LINEARIZATION OF
AN ABSTRACT CONVEXITY SPACE

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ABSTRACT

Axiomatic convexity space, introduced by Kay and Womble [22], will be the main topic discussed in this thesis.

An axiomatic convexity space (X,C), which is domain finite and has regular straight segments, is called a basic convexity space. A weak complete basic convexity space is a basic convexity space which is complete and has C-isomorphic property. If in addition, it is join-hull commutative then it is called (strong) complete basic convexity space.

The main results presented are: a generalized line space is a weak complete basic convexity space, a complete basic convexity space is equivalent to a line space; and a complete basic convexity space whose dimension is greater than two or desarguesian and of dimension two, is a linearly open convex subset of a real affine space.

Finally, we develop a linearization theory by following an approach given by Bennett [3]. A basic convexity space whose dimension is greater than two, which is join-hull commutative and has a parallelism property, is an affine space. It can be made into a vector space over an ordered division ring and the members of C are precisely the convex subsets of the vector space.

BASIC NOTATION

The following terminlogy will be generally used in this thesis. Symbols other than these will be defined individually.

X	any arbitrary nonempty set
$\emptyset$	empty set
A,B	upper case letters usually will represent the subsets of X
a,b,x	lower case letters usually will be the elements of X
$\mathcal{P}(X)$	power set (all subsets of X)
$\subset$	usual set containment (eg. $A \subset B$ means A is contained in B)
$\cup$	usual set union operation
$\cap$	usual set intersection operation
$\backslash$	usual set different operation
$\in$	belongs to (eg. $x \in A$ means x belongs to A)
iff	if and only if
i.e.	that is to say
$\mathcal{R}$	real line
$\mathcal{R}^N$	N-dimensional real vector space
$\square$	end of the proof

Note that we do not distinguish singleton and the element of X.

CONTENTS

CHAPTER ONE

INTRODUCTION

1.1. Historical Review

Non-euclidean geometry or abstract geometry has been studied since the early 19th century. That is, to establish a geometry axiomatically without considering a real metric. Given a nonempty set, one defines points, lines, segments and etc., by imposing some axioms on them. For instance, the elements of a nonempty set may be called points, subsets are referred to as lines, segments and planes which satisfy certain properties. At first, we do not have any concept of distance and direction. However, we could define distance by means of a mapping to the real line and direction could be defined in a way similar to that in a vector space. Abstract projective space and affine space are the most common non-euclidean spaces which have been studied for the last hundred years.

In the usual sense, a convex is a subset of a vector space which, together with any two of its points, also contains the segment joining them. Naturally we may ask a question, what would a convex set look like in an arbitrary space ?

In 1951, F.W. Levi published a paper, 'On Helly's Theorem and the Axioms of Convexity' [23]. In this paper, he defines a semiconvex set and convex set for an arbitrary nonempty set. He says, given a set S, of elements called points in which a class C of subsets is distinguished. If C satisfies the following axioms C1 and C2, then the elements of C will be called semiconvex sets and in addition of C3 and C4, they will be called convex sets. The four axioms are:

C1 : if $K_i \in C$, then $\cap K_i \in C$ for every (not necessarily countable) family $\{K_i\}$,

C2 : let $m > n$ and $Q_1, Q_2, \ldots, Q_m \in |P_1P_2\ldots P_n|$ (where $|S'|$ is the smallest member of class C containing S'. If S' is finite with elements $\{P_1, P_2, \ldots, P_n\}$, then it is denoted by $|P_1P_2\ldots P_n|$), then there exist a permutation $i_1, i_2, \ldots, im$ of $1, 2, \ldots, m$ and a positive integer number $r < m$, such that $|Q_{i_1}\ldots Q_{i_r}| \cap |Q_{i_{r+1}}\ldots Q_{i_m}| = \emptyset$,

C3 : let $Q \in |P_1\ldots P_n| = K$ and for $i = 1, 2, \ldots, n$, $K_i = |P_1\ldots P_{i-1}QP_{i+1}\ldots P_n|$, then $K = \cup K_i$,

C4 : let $R \in |P_1P_2| \subseteq |P_1P_2P_3|$, then $|RP_3| = |RP_1P_3| \cap |RP_2P_3|$.

This gives a more general view than the definition given by G. Birkhoff [4] in early (1948). He defines a convex set of a poset $(P,\leq)$, as a subset which contains $[a,b]$ whenever it contains a, b, where $[a,b] = \{x \in P : a \leq x \leq b\}$ [cf. 2.1.2(4)]. Later, S.P. Franklin [15] (1962) developed this and called such convexity an order convexity.

W. Prenowitz [25] (1961), introduced the concept of join system in an arbitrary space. He gives the definition of a subset to be convex if the join of two elements belongs to it. This motivated V.W. Bryant and R.J. Webster [6] (1972) to obtain a convexity space based on this join system. They consider a pair $(X,\cdot)$, where $\cdot : X\times X \to P(X)$ is a function which associates with each ordered pair of elements a, b of X , a subset of X called the join of a and b satisfying the following axioms : (1) $ab \neq \emptyset$, $a/b \neq \emptyset$, (2) $a(bc) = (ab)c$, (3) $a/b \sim c/d$ implies $ad \sim bc$, (4) $aa = a = a/a$, (5) $ab \sim ac$ implies $b = c$ or $b \sim ac$ or $c \sim ab$, where $ab = a \cdot b$, $a/b = \{x : a \subseteq bx\}$ and $ab \sim ac$ means $ab \cap ac \neq \emptyset$. They define a subset A to be convex, if $A \cdot A \subset A$, where $A \cdot B = \cup\{ab : (a,b) \in A \times B\}$. A convexity space of type-2 was brought out by V.V. Tuz [31] (1974). This is a similar convexity space to that given by V.W. Bryant and R.J. Webster, but it satisfies only four axioms which are : (1) $\ell(a,b) = \ell(b,a)$, (2) $\ell(a,a) = \{a\}$, (3) $\{a,b\} \subseteq \ell(a,b)$ if $a \neq b$, (4) $c, d \in \ell(a,b)$ then $\ell(c,d) \subseteq \ell(a,b)$, where ℓ is a mapping from an ordered pair of a nonempty set into its power set.

Recently, J.P. Doignon [14] (1976) proves that the Pasch-Peano space without bound is a convexity space as defined by V.W. Bryant and R.J. Webster. The Pasch-Peano space is a linear space in which lines contain at least three points, are linearly ordered without bound and satisfy the Pasch-Peano's axiom, where a linear space is a set of points together with family of subsets called lines such that the following conditions hold, (1) two points belong to one and only one line, (2) each line contains at least two points.

A version of lattice characterization of convexity space was used by M.K. Bennett [1] in 1968. Recently, she obtained a convexity closure space [3] (1976) by taking a closure operator from a power set of a nonempty set X into itself, satisfying the conditions of : (1) point-closed, (2) additive, (3) locally finite, (4) extensive, (5) parallel, (6) skew, (7) $b, c \notin A = s(A)$, $s(A \cup b) = s(A \cup c)$, then $b = c$, and (8) $A = s(A)$, $B = s(B)$, $A \cap B = \emptyset$, $x \notin A \cup B$, then $s(A \cup x) \cap B = \emptyset$ or $s(B \cup x) \cap A = \emptyset$, where s is a closure operator on X , and A, B are subsets of X .

Interval convexity was introduced by J.R. Calder [9] (1971). T is an interval convexity on a nonempty point set S means that T is a transformation from $S \times S$ into $\mathcal{P}(S)$. He called a subset of S T-convex iff $T(x,y)$ belongs to it whenever it contains x and y. In his paper, he mentioned that a generalized convexity due to T.S. Motzken (Linear Inequality, Mincograph lecture notes, Univ. Calfornia, Los Angeles, 1951) derives from observation that the semispaces in $\mathcal{R}^N$ form a intersection base for the convex subsets of $\mathcal{R}^N$.

d-convex is another type of convexity which was introduced by P.S. Soltan [28] (1972). This convexity space is defined in a metric space. A set M of a metric space is called d-convex , if $d(x_1,x_2) = d(x_1,x_3) + d(x_3,x_2)$ with x_1, $x_2 \in M$, then $x_3 \in M$, where d is the metric in usual sense. This quite similar to the order-convexity space.

In about 1969-70, Kay and Womble gave a most general definition of an axiomatic convexity space. For any nonempty set X , $\mathcal{C}$ is a collection of the subsets of X closed under arbitrary intersection which includes $\emptyset$ and X , then $(X,\mathcal{C})$ is called a convexity space.

During the conference on convexity at Michigan State University in 1974, Kay proposed a linearization problem. He asked the question, " What is an algebraic structure for a given convexity space $(X,\mathcal{C})$ which makes X a vector space whose convex sets are precisely the members of $\mathcal{C}$?".

Recently, there are couple of papers dealing with this problem. Mah, Naimpally and Whitfield [24] (1974) first gave an external solution. They show that a convexity space (X,C) which satisfies domain finite, join-hull commutative and with the cancellation property (i.e. if $C(x,y) = C(z,y)$ then $x = z$, for all $x, y, z \in X$) that is the family of all convex sets generated by a real linear structure of X iff X has a linearization family X^*. X^* is a family of real-valued convexity-preserving functions on X and is called linearization family for X.

D.A. Szafron and J.H. Weston [30] (1977) obtained a linearization by considering a convexity space gridable over a field. Both of their convexity space follow the definition given by Kay and Womble.

M.K. Bennett [3] (1976) presents a linearization theory based on convexity closure space (see above). She proves that such a convexity closure space essentially is an affine space over an ordered division ring.

Doignon [14] (1976) was working on this problem too. He shows that the Pasch-Peano space without bound which is complete and whose dimension is greater than two, is a line space and it is a linearly open convex subset of a real affine space.

Besides these, there are quite a number of papers related to the axiomatic convexity space. They either contribute some new approach or extend the ideas. But most of them are working on the classical side, such as Helly's Theorem, Separation Theorem, etc. [5,6,7,8,26, 29,31].

In this thesis, we will follow Kay and Womble's definition for a convexity space and relate it to generalized line space [21], line space [10,11] and affine space. Finally, we reach the point of linearization.

1.2. Basic Definitions and Theorems

The following definitions and theorems will be used in this thesis when the terms is used in its usual sense. They can be found easily in most of the elementary set theory, algebra and analysis text books. The proofs of the theorems are omitted.

1.2.1. Definition : A set A is said to be a proper subset of a set B iff every element of A belongs to B and there are some elements of B which are not in A, it will be denoted by $A \subset B$. $A = B$ means $A \subseteq B$ and $B \subseteq A$.

1.2.2. Definition : A set X is said to be a finite set iff there is a positive integer number N, such that the elements of X are in one-to-one correspondence to the set of $\{1,2,3,\ldots\ldots.,N\}$ and we say X has cardinal number N. It is denoted by $|X| = N$. Otherwise, we say X is infinite.

1.2.3. Definition : A relation π on X is said to be a partial order relation, iff it satisfies,

(1) $x \,\pi\, x$ for $x \in X$ (reflexive),

(2) if $x \,\pi\, y$ and $y \,\pi\, x$, then $x = y$ (antisymmetry),

(3) if $x \,\pi\, y$ and $y \,\pi\, z$, then $x \,\pi\, z$ (transitivity),

for all $x, y, z \in X$.

__1.2.4. Definition__ : A nonempty set X with a partial order relation $\mathcal{R}$, $(X,\mathcal{R})$ is said to be a __partial ordered set__ or __poset__.

__1.2.5. Definition__ : A poset $(X,\mathcal{R})$ is said to be a __totally ordered set__ or __linearly ordered set__ or __chain__ iff for any $x, y \in X$, either $x \mathcal{R} y$ or $y \mathcal{R} x$ (i.e. any two elements are comparable).

__1.2.6. Definition__ : An element u of a poset $(X,\leq)$ is called an __upper bound__ iff for all $x \in X$, $x \leq u$. u is called a __least upper bound__, if for all upper bounds c , $u \leq c$. Similarly, an element ℓ of X is called a __lower bound__ iff for all $x \in X$, $\ell \leq x$ and ℓ is called a __greatest lower bound__, if for all lower bounds d , $d \leq \ell$.

__1.2.7. Theorem__ : Any bounded nonempty subset of $\mathbb{R}$ has a least upper bound and a greatest lower bound.

__1.2.8. Lemma (Zorn's Lemma)__ : For any chain C of the poset X , if C has an upper bound, then there exists a maximal element in X .

__1.2.9. Definition__ : A poset $(P,\leq)$ is said to be __dense-in-itself__ iff for any $a < b$ belonging to P , there is $c \in P$, so that $a < c < b$.

__1.2.10. Definition__ : A subset S of a poset $(P,\leq)$ is said to be __order-dense__ or __everywhere-dense__ iff for any $a < b$ belonging to P, there is $c \in S$, so that $a \leq c \leq b$.

<u>1.2.11. Definition</u> : Let (A, π_1) and (B, π_2) be two posets. A mapping $\alpha : A \to B$ is called an <u>order-isomorphism</u> (or <u>isomorphism</u>) iff α is one-to-one and preserves the order. More precisely, if $a_1 \pi_1 a_2$ for $a_1, a_2 \in A$, then $\alpha(a_1) \pi_2 \alpha(a_2)$, where $\alpha(a_1)$ and $\alpha(a_2)$ belong to B.

<u>1.2.12. Definition</u> : A totally ordered set is said to be <u>conditionally complete</u> iff every bounded subset has a greatest lower bound and least upper bound.

<u>1.2.13. Theorem</u> : A totally ordered set is isomorphic to a subset of the reals iff it contains a countable order-dense subset [4].

<u>1.2.14. Lemma</u> : A conditionally complete totally ordered set without maximal or minimal elements is isomorphic to the reals iff (i) it is dense-in-itself and (ii) it contains a countable dense subset [4].

<u>1.2.15. Definition</u> : A mapping $c : P(X) \to P(X)$, where X is a nonempty set, is called a <u>closure operator</u> of X iff it satisfies the following conditions :

(1) for each $A \in P(X)$, $A \subseteq c(A)$,

(2) if $A \subseteq B$ then $c(A) \subseteq c(B)$, for all $A, B \in P(X)$,

(3) for each $A \in P(X)$, then $c(A) = c(c(A)) = c^2(A)$.

<u>1.2.16. Definition</u> : A collection C , of subsets of an arbitrary nonempty set X , is called a <u>closure system</u> on X iff for all $F_i \in C$, $\cap F_i \in C$, $i = 1, 2, 3, \ldots$.

<u>1.2.17. Theorem</u> : A closure operator of X induces a closure system on X , and conversely.

<u>1.2.18. Definition</u> : $(G,*)$, a set G associated with an operator, $* : G \times G \to G$, is called a <u>group</u> iff $*$ satisfies the following conditions :

(1) if $g_1, g_2 \in G$, then $g_1 * g_2 \in G$,

(2) if $g_1, g_2, g_3 \in G$, then $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$,

(3) there is $e \in G$, so that $e * g = g = g * e$, for all $g \in G$,

(4) for each $g \in G$, there is a g^{-1} so that $g * g^{-1} = e = g^{-1} * g$.

Every subset of G with the operator $*$ which satisfies the above conditions, is called a <u>subgroup</u> of G .

<u>1.2.19. Theorem</u> : The intersection of an arbitrary family of subgroups is a subgroup.

<u>1.2.20. Definition</u> : A group $(G,*)$, is called <u>commutative group</u> or <u>abelian group</u> iff it satisfies the condition that $a*b = b*a$, for all a, b of G.

<u>1.2.21. Definition</u> : A <u>ring</u> is a set R associated with two operators $+$ and $*$ satisfying the following conditions :

(1) $(R,+)$ is an abelian group,

(2) if $r_1, r_2 \in R$ then $r_1 * r_2 \in R$,

(3) if $r_1, r_2, r_3 \in R$, then $(r_1 * r_2) * r_3 = r_1 * (r_2 * r_3)$,

(4) for $r_1, r_2, r_3 \in R$, the following equalities hold;

(i) $r_1 * (r_2 + r_3) = (r_1 * r_2) + (r_1 * r_3)$, and

(ii) $(r_1+r_2)*r_3 = (r_1*r_3)+(r_2*r_3)$.

Furthermore, if $r_1*r_2 = r_2*r_1$ for all $r_1, r_2 \in R$, then $(R,+,*)$ is called a commutative ring.

1.2.22. Definition : A ring, $(R,+,*)$ is said to be ring with identity if there is an element $i \in R$, so that $i*r = r = r*i$, for all $r \in R$.

1.2.23. Definition : A ring with identity $(R,+,*)$ is called a division ring, if for all $r \in R$ and $r \neq 0$ (identity of + , or zero element), there is $r^{-1} \in R$ such that $r*r^{-1} = i = r^{-1}*r$.

1.2.24. Definition : A commutative division ring is called a field.

1.2.25. Definition : A nonempty set V with two operators + and * , $(V,+,*)$ is called a vector space over a division ring R , if two operators satisfy the following conditions :

(1) $(V,+)$ is an abelian group ,

(2) if $u \in V$, $\lambda \in R$ then $\lambda*u \in V$,

(3) if $u, v \in V$, $\lambda \in R$ then $\lambda*(u+v) = (\lambda*u) + (\lambda*v)$,

(4) if $u \in V$, $\lambda, \mu \in R$ then $(\lambda+\mu)*u = (\lambda*u) + (\mu*u)$, and $\lambda*(\mu*u) = (\lambda*\mu)*u$,

(5) there is a $1 \in R$ so that $1*u = u = u*1$, for all $u \in V$.

1.2.26. Definition : A subset V_0 of a vector space $(V,+,*)$ over a division ring is said to be a subspace if and only if (1) for all $u, v \in V_0$ then $u+v \in V_0$ and (2) for all $u \in V_0$ and $\lambda \in R$, then $\lambda*u \in V_0$.

It is noted that a vector space is also called a <u>linear space</u>. We will use 'vector space' instead of 'linear space' to differentiate the abstract linear space defined in chapter four. If there is not ambiguity, we may use linear vector space or linear space in the usual sense.

A real vector space is a vector space over $\mathcal{R}$ (reals).

<u>1.2.27. Definition</u> : A subset S of a vector space $(V,+,*)$ is called a <u>convex set</u> iff for all $x, y \in S$, then $\lambda*x+(1-\lambda)*y \in S$, where $\lambda \in R$ (ordered division ring) and $0 \leq \lambda \leq 1$.

<u>1.2.28. Definition</u> : Let $d : X \times X \to \mathcal{R}$ be a mapping, where X is a nonempty set. (X,d) is called a <u>metric space</u> iff d satisfies the following axioms for all elements $x, y, z \in X$,

(1) $d(x,y) \geq 0$,

(2) $d(x,y) = 0$ if and only if $x = y$,

(3) $d(x,y) = d(y,x)$,

(4) $d(x,z) \leq d(x,y) + d(y,z)$.

CHAPTER TWO

BASIC PROPERTIES OF AN AXIOMATIC CONVEXITY SPACE

2.1. An Axiomatic Convexity Space

Let X be an arbitrary nonempty set and C be a collection of subsets of X . Following Kay and Womble's [22] approach, we have the following definition for an axiomatic convexity space.

2.1.1. Definition : (X,C) is said to be a convexity space, if C is closed under arbitrary intersection and includes $\emptyset$ and X .

2.1.2. Examples :

(1) Let G be a group [cf. 1.2.18]. C is the collection of all the subgroups of G including $\emptyset$ and G . It is easy to see that C is closed under arbitrary intersection, since an intersection of subgroups is a subgroup. Thus (X,C) is a convexity space.

(2) If $< S , \tau >$ is a topological space, we mean that τ is a collection of closed sets of S in the usual sense which satisfies :

(i) $\emptyset$, $S \in \tau$

(ii) $\cap F_i \in \tau$, for all $F_i \in \tau$

(iii) $\cup_1^n F_i \in \tau$, for $F_i \in \tau$, $i = 1, 2, \ldots, n$.

By the definition, $< S,\tau >$ is a convexity space. But if (S,C) is a convexity space, it is not necessarily a topological space, since (iii) is not always true.

(3) Let L be a real vector space. $\mathcal{C}$ is the collection of all the usual convex sets [cf. 1.2.27] in L . Then $(L,\mathcal{C})$ is a convexity space, since the intersection of convex sets is a convex set.

(4) Let $(P,\leq)$ be a poset [cf. 1.2.4] .

$\mathcal{C} = \{A : A \subseteq P$, for $a, b \in A$, $a \leq x \leq b$ implies $x \in A\}$.

It is easily seen that $(P,\mathcal{C})$ is a convexity space.

(5) Let $X = \{f : f : \mathcal{R} \to \mathcal{R}$ is continous function$\}$. Suppose $\mathcal{C}$ is the collection of the following sets, $F \subseteq X$, if $f, g \in F$, implies $f \circ g$ and $g \circ f \in F$, where $f \circ g$ is the usual composition function. Clearly $\emptyset, X \in \mathcal{C}$. If $f, g \in \cap F_i$ for $i \in I$, then $f, g \in F_i$, for all i . This implies $f \circ g$ and $g \circ f \in F_i$, for all i and thus $f \circ g$ and $g \circ f \in \cap F_i \in \mathcal{C}$ for $i \in I$. Hence $(X,\mathcal{C})$ is a convexity space.

As we go on further, we will give more examples of convexity spaces for which certain properties hold.

From Definition 2.1.1 and the examples shown above, we see that $\mathcal{C}$ is a closure system on X [cf. 1.2.16] . The hull operator can be defined naturally as follows.

<u>2.1.3. Definition</u> : Let $(X,\mathcal{C})$ be a convexity space. For $A \subseteq X$, the <u>convex hull</u> of A is defined as: $\mathcal{C}(A) = \cap\{C : C \in \mathcal{C}$ and $A \subseteq C\}$. The set $\mathcal{C}(A)$ is called the <u>$\mathcal{C}$-hull</u> of A or <u>convex hull</u> of A .

<u>2.1.4. Examples</u> :

(1) Take $(G,\mathcal{C})$ be a convexity space in Example 2.1.2(1). Then for each $A \subseteq G$, $A = \phi$ or $\mathcal{C}(A)$ convex hull of A is the smallest subgroup generated by A.

(2) Let E be a metric space [cf. 1.2.28]. $\mathcal{C}$ be the collection of all the closed subsets of E . Then $(E,\mathcal{C})$ is a convexity space. For $F \subseteq E$, $\mathcal{C}(F)$ will be the smallest closed subset of E which contains F or $\mathcal{C}(F) = F \cup \{\text{all limit points of } F\}$ or as we usually say $\mathcal{C}(F)$ is the closure of F .

(3) Let $X \equiv \mathcal{R}^2$, 2-dimensional euclidean plane. $\mathcal{C}$ is the collection of all the points and straight lines, including $\emptyset$ and X itself. $(X,\mathcal{C})$ is a convexity space. Take any two points in X , then the $\mathcal{C}$-hull of these points is the straight line passing through them. For any three non-collinear points (i.e. three point do not belong to the same line), then the $\mathcal{C}$-hull is $X \equiv \mathcal{R}^2$.

<u>2.1.5. Lemma</u> : Let $(X,\mathcal{C})$ be a convexity space. For each $A \subset X$, $A \in \mathcal{C}$ iff $\mathcal{C}(A) = A$.

Proof : If $A \in \mathcal{C}$, by Definition 2.1.3, we have

$$\mathcal{C}(A) = \bigcap\{C : C \in \mathcal{C} \text{ and } A \subseteq C\}.$$

Since $A \in \mathcal{C}$ and A is the smallest set containing itself. Hence

$$\mathcal{C}(A) = A .$$

If $A = \mathcal{C}(A)$, since $\mathcal{C}(A) \in \mathcal{C}$, $A \in \mathcal{C}$.

