2,\ g_3 + h_3 + g_1h_3 + 2g_2h_1h_2 + 2g_2h_2 + 3g_3h_1 + 3g_3h_1^2 + g_3h_1^3\big), \tag{4.3.5}$$

where $\mathbf{g} = (g_1, g_2, g_3)$ and $\mathbf{h} = (h_1, h_2, h_3)$ are elements of G.

From the definition of $\varphi$ and Theorem 1.3.1 it follows that the Lie algebra of G, L(G), is the vector space $\mathbb{C}^3$ together with the commutator $[\alpha, \beta]$ such that

$$\left.\begin{array}{l} 1.\ [\alpha, \beta]_1 = 0, \\ 2.\ [\alpha, \beta]_2 = \alpha_2\beta_1 - \alpha_1\beta_2, \\ 3.\ [\alpha, \beta]_3 = 2(\alpha_3\beta_1 - \alpha_1\beta_3) \end{array}\right\}. \tag{4.3.6}$$

From Eq.(4.3.6) and Corollary 1.3.1 we find that the following three relations

$$\left.\begin{array}{l} 1.\ [(1, 0, 0), (0, 1, 0)] = -(0, 1, 0), \\ 2.\ [(1, 0, 0), (0, 0, 1)] = -2(0, 0, 1), \\ 3.\ [(0, 1, 0), (0, 0, 1)] = (0, 0, 0), \end{array}\right\} \tag{4.3.7}$$

completely determine the commutator for L(G). We shall now find the one-parameter subgroups of G.

It follows from Theorem 1.4.1 and Eq.(4.3.5) that the one-parameter subgroups $\mathbf{g}(t) = \mathbf{EXP}(\alpha t)$ are as follows:

If $\alpha_1 \neq 0$, then

$$\mathbf{g}(t) = \Big(\exp(\alpha_1 t) - 1,\ \frac{\alpha_2}{\alpha_1}[\exp(2\alpha_1 t) - \exp(\alpha_1 t)],\ \frac{\alpha_2^2}{\alpha_1^2}\Big[\frac{\exp(3\alpha_1 t)}{2} - \exp(2\alpha_1 t) + \frac{\exp(\alpha_1 t)}{2}\Big] + \frac{\alpha_3}{2\alpha_1}[\exp(3\alpha_1 t) - \exp(\alpha_1 t)]\Big).$$

If $\alpha_1 = 0$, then $\mathbf{g}(t) = \big(0,\ \alpha_2 t\ ,\ \alpha_2^2 t^2 + \alpha_3 t\big)$.

Writing the one-parameter subgroups in terms of the basis of L(G), we have

$$\left.\begin{array}{l} \mathbf{EXP}((1, 0, 0)t) = (e^t - 1, 0, 0), \\ \mathbf{EXP}((0, 1, 0)t) = (0, t, t^2), \\ \mathbf{EXP}((0, 0, 1)t) = (0, 0, t) \end{array}\right\}. \tag{4.3.8}$$

This leads to the following parameterization of G.

**Lemma 4.3.1** *There exists an open neighborhood, W, of* $\mathbf{e} \in G$ *such that every* $\mathbf{g} = (g_1, g_2, g_3) \in W$ *can be uniquely written*

$$\mathbf{g} = (g_1, g_2, g_3) = (e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau}),$$

for some $\tau, b, c \in \mathbb{C}$.

**Proof**: It is a simple matter to solve the three equations for the three unknowns $\tau$, b, and c. Since $\tau = \ln(1 + g_1)$, then we must restrict $g_1$ in order for $\tau$ to be uniquely defined.

Q.E.D.

The significance of this result follows from the fact that, as the reader can check,

$$(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau}) = \varphi\big(\mathbf{EXP}(0, 0, c), \varphi(\mathbf{EXP}(0, b, 0), \mathbf{EXP}(\tau, 0, 0))\big), \tag{4.3.9}$$

where $\varphi$ is defined by Eq.(4.3.5), and the one-parameter subgroups are obtained from Eq.(4.3.8).

Thus, Lemma 4.3.1 allows us to write elements in G in terms of the three one-parameter subgroups associated with the usual basis of L(G). We shall now discuss an algebra of Lie derivatives isomorphic to L(G). Let

$$D_1 = -u - x\frac{d}{dx}, \quad D_2 = \omega x \text{ and } D_3 = \omega x^2, \tag{4.3.10}$$

where u and $\omega$ are elements of $\mathbb{C}$, $\omega \neq 0$. Then the three differential operators are linearly independent and

$$\begin{array}{ll} (1) & [D_1, D_2] = -D_2, \\ (2) & [D_1, D_3] = -2D_3, \\ (3) & [D_2, D_3] = \phi. \end{array}$$

By Eq.(4.3.7) L(G) is clearly isomorphic to the Lie algebra generated by

$\{D_1, D_2, D_3\}$ with isomorphism defined by $(1, 0, 0) \to D_1$, $(0, 1, 0) \to D_2$, $(0, 0, 1) \to D_3$. Thus we can apply Theorem 4.1.2 to the current situation.

