TRANSFORMING ORDINARY LINEAR DIFFERENTIAL EQUATIONS TO CONSTANT COEFFICIENT DIFFERENTIAL EQUATIONS

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Contents

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Abstract

The main purpose of this thesis is to give a characterization of nth order linear homogeneous differential equations that can be transformed into constant coefficient differential equations. The characterization given makes use of invariance theory originated by G. H. Halphen around 1880.

Chapter 1 gives an introduction to the problem of transforming ordinary linear homogeneous differential equations into constant coefficient differential equations. Chapter 1 includes an example as well as enough of the theory of invariants to give a proof of the main theorem of this thesis.

Chapter 2 gives a generalization of the chain rule of elementary calculus. We make extensive use of this generalization throughout the thesis.

Chapter 3 is a development of the transform equations that we make use of throughout this thesis. That is, we make changes of the dependent variable and/or independent variable of a given differential equation. The transformed equation is expressed in terms of the coefficients of the original differential equation and the functions used to define the changes of variables that we have made.

In the last section of Chapter 4 we prové an important invariance relation that is used in Chapter 5. The rest of Chapter 4 contains invariance results that are of historical interest as well as being a prelude to the results of Chapter 5.

Chapter 5, the most important chapter of this thesis, concerns the invariance theory of Halphen. This invariance theory is needed to give the characterization of nth order differential equations that can be transformed into constant coefficient differential equations.

Chapter 6 is devoted to applying the preceeding theory to the solution of differential equations that can be transformed into constant coefficient differential equations.

The appendix contains some interchange of summation formulas that we use throughout the thesis.

Notations

Throughout this thesis we will use the reference system (a.b.c), where a gives the chapter number, b gives the section number and c gives the number of a particular formula, equation, theorem or lemma. For example equation (1.1.1) refers to the 1st equation of section 1 of Chapter 1 while Theorem (1.3.1) refers to the 1st theorem of the 3rd section of Chapter 1. When results from the appendix are referenced, this is denoted by putting $a \equiv A$. For example Lemma (A.1.1) refers to the first lemma of the appendix.

When the dependent variable of a given function is obviously x, we shall not always explicitly specify it. For example $u(x)$ is often referred to by just u.

Phrases like "nth order homogeneous differential equation" are often replaced by "differential equation" or just "equation" when no loss of clarity will result.

Throughout this thesis we will denote the kth derivative of $f(x)$ by

$$\frac{d^k}{dx^k} f(x) = (f(x))^{(k)} = f^{(k)} ,$$

and similarily

$$\frac{d^k}{dt^k} f(x) = (f(x))_t^{(k)} .$$

That is, subscripting by a variable will denote the variable that differentiation is with respect to if it is other than x.

$C^n[a, b]$ shall denote the set of all functions that are n times continuously differentiable with respect to x on $[a, b]$.

The coefficient of the highest order derivative of the dependent variable in a differential equation shall be called the <u>leading coefficient of the differential equation</u>.

$\binom{n}{k}$ shall denote the usual binomial coefficients $\frac{n!}{(n-k)!k!}$.

The characteristic equation (in λ) of the constant coefficient differential equation

$$\sum_{k=0}^{n} \binom{n}{k} c_{n-k} \frac{d^k}{dx^k} y(x) = 0 ,$$

is

$$\sum_{k=0}^{n} \binom{n}{k} c_{n-k} \lambda^k = 0 .$$

The left side of this equation is referred to as the characteristic polynomial.

When referring to a differential equation of order n, we denote this by saying it is of nth order.

Chapter 1

Fundamental Concepts

(1.1) Introduction. Let

$$\sum_{k=0}^{n} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \tag{1.1.1}$$

be an nth order linear homogeneous differential equation with coefficients $a_i(x)$, where $a_i(x)$ is a complex valued function. We shall always assume that equation (1.1.1) is valid on a real interval $[a, b]$ and that $a_0(x) \neq 0$ on $[a, b]$. We will also assume that each $a_i(x)$ is $2n - i$ times continuously differentiable on $[a, b]$. That is, $a_i(x) \in C^{2n-i}[a, b]$. Without loss of generality we can assume that $a_0(x) \equiv 1$, since if this were not the case we could divide equation (1.1.1) through by $a_0(x)$ getting a differential equation having one as its leading coefficient.

In this thesis we are interested in solving equation (1.1.1) by transforming its dependent and independent variables. The transform of the independent variable that we will use has the form

$$\frac{dt}{dx} = u(x) \tag{1.1.2}$$

and the transform of the dependent variable has the form

$$y(x) = v(x) z(t) , \tag{1.1.3}$$

where $u(x)$ and $v(x)$ are non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^{n}[a, b]$. The $z(t)$ in equation (1.1.3) is the new dependent variable of the new independent variable t. In particular we are interested in developing necessary and sufficient conditions that will characterize those nth order differential equations (1.1.1) that can be transformed, by means of equations (1.1.2) and (1.1.3), into constant coefficient differential equations. These necessary and sufficient conditions will explicitly give $u(x)$ and $v(x)$ in terms of the coefficients $a_i(x)$ of the original differential equation.

(1.2) An Example. By adjusting the coefficients of equation (1.1.1) by numeric factors it is easy to see that equation (1.1.1) can be cast in the form

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 \;, \qquad (a_0(x) \equiv 1) \;, \tag{1.2.1}$$

where the $a_i(x)$'s of equation (1.2.1) differ from those of equation (1.1.1) by the constant factors $\binom{n}{k}$.

Definition (1.2.1). Any differential equation of the form of equation (1.2.1) will called a differential equation in its normal form.

To facilitate comparison with work done by other authors, we will usually work with the normal form of equation (1.1.1).

In order to illustrate the transformations we are interested in we consider the case $n = 3$. We transform the third order differential equation

$$\sum_{k=0}^{3} \binom{3}{k} a_{3-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \quad (x \in [a, b], a_0(x) \equiv 1) , \tag{1.2.2}$$

by means of the transformations

$$\frac{dt}{dx} = u(x) \tag{1.2.3}$$

and

$$y(x) = v(x)z(t) . \tag{1.2.4}$$

By using the chain rule of elementary calculus, equation (1.2.3) and equation (1.2.4), we have that

$$\begin{aligned} \frac{dy(x)}{dx} &= u(x) \frac{dy(x)}{dt} \\ &= u(x) \frac{d}{dt} (v(x)z(t)) \\ &= u(x)\left[v(x) \frac{dz(t)}{dt} + \frac{dv(x)}{dx} \frac{z(t)}{u(x)}\right] . \end{aligned}$$

That is,

$$\frac{dy(x)}{dx} = (uv)(z(t))_t^{(1)} + v^{(1)} z(t) , \tag{1.2.5}$$

where we have used the notation $(z(t))_t^{(k)} = \frac{d^k}{dt^k} z(t)$. In a similar manner we obtain

$$(1.2.6) \qquad \frac{d^2y(x)}{dx^2} = (u^2v)(z(t))_t^{(2)} + (2uv^{(1)} + u^{(1)}v)(z(t))_t^{(1)} + v^{(2)}z(t) ,$$

and

$$(1.2.7) \qquad \frac{d^3y(x)}{dx^3} = (u^3v)(z(t))_t^{(3)} + (3u^2v^{(1)} + 3uu^{(1)}v)(z(t))_t^{(2)}$$
$$+ (3uv^{(2)} + 3u^{(1)}v^{(1)} + u^{(2)}v)(z(t))_t^{(1)} + v^{(3)}z(t) .$$

When equations (1.2.4), (1.2.5), (1.2.6) and (1.2.7) are substituted into equation (1.2.2) we obtain

$$(1.2.8) \qquad (u^3v)(z(t))_t^{(3)} + (3a_1u^2v + 3u^2v^{(1)} + 3uu^{(1)}v)(z(t))_t^{(2)}$$
$$+ (3uv^{(2)} + 3u^{(1)}v^{(1)} + u^{(2)}v + 6a_1uv^{(1)} + 3a_1u^{(1)}v + 3a_2uv)(z(t))_t^{(1)}$$
$$+ (v^{(3)} + 3a_1v^{(2)} + 3a_2v^{(1)} + a_3v)z(t) = 0 .$$

From equation (1.2.8) we see that if

$$3a_1u^2v + 3u^2v^{(1)} + 3uu^{(1)}v = k_1u^3v ,$$

$$3uv^{(2)} + 3u^{(1)}v^{(1)} + u^{(2)}v + 6a_1uv^{(1)} + 3a_1u^{(1)}v + 3a_2uv = k_2u^3v$$

and

$$v^{(3)} + 3a_1v^{(2)} + 3a_2v^{(1)} + a_3v = k_3u^3v ,$$

where k_1, k_2 and k_3 are constants, then the solutions of the differential equation (1.2.8) are the same as the solutions of the constant coefficient differential equation

$$(z(t))_t^{(3)} + k_1(z(t))_t^{(2)} \; k_2(z(t))_t^{(1)} + k_3 z(t) = 0 \; ,$$

where t and x are related by $\dfrac{dt}{dx} = u(x)$.

As a specific example of equation (1.2.2) let $a_1(x) = \dfrac{3x}{(1-x)^2}$, $a_2(x) = \dfrac{6x^2}{(1-x)^4}$, $a_3(x) = \dfrac{6x^3}{(1-x)^6}$ and $[a, b] = [2, 3]$. That is, let us consider the differential equation.

$$\text{(1.2.9)} \qquad \sum_{k=0}^{3} \binom{3}{k} \frac{3!x^{3-k}}{k!(1-x)^{6-2k}} \frac{d^k}{dx^k} y(x) = 0 \; , \qquad (x \in [2, 3]) \; .$$

If we use the transformations

$$\text{(1.2.10)} \qquad \frac{dt}{dx} = (1-x)^{-2}$$

and

$$\text{(1.2.11)} \qquad y(x) = \exp\left(\frac{-3}{1-x}\right) \frac{z(t)}{1-x} \; ,$$

the differential equation (1.2.9) transforms into

$$\text{(1.2.12)} \qquad \exp\left(\frac{-3}{1-x}\right)(1-x)^{-7}[(z(t))_t^{(3)} - 9(z(t))_t^{(1)} + 6z(t)] = 0 \; .$$

Equation (1.2.12) has the linearly independent solutions

$$z_i(t) = \exp(\lambda_i t) \; , \qquad i = 1, 2, 3,$$

where λ_1 , λ_2 and λ_3 are the distinct real roots of

(1.2.13) $$\lambda^3 - 9\lambda + 6 = 0 .$$

The solutions $y_i(x)$ of equation (1.2.9) are related to the solutions $z_i(t)$ of equation (1.2.12) by equation (1.2.11), where t is related to x by equation (1.2.10). Since the $z_i(t)$'s are given by

$$z_i(t) = \exp(\lambda_i t) \ , \quad i = 1, 2, 3,$$

we have that

$$\begin{aligned} y_i(x) &= (1 - x)^{-1}\exp\left(\frac{-3}{1 - x}\right)z_i(t) \\ &= (1 - x)^{-1}\exp\left(\frac{-3}{1 - x}\right)\exp(\lambda_i t) \\ &= (1 - x)^{-1}\exp\left(\frac{-3}{1 - x}\right)\exp(\lambda_i\int(1 - x)^{-2}dx) \\ &= C(1 - x)^{-1}\exp\left(\frac{\lambda_i - 3}{1 - x}\right) , \end{aligned}$$

where C is a constant of integration. Without loss of generality C can be taken to be one, hence the solutions of equation (1.2.9) are

(1.2.14) $$y_i(x) = (1 - x)^{-1}\exp\left(\frac{\lambda_i - 3}{1 - x}\right) , \quad i = 1, 2, 3,$$

where the λ_i's are the roots of equation (1.2.13).

(1.3) The Nth Order Case. We wish to generalize the procedure used to solve the differential equation (1.2.9), to solve nth order linear differential equations. In order to do this we must transform the differential equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \quad (x \in [a, b], a_0(x) \equiv 1) , \tag{1.3.1}$$

by means of

$$\frac{dt}{dx} = u(x) \tag{1.3.2}$$

and

$$y(x) = v(x) z(t) , \tag{1.3.3}$$

to obtain

$$\sum_{\ell=0}^{n} b_{n-\ell}(t) (z(t))_t^{(\ell)} = 0 . \tag{1.3.4}$$

As usual $u(x)$ does not vanish on $[a, b]$, hence $t(x) = \int u(x)dx$ is a monotone increasing or decreasing function on $[a, b]$. Letting $h(x) = \int u(x)dx$ we see that the inverse of t exists and it can be written as

$$x = h^{-1}(t) . \tag{1.3.5}$$

In order to find out how the $b_i(t)$'s are related to the $a_i(x)$'s, $u(x)$ and $v(x)$, we require a result known as Faà de Bruno's Formula.

<u>Theorem (1.3.1)</u>. (Faà de Bruno's Formula). Let n be a positive integer and let k be such that $0 \le k \le n$. If $\frac{dt}{dx} = u(x)$ where $u(x) \in C^{n-1}[a, b]$ and $u(x)$ is non-vanishing on $[a, b]$ then for all $x \in [a, b]$

$$\frac{d^k}{dx^k} = \sum_{m=0}^{k} \phi(k, m; u(x)) \frac{d^m}{dt^m}, \tag{1.3.6}$$

where

$$\phi(k, m; u(x)) = \begin{cases} 1 & m = k = 0 \\ 0 & m = 0,\ k > 0 \\ \sum \dfrac{k!}{\prod_{i=1}^{k} (m_i!)} \prod_{i=1}^{k} \left(\dfrac{u^{(i-1)}}{i!}\right)^{m_i} & \text{otherwise.} \end{cases} \tag{1.3.7}$$

The sum in equation (1.3.7) is over all partitions of m such that

$$\sum_{i=1}^{k} m_i = m$$

and

$$\sum_{i=1}^{k} im_i = k,$$

where the m_i's are all integers greater than or equal to zero.

A proof of Theorem (1.3.1) is given in Chapter 2.

Using Theorem (1.3.1) and equations (1.3.2) and (1.3.3), we simultaneously transform the independent and dependent variables of the differential equation (1.3.1), to obtain

$$\text{(1.3.8)} \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \sum_{j=0}^{k} \phi(k,\ j;\ u(x)) \frac{d^j}{dt^j} (v(x)z(t)) = 0 \ ,$$

where $\phi(k,\ j;\ u(x))$ is given by equation (1.3.7).

We wish to write the differential equation (1.3.8) in the form of equation (1.3.4). By using formula (A.1.4) to rearrange the sums in equation (1.3.8) we find that

$$\sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} a_{n-k}(x) \ \phi(k,\ j;\ u(x)) \frac{d^j}{dt^j} (v(x)z(t)) = 0 \ .$$

By using Leibnitz's rule for the differentiation of the product of two functions this equation becomes

$$\sum_{j=0}^{n} \sum_{\ell=0}^{j} \sum_{k=j}^{n} \binom{n}{k} a_{n-k}(x) \ \phi(k,\ j;\ u(x)) \binom{j}{\ell} (v(x))_t^{(j-\ell)} (z(t))_t^{(\ell)} = 0 \ .$$

By rearranging the sums of this equation using formula (A.1.2) we obtain

$$\text{(1.3.9)} \qquad \sum_{\ell=0}^{n} \sum_{j=0}^{n-\ell} \sum_{k=j+\ell}^{n} \binom{n}{k} a_{n-k}(x) \phi(k,\ j+\ell;\ u(x)) \binom{j+\ell}{\ell} (v(x))_t^{(j)} (z(t))_t^{(\ell)} = 0,$$

which is of the same form as equation (1.3.4).

In order that equations (1.3.4) and (1.3.9) are the same differential equations, their coefficients of $(z(t))_t^{(\ell)}$, $\ell = 0, 1, \ldots, n$, must be equal. For $\ell = 0, 1, \ldots, n$ we have that

$$(1.3.10) \qquad b_{n-\ell}(t) = \sum_{j=0}^{n-\ell} \sum_{k=j+\ell}^{n} \binom{n}{k} a_{n-k}(x)\phi(k, j+\ell; u(x)) \binom{j+\ell}{\ell} (v(x))_t^{(j)} ,$$

where t and x are related by $\frac{dt}{dx} = u(x)$.

As will be shown in Chapter 2 $\phi(n, n; u(x)) = (u(x))^n$, hence recalling that $a_0(x) \equiv 1$ we see that the coefficient of $(z(t))_t^{(n)}$ in equation (1.3.9) is $(u(x))^n v(x)$. We have the following theorem.

<u>Theorem (1.3.2)</u>. Let $x \in [a, b]$, then the differential equation (1.3.1) can be transformed via equations (1.3.2) and (1.3.3) into a differential equation of the form

$$A(x) \sum_{\ell=0}^{n} c_{n-\ell} \frac{d^{\ell}}{dt^{\ell}} z(t) = 0 , \qquad (c_0 = 1) ,$$

where the c_k's are constants, if and only if there exist $u(x)$ and $v(x)$ such that

$$\sum_{j=0}^{n-\ell} \sum_{k=j+\ell}^{n} \binom{n}{k} a_{n-k}(x)\phi(k, j+\ell; u(x)) \binom{j+\ell}{\ell} (v(x))_t^{(j)} = c_{n-\ell}(u(x))^n v(x) ,$$

for $\ell = 0, 1, \ldots, n-1$.

Even though Theorem (1.3.2) characterizes those differential equations which can be transformed into essentially constant coefficient differential equations, it does not tell us what $u(x)$ and $v(x)$ must be to accomplish this. By using the theory of invariants we will not only be able to tell whether a given differential equation can be transformed into a constant coefficient differential equation, but we will also be able to explicitly specify the required transformation functions $u(x)$ and $v(x)$ in terms of the coefficients of the original differential equation.

(1.4) Theory of Invariants. Theorem (1.3.2) gives necessary and sufficient conditions that the differential equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (a_0(x) \equiv 1) , \tag{1.4.1}$$

transforms into essentially a constant coefficient differential equation by means of the transform equations

$$\frac{dt}{dx} = u(x) \tag{1.4.2}$$

and

$$y(x) = v(x)z(t) . \tag{1.4.3}$$

Fayet [16] and Berkovic [3] have also found necessary and sufficient conditions for transforming ordinary linear homogeneous differential equations into constant coefficient differential equations via equations (1.4.2) and (1.4.3). In all these cases the required transform functions $u(x)$ and $v(x)$ are not explicitly given in terms of the coefficients of the original equation.

Breuer and Gottlieb [5] have considered the special case when the transform equations (1.4.2) and (1.4.3) are of the form $\frac{dt}{dx} = u(x)$ and $y(x) = z(t)$. That is, only the independent variable is transformed. They obtain essentially the following theorem.

Theorem (1.4.1). The differential equation

$$\sum_{k=0}^{n} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (x \in [a, b], a_0(x) \equiv 1) , \tag{1.4.4}$$

where $a_n(x)$ is non-vanishing on $[a, b]$, can be transformed via $\frac{dt}{dx} = u(x)$ and $y(x) = z(t)$ into a differential equation of the form

$$A(x) \sum_{\ell=0}^{n} c_{n-\ell} \frac{d^\ell}{dt^\ell} z(t) = 0 , \qquad (c_0 \equiv 1) , \tag{1.4.5}$$

where the c_k's are constants, if and only if

$$\sum_{k=\ell}^{n} a_{n-k}(x) \phi(k, \ell; u(x)) = c_{n-\ell} a_n(x) , \qquad (\ell = 1, \ldots, n) , \tag{1.4.6}$$

where the c_ℓ's are constants and $\phi(k, \ell; u(x))$ is given by equation (1.3.7). Moreover if a $u(x)$ exists such that these conditions hold then $u(x)$ can be taken to be $u(x) = (a_n(x))^{1/n}$.

Proof: Using equation (1.3.6) of Theorem (1.3.1) we transform equation (1.4.4) by means of the equations $\frac{dt}{dx} = u(x)$ and $y(x) = z(t)$, to obtain

$$\sum_{k=0}^{n} a_{n-k}(x) \sum_{\ell=0}^{k} \phi(k, \ell; u(x)) (z(t))_t^{(\ell)} = 0 .$$

By rearranging the sums of this equation using formula (A.1.4) we obtain

$$(1.4.7) \qquad \sum_{\ell=0}^{n} \sum_{k=\ell}^{n} a_{n-k}(x)\phi(k,\ \ell;\ u(x))(z(t))_t^{(\ell)} = 0 \ .$$

As we will see in Chapter 2,

$$\phi(k,\ 0;\ u(x)) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

and

$$\phi(n,\ n;\ u(x)) = (u(x))^n \ ,$$

hence the coefficient of $z(t)$ in equation (1.4.7) is $a_n(x)$ and the coefficient of $(z(t))_t^{(n)}$ is $(u(x))^n$. The theorem now follows immediately since if equation (1.4.7) is to be an equation of the form of equation (1.4.5), then each of its coefficients must be a constant times $a_n(x)$. It is obvious that c_0 can be taken to be one without loss of generality.

Q.E.D.

Breuer and Gottlieb's conditions given by equations (1.4.6) have been obtained previously, for some of the lower order cases, by Peyovitch [34], Fayet [17] and Mangeron [29] (see also [32], [33]). In Chapter 4 we will use invariance considerations to prove Theorem (1.4.1), for the case $n = 2$, as Peyovitch [34] did.

The other special case of the transform equations (1.4.2) and (1.4.3), where only the dependent variable is transformed by means of the equation $y(x) = v(x)z(x)$, will also be considered in Chapter 4. Using invariance considerations we will show that there exists a $v(x)$ such that the transformation $y(x) = v(x)z(x)$ takes equation (1.4.1) to a constant coefficient differential equation if and only if the transformation $y(x) = \exp(-\int a_1(x)dx)z(x)$ takes equation (1.4.1) to a constant coefficient differential equation.

Considering the general case where $n \geq 3$ we shall use invariance considerations, originally studied by Halphen [19], to find necessary and sufficient conditions that characterize those nth order differential equations (1.4.1) that can be transformed, by means of equations (1.4.2) and (1.4.3), into constant coefficient differential equations. These necessary and sufficient conditions will explicitly give $u(x)$ and $v(x)$ in terms of the coefficients $a_i(x)$ of the original differential equation (1.4.1).