<u>2.1.6. Definition</u> : Let $(X,\mathcal{C})$ be a convexity space. A subset $A \subset X$ is called <u>$\mathcal{C}$-convex</u> or simply a <u>convex set</u> of X iff $\mathcal{C}(A) = A$ or A is a member of $\mathcal{C}$.

Since $\mathcal{C}(A)$ always contains A , we need only show $\mathcal{C}(A) \subseteq A$, for A to be a $\mathcal{C}$-convex set.

2.1.7. Examples :

(1) Consider example 2.1.2(1) again, the convex sets of the convexity $(G,\mathcal{C})$ are the subgroups of G .

(2) For example 2.1.4(2) , the convex sets of $(E,\mathcal{C})$ are the closed subsets of E .

2.2. Join-hull Commutative

For any members $x, y \in X$, $\mathcal{C}(x,y)$ is called the join of x and y or $\mathcal{C}$-join. In general, we define the $\mathcal{C}$-join of a point (or singleton) x and a subset A (of X) to be the set as follows.

2.2.1. Definition : $\mathcal{C}$-join (or simply join) of x and A in a convexity space $(X,\mathcal{C})$ is the set, $x \mathcal{J} A = \cup\{\mathcal{C}(x,a) : a \in A\}$, where $x \in X$ and $A \subset X$.

For any two subsets of X their $\mathcal{C}$-join will be

$A \mathcal{J} B = \cup\{\mathcal{C}(a,b) : (a,b) \in A\times B\}$.

2.2.2. Examples :

(1) Let X be a 3-dimensional real vector space. $\mathcal{C}$ be the collection of usual convex sets. For any two points, $x, y \in X$, $\mathcal{C}(x,y)$ will be the closed segment between x and y . If A is a convex set in X and x is a point which does not belong to A , then $x \mathcal{J} A$ is a convex cone (in the usual sense) with the vertex x .

Sometimes we will refer to $x \mathcal{J} A$ as a convex cone when A is a convex set in $(X,\mathcal{C})$. In general, where A is not a convex set, we just call $x \mathcal{J} A$ the $\mathcal{C}$-join or join of x and A .

(2) Consider Z_8 under the multiplication mod 8 (refer to the table for the operation). Let C be the collection of all the subsets of Z_8 which are closed under the operation and includes $\emptyset$ and Z_8. (Z_8,C) is a convexity space. Let $A = \{0,1\}$ and $B = \{5,6,7\}$. $A \mathcal{J} B$ is the set of $C(0,5)$, $C(0,6)$, $C(1,5)$, $C(1,6)$ and $C(1,7)$ where $C(0,5) = \{0,5,1\}$, $C(1,5) = \{1,5\}$ and etc.

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

eg. $1 * 5 = 5$, $5 * 7 = 3$ and etc.

$C(3,7) = \{1, 3, 5, 7\}$ and etc.

<u>2.2.3. Lemma</u> : Let (X,C) be a convexity space. For any $x \in X$, $A \subset X$, $x \mathcal{J} C(A) \subseteq C(x \cup A)$.

Proof : Let $y \in x \mathcal{J} C(A)$. Then $y \in C(x,a)$, for some $a \in C(A)$, since $x \mathcal{J} C(A) = \cup\{C(x,a) : a \in C(A)\}$. Since C is a closure operator on X, $C(A) \subseteq C(x A)$. Since $x, a \in C(x \cup A)$, $C(x,a) \subseteq C(x \cup A)$. Hence $y \in C(x \cup A)$ and $x \mathcal{J} C(A) \subseteq C(x \cup A)$. □

It is noted that the reverse inclusion is not always true.

<u>2.2.4. Example</u> : In R^3, 3-dimensional euclidean space, let C be the collection of 1- and 2-dimensional convex subsets (in usual sense) including $\emptyset$ and R^3. Let A be a 2-dimensional subset

of R^3 and x is a point which is not on the plane generated by A . Then $x \mathcal{J} C(A)$ is a convex cone [cf. 2.2.2] , but $C(x \cup A)$ is R^3 which is the smallest member of C containing x and A .

Throughout this chapter we will assume (X,C) to be a convexity space given by Definition 2.1.1, unless otherwise specified.

<u>2.2.5. Definition</u> : (X,C) is said to be <u>join-hull commutative at x</u> $\in X$ iff $x \mathcal{J} C(A) = C(x \cup A)$ for each $A \subset X$. It will be called <u>join-hull commutative</u> if the above property holds for each $x \in X$.

<u>2.2.6. Examples</u> :

(1) An n-dimensional euclidean space with the collection of usual convex sets is a join-hull commutative convexity space.

(2) Let L be a linear space [cf. 1.2.25] and C be the collection of all the linear subspace of L , $\emptyset$ and L . Then (L,C) is a join-hull commutative convexity space.

2.3. Domain Finite

<u>2.3.1. Definition</u> : (X,C) is said to be <u>domain finite</u> iff for each $A \subseteq X$, $C(A) = \cup\{C(F) : F \subseteq A$ and F is finite$\}$. In other words, for every $x \in C(A)$, there is a finite subset F of A , such that $x \in C(F)$.

For a finite subset A of X consisting of points $a_1, a_2, \ldots, a_n$, the C-hull of A will be denoted by $C(a_1, a_2, \ldots, a_n)$.

2.3.2. Examples :

(1) Let R be a finite ring. C is the collection of all the subrings of R and $\emptyset$. Then (R,C) is a convexity space which is domain finite, since every member of C is finite .

(2) Let X be a n-dimensional euclidean space. C is the collection of convex subsets in the usual sense. Then (X,C) is domain finite.

Remark : From the first example, we may notice that if X is a finite set, then it always is domain finite.

2.3.3. Theorem : If (X,C) is domain finite and join-hull commutative, then a subset A of X is C-convex iff $C(x,y) \subseteq A$ for all $x, y \in A$.

Proof : It is easy to see that if A is C-convex, then for $x, y \in A$, we have $C(x,y) \subseteq C(A) = A$.

Conversely, it suffices to show that $C(A) \subseteq A$. First, we will show by induction, that for each finite subset F of A , $C(F) \subseteq A$. For $|F| = 1$, i.e. $F = \{a\}$, then by assumption, $C(a) = C(a,a) \in A$. Hence $C(F) \subseteq A$.

Suppose it is true for $C(F) \subseteq A$ where $|F| \leq n-1$ and $F \subseteq A$. Let $t \in C(F)$ where $|F| = n$. Then $t \in C(a_1, a_2, \ldots\ldots, a_n)$. By join-hull commutativity, there is $b \in C(a_2, a_3, \ldots\ldots, a_n)$ such that $t \in C(a_1, b)$. From the induction hypothesis, $b \in A$, and thus $t \in C(a_1, b) \subseteq A$. Hence, $C(F) \subseteq A$ for any finite set $F \subseteq A$.

Finally, by domain finiteness,

$$C(A) = \cup\{C(F) : F \subseteq A , |F| < \infty\} \subseteq A. \qquad \square$$

<u>2.3.4. Lemma</u> : If (X,C) is a convexity space which is domain finite and join-hull commutative, then $C(A \cup B) = C(A) \mathcal{J} C(B) = C(B) \mathcal{J} C(A)$.

Proof : By Definition 2.2.1, we have ,

$$C(A) \mathcal{J} C(B) = \cup\{C(a,b) : a \in C(A) , b \in C(B)\}.$$

Let $a \in C(A)$ and $b \in C(B)$, then $a, b \in C(A) \cup C(B) \subseteq C(A \cup B)$. Therefore $C(a, b) \subseteq C(A \cup B)$ and $\cup C a, b) \subseteq C(A \cup B)$. Hence $C(A) \mathcal{J} C(B) \subseteq C(A \cup B)$.

By domain finiteness, for $x \in C(A \cup B)$, we have $x \in C(p_1, p_2, \ldots\ldots, p_n)$ for $p_1, p_2, \ldots\ldots, p_n \in A \cup B$. If $p_1, p_2, \ldots\ldots, p_n$ belong to either A or B we are done. Let $p_1, p_2, \ldots\ldots, p_m \in A$ and $p_{m+1}, p_{m+2} , \ldots\ldots, p_n \in B$ Then $x \in C(p_1, p_2, \ldots\ldots, p_m, p_{m+1}, \ldots\ldots, p_n)$. By join-hull commutative, we have $C(p_1, p_2, \ldots\ldots, p_m, p_{m+i}, \ldots\ldots, p_n)$

$$= p_1 \mathcal{J} p_2 \mathcal{J} p_3 \cdots\cdots \mathcal{J} p_m \mathcal{J} C(p_{m+1}, \ldots\ldots, p_n)$$

$$\subseteq C(p_1, p_2, \ldots\ldots, p_m) \mathcal{J} C(p_{m+1}, \ldots\ldots, p_n) \subseteq C(A) \mathcal{J} C(B).$$

hence $C(A \cup B) \subseteq C(A) \mathcal{J} C(B)$. □

<u>2.3.5. Corollary</u> : Suppose (X,C) is domain finite and join-hull commutative convexity space. If A, B are convex subsets of X , then for each $x \in C(A \cup B)$, there exist $a \in A$ and $b \in B$ such that $x \in C(a,b)$.

Proof : The result is immediate from the previous lemma. □

This is an important result, that we are going to use very often in the later chapter.

2.4. Regularity

2.4.1. Definition : (X,C) is regular iff the C-join of two ponits satisfies the following properties :

(1) non-discrete ; for every $x, y \in X$, $x \neq y$, then
$C(x,y) \setminus \{x,y\} \neq \emptyset$,

(2) decomposable ; for any $z \in C(x,y)$, then
(i) $C(x,z) \cap C(z,y) = \{z\}$ and (ii) $C(x,z) \cup C(z,y) = C(x,y)$,

(3) extendible ; for every $x \neq y$, then there exist $u, v \in X$, so that $C(x,y) \subseteq C(x,v) \setminus \{v\}$ and $C(x,y) \subseteq C(u,y) \setminus \{u\}$.

We will refer to the set

$\{z \in X : C(x,y) \subseteq C(x,z)\setminus\{z\}$ or $C(x,y) \subseteq C(z,y)\setminus\{z\}\}$ as the extension of $C(x,y)$ and use x/y and y/x for the sets $\{z \in X : C(x,y) \subseteq C(z,y)\setminus\{z\}\}$ and $\{w \in X : C(x,y) \subseteq C(x,w)\setminus\{w\}\}$ respectively. The extension of $C(x,x)$ will be defined as $\{x\}$.

Note that the statement that , for each $x \neq y$, there exist $u, v \in X$, so that $x \in C(y,v)\setminus\{v\}$ and $y \in C(x,u)\setminus\{u\}$, is not equivalent to the extendible property. However, the extendible property does not hold if this property fails.

2.4.2. Examples :

(1) Consider $X \equiv R^2\setminus S$, where S is a strip (see Figure 2.4.1) perpendicular to the x-axis in the two dimensional euclidean plane R^2 . Let C be the collection of convex sets in usual sense. Then it is easy to see that (X,C) is a convexity space. For any two points, $a, b \in X$, $C(a,b)$ will be the usual segment joined by them. Clearly the segment is decomposable

and extendible. But if we take a, b at each side of the strip, then $C(a,b) = \{a,b\}$. Hence it is discrete for those points on the oppsite edges of the strip.

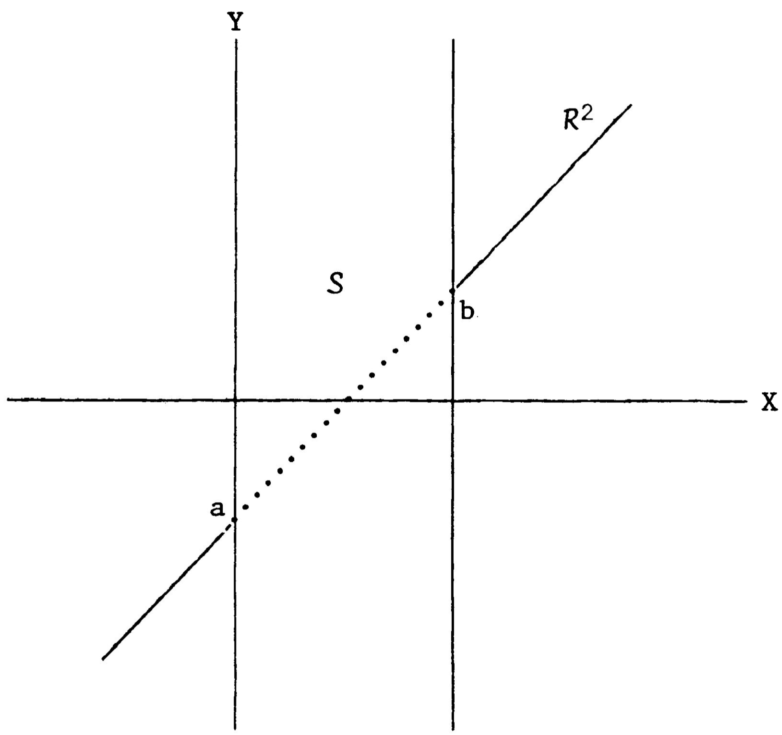


Figure 2.4.1

(2) Let X be a closed unit circle in R^2 (i.e. all the points inside the circle and on the circle itself, see Figure 2.4.2). Let C be the collection of convex sets (in usual sense) inside the circle. For any two points, $x, y \in X$, $C(x,y)$ is the usual line segment joined by them. It is easy to see (X,C) is a convexity space which is non-discrete and decomposable. But if we take a, b on the circle, $C(a,b)$ is the chord joined by them, which is not extendible.

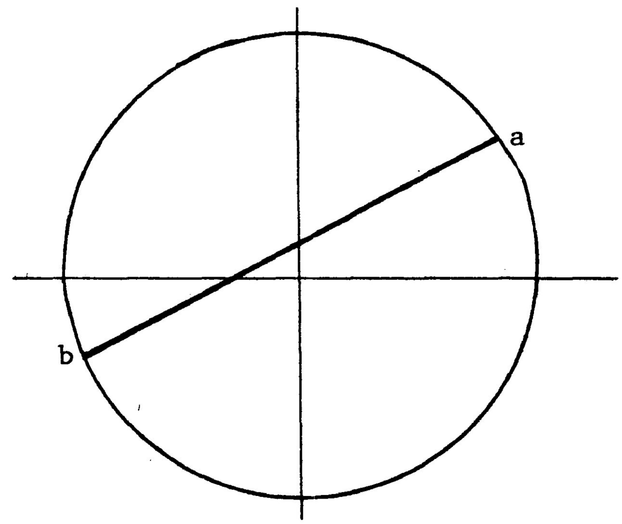


Figure 2.4.2

(3) Let $X \equiv R^2$. Define $C(a,b)$ as follows: (i) if the line joined by a and b is perpendicular to x-axis and y-axis, then $C(a,b)$ is a segment between them; (ii) otherwise, $C(a,b)$ is a rectangle formed by them (see Figure 2.4.3). $A \subseteq X$, is convex iff $C(a,b) \subseteq A$, whenever $a, b \in A$. Then (X,C) is a convexity space. It is clear that (X,C) is non-discrete and extendible. But if $c \in C(a,b)$, where $C(a,b)$ is a rectangle, then $C(a,c) \cup C(c,b) \neq C(a,b)$. Hence it is not decomposable.

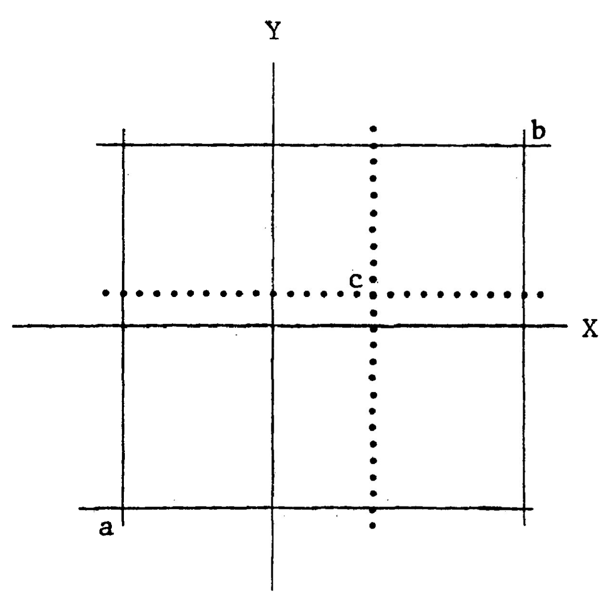


Figure 2.4.3

These examples show that the three properties of a regular convexity space are independent and Examples 2.4.2(3) and 2.4.6(2) show that the two conditions of decomposable are independent.

<u>2.4.3. Lemma</u> : If $(X,\mathcal{C})$ is a regular convexity space, then $C(x,x) = \{x\}$, for each $x \in X$.

Proof : Clearly $C(x,x) \neq \emptyset$, since at least $x \in C(x,x)$. By the non-discrete property, we have $C(x,y) \setminus \{x,y\} \neq \emptyset$, for $x \neq y$. Using the decomposable property, since $x \in C(x,y)$, then we have

(a) $C(x,x) \cap C(x,y) = \{x\}$ and

(b) $C(x,x) \cup C(x,y) = C(x,y)$

From (b) , it implies $C(x,x) \subseteq C(x,y)$. Hence $C(x,x) \cap C(x,y) = C(x,x)$. Therefore $C(x,x) = \{x\}$. □

Usually, we call this property as point-closed or T_1.

<u>2.4.4. Lemma</u> : Let $(X,\mathcal{C})$ be a regular convexity space. If $a \in C(b,c)$ and $b \in C(a,c)$, where $a, b \neq c$, then $a = b$.

Proof : $a \in C(b,c)$ implies $C(b,c) = C(a,b) \cup C(a,c)$ and $\{a\} = C(a,b) \cap C(a,c)$. But $b \in C(a,c)$ implies $\{b\} = C(a,b) \cap C(b,c)$ $= C(a,b) \cap [C(a,b) \cup C(a,c)] = C(a,b) \cup [C(a,b) \cap C(a,c)] = C(a,b) \cup \{a\}$ $= C(a,b)$. Similarly, $\{a\} = C(a,b)$. Hence $a = b$. □

<u>2.4.5. Lemma</u> : In a regular convexity space $(X,\mathcal{C})$, if $a \in C(b,c)$, then $b \notin C(a,c)$ and $c \notin C(a,b)$, for any $a, b, c \in X$ and $a \neq b \neq c$.

Proof : Suppose $b \in C(a,c)$. Then $C(b,c) \subseteq C(a,c)$. Since $a \in C(b,c)$, then $C(a,c) \subseteq C(b,c)$. Hence $C(a,c) = C(b,c)$. By the last lemma , $a = b$. Similarly, $c \notin C(a,b)$. □

<u>2.4.6. Examples</u> : The following examples show that domain finiteness, join-hull commutativdity and regularity are independent.

(1) Consider an annulus A, which is open at its outer cicle and closed at its inner circle. C is the collection of convex sets (in usual sense) of A. Then (A,C) is a convexity space that is domain finite, join-hull commutative, decomposable and extendible. But it is discrete, when we take two points x, y on the inner circle. $C(x,y) = \{x,y\}$.

(2) Let C be the collection of annuli with centre at origin in a two dimensional euclidean plane R^2. (R^2,C) is a convexity space which is domain finite, join-hull commutative, non-discrete and extendible. But which is not decomposable, since for $z \in C(x,y)$, $C(x,z) \cap C(z,y)$ is a circle with centre at origin, containing z.

(3) Example 2.4.2(2) is a convexity space which is domain finite, join-hull commutative, non-discrete and decomposable, but which is not extendible.

(4) In general, a generalized line space [cf. 3.3], with its collection of convex sets is a convexity space, which is domain finite and has regular segments, but is not join-hull commutative. For details of the proof, refer to 3.3, next chapter.

(5) Let X be a linear metric space. Any subset of X is said to be closed convex if it is a closed subset as well as a convex subset (in usual sense) in X. C is the collection of closed convex sets and $\emptyset$, X. It is easy to show that (X,C) is a convexity space since intersection of closed andconvex sets is closed and convex.

For any two distinct points, the smallest closed convex containing them is a usual closed segment. Clearly, the segments are regular (i.e. non-discrete, decomposable and extendible). Also (X,C) is join-hull commutative. But (X,C) is not domain finite. To see this, take A to be an open convex set in X. Then $C(A)$ is a closed convex set. Choose a point $x \in C(A)$, so that x is a boundary point (in usual sense). We can not find a finite subset F, of A, so that $x \in C(F)$, since $C(F)$ will be totally contained in A (i.e. $C(F)$ does not meet the boundary of A).

2.5. Others

2.5.1. Pasch's Axiom : A convexity space (X,C) satisfies Pasch's axiom, if for $y \in C(a,c)$ and $z \in C(b,y)$, then there is a $x \in C(a,b)$, such that $z \in C(c,x)$.

2.5.2. Lemma : A join-hull commutative convexity space (X,C) satisfies Pasch's axiom.

Proof : Let $y \in C(a,c)$ and $z \in C(b,y)$. We wish to show that there is a $x \in C(a,b)$, such that $z \in C(c,x)$. Since $z \in C(b,y)$ and $y \in C(a,c)$, therefore $z \in C(a,b,c)$. By join-hull commutative, we have $C(a,b,c) = c \, J \, C(a,b)$. Hence $z \in C(c,x)$, for some $x \in C(a,b)$. The lemma is proved. □

Note that the converse is not always true. Consider $X \equiv R^2$ (2-dimensional euclidean plane) and C is the collection of 1-dimensional linear convex sets (i.e. segments and lines in usual sense), points, R^2 and $\emptyset$. (X,C) is a convexity space satisfying Pasch's axiom but which is not join-hull commutative. Since take any three non-collinear points, the smallest convex set containing them is R^2 itself, and the join are the segments in the triangle formed by them.

<u>2.5.3. Example</u> : We also notice that a convexity space which is domain finite and regular, may not satisfy Pasch's axiom.

Consider the Moulton plane, M which is a two dimensional euclidean plane where all the positive slope lines are broken at x-axis at an angle (refer to Figure 2.5.1). The equation of lines can be written in the form of : $y = \delta(x - a)\tan\Theta$ where $\delta = 1$ if $\frac{\pi}{2} \leq \delta \leq \pi$ and $\delta = c \neq 1$ if $0 < \delta < \frac{\pi}{2}$ and c is a constant.

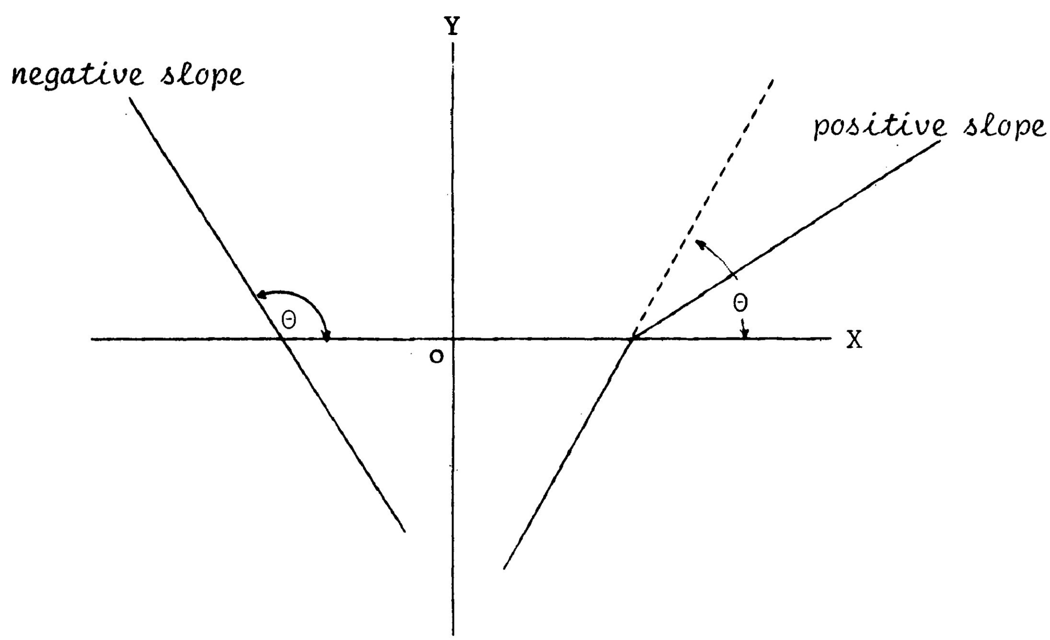


Figure 2.5.1

Call those broken lines, 'K' lines. Now, take $X \equiv M \times R$. Let C be the collection of usual convex sets and X, $\emptyset$. It is not hard to see that (X,C) is domain finite and regular.