**Lemma 4.3.2**: *Let* $D_1$, $D_2$ *and* $D_3$ *be the three general differential operators defined by* Eq.(4.3.10). *Then*

$$[T(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau})f](z) = \exp[\omega(zb + z^2c) - u\tau]f(ze^{-\tau}) \tag{4.3.11}$$

*is a local multiplier representation on the local Lie group* G, *with Lie product defined by* Eq.(4.3.5).

**Proof**: The proof follows from Theorem 4.1.2 and Eq.'s(4.3.8) and (4.3.9). The Lie algebra generated by $D_1$, $D_2$ and $D_3$ is isomorphic to L(G), where G is the local Lie group with Lie product defined by Eq.(4.3.5). By Theorem 4.1.2, $D_1$, $D_2$ and $D_3$ form the basis of a Lie Algebra of generalized Lie Derivatives of a local multiplier representation **T**, obtained by integrating

$$\frac{d}{dt}\mathbf{x}(t) = -\alpha_1\mathbf{x}(t); \qquad \frac{d}{dt}\ln v(\mathbf{x}^0, \mathbf{EXP}(\alpha t)) = -\alpha_1 u + \alpha_2\,\omega\mathbf{x}(t) + \alpha_3\omega(\mathbf{x}(t))^2,$$

with the initial conditions $\mathbf{x}(0) = \mathbf{x}^0$, and $v(\mathbf{x}^0, 0) = 1$ and $\mathbf{EXP}(\alpha t)$ determined by Eq.(4.3.8). Instead of solving these differential equations directly for the multiplier $\mathbf{T}(\mathbf{EXP}(\alpha t))$, we do the following. Consider $[\mathbf{T}(\mathbf{EXP}((1,0,0)t))f](z)$. To find $[\mathbf{T}(\mathbf{EXP}((1,0,0)t))f](\mathbf{x}^0)$ we must integrate

$$\frac{d}{dt}\mathbf{x}(t) = -\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}^0; \quad \frac{d}{dt}\ln v(\mathbf{x}^0, \mathbf{EXP}(\alpha t)) = -u,\ v(\mathbf{x}^0, \mathbf{e}) = 1.$$

Clearly, $\mathbf{x}(t) = \mathbf{x}^0e^{-t}$ and $v(\mathbf{x}^0, \mathbf{EXP}((1,0,0)t)) = e^{-ut}$. Therefore,

$$[\mathbf{T}(\mathbf{EXP}((1, 0, 0)\tau))f](z) = e^{-u\tau} f(ze^{-\tau}). \tag{4.3.12}$$

Similarly,

$$[\mathbf{T}(\mathbf{EXP}((0, 1, 0)b))\, f](z) = \exp(\omega zb)f(z), \tag{4.3.13}$$

and

$$[\mathbf{T}(\mathbf{EXP}((0, 0, 1)c))\, f](z) = \exp(\omega z^2c)f(z). \tag{4.3.14}$$

Now consider $\left[T(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau})f\right](z)$. By Eq.'s (4.1.15), (4.3.12), (4.3.13) and (4.3.14),

$$[T(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau})f](z) = \left[\mathbf{T}\big((\mathbf{EXP}(0, 0, c)\mathbf{EXP}(0, b, 0))\mathbf{EXP}(\tau, 0, 0)\big)f\right](z)$$

$$= \{T(EXP(0, 0, c)) [T(EXP(0, b, 0)) (T(EXP(\tau, 0, 0))f)]\}(z)$$

$$= \exp(\omega z^2 c)\exp(\omega zb)e^{-u\tau} f(ze^{-\tau}),$$

which simplifies to the required result.

Q.E.D.

Note that, by Lemma 4.3.1, this means that we have completely defined the effect of the multiplier rep on an open neighborhood $W$ of **e**. Assume **g** $= (e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau}) \in W$.