In section (1.3) we transformed the differential equation (1.4.1), using equations (1.4.2) and (1.4.3), into the differential equation

$$\sum_{\ell=0}^{n} b_{n-\ell}(t)(z(t))_t^{(\ell)} = 0 , \tag{1.4.8}$$

where the $b_\ell(t)$'s are given by equation (1.3.10). We also saw that $b_0(t) = (u(x))^n v(x)$, which is non-vanishing on $[a, b]$ since $u(x)$ and $v(x)$ are non-vanishing on $[a, b]$. Since $b_0(t)$ is non-vanishing on $[a, b]$ we can divide equation (1.4.8) through by $b_0(t)$ to obtain, after adjusting the coefficients by appropriate numeric factors, an equation of the form

$$(1.4.9) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)\,(z(t))_t^{(k)} = 0 \;, \qquad (b_0(t) \equiv 1) \;.$$

That is, each $b_{n-k}(t)$ of equation (1.4.9) is obtained from the corresponding $b_{n-k}(t)$ of equation (1.4.8) by dividing by $(u(x))^n v(x)\binom{n}{k}$. In Chapter 3 we will show that the $b_{n-k}(t)$'s of equation (1.4.9) are given by

$$(1.4.10) \quad b_{n-s}(t) = (u^n v)^{-1}(n-s)!\sum_{k=0}^{n-s}\sum_{j=0}^{n-s-k} \frac{a_k(x)\,s!}{k!\,j!\,(n-k-j)!}\,\phi(n-k-j,s;u)v^{(j)} \;,$$

$$(s = 0, 1, \ldots, n) \;,$$

where $\phi(k, j; u(x))$ is given by equation (1.3.7).

<u>Definition (1.4.1)</u>. The differential equation (1.4.9), with its $b_k(t)$'s given by equation (1.4.10), is called the <u>$P(u(x), v(x))$ transform</u> of equation (1.4.1).

<u>Definition (1.4.2)</u>. The equations $\frac{dt}{dx} = u(x)$ and $y(x) = v(x)z(t)$ are called the <u>defining equations</u> of the $P(u(x), v(x))$ transform of equation (1.4.1).

Note that the $P(u(x), v(x))$ transform of equation (1.4.1) was obtained from equation (1.4.1) by simultaneously transforming its independent and dependent variables by letting $\frac{dt}{dx} = u(x)$ and $y(x) = v(x)z(t)$.

When it is obvious what the defining equations of a $P(u(x), v(x))$ transform are, we may not always specify them.

Note that the original differential equation (1.4.1) and its $P(u(x), v(x))$ transform, equation (1.4.9), are both in normal forms. This is important for invariance theory considerations. In particular notice that the equations (1.4.1) and (1.4.9) both have leading coefficient one.

Let n be some fixed positive integer and let $[a, b]$ be a closed interval of the real line. We will denote by $\mathcal{D}$ the set of all nth order ordinary linear homogeneous differential equations of the normal form

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (a_0(x) \equiv 1) .$$

<u>Definition (1.4.3)</u>. The differential equations

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0$$

and

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t) \frac{d^k}{dt^k} z(t) = 0 ,$$

belonging to $\mathcal{D}$, are said to be P <u>equivalent</u>, if there exist $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$, both non-vanishing on $[a, b]$, such that the $P(u(x), v(x))$ transform of the first differential equation is the second differential equation.

We may sometimes only say equivalent when we mean P equivalent.

From the example we did in section (1.2) it is obvious that on the interval $[2, 3]$,

$$\sum_{k=0}^{3} \binom{3}{k} \frac{3!x^{3-k}}{k!(1-x)^{6-2k}} \frac{d^k}{dx^k} y(x) = 0$$

is P equivalent to

$$(z(t))_t^{(3)} - 9(z(t))_t^{(1)} + 6z(t) = 0 \ .$$

In this case we know $u(x) = (1 - x)^{-2}$ and $v(x) = \exp\left(\frac{-3}{1 - x}\right)(1 - x)^{-1}$.

Let G denote the set of all maps from $\mathcal{D}$ to $\mathcal{D}$ given by $P(u, v)$ transforms. Let us consider an arbitrary $P(u(x), v(x))$ transform of an arbitrary equation (say equation (1.4.1)) in G . This $P(u(x), v(x))$ transform of equation (1.4.1), defined by $\frac{dt}{dx} = u(x)$ and $y(x) = v(x)z(t)$, is given by equation (1.4.9). As in section (1.3) $t(x) = \int u(x)dx$ is a monotone increasing or decreasing function on $[a, b]$. Letting $h(x) = \int u(x)dx$ we see that the inverse of t exists, thus $x = h^{-1}(t)$. We now consider the $P\left(\frac{1}{u(x)}, \frac{1}{v(x)}\right) = P\left(\frac{1}{u(h^{-1}(t))}, \frac{1}{v(h^{-1}(t))}\right)$ transform of equation (1.4.9), defined by

$$\frac{dx}{dt} = \frac{1}{u(x)} = \frac{1}{u(h^{-1}(t))}$$

and

$$z(t) = \frac{1}{v(x)}\, y(x) = \frac{1}{v(h^{-1}(t))}\, y(x) \ .$$

Since $y(x) = v(x)z(t)$, it is obvious that the $P\left(\frac{1}{u(x)}, \frac{1}{v(x)}\right)$ transform of equation (1.4.9) has the solutions $y_i(x)$, $i=1,\ldots,n$, where each $y_i(x)$ is a solution of equation (1.4.1). All P transforms have leading coefficient one, hence we must have that the $P\left(\frac{1}{u(x)}, \frac{1}{v(x)}\right)$ transform of equation (1.4.9) is equation (1.4.1) (see Ross [37], p. 385). We have in effect shown that the inverse of every mapping in G is in G. This also shows that the relation defined by definition (1.4.3) is symmetric. This relation can also be shown to be reflexive and transitive, hence the relation is in fact a true equivalence relation.

Clearly the identity mapping is in G, it has defining equations of the form $t = x$ and $y(x) = z(t)$.

It can be shown that the composition of two maps in G is in G and that composition of mappings in G is associative, hence we have that G forms a group. The group elements are the mappings given by $P(u, v)$ transforms and the group multiplication is the composition of these mappings. We now see that the problem of solving linear differential equations can be approached by the invariant theory of groups. Lie [27] was the first author to consider differential equations from this point of view.

We now make some definitions and introduce the idea of an invariant of a differential equation.

Definition (1.4.4). Let $\underline{M}$ be the set of all matrices of the form

$$\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)} = \begin{pmatrix} a_0(x) & a_1(x) & \cdots & a_n(x) \\ \frac{da_0(x)}{dx} & \frac{da_1(x)}{dx} & \cdots & \frac{da_n(x)}{dx} \\ \vdots & & & \vdots \\ \frac{d^n a_0(x)}{dx^n} & & & \frac{d^n a_n(x)}{dx^n} \end{pmatrix},$$

where the $a_j(x)$'s are functions of x that are sufficiently differentiable.

Definition (1.4.5). Let $u(x)$ and $v(x)$ be arbitrary non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$. Let I be a map from M to the set of all complex valued functions with domain $[a, b]$. Let $a_j(x)$, $j = 0, 1, \ldots, n$, be the coefficients of equation (1.4.1) and let $b_j(t)$, $j = 0, 1, \ldots, n$, be the coefficients of the $P(u(x), v(x))$ transform of equation (1.4.1). If for all $x \in [a, b]$ and for all $u(x)$ and $v(x)$ as defined above we have the identity

$$(1.4.13) \qquad I\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv I\left(\left(\frac{d^i}{dt^i} b_j(t)\right)_{(n+1)\times(n+1)}\right),$$

where $\frac{dt}{dx} = u(x)$, then the function I is called an absolute invariant of equation (1.4.1).

From this definition we see that an absolute invariant of equation (1.4.1) is a function of the coefficients $a_i(x)$ of equation (1.4.1) and their derivatives with respect to x . This function has the property that for all $x \in [a, b]$ it has the same value as the same function formed from the coefficients of any arbitrary equation which is P equivalent to equation (1.4.1).

Note that in the left hand side of the identity (1.4.13) the derivatives are with respect to x while in the right hand side they are with respect to t .

Let us again refer back to section (1.2). An absolute invariant, call it I , of the differential equation

$$\sum_{k=0}^{3} \binom{3}{k} a_{3-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (a_0(x) \equiv 1) ,$$

is

$$I = [V_3^2(a_2 - a_1^2 - a_1^{(1)}) + \frac{7}{27}(V_3^{(1)})^2 - \frac{6}{27} V_3 V_3^{(2)}] V_3^{-8/3} ,$$

where

(1.4.14) $$V_3 = V_3(a_i(x)) = -a_1^{(2)} + 3(a_2^{(1)} - 2a_1 a_1^{(1)}) - 2(a_3 - 3a_1 a_2 + 2a_1^3)$$

As a specific example consider the differential equation (1.2.9), that is

$$\sum_{k=0}^{3} \binom{3}{k} \frac{3!x^{3-k}}{k!(1-x)^{6-2k}} \frac{d^k}{dx^k} y(x) = 0 . \tag{1.4.15}$$

For this differential equation $V_3(a_i(x)) = -12(1-x)^{-6}$ and $I = -3(-12)^{-2/3}$. That is, in this case the absolute invariant I is a constant.

We now come to one of the most important ideas of this thesis, that is, the idea of a canonical transform.

<u>Definition (1.4.6)</u>. A differential equation

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t) \frac{d^k}{dt^k} z(t) = 0 , \qquad (b_0(t) \equiv 1) , \tag{1.4.16}$$

that is P equivalent to equation (1.4.1), is called a <u>canonical transform</u> of equation (1.4.1) if each $b_i(t)$ of equation (1.4.16) is an absolute invariant of equation (1.4.1).

We now assume that the order of equation (1.4.1) is three or greater. We also assume that $V_3(a_i(x))$, given by equation (1.4.14), is non-vanishing on $[a, b]$ and that C is an arbitrary non-zero constant. Under these assumptions we will show in Chapter 5 that the $P\left(\left[C^{-1}V_3(a_i(x))\right]^{1/3}, \exp(-\int a_1(x)dx)\left[C^{-1}V_3(a_i(x))\right]^{\frac{1-n}{6}}\right)$ transform of equation (1.4.1) is a canonical transform of equation (1.4.1). That is, a canonical transform of equation (1.4.1) is

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t) \frac{d^k}{dt^k} z(t) = 0 , \qquad (b_0(t) \equiv 1) ,$$

where

$$(1.4.17)\qquad b_{n-s}(t) = (n-s)!\left(C^{-1}V_3(a_i(x))\right)^{\frac{-n-1}{6}} \sum_{k=0}^{n-s}\sum_{j=0}^{n-s-k} \frac{a_k(x)s!}{k!j!(n-k-j)!}$$

$$\cdot\ \phi\left(n-k-j,\ s;\left(C^{-1}V_3(z_i(x))\right)^{1/3}\right)\exp\left(\int a_1(x)dx\right)$$

$$\cdot\ \left(\exp\left(-\int a_1(x)dx\right)\left(C^{-1}V_3(a_i(x))\right)^{\frac{1-n}{6}}\right)^{(j)}$$

for $s = 0, 1, \ldots, n$.

Note that the coefficients $b_k(t)$, given by equation (1.4.17), are functions of the $a_i(x)$'s of equation (1.4.1) and their derivatives with respect to x , that contain no integrations. That is, for each j the term

$$\exp\left(\int a_1(x)dx\right)\left(\exp\left(-\int a_1(x)\right)\left(C^{-1}V_3(a_i(x))\right)^{\frac{1-n}{6}}\right)^{(j)}$$

of equation (1.4.17), results in an expression that contains no integrations. For example if $j = 1$ the expression is

$$\frac{1-n}{6C}\left(C^{-1}V_3(a_i(x))\right)^{\frac{-5-n}{6}}\left(V_3(a_i(x))\right)^{(1)} - a_1(x)\left(C^{-1}V_3(a_i(x))\right)^{\frac{1-n}{6}} .$$

Moreover the ϕ function in equation (1.4.17) contributes no integrations (see equation (1.3.7)).

Laguerre [25] gave the above canonical transform for the case $n = 3$ and $C = 1$. At the time Laguerre published the paper [25] he was not looking for a canonical transform and in fact he was not aware that the transform he gave was a canonical transform. Halphen ([19], p. 219) gave the above canonical transform assuming that $C = 1$, however he omitted the proof saying only that it was obvious.

For the present, we shall assume that the $P\left(\left(C^{-1}V_3(a_i)\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(C^{-1}V_3(a_i)\right)^{\frac{1-n}{6}}\right)$ transform of equation (1.4.1) is in fact a canonical transform of equation (1.4.1). Under this assumption we prove the following important theorem which was originally done by Halphen ([19], p. 142-143). Halphen merely stated the theorem (with $C = 1$) saying that it was obvious. In the proof we will use the usual notation that subscripting by a variable indicates the variable that differentiation is with respect to if it is other than x. For example ϕ_t indicates that derivatives in Faà de Bruno's Formula (1.3.7) are to be taken with respect to t rather than x.

<u>Theorem (1.4.2)</u>. Let C be an arbitrary non-zero constant and assume that the order of equation (1.4.1) is three or greater. Moreover, assume that the function

$$V_3(a_i(x)) = -a_1^{(2)} + 3(a_2^{(1)} - 2a_1a_1^{(1)}) - 2(a_3 - 3a_1a_2 + 2a_1^3)$$

does not vanish on $[a, b]$. There exists a constant coefficient differential equation of the form

$$\sum_{k=0}^{n} \binom{n}{k} c_{n-k} \frac{d^k}{dt^k} z(t) = 0 , \qquad (c_0 = 1) \tag{1.4.18}$$

that is P equivalent to equation (1.4.1), if and only if the $P\left(\left(C^{-1}V_3(a_i(x))\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(C^{-1}V_3(a_i(x))\right)^{\frac{1-n}{6}}\right)$ transform of equation (1.4.1) is a constant coefficient differential equation of the form of equation (1.4.18).

<u>Proof</u>: The sufficiency is obvious from the definition of P equivalent. The necessity follows directly from the fact that the $P\left(\left(C^{-1}V_3(a_i(x))\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(C^{-1}V_3(a_i(x))\right)^{\frac{1-n}{6}}\right)$ transform of equation (1.4.1) is a canonical transform of equation (1.4.1). In fact it is

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t) \frac{d^k}{dt^k} z(t) = 0 , \qquad (b_0(t) \equiv 1) , \tag{1.4.19}$$

where the $b_k(t)$'s are absolute invariants of equation (1.4.1) given by

$$b_{n-s}(t) = \frac{(n-s)!}{\exp(-\int a_1(x)dx)} \left(C^{-1}V_3(a_i(x))\right)^{\frac{-n-1}{6}} \sum_{k=0}^{n-s} \sum_{j=0}^{n-s-k} a_k(x) \tag{1.4.20}$$

$$\cdot \frac{s!}{k!j!(n-k-j)!} \phi\left(n-k-j, s; \left(C^{-1}V_3(a_i(x))\right)^{1/3}\right)$$

$$\cdot \left(\exp\left(-\int a_1(x)dx\right)\left(C^{-1}V_3(a_i(x))\right)^{\frac{1-n}{6}}\right)^{(j)}$$

for $s = 0, 1, \ldots, n$. Now suppose that there is a $P(u(x), v(x))$

transform of equation (1.4.1) that is a constant coefficient differential equation of the form of equation (1.4.18). That is, we are supposing that equation (1.4.18) is P equivalent to equation (1.4.1). By definition of absolute invariant, definition (1.4.5), we have that for each $s = 0, 1, \ldots, n$, the function $b_{n-s}(t)$, given by equation (1.4.20), has the same value as the same function formed from the constant coefficients of equation (1.4.18). That is, we have that

$$(1.4.21)\quad b_{n-s}(t) = \frac{(n-s)!}{\exp(-\int c_1 dt)} \left(C^{-1} V_3(c_i)\right)^{\frac{-n-1}{6}} \sum_{k=0}^{n-s} \sum_{j=0}^{n-s-k} \frac{c_k s!}{k!j!(n-k-j)!}$$

$$\cdot\, \phi_t\left(n-k-j, s; \left(C^{-1} V_3(c_i)\right)^{1/3}\right) \left(\exp\left(-\int c_1 dt\right) \left(C^{-1} V_3(c_i)\right)^{\frac{1-n}{6}}\right)_t^{(j)}$$

for $s = 0, 1, \ldots, n$, where the c_i's are the constant coefficients of equation (1.4.18). The integrations and differentiations in the right hand side of equation (1.4.21) are now with respect to t since the independent variable of equation (1.4.18) is t.

Since the c_i's are constants and

$$V_3(c_i) = -(c_1)_t^{(2)} + 3\left((c_2)_t^{(1)} - 2c_1 (c_1)_t^{(1)}\right) - 2(c_3 - 3c_1 c_2 + 2c_1^3),$$

we have that

$$V_3(c_i) = c,$$

where c is a constant. In Chapter 4 we will show that our assumption that $V_3(a_i(x))$ is non-vanishing on $[a, b]$ guarantees that this constant c is non-zero, hence we can write

$$\frac{V_3(a_i(x))}{C} = \frac{c}{C} = D \,, \tag{1.4.22}$$

where D is a non-zero constant. In Chapter 2 we will show that

$$\phi_t(k, m; c) = \begin{cases} c^m & k = m \\ 0 & \text{otherwise}\,, \end{cases}$$

where c is a constant. Using this equation we obtain

$$\phi_t(n - k - j, s; D^{1/3}) = \begin{cases} D^{s/3} & j = n - s - k \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.4.23}$$

Using equations (1.4.22) and (1.4.23) in equation (1.4.21) we obtain

$$b_{n-s}(t) = \exp\left(\int c_1 dt\right) D^{\frac{-n-1}{6}} \sum_{k=0}^{n-s} \binom{n-s}{k} c_k D^{s/3} \left(\exp\left(-\int c_1 dt\right) D^{\frac{1-n}{6}}\right)_t^{(n-s-k)}$$

for $s = 0, 1, \ldots, n$. Since c is a constant we find that

$$\left(\exp\left(-\int c_1 dt\right) D^{\frac{1-n}{6}}\right)_t^{(n-s-k)} = D^{\frac{1-n}{6}} (-c_1)^{n-s-k} \exp\left(-\int c_1 dt\right) \,,$$

hence

$$(1.4.24)\quad b_{n-s}(t) = \sum_{k=0}^{n-s} \binom{n-s}{k} c_k(-c\ )^{n-s-k} D^{\frac{s-n}{3}}, \qquad s = 0, 1, \ldots, n .$$

By equation (1.4.24) it is clear that $b_{n-s}(t)$ is a constant for $s = 0, 1, \ldots, n$. To be done we need only show that $b_0(t) \equiv 1$. This follows immediately from equation (1.4.24) since $c_0 = 1$.

Q.E.D.

Halphen [19] and Chiellini [11] stated Theorem (1.4.2) in roughly the following manner.

Necessary and sufficient conditions that equation (1.4.1) can be transformed into a constant coefficient differential equation are that the coefficients of its canonical transform, which are absolute invariants of equation (1.4.1), are constants.

As an example consider the differential equation (1.4.15), which, as we saw in section (1.2), can be transformed into a constant coefficient differential equation. For this differential equation $V_3(a_i(x)) = -12(1 - x)^{-6}$, which does not vanish on $[2, 3]$. Since $a_1(x)$ of this equation is $a_1(x) = \dfrac{3x}{(1 - x)^2}$, we easily find that

$$\exp\left(-\int a_1 dx\right) = (1 - x)^{-3} \exp\left(\frac{-3}{1 - x}\right)$$

By Theorem (1.4.2), with $n = 3$ and $C = -12$, the $P\left((1 - x)^{-2}, (1 - x)^{-1} \exp\left(\frac{-3}{1 - x}\right)\right)$ transform of equation (1.4.15), defined by $\dfrac{dt}{dx} = (1 - x)^{-2}$ and $y(x) = (1 - x)^{-1} \exp\left(\dfrac{-3}{1 - x}\right) z(t)$, must be a constant coefficient differential equation. In fact it is easily found to be

$$(z(t))_t^{(3)} - 9(z(t))_t^{(1)} + 6z(t) = 0 \ .$$

This is no surprise since the defining equations $\frac{dt}{dx} = (1 - x)^{-2}$ and $y(x) = (1 - x)^{-1} \exp\left(\frac{-3}{1 - x}\right) z(t)$ are the same defining equations we used in section (1.2).

As we shall show later, if the order of the differential equation (1.4.1) is $n = 3$ and $V_3(a_i(x)) \equiv 0$ on $[a, b]$, then the problem of solving equation (1.4.1) can be reduced to that of solving a second order linear differential equation. Laguerre [25], Brioschi ([6], [7]) and Halphen ([19], [20]) were aware of this. Halphen also knew that the theory of invariants could be extended to handle the other exceptional cases to Theorem (1.4.2) where $n > 3$ and $V_3(a_i(x)) \equiv 0$ on $[a, b]$.

In Chapters 4 and 5 we will give a brief indication of how to handle these cases.

Although geometric interpretation can be given to invariants, we do not go into this here (see Wilczynski [41]). We only mention that the n linearly independent solutions of equation (1.4.1) can be interpreted as homogeneous coordinates of a point in $n - 1$ dimensional space. When x is varied this point moves along a curve in the $n - 1$ dimensional space that is referred to as the <u>integral curve</u> of equation (1.4.1). Halphen ([19], [20]) called such a curve a "courbe attachée". It turns out (see Wilczynski ([41], p. 48-53)) that the invariants of equation (1.4.1), under $P(u(x), v(x))$ transforms, characterize the projective properties of the integral curve of equation (1.4.1), hence the maps that define $P(u(x), v(x))$ transforms are said to form a <u>projective group</u>.

It is interesting to note that the main result of this thesis, namely Theorem (1.4.2), was given by Halphen in his prize-winning paper [19]. Although this paper was not published until later, Halphen in 1880 won the Ormay Prize (Grand Prix des Sciences Mathématiques) of the Academy of Sciences in Paris for the results it contained. Halphen's results [19] seem to have been misunderstood and/or forgotten. Several authors, including Forsyth [18], Brioschi [7], Fayet [16], Peyovitch [34] and Berkovic [3], have referred to Halphen's work [19], yet they have failed to even give a statement of Theorem (1.4.2). In the little known paper [11], Chiellini does give a statement of Theorem (1.4.2). In [9] and [10] Chiellini considers the special cases where $n = 3$ and $n = 4$. Halphen's collected works [21] have been published in four volumes. Bibliographical material on Halphen is available in [35].