Construct a triangluar, shown as in Figure 5.2.2, in a Moulton plane. The three lines $L(a,b)$, $L(a,c)$ and $L(c,x)$ will be on the same plane in $M \times R$, where the segment (b,y,w) is on the plane of $\{a,b,c\}$, but the segment (w,z) will be out of that plane. Therefore the Pasch's axiom does not hold in $M \times R$. We also note that this is another example of a convexity space which is domain finite and regular with the collection of convex sets (in usual sense), and it is not join-hull commutative.

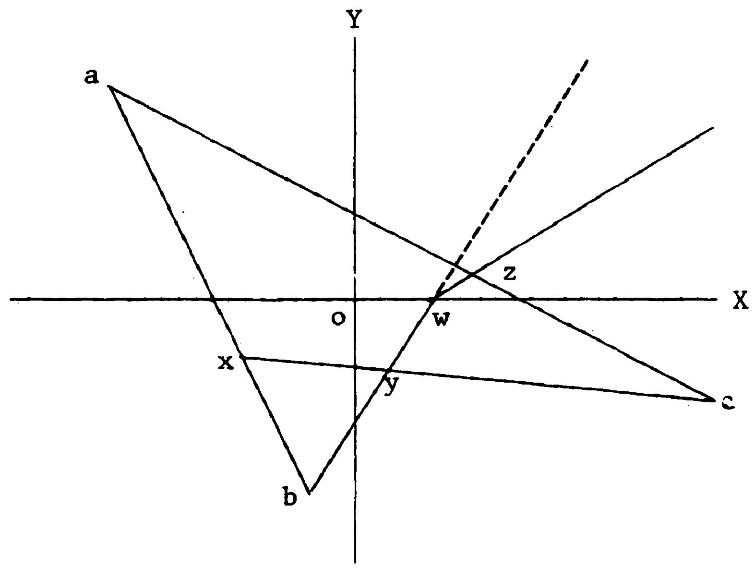


Figure 2.5.2

To simplify our notation, the following terminology will be used from next chapter onwards.

$C(a,b)$ will be denoted by ab. For $C(a,b)\setminus\{a\}$, $C(a,b)\setminus\{b\}$ and $C(a,b)\setminus\{a,b\}$, we will use $(ab]$, $[ab)$ and (ab) respectively. If $a = b$, define $[aa) = (aa] = (aa) = \{a\}$. Further a singleton $\{x\}$ will be x and always is referred to as a point in (X,C). Generally, we will call ab a segment and (ab) an open segment.

CHAPTER THREE

COMPLETE BASIC CONVEXITY SPACE

In this chapter, we are looking for necessary and sufficient conditions for an axiomatic convexity space [cf. 2.1] to be a line space [cf. 3.5, 21]. We refer to such a space as a complete basic convexity space which is a weak complete basic convexity space that is join-hull commutative. A generalized line space [cf. 3.3, 10] is a weak complete basic convexity space.

3.1. Basic Convexity Space

3.1.1 Definition : An axiomatic convexity space $(X,\mathcal{C})$ is said to be straight iff the union of two segments having more than one point in common is a segment.

3.1.2. Definition : An axiomatic convexity space $(X,\mathcal{C})$ is called a basic convexity space iff it is domain finite and has regular straight segments.

3.1.3. Examples.

(1) An euclidean n-space is a typical basic convexity space with $\mathcal{C}$ is the collection of all the convex subsets in usual sense.

(2) Let $(P,\leq)$ be a linearly ordered set satisfying the following conditions for all $a, b \in P$ and $a \neq b$;

(a) there is $c \in P$, such that $a \leq c \leq b$,

(b) there are $c, d \in P$, such that $c \leq a \leq b$ and $a \leq b \leq d$ with $a \neq b \neq c \neq d$.

$\mathcal{C}$ is the collection of all the subsets A of P, so that for all $a, b \in A$, if $a \leq x \leq b$ implies $x \in A$. In example 2.1.3. (4), we have seen that $(P,\mathcal{C})$ is a convexity space. The segment $\mathcal{C}(a,b)$ will be the set of $x \in P$, such that $a \leq x \leq b$. It is easy to see that the segment is regular and straight. To check that $(P,\mathcal{C})$ is domain finite, we will show

$$\mathcal{C}(A) = \bigcup \{\mathcal{C}(a,b) : a, b \in A\} = U$$

Clearly, $U \subseteq \mathcal{C}(A)$. We need to show that $\mathcal{C}(x,y) \subseteq U$ whenever $x, y \in U$. This implies that $x \in \mathcal{C}(a_1,b_1)$ and $y \in \mathcal{C}(a_2,b_2)$, for some $a_1, b_1, a_2, b_2 \in A$. Let $t \in \mathcal{C}(x,y)$, then $a_1 \leq x \leq t \leq y \leq b_2$. Hence $t \in U$. Therefore $(P,\mathcal{C})$ is a basic convexity space.

<u>3.1.4. Definition</u> : Let $(X,\mathcal{C})$ be a basic convexity space. For $a, b \in X$ and $a \neq b$, the <u>line</u> determined by a, b is the set of ab and its extension [cf. 2.4.1], which will be denoted by $L(a,b)$.

From definition 2.4.1. of extension, we see that $L(a,b) = ab \cup a/b \cup b/a$, for $a \neq b$. If $a = b$, then $L(a,a) = a$.

In future, $L(a,b)$ will be referred to as a line determined by two distinct points a and b. For any line in $(X,\mathcal{C})$, we will use ℓ. The following lemma and theorem will show that each line in a basic convexity space is uniquely determined by two distinct points.

<u>3.1.5. Lemma</u> : Let $(X,\mathcal{C})$ be a basic convexity space. For each $x \in L(a,b)$, $xa \subseteq L(a,b)$ and $xb \subseteq L(a,b)$ where $a, b \in X$ and $a \neq b$.

Proof : Let $x \in L(a,b) = ab \cup a/b \cup b/a$. If $x \in ab$, it is easily seen that $xa \subseteq ab \subseteq L(a,b)$ and $xb \subseteq ab \subseteq L(a,b)$. Now if $x \in a/b$, that is $a \in xb$, let $t \in xa \subseteq xb$. By regularity, $xb = xt \cup tb$. So $a \in tb$, since by Lemma 2.4.5, $a \notin xt$. Thus $t \in L(a,b)$ and $xa \subseteq L(a,b)$. Similarly, $xb \subseteq L(a,b)$. Finally, if $x \in b/a$, the above argument works by switching the role of a and b . □

<u>3.1.6. Lemma</u> : Let (X,C) be a basic convexity space. For $a, b, c \in X$, if $ab \cup bc$ is a segment , then $c \in ab$ or $b \in ac$ or $a \in bc$.

Proof : The result follows immediately from the properties of regularity and straightness. □

<u>3.1.7. Lemma</u> : Let (X,C) be a basic convexity space. If $y \in L(a,b)$, with $y \neq a$, then $L(a,y) = L(a,b)$, where $a, b, y \in X$ and $a \neq b$.

Proof : It is trival when $y = b$. We may assume that $y \neq b$ and we have $y \in L(a,b) = ab \cup a/b \cup b/a$. Let $t \in L(a,y) = ay \cup a/y \cup y/a$.

Suppose $t \in ay$. If $y \in ab$, it is obvious that $t \in ay \subseteq ab \subseteq L(a,b)$. Secondly, if $y \in a/b$, that is $a \in by$, by regularity, $by = ay \cup ab$. This implies that $t \in by$ and it follows that $by = yt \cup tb$. Hence $a \in tb$ since $a \notin yt$. Therefore $t \in L(a,b)$.

Now assume $t \in a/y$, that is $a \in ty$. If $y \in ab$, by regularity, $ty = at \cup ay$ and $ab = ay \cup yb$. It follows from straightness, $ty \cup ab = at \cup ab$ is a segment. By lemma 3.1.6, we have $t \in L(a,b)$. If $y \in a/b$, that is $a \in by$, since $ay \subseteq by$ and $ay \subseteq ty$, therefore by straightness, $ty \cup by$ is a segment .

By Lemma 3.1.6 and Lemma 3.1.5, if $t \in by \subseteq L(a,b)$. If $b \in ty$, then $b \in ay \subseteq at$. Since $b \in ay$, hence $b \in at$. And if $y \in tb$, then $a \in by \subseteq tb$. Therefore $t \in L(a,b)$.

By switching the role of a and b , y and a we have the other cases, and hence $L(a,y) \subseteq L(a,b)$. Finally, using an argument similar to that above, by changing y and b , we have $L(a,b) \subseteq L(a,y)$. Hence $L(a,y) = L(a,b)$. □

<u>3.1.8. Theorem</u> : Let $(X,\mathcal{C})$ be a basic convexity space. For $a, b \in X$, if $a \neq b$, then $L(a,b)$ is uniquely determined by two points.

Proof : We need to show that $L(x,y) = L(a,b)$ for any $x, y \in L(a,b)$. By Lemma 3.1.7, we have $y \in L(a,b)$ yields $L(a,y) = L(a,b)$ and $x \in L(a,y)$ implies $L(x,y) = L(a,b)$. Hence $L(x,y) = L(a,b)$. □

<u>3.1.9. Definition</u> : Any three distinct points in a basic convexity space are said to be <u>collinear</u>, iff they belong to an unique line. Otherwise, we say they are <u>non-collinear</u>.

<u>3.1.10. Lemma</u> : If any two distinct lines in a basic convexity space intersect, their intersection is a point.

Proof : Let ℓ and m be two distinct lines. Suppose $x, y \in \ell \cap m$. Then by Theorem 3.1.8, $\ell = L(x,y) = m$. □

<u>3.1.11. Definition</u> : In a basic convexity space $(X,\mathcal{C})$, $\overline{} : P(X) \to P(X)$ is an operator defined by $\overline{A} = \bigcap\{F : F \supseteq A \text{ and } L(x,y) \subseteq F \text{ whenever } x,y \in F\}$.

<u>3.1.12. Definition</u> : For any subset A of a basic convexity space, $\overline{A}$ is called the <u>flat</u> generated by A .

<u>3.1.13. Definition</u> : A subset A of a basic convexity space is said to be <u>independent</u> iff $a \notin \overline{\{A\backslash a\}}$ for all $a \in A$.

<u>3.1.14. Definition</u> : A basic convexity space (X,C) is said to have <u>finite dimension n</u> iff there exists a finite point subset $\{ x_0, x_1, \ldots\ldots, x_n \}$, such that it is independent and spans the whole space (i.e. $X = \overline{\{x_0, x_1, \ldots\ldots., x_n\}}$).

We will often call $\overline{\{x_0, x_1, x_2\}}$ a plane, where $\{x_0, x_1, x_2\}$ is an independent subset of X .

3.2. Ordering of a Line

From the previous section, we have seen every two distinct points in a basic convexity space uniquely determine a line. To place an ordering on a line ℓ , we will choose two points, 0 and 1 on ℓ . Once these two points are chosen, they will be fixed.

<u>3.2.1.</u> Let ℓ be a line in a basic convexity space, (X,C) , define the following two sets ;

$$R_- = \{x \in \ell : 0 \in x1\} ,$$

$$R_+ = \{x \in \ell : x \in 01 \text{ or } 1 \in 0x\} ,$$

for two fixed elements $0, 1 \in \ell$.

Set a relation $\leq$ on ℓ as follows :

for $a, b \in \ell$, we say $a \leq b$ iff one of the following conditions holds;

(1) $a \in R_-$ and $b \in R_+$,

(2) $b \in a0$, for $a, b \in R_-$,

(3) $a \in 0b$, for $a, b \in R_+$.

3.2.2. Lemma : Let $(X,\mathcal{C})$ be a basic convexity space and ℓ be a line. If $a, b, c \in \ell$, then $a \in bc$ or $b \in ac$ or $c \in ab$.

Proof : This is an immediate result of Theorem 3.1.8. □

3.2.3. Lemma : Let $(X,\mathcal{C})$ be a basic convexity space and ℓ be a line. Then $(\ell,\leq)$ is a poset, where $\leq$ is defined as in 3.2.1.

Proof : To show $(\ell,\leq)$ is a poset [cf. 1.2.4], we need to check the following :

(I) $a \leq a$ for all $a \in \ell$, which is immediate .

(II) If $a \leq b$ and $b \leq a$, then $a = b$. For $a, b \in R_+$, $a \leq b$ and $b \leq a$ then $b \in a0$ and $a \in b0$. Thus by Lemma 2.4.4, $a = b$. Similarly for $a, b \in R_-$, $a = b$. If $a \leq b$ for $a \in R_-$ and $b \in R_+$ and $b \leq a$ for $a \in R_+$ and $b \in R_-$, then $a = b = 0$.

(III) If $a \leq b$, $b \leq c$ show $a \leq c$. (i) If $a, b, c \in R_-$, then $b \in a0$, $c \in b0$. So $c \in a0$ and $a \leq c$. (ii) If $a, b \in R_-$, $c \in R_+$ or $a \in R_-$, $b, c \in R_+$, then $a \leq c$. (iii) If $a, b, c \in R_+$, then $a \in 0b$, $b \in 0c$. So $a \in 0c$ and $a \leq c$. Hence $(\ell,\leq)$ is a poset. □

3.2.4. Theorem : Let $(X,\mathcal{C})$ be a basic convexity space and ℓ be a line. Then $(\ell,\leq)$ is a totally ordered set.

Proof : We only need to show that for any $a, b \in \ell$ and $a \neq b$, either $a \leq b$ or $b \leq a$.

(I) If $a \in R_-$ and $b \in R_+$, then $a \leq b$. If $b \in R_-$ and $a \in R_+$, then $b \leq a$.

(II) If $a, b \in R_-$, then $0 \in a1$ and $0 \in b1$. Now we have $\ell = L(a,b)$. In this case, $0 \in L(a,b)$, so $a \in 0b$ or $b \in 0a$ or

$0 \in ab$. Clearly, $0 \notin ab$. Therefore, $a \in 0b$ in which case $b \leq a$ or $b \in 0a$ and $a \leq b$.

(III) If $a, b \in R_+$, we have by an argument similar to that in II that $a \leq b$ or $b \leq a$.

Therefore any two elements in ℓ are comparable. Hence $(\ell,\leq)$ is totally ordered set. □

In future, we will assume $\leq$ be an order relation on ℓ.

3.3. Generalized Line Space

3.3.1. Let X be a nonempty set and L be a collection of subsets of X . The members of X and L are called points and lines respectively. Suppose every line has a given ordering (say $<$). Following Kay and Sandstrom [21], we have the following definition for a generalized line space.

3.3.2. Definition : The pair (X,L) is called a generalized line space if the members of L satisfy the following conditions :

(1) every line is order isomorphic to the reals ,

(2) every line is uniquely determined by two distinct points of X .

The term order isomorphic means that a mapping from an ordered set to an ordered set is one-to-one, onto and preserves the order [cf. 1.2.11]. Denote the line determined by a and b by $L(a,b)$.

3.3.3. Definition : If a, b are distinct points and $a \leq b$, then the (open) segment determined by a and b is

$$(a,b) = \{ c \in L(a,b) : a < c < b \}.$$

$(a,b] = (a,b) \cup \{b\}$, $[a,b) = (a,b) \cup \{a\}$ and $[a,b] = (a,b) \cup \{a,b\}$ are referred to a left-open (or right-closed), right-open (or left-closed) and closed segments, respectively. Define $(a,a) = [a,a) = (a,a] = [a,a] = \{a\}$.

By Definition 3.3.3, we can see that the set,

$$L(a,b) = \{ c \in X : c \in [a,b] \text{ or } a \in (c,b) \text{ or } b \in (a,c) \}.$$

If $a = b$, then $L(a,a) = \{a\}$.

3.3.4. Lemma : In a generalized line space, every open segment is order isomorphic to an open segment of the reals.

Proof : Let (a,b) be an open segment contained in the line ℓ. It is easily shown that (a,b) is isomorphic to $(\phi_\ell(a),\phi_\ell(b))$, where ϕ_ℓ is an order isomorphism of ℓ onto R. □

3.3.5. Lemma : Every open segment in a generalized line space (X,L) is order isomorphic to a line ℓ in L.

Proof : This follows from Lemma 3.3.4 and the fact that each open interval in R is order isomorphic to R itself. □

3.3.6. Lemma : The following properties hold in a generalized line space (X,L) , for $w, x, y, z \in X$,

(1) (x,y) is not empty for $x \neq y$;

(2) if $z \in (x,y)$, then $(x,z] \cap [z,y) = \{z\}$ and $(x,z] \cup [z,y) = (x,y)$;

(3) for each (x,y), there are z, w, so that $x \in (z,y)$ and $y \in (x,w)$.

Proof : (1) Suppose (x,y) is empty for $x \neq y$, then $(\phi(x),\phi(y))$ is empty, where ϕ is an isomorphism from $L(x,y)$ onto $\mathcal{R}$. Thus $\phi(x) = \phi(y)$ and $x = y$.

(2) and (3) follow from the properties of the real line. □

<u>3.3.7. Lemma</u> : The union of two segments in a generalized line space having more than one point in common is a segment.

Proof : Let $[a,b]$ and $[c,d]$ be two segments which have more than one point in common. Let $x, y \in [a,b] \cap [c,d]$. Since $L(x,y) = L(a,b)$ and $L(x,y) = L(c,d)$, then $L(a,b) = L(c,d)$. Let $\leq$ be the ordering on the line. We may assume $a \leq b$ and $c \leq d$, $a \leq x \leq y \leq b$, and $c \leq x \leq y \leq d$. So $a \leq d$ and $c \leq b$. If $a \leq c$ and $b \leq d$ then $a \leq c \leq b \leq d$ thus $c, b \in [a,d]$ and $[a,b] \cup [c,d] = [a,d]$. Other cases will give similar results. □

<u>3.3.8. Definition</u> : A subset A of a generalized line space is said to be <u>convex</u> iff whenever $x, y \in A$, $(x,y) \subseteq A$.

<u>3.3.9. Lemma</u> : Let A be a convex subset of a generalized line space $(X,\mathcal{L})$. If $a \in A$ and $x \in X \backslash A$, then there exists a $c \in [a,x]$, such that $(a,c) \subseteq A$ and $(c,x) \cap A = \emptyset$.

Proof : Let $a \in A$, $x \in X \backslash A$, $\ell = L(a,x)$ and $\phi : \ell \to \mathcal{R}$ be an order isomorphism. Then $[a,x]$ is isomorphic to $[\phi(a),\phi(x)]$. Choose $e = \text{l.u.b.}\{\phi(d) : d \in [a,x] \text{ and } (a,d) \subseteq A\}$, where l.u.b. means least upper bound. Let $c = \phi^{-1}(e) \in [a,x]$. Then $(a,c) \subseteq A$ and $(c,x) \cap A = \emptyset$. □

<u>3.3.10. Definition</u> : The <u>convex-hull</u> of a subset A of a generalized line space (X,L) , is the smallest convex subset in (X,L) containing A and is denoted by $C(A)$.

<u>3.3.11. Lemma</u> : Let A be a subset of a generalized line space (X,L), then $C(A) = \cup\{C(F) : F \subseteq A$ and F is finite$\}$.

Proof : It is obvious that $D = \cup\{C(F) : F \subseteq A$ and $|F| < \infty\} \subseteq C(A)$, and $A \subseteq D$. It suffices to show that D is convex. Let $x, y \in D$, then $x \in C(F_1)$ and $y \in C(F_2)$ for some $F_1, F_2 \in A$ and F_1, F_2 are finite. Clearly $F_1 \cup F_2 \subseteq A$ and $F_1 \cup F_2$ is finite, so $x, y \in C(F_1 \cup F_2)$. Now since $C(F_1 \cup F_2)$ is convex, $[x,y] \in C(F_1 \cup F_2) \subseteq D$. Hence D is convex and $C(A) = D$. □

<u>3.4. Weak Complete Basic Convexity Space</u>

<u>3.4.1. Definition</u> : Let (X,C) be a convexity space and A be any C-convex subset of X [cf. 2.1.6]. (X,C) is said to be <u>complete</u> iff for each $a \in A$ and $x \in X \backslash A$, there is $c \in ax$, so that $(ac) \subseteq A$ and $(cx) \cap A = \emptyset$.

<u>3.4.2. Remarks</u> : The point $c \in ax$ may mean that $c = a$ or $c = x$. In these cases $(aa) = a \in A$ and $(ax) \cap A = \emptyset$ or $(ax) \subseteq A$ and $(xx) \cap A = x \cap A = \emptyset$.

<u>3.4.3. Example</u> : $X \equiv Q \times Q$, where Q is the rational numbers, is a basic convexity space but not complete, with C as the collection of convex set in usual sense.

This example shows that completeness is independent of the basic properties (domain finite and regular straight segments).

<u>3.4.4. Definition</u> : An order isomorphism between an open segment and a line of a convexity space $(X,\mathcal{C})$ is called <u>$\mathcal{C}$-isomorphism</u>. We say $(X,\mathcal{C})$ has the <u>$\mathcal{C}$-isomorphic property</u> iff for each segment, there is an $\mathcal{C}$-isomorphism to a line.

<u>3.4.5. Example</u> : Consider a set $S = \{x : x > 0$ and rational, $x < 0$ and irrational or $x = 0\}$. Let $X \equiv S\times S$ and $\mathcal{C}$ be collection of the convex set in usual sense. Then $(X,\mathcal{C})$ is a convexity space that open segment is not isomorphic to the line itself.

<u>3.4.6. Definition</u> : A basic convexity space $(X,\mathcal{C})$ is said to be a <u>weak complete basic convexity space</u> if it is complete and has the $\mathcal{C}$-isomorphic property.

We will assume $(X,\mathcal{C})$ to be a weak complete basic convexity space for this section.

<u>3.4.7. Lemma</u> : Let $a, b \in \ell$ and $a \le b$ where ℓ in $(X,\mathcal{C})$. Then $c \in ab$ iff $a \le c \le b$, where $\le$ is the order on ℓ.

Proof: Let $a, b \in \ell$ and $a \le b$. There are several cases, but we will only do one. The arguments in the other cases are similar.

Let $a, b \in R_-$. Since $a \le b$, the $b \in a0$. If $c \in ab$ we have $c \in a0$, so $a \le c$. Also $a0 = ac \cup c0$. Thus $b \in ac$ in which case $b = c$ or $b \in c0$ and $c \le b$. Therefore $a \le c \le b$.