**Lemma 4.3.3** *If* $c \neq 0$, *then*

$$[T(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau})f](z) = e^{-u\tau} \sum_{n=0}^{\infty} \frac{H_n\left(\frac{-b}{2}\sqrt{\frac{-\omega}{c}}\right)\left(z\sqrt{-\omega c}\right)^n}{n!} f(ze^{-\tau}). \quad (4.3.15)$$

**Proof:** From Eq.(4.3.11) we know that

$$[T(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau})f](z) = \exp[\omega(z^2 c + zb) - u\tau]f(ze^{-\tau}).$$

Let

$$x = \frac{-b}{2}\sqrt{\frac{-\omega}{c}} \text{ and let } t = z\sqrt{-\omega c}, \; c \neq 0.$$

Then

$$\exp[\omega(zb + z^2c)] = \exp(2xt - t^2) = \exp\left(2\left(\frac{-b}{2}\right)\sqrt{\frac{-\omega}{c}}\, z\sqrt{-\omega c} - z^2(-\omega c)\right)$$

Thus, by Eq.(4.3.1), Eq.(4.3.11) becomes

$$[T(e^{\tau} - 1, be^{2\tau}, (b^2 + c)e^{3\tau})f](z) = \exp[\omega(z^2 c + zb) - u\tau]f(ze^{-\tau})$$

$$= e^{-u\tau} \sum_{n=0}^{\infty} \frac{H_n\left(\frac{-b}{2}\sqrt{\frac{-\omega}{c}}\right)\left(z\sqrt{-\omega c}\right)^n}{n!} f(ze^{-\tau}),$$

as required.

Q.E.D.

Now that we can express our multiplier representation in terms of the Hermite polynomials we are ready to prove Eq.(4.3.2). Let $\mathbf{g}_1, \mathbf{g}_2, \varphi(\mathbf{g}_1, \mathbf{g}_2) \in G$. Then by Lemma 4.3.1 there exists $\tau_j$, $b_j$ and $c_j$, $j = 1, 2, 3$, such that

$$\mathbf{g}_j = (\exp(\tau_j) - 1, b_j\exp(2\tau_j), (b_j^2 + c_j)\exp(3\tau_j)), \qquad j = 1 \text{ or } j = 2,$$

and
$$\varphi(\mathbf{g}_1, \mathbf{g}_2) = (\exp(\tau_3) - 1, b_3\exp(2\tau_3), (b_3^2 + c_3)\exp(3\tau_3)).$$

By direct calculation from Eq.(4.3.5) we find that

$$\tau_3 = \tau_1 + \tau_2, \quad b_3 = (b_1 + b_2\exp(-\tau_1)) \text{ and } c_3 = (c_1 + c_2\exp(-2\tau_1)). \tag{4.3.16}$$

Now, let $\mathcal{A}_0$ be the space of all functions analytic in some neighborhood of zero. Then $\mathcal{A}_0$ has basis $h_k(z) = z^k$, $k = 0, 1, 2, \ldots$, and by Eq.(4.3.11), for $\mathbf{g}$ in some neighborhood of $\mathbf{e}$ and $|z|$ close to zero, $\mathbf{T}(\mathbf{g})\colon \mathcal{A}_0 \to \mathcal{A}_0$. Let $\mathbf{g}_1, \mathbf{g}_2, \varphi(\mathbf{g}_1, \mathbf{g}_2)$ be close to $\mathbf{e}$ and $|z|$ be close to zero. By Eq.(4.1.15)

$$[\mathbf{T}(\varphi(\mathbf{g}_1, \mathbf{g}_2))h_n](z) = [\, \mathbf{T}(\mathbf{g}_1) [\mathbf{T}(\mathbf{g}_2)\, h_n]\, ](z).$$

It follows from Eq.'s (4.2.3) and (4.3.15) that for $c_1$, $c_2$ both non-zero,

$$\exp(-u\tau_3)\sum_{k=0}^{\infty} \frac{H_k\left(\frac{-b_3}{2}\sqrt{\frac{-\omega}{c_3}}\right)\left(z\sqrt{-\omega c_3}\right)^k}{k!}(z\exp(-\tau_3))^n$$

$$= \exp(-u\tau_2)\sum_{j=0}^{\infty} \frac{H_j\left(\frac{-b_2}{2}\sqrt{\frac{-\omega}{c_2}}\right)\left(\sqrt{-\omega c_2}\right)^j}{j!} \exp(-n\tau_2)[\mathbf{T}(\mathbf{g}_1)h_{n+j}](z)$$

$$= \exp((-u-n)(\tau_2+\tau_1)) \sum_{k,j=0}^{\infty} \frac{H_j\left(\frac{-b_2}{2}\sqrt{\frac{-\omega}{c_2}}\right)\left(\sqrt{-\omega c_2}\right)^j}{j!} \frac{H_k\left(\frac{-b_1}{2}\sqrt{\frac{-\omega}{c_1}}\right)\left(\sqrt{-\omega c_1}\right)^k}{k!} \exp(-j\tau_1)z^{n+k+j}$$