Chapter 2

A Generalization of the Chain Rule of Elementary Calculus

(2.1) Introduction. As we have seen in Chapter 1, when the differential equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (a_0(x) \equiv 1) , \tag{2.1.1}$$

is transformed by means of

$$\frac{dt}{dx} = u(x) \tag{2.1.2}$$

and

$$y(x) = v(x) z(t) , \tag{2.1.3}$$

we require a formula of the form

$$\frac{d^k}{dx^k} = \sum_{m=0}^{k} \phi(k, m; u(x)) \frac{d^m}{dt^m} , \tag{2.1.4}$$

or more generally one of the form

$$\frac{d^k}{dx^k} = \sum_{m=0}^{\infty} \phi(k, m; u(x)) \frac{d^m}{dt^m} . \tag{2.1.5}$$

In this chapter we study in some detail the properties of the function $\phi(k, m; u(x))$. We will prove a number of results, concerning the $\phi(k, m; u(x))$ functions, that are not only of interest in their own right but also have application in proving invariance results.

(2.2) Schlömilch's Formula. In 1858 Schlömilch [39] showed that for $0 \le m \le k \le n$

$$\phi(k, m; u(x)) = (m!)^{-1} \lim_{\rho \to 0} \frac{d^k}{d\rho^k} (t(x + \rho) - t(x))^m , \tag{2.2.1}$$

where $\frac{dt}{dx} = u(x)$. We will call equation (2.2.1) Schlömilch's Formula. We now prove that Schlömilch's Formula is true using Schlömilch's original proof.

Theorem (2.2.1). Let n be a positive integer and let k be such that $0 \le k \le n$. If $\frac{dt}{dx} = u(x)$ where $u(x) \in C^{n-1}[a, b]$ and $u(x)$ does not vanish on $[a, b]$ then for all $x \in [a, b]$

$$\frac{d^k}{dx^k} = \sum_{m=0}^{k} (m!)^{-1} \lim_{\rho \to 0} \frac{d^k}{d\rho^k} (t(x + \rho) - t(x))^m \frac{d^m}{dt^m} .$$

Proof: Let $y(t(x))$ be any function of $t(x)$ where $\frac{dt}{dx} = u(x)$ and $y(t(x)) \in C^n[a, b]$. By the usual chain rule of elementary calculus we have

$$\frac{dy(t)}{dx} = \frac{dt}{dx} \frac{dy(t)}{dt} = u(x)(y(t))_t^{(1)} .$$

Similarily we find that

$$(y(t))^{(2)} = (u(x))^{(1)}(y(t))_t^{(1)} + (u(x))^2 (y(t))_t^{(2)}$$

and

$$(y(t))^{(3)} = (u(x))^{(2)}(y(t))_t^{(1)} + 3u(x)(u(x))^{(1)}(y(t))_t^{(2)} + (u(x))^3 (y(t))_t^{(3)} .$$

It is easy to show by mathematical induction on k that

$$(2.2.2) \qquad (y(t))^{(k)} = \sum_{m=0}^{k} (m!)^{-1} A(k, m; u(x)) (y(t))_t^{(m)} ,$$

where $A(k, m; u(x))$ is a function only of $(u(x))^{(\ell)}$, $\ell = 0, 1, \ldots, k-1$. Clearly all the $A(k, m; u(x))$'s are independent of $y(t)$, hence we can specify a convenient $y(t)$ in order to determine them. By letting $y(t) = t^k$ in equation (2.2.2) we obtain

$$(2.2.3) \qquad (t^k)^{(k)} = \sum_{m=0}^{k} A(k, m; u(x)) \binom{k}{m} t^{k-m} .$$

Since $\left((t(x+\rho))^k\right)^{(k)} = \left((t(x+\rho))^k\right)_\rho^{(k)}$ and $\left((t(x+\rho))^k\right)^{(k)}$ is continuous by hypothesis, we have for $x \in [a, b]$ that

$$\left((t(x))^k\right)^{(k)} = \left((t(x+\rho))^k\right)_\rho^{(k)}\Big|_{\rho=0} .$$

Letting $t(x+\rho) = H + t(x)$ we obtain

$$\begin{aligned}\left((t(x))^k\right)^{(k)} &= \left((H+t(x))^k\right)_\rho^{(k)}\Big|_{\rho=0} \\ &= \left(\sum_{m=0}^{k} \binom{k}{m} H^m (t(x))^{k-m}\right)_\rho^{(k)}\Big|_{\rho=0} \\ &= \sum_{m=0}^{k} \binom{k}{m} (t(x))^{k-m} (H^m)_\rho^{(k)}\Big|_{\rho=0} ,\end{aligned}$$

that is

$$(2.2.4) \qquad \left((t(x))^k\right)^{(k)} = \sum_{m=0}^{k} \binom{k}{m} (t(x))^{k-m} (H^m)_\rho^{(k)}\Big|_{\rho=0} .$$

Comparing equations (2.2.3) and (2.2.4) we obtain

$$A(k,\ m;\ u(x)) = (H^m)_\rho^{(k)}\Big|_{\rho=0}$$

$$= \lim_{\rho\to 0} \frac{d^k}{d\rho^k} (t(x+\rho) - t(x))^m ,$$

where $\frac{dt}{dx} = u(x)$ and $0 \le m \le k \le n$.

Q.E.D.

Equation (2.2.1) follows immediately from Theorem (2.2.1).

Jordan ([22], p. 31) gives a proof of Schlömilch's formula that is similar to the one we have given. The s index of summation in Jordan's equation (8) ([22], p. 32) should only run up to n .

In that which follows, equation (2.1.5), rather than equation (2.1.4), will be taken as the definition of $\phi(k,\ m;\ u(x))$. It is obvious from the proof of Theorem (2.2.1) that we have

$$(2.2.5) \qquad \phi(k,\ m;\ u(x)) = \begin{cases} 0 & m > k \\ (m!)^{-1} \lim\limits_{\rho\to 0} \frac{d^k}{d\rho^k} (t(x+\rho) - t(x))^m & 0 \le m \le k . \end{cases}$$

(2.3) <u>Faà de Bruno's Formula</u>. We now find some alternate expressions for the function $\phi(k,\ m;\ u(x))$. The following lemma was proven by Forsyth [18] under the more restrictive condition that $u(x)$ is infinitely differentiable.

Lemma (2.3.1). Let n be a positive integer and let k be such that $0 \le k \le n$. If $u(x)$ is a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $\frac{dt}{dx} = u(x)$ for $x \in [a, b]$, then

$$(2.3.1) \qquad m!(k!)^{-1}\phi(k, m; u(x)) = \left(\text{coefficient of } \rho^k \text{ in } \left(\sum_{i=1}^{n} \frac{\rho^i}{i!} (t(x))^{(i)} \right)^m \right).$$

Proof: Since $u(x) \in C^{n-1}[a, b]$ we have that $t(x) \in C^n[a, b]$. Letting $x + \rho \in [a, b]$ we use Taylor's series with Lagrange's form of the remainder to expand $t(x + \rho)$ about $\rho = 0$. We obtain

$$t(x + \rho) = \sum_{k=0}^{n-1} \frac{\rho^k}{k!} \left((t(x + \rho))_\rho^{(k)} \Big|_{\rho=0} \right) + \frac{\rho^n}{n!} \left((t(x + \rho))_\rho^{(n)} \Big|_{x+\rho=c} \right),$$

where c is a constant between x and $x + \rho$. As in Theorem (2.2.1) we have that

$$\left(t(x + \rho) \right)_\rho^{(k)} \Big|_{\rho=0} = \left(t(x) \right)^{(k)},$$

hence we obtain

$$t(x + \rho) - t(x) = \sum_{k=1}^{n} \frac{\rho^k}{k!} (t(x))^{(k)} + \frac{\rho^n}{n!} \left((t(x + \rho))_\rho^{(n)} \Big|_{x+\rho=c} - (t(x))^{(n)} \right)$$

By using the notation

$$P(\rho) = \sum_{i=1}^{n} \frac{\rho^i}{i!} (t(x))^{(i)}$$

and

$$R(\rho) = (t(x + \rho))_\rho^{(n)} \Big|_{x+\rho=c} - (t(x))^{(n)},$$

this equation becomes

$$t(x+\rho) - t(x) = P(\rho) + \frac{\rho^n}{n!} R(\rho) .$$

$P(\rho)$ is a polynomial of degree n and since $t(x) \in C^n[a, b]$ we have that $R(\rho)$ is n times continuously differentiable with respect to ρ for $x + \rho \in [a, b]$. Since $m \geq 0$ it follows that

$$\left((t(x+\rho) - t(x))^m\right)_\rho^{(k)}\Big|_{\rho=0} = \left(\sum_{i=0}^{m} \binom{m}{i} (P(\rho))^{m-i} \left(\frac{\rho^n}{n!} R(\rho)\right)^i\right)_\rho^{(k)}\Big|_{\rho=0}$$

$$= \left(\sum_{i=0}^{m} \binom{m}{i} \sum_{j=0}^{k} \binom{k}{j} \left((P(\rho))^{m-i}\right)_\rho^{(k-j)} \left(\left(\frac{\rho^n}{n!} R(\rho)\right)^i\right)_\rho^{(j)}\right)\Big|_{\rho=0} .$$

For $0 \leq j \leq n$ it is obvious that

$$\left(\left(\rho^n R(\rho)\right)^i\right)_\rho^{(j)}\Big|_{\rho=0} = 0 \quad \text{if} \quad i > 0 ,$$

hence the only non-zero term in the right side of the above equation is the one where $i = 0$ and $j = 0$. We have that

$$\text{(2.3.2)} \qquad \left((t(x+\rho) - t(x))^m\right)_\rho^{(k)}\Big|_{\rho=0} = \left((P(\rho))^m\right)_\rho^{(k)}\Big|_{\rho=0}$$

$$= \left(\left(\sum_{i=1}^{n} \frac{\rho^i}{i!} (t(x))^{(i)}\right)^m\right)_\rho^{(k)}\Big|_{\rho=0}$$

$$= k!C_k(x) ,$$

where $C_k(x)$ is the coefficient of ρ^k in $\left(\sum_{i=1}^{n} \frac{\rho^i}{i!} (t(x))^{(i)}\right)^m$.
Equation (2.3.1) now follows immediately from equations (2.3.2) and (2.2.5).

Q.E.D.

We now prove a result that we shall call Faà de Bruno's Formula. It was first published by Faà de Bruno in 1855 (see [8]).

Theorem (2.3.1). (Faà de Bruno's Formula). Let n be a positive integer and let k be such that $0 \le k \le n$. If $u(x)$ is a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $\frac{dt}{dx} = u(x)$ for $x \in [a, b]$, then

$$(2.3.3) \quad \phi(k, m; u(x)) = \begin{cases} 1 & m = k = 0 \\ 0 & m > k \text{ or } m = 0,\ k > 0 \\ \sum k! \left(\prod_{i=1}^{k} (m_i!)\right)^{-1} \prod_{i=1}^{k} \left((u(x))^{(i-1)} (i!)^{-1}\right)^{m_i} & \text{otherwise.} \end{cases}$$

The sum in equation (2.3.3) is taken over all partitions of m such that

$$(2.3.4) \qquad \sum_{i=1}^{k} m_i = m$$

and

$$(2.3.5) \qquad \sum_{i=1}^{k} i m_i = k ,$$

where the m_i's are integers greater than or equal to zero.

Proof: Expanding $\left(\sum_{i=1}^{n} \frac{\rho^i}{i!} (t(x))^{(i)}\right)^m$ using the multinomial theorem ([2], p. 33) we find that

$$\sum m! \left(\prod_{i=1}^{k} (m_i!)\right)^{-1} \prod_{i=1}^{k} \left((u(x))^{(i-1)} (i!)^{-1}\right)^{m_i} = \left(\text{coefficient of } \rho^k \text{ in } \left(\sum_{i=1}^{n} \frac{\rho^i}{i!} (t(x))^{(i)}\right)^m\right) ,$$

where the sum in the left side of the above equation is over all partitions of m such that $\sum_{i=1}^{n} m_i = m$, $\sum_{i=1}^{n} im_i = k$ and $m_i \geq 0$ for all i . Since $0 \leq k \leq n$ it is obvious that $m_i = 0$ for $i > k$, hence the sum is over all partitions of m such that equations (2.3.4) and (2.3.5) hold, where $m_i \geq 0$ for all i . Using equation (2.3.1) we immediately obtain equation (2.3.3).

Q.E.D.

Jordan ([22], p. 33, 34) gives a proof of Theorem (2.3.1) assuming that $u(x)$ is infinitely differentiable. Jordan's index of summation ([22], p. 34) should only run up to n .

When $m > k$ equations (2.3.4) and (2.3.5) cannot be satisfied by any partition $\{m_i\}$ of m where $m_i \geq 0$ for all i . We interpret this as $\phi(k, m; u(x)) = 0$ for $m > k$. Similarly we find that $\phi(k, 0; u(x)) = 0$ for $k > 0$. These interpretations are consistent with Lemma (2.3.1) and equation (2.2.5).

(2.4) <u>Some Elementary Formulas for $\phi(k, m; u(x))$</u>. We have defined $\phi(k, m; u(x))$ by the formula

$$\frac{d^k}{dx^k} = \sum_{k=0}^{\infty} \phi(k, m; u(x)) \frac{d^m}{dt^m} , \tag{2.4.1}$$

where

$$\frac{dt}{dx} = u(x) \tag{2.4.2}$$

and $u(x)$ is a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$. In this section we find, for specific values of k and m, explicit formulas for $\phi(k, m; u(x))$.

Equations (2.2.5), (2.3.1) and (2.3.3) give closed form expressions for $\phi(k, m; u(x))$. It is easy to show directly from Faà de Bruno's Formula, equation (2.3.3), that for $0 \le k \le n$

$$\phi(k, k; u(x)) = (u(x))^k , \tag{2.4.3}$$

$$\phi(k, k-1; u(x)) = \binom{k}{2} (u(x))^{k-2} (u(x))^{(1)} , \tag{2.4.4}$$

$$\phi(k, k-2; u(x)) = 3\binom{k}{4} ((u(x))^{(1)})^2 + \binom{k}{3} (u(x))^{k-3} (u(x))^{(2)} \tag{2.4.5}$$

and

$$\begin{aligned} \phi(k, k-3; u(x)) = {} & \binom{k}{4} (u(x))^{k-4} (u(x))^{(3)} \\ & + 10\binom{k}{5} (u(x))^{k-5} (u(x))^{(1)} (u(x))^{(2)} \\ & + 15\binom{k}{6} (u(x))^{k-6} ((u(x))^{(1)})^3 . \end{aligned} \tag{2.4.6}$$

In formulas (2.4.4), (2.4.5) and (2.4.6) we define $(-j!)^{-1} = 0$, where j is a positive integer. This interpretation follows naturally as is seen in the following derivation of formula (2.4.6).

To find $\phi(k, k-3; u(x))$ from Faà de Bruno's Formula we let $m = k - 3$ in equation (2.3.3) and (2.3.4). Equation (2.3.4) becomes

$$\sum_{i=1}^{k} m_i = k - 3 \tag{2.4.7}$$

while equation (2.3.5) is

$$\sum_{i=1}^{k} im_i = k \ . \tag{2.4.8}$$

Subtracting equation (2.4.7) from (2.4.8) we get

$$\sum_{i=2}^{k} (i - 1)m_i = 3 \ . \tag{2.4.9}$$

Since $m_i \geq 0$ for all i we must have $m_5 = m_6 = \ldots = m_k = 0$, hence equation (2.4.9) becomes

$$m_2 + 2m_3 + 3m_4 = 3 \ , \tag{2.4.10}$$

while equations (2.4.7) and (2.4.8) become

$$m_1 + m_2 + m_3 + m_4 = k - 3 \tag{2.4.11}$$

and

$$m_1 + 2m_2 + 3m_3 + 4m_4 = k \ . \tag{2.4.12}$$

Since $m_i \geq 0$ for all i equation (2.4.10) shows that $m_4 = 1$ or $m_4 = 0$. If $m_4 = 1$ we see from equation (2.4.10) that $m_2 = m_3 = 0$, hence using equation (2.4.11), or equation (2.4.12), we find that $m_1 = k - 4$. We have found one partition $\{m_i\}$ of $k - 3$ to be

$$m_1 = k - 4, \ m_2 = m_3 = 0, \ m_4 = 1, \ m_5 = m_6 = \ldots = m_k = 0 \ . \tag{2.4.13}$$

We know that $m_5 = m_6 = \ldots = m_k = 0$ and that if $m_4 \neq 1$ then $m_4 = 0$. When $m_4 = 0$ we again use the fact that $m_i \geq 0$ for all i and we find from equation (2.4.10) that $m_3 = 1$ or $m_3 = 0$.

When $m_4 = 0$ and $m_3 = 1$ equation (2.4.10) gives $m_2 = 1$, hence by equation (2.4.11), or equation (2.4.12), we find that $m_1 = k - 5$. We have found a second partition $\{m_i\}$ of $k - 3$ to be

(2.4.14) $$m_1 = k - 5,\ m_2 = m_3 = 1,\ m_4 = m_5 = \dots = m_k = 0 .$$

When $m_4 = 0$ and $m_3 = 0$ we find, similarily to above, that a third partition $\{m_i\}$ of $k - 3$ is

(2.4.15) $$m_1 = k - 6,\ m_2 = 3,\ m_3 = m_4 = \dots = m_k = 0 .$$

Clearly these three partitions $\{m_i\}$ of $k - 3$ are the only partitions that satisfy equations (2.4.7) and (2.4.8) since $m_i \geq 0$ for all i and equation (2.4.9) must also hold. We sum equation (2.3.3) over these partitions and find that $\phi(k, k - 3; u(x))$ is given by formula (2.4.6).

Letting $(-j!)^{-1} = 0$, where j is a positive integer, we find from formula (2.4.6) that $\phi(3, 0; u(x)) = 0$, which agrees with equation (2.3.3). The interpretation that $(-j!)^{-1} = 0$ for positive integer j amounts to excluding from the sum in equation (2.3.3) all those partitions $\{m_i\}$ of m that do not satisfy the condition $m_i \geq 0$ for all i.

As a further example consider $\phi(4, 1; u(x))$. The only possible partition $\{m_i\}$, $m_i \geq 0$ for all i, of 1 that satisfies the equations $\sum_{i=1}^{4} m_i = 1$ and $\sum_{i=1}^{4} im_i = 4$, is $m_1 = m_2 = m_3 = 0$, $m_4 = 1$. Summing equation (2.3.3) over this partition we find

$\phi(4, 1; u(x)) = (u(x))^{(3)}$, which follows directly from formula (2.4.6) on the interpretation that $(-j!)^{-1} = 0$ for positive integer j .

The formulas (2.4.3), (2.4.4) and (2.4.5) are easy to verify in the same way that formula (2.4.6) was verified. Equation (2.3.3) can be used to find any $\phi(k, m; u(x))$ where $m \geq 0$, $0 \leq k \leq n$ and $u(x)$ is a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$.

Let c be a constant. By equation (2.3.3) we obtain that

$$(2.4.16) \quad \phi(k, m; c) = \begin{cases} 1 & m = k = 0 \\ 0 & m > k \text{ or } m = 0,\ k > 0 \\ \sum k! \left(\prod_{i=1}^{k} (m_i!) \right)^{-1} \prod_{i=1}^{k} \left(c^{(i-1)} (i!)^{-1} \right)^{m_i} & \text{otherwise ,} \end{cases}$$

where the sum is over all partitions $\{m_i\}$ of m such that

$$(2.4.17) \qquad \sum_{i=1}^{k} m_i = m ,$$

$$(2.4.18) \qquad \sum_{i=1}^{k} i m_i = k$$

and $m_i \geq 0$ for all i . Subtracting equation (2.4.17) from equation (2.4.18) we have that

$$(2.4.19) \qquad \sum_{i=2}^{k} (i - 1) m_i = k - m .$$

Assuming that $\phi(k, m; c) \neq 0$, we see from equation (2.4.16) that $m_i = 0$ for $i > 1$ (since c is a constant). Equation (2.4.19) now gives that $m = k$, hence using equation (2.4.17) we find that

$m_1 = k$. Equation (2.4.16) now gives

$$(2.4.20) \qquad \phi(k, m; c) = \begin{cases} c^k & m = k \\ 0 & \text{otherwise} . \end{cases}$$

Similarly we obtain that

$$\phi_t(k, m; c) = \begin{cases} c^k & m = k \\ 0 & \text{otherwise} , \end{cases}$$

where

$$\phi_t(k, m; c) = \begin{cases} 1 & m = k = 0 \\ 0 & m > k \text{ or } m = 0,\ k > 0 \\ \sum k! \left(\prod_{i=1}^{k} (m_i!) \right)^{-1} \prod_{i=1}^{k} \left(c_t^{(i-1)} (i!)^{-1} \right)^{m_i} & \text{otherwise} . \end{cases}$$

In the last equation $c_t^{(i-1)} = \frac{d^{i-1}}{dt^{i-1}} c$ and the summation is over all partitions $\{m_i\}$ of m such that equation (2.4.17) and (2.4.18) hold.

(2.5) Some Convolution Type Equations Involving $\phi(k, m; u(x))$.

We now study in some detail the $\phi(k, m; u(x))$ function defined by equation (2.1.5).

The first result we wish to prove is a convolution type formula that is satisfied by $\phi(k, m; u(x))$.

Theorem (2.5.1). Let $u(x)$ be a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$. Let $t = h(x)$ be any function such that $(t(x))^{(1)} = (h(x))^{(1)} = u(x)$. Also let $g(t)$ be a non-vanishing function on T such that $g(t)$ is $n - 1$ times continuously differentiable with respect to t on T where

$$T = \{t \mid t = h(x) \quad \text{and} \quad x \in [a, b]\} .$$

If s is any function of t such that $\frac{ds}{dt} = g(t)$ then for $0 \le k \le n$

$$\frac{d^k}{dt^k} = \sum_{m=0}^{k} \phi_t(k, m; g(t)) \frac{d^m}{ds^m} , \tag{2.5.1}$$

$$\frac{d^k}{dx^k} = \sum_{m=0}^{k} \phi(k, m; u(x)g(h(x))) \frac{d^m}{ds^m} , \tag{2.5.2}$$

and

$$\phi(k, m; u(x)g(h(x))) = \sum_{\ell=0}^{k} \phi(k, \ell; u(x))\phi_t(\ell, m; g(t)) , \tag{2.5.3}$$

where the subscript t denotes that derivatives are to be taken with respect to t rather than x .

Proof: Equation (2.5.1) follows immediately from the defining equation (2.1.5) and equation (2.2.5).

From $\frac{ds}{dx} = \frac{dt}{dx}\frac{ds}{dt}$ it follows that

$$\frac{ds}{dx} = u(x)g(h(x)) .$$

It is easy to show from the hypothesis of the theorem that $u(x)g(h(x)) \in C^{n-1}[a, b]$ and that $u(x)g(h(x))$ does not vanish on $[a, b]$. By equation (2.1.5) and (2.2.5) we obtain equation (2.5.2).