Suppose $a \leq c \leq b$. Again, we will only show one case, the others are similar. Let $a, b, c \in R_-$. Then $b \in a0$, so $a0 = ab \cup b0$. Also $c \in a0$, which implies $c \in ab$ or $b = c$. □

3.4.8. Lemma : For each $A \subseteq \ell$ in $(X,\mathcal{C})$, $\mathcal{C}(A) = \cup\{\mathcal{C}(F) : F \subseteq A, |F| \leq 2\}$.

Proof : By domain finiteness, $\mathcal{C}(A) = \cup\{\mathcal{C}(F) : F \subseteq A, |F| < \infty\}$

Let $|F| = n$, and $F = \{f_1, f_2, \ldots\ldots, f_n\}$. Since $F \subseteq A \subseteq \ell$, the f_i are comparable. Suppose $f_{i_1} \leq f_{i_2} \leq \cdots\cdots\cdots \leq f_{i_n}$, so $f_{i_2}, f_{i_3}, \ldots\ldots, f_{i_{n-1}} \in f_{i_1} f_{i_n}$ This implies $\mathcal{C}(f_{i_2}, \ldots\ldots, f_{i_{n-1}}) \in f_{i_1} f_{i_n}$. Hence the lemma is proved. □

3.4.9. Lemma : $A \subseteq \ell$ is convex in $(X,\mathcal{C})$ iff $xy \subseteq A$, for all $x, y \in A$.

Proof : It is obvious that $xy \subseteq A$, if A is convex.

Conversely, since $\mathcal{C}(A) = \cup\{\mathcal{C}(x,y) : x, y \in A\}$, it is easy to see that $\mathcal{C}(A) \subseteq A$. Hence A is convex . □

3.4.10. Definition : A subset C , of an open segment (ab) is said to be bounded above iff there is $x \in (ab)$ such that for all $c \in C$, $c \leq x$, and bounded below iff there is $y \in (ab)$, such that for all $c \in C$, $y \leq c$. C is said to be a bounded subset iff it is bounded above and below.

3.4.11. Lemma : Any open segment in a weak complete basic convexity space is a conditionally complete linear ordered set (i.e. every bounded subset has a greatest lower bound and a least upper bound).

Proof : This follows by the Definition 3.4.10 and the regularity, completeness properties. □

<u>3.4.12. Theorem</u> : If (X,L) is a generalized line space and C_L is the collection of the convex sets, then (X,C_L) is a weak complete basic convexity space.

Proof: Let (X,L) be a generalized line space and C_L be the collection of convex sets [cf. 3.3.8]. Then (X,C_L) is a weak complete basic convexity space, since domain finiteness, regularity, straightness, C-isomorphic property and completeness are shown by Lemmas 3.3.11, 3.3.6, 3.3.7, 3.3.5 and 3.3.9 respectively. □

<u>3.5. Complete Basic Convexity Space and Line Space</u>

<u>3.5.1. Definition</u> : A <u>complete basic convexity space</u> or <u>strong complete basic convexity space</u> is a basic convexity space which satisfies join-hull commutativity and is complete.

Before proving the following lemmas, we recall a lemma in 1.2.14 which states: a conditionally complete linearly ordered set without maximal or minimal elements, is isomorphic to the reals iff (i) it is dense-in-itself and (ii) contains a countable dense subset [4].

<u>3.5.2. Lemma</u> : If (X,C) is a basic convexity space which satisfies join-hull commutativity, the lines are members of C.

Proof: By Theorem 2.3.3, to show a line is C-convex, we need to check $xy \subseteq \ell$ whenever $x, y \in \ell$. If $x, y \in \ell$, then by Theorem 3.1.8, $\ell = L(x,y)$ and hence $xy \subseteq \ell$. □

<u>3.5.3. Lemma (Peano's axiom)</u> : Let a, b, c be distinct non-collinear points of a complete basic convexity space, $(X,\mathcal{C})$. If $x \in ab$ and $y \in ac$, then $by \cap cx \neq \emptyset$.

Proof : If $x = a$ or b and $y = a$ or c, the results are trivial. So we may assume $x \neq a, b$ and $y \neq a, c$.

Choose d, such that $a \in xd$. Since $y \in ac$, $y \in \mathcal{C}(c,x,d)$. By join-hull commutativity, we have $y \in de$, for some $e \in cx$. Since $x \in ab$ and $a \in bd$, $cx \subseteq \mathcal{C}(b,c,d)$. But $e \in cx$, so $e \in df$, for some $f \in bc$. If $f \in ey$, then $f \in de$ and $df \subseteq de$. But $de \subseteq df$, so $de = df$ and $e = f$. Thus $e \in bc$ which shows $bc = cx$, so $b = x$. Hence $e \in yf$ and since $f \in bc$, $e \in \mathcal{C}(y,b,c)$. So we have $e \in cz$, for some $z \in by$.

A similar argument shows that $z \in cx$. □

<u>3.5.4. Lemma</u> : Any two open segments in a complete basic convexity space are isomorphic to each other.

Proof : Let (ab) and (bc) be two segments and choose x so that $c \in (ax)$. Define $\alpha : (ab) \to (bc)$ as follows : $\alpha(z) = y$ iff $y \in (xz)$, for $z \in (ab)$ and $y \in (bc)$. By the properties of Pasch's axiom and Peano's axiom, it is not hard to see that α is well-defined and one-to-one. Next, we will check that α preserves order. Suppose $<_1$ is the order on (ab) and $<_2$ is the order on (bc). We may assume that $a <_1 b$ and $b <_2 c$. For $z, u \in (ab)$, suppose $a <_1 z <_1 u <_1 b$. Let $\alpha(z) = y$ and $\alpha(u) = v$. We will show $y <_2 v$, that is $b <_2 y <_2 v <_2 c$. Suppose $b <_2 v <_2 y$. By Lemma 3.5.3, there is $w \in (zy)$ such that $w \in (uv)$. Thus $L(u,v) \cap L(z,y) = \{w,x\}$. By Lemma 3.1.10, this is a contradiction. Hence $b <_2 y <_2 v <_2 c$.

If (ab) and (cd) are any two segments which do not intersect or are not on the same plane (i.e. $(ab) \subseteq \overline{\{a,c,d\}}$). We will use the third segment, which is joined by a and c . The result is immediate. □

<u>3.5.5. Corollary</u> : In a complete basic convexity space, each open segment is order isomorphic to a line.

Proof : Consider a line ℓ and a point $x \notin \ell$. Let $0 \in \ell$, by the Theorem 12.61 of Coxeter [12], there is a ray through x which does not meet ℓ and that separates all the rays from x that meet ℓ from all other rays that do not. Let m be the ray. Choose $a \in m$ and join 0, a . Clearly, all the lines between x0 and xa will meet 0a and meet ℓ . Let the ray from 0 of ℓ which meets the lines between x0 and xa be R . Consider the mapping $\alpha : (0a) \to R\backslash\{0\}$. $\alpha(u) = v$ iff x, u, v are collinear, where $u \in (0a)$ and $v \in R\backslash\{0\}$. It is easy to see that α is well-defined and one-to-one onto. Using an argument similar to that in the last lemma, we can easily show that α preserves order. By extending, 0a so that $0 \in ab$. Using a similar argument, we can show that (0b) is order isomorphic to the other side of ℓ To complete the proof, we map $0 \in \ell$ to $0 \in (ab)$. Hence (ab) is order isomorphic to ℓ

By the previous lemma, each segment is order isomorphic to ℓ . □

<u>3.5.6. Corollary</u> : (X,C) is a complete basic convexity space iff it is a weak complete basic convexity space that is join-hull commutative.

Using a construction similar to the one given by Doignon [14], we have the following lemma.

<u>3.5.7. Lemma</u> : Any line in a complete basic convexity space (X,C) of dimension two or greater is order isomorphic to the reals.

Proof : It follows from Corollary 3.5.5 that we need only show that there is a segment order isomorphic to the reals.

Let $a, b \in X$, $a \neq b$. Define a sequence $\{x_n\}$ by $x_1 = a$ and, assuming $x_1, x_2, \ldots\ldots, x_n$ have been defined, $x_{n+1} \in (x_n b)$. By completeness, there is $d \in (ab)$, such that for each $t \in (ad)$, $x_n \in (td)$, for all n sufficiently large. That is, d is the limit point of the sequence. Further, $x_n \neq d$ for all n . We assume that the line $L(a, b)$ is ordered so $d < a$.

Choose $c \notin L(a,b)$ and choose a point $e \in (dc)$. We will define a binary operation $+$ on (da) as follows: for $u, v \in (da)$, by Lemmas 2.5.2 and 3.5.3, each of the following points exist: $y = (cu) \cap (ea)$, $z = y/d \cap (ca)$ and $w = (ea) \cap (zv)$. Define $u + v = w/c \cap (da)$. It is easily seen that this operation is strictly increasing (with respect to the order on $L(a,b)$) in each of its arguments. In particular, if $u, v, t \in (da)$, $u < v$, then $t + u < t + v$ and $u + t < v + t$. Also, if $u, w \in (da)$ and $u < w$, we have $y = (cu) \cap (ea)$, $z = y/d \cap (ca)$, $x = (ea) \cap (cw)$ and $v = x/z \cap (da)$. Then v is the unique point such that $u + v = w$.

Now, define $n \cdot t$, $t \in (da)$, $n \in N$ as follows: $1 \cdot t = t$, $n \cdot t = (n-1) \cdot t + t$. The operation $+$ is archimedean in the following sense: if $u, v \in (da)$, $u < v$, then there is n such that $n \cdot u > v$. Suppose not, that is $n \cdot u \leq v$ for all $n \in N$. Let $\overline{u} = \sup\{n \cdot u : n \in N\}$ (which exists by completeness) and let w be the unique element of (da) such that $u + w = \overline{u}$. Now $w < \overline{u}$, so there is some n , such that $n \cdot u \in (w\overline{u})$. Consequently, $(n+1) \cdot u = n \cdot u + u > w + u = \overline{u}$, which is a contradiction.

Let $Q = \{m \cdot x_n : m, n \in N\}$. Q is countable and is dense in (da) . For if $f, g \in (da)$, $f < g$, there is a unique h , such that $f + h = g$ and there is $n \in N$, such that $x_n < h$. If $f < x_n$, we are finished. If not, i.e. $x_n \leq f$, choose m such that $m \cdot x_n \leq f$ and $f < (m+1) \cdot x_n$. Then $f < (m+1) \cdot x_n < g$ and we have shown that Q is dense in (da) . It follows from Lemmas 1.2.14 and 3.4.11, (da) is order isomorphic to the reals. □

<u>3.5.8. Definition</u> : A generalized line space is said to be a <u>line space</u> iff it satisfies the following axiom ; if $x \in [a,b]$ and $y \in [c,x]$, then there is a $z \in [a,c]$, so that $y \in [b,z]$.

Since the properties discussed in 3.3 hold in a generalized line space depending on the two axioms, therefore they hold in the line space too. The following lemmas hold in the line space only, since they depend on the above axiom.

Recall $\mathcal{C}(A)$ is the smallest convex set containing A [cf. 3.3.10] .

<u>3.5.9. Lemma</u> : Let a, b, c be distinct points in a line space. If $u \in [a,b]$ and $v \in [a,c]$, then $[b,v] \cap [c,u] \neq \emptyset$.

Proof : The result is obvious if a, b, c are collinear. We may assume that they are non-collinear. The proof is similar to the argument in Lemma 3.5.3 using the last axiom of the line space instead of join-hull commutativity. □

<u>3.5.10. Lemma</u> : Let a, b, c be distinct points in a line space. If $u \in [a,b]$ and $v \in [a,c]$, then for any $t \in [u,v]$, there is a point $w \in [b,c]$ such that $t \in [a,w]$.

Proof : Clearly, it is true if a, b, c are collinear. Suppose they are non-collinear. In Δbva , by the last axiom for a line space, we have a $y \in [b,v]$ such that $t \in [a,y]$. Apply the axiom again in Δabc , there is $w \in [b,c]$, such that $y \in [a,w]$. By the uniqueness of the line and a given order, $t \in [a,w]$ and hence the lemma is proved. □

<u>3.5.11. Lemma</u> : If x, y are points of a line space, then $\mathcal{C}(x,y) = [x,y]$.

Proof : Clearly $[x,y] \subseteq \mathcal{C}(x,y)$ for any x, y . We will show $[x,y]$ is convex. Let $a, b \in [x,y]$. Then $a, b \in L(x,y)$. Let $\leq$ be the order on $L(x,y)$. Suppose $x \leq y$, then $x \leq a \leq y$ and $x \leq b \leq y$. Without loss generality, we may assume that $a \leq b$. Let $t \in [a,b]$, that is $a \leq t \leq b$. Thus $x \leq t \leq y$, that is $t \in [x,y]$, $[a,b] \subseteq [x,y]$. Hence $\mathcal{C}(x,y) = [x,y]$. □

<u>3.5.12. Lemma</u> : In a line space $(X,\mathcal{L})$, if $A \subseteq X$ and $x \in X$, then $\mathcal{C}(x \cup A) = \bigcup \{\mathcal{C}(x,a) : a \in \mathcal{C}(A)\}$.

Proof : By the last lemma, we may rewrite $\mathcal{C}(x \cup A) = \cup\{[x,a] : a \in \mathcal{C}(A)\}$ $= D$. Clearly, $[x,a] \subseteq \mathcal{C}(x,A)$. Need to show that D is convex. Let $u, v \in D$, then $u \in [x,a_1]$ and $v \in [x,a_2]$ for some $a_1, a_2 \in \mathcal{C}(A)$. By Lemma 3.5.9, we have for each $t \in [u,v]$ there is a $a_3 \in [a_1,a_2]$, so that $t \in [x,a_3]$. Thus $[u,v] \subseteq D$. Hence D is convex. □

3.5.13. Theorem : Let $(X,\mathcal{C})$ be a complete basic convexity space. If $\mathcal{L}_\mathcal{C}$ is the collection of lines of $\mathcal{C}$ then $(X,\mathcal{L}_\mathcal{C})$ is a line space. Conversely, if $(X,\mathcal{L})$ is a line space and $\mathcal{C}_\mathcal{L}$ is the collection of convex sets of X , then $(X,\mathcal{C}_\mathcal{L})$ is a complete basic convexity space.

Proof : If $(X,\mathcal{C})$ is a complete basic convexity space, then by Lemma 3.5.7, each line is order isomorphic to the reals. By Lemma 2.5.2, if a convexity space is join-hull commutative, it satisfies Pasch's axiom. Hence $(X,\mathcal{L}_\mathcal{C})$ is a line space, where $\mathcal{L}_\mathcal{C}$ is the collection of lines.

Conversely, if $(X,\mathcal{L})$ is a line space, then it is a generalized line space. By Theorem 3.4.12, if $\mathcal{C}_\mathcal{L}$ is the collection of convex sets, then $(X,\mathcal{C}_\mathcal{L})$ is a weak complete basic convexity space. By Lemma 3.5.12, since $(X,\mathcal{L})$ satisfies Pasch's axiom, therefore $(X,\mathcal{C}_\mathcal{L})$ is join-hull commutative. Hence by Corollary 3.5.6, $(X,\mathcal{C}_\mathcal{L})$ is a complete basic convexity space. □

CHAPTER FOUR

CHARACTERIZATION OF A LINEARLY OPEN CONVEX SUBSET OF A REAL AFFINE SPACE

We will follow the approach given by Doignon [14] to show that a complete basic convexity space is isomorphic to a linearly open convex subset of a real affine space. The affine space and projective space involved in Doignon's characterization may not be complete (or real). However, we will restrict ourself to real projective space and real affine space. The theorems given below may not hold if they are not complete (or real). Throughout the whole chapter, we will assume that $(X,\mathcal{C})$ is a complete basic convexity space.

4.1. Convex Sets in a Projective Space

4.1.1. Definition : A nonempty set L with a nonempty collection $\mathcal{L}$, of the subsets of L is called a linear space iff every member of $\mathcal{L}$ satisfies the following axioms :

(1) any two elements (or points) of L , there is one and only one member of $\mathcal{L}$ containing them ,

(2) each member of $\mathcal{L}$ contains at least two points .

Usually, we call a member of $\mathcal{L}$ a line and an element of L a point.

4.1.2. Definition : A projective space P , is a linear space where (1) any two lines will meet at a point , (2) every line contains at least three points and (3) not all points lie on the same line.

4.1.3. Definition : An ordered projective space is a projective space where every line satisfies the following separation relation and which is preserved under a central projection of one line onto another, where the separation relation is a relation $|$, on four collinear points satisfying the following conditions :

(1) if $ab \mid cd$ then a, b, c, d are distinct ,

(2) if $ab \mid cd$ then $ba \mid cd$ and $cd \mid ab$,

(3) if $ac \mid bd$ then $ab \mid cd$ is not true ,

(4) if a, b, c, d are distinct points then $ab \mid cd$ or $ca \mid bd$ or $ad \mid bc$,

(5) if $ab \mid cd$ and $bc \mid de$ then $cd \mid ea$.

4.1.4. Definition : A real projective space is an ordered projective space over a real ordered field.

4.1.5. Definition : S is said to be a linear subspace of a projective space iff the line determined by any two distinct points of S is contained in S . A hyperplane is a maximal proper linear subspace.

4.1.6. Definition : An affine space A , is a linear space with collection of subsets lines L and planes π , satisfying the following axioms :

(1) any three distinct non-collinear points are contained in an unique member of π ,

(2) for a given point p and a line ℓ which does not contain it, there is a line m which passes through p , does not meet ℓ and ℓ, m belong to the same member of π (denoted by $\ell \parallel m$)

(3) if any two distinct planes intersect, their intersection is a line ,

(4) not all the points are on the same plane.

A <u>real affine space</u> is an affine space whose lines are isomorphic to the reals.

<u>4.1.7. A. Theorem</u> : Let (P,L) be a projective space and H is a hyperplane of (P,L) . Then $A = \{x : x \in P \text{ and } x \notin H\}$ can be made into an affine space.

<u>B. Theorem</u> : Let (A,L,π) be an affine space. H is the set of points at infinity of A . Then $P = A \cup H$ is a projective space and H is the hyperplane of P .

Proof : Omitted [26] .

Remark : The points at infinity of A are the equivalence classes under the relation of parallelism.

<u>4.1.8. Definition</u> : A set $\mathcal{O}$ is said to be a <u>linearly open subset</u> of a real projective space P (or a real affine space A) , if every line of P (or A) either meets $\mathcal{O}$ in an open interval or empty set.

<u>4.1.9. Definition</u> : A set C of a real projective space is said to be a <u>semi-convex subset</u> iff every segment (in the usual sense) joined by two distinct points of C is contained in C . It is said to be <u>convex subset</u> iff C does not contain a whole line. In other words, every segment joined by two distinct points of C is uniquely determined and is contained in C [16] .

4.1.10. Definition : A set C is convex in a real affine space iff the segment joined by two distinct points of C is contained in C .

4.1.11. Theorem : If C is a (an open) convex subset in a real projective space P and H is a hyperplane, so that $C \cap H = \emptyset$, then C is a (an open) convex subset of the real affine space $A = P \setminus H$ and conversely.

Proof : The result follows immediately from the definitions 4.1.8 to 4.1.10 and the Theorem 4.1.7.

4.1.12. Theorem : An open semi-convex subset in a n-dimensional real projective space is an open convex subset iff it does not meet (at least one) hyperplane.

Proof : Omitted [13,16] . □

We note that the Pasch and Peano's axioms mentioned in the last chapter, generally hold in the real projective space and affine space.

4.1.13. Lemma : Let C be a linearly open convex subset and H be a hyperplane of a real projective space. If $C \cap H = \emptyset$, then H divides C into two disjoint linearly open convex subset.

Proof : Let $H' = H \cap C$. Fix a point $p \in C$ and $p \notin H'$. Let K_1 be the set of all the points of $C \setminus H'$ such that $s(p,x) \cap H = \emptyset$, where $s(p,x)$ is the segment joined by p and $x \in C \setminus H'$, in the usual sense, and K_2 are those points in $C \setminus H'$ and not in K_1 . Clearly, $K_1 \cap K_2 = \emptyset$ and $C = K_1 \cup K_2 \cup H$.

First, we claim that, for all k_1, $k_2 \in K_1$, then $s(k_1,k_2) \cap H' = \emptyset$. We may assume that the plane generated by k_1, k_2, and p meets H' is nonempty set, otherwise we are done. Let λ be the line where the plane $\{\overline{k_1,k_2,p}\}$ meets H. Suppose $L(k_1,p) \cap \lambda = t_2$ and $s(k_1,k_2) \cap \lambda = t_1$ where $p \in s(k_1,t_2)$. By Peano's axiom, $s(t_1,t_2) \cap s(p,k_2) \neq \emptyset$, which is a contradiction. Similary, if k_1 belongs to $s(p,t_2)$, we arrive at a contradiction.

Now we show that $s(k_1,k_2) \subseteq K_1$, for all k_1, $k_2 \in K_1$. Let $k \in s(k_1,k_2)$ and suppose that $s(p,k) \cap \lambda = \emptyset$. If $t_1 = L(k_1,k_2) \cap \lambda$ with $k_1 \in s(t_1,k_2)$, then by Peano's axiom, we have $s(k_1,p) \cap \lambda \neq \emptyset$ which is a contradiction. Therefore $s(k_1,k_2) \subseteq K_1$ and K_1 is a semi-convex subset.

To show K_1 is convex, we will check that K_1 does not contain a whole line in the space. But if ℓ is a whole line contained in K_1, then $\ell \subseteq C$ which contradicts the fact that C is convex. Similarly, we can show that K_2 is convex.

To see that K_1 and K_2 are linearly open, let ℓ be any line in P. Consider a plane α which passes through ℓ and meets C in a nonempty set. If α meets C at K_1 or K_2, then it is easy to see that ℓ meets K_1 or K_2 at an open interval, since C is linearly open.

If ℓ meets C is empty, clearly ℓ meets K_1, K_2 and H' in the empty set. Suppose ℓ meets C at an open interval and meets K_1, K_2 and H' in nonempty set. If it is an open segment, $S(c_1,c_2)$ and suppose the open segment $s(c_1,p)$ is contained in K_1

(i.e. $s(c_1,p) \cap \lambda = \emptyset$), then by applying Pasch and Peano's axiom, we can see that $K_1 \cap \ell = (c_1,t)$ and $K_2 \cap \ell = (t,c_2)$, where $t = H' \cap \ell$, (c_1,t) and (t,c_2) are open segments.

For the case K_1 or K_2 is unbounded then we will show that for each point on $K_1 \cap \ell$ and $K_2 \cap \ell$, there is an open segment containing it. Let $c = K_2 \cap \ell$ and $d = K_1 \cap \ell$. By Pasch and Peano's axioms again, it is easy to see that every point in $s(t,c)$ is in K_2 . Choose a point $c_1 \in \ell$, so that $c \in s(t,c_1)$. Clearly, $s(p,c_1) \cap \lambda \neq \emptyset$ and $c_1 \in K_2$, otherwise K_2 is bounded. Hence there is always an open segment containing c . Similarly, for the case that K_1 is unbounded.

Hence K_1 and K_2 are linearly open convex subset in P. $\square$

<u>4.1.14. Theorem</u> : A linearly open semi-convex subset in a real projective space is convex iff it does not meet (at least one) hyperplane.