$$= \exp((-u - n)(\tau_2 + \tau_1))$$
$$\times \sum_{k=0}^{\infty}\sum_{j=0}^{k} \exp(-j\tau_1)\frac{H_j\left(\frac{-b_2}{2}\sqrt{\frac{-\omega}{c_2}}\right)\left(\sqrt{-\omega c_2}\right)^j}{j!} \frac{H_{k-j}\left(\frac{-b_1}{2}\sqrt{\frac{-\omega}{c_1}}\right)\left(\sqrt{-\omega c_1}\right)^{k-j}}{(k-j)!} z^{n+k}.$$

However, by Eq.(4.3.16),

$$\exp(-u\tau_3)\sum_{k=0}^{\infty}\frac{H_k\left(\frac{-b_3}{2}\sqrt{\frac{-\omega}{c_3}}\right)\left(\sqrt{-\omega c_3}\right)^k}{k!}(z\exp(-\tau_3))^n$$

$$= \exp((-u-n)(\tau_1+\tau_2))$$

$$\times\sum_{k=0}^{\infty}\frac{H_k\left(\frac{-(b_1+b_2\exp(-\tau_1))}{2}\sqrt{\frac{-\omega}{(c_1+c_2\exp(-2\tau_1))}}\right)\left(\sqrt{-\omega(c_1+c_2\exp(-2\tau_1))}\right)^k}{k!}z^{n+k}.$$

Compare coefficients of $z^{n+k}$ in the two expressions of $[T(\varphi(g_1, g_2))h_n](z)$ to conclude that

$$H_k\left(\frac{-(b_1+b_2\exp(-\tau_1))}{2}\sqrt{\frac{-\omega}{(c_1+c_2\exp(-2\tau_1))}}\right)\left(\sqrt{c_1+c_2\exp(-2\tau_1)}\right)^k$$

$$=\sum_{j=0}^{k}\binom{k}{j}H_j\left(\frac{-b_2}{2}\sqrt{\frac{-\omega}{c_2}}\right)H_{k-j}\left(\frac{-b_1}{2}\sqrt{\frac{-\omega}{c_1}}\right)\left(\sqrt{c_2\exp(-2\tau_1)}\right)^j\left(\sqrt{c_1}\right)^{k-j}. \qquad (4.3.17)$$

Let

$$x_1 = \frac{-b_1}{2}\sqrt{\frac{-\omega}{c_1}},\quad x_2=\frac{-b_2}{2}\sqrt{\frac{-\omega}{c_2}},\quad a_1 = \sqrt{c_1},\ a_2 = \sqrt{c_2\exp(-2\tau_1)},$$

to transform the addition formula given by Eq.(4.3.17) into Eq.(4.3.2), as required.

If we wanted to, we could use the fact that,

$$[T(\varphi(\varphi(g, h), k))f](z) = [T(\varphi(g, h))\{T(k)f\}](z),$$

to obtain the generalized addition formula for the Hermite polynomials, which is:

$$\left(\sqrt{a_1^2+a_2^2+\ldots+a_n^2}\right)^n H_n\left(\frac{a_1x_1+a_2x_2+\ldots+a_nx_n}{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}\right)$$

$$=\sum_{k_1+k_2+\ldots+k_n=n}\binom{n}{k_1,k_2,\ldots,k_n}\prod_{j=1}^{n}a_j^{k_j}H_{k_j}(x_j),$$

where $\binom{n}{k_1,k_2,\ldots,k_n}=\frac{n!}{k_1!\,k_2!\,\ldots\,k_n!}$.

Both Miller [2, p.104-106, 138, 304-305] and Vilenkin [1, p.560-567] arrive at equivalent addition formulas and other properties for the Hermite polynomials using multiplier representations. Miller discovers the properties while investigating the representations of various Lie algebras and Vilenkin derives his results on Hermite polynomials after studying Gegenbauer polynomials and Bessel functions with the use of multiplier representations.

As the examples of Sections 4.2 and 4.3 indicate, multiplier representations can be a valuable tool in studying the properties of special functions. One starts with a Lie algebra, (or equivalently a local Lie group), examines the different multiplier representations to discover any special functions involved, then derives any possible addition theorems. In order to avoid repetition of results, the study should be done in a logical manner examining non-isomorphic Lie algebras and non-equivalent multiplier representations. The theory of multiplier representations provides a general method to examine special functions as opposed to using different methods specifically tailored for a given special function.

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