In a like manner we obtain

$$\frac{d^k}{dx^k} = \sum_{\ell=0}^{k} \phi(k, \ell; u(x)) \frac{d^\ell}{dt^\ell} .$$

By using equation (2.5.1) to substitute for $\frac{d^\ell}{dt^\ell}$ in this equation we obtain

$$\frac{d^k}{dx^k} = \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \phi(k, \ell; u(x))\phi_t(\ell, m; g(t)) \frac{d^m}{ds^m} .$$

Using formula (A.1.4) to rearrange the sums this is

$$\frac{d^k}{dx^k} = \sum_{m=0}^{k} \sum_{\ell=m}^{k} \phi(k, \ell; u(x))\phi_t(\ell, m; g(t)) \frac{d^m}{ds^m} .$$

By equation (2.2.5) $\phi(\ell, m; u(x)) = 0$ if $\ell < m$, similarily we have $\phi_t(\ell, m; g(t)) = 0$ if $\ell < m$, hence we obtain

$$\frac{d^k}{dx^k} = \sum_{m=0}^{k} \sum_{\ell=0}^{k} \phi(k, \ell; u(x))\phi_t(\ell, m; g(t)) \frac{d^m}{ds^m} .$$

By comparing this equation with equation (2.5.2) we obtain equation (2.5.3).

Q.E.D.

It is important to emphasize that for $0 < m \leq k$

$$\phi(k, m; u(x)g(h(x))) = \sum k!\left(\prod_{i=1}^{k}(m_i!)\right)^{-1} \prod_{i=1}^{k} \left((i!)^{-1}(u(x)g(h(x)))^{(i-1)}\right)^{m_i} ,$$

while

$$\phi_t(k, m; g(t)) = \sum k!\left(\prod_{i=1}^{k}(m_i!)\right)^{-1} \prod_{i=1}^{k} \left((i!)^{-1}(g(t))_t^{(i-1)}\right)^{m_i} .$$

That is, the subscript t indicates that differentiation is with respect to t.

Another convolution type formula that the function $\phi(k, m; u(x))$ satisfies is given in the following theorem.

Theorem (2.5.2). Let $0 \le r \le m \le k \le n$ where n is a positive integer. If $u(x)$ is a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$, then for all $x \in [a, b]$

$$(2.5.4)\quad \binom{m}{r}\phi(k, m; u(x)) = \sum_{j=0}^{k} \binom{k}{j}\phi(j, m-r; u(x))\phi(k-j, r; u(x)) .$$

Proof: Adopting the notation used in Theorem (2.2.1) we let

$$t(x + \rho) - t(x) = H .$$

By Schlömilch's Formula, equation (2.2.1), we obtain

$$\binom{m}{r}\phi(k, m; u(x)) = ((m - r)!r!)^{-1} \lim_{\rho\to 0} (H^m)_\rho^{(k)}$$

$$= ((m - r)!r!)^{-1} \lim_{\rho\to 0} (H^{m-r}H^r)_\rho^{(k)} .$$

By Leibnitz's rule for product differentiation this is

$$\binom{m}{r}\phi(k, m; u(x)) = ((m - r)!r!)^{-1} \lim_{\rho\to 0} \sum_{j=0}^{k} \binom{k}{j} (H^{m-r})^{(j)} (H^r)^{(k-j)} ,$$

which by equation (2.2.1) is precisely equation (2.5.4).

Q.E.D.

Chapter 3

Transforming Nth Order Linear Differential Equations

(3.1) Introduction. Let $u(x)$ and $v(x)$ be non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$, $v \in C^n[a, b]$ and n is a positive integer. In section (1.3) we saw that transforming

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0, \qquad (a_0(x) \equiv 1) , \tag{3.1.1}$$

by means of the equations

$$\frac{dt}{dx} = u(x) \tag{3.1.2}$$

and

$$y(x) = v(x) z(t) , \tag{3.1.3}$$

results in the transformed equation

$$\sum_{\ell=0}^{n} \sum_{j=0}^{n-\ell} \sum_{k=j+\ell}^{n} \binom{n}{k} a_{n-k}(x) \phi(k, j+\ell; u(x)) \binom{j+\ell}{\ell} (v(x))_t^{(j)} (z(t))_t^{(\ell)} \doteq 0 \tag{3.1.4}$$

As indicated in section (1.4) we wish to express equation (3.1.4) in its normal form

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t) (z(t))_t^{(\ell)} = 0 ; \qquad (b_0(t) \equiv 1) . \tag{3.1.5}$$

In section (3.2) we express equation (3.1.4) in the form of equation (3.1.5) where the $b_{n-k}(t)$'s are functions of only the $a_i(x)$'s, $u(x)$, $v(x)$ and their derivatives with respect to x. In section (3.3) we transform equation (3.1.1) by changing only its dependent variable. That is, we let $y(x) = v(x)\overline{y}(x)$. We then find necessary and sufficient conditions so that the transformed equation is a constant coefficient differential equation. We also show that if equation (3.1.1) is transformed by means of the equation $y(x) = \exp(-\int a_1(x)dx)\overline{y}(x)$, then we obtain a transformed equation that has the coefficient of the second highest order derivative of the dependent variable identically equal to zero. That is, we obtain an equation of the form $\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(x) \frac{d^k}{dx^k} \overline{y}(x) = 0$, where $b_1(x) \equiv 0$. In section (3.4) we transform equation (3.1.1) by changing its independent variable. That is, we let $\frac{dt}{dx} = u(x)$ and $y(x) = z(t)$. We then give necessary and sufficient conditions so that the transformed equation is a constant coefficient differential equation. In section (3.5) we transform equation (3.1.1) by changing its dependent variable, then in the resulting equation we transform the independent variable. In section (3.6) we transform equation (3.1.1) by changing its independent variable, then in the resulting equation we transform the dependent variable. An identity between the results of sections (3.2), (3.5) and (3.6) is shown to hold.

In section (3.7) we show that equation (3.1.1) can be transformed into an equation of the form

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (b_0(t) \equiv 1,\ b_1(t) \equiv b_2(t) \equiv 0) .$$

This form has applications relative to invariance theory. Notably for $n = 2$ it means that every second order linear homogeneous differential equation can be reduced to the equation

$$(z(t))_t^{(2)} = 0 .$$

However to actually effect this reduction we must first find a solution of the second order equation in question.

(3.2) Simultaneously Transforming the Independent and Dependent Variables.

Let $u(x)$ and $v(x)$ be non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$, $v(x) \in C^n[a, b]$ and n is a positive integer. As in the previous section we wish to transform

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) , \tag{3.2.1}$$

by means of the equations

$$\frac{dt}{dx} = u(x) \tag{3.2.2}$$

and

$$y(x) = v(x)z(t) , \tag{3.2.3}$$

to obtain

$$(3.2.4) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (b_0(t) \equiv 1) .$$

By equation (3.1.4) equations (3.2.2) and (3.2.3) transform equation (3.2.1) to

$$(3.2.5) \qquad \sum_{\ell=0}^{n} p_{n-\ell}(t)(z(t))_t^{(\ell)} = 0 ,$$

where

$$(3.2.6) \quad p_m(t) = \sum_{j=0}^{n} \sum_{k=j+n-m}^{n} \binom{n}{k} a_{n-k}(x)\phi(k,\ j+n-m;\ u(x)) \binom{j+n-m}{n-m} (v(x))_t^{(j)} .$$

By equation (2.4.3) we have

$$\phi(n,\ n;\ u(x)) = (u(x))^n ,$$

hence since $a_0(x) \equiv 1$ we see that

$$p_0(t) = (u(x))^n v(x) .$$

Since $u(x)$ and $v(x)$ do not vanish on $[a, b]$ neither does $(u(x))^n(v(x))$. To have equation (3.2.5) take on the normal form of equation (3.2.4) we need only let

$$b_m(t) = p_m(t) \left((u(x))^n v(x) \binom{n}{m} \right)^{-1} ,$$

where $p_m(t)$ is given by equation (3.2.6). We have that

$$b_m(t) = \sum_{j=0}^{m} \sum_{k=j+n-m}^{n} \binom{n}{k} a_{n-k}(x)\phi(k,\ j+n-m;\ u(x))$$
$$\cdot \binom{j+n-m}{n-m}\left((u(x))^n v(x)\binom{n}{m}\right)^{-1} (v(x))_t^{(j)} .$$

By dropping the k index of sumation by $n - m$ and then rearranging the sums using formula (A.1.4) this equation becomes

$$\text{(3.2.7)} \qquad b_m(t) = \left((u(x))^n v(x)\binom{n}{m}\right)^{-1} \sum_{k=0}^{m} a_{m-k}(x)\binom{n}{k+n-m}$$
$$\cdot \sum_{j=0}^{k} \phi(k+n-m,\ j+n-m;\ u(x))\binom{j+n-m}{n-m}(v(x))_t^{(j)} .$$

We wish to express the $b_m(t)$'s in terms of the $a_i(x)$'s , $u(x)$, $v(x)$ and their derivatives with respect to x , hence we wish to express the operator

$$\sum_{j=0}^{k} \phi(k+n-m,\ j+n-m;\ u(x))\binom{j+n-m}{n-m}\frac{d^j}{dt^j}$$

in terms of an operator containing only derivatives with respect to x . Towards this end in equation (2.5.4) we let

$$m \to \ell + n - m,\ r \to n - m \quad \text{and} \quad k \to k + n - m ,$$

where the new k, m and n are as in equation (3.2.7) and $0 \le \ell \le k$. Doing this we obtain

$$\binom{\ell+n-m}{n-m}\phi(k+n-m,\ \ell+n-m;\ u(x))$$
$$= \sum_{j=0}^{k+n-m} \binom{k+n-m}{j}\phi(j,\ \ell;\ u(x))\phi(k+n-m-j,\ n-m;\ u(x)) .$$

After multiplying this equation through by $(v(x))_t^{(\ell)}$ we sum both sides of the resulting equation from $\ell = 0$ to $\ell = k$ and use equation (2.2.5) to obtain

$$(3.2.8)\qquad \sum_{\ell=0}^{k} \binom{\ell+n-m}{n-m} \phi(k+n-m,\ \ell+n-m;\ u(x))(v(x))_t^{(\ell)} = \sum_{\ell=0}^{k} \sum_{j=\ell}^{k} \binom{k+n-m}{j}$$

$$\cdot\ \phi(j,\ \ell;\ u(x))\phi(k+n-m-j,\ n-m;\ u(x))(v(x))_t^{(\ell)} .$$

Using formula (A.1.4) to rearrange the sums in the right hand side of equation (3.2.8) we obtain

$$R(3.2.8) = \sum_{j=0}^{k} \sum_{\ell=0}^{j} \binom{k+n-m}{j} \phi(j,\ \ell;\ u(x))\phi(k+n-m-j;\ n-m;\ u(x))(v(x))_t^{(\ell)} ,$$

where we have let R(3.2.8) stand for the right hand side of equation (3.2.8). Using equations (2.1.5) and (2.2.5) we obtain

$$R(3.2.8) = \sum_{j=0}^{k} \binom{k+n-m}{j} \phi(k+n-m-j,\ n-m;\ u(x))(v(x))^{(j)} .$$

By reversing the order of summation we obtain

$$R(3.2.8) = \sum_{j=0}^{k} \binom{k+n-m}{k-j} \phi(j+n-m,\ n-m;\ u(x))(v(x))^{(k-j)} .$$

Equation (3.2.8) now becomes

$$(3.2.9)\qquad \sum_{j=0}^{k} \binom{j+n-m}{n-m} \phi(k+n-m,\ j+n-m;\ u(x))(v(x))_t^{(j)}$$

$$= \sum_{j=0}^{k} \binom{k+n-m}{k-j} \phi(j+n-m,\ n-m;\ u(x))(v(x))^{(k-j)}$$

Equation (3.2.9) expresses the operator

$$\sum_{j=0}^{k} \phi(k+n-m,\ j+n-m;\ u(x)) \binom{j+n-m}{n-m} \frac{d^j}{dt^j}$$

in terms of an operator containing only derivatives with respect to x as we wanted.

Using equation (3.2.9) in equation (3.2.7) we obtain

$$b_m(t) = \left((u(x))^n v(x) \binom{n}{m} \right)^{-1} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{n}{k+n-m}$$
$$\cdot \binom{k+n-m}{j+n-m} \phi(j+n-m,\ n-m;\ u(x)) (v(x))^{(k-j)} a_{m-k}(x) .$$

By reversing the order of the k summation, then reversing the order of the j summation and replacing m by $n-s$ we obtain

$$(3.2.10) \quad b_{n-s}(t) = ((u(x))^n v(x))^{-1} (n-s)! \sum_{k=0}^{n-s} \sum_{j=0}^{n-s-k} (k!j!(n-k-j)!)^{-1}$$
$$\cdot a_k(x) s!\ \phi(n-k-j,\ s;\ u(x)) (v(x))^{(j)} .$$

If we express the ϕ function in equation (3.2.10) by Schlömilch's formula, we obtain an expression for $b_{n-s}(t)$ that is the same as the one used by Forsyth ([18], p. 389). Forsyth derived his expression for $b_{n-s}(t)$ by another method; we shall return to this later.

We now have that equation (3.2.1) transforms, by means of the equations (3.2.2) and (3.2.3), into the normal form equation (3.2.4), where the $b_i(t)$'s are given by equation (3.2.10). Recalling section (1.4) we see that equation (3.2.4) is the $P(u(x), v(x))$ transform of equation (3.2.1), defined by the equations (3.2.2) and (3.2.3). Clearly the $P(u(x), v(x))$ transform of equation (3.2.1) was obtained by simultaneously transforming its independent and dependent variables.

(3.3) Transforming the Dependent Variable. Let $v(x)$ be non-vanishing on $[a, b]$ such that $v(x) \in C^n[a, b]$ where n is a positive integer. Transforming the equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) , \tag{3.3.1}$$

by means of the equation

$$y(x) = v(x)\bar{y}(x) , \tag{3.3.2}$$

gives

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(v(x)\bar{y}(x))^{(k)} = 0 ,$$

which is

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \sum_{j=0}^{k} \binom{k}{j} (v(x))^{(k-j)} (\bar{y}(x))^{(j)} = 0 .$$

By rearranging the sums of this equation using formula (A.1.2) we obtain

$$\sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n}{k+j} \binom{k+j}{j} a_{n-k-j}(x)(v(x))^{(k)} (\bar{y}(x))^{(j)} = 0 .$$

This equation can be written in the form

$$\sum_{j=0}^{n} p_{n-j}(x)(\bar{y}(x))^{(j)} = 0 , \tag{3.3.3}$$

where

$$p_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n}{j}\binom{n-j}{k} a_{n-k-j}(x)(v(x))^{(k)} = 0 . \tag{3.3.4}$$

We wish to express equation (3.3.3) in the normal form

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x)(\bar{y}(x))^{(j)} = 0 , \qquad (b_0(x) \equiv 1) . \tag{3.3.5}$$

Equation (3.3.4) gives that $p_0(x) = v(x)$, which by hypothesis is non-vanishing on $[a, b]$. To obtain the normal form of equation (3.3.5) from equation (3.3.3) we need only let

$$b_{n-j}(x) = \left(\binom{n}{j} v(x)\right)^{-1} p_{n-j}(x) ,$$

where the $p_i(x)$'s are given by equation (3.3.4). Doing this we obtain

$$b_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x)(v(x))^{-1}(v(x))^{(k)} . \tag{3.3.6}$$

<u>Definition (3.3.1)</u>. The differential equation (3.3.5), with its $b_i(x)$'s given by equation (3.3.6), is called the <u>$S(v(x))$ transform</u> of equation (3.3.1).

<u>Definition (3.3.2)</u>. The equation $y(x) = v(x)\bar{y}(x)$ is called the <u>defining equation</u> of the $S(v(x))$ transform of equation (3.3.1).

Note that the $S(v(x))$ transform of equation (3.3.1) is obtained from equation (3.3.1) by transforming its dependent variable by letting $y(x) = v(x)\bar{y}(x)$.

When it is obvious what the defining equation of a $S(v(x))$ transform is, we may not always specify it.

The following theorem is an immediate consequence of definition (3.3.1).

Theorem (3.3.1). Let n be a positive integer and $x \in [a, b]$. There exists a constant coefficient differential equation of the form

$$\sum_{j=0}^{n} \binom{n}{j} c_{n-j} \frac{d^j}{dx^j} \bar{y}(x) = 0 , \qquad (c_0 = 1) ,$$

that is a $S(v(x))$ transform of equation (3.3.1), if and only if there exists a $v(x)$ such that

$$\sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x)(v(x))^{-1}(v(x))^{(k)} = c_{n-j} , \tag{3.3.7}$$

$$j = 0, 1, \ldots, n-1 ,$$

where the c_{n-j}'s are constants.

Assuming that the conditions given by Theorem (3.3.1) hold, we wish to determine $v(x)$. Letting $j = n-1$ in equation (3.3.7) we find

$$a_1(x) + (v(x))^{-1}(v(x))^{(1)} = c_1 ,$$

where c_1 is some constant. Integrating this equation we find that

$$v(x) = C \exp\left(-\int a_1(x)dx\right) \exp(c_1 x) ,$$

where C is a non-zero constant. It is easy to see from equation (3.3.7) that we can assume without loss of generality that $C = 1$, hence

$$v(x) = \exp\left(-\int a_1 dx\right) \exp(c_1 x) .$$

In the next chapter we show that the constant c_1 can be taken to be zero without loss of generality. That is, if there exists a $S(v(x))$ transform of equation (3.3.1) that is a constant coefficient differential equation, then the defining equation of the transform can be taken as

$$(3.3.8) \qquad y(x) = \exp\left(-\int a_1 dx\right) \bar{y}(x) .$$

The transform defined by equation (3.3.8) is known as the Liouville transform (see for example [27], p. 180).

We now show that the $S(v(x))$ transform of equation (3.3.1) is the special case of the $P(u(x), v(x))$ transform of equation (3.3.1) where $u(x) = 1$.

By equations (3.2.4) and (3.2.10) we see that the $P(1, v(x))$ transform of equation (3.3.1), defined by $\frac{dt}{dx} = 1$ and $y(x) = v(x)z(t)$, is

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(t)(z(t))_t^{(j)} = 0 , \qquad (b_0(t) \equiv 1) , \tag{3.3.9}$$

where

$$b_{n-j}(t) = (n-j)!(v(x))^{-1} \sum_{k=0}^{n-j} \sum_{\ell=0}^{n-j-k} (k!\ell!(n-k-\ell)!)^{-1} \cdot a_k(x) j! \, \phi(n-k-\ell, j; 1)(v(x))^{(\ell)} . \tag{3.3.10}$$

Since $\frac{dt}{dx} = 1$ we find that

$$t(x) = x + c , \tag{3.3.11}$$

where c is a constant of integration. We also have that

$$\frac{d^j}{dt^j} = \frac{d^j}{dx^j} . \tag{3.3.12}$$

Using equation (2.4.20) we have that

$$\phi(k, j; 1) = \begin{cases} 1 & j = k \\ 0 & j < k , \end{cases}$$

hence equation (3.3.10) reduces to

$$b_{n-j}(t) = (n-j)!(v(x))^{-1} \sum_{k=0}^{n-j} (k!(n-j-k)!j!)^{-1} a_k(x) j! (v(x))^{(n-j-k)} .$$

By reversing the summation in this equation we obtain

$$b_{n-j}(t) = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x)(v(x))^{-1}(v(x))^{(k)} . \tag{3.3.13}$$

By equations (3.3.9), (3.3.11), (3.3.12) and (3.3.13) we have that the $P(1, v(x))$ transform of equation (3.3.1), defined by $\frac{dt}{dx} = 1$ and $y(x) = v(x)z(t)$, is

$$(3.3.14) \qquad \sum_{j=0}^{n} \binom{n}{j} b_{n-j}(t) \frac{d^j}{dx^j} z(x + c) = 0 ,$$

where the $b_{n-j}(t)$'s are given by equation (3.3.13).

Lemma (3.3.1). Let n be a positive integer and let $v(x)$ be a non-vanishing function on $[a, b]$ such that $v(x) \in C^n[a, b]$. The $P(1, v(x))$ transform of equation (3.3.1) is the same as the $S(v(x))$ transform of equation (3.3.1).

Proof: The proof follows immediately by comparing equations (3.3.5) and (3.3.14). The operators of these differential equations are the same, hence it follows that they are the same differential equations.

Q.E.D.

The $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (3.3.1) is

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x) (\overline{y}(x))^{(j)} = 0 ,$$

where

$$(3.3.15) \qquad b_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x) \exp\left(\int a_1 dx\right) \left(\exp\left(-\int a_1 dx\right)\right)^{(k)} .$$

We easily find from equation (3.3.15) that

$$b_0(x) \equiv 1 ,$$

$$b_1(x) \equiv 0 ,$$

$$b_2(x) = a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}$$

and

$$b_3(x) = a_3(x) - 3a_1(x)a_2(x) + 2(a_1(x))^3 - (a_1(x))^{(2)} .$$

We immediately have the following well-known result (see [12], [41]).

<u>Lemma (3.3.2)</u>. The $S\left(\exp\left(-\int a_1 dx\right)\right)$ transform

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x)(\overline{y}(x))^{(j)} = 0 ,$$

of equation (3.3.1), has the property that $b_0(x) \equiv 1$ and $b_1(x) \equiv 0$ on $[a, b]$.

<u>Definition (3.3.3)</u>. A differential equation

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x)(y(x))^{(j)} = 0 ,$$

is said to be in <u>reduced normal form</u> if $b_0(x) \equiv 1$ and $b_1(x) \equiv 0$.

In view of Lemma (3.3.1) and Lemma (3.3.2) we can reduce every equation (3.3.1) to a P equivalent equation which is in reduced normal form.

(3.4) <u>Transforming the Independent Variable</u>. Let $u(x)$ be a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ where n is a positive integer. Transforming the independent variable of the equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) , \tag{3.4.1}$$

by means of the equations

$$\frac{dt}{dx} = u(x) \tag{3.4.2}$$

and

$$y(x) = z(t) , \tag{3.4.3}$$

gives

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \sum_{m=0}^{k} \phi(k, m; u(x))(z(t))_t^{(m)} = 0 . \tag{3.4.4}$$

Note that we have used equations (2.1.5) and (2.3.3) of Chapter 2 to replace the operator $\frac{d^k}{dx^k}$ by $\sum_{m=0}^{k} \phi(k, m; u(x)) \frac{d^m}{dt^m}$. The dependent variable $y(x)$ has been replaced by $z(t)$ to keep the dependent variable of the transformed equation consistent with the operator acting on it.