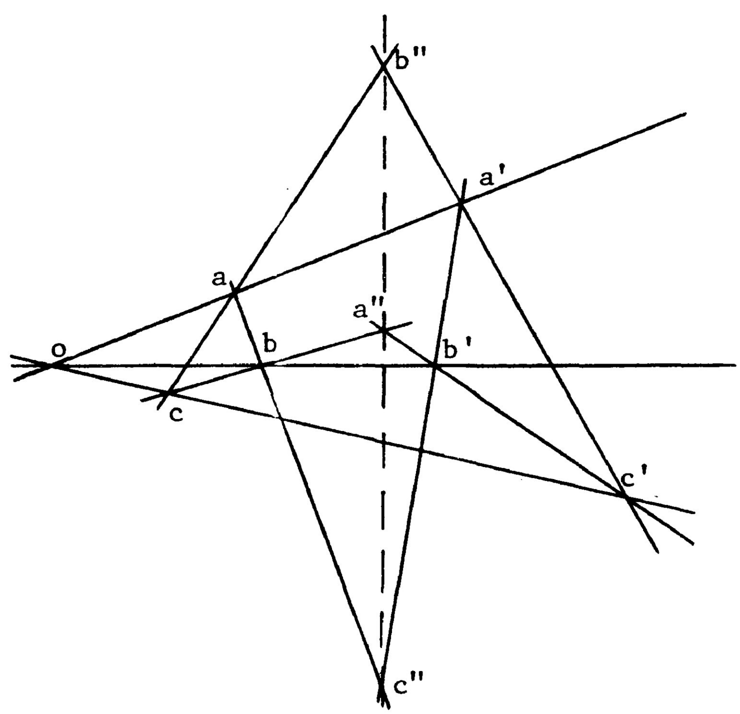
Proof : The necessary condition is obvious. Since if C is a semi-convex subset which does not meets a hyperplane H , then C does not contain a whole line.

To prove the sufficiency, let H be a hyperplane and C be a linearly open convex set. If $C \cap H = \emptyset$, we are done. Suppose $C \cap H \neq \emptyset$. By the previous lemma, H divides C into two disjoint linearly open convex subsets, say K_1 and K_2 . From Theorem 4.1.10, K_1 and K_2 are linearly open convex subsets of the real affine space $A = P \setminus H$. Clearly by linearly open property K_1 and K_2 have nonempty interior. For an interior point p , of a set K , we mean that any line ℓ passes through p, there is an open segment of $K \cap \ell$ containing p. Since $K_1 \cap K_2 = \emptyset$, then by the Separation Theorem in a real affine space, there is a hyperplane I, such that I separates them. Since $I \subseteq A$, therefore $I \cap H = \emptyset$ and hence $I \cap C = \emptyset$ in the real projective space $P = A \cup H$ and the theorem is proved. □

<u>4.2. Simplices in a Complete Basic Convexity Space</u>

<u>4.2.1. Definition</u> : A complete basic convexity space is <u>desarguesian</u>, if o, a, b, c are non-collinear and each of the following triples are collinear, oaa', obb', occ', abc'', $a'b'c''$, bca'', $b'c'a''$, cab'' and $c'a'b''$, then a'', b'', c'' are collinear.

Refer to the following diagram, and note that this is one of the cases.



<u>4.2.2. Theorem</u> : (X,C) is desarguesian if its dimension is greater or equal to three.

Proof : Let o, a, b, c, a', b', c', a", b" and c" satisfy the hypothesis of 4.2.1. We may assume that $a \in oa'$, $b \in ob'$, $c \in oc'$ $b \in ac''$, $b' \in a'c''$, $c \in ba''$ and $c' \in b'a''$. For the sake of convenience, we will use diagrams to help the proof.

First, we will consider the case that $b \notin \{\overline{o,a,c}\}$ (refer to Fig. 4.2.1). In $\Delta a''c''a'$, by Pasch's axiom, there is $b_1 \in a''c''$ so that $c' \in b_1a'$. Similarly, in $\Delta a''c''a$, there is $b_2 \in a''c''$, so that $c \in b_2a'$. By hypothesis , $L(a,c) \cap L(a',c') = b'$. Suppose b_1, b_2 and b" are distinct points. By assumption, we know that $L(a,c)$, $L(a',c')$ are on the $\{\overline{o,a,c}\}$. Since $b_1 \in L(a',c')$ and $b_2 \in L(a,c)$, therefore $b_1, b_2 \in \{\overline{o,a,c}\}$. This implies $a'', c'' \in \{\overline{o,a,c}\}$. It follows that $b', b \in \{\overline{o,a,c}\}$, which is a contradiction. Other cases have a similar argument and yield the same result.

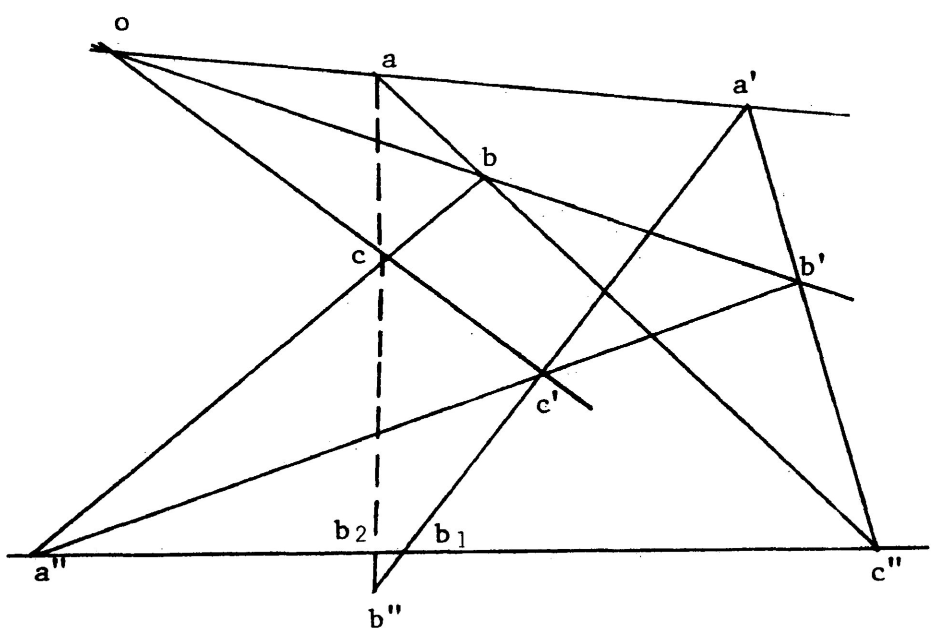


Figure 4.2.1

Now, we consider that $b \notin \overline{\{o,a,c\}}$ (refer to Fig. 4.2.2). Since $\dim X \geq 3$, we select a point $x \notin \overline{\{o,a,c\}}$. Let $y \in L(o,x)$, such that $x \in oy$. By Peano's axiom, we have the following points : $a_1 = xa' \cap ya$, $b_1 = xb' \cap yb$ and $c_1 = xc' \cap yc$. It is easy to see that a_1, b_1, and c_1 are not in $\overline{\{o,a,c\}}$. Considering $\Delta yab''$, by Peano's axiom, we have $a_1b'' \cap yc = c_2$. Similarly, $\Delta xa'b''$, $a_1b'' \cap xc = c_3$. If c_1, c_2 and c_3 are distinct points, then a_1 and c_1 are on $\overline{\{o,a,c\}}$. Therefore, we have $L(a_1,b_1) \cap L(a,b) = c''$, $L(b_1,c_1) \cap L(b,c) = a''$ and $L(a_1,c_1) \cap L(a,c) = b''$, which satisfy the hypothesis of desarguesian. From the first case, thus we have a'', b'', c'' are collinear. Other cases will yield a same result and hence the theorem is proved. $\square$

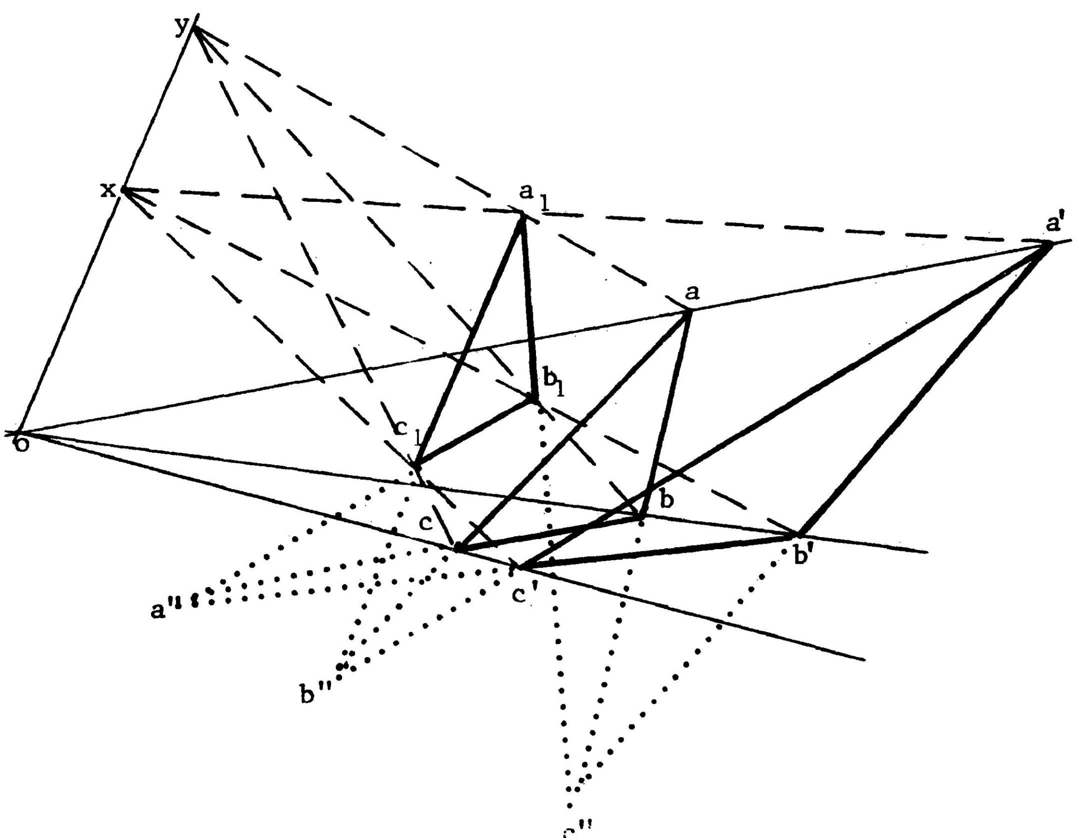


Figure 4.2.2

4.2.3. Definition : A subset S of (X,C) is a simplex, if $S = C(A)$, where A is an independent subset of X [cf. 3.1.13] and the dimension of S ($\dim S$) $= |A| - 1$ [cf. 1.2.2] .

4.2.4. Theorem : If the dimension of (X,C) is finite and is greater than two or desarguesian and of dimension two, then each simplex of (X,C) is isomorphic to a simplex of a real affine space.

Proof : We will apply induction on the dimension of the simplices.

Let S be a simplex of X of dimension n .

When $n = 1$, S is a segment. Since the lines of both spaces, X and the real affine space are isomorphic to the reals, hence S is isomorphic to some segment of a real affine space.

Suppose it is true for $n \neq k-1$. We will show that it is true for $n = k$. Let $B = \{a_i : i = 1, 2, \ldots, k\}$ be independent and $A = B \setminus a_k$. Suppose $T = C(A)$ and f is the isomorphism from T onto a simplex S ($= f(T)$) of a real affine space. Let $T' = C(B)$ and $S' = \mathrm{conv}(f(A) \cup c)$, where $f(A) \cup c$ is an independent set in the real affine space, $\mathrm{conv}(f(A) \cup c)$ means the smallest convex set in the real affine space containing $f(A) \cup c$. Choose $a \in A$ and let p be a mapping from $C = C(B \setminus a)$ onto $C' = \mathrm{conv}(f(A \setminus a) \cup c)$. By using the induction hypothesis and the fact that simplices of the same dimension are isomorphic to each other in a real affine space, p is an isomorphism.

By join-hull commutativity, for each $x \in T' \setminus a_k a$, we have $x \in a x_2$ for some $x_2 \in C(B \setminus a)$ and similarly, $x \in a_k x_1$ for some $x_1 \in C(A)$. Therefore each point $x \in T' \setminus a_k a$ can be represented by (x_1 , x_2) . Define a map $f' : T' \setminus a_k a$ onto $S' \setminus [c, f(a)]$,

$f'(x_1,x_2) = (f(x_1),p(x_2))$, where $[c,f(a)]$ is a closed segment (in the usual sense) joining c and $f(a)$ in the real affine space.

Now, we claim that $y = (f(a),p(x_2)) \cap (f(x_1),c)$ is well-defined and unique. To see this, we know by join-hull commutativity , $x_2 \in C(B\backslash a) = C(a_k \cup (A\backslash a))$ and $x_1 \in C(A)$ imply $x_2 \in a_k x_3$ and $x_1 \in ax_4$, for some $x_3, x_4 \in C(A\backslash a)$. If $a \in L(x_3,x_4)$, we are done. Suppose $a \notin L(x_3,x_4)$, we will show $x_3 = x_4$. Suppose $x_3 \neq x_4$. By join-hull commutativity, there is $x_1' \in ax_3$ so that $x \in a_k x_1'$. But $x_1 \in L(a_k,x)$ and $x_1' \in L(a_k,x)$. hence $a_k \in L(a_k,x) \subseteq \{a,x_3,x_4\} \subseteq \overline{A}$, which contradicts that $B = A \cup a_k$ is independent set. Under the isomorphisms f and p , $f(x_3)$ is uniquely determined in the real affine space. Applying Peano's axiom, it is easy to see that y is well-defined and uniquely determined. A simple routine will show that f' is an isomorphism.

Obviously, $a_k a$ is isomorphic to $[c,f(a)]$ and let p' be the isomorphism. Therefore define $\hat{f} : T' \to S'$ as follows ; $\hat{f}|_{C(A)} = f$, $\hat{f}|_{C(B\backslash a)} = p$, $\hat{f}|_{a_k a} = p'$ and for all $x \in T'\backslash D$, $\hat{f}(x) = f'(x_1,x_2)$ defined as above, where $D = C(A) \cup C(B\backslash a) \cup a_k a$
Hence $\hat{f}$ is an isomorphism from T' onto S' . $\square$

<u>4.2.5. Theorem</u> : If the dimension of (X,C) is greater than two or desarguesian and of dimension two, then each simplex of (X,C) is isomorphic to a simplex of a real affine space.

Proof : Consider triples $S = (T,f,A)$ of (X,C) , where $T = \overline{B}$, $B = \{a_i : i = 1, 2, \ldots, n\}$ is an independent subset of X ,

$\overline{B}$ means the space spanned by B [cf. 3.1.11] ; f is an isomorphism from $C(B)$ onto a simplex which generates the real affine space A.

Order S as follows :

$(T, f, A) \leq (T', f', A')$,

iff T is a subspace of T' , A is a subspace of A' and $f|_{C(B)} = f'$, where $T = \overline{B}$. Suppose (T_j, f_j, A_j) , $j \in I$, is a chain in S. Let $T_u = \overline{\cup B_j}$, A_u be the space generated by $\cup A_j$ and f_u be the isomorphism generated by the f_j's . Then (T_u, f_u, A_u) is an upper bound for the chain. By Zorn's Lemma [cf. 1.2.8] , there is a maximal element in S , say (T, f, A) .

Claim that $T = X$. If not, by using a similar argument to that in the previous theorem, we contradict that (T, f, A) is a maximal element.

Hence each simplex of X is isomorphic to a simplex of a real affine space. □

4.3. Characterization Theorem

4.3.1. Theorem : (X,C) is isomorphic to a semi-convex subset of a real projective space, if the dimension of X is greater than two or equal to two and desarguesian.

Proof : Let $B = \{a_i : i \in I\}$ be a base of (X,C) , $B' = \{a_i' : i \in I\}$ be a base of a real affine space A , and f be the isomorphism from $C(B)$ onto $\mathrm{conv}(B')$.

Consider the real projective space $P = A \cup H$ [cf. 4.1.7] . For each $E \subseteq P$, we represent $L(E)$ as the smallest linear subset of P containing E. Each point $z \in X \backslash C(B)$ is the intersection of a pair of lines $\{C,D\}$, where C, D meet $\mathcal{C}(B)$ at more than one point. Let C' and D' be the image under f of $C \cap \mathcal{C}(B)$ and $D \cap \mathcal{C}(B)$ respectively. Let $z' = L(C') \cap L(D')$. Define the map $g : X \to P$, $g(z) = z'$, for all $z \in X \backslash \mathcal{C}(B)$ and $g|_{\mathcal{C}(B)} = f$.

First, we show that g is well-defined. To see this, let $\{C'',D''\}$ be another lines pair meet $\mathcal{C}(B)$ more than one point and meet at z . We will show that $z' = L(f(C'') \cap \mathcal{C}(B))) \cap L(f(D'') \cap \mathcal{C}(B)))$. Suppose $\{C,D\}$ and $\{C'',D''\}$ do not belong to the same plane. Under f , $\{C',D'\}$ and $\{f(C'' \cap \mathcal{C}(B)), f(D'' \cap \mathcal{C}(B))\}$ will not be on the same plane. Let $\ell_1 = L(C')$, $\ell_2 = L(D')$, $\ell_3 = L(f(C'' \cap \mathcal{C}(B)))$ and $\ell_4 = L(f(D'' \cap \mathcal{C}(B)))$. Suppose they do not meet at z' , then all these lines are on the same plane, which is a contradiction.

Suppose $\{C,D\}$ and $\{C'',D''\}$ are on the same plane, and $\dim X \geq 3$. We select a third pair which is not on the same plane, then using the transitive property, $z' = \ell_3 \cap \ell_4$ as above.

If $\dim X = 2$, consider three lines ℓ_1, ℓ_2 and ℓ_3 which meet z in X . By the desarguesian property, it follows that the image under f of these three lines

$$z' = L(f(\ell_1 \cap \mathcal{C}(B))) \cap L(f(\ell_2 \cap \mathcal{C}(B))) \cap L(f(\ell_3 \cap \mathcal{C}(B))) .$$

Finally, we show that the image of a segment in X under g is a segment in P . Let $a, b \in X$, we want to show that ab maps onto $s(g(a), g(b))$, i.e. for any $c \in ab$, $g(c) \in s(g(a), g(b))$. Consider the following cases individually :

Case I. It is clear that if $ab \subseteq \mathcal{C}(B)$, then for each $c \in ab$, $f(c) \in s(f(a), f(b))$ since $g|_{\mathcal{C}(B)} = f$.

Case II. $L(a,b)$ meets $C(B)$ in more than one point, but $ab \cap C(B) = \emptyset$. Select an interior point x of $C(B)$, that is $x \in C(B\backslash a_i)$ for all $a_i \in B$. Let $y \in ax$, such that $y \in C(B)$ and $u, v \in L(a,b) \cap C(B)$. Without loss of generality, we may assume $b \in au$, $v \in ub$. By Peano's axiom , we have $z = xb \cap yu$. Let $c \in ab$, similarly, $w = xc \cap yz$. Now every point $c \in ab$ maps by g to a point determined as above by $L(u,v) \cap L(x,w)$. But the segment yz is preserved by $f = g|_{C(B)}$, so every point in ab will maps onto the segment between $g(a)$ and $g(b)$. A similar argument will yield the same result for the case where $L(a,b)$ meets $C(B)$ in more than one point and $ab \cap C(B) \neq \emptyset$.

Case III. $L(a,b)$ meets $C(B)$ is a single point or empty. Using the result in II and the fact that segments are isomorphic to each other in the real projective space, therefore ab maps to $s(g(a), g(b))$.

We have shown that each point of X is uniquely mapped onto P under g and each segment maps onto a segment of P . Hence X is isomorphic to a semi-convex subset of P. $\square$

<u>4.3.2. Lemma</u> : (X,C) , considered as a semi-convex subset of the real projective space P , does not contain a whole line in P.

Proof : Suppose X contains a whole line ℓ , x is a point such that $x \notin \ell$. Then all the lines passing through x will meet ℓ , since every two lines in a projective space will meet. This contradicts Corollary 12.62 of Coxeter [12], that there is at least one line through x which does not meet ℓ.

Hence (X,C) does not contain a whole line. $\square$

<u>4.3.3. Corollary</u> : $(X,\mathcal{C})$ is isomorphic to a linearly open convex subset of a real projective space, if the dimension of X is greater than two or two and X is desarguesian.

Proof : By Theorem 4.3.1, Definition 4.1.9 and the previous lemma, we can see that $(X,\mathcal{C})$ is isomorphic to a convex subset of a real projective space.

Let C be a convex subset in the real projective P, such that X is isomorphic to C and f be the isomorphism.

It is enough to show that C is linearly open. Suppose that ℓ is a line which meets C at a closed interval with an end-point c. Let $x \in \ell \cap C$. Then $L(f^{-1}(c), f^{-1}(x))$ in X will violate the extendible property. Hence every line meets C in an open interval. □

<u>4.3.4. Theorem</u> : Let $(X,\mathcal{C})$ be a complete basic convexity space whose dimension is greater than two or two and desarguesian. Then $(X,\mathcal{C})$ is isomorphic to a linearly open convex subset of a real affine space.

Proof : Applying Theorem 4.1.14, together with the result of the previous corollary and we have our characterization theorem immediately. □

CHAPTER FIVE

LINEARIZABLE CONVEXITY SPACE

In this chapter, we will use the axioms of an affine space given by Sasaki [26] and follow the approach of Bennett [3] to show that a basic convexity space [cf. 3.1] (domain finite, regular and straight) which is join-hull commutative [cf. 2.2], has a parallelism property [cf. 5.1] and whose dimension is greater than two, is an affine space. Finally, using the coordinatization theorem of Bennett [2], a linearizable convexity space is a vector space over a division ring. The members of C in such a space are precisely the convex subsets of the vector space.

5.1. Linearizable Convexity Space

5.1.1. Definition : Let (X,C) be a convexity space. Two segments ab and cd are parallel (denoted by $ab \parallel cd$) iff (1) $ab \subseteq xy$ and $cd \subseteq uv$, then $xy \cap uv = \emptyset$ and (2) $ac \cap bd \neq \emptyset$ or $ad \cap bc \neq \emptyset$.

5.1.2. Definition : A convexity space (X,C) is said to have the parallel property iff (1) given ab and $c \notin L(a,b)$, there is $d \in X$, such that $ab \parallel cd$; (2) if $cd \parallel ab$ and $ce \parallel ab$, then c, d, e are collinear.

5.1.3. Definition : A basic convexity space (X,C) [cf. 3.1] is said to be a linearizable convexity space iff it is join-hull commutative and has the parallel property.

<u>5.1.4. Lemma</u> : Let (X,C) be a linearizable convexity space. If $A = C(A)$, $B = C(B)$, $A \cap B = \emptyset$ and $x \notin A \cup B$, then $C(A \cup x) \cap B = \emptyset$ or $C(B \cup x) \cap A = \emptyset$.

Proof : Suppose $C(A \cup x) \cap B \neq \emptyset$ and $C(B \cup x) \cap A \neq \emptyset$. Let $u \in C(A \cup x) \cap B$ and $v \in C(B \cup x) \cap A$. By join-hull commutativity, we have $u \in ax$ and $v \in bx$, for some $a \in A$ and $b \in B$. By lemma 3.5.3 (Peano's axiom), we have $va \cap ub \neq \emptyset$. But $v, a \in A$ and $u, b \in B$, since A, B are convex $va \subseteq A$ and $ub \subseteq B$. Thus $A \cap B \neq \emptyset$, which is a contradiction. □

<u>5.1.5. Lemma</u> : Let (X,C) be a linearizable convexity space. If $A = C(A)$, then $\overline{A} = \bigcup\{L(a,b) : a, b \in A\}$.

Proof : Recall the Definition 3.1.11, $A = \overline{A}$ iff $L(a,b) \subseteq A$, whenever $a, b \in A$. Let $D = \bigcup\{L(a,b) : a, b \in A\}$. It is easy to see that $D \subseteq \overline{A}$ and $A \subseteq D$. Now we need to show that $D = \overline{D}$. That is to show $b \in L(x,y) \subseteq D$, whenever $x, y \in D$.