Rearranging the sums of equation (3.4.4) using formula (A.1.4) we obtain the equation

$$\sum_{m=0}^{n} p_{n-m}(t)(z(t))_t^{(m)} = 0 , \tag{3.4.5}$$

where

$$p_{n-m}(t) = \sum_{k=m}^{n} \binom{n}{k} a_{n-k}(x)\ \phi(k,\ m;\ u(x)) \ . \tag{3.4.6}$$

Using equation (2.4.3) we find that $p_0(t) = (u(x))^n$, which by hypothesis cannot vanish on $[a,\ b]$. We wish to express equation (3.4.5) in the normal form

$$\sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t)\,(z(t))_t^{(m)} = 0\ , \qquad (b_0(t) \equiv 1)\ . \tag{3.4.7}$$

To obtain the normal form of equation (3.4.7) from equation (3.4.5) we need only let

$$b_{n-m}(t) = \left(\binom{n}{m} (u(x))^n\right)^{-1} p_{n-m}(t)\ ,$$

where $p_{n-m}(t)$ is given by equation (3.4.6). Doing this we obtain

$$b_{n-m}(t) = \sum_{k=m}^{n} \left(\binom{n}{m} (u(x))^n\right)^{-1} \binom{n}{k} a_{n-k}(x)\phi(k,\ m;\ u(x)) \ . \tag{3.4.8}$$

<u>Definition (3.4.1)</u>. Equation (3.4.7), where the $b_{n-m}(t)$'s are given by equation (3.4.8), is called the <u>T(u(x)) transform</u> of equation (3.4.1).

<u>Definition (3.4.2)</u>. The equations $\frac{dt}{dx} = u(x)$ and $y(x) = z(t)$ are called the <u>defining equations</u> of the T(u(x)) transform of equation (3.4.1).

Note that the $T(u(x))$ transform of equation (3.4.1) was obtained by changing the independent variable of equation (3.4.1) on letting $\frac{dt}{dx} = u(x)$.

When it is obvious what the defining equations of a $T(u(x))$ transform are, we may not always specify them.

<u>Theorem (3.4.1)</u>. Let $a_n(x)$ of equation (3.4.1) be non-vanishing on $[a, b]$. There exists a $T(u(x))$ transform of equation (3.4.1) that is a constant coefficient differential equation of the form

$$\sum_{m=0}^{n} \binom{n}{m} c_{n-m} \frac{d^m}{dt^m} z(t) = 0 , \qquad (c_0 = 1) ,$$

if and only if there exists a $u(x)$ such that

$$(3.4.9) \qquad \sum_{k=m}^{n} \left(\binom{n}{m} (u(x))^n \right)^{-1} \binom{n}{k} a_{n-k}(x) \phi(k, m; u(x)) = c_{n-m} ,$$

$$m = 0, 1, \ldots, n-1 ,$$

where the c_{n-m}'s are constants. Moreover if a $u(x)$ exists such that the conditions given by equation (3.4.9) hold, then $u(x)$ can be taken to be $(a_n(x))^{1/n}$.

<u>Proof:</u> The necessary and sufficient conditions given by equation (3.4.9) follow immediately from definition (3.4.1). We now assume that the conditions given by equation (3.4.9) hold and we show that $u(x)$ can be taken as $\left[a_n(x)\right]^{1/n}$. Using equations (3.4.8), (2.3.3) and (2.4.3) we find that the coefficient of $z(t)$,

in the $T(u(x))$ transform of equation (3.4.1), is $(u(x))^{-n}a_n(x)$, and that the coefficient of $(z(t))_t^{(n)}$ is 1. Multiplying the $T(u(x))$ transform of equation (3.4.1) through by $(u(x))^n$ we obtain an equation of the form

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)(z(t))_t^{(k)} = 0 ,$$

where $b_0(t) = (u(x))^n$ and $b_n(t) = a_n(x)$. Since we are assuming that the conditions given by equation (3.4.9) hold, this differential equation must be proportional to a constant coefficient differential equation with leading coefficient 1. It is now clear that we can take $u(x)$ to be $(a_n(x))^{1/n}$.

Q.E.D.

Note that Theorem (3.4.1) is just the normalized version of Breuer and Gottlieb's [5] result given by Theorem (1.4.1).

Assuming that the conditions given by equation (3.4.9) hold, we let $m = 1$ in equation (3.4.9) and we obtain

$$\frac{a_1(x)}{u(x)} + \frac{n-1}{2}(u(x))^{-2}(u(x))^{(1)} = c_1 ,$$

where c_1 is a constant. If we assume that $c_1 = 0$ this equation integrates to give

$$u(x) = \exp\left(\frac{2}{1-n}\int a_1(x)dx\right) ,$$

where we have let the constant of integration be zero. In the next chapter we will show that the coefficients of the $T\left(\exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right)$ transform of equation (3.4.1) are a type of invariant of equation (3.4.1).

Lemma (3.4.1). It is always possible to find an equation

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(x)(\overline{y}(x))^{(k)} = 0 , \qquad (b_0(x) \equiv 1) ,$$

which is S equivalent to equation (3.4.1), that has the property that $b_n(x)$ is non-vanishing on $[a, b]$.

Proof: The $S(v(x))$ transform of equation (3.4.1) is

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(x)(\overline{y}(x))^{(k)} = 0 , \qquad (b_0(x) \equiv 1) ,$$

where

$$b_{n-k}(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} a_{n-j-k}(x)(v(x))^{-1}(v(x))^{(j)} ,$$

hence

$$b_n(x) = \sum_{j=0}^{n} \binom{n}{j} a_{n-j}(x)(v(x))^{-1}(v(x))^{(j)} .$$

It is now obvious that we can always pick a $v(x)$ such that $b_n(x)$ is non-vanishing on $[a, b]$.

Q.E.D.

The following example illustrates an application of Lemma (3.4.1).

Consider the differential equation

$$y^{(2)} + \frac{1}{x} y^{(1)} = 0 . \qquad (3.4.10)$$

Since the coefficient of $y(x)$ of equation (3.4.10) is identically zero we cannot apply Theorem (3.4.1) to equation (3.4.10). However the $S(x)$ transform, defined by $y(x) = x\overline{y}(x)$, of equation (3.4.10) is

$$\overline{y}^{(2)} + \frac{3}{x}\,\overline{y}^{(1)} + \frac{1}{x^2}\,\overline{y}(x) = 0 . \tag{3.4.11}$$

Applying theorem (3.4.1) to equation (3.4.11) we find that the $T\left(\frac{1}{x}\right)$ transform, defined by $\frac{dt}{dx} = \frac{1}{x}$ and $\overline{y}(x) = z(t)$, of equation (3.4.11) is

$$(z(t))_t^{(2)} + 2(z(t))_t^{(1)} + z(t) = 0 .$$

We have that a solution of equation (3.4.10) is

$$y(x) = x\overline{y}(x)$$

$$= xz(t)$$

$$= x\exp(\lambda t) ,$$

where λ is a root of $\lambda^2 + 2\lambda + 1 = 0$ and $t = \int \frac{1}{x}\,dx = \ell n\, x$.

That is, a solution of equation (3.4.10) is given by

$$y(x) = x^{\lambda+1} ,$$

where λ is a root of $\lambda^2 + 2\lambda + 1 = 0$.

We now show that the $T(u(x))$ transform of equation (3.4.1) is the special case of the $P(u(x), v(x))$ transform where $v(x) = 1$. The $P(u(x), 1)$ transform of equation (3.4.1) is

$$(3.4.12) \qquad \sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t)(z(t))_t^{(m)} = 0 \;, \qquad (b_0(t) \equiv 1) \;,$$

where

$$b_{n-m}(t) = (n-m)!(u(x))^{-n} \sum_{j=0}^{n-m} (j!(n-j)!)^{-1} a_j(x) m! \phi(n-j, m; u(x)) \;.$$

By reversing the order of summation of this equation and then raising the index of summation by m we obtain

$$(3.4.13) \qquad b_{n-m}(t) = \sum_{j=m}^{n} \left(\binom{n}{m} (u(x))^n \right)^{-1} \binom{n}{j} a_{n-j}(x) \phi(j, m; u(x)) \;.$$

We have that the $P(u(x), 1)$ transform of equation (3.4.1) is equation (3.4.12) where the $b_{n-m}(t)$'s are given by equation (3.4.13).

<u>Lemma (3.4.2)</u>. Let n be a positive integer and let $u(x)$ be a non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$. The $P(u(x), 1)$ transform of equation (3.4.1) is the same as the $T(u(x))$ transform of equation (3.4.1).

<u>Proof</u>: The proof follows immediately by comparing equations (3.4.7) and (3.4.12). These equations are the same.

Q.E.D.

(3.5) The $T(u(x)) \circ S(v(x))$ Transform. We now show that the $P(u(x),\ v(x))$ transform of

$$(3.5.1) \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 \ , \qquad (a_0(x) \equiv 1) \ ,$$

can be decomposed into an S transform of the dependent variable, followed by a T transform of the independent variable. That is, $P(u(x),\ v(x)) = T(u(x)) \circ S(v(x))$.

The $S(v(x))$ transform of equation (3.5.1) is

$$(3.5.2) \qquad \sum_{j=0}^{n} \binom{n}{j} p_{n-j}(x)(y(x))^{(j)} = 0 \ , \qquad (p_0(x) \equiv 1) \ ,$$

where

$$(3.5.3) \qquad p_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x)(v(x))^{-1}(v(x))^{(k)} \ .$$

The $T(u(x))$ transform of equation (3.5.2) is

$$(3.5.4) \qquad \sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t)(z(t))_t^{(m)} = 0 \ , \qquad (b_0(t) \equiv 1) \ ,$$

where

$$b_{n-m}(t) = \sum_{j=m}^{n} \left(\binom{n}{m} (u(x))^n \right)^{-1} \binom{n}{j} p_{n-j}(x) \phi(j,\ m;\ u(x)) \ .$$

By dropping the j index of summation by m and using equation (3.5.3) we obtain

$$b_{n-m}(t) = \sum_{j=0}^{n-m} \sum_{k=0}^{n-j-m} \left(\binom{n}{m} (u(x))^n v(x) \right)^{-1} \binom{n}{j+m} \binom{n-j-m}{k}$$

$$\cdot\ a_{n-k-j-m}(x) (v(x))^{(k)} \phi(j+m,\ m;\ u(x))\ .$$

By rearranging the sums using formula (A.1.2) this gives

$$b_{n-m}(t) = \sum_{k=0}^{n-m} \sum_{j=0}^{k} \left(\binom{n}{m} (u(x))^n v(x) \right)^{-1} \binom{n}{j+m} \binom{n-j-m}{k-j}$$

$$\cdot\ a_{n-k-m}(x) (v(x))^{(k-j)} \phi(j+m,\ m;\ u(x))\ .$$

By reversing the order of the k summation and then reversing the order of the j summation we obtain

$$(3.5.5)\quad b_{n-m}(t) = (n-m)! \left(v(x) (u(x))^n \right)^{-1} \sum_{k=0}^{n-m} \sum_{j=0}^{n-m-k} (k!\,j!\,(n-k-j)!)^{-1}$$

$$\cdot\ a_k(x) m! (v(x))^{(j)}\ \phi(n-k-j,\ m;\ u(x))\ .$$

<u>Lemma (3.5.1)</u>. Let n be a positive integer and let $u(x)$ and $v(x)$ both be non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$. The $P(u(x), v(x))$ transform of equation (3.5.1) is the same as the $T(u(x)) \circ S(v(x))$ transform of equation (3.5.1).

<u>Proof</u>: The proof follows immediately by comparing equation (3.2.4) with equation (3.5.4). These equations are the same.

Q.E.D.

Lemma (3.5.1) is historically significant. Some authors have used $T(u(x))\circ S(v(x))$ transforms to obtain their results while others have used $P(u(x), v(x))$ transforms. For example Forsyth [18] used the $T(u(x))\circ S(v(x))$ transform of equation (3.5.1) which he derived in roughly the same way as we have. His ϕ function was expressed in terms of Schlömilch's formula.

(3.6) <u>The $S(v(x))\circ T(u(x))$ Transform.</u> We now show that the $P(u(x), v(x))$ transform of

$$(3.6.1) \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)\,(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

can be decomposed into a T transform of the independent variable, followed by an S transform of the dependent variable. That is, $P(u(x), v(x)) = S(v(x))\circ T(u(x))$.

The $T(u(x))$ transform of equation (3.6.1) is

$$(3.6.2) \qquad \sum_{m=0}^{n} \binom{n}{m} p_{n-m}(t)\,(\bar{y}(t))_t^{(m)} = 0 , \qquad (p_0(t) \equiv 1) ,$$

where

$$(3.6.3) \qquad p_{n-m}(t) \sum_{k=m}^{n} \left(\binom{n}{m} (u(x))^n \right)^{-1} \binom{n}{k} a_{n-k}(x)\phi(k, m; u(x)) .$$

As usual $u(x)$ is non-vanishing on $[a, b]$, hence

$$t = \int u(x)dx = h(x) ,$$

where $h(x)$ is a continuous monotone increasing or decreasing function. Obviously the inverse of t exists, that is

$$x = h^{-1}(t) \ .$$

We see that the $S(v(x))$ transform of equation (3.6.2) is the same as the $S(v(h^{-1}(t)))$ transform of equation (3.6.2). It is given by

$$(3.6.4) \qquad \sum_{j=0}^{n} \binom{n}{j} b_{n-j}(t)(z(t))_t^{(j)} = 0 \ , \qquad (b_0(t) \equiv 1) \ ,$$

where

$$b_{n-j}(t) = \sum_{m=0}^{n-j} \binom{n-j}{m} p_{n-m-j}(t)(v(x))^{-1}(v(x))_t^{(m)} \ .$$

Using equation (3.6.3) we obtain

$$b_{n-j}(t) = \sum_{m=0}^{n-j} \sum_{k=m+j}^{n} \left(\binom{n}{m+j} (u(x))^n v(x) \right)^{-1} \binom{n-j}{m} \binom{n}{k}$$

$$\cdot \ a_{n-k}(x)\phi(k, \ m+j; \ u(x))(v(x))_t^{(m)} \ ,$$

which on dropping the k index of summation by j gives

$$b_{n-j}(t) = \sum_{m=0}^{n-j} \sum_{k=m}^{n-j} \left((u(x))^n v(x) \binom{n}{n-j} \right)^{-1} \binom{n}{k+j} \binom{m+j}{j}$$

$$\cdot \ a_{n-j-k}(x)\phi(k+j, \ m+j; \ u(x))(v(x))_t^{(m)} \ .$$

By rearranging the sum's using formula (A.1.4) this is

$$(3.6.5) \qquad b_{n-j}(t) = \sum_{k=0}^{n-j} \sum_{m=0}^{k} \left((u(x))^n v(x) \binom{n}{n-j} \right)^{-1} \binom{n}{k+j} \binom{m+j}{j}$$

$$\cdot \ a_{n-j-k}(x)\phi(k+j, \ m+j; \ u(x))(v(x))_t^{(m)} \ .$$

It is easy to see that equation (3.6.5) is the same as equation (3.2.7). In exactly the same way equation (3.2.7) was shown to be equation (3.2.10), we obtain that equation (3.6.5) is

$$(3.6.6)\qquad b_{n-j}(t) = \sum_{k=0}^{n-j}\sum_{m=0}^{n-j-k}\left((u(x))^n v(x)\right)^{-1}(n-j)!\,(k!m!(n-k-m)!)^{-1}$$

$$\cdot\ a_k(x)j!\ \phi(n-k-m,\ j;\ u(x))(v(x))^{(m)}\ .$$

Lemma (3.6.1). Let n be a positive integer and let $u(x)$ and $v(x)$ both be non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$. The $P(u(x), v(x))$ transform of equation (3.6.1) is the same as the $S(v(x))oT(u(x))$ transform of equation (3.6.1).

Proof: The proof follows immediately by comparing equation (3.2.4) with equation (3.6.4). These equations are the same since the $b_{n-j}(t)$'s of equation (3.6.4) are given by equation (3.6.6).

Q.E.D.

Some authors have used $S(v(x))oT(u(x))$ transforms to obtain their results while others have used $P(u(x), v(x))$ transforms or $T(u(x))oS(v(x))$ transforms. For example Brioschi [6] used $P(u(x), v(x))$ transforms while Laguerre [25] used $S(v(x))oT(u(x))$ transforms. As we saw earlier $t = \int u(x)dx = h(x)$, where $h(x)$ is a monotone increasing or decreasing function on $[a, b]$. The inverse of t exists and can be written as $x = h^{-1}(t)$. We have that $v(x) = v(h^{-1}(t))$. Rather than writing $v(h^{-1}(t))$ Laguerre [25] misleadingly writes $v(t)$.

(3.7) <u>The Laguerre-Forsyth Form</u>. In section (3.3) we showed that every differential equation of the type we are considering can be reduced to a P equivalent differential equation which is in reduced normal form, that is $b_1(x) \equiv 0$. We now assume that this reduction has been effected and we consider the equation

$$(3.7.1) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (b_0(x) \equiv 1,\ b_1(x) \equiv 0) .$$

We show that equation (3.7.1) can be transformed, by a $P(u(x), v(x))$ transformation, into an equation of the form

$$(3.7.2) \qquad \sum_{k=0}^{n} \binom{n}{k} c_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (c_0(t) \equiv 1,\ c_1(t) \equiv c_2(t) \equiv 0)$$

That is, we show that the coefficients of the second and third highest order derivatives of the dependent variable can be made equal to zero. Cockle [13] first discovered this result for the case $n = 3$. Laguerre [26] gave the general case and Forsyth [18] gave a clearer presentation of it (see also Wilczynski [41]).

<u>Definition (3.7.1)</u>. A differential equation

$$\sum_{k=0}^{n} \binom{n}{k} c_{n-k}(t)(z(t))_t^{(k)} = 0 ,$$

is said to be in Laguerre-Forsyth form if $c_0(t) \equiv 1$ and $c_1(t) \equiv c_2(t) \equiv 0$.

The $P(u(x), v(x))$ transform of equation (3.7.1) is

$$\sum_{k=0}^{n} \binom{n}{k} c_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (c_0(t) \equiv 1) , \tag{3.7.3}$$

where

$$c_{n-s}(t) = (n-s)! \left((u(x))^n v(x)\right)^{-1} \sum_{j=0}^{n-s} \sum_{k=0}^{n-s-j} \left(j!k!(n-j-k)!\right)^{-1} \cdot b_j(x) s! \, \phi(n-j-k, s; u(x))(v(x))^{(k)} , \tag{3.7.4}$$

for $s = 0, 1, \ldots, n$. Using the facts that $b_0(x) \equiv 1$ and $b_1(x) \equiv 0$, we apply equations (3.7.4), (2.4.3), (2.4.4), (2.4.5) and (2.4.6) to find that

$$c_0(t) \equiv 1 , \tag{3.7.5}$$

$$c_1(t) = \frac{n-1}{2} u^{-2} u^{(1)} + (uv)^{-1} v^{(1)} , \tag{3.7.6}$$

$$c_2(t) = u^{-2} b_2(x) + \frac{(n-2)(n-3)}{4} u^{-4} \left(u^{(1)}\right)^2 + \frac{n-2}{3} u^{-3} u^{(2)} + u^{-2} v^{-1} v^{(2)} + \frac{n-2}{v} u^{-3} u^{(1)} v^{(1)} \tag{3.7.7}$$

and

$$\begin{aligned} c_3(t) = {} & u^{-3} b_3(x) + \frac{3(n-3)}{2} b_2(x) u^{-4} u^{(1)} + \frac{n-3}{4} u^{-4} u^{(3)} \\ & + \frac{(n-3)(n-4)}{2} u^{-5} u^{(1)} u^{(2)} + \frac{(n-3)(n-4)(n-5)}{8} u^{-6} \left(u^{(1)}\right)^3 \\ & + \frac{3}{v} b_2(x) u^{-3} v^{(1)} + \frac{3(n-3)(n-4)}{4v} u^{-5} v^{(1)} \left(u^{(1)}\right)^2 \\ & + \frac{n-3}{v} u^{-4} u^{(2)} v^{(1)} + \frac{3(n-3)}{2v} u^{-4} u^{(1)} v^{(2)} + u^{-3} v^{-1} v^{(3)} . \end{aligned} \tag{3.7.8}$$

If $c_1(t) \equiv 0$ we find from equation (3.7.6) that

$$v^{-1}v^{(1)} = \frac{1-n}{2u}u^{(1)} ,$$

where as usual $v(x)$ is not zero on $[a, b]$. Integrating this equation we obtain that

$$\ell n\ v(x) = \ell n\left((u(x))^{\frac{1-n}{2}}\right) + c ,$$

where c is a constant of integration. We have that

$$\begin{aligned} v(x) &= (u(x))^{\frac{1-n}{2}} \exp(c) \\ &= C(u(x))^{\frac{1-n}{2}} , \end{aligned}$$

where C is a non-zero constant. From equation (3.7.4) it is obvious that we can assume that $C = 1$ without loss of generality. We have that $c_1(t) \equiv 0$ if and only if

(3.7.9) $$v(x) = (u(x))^{\frac{1-n}{2}} ,$$

which is

(3.7.10) $$u(x) = (v(x))^{\frac{-2}{n-1}} .$$

Since $b_0(x) \equiv 1$ and $b_1(x) \equiv 0$ the case for $n = 1$ is trivial. For $n = 1$ equation (3.7.1) is

$$\frac{dy}{dx} = 0 .$$

By equations (3.7.3), (3.7.5) and (3.7.6) the $P(u(x), v(x))$ transform of $\frac{dy}{dx} = 0$ is

$$\frac{dz(t)}{dt} + (u(x)v(x))^{-1}(v(x))^{(1)}z(t) = 0 .$$

We want the coefficient of $z(t)$ to be zero, hence we let $v(x)$ be any non-zero constant. This agrees with equation (3.7.9) which for $n = 1$ gives $v(x) = 1$.

For $n > 1$ $c_1(t) \equiv 0$ if and only if equation (3.7.10) holds, hence we find that if $c_1(t) \equiv 0$ then

$$(3.7.11) \qquad (u(x))^{(1)} = \frac{-2}{n-1}(v(x))^{\frac{-n-1}{n-1}}(v(x))^{(1)} ,$$

$$(3.7.12) \qquad u^{(2)} = \frac{-2}{n-1}\left[v^{\frac{-n-1}{n-1}}v^{(2)} + \frac{n-1}{-n-1}v^{\frac{-2n}{n-1}}\left(v^{(1)}\right)^2\right]$$

and

$$(3.7.13) \qquad u^{(3)} = \frac{-2}{n-1}\left[v^{\frac{-n-1}{n-1}}v^{(3)} + \frac{3(n-1)}{-n-1}v^{\frac{-2n}{n-1}}v^{(1)}v^{(2)} + 2(n^2+n)(n-1)^{-2}v^{\frac{-3n+1}{n-1}}\left(v^{(1)}\right)^3\right] .$$

For $n > 1$ $c_1(t) \equiv 0$ implies that equations (3.7.9) to (3.7.13) hold. Assuming that $c_1(t) \equiv 0$ these equations show that

$$(3.7.14) \qquad c_2(t) = v^{\frac{4}{n-1}}b_2(x) + (n^2 - n - 2)(-3(n-1)^2)^{-1}\left(v^{(1)}\right)^2 v^{\frac{6-2n}{n-1}} + \frac{n+1}{3(n-1)}v^{\frac{5-n}{n-1}}v^{(2)}$$

and

$$(3.7.15)\quad c_3(t) = v^{\frac{6}{n-1}} b_3(x) + \frac{6}{n-1} b_2(x) v^{\frac{-n+7}{n-1}} v^{(1)} + \frac{n+1}{2(n-1)} v^{\frac{-n+7}{n-1}} v^{(3)}$$
$$+ \frac{-3}{2}(n-3)(n+1)(n-1)^{-2} v^{\frac{-2n+8}{n-1}} v^{(1)} v^{(2)}$$
$$+ (n-1)^{-3}(n-2)(n-3)(n+1) v^{\frac{-3n+9}{n-1}} v^{(1)\,3} .$$

We now find a $u(x)$ and $v(x)$ such that both $c_1(t)$ and $c_2(t)$ are zero.

If $c_1(t) \equiv 0$ then $c_2(t)$ is given by equation (3.7.14), hence we wish to find a $v(x)$ such that $c_2(t)$ given by equation (3.7.14) is zero. As Forsyth [18] did, we make the substitution

$$(3.7.16)\qquad v(x) = \left|\xi(x)\right|^{n-1} ,$$

where $\xi(x)$ is some non-vanishing function on $[a, b]$. From equation (3.7.16) we easily obtain

$$(3.7.17)\qquad (v(x))^{(1)} = (n-1)(\xi(x))^{n-2}(\xi(x))^{(1)} ,$$

$$(3.7.18)\qquad (v(x))^{(2)} = (n-1)\xi^{n-2}\xi^{(2)} + (n-1)(n-2)\xi^{n-3}\left|\xi^{(1)}\right|^2$$

and

$$(3.7.19)\qquad (v(x))^{(3)} = (n-1)\xi^{n-2}\xi^{(3)} + 3(n-1)(n-2)\xi^{n-3}\xi^{(1)}\xi^{(2)}$$
$$+ (n-1)(n-2)(n-3)\xi^{n-4}\left|\xi^{(1)}\right|^3 .$$

Substituting equations (3.7.16), (3.7.17) and (3.7.18) into equation (3.7.14) we easily find

$$(3.7.20) \qquad c_2(t) = (\xi(x))^4 b_2(x) + (n+1)3^{-1}(\xi(x))^3(\xi(x))^{(2)} ,$$

provided that $c_1(t) \equiv 0$. Since $\xi(x)$ is non-vanishing on $[a, b]$ we see from equation (3.7.20) that $c_2(t) \equiv 0$ if and only if

$$(3.7.21) \qquad (\xi(x))^{(2)} + 3(n+1)^{-1} b_2(x)\xi(x) = 0$$

and $c_1(t) \equiv 0$. That is, if $c_1(t) \equiv 0$ we can make $c_2(t) \equiv 0$ by letting $v(x) = (\xi(x))^{n-1}$, where $\xi(x)$ is any non-trivial solution of (3.7.21) on $[a, b]$. Since $c_1(t) \equiv 0$ if and only if equation (3.7.10) holds, we need only let

$$(3.7.22) \qquad v(x) = (\xi(x))^{n-1}$$

and

$$(3.7.23) \qquad u(x) = (\xi(x))^{-2} ,$$

to make $c_1(t) \equiv c_2(t) \equiv 0$, where $\xi(x)$ is any non-trivial solution of (3.7.21) on $[a, b]$.

Definition (3.7.2). Let $\xi(x)$ be a non-trivial solution of equation (3.7.21) on $[a, b]$. The $P\left((\xi(x))^{-2}, (\xi(x))^{n-1}\right)$ transform of equation (3.7.1) is called the Laguerre-Forsyth transform of equation (3.7.1).

Theorem (3.7.1). Let the order of the differential equation (3.7.1) be greater than 1. If $\xi(x)$ is a solution of the differential equation

$$(\xi(x))^{(2)} + \frac{3}{n+1} b_2(x)\xi(x) = 0 ,$$

such that $\xi(x)$ does not vanish on $[a, b]$ and $\xi(x) \in C^n[a, b]$, then the Laguerre-Forsyth transform of equation (3.7.1) is a differential equation having the Laguerre-Forsyth form.

Proof: The proof follows immediately from the definition of Laguerre-Forsyth form, definition (3.7.1).

Q.E.D.

We now prove the following theorem which was done by Combescure [15] for the case $n = 3$.

Theorem (3.7.2). Let $n > 1$ and let $u(x)$ and $v(x)$ be non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$. Moreover let $u(x)$ and $v(x)$ be related by

$$u(x) = (v(x))^{\frac{-2}{n-1}} . \tag{3.7.24}$$

The $P(u(x), v(x))$ transform of equation (3.7.1) is given by the equations (3.7.3) and (3.7.4), where in particular we have

$$c_1(t) \equiv 0$$

$$c_2(t) = v^{\frac{4}{n-1}} b_2(x) + \left(-3(n-1)^2\right)^{-1} (n^2-n-2)\left(v^{(1)}\right)^2 v^{\frac{6-2n}{n-1}} + \frac{n+1}{3(n-1)} v^{(2)} v^{\frac{5-n}{n-1}}$$

and

$$(3.7.25) \qquad c_3(t) = \frac{3}{2} (c_2(t))_t^{(1)} - v^{\frac{6}{n-1}} \left(\frac{3}{2} (b_2(x))^{(1)} - b_3(x) \right) .$$

<u>Proof</u>: Since equation (3.7.24) holds, the $P(u(x), v(x))$ transform of equation (3.7.1) has $c_1(t) \equiv 0$, and $c_2(t)$ and $c_3(t)$ are given by equations (3.7.14) and (3.7.15) respectively. It remains to show that $c_3(t)$ is given by equation (3.7.25). Differentiating equation (3.7.14) with respect to x we find that

$$\begin{aligned} (c_2(t))^{(1)} &= v^{\frac{4}{n-1}} (b_2(x))^{(1)} + 4(n-1) v^{\frac{5-n}{n-1}} v^{(1)} b_2(x) \\ &+ \frac{6-2n}{3} (-n^2+n+2)(n-1)^{-3} v^{\frac{7-3n}{n-1}} \left(v^{(1)}\right)^3 \\ &+ \frac{n+1}{3(n-1)} v^{\frac{5-n}{n-1}} v^{(3)} - (n-1)^{-2}(n-3)(n+1) v^{\frac{6-2n}{n-1}} v^{(1)} v^{(2)} . \end{aligned}$$

Using this equation we express equation (3.7.15) as

$$c_3(t) = v^{\frac{2}{n-1}} \left[\frac{3}{2} (c_2(t))^{(1)} - \frac{3}{2} v^{\frac{4}{n-1}} (b_2(x))^{(1)} + v^{\frac{4}{n-1}} b_3(x) \right] .$$

Recalling that $(t(x))^{(1)} = u(x)$, equation (3.7.24) gives

$$(t(x))^{(1)} = v^{\frac{-2}{n-1}} ,$$

hence we easily find that

$$c_3(t) = \frac{3}{2} (c_2(t))_t^{(1)} - v^{\frac{6}{n-1}} \left(\frac{3}{2} (b_2(x))^{(1)} - b_3(x) \right) .$$

Q.E.D.

In view of Theorem (3.7.1) an immediate corollary to Theorem (3.7.2) is the following.

Corollary. Let $n > 1$ and let $\xi(x)$ be any solution of the differential equation

$$(\xi(x))^{(2)} + 3(n+1)^{-1} b_2(x)\xi(x) = 0 ,$$

such that $\xi(x)$ does not vanish on $[a, b]$ and $\xi(x) \in C^n[a, b]$. The Laguerre-Forsyth transform of equation (3.7.1) is given by equations (3.7.3) and (3.7.4) with $u(x) = (\xi(x))^{-2}$ and $v(x) = (\xi(x))^{n-1}$. In particular we have

$$c_1(t) \equiv c_2(t) \equiv 0$$

and

$$c_3(t) = -(\xi(x))^6 \left[\frac{3}{2} (b_2(x))^{(1)} - b_3(x) \right] .$$

For $n = 2$ the above corollary can be stated as the following theorem which was originally given by Laguerre [25].

Theorem (3.7.3). Let $\xi(x)$ be any solution of

$$(\xi(x))^{(2)} + b_2(x)\xi(x) = 0 ,$$

such that $\xi(x)$ does not vanish on $[a, b]$ and $\xi(x) \in C^2[a, b]$. The Laguerre-Forsyth transform, with $n = 2$, of the equation

$$(y(x))^{(2)} + b_2(x)y(x) = 0 \tag{3.7.26}$$

is

$$(z(t))_t^{(2)} = 0 \ .$$

Consider an arbitrary second order differential equation

$$\sum_{k=0}^{2} \binom{2}{k} a_{2-k}(x)(y(x))^{(k)} = 0 \ , \qquad (a_0(x) \equiv 1) \ . \tag{3.7.27}$$

By Lemma (3.3.2), equation (3.7.27) is reduced, by the transformation $y(x) = \exp(-\int a_1(x)dx)\bar{y}(x)$, to an equivalent equation of the form

$$\bar{y}^{(2)}(x) + b_2(x)\bar{y}(x) = 0 \ . \tag{3.7.28}$$

We also have that $y(x)$ of equation (3.7.27) is connected to $\bar{y}(x)$ of equation (3.7.28) by

$$y(x) = \exp(-\int a_1(x)dx)\bar{y}(x) \ . \tag{3.7.29}$$

Suppose that a non-trivial solution $y_1(x)$ of equation (3.7.27) is known. By equation (3.7.29) a non-trivial solution $\xi(x)$ of equation (3.7.28) is

$$\xi(x) = \exp(\int a_1(x)dx)y_1(x) \ . \tag{3.7.30}$$

It follows from Theorem (3.7.3) and the definition of Laguerre-Forsyth transform (for $n = 2$) , that the solutions of equation (3.7.28) are related to those of $(z(t))_t^{(2)} = 0$, by $\bar{y}(x) = \xi(x)z(t)$, where $t = \int (\xi(x))^{-2} dx$. Two linearly independent solutions of $(z(t))_t^{(2)} = 0$ are c_1 and $c_2 t$, where c_1 and c_2 are arbitrary non-zero constants, hence the general solution of equation (3.7.28) is

$$\begin{aligned}\bar{y}(x) &= \xi(x) z(t) \\ &= \xi(x)(c_1 + c_2 t) \\ &= c_1 \xi(x) + c_2 \xi(x) \int (\xi(x))^{-2} dx .\end{aligned}$$

Using this equation in equation (3.7.29) we find that the general solution of equation (3.7.27) is

$$y(x) = \exp\left(-\int a_1(x) dx\right)\left[c_1 \xi(x) + c_2 \xi(x) \int (\xi(x))^{-2} dx\right] .$$

By equation (3.7.30) this is

$$y(x) = c_1 y_1(x) + c_2 y_1(x) \int \exp\left(-2\int a_1(x) dx\right)(y_1(x))^{-2} dx .$$

We have proven the following theorem.

<u>Theorem (3.7.4)</u>. Let $y_1(x)$ be a non-trivial solution of equation (3.7.27), then a second linearly independent solution of equation (3.7.27) is

$$y_2(x) = y_1(x) \int \exp\left(-2\int a_1(x) dx\right)(y_1(x))^{-2} dx .$$

Remark. Theorem (3.7.4) is a well known result (see for example Ross [37] , p. 91).

Chapter 4

Theory of Invariants

(4.1) Introduction. This chapter is a prelude to the next chapter where we find a canonical transform (see definition (1.4.6)) for the equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 \ , \qquad (a_0(x) \equiv 1) \ . \tag{4.1.1}$$

In sections (4.2) and (4.3) we find semi-canonical $S(v(x))$ and $T(u(x))$ transforms of equation (4.1.1). That is, we find canonical transforms for the special cases where only the dependent variable, or independent variable is transformed. In section (4.4) we show that the function $V_3(a_i(x))$, mentioned in section (1.4), obeys a certain invariance relation. This invariance relation will enable us to find the canonical transform in Chapter 5.

(4.2) Invariants Under Changes of the Dependent Variable. In section (3.3) we saw that the $S(v(x))$ transform of

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 \ , \qquad (a_0(x) \equiv 1) \ , \tag{4.2.1}$$

is

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x) (\overline{y}(x))^{(j)} = 0 \ , \qquad (b_0(x) \equiv 1) \ , \tag{4.2.2}$$

where

$$b_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x) v^{-1} v^{(k)} \ . \tag{4.2.3}$$

We also saw by Lemma (3.3.1) that the $S(v(x))$ transform of equation (4.2.1) is the special case of the $P(u(x), v(x))$ transform of equation (4.2.1) where $u(x) = 1$. By definition (1.4.3) every equation that is a $S(v(x))$ transform of equation (4.2.1) is P equivalent to equation (4.2.1).

Definition (4.2.1). Any equation

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(x)(\bar{y}(x))^{(k)} = 0, \qquad (b_0(x) \equiv 1),$$

that is a $S(v(x))$ transform of equation (4.2.1), is called S equivalent to equation (4.2.1).

Note that every equation that is S equivalent to equation (4.2.1) is obtainable from equation (4.2.1) by transforming the dependent variable of equation (4.2.1), by means of a transform of the form $y(x) = v(x)\bar{y}(x)$.

Recalling equation (1.4.12) of definition (1.4.4) we have that M is the set of all matrices of the form

$$\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)} = \begin{pmatrix} a_0(x) & a_1(x) & \dots & a_n(x) \\ \frac{da_0(x)}{dx} & \frac{da_1(x)}{dx} & \dots & \frac{da_n(x)}{dx} \\ \vdots & & & \vdots \\ \frac{d^n a_0(x)}{dx^n} & & & \frac{d^n a_n(x)}{dx^n} \end{pmatrix}.$$

We now make the following definition.

Definition (4.2.2). Let $v(x)$ be an arbitrary non-vanishing function on $[a, b]$ such that $v(x) \in C^n[a, b]$. Let I be a map from M to the set of all complex valued functions with domain $[a, b]$. Let $a_j(x)$, $j = 0, 1, \ldots, n$, be the coefficients of equation (4.2.1) and let $b_j(x)$, $j = 0, 1, \ldots, n$, be the coefficients of the $S(v(x))$ transform of equation (4.2.1). If for all $x \in [a, b]$ and for all $v(x)$ as defined above we have the identity

$$I\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv I\left(\left(\frac{d^i}{dx^i} b_j(x)\right)_{(n+1)\times(n+1)}\right),$$

then the function I is called an absolute S semi-invariant of equation (4.2.1).

From this definition we see that an absolute S semi-invariant of equation (4.2.1) is a function of the coefficients $a_i(x)$ of equation (4.2.1) and their derivatives with respect to x. This function has the property that for all $x \in [a, b]$ it has the same value as the same function formed from the coefficients of any arbitrary equation which is S equivalent to equation (4.2.1). An absolute S semi-invariant is a function that is invariant only with respect to changes of the dependent variable, hence the prefix "semi" (see [17], [26], [34] and [41]).

Definition (4.2.3). A differential equation

$$(4.2.4) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(x)(\overline{y}(x))^{(k)} = 0 , \qquad (b_0(x) \equiv 1) ,$$

that is S equivalent to equation (4.2.1), is called a semi-canonical S transform of equation (4.2.1) if each $b_i(x)$ of equation (4.2.4) is an absolute S semi-invariant of equation (4.2.1).

A semi-canonical S transform of equation (4.2.1) is an equation that is obtainable from equation (4.2.1) by transforming only its dependent variable.

We now prove the following theorem originally done by Cockle [12] in 1870 (see also [4], [26], [28], [32], [38] and [41]). That is, we show that the $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (4.2.1) is a semi-canonical S transform of equation (4.2.1). First we make a comment about constants of integration. If, for example, $a_1(x) = x$, then the $S\left(\exp\left(-\int a_1(x)dx\right)\right) = S\left(\exp\left(-\int x\, dx\right)\right)$ transform of equation (4.2.1) is defined by

$$\begin{aligned} y(x) &= \exp\left(-\int a_1(x)dx\right)\overline{y}(x) \\ &= \exp\left(-\int x\, dx\right)\overline{y}(x) \\ &= \exp\left(-\frac{x^2}{2} - c\right)\overline{y}(x) , \end{aligned}$$

where c is a constant of integration. We always take the constant of integration c, that stems from integrating $a_1(x)$ with respect to x, to be zero. That is, in the above example we have

$$y(x) = \exp\left(-\frac{x^2}{2}\right)\bar{y}(x) .$$

Taking the constant of integration to be zero results in no loss of generality since we are only interested in finding one particular semi-canonical S transform.

<u>Theorem (4.2.1)</u>. Let $a_1(x)$ of equation (4.2.1) be non-vanishing on $[a, b]$. The $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (4.2.1) is a semi-canonical S transform of equation (4.2.1).

<u>Proof</u>: Let the defining equation of the $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (4.2.1) be

$$(4.2.5) \qquad y(x) = \exp\left(-\int a_1(x)dx\right)\bar{y}(x) .$$

Note that we are taking the constant of integration, that stems from integrating $a_1(x)$ with respect to x, to be zero.

The $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (4.2.1) is

$$(4.2.6) \qquad \sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x) \frac{d^j}{dx^j} y(x) = 0 , \qquad (b_0(x) \equiv 1) ,$$

where

$$(4.2.7)\qquad b_{n-j}(x) = \sum_{k=0}^{n-j}\binom{n-j}{k}a_{n-k-j}(x)\exp\left(\int a_1(x)\,dx\right)\left(\exp\left(-\int a_1(x)\,dx\right)\right)^{(k)} .$$

Each $b_{n-j}(x)$ is a function of the $a_j(x)$'s and their derivatives with respect to x, hence in the notation of definition (1.4.4) we can write

$$(4.2.8)\qquad b_k(x) = B_k\left(\left(\frac{d^i}{dx^i}a_j(x)\right)_{(n+1)\times(n+1)}\right), \qquad k=0,1,\ldots,n .$$

To be finished we need to show for $k = 0,1,\ldots,n$, that for all $x \in [a, b]$ the function $B_k\left(\left(\frac{d^i}{dx^i}a_j(x)\right)_{(n+1)\times(n+1)}\right)$ has the same value as the same function formed from the coefficients of any arbitrary equation which is S equivalent to equation (4.2.1).

Let $v(x)$ be an arbitrary non-vanishing function on $[a, b]$ such that $v(x) \in C^n[a, b]$. The $S(v(x))$ transform of equation (4.2.1), defined by

$$y(x) = v(x)z(x) ,$$

is

$$(4.2.9)\qquad \sum_{j=0}^{n}\binom{n}{j}c_{n-j}(x)(z(x))^{(j)} = 0 , \qquad (c_0(x) \equiv 1) ,$$

where

$$(4.2.10)\qquad c_{n-j}(x) = \sum_{k=0}^{n-j}\binom{n-j}{k}a_{n-k-j}(x)(v(x))^{-1}(v(x))^{(k)} .$$

The coefficients $c_j(x)$ of equation (4.2.9) represent the coefficients of any arbitrary equation that is S equivalent to equation (4.2.1). Hence to complete the proof we need to show that for all $x \in [a, b]$

$$(4.2.11) \qquad B_k\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv B_k\left(\left(\frac{d^i}{dx^i} c_j(x)\right)_{(n+1)\times(n+1)}\right), \quad k=0,1,\ldots,n,$$

where the function B_k is defined by equations (4.2.7) and (4.2.8). Clearly the $B_k\left(\left(\frac{d^i}{dx^i} c_j(x)\right)_{(n+1)\times(n+1)}\right)$'s are just the coefficients of the $S\left(\exp\left(-\int c_1(x)dx\right)\right)$ transform of equation (4.2.9). Letting the defining equation of this transform be

$$z(x) = \exp\left(-\int c_1(x)dx\right)\overline{z}(x) ,$$

we find that the $S\left(\exp\left(-\int c_1(x)dx\right)\right)$ transform of equation (4.2.9) is

$$(4.2.12) \qquad \sum_{j=0}^{n} \binom{n}{j} d_{n-j}(x)\,(\overline{z}(x))^{(j)} = 0 , \qquad (d_0(x) \equiv 1) ,$$

where

$$(4.2.13) \quad d_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} c_{n-k-j}(x)\exp\left(\int c_1(x)dx\right)\left(\exp\left(-\int c_1(x)dx\right)\right)^{(k)} .$$

Note that we are taking the constant of integration, that stems from integrating $c_1(x)$ with respect to x, to be zero. It is important that this constant be taken to be zero since we took the constant of integration when integrating $a_1(x)$ to be zero. By equation (4.2.10) we have that

$$c_1(x) = a_1(x) + (v(x))^{-1}(v(x))^{(1)} .$$

Integrating this equation, taking the constant of integration to be zero, we obtain that

$$c_1(x) = \int a_1(x)dx + \ell n\, v(x) . \tag{4.2.14}$$

By equations (4.2.7), (4.2.8) and (4.2.13) we have as expected that

$$B_k\left(\left(\frac{d^i}{dx^i} c_j(x)\right)_{(n+1)\times(n+1)}\right) = d_k(x) , \qquad k = 0,1,\ldots,n . \tag{4.2.15}$$

To show that the required identity (4.2.11) is true, it suffices to show that for all $x \in [a, b]$

$$b_k(x) = d_k(x) , \qquad k = 0,1,\ldots, n ,$$

(see equations (4.2.8) and (4.2.15)). That is, to be finished we need only show that the differential equations (4.2.6) and (4.2.12) are the same. First we show that they have the same linearly independent solutions.