Suppose $x, y \in D$. then $x \in L(a_1,a_2)$ and $y \in L(a_3,a_4)$, for some $a_1, a_2, a_3, a_4 \in A$. Considering the various positions of $b \in L(x,y)$, $x \in L(a_1,a_2)$ and $y \in L(a_3,a_4)$, we will have $a_5, a_6 \in A$ so that $b \in L(a_5,a_6)$. Here we will show one of the cases, other cases will have a similar argument by applying join-hull commutativity and Peano's axiom (Lemma 3.5.3).

Now take the case that $x \in a_1a_2$ and $y \in a_3/a_4$. (i) If $b \in xy$, by Lemma 3.5.3, we have $a_5 = xa_3 \cap ba_4$. Since $a_5 \in xa_3 \subseteq A$, therefore $b \in L(a_5,a_4) \subseteq D$. (ii) If $b \in x/y$, that is $x \in by$. By Lemma 3.5.3, we have $a_5 = xa_4 \cap ba_3$. Hence $b \in L(a_5,a_3) \subseteq D$.

(iii) If $b \in y/x$, that is $y \in bx$. Since $a_3 \in ya_4 \subseteq \mathcal{C}(b,x,a_4)$, then by join-hull commutativity, $a_3 \in ba_5$, for some $a_5 \in xa_4$. This implies $b \in L(a_3,a_5) \subseteq D$. $\square$

<u>5.1.6. Lemma</u> : Let $(X,\mathcal{C})$ be a linearizable convexity space. If $a, b, c \in X$ and $c \notin L(a,b)$, then $\overline{\{a,b,c\}} = \bigcup \{L(x,y) : x, y \in \mathcal{C}(a,b,c)\}$.

Proof : First, we claim that for each $A \subseteq X$, then $\overline{\mathcal{C}(A)} = \overline{A}$. Since $A \subseteq \mathcal{C}(A)$, then $\overline{A} \subseteq \overline{\mathcal{C}(A)}$. But $\mathcal{C}(A) \subseteq \overline{A}$ and hence $\overline{\mathcal{C}(A)} \subseteq \overline{\overline{A}} = \overline{A}$. Therefore $\overline{\mathcal{C}(a,b,c)} = \bigcup\{L(x,y) : x, y \in \mathcal{C}(a,b,c)\}$. Hence $\overline{\{a,b,c\}} = \bigcup\{L(x,y) : x, y \in \mathcal{C}(a,b,c)\}$. $\square$

<u>5.1.7. Lemma</u> : Let $(X,\mathcal{C})$ be a linearizable convexity space. If $d \in \overline{\{a,b,c\}}$ and $c, d \notin L(a,b)$, then $\overline{\{a,b,c\}} = \overline{\{a,b,d\}}$.

Proof : It is obvious that $\overline{\{a,b,d\}} \subseteq \overline{\{a,b,c\}}$.

To show $\overline{\{a,b,c\}} \subseteq \overline{\{a,b,d\}}$, we need to check $c \in \overline{\{a,b,d\}}$. Since $d \in \overline{\{a,b,c\}}$, then $d \in L(u,v)$, for some $u, v \in \mathcal{C}(a,b,c)$ and $u \in cx$, $v \in cy$ for some $x, y \in ab$. Consider the following cases :

(i) $d \in uv$, since $u,v \in \mathcal{C}(a,b,c)$, then $d \in \mathcal{C}(a,b,c)$. By join-hull commutativity, we have $d \in cz$ for some $z \in ab$. Hence $c \in L(d,z) \subseteq \overline{\{a,b,d\}}$.

(ii) $d \in u/v$, that $u \in vd$. Since $v \in cy$ then $u \in \mathcal{C}(c,y,d)$. This implies $u \in cz$, for some $z \in yd$. But $u \in cx$, then $L(c,z) = L(c,u) = L(c,x)$. Thus $c \in L(x,z)$. Since $y \in ab$, $x \in ab$ and $z \in yd$, we have $x, z \in \mathcal{C}(a,b,d)$. Therefore $c \in \overline{\{a,b,d\}}$.

(iii) $d \in v/u$, switching u, v, we have an analogous argument.

Hence the lemma is proved. $\square$

<u>5.1.8. Definition</u> : Let $(X,\mathcal{C})$ be a linearizable convexity space. For $a, b, c \in X$, if $c \notin L(a,b)$, then $\overline{\{a,b,c\}}$ will be referred to as a <u>plane</u>.

<u>5.1.9. Lemma</u> : In a linearizable convexity space $(X,\mathcal{C})$, a plane is uniquely determined by three non-collinear points.

Proof : Let $a, b, c \in X$ with $c \notin L(a,b)$. If $x, y, z \in \overline{\{a,b,c\}}$ are non-collinear points, we will show $\overline{\{x,y,z\}} = \overline{\{a,b,c\}}$. Following the Lemma 5.1.7, we have $\overline{\{a,b,c\}} = \overline{\{a,b,x\}} = \overline{\{a,x,y\}} = \overline{\{x,y,z\}}$ □

<u>5.1.10. Definition</u> : Two distinct lines ℓ and m in a linearizable convexity space $(X,\mathcal{C})$ are <u>parallel</u> (denoted by $\ell \parallel m$) iff (i) $\ell \cap m = \emptyset$ and (ii) $\ell \cup m \subseteq \overline{\{a,b,c\}}$ where $c \notin L(a,b)$, for some $a, b, c \in X$.

<u>5.1.11. Lemma</u> : If a, b, c, d are distinct points in a linearizable convexity space $(X,\mathcal{C})$, then $ab \parallel cd$ iff $L(a,b) \parallel L(c,d)$.

Proof : (I) Suppose $ab \parallel cd$. (i) Suppose $L(a,b) \cap L(c,d) = x$. Then $x \notin ab \cap cd$, since $ab \cap cd = \emptyset$. If $x \in a/b \cap cd$, then $ab \subseteq bx$ and $x \in bx \cap cd$, which is a contradiction. Other cases will yield a contradiction that similar to the above. Therefore we may assume $L(a,b) \cap L(c,d) = \emptyset$. (ii) Since $ab \parallel cd$, either $ac \cap bd \neq \emptyset$ or $ad \cap bc \neq \emptyset$. We may assume $ac \cap bd = x$. Then $c \in L(a,x) \subseteq \overline{\{a,b,x\}}$ and $d \in L(b,x) \subseteq \overline{\{a,b,x\}}$. Thus $L(a,b)$ and $L(c,d)$ are contained in $\overline{\{a,b,x\}}$. Hence $L(a,b) \parallel L(c,d)$.

(II) Conversely, suppose $L(a,b) \parallel L(c,d)$. It is obvious that if $ab \subseteq xy$ and $cd \subseteq uv$, then $xy \cap uv = \emptyset$, for some $x, y, u, v \in X$.

Suppose $ac \parallel bd$. Then $bc \cap ad \neq \emptyset$ and we are done. If $ac \nparallel bd$ (is not parallel), then $L(a,c) \cap L(b,d) = x$. If $x = ac \cap bd$, we are done. If $x = a/c \cap b/d$, then $a \in cx$ and $b \in xd$. By Lemma 3.5.3, we have $ad \cap cb \neq \emptyset$. Similarly, $x = c/a \cap d/b$. Both cases imply $ab \parallel cd$. Hence $L(a,b) \parallel L(c,d)$ implies $ab \parallel cd$. □

<u>5.1.12. Theorem</u> : If (X,C) is a linearizable convexity space and $c \notin L(a,b)$, then there exists one and only one line which passes through c and is parallel to $L(a,b)$.

Proof : By Definition 5.1.3, we have, for $c \notin L(a,b)$, there exists $d \in X$, such that $ab \parallel cd$. By the last lemma, we have $ab \parallel cd$ iff $L(a,b) \parallel L(c,d)$.

Suppose $L(c,e) \parallel L(a,b)$. Then $ce \parallel ab$ and therefore $e \in L(c,d)$. This implies $L(c,e) = L(c,d)$ and the proof is complete. □

<u>5.2. Affine Space</u>

This section shows that if two distinct planes in a 3-space of a linearizable convexity space intersect, their intersection is a line and therefore a linearizable convexity space is an affine space.

Throughout the whole chapter, we will assume that (X,C) is a linearizable convexity space, unless otherwise specified.

<u>5.2.1. Lemma</u> : For $a, b, c, d \in X$, if $d \notin \overline{\{a,b,c\}}$ and $c \notin L(a,b)$, then $x \in \overline{\{a,b,c,d\}}$ with $x \notin \overline{\{a,b,c\}}$ implies $\overline{\{a,b,c,d\}} = \overline{\{a,b,c,x\}}$.

Proof : It is enough to show that $\overline{\{a,b,c,d\}} \subseteq \overline{\{a,b,c,x\}}$ or

$d \in L(u',v')$, for some $u', v' \in C(a,b,c,x)$. Since $x \in \{\overline{a,b,c,d}\}$, $x \in L(u,v)$, for some $u, v \in C(a,b,c,d)$, and $u \in ds$, $v \in dt$, for some $s, t \in C(a,b,c)$. (i) If $x \in uv$, then $x \in C(a,b,c,d)$ and $x \in dr$, for some $r \in C(a,b,c)$. Thus $d \in L(x,r)$ and hence $d \in \{\overline{a,b,c,x}\}$. (ii) If $x \in v/u$, that is $v \in ux$. Then $v \in C(d,s,x)$ and $v \in dw$, for some $w \in xs$. But $v \in dt$, implies $d \in L(w,t)$. Since $w \in xs$ and $s \in C(a,b,c)$, therefore $w \in C(x,a,b,c)$ and hence $d \in \{\overline{a,b,c,x}\}$ (iii) If $x \in u/v$, switching the role u and v , we have the same argument and conclusion. □

<u>5.2.2. Definition</u> : If $a, b, c, d \in (X,C)$ are distinct points . $d \notin \{\overline{a,b,c}\}$, and $c \notin L(a,b)$, we call $\{\overline{a,b,c,d}\}$ a <u>3-space</u>.

<u>5.2.3. Lemma</u> : If a, b, c, d are distinct points of (X,C) , where $d \notin \{\overline{a,b,c}\}$ and $c \notin L(a,b)$, then the 3-space of $\{\overline{a,b,c,d}\}$ is uniquely determined.

Proof : The result follows from the application of Lemma 5.2.1. □

<u>5.2.4. Lemma</u> : Let ℓ be a line in the plane $\{\overline{a,b,c}\}$ of (X,C) . If $\ell \cap ab = d$, then $\ell \cap bc \neq \emptyset$ or $\ell \cap ac \neq \emptyset$.

Proof : If $d = a$ or b , then the result is trvial.

Since $L(a,c) \cap L(b,c) = c$ and by Theorem 5.1.12, there is one and only one line passing d and parallel to either $L(a,c)$ or $L(b,c)$ (i.e. $\ell \cap L(a,c) \neq \emptyset$ or $\ell \cap L(b,c) \neq \emptyset$). Suppose $\ell \cap L(a,c) = e$. If $e \in ac$, we are done . If $e \in c/a$, that is $c \in ae$. Since $d \in ab$, by Lemma 3.5.3, we have $bc \cap de \neq \emptyset$ and $\ell \cap bc \neq \emptyset$. If $e \in a/c$, that is $a \in ce$. Since $d \in ab$, we have

$d \in C(e,b,c)$. By join-hull commutativity, $d \in ef$, for some $f \in bc$. Hence $\ell \cap bc = f$. □

5.2.5. Corollary : If $a, b, c \in X$ are non-collinear points, then $\overline{\{a,b,c\}} = \cup\{L(x,y) : x, y \in ab \cup bc \cup ac\}$.

Proof : This result follows from the last lemma immediately. □

5.2.6. Definition : Four distinct points a, b, c, d in a 3-space of (X,C) , are non-coplanar means $d \notin \overline{\{a,b,c\}}$ where $c \notin L(a,b)$. They are coplanar means $d \in \overline{\{a,b,c\}}$ where $c \notin L(a,b)$.

5.2.7. Definition : If $a, b, c, d \in X$ are non-coplanar, $C(a,b,c,d)$ is called a tetrahedron with vertices a, b, c and d ; ab, ac, ad, bc, bd and cd are called sides; $C(a,b,c)$, $C(a,b,d)$, $C(a,c,d)$ and $C(b,c,d)$ faces. The union of the faces is called surface; the elements that do not belong to surface are called interior points.

5.2.8. Lemma : Let $x, y \in C(a,b,c)$ of (X,C) , where a, b, c are non-collinear points. Then $L(x,y)$ meets two sides of $C(a,b,c)$.

Proof : By Theorem 5.2.12, we have $L(x,y) \cap L(a,b) \neq \emptyset$ or $L(x,y) \cap L(a,c) \neq \emptyset$ or $L(x,y) \cap L(b,c) \neq \emptyset$. Suppose $L(x,y) \cap L(b,c) \neq \emptyset$. If $L(x,y) \cap bc \neq \emptyset$, by Lemma 5.2.4, we are done. If $L(x,y) \cap b/c = w$, since $x \in C(a,b,c)$, we have $x \in au$, for some $u \in bc$. By Lemma 3.5.3, $b \in wu$ implies $ab \cap xw \neq \emptyset$. Since $w \in L(x,y)$, by Lemma 5.2.4, we are done. Switching b and c , we have the same result. □

5.2.9. Lemma : Let x be an interior point of the tetrahedron $C(a,b,c,d)$ in (X,C) . If $p \in C(a,b,c,d)$, then $L(p,x)$ meets two surface of $C(a,b,c,d)$.

Proof : Since $x \in C(a,b,c,d)$, by join-hull commutativity $x \in as$, for some $s \in C(b,c,d)$. If $p \in C(b,c,d)$, by the previous lemma, we have $L(s,p)$ meets two sides of $C(b,c,d)$ (say u, v). Then x and p belong to $C(a,u,v)$ and $L(x,p)$ meets two sides of $C(a,u,v)$. Since u, v are on the sides of $C(b,c,d)$, thus au, av and uv are on the surfaces of $C(a,b,c,d)$. Hence $L(p,x)$ meets two surfaces of $C(a,b,c,d)$.

Suppose $p \notin C(b,c,d)$. Since $p \in C(a,b,c,d)$, we have $p \in at$, for some $t \in C(b,c,d)$. By the previous lemma, we have $L(s,t)$ meets two sides of $C(b,c,d)$ (say u, v). Then $x, p \in C(a,u,v)$. Similarly, $L(p,x)$ meets two surfaces of $C(a,b,c,d)$ and hence the lemma is proved. □

<u>5.2.10. Lemma</u> : Let $C(a,b,c,d)$ be a tetrahedron of (X,C) and p be an interior of the surface $C(a,b,c)$. Let k be a point of $\overline{\{a,b,c,d\}}$. If $k \notin \overline{\{a,b,c\}}$ and $d \notin L(k,p)$, then $L(k,p)$ meets a surface other than $C(a,b,c)$.

Proof : Since $k \in \overline{\{a,b,c,d\}}$, by Lemma 5.1.5, there exist u, v in $C(a,b,c,d)$ such that $k \in L(u,v)$. By Lemma 5.2.9, we have two points m, n on the surface of $C(a,b,c,d)$, so that $k \in L(m,n)$. If $k \in C(a,b,c,d)$, by last lemma, we are done. We may assume $k \notin C(a,b,c,d)$, and $n \in km$.

Consider the following cases :

(i) $m \in C(a,b,c)$. Since p is an interior point of $C(a,b,c)$, then there is $x \in C(a,b,c)$ so that $p \in mx$. Since $n \in km$, by Lemma 3.5.3, we have $nx \cap ky = y$ and y is an interior point. By last lemma, $L(p,x)$ meets two surfaces of $C(a,b,c,d)$. Since p is on

$C(a,b,c)$, therefore other point will be on some other surface than $C(a,b,c)$.

(ii) $n \in C(a,b,c)$. Similarly, we have $p \in nx$, and then $p \in C(m,x,k)$. Thus $p \in ky$, for some $y \in mx$. Unlikely, $p = n$, if so we are done. Therefore y is an interior point. By last lemma, $L(k,p)$ meets two surfaces of $C(a,b,c,d)$.

(iii) m and $n \notin C(a,b,c)$. Suppose $m \in C(a,b,d)$ and $n \in C(a,c,d)$ and $n \in km$. By join-hull commutativity, $m \in dm'$, $n \in dn'$ for some $m' \in ac$ and $n' \in ab$. Since $k \in L(m,n)$, then $k \in \overline{\{m',n',d\}}$. By Lemma 5.2.5, there are u, v on the sides of $C(m',n',d)$, such that $k \in L(u,v)$. Since m', $n' \in C(a,b,c)$, then u or $v \in C(a,b,c)$. Using the same argument as in (i) and (ii) we are done.

Other cases have similar argument. $\square$

<u>5.2.11. Lemma</u> : If two different planes in 3-space of (X,C) intersect, then their intersection is a line.

Proof : Let the 3-space be generated by $C(a,b,c,d)$, where $d \notin \overline{\{a,b,c\}}$ and $c \notin L(a,b)$. Without loss of generality, we may take one of the planes generated by $C(a,b,c)$ and d is on the other plane α . Again we may assume that x is on their intersection and x is an interior point of $C(a,b,c)$.

Choose a point $e \in \alpha$ so that $e \notin L(d,x)$. If $e \in C(a,b,c)$, we are done.

By the previous lemma $L(e,x)$ will meet another surface of $C(a,b,c,d)$, say at $f \in C(a,c,d)$. Other positions of f will yield a similar result.

By join-hull commutativity, we have $f \in dg$ for some $g \in ac$. Since $x, e \in \alpha$, then $L(x,e) \subseteq \alpha$ and $f \in \alpha$. Since $d \in \alpha$, we have $L(f,d) \subseteq \alpha$ and $g \in \alpha$. But $g \in ac$ implies $g \in \overline{\{a,b,c\}}$. Now $x, g \in \overline{\{a,b,c\}}$ and α. Hence $L(x,g) \subseteq \overline{\{a,b,c\}} \cap \alpha$.

Suppose $k \notin L(x,g)$ and $k \in \alpha \cap \overline{\{a,b,c\}}$. Since $x, g \in \overline{\{a,b,c\}}$ and α, therefore $\overline{\{k,x,g\}} = \overline{\{a,b,c\}}$ and $\overline{\{k,x,g\}} = \alpha$. This implies $\alpha = \overline{\{a,b,c\}}$ and $d \in \overline{\{a,b,c\}}$, which contradicts our assumption. Hence $\overline{\{a,b,c\}} \cap \alpha = L(x,g)$. $\square$

For the sake of convenience, we rewrite the definition of an affine space which has been defined in 4.1.6.

<u>5.2.12. Definition</u> : Let X be a nonempty set. The elements of X are called points. L and π are the collection of the subsets of X, whose elements are called lines and planes respectively. (X, L, π) is said to be an <u>affine space</u> iff it satisfies the following axioms :

(1) for any two distinct points, there is one and only one line containing them ,

(2) for any three non-collinear (i.e. not all of them belong to the same line) points, there is one and only one plane containing them,

(3) given a line ℓ and a point which does not belong to ℓ, then there is an unique line m passes through the point, such that $\ell \quad m = \emptyset$ and they are contained in the same plane (refer to as ℓ parallel to m, written $\ell \parallel m$),

(4) if ℓ_1, ℓ_2 and ℓ_3 are distinct lines, then $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$ imply $\ell_1 \parallel \ell_3$.

The axiom (4) is equivalent to the following axiom too,

(4') if two distinct planes in 3-space intersect, then their intersection is a line.

Sasaki [26] gives a proof of the equivalence of the above axioms.

<u>5.2.13. Theorem</u> : If (X,C) is a linearizable convexity space whose dimension is greater than two, L is the collection of lines (i.e. $L(a,b) = ab \cup a/b \cup b/a$, $a \neq b$, $a, b \in X$) and π is the collection of planes (i.e. $\overline{\{a,b,c\}}$ with $c \notin L(a,b)$, $a, b, c \in X$) , then (X, L, π) is an affine space.

Proof : (1) Theorem 3.1.8 shows each line is uniquely determined by two distinct points. (2) Lemma 5.1.9 shows each plane is uniquely determined by three non-collinear points. (3) Thorem 5.1.12 gives the propterty of axiom (3) of the Definition 5.2.12 and (4) Lemma 5.2.11 shows that if two different planes in 3-space intersect, their intersection is a line. Hence (X, L, π) is an affine space. □

5.3. Coordinatization of a Linearizable Convexity Space

In the last section a linearizable convexity space has been shown to be an affine space. From this point, we will follow the approach given by Bennett [2] to coordinatize a linearizable convexity space.

<u>5.3.1. Theorem (Desargues' Theorem for Affine Space)</u> [2,18] : Let (X,L,π) be an affine space whose dimension is greater than two. Let $L(a,a') \parallel L(b,b') \parallel L(c,c')$ or $L(a,a') \cap L(b,b') \cap L(c,c') = p$, for some $p \in X$. If $L(a,b) \parallel L(a',b')$ and $L(a,c) \parallel L(a',c')$, then $L(b,c) \parallel L(b',c')$

This theorem for an affine space whose dimension is greater than two, will play an important role for defining addition and multiplication on the space. If no ambiguity will arise, we will assume $(X,\mathcal{C})$ to be a linearizable convexity space whose dimension is greater than two or of dimension two with Desargues' Theorem holding; L and π are the collection of lines and planes respectively. We suppose the 0_ℓ and 1_ℓ are two fixed points on line ℓ, for defining addition, multiplication or other operations done on it [cf. 3.2]. For only a line, we will simply use 0 and 1.

<u>5.3.2. Definition</u> : (<u>Addition</u>) For $a, b \in \ell$, define $a+b \in \ell$ as follows :

choose a point $c \notin \ell$. Let d be the point so that $L(c,d) \parallel \ell$ and $L(a,d) \parallel L(b,c)$. Then $e = a+b$ iff $L(d,e) \parallel L(b,c)$. Refer to the Figure 5.3.1.

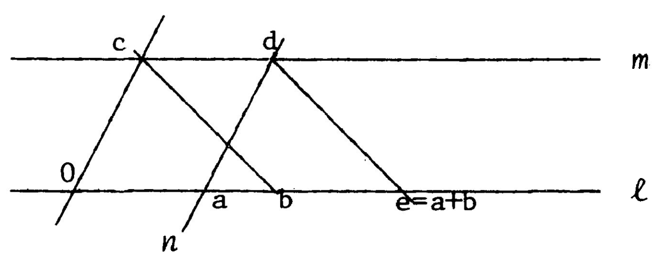


Figure 5.3.1

The addition is well-defined. Since we know that there exists m which passes through c and $m \parallel \ell$. Similarly, n passes through a and $n \parallel L(0,c)$. m and n are on the same plane. Since $L(0,c)$ meets m, hence n meets m at point d. For the same reason, the point e exists. The application of Desargues' Theorem shows the independence of the choice of c . Let c' be another point, and d' be the intersection point. Since $L(c,d) \parallel \ell$ and $L(c',d') \parallel \ell$, then by axiom 4 of affine space $L(c,d) \parallel L(c',d')$. Since $L(0,c) \parallel L(a,d)$ and $L(0,c') \parallel L(a,d')$, by Desargues' Theorem , $L(c,c') \parallel L(d,d')$. Similarly, $L(c,d) \parallel L(d,e)$ implies $L(c',b) \parallel L(d',e)$. Hence $e = a + b$ is the same point.