From equation (4.2.5) we have that the n linearly independent solutions of equation (4.2.6) are related to those of equation (4.2.1) by

$$\bar{y}_i(x) = \exp\left(\int a_1(x)dx\right) y_i(x) , \qquad i = 1,\ldots, n . \tag{4.2.16}$$

We also have that $y(x) = v(x)z(x)$ and $z(x) = \exp\left(-\int c_1(x)dx\right)\overline{z}(x)$, hence we easily find that

$$\overline{z}(x) = \exp\left(\int c_1(x)dx\right)(v(x))^{-1}y(x) \ . \tag{4.2.17}$$

Making use of equation (4.2.14) we find from equation (4.2.17) that the n linearly independent solutions of equation (4.2.12) are related to those of equation (4.2.1) by

$$\overline{z}_i(x) = \exp\left(\int a_1(x)dx\right)y_i(x) \ , \qquad i = 1,\ldots,n \ . \tag{4.2.18}$$

Comparing equation (4.2.18) with equation (4.2.16) we see that the differential equations (4.2.6) and (4.2.12) have the same n linearly independent solutions. We now use this fact to show that the differential equations (4.2.6) and (4.2.12) are the same. Subtracting equation (4.2.12) from equation (4.2.6) we obtain that

$$\sum_{j=0}^{n-1} \binom{n}{j}\left(b_{n-j}(x) - d_{n-j}(x)\right)\frac{d^j}{dx^j}z(x) = 0 \ , \tag{4.2.19}$$

where we have let $z(x)$ be the dependent variable of both the differential equations (4.2.6) and (4.2.12). Since equations (4.2.6) and (4.2.12) have the same n linearly independent solutions, call them $z_i(x)$, $i = 1,\ldots,n$, it is obvious that the differential equation (4.2.19) also has the n linearly independent solutions $z_i(x)$, $i=1,\ldots,n$, for $x \in [a, b]$. However, equation (4.2.19) is of order at most $n - 1$, hence it can have at most $n - 1$ linearly independent solutions on $[a, b]$. That is, we have a contradiction unless the left side of equation (4.2.19) is identically zero, hence we have that $b_k(x) = d_k(x)$, $k = 1,2,\dots,n$, for all $x \in [a, b]$. Recalling that $b_0(x) \equiv d_0(x) \equiv 1$, we have for all $x \in [a, b]$ that

$$b_k(x) = d_k(x) , \qquad k = 0,1,\dots,n .$$

Q.E.D.

Note that we can also prove Theorem (4.2.1) by directly showing that for all $x \in [a, b]$

$$b_k(x) = d_k(x) , \qquad k = 0,1,\dots,n .$$

Let k be an arbitrary integer such that $0 \le k \le n$. We must show that for all $x \in [a, b]$

$$(4.2.20) \qquad \sum_{\ell=0}^{k} \binom{k}{\ell} a_{k-\ell}(x)\left(\exp\int a_1(x)\,dx\right)\left(\exp\left(-\int a_1(x)\,dx\right)\right)^{(\ell)}$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} c_{k-\ell}(x)\left(\exp\left(\int c_1(x)\,dx\right)\right)\left(\exp\left(-\int c_1(x)\,dx\right)\right)^{(\ell)} ,$$

where the $c_i(x)$'s are given by equation (4.2.10). By equation (4.2.14) we have that

$$\exp\left(-\int c_1 dx\right) = (v(x))^{-1}\exp\left(-\int a_1(x)\,dx\right)$$

Using this equation and equation (4.2.10) in equation (4.2.20) it follows that we need to show that

(4.2.21)
$$\sum_{\ell=0}^{k} \binom{k}{\ell} a_{k-\ell}(x)\exp\left(-\int a_1(x)dx\right)^{(\ell)}$$

$$= \sum_{\ell=0}^{k}\sum_{j=0}^{k-\ell} \binom{k}{\ell}\binom{k-\ell}{j} a_{k-\ell-j}(x)v^{(j)}\left(v^{-1}\exp\left(-\int a_1(x)dx\right)\right)^{(\ell)} .$$

Denote the left hand side of equation (4.2.21) by L(4.2.21). We have that

$$L(4.2.21) = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{k-\ell}(x)\left(v\, v^{-1}\exp\left(-\int a_1(x)dx\right)\right)^{(\ell)} .$$

Using Leibnitz's rule for product differentiation this is

$$L(4.2.21) = \sum_{\ell=0}^{k}\sum_{j=0}^{\ell} \binom{k}{\ell} a_{k-\ell}(x)\binom{\ell}{j} v^{(\ell-j)}\left(v^{-1}\exp\left(-\int a_1(x)dx\right)\right)^{(j)} .$$

By rearranging the sums using formula (A.1.2) this is

$$L(4.2.21) = \sum_{j=0}^{k}\sum_{\ell=0}^{k-j} \binom{k}{\ell+j}\binom{\ell+j}{j} a_{k-\ell-j}(x)v^{(\ell)}\left(v^{-1}\exp\left(-\int a_1(x)dx\right)\right)^{(j)} .$$

Letting $j \to \ell$ and $\ell \to j$ we easily find that

$$L(4.2.21) = \sum_{\ell=0}^{k}\sum_{j=0}^{k-\ell} \binom{k}{\ell}\binom{k-\ell}{j} a_{k-\ell-j}(x)v^{(j)}\left(v^{-1}\exp\left(-\int a_1(x)dx\right)\right)^{(\ell)} .$$

This equation shows that equation (4.2.21) is true. We are done since k was arbitrary.

Note that the coefficients $b_k(x)$, given by equation (4.2.7), are functions of the $a_i(x)$'s of equation (4.2.1) and their derivative with respect to x, that contain no integrations. That is, for each k the term $\exp\left(\int a_1(x)dx\right)\left(\exp\left(-\int a_1(x)dx\right)\right)^{(k)}$, of equation (4.2.7), results in an expression containing no integrations. For example if $k = 1$ the expression is just $-a_1(x)$.

Theorem (4.2.2). Let $a_1(x)$ of equation (4.2.1) be non-vanishing on $[a, b]$. There exists a constant coefficient differential equation of the form

$$(4.2.22) \qquad \sum_{j=0}^{n} \binom{n}{j} c_{n-j} \frac{d^j}{dx^j} \overline{y}(x) = 0 , \qquad (c_0 = 1)$$

that is S equivalent to equation (4.2.1), if and only if the $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (4.2.1) is a constant coefficient differential equation of the form of equation (4.2.22).

Proof: The sufficiency is obvious from the definition of S equivalent. The necessity follows immediately from Theorem (4.2.1). The $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (4.2.1) is

$$(4.2.23) \qquad \sum_{j=0}^{n} \binom{n}{j} b_{n-j}(x)(\overline{y}(x))^{(j)} = 0 , \qquad (b_0(x) \equiv 1) ,$$

where the $b_j(x)$'s are absolute S semi-invariants of equation (4.2.1) given by

$$(4.2.24)\quad b_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x)\exp\left[\int a_1(x)dx\right]\left(\exp -\int a_1(x)dx\right)^{(k)} ,$$

$j = 0,1,\ldots,n$. Now suppose that there is a $S(v(x))$ transform of equation (4.2.1) that is a constant coefficient differential equation of the form of equation (4.2.22). That is, we are supposing that equation (4.2.22) is S equivalent to equation (4.2.1). By the definition of absolute S semi-invariant, definition (4.2.1), we have that the function $b_{n-j}(x)$ given by equation (4.2.24) has the same value as the same function formed from the constant coefficients of equation (4.2.22). That is, we have that

$$(4.2.25)\quad b_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} c_{n-k-j}\left(\exp\int c_1 dx\right)\left(\exp -\int c_1 dx\right)^{(k)} ,$$

for $j = 0,1,\ldots,n$, where the c_k's are the constant coefficients of equation (4.2.22). Equation (4.2.25) easily reduces to

$$b_{n-j}(x) = \sum_{k=0}^{n-j} \binom{n-j}{k} c_{n-k-j}(-c_1)^k , \quad j = 0,1,\ldots,n .$$

From this equation it is obvious that $b_{n-j}(x)$ is a constant for $j = 0,1,\ldots,n$. Moreover since $c_0 = 1$ we have that $b_0(x) \equiv 1$.

Q.E.D.

<u>Theorem (4.2.3)</u>. Let $a_1(x)$ of equation (4.2.1) be identically zero on $[a, b]$. There exists a constant coefficient differential equation, that is S equivalent to equation (4.2.1), if and only if equation (4.2.1) is a constant coefficient differential equation.

Proof: The sufficiency is obvious since equation (4.2.1) is equivalent to itself. To prove the necessity we suppose that there exists a constant coefficient differential equation that is S equivalent to equation (4.2.1). That is, we suppose that there exists a $v(x)$ such that the $S(v(x))$ transform of equation (4.2.1) is a constant coefficient differential equation. By equation (3.3.7) of Theorem (3.3.1) we must have that

$$\sum_{k=0}^{n-j} \binom{n-j}{k} a_{n-k-j}(x)(v(x))^{-1}(v(x))^{(k)} = c_{n-j} , \qquad j = 0,1,\ldots,n-1 , \tag{4.2.26}$$

where the c_{n-j}'s are constants. Letting $j = n - 1$ in equation (4.2.26) and recalling that $a_1(x) = 0$ by hypothesis, we have that

$$(v(x))^{-1}(v(x))^{(1)} = c_1 ,$$

where c_1 is some constant. If $c_1 = 0$ then we are done since $v(x)$ must be a constant, which says that equation (4.2.1) must have been a constant coefficient differential equation to start with. If $c_1 \neq 0$ then

$$v(x) = C \exp(c_1 x) ,$$

where C is a non-zero constant. It is easy to see from equation (4.2.26) that we can assume without loss of generality that $C = 1$, hence

$$v(x) = \exp(c_1 x) . \tag{4.2.27}$$

Letting $j = n - 2$ in equation (4.2.6) we use equation (4.2.7) and the fact that $a_1(x) = 0$ to obtain

$$a_2(x) + c_1^2 = c_2 ,$$

which is

(4.2.8) $$a_2(x) = \overline{c}_2 ,$$

where $\overline{c}_2$ is the constant $c_2 - c_1^2$. Letting $j = n - 3$ in equation (4.2.6) we use equations (4.2.7) and (4.2.8), and the fact that $a_1(x) = 0$, to find that $a_3(x)$ is also a constant. Continuing in this manner we find that each $a_j(x)$ is a constant.

Q.E.D.

(4.3) <u>Invariants Under Changes of the Independent Variable.</u>

In section (3.4) we saw that the $T(u(x))$ transform of

(4.3.1) $$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (a_0(x) \equiv 1) ,$$

is

(4.3.2) $$\sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t) (z(t))_t^{(m)} = 0 , \qquad (b_0(t) \equiv 1) ,$$

where

(4.3.3) $$b_{n-m}(t(x)) = \sum_{k=m}^{n} \left(\binom{n}{m} (u(x))^n \right)^{-1} \binom{n}{k} a_{n-k}(x) \phi(k, m; u(x)) .$$

We also saw by Lemma (3.4.1) that the $T(u(x))$ transform of equation (4.3.1) is the special case of the $P(u(x), v(x))$ transform of equation (4.3.1) where $v(x) = 1$. By definition (1.4.3) every equation that is a $T(u(x))$ transform of equation (4.3.1) is P equivalent to equation (4.3.1).

Definition (4.3.1). Any equation

$$\sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t)(z(t))_t^{(m)} = 0 , \qquad (b_0(t) \equiv 1) ,$$

that is a $T(u(x))$ transform of equation (4.3.1), is called T equivalent to equation (4.3.1).

Note that every equation that is T equivalent to equation (4.3.1) is obtainable from equation (4.3.1) by transforming the independent variable of equation (4.3.1), by means of a transform of the form $\frac{dt}{dx} = u(x)$.

Recalling equation (1.4.12) of definition (1.4.4) we have that M is the set of all matrices of the form

$$\left(\frac{d^i}{dx^i} a_j(x) \right)_{(n+1)\times(n+1)} = \begin{pmatrix} a_0(x) & a_1(x) & \cdots & a_n(x) \\ \frac{da_0(x)}{dx} & \frac{da_1(x)}{dx} & \cdots & \frac{da_n(x)}{dx} \\ \vdots & & & \vdots \\ \frac{d^n a_0(x)}{dx^n} & & & \frac{d^n a_n(x)}{dx^n} \end{pmatrix} .$$

We now make the following definition.

<u>Definition (4.3.2)</u>. Let $u(x)$ be an arbitrary non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$. Let I be a map from M to the set of all complex valued functions with domain $[a, b]$. Let $a_j(x)$, $j = 0,1,\ldots,n$, be the coefficients of equation (4.3.1) and let $b_j(t)$, $j = 0,1,\ldots,n$, be the coefficients of the $T(u(x))$ transform of equation (4.3.1). If for all $x \in [a, b]$ and for all $u(x)$ as defined above we have the identity

$$I\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv I\left(\left(\frac{d^i}{dt^i} b_j(t)\right)_{(n+1)\times(n+1)}\right),$$

where $\frac{dt}{dx} = u(x)$, then the function I is called an <u>absolute T semi-invariant</u> of equation (4.3.1).

From this definition we see that an absolute T semi-invariant of equation (4.3.1) is a function of the coefficients $a_i(x)$ of equation (4.3.1) and their derivatives with respect to x. This function has the property that for all $x \in [a, b]$ it has the same value as the same function formed from the coefficients of any arbitrary equation which is T equivalent to equation (4.3.1). An absolute T semi-invariant function is a function that is invariant only with respect to changes of the independent variable, hence the prefix "semi" (see [17], [26], [34], [41]).

Note that in the identity given in definition (4.3.2) the derivatives in the left hand side of the identity are taken with respect to x , while the derivatives in the right hand side are taken with respect to t .

Definition (4.3.3). A differential equation

$$\sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t)(z(t))_t^{(m)} = 0 , \qquad (b_0(t) \equiv 1) , \tag{4.3.4}$$

that is T equivalent to equation (4.3.1), is called a semi-canonical T transform of equation (4.3.1) if each $b_i(t)$ of (4.3.4) is an absolute T semi-invariant of equation (4.3.1).

A semi-canonical T transform of equation (4.3.1) is an equation that is obtainable from equation (4.3.1) by transforming only the independent variable.

We now prove the following theorem done originally by Peyovitch [34] in 1923 (see also [32]). The proof is similar to that of Theorem (4.2.1). In the proof we will use the usual notation that subscripting by a variable indicates the variable that differentiation is with respect to, if it is other than x . For example ϕ_τ indicates that the derivatives in Faà de Bruno's Formula, given by equation (1.3.7), are to be taken with respect to τ rather than x . The theorem says that the $T\left(\exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right)$ transform of equation (4.3.1) is a semi-canonical T transform of equation (4.3.1). Before proving the theorem we make a comment about constants of integration. If, for example, $a_1(x) = x$ and $n = 3$ then the $T\left(\exp\left(-\int a_1(x)dx\right)\right) = T\left(\exp\left(-\int x\ dx\right)\right)$ transform of equation (4.3.1) is defined by

$$\begin{aligned}\frac{dt}{dx} &= \exp\left(-\int a_1(x)dx\right)\\ &= \exp\left(-\int x\ dx\right)\\ &= \exp\left(-\frac{x^2}{2} - c\right),\end{aligned}$$

where c is a constant of integration. We always take the constant of integration c, that stems from integrating $a_1(x)$ with respect to x, to be zero. That is, in the above example we have

$$\frac{dt}{dx} = \exp\left(-\frac{x^2}{2}\right).$$

Taking the constant of integration to be zero results in no loss of generality since we are only interested in finding one particular semi-canonical T transform.

<u>Theorem (4.3.1)</u>. Let the order of equation (4.3.1) be greater than one and let $a_1(x)$ of equation (4.3.1) be non-vanishing on $[a, b]$. The $T\left(\exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right)$ transform of equation (4.3.10) is a semi-canonical T transform of equation (4.3.1).

<u>Proof</u>: Let the defining equations of the $T\left(\exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right)$ transform of equation (4.3.1) be

$$\frac{dt}{dx} = \exp\left(\frac{2}{1-n}\int a_1(x)dx\right) \tag{4.3.5}$$

and

$$y(x) = \bar{y}(t) \ .$$

Note that we are taking the constant of integration, that stems from integrating $a_1(x)$ with respect to x, to be zero.

The $T\left(\exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right)$ transform of equation (4.3.1) is

$$\sum_{m=0}^{n} \binom{n}{m} b_{n-m}(t)\,(\bar{y}(t))_t^{(m)} = 0 \ , \qquad (b_0(t) \equiv 1) \ , \tag{4.3.6}$$

where

$$b_{n-m}(t) = \sum_{k=m}^{n} \left(\binom{n}{m}\exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right)^{-1} \binom{n}{k} a_{n-k}(x) \cdot \phi\left(k,\ m;\ \exp\left(\frac{2}{1-n}\int a_1(x)dx\right)\right) \tag{4.3.7}$$

Each $b_k(t)$ is a function of the $a_i(x)$'s and their derivatives with respect to x, hence in the notation of definition (1.4.4) we can write

$$b_k(t) = B_k\left(\left(\frac{d^i}{dx^i}\,a_j(x)\right)_{(n+1)\times(n+1)}\right) \ , \qquad k = 0,1,\ldots,n \ . \tag{4.3.8}$$

To be done we need to show for $k = 0,1,\ldots,n$, that for all $x \in [a, b]$ the function $B_k\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right)$ has the same value as the same function formed from the coefficients of any arbitrary equation which is T equivalent to equation (4.3.1).

Let $u(x)$ be an arbitrary non-vanishing function on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$. The $T(u(x))$ transform, defined by

$$\frac{d\tau}{dx} = u(x)$$

and

$$y(x) = z(\tau) ,$$

of equation (4.3.1) is

$$(4.3.9) \qquad \sum_{m=0}^{n} \binom{n}{m} c_{n-m}(\tau)(z(\tau))_{\tau}^{(m)} = 0 , \qquad (c_0(\tau) \equiv 1) ,$$

where

$$(4.3.10) \quad c_{n-m}(\tau) = \sum_{k=m}^{n} \left(\binom{n}{m}(u(x))^n\right)^{-1} \binom{n}{k} a_{n-k}(x)\phi(k,\ m;\ u(x)), \quad m=0,1,\ldots,n .$$

The coefficients $c_m(\tau)$ of equation (4.3.9) represent the coefficients of any arbitrary equation that is T equivalent to equation (4.3.1). To complete the proof we need to show that for all $x \in [a, b]$

$$(4.3.11) \quad B_k\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv B_k\left(\left(\frac{d^i}{d\tau^i} c_j(\tau)\right)_{(n+1)\times(n+1)}\right) ,$$
$$k = 0,1,\ldots,n ,$$

where $\frac{d\tau}{dx} = u(x)$ and the function B_k is defined by equations (4.3.7) and (4.3.8).

Note that derivatives and integrations are taken with respect to the independent variable of the function in question. Clearly the $B_k\left(\left(\frac{d^i}{d\tau^i} c_j(\tau)\right)_{(n+1)\times(n+1)}\right)$'s are just the coefficients of the $T\left(\exp\left(\frac{2}{1-n}\int c_1(\tau)d\tau\right)\right)$ transform of equation (4.3.9). Letting the defining equations of this transform be

$$\frac{ds}{d\tau} = \exp\left(\frac{2}{1-n}\int c_1(\tau)d\tau\right)$$

and

$$z(\tau) = \bar{z}(s) ,$$

we find that the $T\left(\exp\left(\frac{2}{1-n}\int c_1(\tau)d\tau\right)\right)$ transform of equation (4.3.9) is given by

$$\text{(4.3.12)} \qquad \sum_{m=0}^{n} \binom{n}{m} d_{n-m}(s)\,(\bar{z}(s))_s^{(m)} = 0 , \qquad (d_0(s) \equiv 1) ,$$

where

$$\text{(4.3.13)} \qquad d_{n-m}(s) = \sum_{k=m}^{n} \left(\binom{n}{m}\exp\left(\frac{2n}{1-n}\int c_1(\tau)d\tau\right)\right)^{-1} \binom{n}{k} c_{n-k}(\tau) \cdot \phi_\tau\left(k,\ m;\ \exp\left(\frac{2}{1-n}\int c_1(\tau)d\tau\right)\right) ,$$

for $m = 0,\ldots,n$.

Note that we are taking the constant of integration, that stems from integrating $c_1(\tau)$ with respect to τ , to be zero. It is important that this constant of integration be taken to be zero since we took the constant of integration when integrating $a_1(x)$ with respect to x to be zero. By equations (4.3.10), (2.4.3) and (2.4.4) we have that

$$c_1(\tau) = \frac{a_1(x)}{u(x)} + \frac{n-1}{2}(u(x))^{-2}(u(x))^{(1)} .$$

Recalling that $\frac{d\tau}{dx} = u(x)$ we integrate $c_1(\tau)$ with respect to τ, taking the constant of integration to be zero, to obtain

$$\text{(4.3.14)} \qquad \int c_1(\tau)d\tau = \int c_1(\tau)u(x)dx$$
$$= \int a_1(x)dx + \frac{n-1}{2}\ell n(u(x)) .$$

By equations (4.3.7), (4.3.8) and (4.3.13) we have as expected that

$$\text{(4.3.15)} \qquad B_k\left(\left(\frac{d^i}{d\tau^i}c_j(\tau)\right)_{(n+1)\times(n+1)}\right) = d_k(s) , \qquad k = 0,1,\ldots,n .$$

To show that the required identity (4.3.11) is true, it suffices to show that for all $x \in [a, b]$

$$b_k(t) = d_k(s) , \qquad k = 0,1,\ldots,n ,$$

(see equations (4.3.8) and (4.3.15)). That is, to be finished we need only show that the differential equations (4.3.6) and (4.3.12) are the same. First we show that they have the same linearly independent solutions.

Recall that $y(x) = \overline{y}(t)$, hence the n linearly independent solutions of equation (4.3.6) are related to those of equation (4.3.1) by

$$\text{(4.3.16)} \qquad \overline{y}_i(t) = y_i(x) , \qquad i = 1,\ldots,n .$$

We also have that $y(x) = z(\tau)$ and $z(\tau) = \overline{z}(s)$, hence we have that $y(x) = \overline{z}(s)$. That is, the n linearly independent solutions of equation (4.3.12) are related to those of equation (4.3.1) by

$$\overline{z}_i(s) = y_i(x) \ , \qquad\qquad i = 1,\ldots,n \ .$$

Comparing this equation with equation (4.3.16) we see that the differential equations (4.3.6) and (4.3.12) have the same n linearly independent solutions.