5.3.3. Lemma : For each $a \in \ell$, there is $-a \in \ell$ so that $a + (-a) = 0$ and $(-a) + a = 0$. Further, $a + 0 = a = 0 + a$, for each $a \in \ell$.

Proof : Let $a, 0 \in \ell$ and $b \in m$, where $m \parallel \ell$. Let $c = L(a,c) \cap m$ where $L(a,c) \parallel L(0,b)$ and $-a \in \ell$, where $L(b,-a) \parallel L(0,c)$. It is easy to see that $a + (-a) = 0$ and $(-a) + a = 0$. Also, it easily follows that $a + 0 = a$ and $0 + a = a$. □

5.3.4. Definition : (Multiplication) For $a, b \in \ell$, define $a \cdot b \in \ell$, as follows :

choose a point $c \notin \ell$. Let d be the point on $L(0,c)$ such that $L(1,c) \parallel L(a,d)$. Then $e = a \cdot b \in \ell$ where $L(b,c) \parallel L(d,e)$. Refer to the Figure 5.3.2.

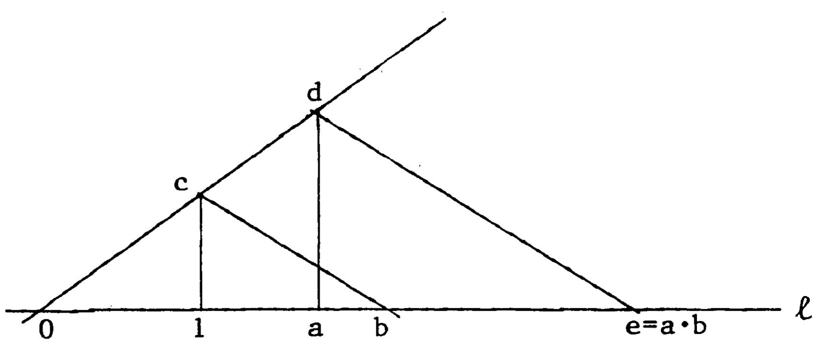


Figure 5.3.2

Using the same reason as for addition, it is easy to see that multiplication is well-defined and is independent of the choice of c .

From the definition, it is easy to see that 1 is the multiplicative identity, that is $a \cdot 1 = a = 1 \cdot a$, for each $a \in \ell$. Also note that $a \cdot 0 = 0 = 0 \cdot a$.

<u>5.3.5. Theorem (Converse Desargues' Theorem for Affine Space)</u> :
Let (X, L, π) be an affine space whose dimension is greater than two. If $L(a,b) \parallel L(a',b')$, $L(a,c) \parallel L(a',c')$ and $L(b,c) \parallel L(b',c')$, then either $L(a,a') \parallel L(b,b') \parallel L(c,c')$ or $L(a,a')$, $L(b,b')$, $L(c,c')$ meet at a point in X. [18]

Using Desargues' Theorem and its converse, one can prove that addition is commutative, addition and multiplication are associative and multiplication is distributive over addition.

<u>5.3.6. Lemma</u> : Addition on a line of (X,C), is commutative (i.e. $a + b = b + a$, for $a, b \in \ell$).
Proof : Using Definition 5.3.2, we have the point $a+b$ on ℓ, such that $L(a,c) \parallel L(d,a+b)$, $\ell \parallel L(c,d)$ and $L(0,c) \parallel L(b,d)$ (refer to Figure 5.3.3). Let $\ell_1 = L(c,d)$ and $\ell_2 = L(e,f)$, where $L(b,e) \parallel L(a,c)$ $L(e,f) \parallel L(c,d)$ and $L(b,f) \parallel L(0,c)$. $g = \ell_2 \cap L(a,h)$ where $L(a,h) \parallel L(0,c)$. For Δcag and Δebf , applying the Converse Desargues' Theorem, we have either $L(e,0) \parallel L(0,f) \parallel L(0,a)$ or $L(e,0)$, $L(0,f)$, and $L(0,a)$ meet at one point. But since $L(0,e) \cap L(0,a) = 0$, hence 0, g, f are collinear. Since $L(h,f) \parallel L(c,d)$, $L(d,f) \parallel L(a,g)$, then

$L(c,a) \parallel L(h,d)$ (i.e. $L(c,a) \parallel L(h,b+a)$). By Definition 5.3.2, through e and h, we have the point $a+b \in \ell$. Hence $a + b = b + a$. □

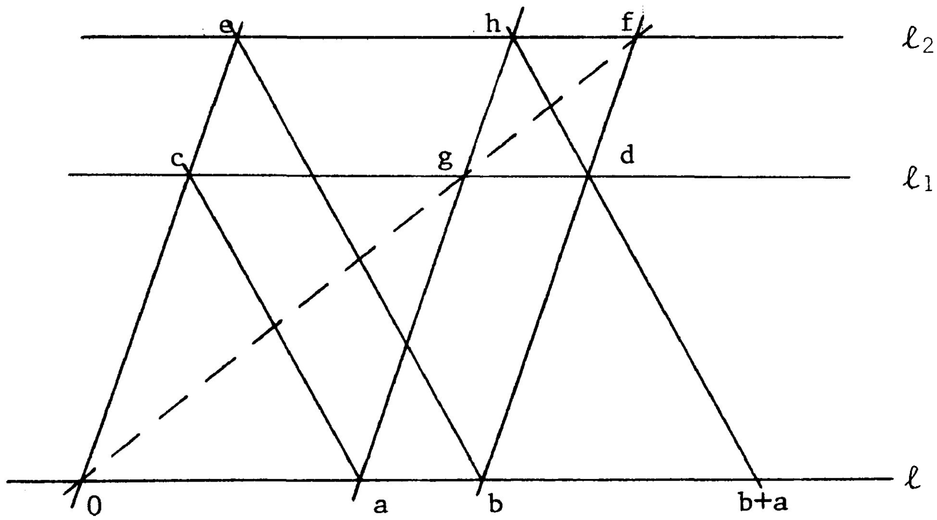


Figure 5.3.3

5.3.7. Lemma : The associative law holds on a line of (X,C) under the addition and multiplication defined in 5.3.2 and 5.3.4 respectively.

Proof : (I) Addition , i.e. $(a + b) + c = a + (b + c)$.

First, we have the following construction (refer to Figure 5.3.4).

(1) Obtain $R = a + b$ through b' and e' . Let $e = L(e',R) \cap L(0,b')$ and through e obtain $\ell_2 \parallel \ell$.

(2) Obtain $T = (a + b) + c = R + c$ through e and g' and let $a' = \ell_2 \cap L(a,e')$.

(3) Obtain $S = b + c$ through b' and c' (since $b+c = c+b$). Let $f = L(S,c') \cap L(0,b')$. Obtain ℓ_3 through f , so that $\ell_3 \parallel \ell$.

(4) $g = L(a,e') \cap \ell_3$, $h = L(c,c') \cap \ell_3$. Since

$L(b,b') \parallel L(e',R)$ $(\equiv L(e,e'))$ and $L(b,b') \parallel L(c',S)$ $(\equiv L(f,c'))$, then $L(e,e') \parallel L(f,c)$.

$\ell_3 \parallel \ell_2$ since $\ell_3 \parallel \ell$ and $\ell_2 \parallel \ell$ $L(a',e') \parallel L(c,h)$, since they both are parallel to $L(0,b')$. By the Converse Desargues' Theorem, we have b', a' and h are collinear. $L(g,h) \parallel L(e,a')$ and $L(a',e') \parallel L(h,g)$ implies $L(e,e') \parallel L(g,g')$ and $L(g,g') \parallel L(f,S)$. Since both are parallel to $L(b,b')$, but $g' \in L(g',T)$ and $L(g',T) \parallel L(R,e)$, implies $L(g,T) \parallel L(f,S)$. Hence $(a + b) + c = a + (b + c)$.

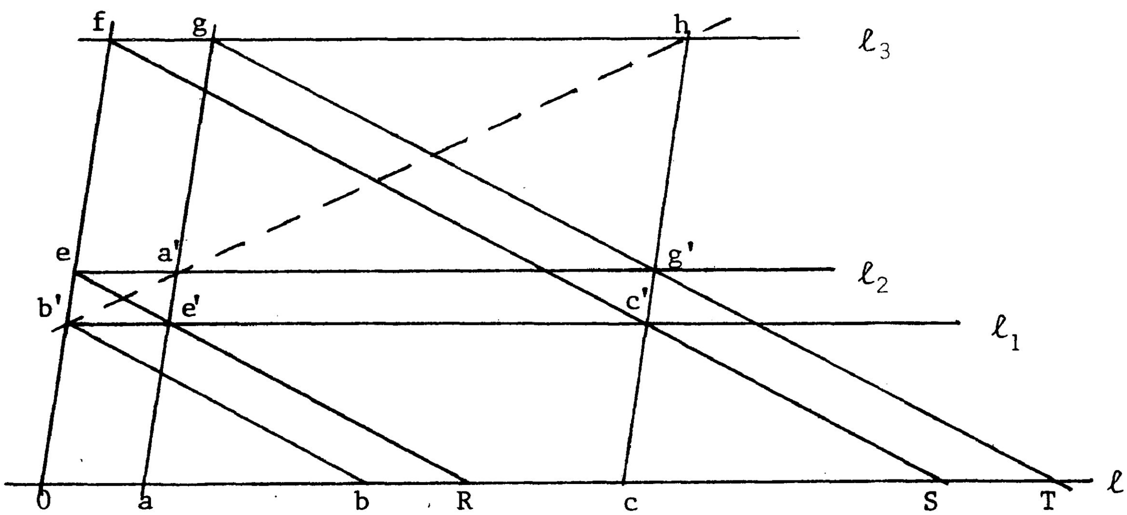


Figure 5.3.4

(II) Multiplication , i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Do the following construction (refer to Figure 5.3.5).

(1) Obtain $R = a \cdot b$ through b' and a' .

(2) $L(c',1) \parallel L(b,b')$.

(3) Obtain $S = b \cdot c$ through c' and b'.

(4) Obtain $T = (a \cdot b) \cdot c = R \cdot c$ through c' and a'.

Need to show $T = a \cdot (b \cdot c) = a \cdot S$ through b' and a'.

We have the following results from the above ;

(1) implies $L(1,b') \parallel L(a,a')$ and $L(b,b') \parallel L(a',R)$,

(3) implies $L(c',1) \parallel L(b,b')$ and $L(c,c') \parallel L(b',S)$ --(A),

(4) implies $L(c',1) \parallel L(a',R)$ and $L(c,c') \parallel L(a',T)$ --(B).

Clearly, (A) and (B) imply $L(b',S) \parallel L(a',T)$. Hence $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. □

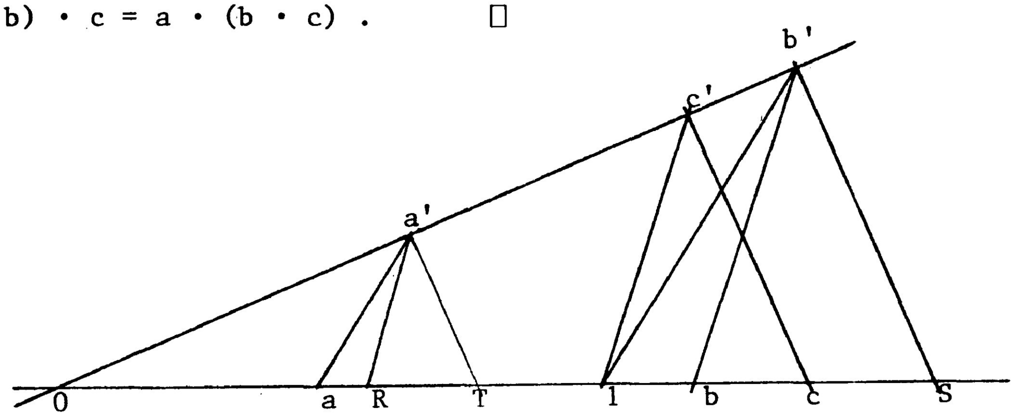


Figure 5.3.5

<u>5.3.8. Lemma</u> : The distributive law holds on a line of (X,C) under the addition and multiplication defined in 5.3.2 and 5.3.4 respectively.

Proof : The proof is similar in construction to the previous two lemmas and hence is omitted.

<u>5.3.9. Lemma</u> : If ℓ, m, n are distinct coplanar parallel lines, then $\ell \subseteq C(m \cup n)$ or $m \subseteq C(\ell \cup n)$ or $n \subseteq C(\ell \cup m)$.

Proof : Since $\ell \parallel m \parallel n$ and are coplanar, let k be a line which meets ℓ, m, n at a, b, c respectively. Consider the case where $b \in ac$. Let $x \in m$ and $d = L(c,x) \cap \ell$. It is easily seen, using Lemma 3.5.3 and join-hull commutativity that $x \in cd$. Hence $m \subseteq C(n \cup \ell)$.

Other cases lead to similar results. □

<u>5.3.10. Corollary</u> : If $c \in ab$, then $c+x \in C(a+x,b+x)$, for each $x \in L(a,b)$. Similarly, $c \cdot x \in C(a \cdot x, b \cdot x)$.

Proof : This follows by applying the previous lemma. $\square$

<u>5.3.11. Theorem</u> : Every line of a linearizable convexity space is an ordered division ring.

Proof : Let $\ell \in L$, where (X,C) is a linearizable convexity space and L is the collection of the lines.

That $(\ell,+)$ is a commutative group under the addition given by Definition 5.3.2, follows by Lemmas 5.3.3, 5.3.6 and 5.3.7.

Also Lemma 5.3.7 shows that multiplication is associative. Further 1 is the multiplicative identity.

It remains to show that for each $0 \neq a \in \ell$, there is $a^{-1} \in \ell$ such that $a \cdot a^{-1} = 1 = a^{-1} \cdot a$. Let $B \notin \ell$ and $C \in L(0,B)$ such that $L(B,1) \parallel L(a,C)$. Let $a^{-1} \in \ell$ such that $L(B,a^{-1}) \parallel L(C,1)$. By Definition 5.3.4, we see that $a \cdot a^{-1} = 1$ through B and similarly, $a^{-1} \cdot a = 1$ through C .

By Theorem 3.2.4, we know that every line is a totally ordered set and Lemma 5.3.10 shows that addition and multiplication preserve order.

Hence $(\ell, +, \cdot)$ is an ordered division ring. $\square$

<u>5.3.12. Lemma</u> : Lines in (X,C) are ordered ring-isomorphic to each other.

Proof : Let ℓ, m be two different lines in (X,C) . Then $\ell \cap m = \emptyset$ or $\ell \parallel m$ or $\ell \cap m = x$, for some $x \in X$. For the first case, we may choose a line n such that $\ell \parallel n$ where n, m are on the same plane.

Therefore we only need to consider the other two cases.

There are number of ways to define an isomorphism. We will choose an isomorphism described below, since that will be useful for us to do the coordinatization on the space.

Let 1_ℓ, 1_m are the fixed points of ℓ and m respectively, where ℓ, m are on the same plane and $\ell \cap m = x$. Let x to be $0_\ell = 0_m$. Define $\alpha : \ell \to m$ as follows : $\alpha(0_\ell) = 0_m$, $\alpha(1_\ell) = 1_m$, for $a \in \ell$, $b \in m$, $\alpha(a) = b$ iff $L(a,b) \parallel L(1_\ell, 1_m)$. It is easy to see that : (1) by the property of parallelism, α is well-defined and one-to-one onto, (2) by Lemma 5.3.9, α preserves the order.

To see α is a ring-isomorphism, we will check $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$ for $a, b \in \ell$, and $\alpha(a)$, $\alpha(b) \in m$.

(i) $\alpha(a + b) = \alpha(a) + \alpha(b)$. Refer to the Figure 5.3.6. Obtain b+a (= a+b) through $\alpha(a)$ and c. Let $e = m \cap L(c, a+b)$. Clearly, $e = \alpha(a + b)$, since $L(c, a+b) \parallel L(\alpha(a), a)$. It is not hard to see that $e = \alpha(a) + \alpha(b)$ through b and c. Hence $\alpha(a + b) = \alpha(a) + \alpha(b)$.

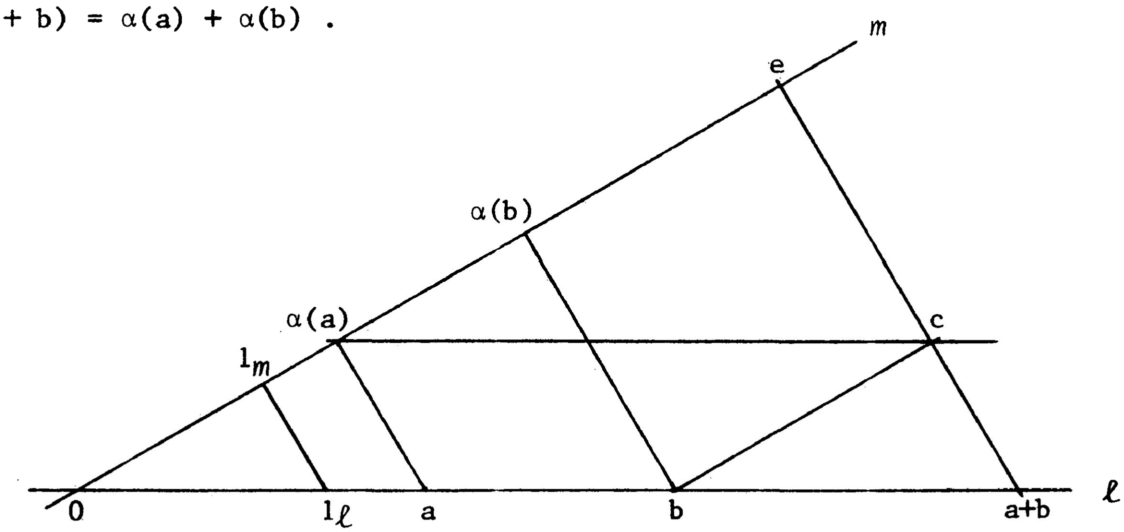


Figure 5.3.6

(ii) $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$. Refer to the Figure 5.3.7. Obtain $a \cdot b$ through 1_m and $\alpha(a)$. It is easy to see that $e = \alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$ can be obtained through $a \cdot b$ and b. Hence $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$.

Therefore α is an ordered ring-isomorphism.

For the case $\ell \parallel m$, we will use the third line which is joined by 0_ℓ and 0_m . The result is immediate. $\square$

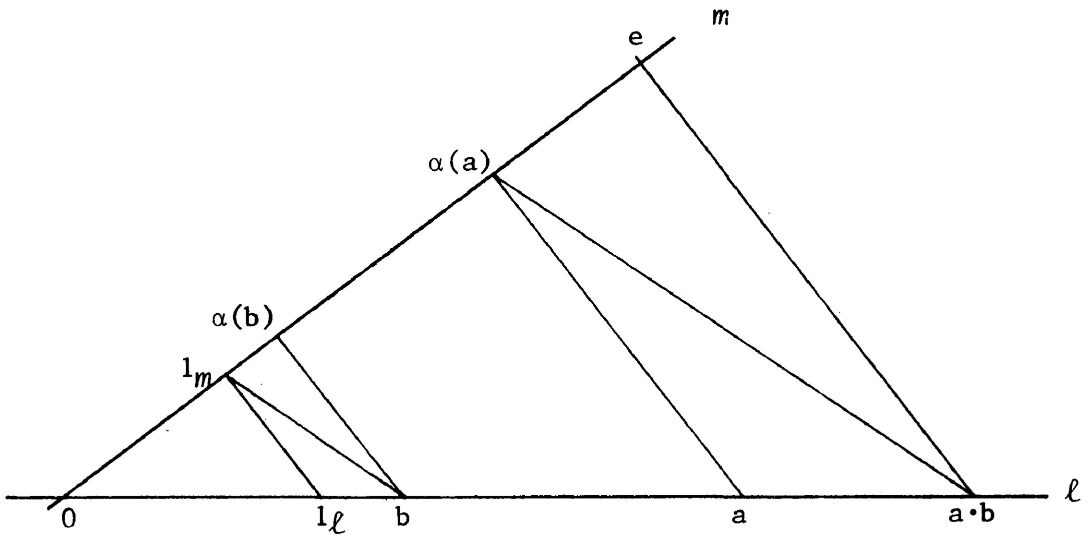


Figure 5.3.7

Since every line is an ordered division ring, we may choose an ordered division ring R so that every line is isomorphic to it. Generally, we will use ψ_ℓ for the isomorphism from R onto ℓ , if no confusion arises, we simply use ψ . Select a fixed point $0 \in X$. We will refer to this point as an origin point for the space. Once the point has been selected, it will be fixed for all operations done on the space.

<u>5.3.13. Definition</u> : For $a, b \in X$, define $a + b$ to be the point such that, $L(0,a) \parallel L(a,a+b)$ and $L(0,b) \parallel L(a,a+b)$. If $0 \in L(a,b)$ then $a + b$ will be defined in Definition 5.3.2.

<u>5.3.14. Lemma</u> : X together with $+$ as defined in 5.3.13, is a commutative group.

Proof : It is enough to show that the associative law holds, i.e. $(a + b) + c = a + (b + c)$, for $a, b, c \in X$. Lemma 5.3.7 covers the case where a, b, c and 0 are collinear. We may assume that a, b, c and 0 are non-collinear. Refer to the Figure 5.3.8. First, by Definition 5.3.13, we have the point $a+b$ and $b+c$, such that : (i) $L(a,a+b) \parallel L(0,b)$, and $L(b,a+b) \parallel L(0,a)$, (ii) $L(0,b) \parallel L(c,b+c)$ and $L(0,c) \parallel L(b,b+c)$. Now using the Definition 5.3.13 again, we have the point $a+(b+c)$ such that $L(b+c,a+(b+c)) \parallel L(0,a)$ and $L(a,a+(b+c)) \parallel L(0,b+c)$. By Definition 5.3.13, to give $(a+b)+c$, we need to show $L(c,a+(b+c)) \parallel L(0,a+b)$.

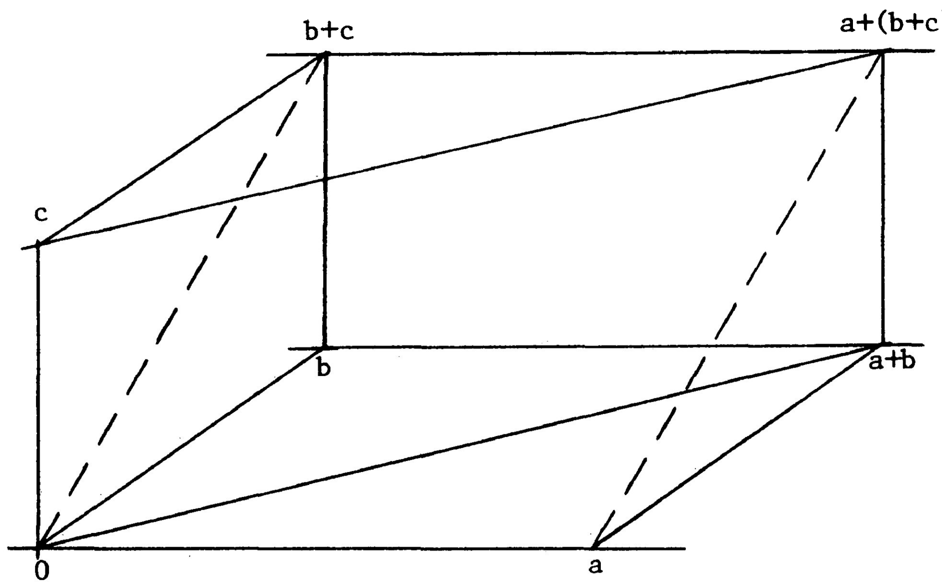


Figure 5.3.8

Since $L(b,a+b) \parallel L(0,a)$ and $L(b+c,a+(b+c)) \parallel L(0,a)$, then $L(b+c,a+(b+c)) \parallel L(b,a+b)$. By Desargues' Theorem, since $L(a,a+b) \parallel L(0,b)$ and $L(a,a+(b+c)) \parallel L(b,b+c)$ imply $L(a+b,a+(b+c)) \parallel L(b,b+c)$. Now applying Desargues' Theorem again, where $L(a+b,a+(b+c)) \parallel L(0,c) \parallel L(b,b+c)$, $L(c,b+c) \parallel L(o,b)$ and $L(b+c,a+(b+c)) \parallel L(b,a+b)$ imply $L(c,a+(b+c)) \parallel L(0,a+b)$. This completes the proof. $\square$

<u>5.3.15. Definition</u> : For $a \in X$ and $\lambda \in R$ (an ordered division ring), define $\lambda \cdot a$ to be the point $\psi(\lambda) \cdot a$ where ψ is an isomorphism from R onto $L(0,a)$. This operation will be referred to as <u>scalar multiplication</u>.

<u>5.3.16. Lemma</u> : The following equalities hold under the addition and scalar multiplication defined in 5.3.13 and 5.3.15 respectively.

(1) $\lambda \cdot (a + b) = (\lambda \cdot a) + (\lambda \cdot b)$,

(2) $(\lambda + \mu) \cdot a = (\lambda \cdot a) + (\mu \cdot a)$,

(3) $(\lambda \cdot \mu) \cdot a = \lambda \cdot (\mu \cdot a)$,

for $a, b \in X$ and $\lambda, \mu \in R$, where R is an ordered division ring.

Proof : (1) The case where $a, b, 0$ are collinear has been covered by Lemma 5.3.8. We may assume a, b and 0 are non-collinear. Using Definition 5.3.13, we have $a+b$, such that $L(a,a+b) \parallel L(0,b)$ and $L(b,a+b) \parallel L(0,a)$ (refer to the Figure 5.3.9). Let $\ell = L(0,a)$, $m = L(0,b)$ and $n = L(0,a+b)$ and the isomorphism from R onto ℓ, m, n be $\psi_\ell, \psi_m, \psi_n$ respectively. Let $A \in \ell$, $B \in m$, $C \in n$ such that $A = \psi_\ell(\lambda)$, $B = \psi_m(\lambda)$ and $C = \psi_n(\lambda)$. By Definition 5.3.15, $\lambda \cdot a = A \cdot a$, $\lambda \cdot b = B \cdot b$ and $\lambda \cdot (a + b) = C \cdot (a + b)$.

Using the Definition 5.3.4 of multiplication on lines ℓ, m, n to obtain $A \cdot a$, $B \cdot b$ and $C \cdot (a+b)$ respectively, we have the following : $L(1_n, b) \parallel L(C, B\,b)$, $L(1_n, a) \parallel L(C, A\,a)$. Since $\ell \cap m \cap n = 0$, by Desargues' Theorem we have $L(A \cdot a, B \cdot b) \parallel L(a,b)$. But $C \cdot (a+b)$ through b and $B \cdot b$ gives $L(B \cdot b, C \cdot (a+b)) \parallel L(b, a+b)$. Applying Desargues' Theorem again, we have $L(A \cdot a, C \cdot (a+b)) \parallel L(a, a+b)$. Hence $L(B \cdot b, C \cdot (a+b)) \parallel L(0, A \cdot a)$ and $L(A \cdot a, C \cdot (a+b)) \parallel L(0, B \cdot b)$ imply $C \cdot (a + b) = (A \cdot a) + (B \cdot b)$. Therefore $\lambda \cdot (a + b) = (\lambda \cdot a) + (\lambda \cdot b)$.

Similarly, we can show (2) and (3). □

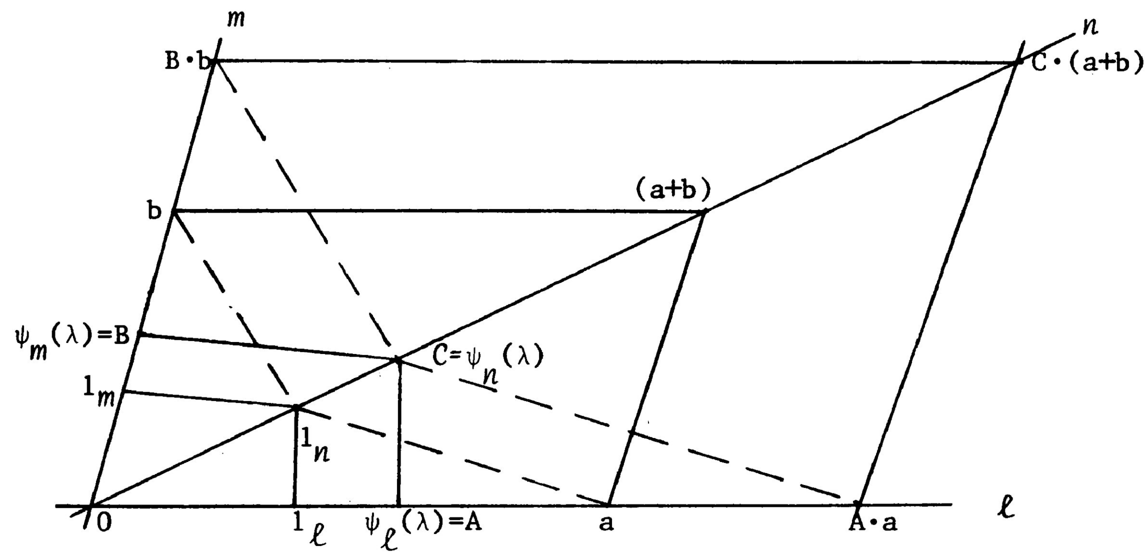


Figure 5.3.9

5.4. Linearization Theory

From the result of last three sections, we see that a linearizable convexity $(X, \mathcal{C})$, whose dimension is greater than two, essentially is an affine space and it can be made into a vector space V, over an ordered division ring, where the members of $\mathcal{C}$ are precisely the convex sets of V.

<u>5.4.1. Theorem</u> : If (X,C) is a linearizable convexity space whose dimension is greater than two, then it can be made into a vector space over an ordered division ring under the addition, multiplication and scalar multiplication defined in 5.3.13, 5.3.4 and 5.3.15 respectively.

Proof : The theorem follows immediately from the Lemmas 5.3.14 and 5.3.16. $\square$

<u>5.4.2. Lemma</u> : For each member of L, where L is the collection of (X,C), we have $L(a,b) = \{\lambda\cdot a + (1+(-\lambda))\cdot b : \lambda \in R\}$, where R is the ordered division ring, 1 is the multiplicative identity of R and $a, b \in X$.

Proof : Let $U = \{\lambda\cdot a + (1+(-\lambda))\cdot b : \lambda \in R\}$. If $u \in U$, then $u = \lambda\cdot a + (1+(-\lambda))\cdot b = b + \lambda\cdot(a-b)$, for some $\lambda \in R$. By Definition 5.3.13, we have $L(0,b) \parallel L(u,v)$ and $L(0,v) \parallel L(b,v)$, where $v = \lambda\cdot(a-b)$. Since for a fixed $\lambda \in R$, we have a point $\psi_\ell(\lambda)$ on $L(0,v) = \ell$, where ψ_ℓ is an isomorphism from R onto ℓ. Then there is a point $(a-b)$ on ℓ, such that $\psi_\ell(\lambda)\cdot(a-b) = \lambda\cdot(a-b) = v$. Adding point b and $(a-b)$, we have a point, such that $L(b,b+(a-b)) \parallel L(0,a-b)$ and $L(a-b, b+(a-b)) \parallel L(0,b)$. But $b + (a - b) = a$. Therefore $L(0,a-b) \parallel L(b,a)$. Since $L(b,u) \parallel L(0,a-b)\ (\equiv \ell)$, then either $L(b,a) \parallel L(b,u)$ or $L(b,a) = L(b,u)$. But $b \in L(b,a) \cap L(b,u)$. Hence $L(a,b) = L(b,u)$ and $u \in L(a,b)$.

Conversely, suppose $u \in L(a,b)$. Case (I), if $a, b, 0$ are collinear (say ℓ), obtain $u-b$, $a-b$ on the line ℓ. If $u-b = a-b$, then take $\lambda = \psi_\ell^{-1}(1_\ell) = 1 \in R$, where ψ_ℓ is an isomorphism from R onto ℓ. Otherwise, we can obtain $v \in \ell$, such that $u = b + v\cdot(a-b)$

since ℓ is an ordered division ring. Take $\lambda = \psi_{\ell}^{-1}(v) \in R$. Hence $u \in U$.
Case (II), $a, b, 0$ are non-collinear. By the property of a vector space, we can obtain a point p, such that $b + p = a$ where $L(a,b) \parallel L(0,p)$ and $L(0,a) \parallel L(b,p)$. Similarly, we have $x \in L(0,p)$ such that $b + x = u$. Further, there is a point $v \in L(0,p)$, such that $x = v \cdot p$. Hence $u = b + x = b + v \cdot p = b + v \cdot (a-b)$. Thake $\lambda = \psi_{\ell}^{-1}(v)$, we have $u = b + \lambda \cdot (a-b) \in U$. □

<u>5.4.3. Lemma</u> : Let V be a vector space determined by $(X,\mathcal{C})$. Then $A = \mathcal{C}(A)$ iff A is a convex subset of V.

Proof : We need to show, for $x, y \in A = \mathcal{C}(A)$, $z \in \mathcal{C}(x,y)$ iff $z = \lambda \cdot x + (1-\lambda) \cdot y$, for $0 \le \lambda \le 1$, $\lambda \in R$.

Let 0_{ℓ} and 1_{ℓ} be the two fixed elements of $L(x,y) = \ell$. If $z \in \mathcal{C}(x,y)$, by Lemma 5.4.2, we have $z = \lambda \cdot x + (1-\lambda) \cdot y$, for some $\lambda \in R$. It is sufficient to show $0 \le \lambda \le 1$, i.e. $0_{\ell} \le v \le 1_{\ell}$, where $\psi_{\ell}(\lambda) = v$, ψ_{ℓ} is an isomorphism from R onto ℓ. By Corollary 5.3.10, we have $0_{\ell} = z + (-z) \in \mathcal{C}(x-z, y-z)$, then $0_{\ell} \in \mathcal{C}(x-(\lambda \cdot x + (1-\lambda) \cdot y), y-(\lambda \cdot x + (1-\lambda) \cdot y)) = \mathcal{C}((1_{\ell}-v) \cdot (x-y), (-v) \cdot (x-y))$. By Definition 5.3.15, $\lambda \cdot x = \psi_{\ell}(\lambda) \cdot x = v \cdot x$. This implies $0_{\ell} \in \mathcal{C}((1_{\ell}-v), (-v))$ and thus $v \in \mathcal{C}(1_{\ell}, 0_{\ell})$. By Lemma 3.4.7, we have $0_{\ell} \le v \le 1_{\ell}$

Conversely, reversing the steps, we have if $z = \lambda \cdot x + (1-\lambda) \cdot y$ $0 \le \lambda \le 1$, then $z \in \mathcal{C}(x,y)$.

Hence $A = \mathcal{C}(A)$ iff A is a convex subset of the vector space determined by $(X,\mathcal{C})$. □

<u>5.4.4. Theorem</u> : $(X,\mathcal{C})$ is a linearizable space iff X is a vector space of dimension greater than two, over an ordered division ring and $A \in \mathcal{C}$ iff A is a convex subset of X .

Proof : Theorem 5.2.13 and Theorem 5.4.1 show a linearizable convexity space $(X,\mathcal{C})$ whose dimension is greater than two, is a vector space over an ordered division ring.

Conversely, if X is a vector space, with the collection $\mathcal{C}$, of convex subsets of X , then $(X,\mathcal{C})$ is a linearizable convexity space. This follows immediately by the properties of a vector space that satisfies that all the conditions of a linearizable convexity space.

Finally, Lemma 5.4.3 completes the proof. $\square$

CHAPTER SIX

CONCLUSION

6.1. Summary and Remarks

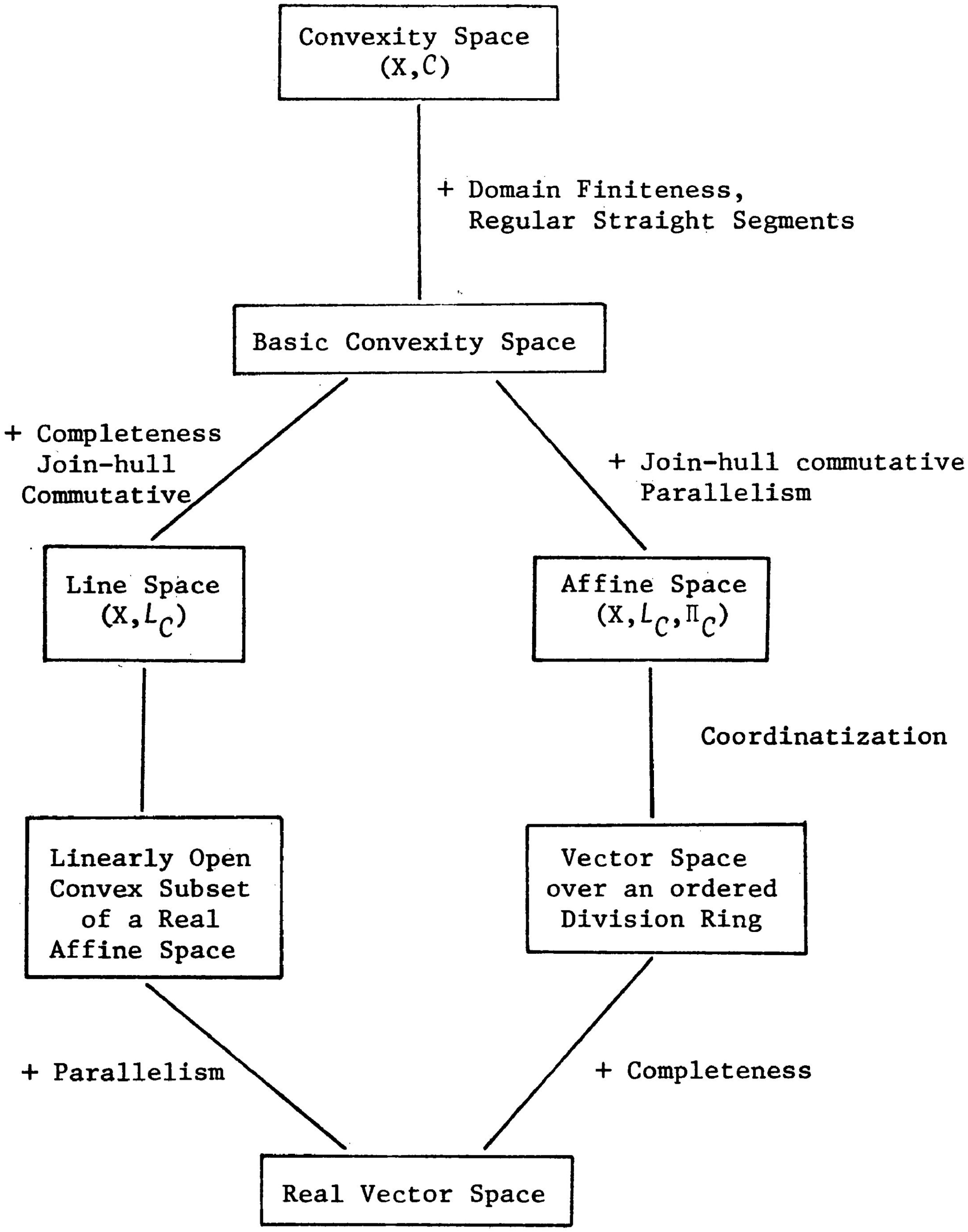
From the last three chapters, we have seen how an axiomatic convexity space plays an important role in modern geometry. It gives an easy way to establish an abstract geometry by imposing some simple axioms. Under the existence of different axioms will arise different type of geometries. By this point, we will be able to see clearly, how these geometries are related.

We summarize main results we have proved in this thesis into a diagram (see next page).

From this diagram, we see that if we combine the two approaches together, then a convexity space with all the properties stated in the diagram, can be made into a real vector space and the members of $\mathcal{C}$ are precisely the convex subsets of the real vector space.

For a generalized line space to be a line space, we need Pasch's axiom rather than join-hull commutativity which is a strong axiom (see 2.5.2). We also notice that our characterization theorem can be proved by replacing join-hull commutativity by Pasch's axiom. In view of this point, we have a question.

Given a convexity space $(X,\mathcal{C})$, what are the required properties, so that it is a generalized line space? In particular, does the converse of Theorem 3.4.12 hold? Now, let $\mathcal{C}_{L_{\mathcal{C}}}$ be the collection of all the convex subsets of $(X,L_{\mathcal{C}})$ [cf. 3.3.8]. Then $(X,\mathcal{C}_{L_{\mathcal{C}}})$ will satisfy all the properties


Convexity Space
(X,C)
+ Domain Finiteness,
Regular Straight Segments
Basic Convexity Space
+ Completeness
Join-hull
Commutative
+ Join-hull commutative
Parallelism
Line Space
(X, L_C)
Affine Space
(X, L_C, Π_C)
Coordinatization
Linearly Open
Convex Subset
of a Real
Affine Space
Vector Space
over an ordered
Division Ring
+ Parallelism
+ Completeness
Real Vector Space

of a convexity space. Are the members of C, precisely the members of C_{L_C}? The answer is no in general. We may ask when are they equal, or what are the porperties of C so that $C = C_{L_C}$?

As we know, the convexity structure in a vector space is very useful in optimization, convex functions are important in approximation theory, etc. In [17], Gudder introduces an abstract convexity structure and gives a lot of examples in how to apply it to the behavioral, social and physical science. However, it is still unknown whether the abstract convexity space that we have studied can be used in any significant applications.

BIBLIOGRAPHY

[1] Bennett, M.K., A LATTICE CHARACTERIZATION OF CONVEXITY, J. Combinatorial Theory, Vol. 5, 1968, pp. 198-202.

[2] Bennett, M.K., COORDINATIZATION OF AFFINE AND PROJECTIVE SPACE, Dis. Math., Vol. 4, 1973, pp. 219-231.

[3] Bennett, M.K., CONVEXITY CLOSURE OPERATORS, Dept. Math. & Stat., Univ. Masschusetts.

[4] Birkhoff, G., LATTICE THEORY, 3ed. Providence, Rhode Island, Amer. Math. Soc., Vol. 25, 1967.

[5] Bryant, V.W., INEPENDENT AXIOMS FOR CONVEXITY, J. Geometry, Vol. 5, 1974, pp. 95-99.

[6] Bryant, V.W. and Webster, R.J., CONVEXITY SPACE I - THE BASIC PROPERTIES, J. Math. Anal. Appl., Vol. 37, 1972, pp. 206-213.

[7] Bryant, V.W. and Webster, R.J., CONVEXITY SPACE II - SEPARATION, J. Math. Anal. Appl., Vol. 43, 1973, pp. 321-327.

[8] Bryant, V.W. and Webster, R.J., CONVEXITY SPACE III - DIMENSION, J. Math. Anal. Appl., Vol. 57, 1977, pp.382-392.

[9] Calder, J.R., SOME ELEMENTARY PROPERTIES OF INTERVAL CONVEXITY, J. London Math. Soc. (2), Vol. 3, 1971, pp. 422-428.

[10] Cantwell, J., CONVEXITY IN STRAIGHT LINE SPACE, Proc. Conf. Convexity Combination Geometry, Univ. Oklahoma, 1971, pp. 65-102.

[11] Cantwell, J., GEOMETRIC CONVEXITY, Bull. Inst. Acad. Sinica, Vol. 2, No. 2, 1974, pp.289-306.

[12] Coxeter, H.S.M., INTRODUCTION TO GEOMETRY, New York, John Wiley and Sons Inc., 1969.

[13] Dekker, D., CONVEX REGIONS IN PROJECTIVE SPACE, Amer. Math. Monthly, Vol. 62, No. 6, 1955, pp. 430-431.

[14] Doignon, J.P., CARACTERISATION D'ESPACE DE PASCH-PEANO, Bull. de l'Acad. Royale de Belqique, Ser. 5, Vol. 62, 1976, pp. 679-699.

[15] Franklin, S.P., SOME RESULTS ON ORDER-CONVEXITY, Amer. Math. Monthly, Vol. 69, 1962, pp. 357-359.

[16] de Groot, J. and de Vries, H., CONVEX SETS IN PROJECTIVE SPACE, Compositio Math., Vol. 13, 1958, pp. 113-118.

[17] Gudder, S.P., CONVEXITY AND MIXTURE, SIAM Review, Vol. 19, No. 2, 1977, pp. 221-239.

[18] Hilbert, D., FOUNDATION OF GEOMETRY, Open Court, 1971.

[19] Kay, D.C., AXIOMATIC CONVEXITY AND THE LINEARIZATION PROBLEM, Proc. Conf. Convexity Combination Geometry, Univ. Oklahoma, 1977, pp. 57-60.

[20] Kay, D.C., A NON-ALGEBRIC APPROACH TO THE THEORY OF TOPOLOGICAL LINEAR SPACE, Geometriae Dedicata, Vol. 6, 1977, pp. 419-433.

[21] Kay, D.C. and Sandstrom, R., PRODUCTS AND DUALS OF STRAIGHT LINE SPACE, Univ. Oklahoma and Illionis Wesleyan Univ.

[22] Kay, D.C. and Womble, E.W., AXIOMATIC CONVEXITY THEORY AND RELATIONSHIP BETWEEN THE CARATHEODORY HELLY AND RANDON NUMBERS, Pacific J. Math., Vol. 38, 1971, pp. 471-485.

[23] Levi, F.W., ON HELLY'S THEOREM AND THE AXIOMS OF CONVEXITY, J. India Math. Soc., Vol. 15, 1951, pp. 65-76.

[24] Mah, P., Naimpally, S.A. and Whitfield, J.H.M., LINEARIZATION OF A CONVEXITY SPACE, J. London Math. (2), Vol. 13, 1976, pp. 209-214.

[25] Prenowitz, W., A CONTEMPORARY APPROACH TO CLASSICAL GEOMETRY, Amer. Math. Monthly, Vol. 68, No. 1, 1961, pp. 1-67.

[26] Sasaki, U., LATTICE THEORETIC CHARACTERIZATION OF AN AFFINE GEOMETRY OF ARBITRARY DIMENSIONS, Fac. Sci., Hiroshima Univ., Vol. 16, 1952, pp. 223-238.

[27] Sierksma, G., CARATHEODORY AND HELLY-NUMBER OF CONVEX-PRODUCT STRUCTURES, Pacific J. Math., Vol. 61, No. 1, 1975, pp. 275-282.

[28] Soltan, P.S., HELLY'S THEOREM FOR d-CONVEX SETS, Soviet Math. Dokl., Vol. 13, 1972, pp. 975-978.

[29] Soltan, P.S., CERTAIN QUESTIONS OF THE ABSTRACT THEORY OF CONVEXITY, Soviet Math. Dokl., Vol. 17, No. 3, 1976, pp. 730-733.

[30] Szafron, D.A. and Weston, J.H., AN INTERNAL SOLUTION TO THE LINEARIZATION OF A CONVEXITY SPACE, Canad. Math. Bull., Vol. 19(4), 1976, pp. 487-494.

[31] Tuz, V.V., AXIOMATIC THEORY OF CONVEXITY, Mat. Z., Vol. 20, No. 5, 1976, pp. 761-770.

[32] Valentine, F., CONVEX SETS, McGraw Hill.