We now show that the independent variables t and s , of equations (4.3.6) and (4.3.12) respectively, are related by $\frac{ds}{dt} = 1$. Since $\frac{d\tau}{dx} = u(x)$ and $\frac{ds}{d\tau} = \exp\left(\frac{2}{1-n}\int c_1(\tau)d\tau\right)$ we can use equation (4.3.14) to obtain

$$\begin{aligned}\frac{ds}{dx} = \frac{d\tau}{dx}\frac{ds}{d\tau} &= u(x)\exp\left(\frac{2}{1-n}\int c_1(\tau)d\tau\right)\\ &= u(x)\exp\left[\frac{2}{1-n}\left(\int a_1(x)dx + \frac{n-1}{2}\ln u(x)\right)\right]\\ &= \exp\left(\frac{2}{1-n}\int a_1(x)dx\right) \ .\end{aligned}$$

That is,

$$\frac{ds}{dx} = \exp\left(\frac{2}{1-n}\int a_1(x)dx\right) \ . \tag{4.3.17}$$

Comparing equation (4.3.17) with equation (4.3.5), that is

$$\frac{dt}{dx} = \exp\left(\frac{2}{1-n}\int a_1(x)dx\right) \ ,$$

we have that $\frac{ds}{dt} = 1$, which is what we wanted to show.

We now show that the differential equations (4.3.6) and (4.3.12) are the same. Since $\frac{ds}{dt} = 1$, we have that

$$\frac{d^k}{ds^k} = \frac{d^k}{dt^k}, \qquad k = 0,1,\ldots,n .$$

Using the relation $\frac{d^k}{ds^k} = \frac{d^k}{dt^k}$ we can subtract equation (4.3.12) from equation (4.3.6) obtaining

$$\text{(4.3.18)} \qquad \sum_{m=0}^{n-1} \binom{n}{m} \left(b_{n-m}(t) - d_{n-m}(s) \right) \frac{d^m}{dt^m} z(x) = 0 ,$$

where we have let $z(x)$ be the dependent variable of both the differential equations (4.3.6) and (4.3.12). Since equations (4.3.6) and (4.3.12) have the same n linearly independent solutions, call them $z_i(x)$, $i = 1,\ldots,n$, it is obvious that the differential equation (4.3.18) has the n linearly independent solutions $z_i(x)$, $i = 1,\ldots,n$, for all $x \in [a, b]$. Equation (4.3.18) can have at most $n - 1$ linearly independent solutions for $x \in [a, b]$ since it is of order at most $n - 1$. We have a contradiction unless equation (4.3.18) has $b_k(t) = d_k(s)$, $k = 1,\ldots,n$, for $x \in [a, b]$. That is, recalling that $b_0(t) \equiv d_0(s) \equiv 1$, we have for all $x \in [a, b]$ that

$$b_k(t(x)) = d_k(s(\tau(x))) , \qquad k = 0,1,\ldots,n .$$

Q.E.D.

Note that we can also prove Theorem (4.3.1) by directly showing that for all $x \in [a, b]$

$$b_{n-m}(t(x)) = d_{n-m}(s(\tau(x))) \ , \qquad m = 0,1,\ldots,n \ .$$

Let m be an arbitrary integer such that $0 \le m \le n$. We must show that for $x \in [a, b]$

$$\begin{aligned}
(4.3.19) \qquad & \sum_{k=m}^{n} \left(\binom{n}{m} \exp\left(\frac{2n}{1-n} \int a_1(x)dx \right) \right)^{-1} \binom{n}{k} a_{n-k}(x) \\
& \quad \cdot \phi\left(k,\ m;\ \exp\left(\frac{2}{1-n} \int a_1(x)dx\right)\right) \\
&= \sum_{k=m}^{n} \left(\binom{n}{m} \exp\left(\frac{2n}{1-n} \int c_1(\tau)d\tau \right) \right)^{-1} \binom{n}{k} c_{n-k}(\tau) \\
& \quad \cdot \phi_\tau\left(k,\ m;\ \exp\left(\frac{2}{1-n} \int c_1(\tau)d\tau\right)\right) ,
\end{aligned}$$

where the $c_i(\tau)$'s are given by equation (4.3.10). By equation (4.3.14) we have that

$$\int c_1(\tau)d\tau = \int a_1(x)dx + \frac{n-1}{2} \ln u(x) \ .$$

Using this equation and equation (4.3.10) in equation (4.3.19) it follows, after easy simplications, that we need to show that

$$\begin{aligned}
& \sum_{k=m}^{n} \binom{n}{k} a_{n-k}(x) \phi\left(k,\ m;\ \exp\left(\frac{2}{1-n} \int a_1(x)dx\right)\right) \\
&= \sum_{k=m}^{n} \sum_{j=k}^{n} \binom{n}{j} a_{n-j}(x) \phi(j,\ k;\ u(x)) \phi_\tau\left(k,\ m;\ u^{-1} \exp\left(2(1-n)^{-1} \int a_1 dx\right)\right) \ .
\end{aligned}$$

Letting ℓ be an arbitrary integer such that $m \le \ell \le n$, it suffices to show that the coefficients of $a_{n-\ell}(x)$ of each side of this equation are equal. That is, it suffices to show that

$$\phi\left(\ell,\ m;\ \exp\left(2(1-n)^{-1}\int a_1 dx\right)\right)$$
$$= \sum_{k=m}^{n} \phi(\ell,\ k;\ u(x))\phi_\tau\left(k,\ m;\ u^{-1}\exp\left(2(1-n)^{-1}\int a_1 dx\right)\right).$$

By equation (2.2.5) $\phi(\ell,\ k;\ u(x)) = 0$ for $k > \ell$, hence the index of summation in the right hand side of this equation can be stopped at $k = \ell$. Similarily $\phi_\tau\left(k,\ m;\ u^{-1}\exp\left(2(1-n)^{-1}\int a_1 dx\right)\right) = 0$ for $k < m$, hence the same index of summation can be started at $k = 0$. Thus we must show that

$$\phi\left(\ell,\ m;\ \exp\left(\frac{2}{1-n}\int a_1 dx\right)\right)$$
$$= \sum_{k=0}^{\ell} \phi(\ell,\ k;\ u(x))\phi_\tau\left(k,\ m;\ u^{-1}\exp\left(\frac{2}{1-n}\int a_1 dx\right)\right).$$

This equation follows immediately from equation (2.5.3) with $u(x)g(h(x)) = \exp\left(\frac{2}{1-n}\int a_1 dx\right)$, $t = \tau$ and $g(t) = u^{-1}\exp\left(\frac{2}{1-n}\int a_1 dx\right)$, hence we are done.

Note that the coefficients $b_m(t)$, given by equation (4.3.7), do contain integrations. It is the presence of these integrations that prevents us from proving a result analogous to Theorems (4.2.2) and (1.4.2). This is the case since the integral of a non-zero constant is not a constant.

We now prove Theorem (3.4.1), for the case $n = 2$, in a manner similar to what Peyovitch [34] did.

Theorem (4.3.2). Let

$$\sum_{k=0}^{2} \binom{2}{k} a_{2-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) , \tag{4.3.20}$$

be a second order linear differential equation such that $a_1(x)$ and $a_2(x)$ are non-vanishing on $[a, b]$. There exists a $T(u(x))$ transform of equation (4.3.20) that is a constant coefficient differential equation of the form

$$\sum_{k=0}^{2} \binom{2}{k} c_{2-k} \frac{d^k}{d\tau^k} z(\tau) = 0 , \qquad (c_0 \equiv 1) , \tag{4.3.21}$$

if and only if

$$(a_2(x))^{(1)} + 4a_1(x)a_2(x) + \gamma(a_2(x))^{3/2} = 0 , \tag{4.3.22}$$

where γ is some constant. Moreover if the condition given by equation (4.3.22) holds, then $u(x)$ can be taken to be $(a_2(x))^{1/2}$.

Proof: First we show the necessity. Suppose that there exists a $T(u(x))$ transform of equation (4.3.20) that is a constant coefficient differential equation of the form of equation (4.3.21). Thus we are supposing that there exists a $u(x)$ such that the constant coefficients of equation (4.3.21) are given by

$$c_{2-m} = \sum_{k=m}^{2} \left(\binom{2}{m} (u(x))^2 \right)^{-1} \binom{2}{k} a_{2-k}(x)\phi(k, m; u(x)) , \quad m = 0,1,2 . \tag{4.3.23}$$

By Theorem (4.3.1) the $T\left(\exp\left(-2\int a_1(x)dx\right)\right)$ transform of equation (4.3.20), defined by $\frac{dt}{dx} = \exp\left(-2\int a_1(x)dx\right)$ and $y(x) = \overline{y}(t)$, is a semi-canonical T transform of equation (4.3.20). It is given by

$$(4.3.24) \qquad \sum_{m=0}^{2} \binom{2}{m} b_{2-m}(t)\,(\overline{y}(t))_t^{(m)} = 0\,, \qquad (b_0(t) \equiv 1)\,,$$

where

$$(4.3.25) \quad b_{2-m}(t) = \sum_{k=m}^{2}\left(\binom{2}{m}\exp\left(-4\int a_1(x)dx\right)\right)^{-1}\binom{2}{k}a_{2-k}(x)\phi\left(k,m;\exp\left(-2\int a_1(x)dx\right)\right),$$

$m = 0,1,2$. As usual we are taking the constant of integration, that stems from integrating $a_1(x)$, to be zero.

We now find the $T\left(\exp\left(-2\int c_1 d\tau\right)\right)$ transform of equation (4.3.21), defined by

$$(4.3.26) \qquad \frac{ds}{d\tau} = \exp\left(-2\int c_1 d\tau\right)$$

and

$$z(\tau) = \overline{z}(s)\,.$$

As usual we take the constant of integration, that stems from integrating c_1, to be zero. That is, the constant of integration is taken to be zero since we took the constant of integration when integrating $a_1(x)$ to be zero. Since c_1 is a constant by hypothesis, we have that

$$\int c_1 d\tau = c_1\tau\,,$$

hence equation (4.3.26) is

$$\frac{ds}{d\tau} = \exp(-2c_1\tau) \quad . \tag{4.3.27}$$

The $T\left(\exp\left(-2\int c_1 d\tau\right)\right) = T\left(\exp\left(-2c_1\tau\right)\right)$ transform of equation (4.3.21) is given by

$$\sum_{k=0}^{2} \binom{2}{k} d_{2-k}(s)(z(s))_s^{(k)} = 0 \ , \qquad (d_0(s) \equiv 1) \ , \tag{4.3.28}$$

where

$$d_{2-m}(s) = \sum_{k=m}^{2} \left(\binom{2}{m}\exp\left(-4c_1\tau\right)\right)^{-1}\binom{2}{k}c_{2-k}\phi_\tau\left(k,m;\exp\left(-2c_1\tau\right)\right) \ , \tag{4.3.29}$$

$m = 0,1,2$. From Theorem (4.3.1) (see its proof) it is obvious that

$$b_k(t) = d_k(s) \ , \qquad k = 0,1,2 \ , \tag{4.3.30}$$

and

$$\frac{ds}{dt} = 1 \ . \tag{4.3.31}$$

We have shown that the equations (4.3.30) give necessary conditions that there exists a constant coefficient differential equation that is T equivalent to equation (4.3.20). We now use these conditions to derive the condition (4.3.22).

By equations (4.3.25), (4.3.29), (4.3.30), (2.4.3), (2.4.4) and (2.4.5) we have that

$$b_0(t) = d_0(s) = 1 \ ,$$

$$b_1(t) = d_1(s) = 0$$

and

$$b_2(t) = a_2(x)\exp\left(4\int a_1(x)\,dx\right) = d_2(s) = c_2\exp(4c_1\tau) \ . \tag{4.3.32}$$

The first two of these conditions, given by equations (4.3.30), are independent of the coefficients $a_i(x)$ of equation (4.3.20). That is, only the equation $b_2(t) = d_2(s)$ can be used to find a condition, on the $a_i(x)$'s of equation (4.3.20), that must be satisfied if there is to be a constant coefficient differential equation that is a $T(u(x))$ transform of equation (4.3.20). By equation (4.3.32) the equation $b_2(t) = d_2(s)$ is

$$a_2(x)\exp\left(4\int a_1(x)\,dx\right) = c_2\exp(4c_1\tau) \ . \tag{4.3.33}$$

We now use equation (4.3.33) to derive the condition (4.3.22).

Integrating equation (4.3.27) we find that

$$s = (-2c_1)^{-1} \exp(-2c_1\tau) + C \ ,$$

where C is a constant. It follows that

$$\exp(4c_1\tau) = \left(4c_1^2(s - C)^2\right)^{-1} \ ,$$

Hence equation (4.3.33) becomes

$$a_2(x)\exp\left(4\int a_1(x)\,dx\right) = \frac{c_2}{4c_1^2(s - C)^2} \ .$$

That is

$$(4.3.34) \qquad a_2(x)\exp\left(4\int a_1(x)dx\right) = \frac{A}{(\alpha + \beta s)^2} ,$$

where A, α, and β are constants.

By equation (4.3.31) $\frac{ds}{dt} = 1$, hence we can differentiate the left side of equation (4.3.34) with respect to t , and at the same time differentiate its right side with respect to s , to obtain

$$\frac{dx}{dt}\frac{d}{dx}\left(a_2(x)\exp\left(4\int a_1(x)dx\right)\right) = \frac{d}{ds}\left(\frac{A}{(\alpha + \beta s)^2}\right) .$$

Since $\frac{dt}{dx} = \exp\left(-2\int a_1(x)dx\right)$, we obtain that

$$(4.3.35) \qquad \exp\left(6\int a_1(x)dx\right)\left[4a_1(x)a_2(x) + (a_2(x))^{(1)}\right] = -2A\beta(\alpha + \beta s)^{-3} .$$

We can now eliminate s between the equations (4.3.34) and (4.3.35) to find the condition (4.3.22) that we have been trying to derive. From equation (4.3.34) it follows that

$$(\alpha + \beta s)^{-3} = \left(\frac{A}{a_2(x)}\right)^{-3/2}\exp\left(6\int a_1(x)dx\right) .$$

Substituting this expression for $(\alpha + \beta s)^{-3}$ into equation (4.3.35) it follows that s has been eliminated and

$$4a_1(x)a_2(x) + (a_2(x))^{(1)} = -2A^{-1/2}\beta(a_2(x))^{3/2} ,$$

which is

$$(4.3.36) \qquad (a_2(x))^{(1)} + 4a_1(x)a_2(x) + \gamma(a_2(x))^{3/2} = 0 \ ,$$

where

$$\gamma = 2A^{-1/2}\beta \ .$$

Equation (4.3.36) gives the condition (4.3.22) that we have been trying to derive, hence the necessity part of the proof is complete.

We now prove the sufficiency part of the proof.

Suppose that the condition (4.3.22) holds. Integrating this Bernoulli differential equation (4.3.22) gives

$$a_2(x) = \exp\left(-4\int a_1 dx\right)\left(c + \frac{\gamma}{2}\int \exp\left(-2\int a_1 dx\right)dx\right)^{-2} \ ,$$

where c is a constant. That is, equation (4.3.20) is of the form

$$(4.3.37) \qquad \frac{d^2y(x)}{dx^2} + 2a_1(x)\frac{dy}{dx} + \exp\left(-4\int a_1 dx\right)\left(c + \frac{\gamma}{2}\int \exp\left(-2\int a_1 dx\right)dx\right)^{-2} y(x) = 0 \ .$$

It is easy to verify that the

$T\left(\left(a_2(x)\right)^{1/2}\right) = T\left(\exp\left(-2\int a_1 dx\right)\left(c + \frac{\gamma}{2}\int \exp\left(-2\int a_1 dx\right)dx\right)^{-1}\right)$ transform of equation (4.3.37), defined by

$$\frac{d\tau}{dx} = \exp\left(-2\int a_1 dx\right)\left(c + \frac{\gamma}{2}\int \exp\left(-2\int a_1 dx\right)dx\right)^{-1}$$

and $y(x) = z(\tau)$, is a constant coefficient differential equation of the form of equation (4.3.21). The sufficiency is proven and moreover if the condition given by equation (4.3.22) holds, we see that the $u(x)$ of the hypothesis can be taken to be $(a_2(x))^{1/2}$.

Q.E.D.

We now compare Theorem (3.4.1) with Theorem (4.3.2). For the case $n = 2$ we consider the conditions of Theorem (3.4.1) that are given by equation (3.4.9). Recalling that the $u(x)$ of the hypothesis of Theorem (3.4.1) can be taken as $(a_2(x))^{1/2}$ when $n = 2$, we see that the conditions given by equation (3.4.9) are

$$\sum_{k=m}^{2}\left(\binom{2}{m}a_2(x)\right)^{-1}\binom{2}{k}a_{2-k}(x)\phi\left(k,\ m;\ (a_2(x))^{1/2}\right) = c_{2-m}\ ,$$

$$m = 0,1\ ,$$

where c_1 and c_2 are some constants. Using equations (2.4.3), (2.4.4) and (2.4.5), recalling that $(-j!)^{-1} = 0$ for positive integer j, we see that these conditions are

$$1 = c_2$$

and

$$a_1(x)(a_2(x))^{-1/2} + \frac{1}{4}(a_2(x))^{-3/2}(a_2(x))^{(1)} = c_1\ .$$

The first of these conditions is independent of the $a_i(x)$'s, hence vacuously it always holds. Easily the second condition is

$$(a_2(x))^{(1)} + 4a_1(x)a_2(x) - 4c_1(a_2(x))^{3/2} = 0\ ,$$

which is the same as the condition given by equation (4.3.22) of Theorem (4.3.2) since c_1 and γ are just some constants. We have shown that for the case $n = 2$ Theorem (3.4.1) can be derived from invariance considerations. Theorem (4.3.2) can be generalized (see Peyovitch [34]) to find the conditions of Theorem (3.4.1) for higher orders. Since Theorem (3.4.1) is just the normarlized form of Theorem (1.4.1), we see that Breuer and Gottlieb's [5] conditions, for at least the case $n = 2$, were obtained previously using invariance considerations (see also [17], [29], [32] and [33]). Breuer and Gottlieb's direct derivation of these conditions is much simpler than the derivation that makes use of invariance arguments.

(4.4) The Fundamental Relative Invariant. In this section we consider the function $V_3(a_i(x))$ that was mentioned in section (1.4). The function $V_3(a_i(x))$, formed from the coefficients $a_i(x)$ of the differential equation

$$(4.4.1) \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

is defined by

$$(4.4.2) \qquad V_3(a_i(x)) = -(a_1(x))^{(2)} + 3\left[(a_2(x))^{(1)} - 2a_1(x)(a_1(x))^{(1)}\right] - 2\left[a_3(x) - 3a_1(x)a_2(x) + 2(a_1(x))^3\right] .$$

Let $u(x)$ and $v(x)$ be arbitrary non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$, where n is the order of the differential equation (4.4.1) (we are assuming that $n \geq 3$). The $P(u(x), v(x))$ transform of (4.4.1), defined by

$\frac{dt}{dx} = u(x)$ and $y(x) = v(x)z(t)$, is

$$(4.4.3) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (b_0(t) \equiv 1) ,$$

where

$$(4.4.4) \qquad b_{n-s}(t) = (n-s)!(u^n v)^{-1} \sum_{k=0}^{n-s} \sum_{j=0}^{n-s-k} (k!j!(n-k-j)!)^{-1} a_k(x) s!$$

$$\cdot \phi(n-k-j, s; u(x))(v(x))^{(j)} ,$$

and ϕ is defined by equation (1.3.7).

Using equations (2.4.3) to (2.4.6), and equation (4.4.4) we find that

$$b_1(t) = u^{-1}a_1(x) + (2u^2)^{-1}(n-1)u^{(1)} + (uv)^{-1}v^{(1)} ,$$

$$b_2(t) = u^{-2}a_2(x) + u^{-3}(n-2)a_1(x)u^{(1)} + (4u^4)^{-1}(n-2)(n-3)(u^{(1)})^2$$

$$+ (3u^3)^{-1}(n-2)u^{(2)} + (u^2v)^{-1}2a_1(x)v^{(1)} + (u^2v)^{-1}v^{(2)}$$

$$+ (u^3v)^{-1}(n-2)u^{(1)}v^{(1)} ,$$

and

$$b_3(t) = u^{-3}a_3(x) + (2u^4)^{-1}3(n-3)a_2(x)u^{(1)} + (4u^5)^{-1}3(n-3)(n-4)a_1(x)(u^{(1)})^2$$

$$+ u^{-4}(n-3)a_1(x)u^{(2)} + (4u^4)^{-1}(n-3)u^{(3)} + (2u^5)^{-1}(n-3)(n-4)u^{(1)}u^{(2)}$$

$$+ (8u^6)^{-1}(n-3)(n-4)(n-5)(u^{(1)})^3 + (u^3v)^{-1}3a_2(x)v^{(1)}$$

$$+ (4u^5v)^{-1}(n-3)(n-4)3v^{(1)}(u^{(1)})^2 + (u^4v)^{-1}(n-3)u^{(2)}v^{(1)}$$

$$+ (u^3v)^{-1}3a_1(x)v^{(2)} + (2u^4v)^{-1}3(n-3)u^{(1)}v^{(2)}$$

$$+ (u^3v)^{-1}v^{(3)} + (u^4v)^{-1}3(n-3)a_1(x)u^{(1)}v^{(1)} .$$

We now form the same function of these $b_i(t)$'s that $V_3(a_i(x))$ is of the $a_i(x)$'s , that is

$$(4.4.5) \qquad V_3(b_i(t)) = -(b_1(t))_t^{(2)} + 3\left[(b_2(t))_t^{(1)} - 2b_1(t)(b_1(t))_t^{(1)}\right]$$
$$+ (-2)\left[b_3(t) - 3b_1(t)b_2(t) + 2(b_1(t))^3\right] .$$

Recalling that $\frac{dt}{dx} = u(x)$ we see that

$$\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx} = u^{-1}\frac{d}{dx}$$

and

$$\frac{d^2}{dt^2} = u^{-2}\frac{d^2}{dx^2} - u^{(1)}u^{-3}\frac{d}{dx} .$$

Using these formulas in equation (4.4.5) we find that

$$V_3(b_i(t)) = (u(x))^{-3}V_3(a_i(x)) .$$

That is, we have proven the following lemma which was originally published by Laguerre [25] for only the case $n = 3$.

<u>Lemma (4.4.1)</u>. Let

$$V_3(a_i(x)) = -a_1^{(2)} + 3(a_2^{(1)} - 2a_1a_1^{(1)}) - 2(a_3 - 3a_1a_2 + 2a_1^3)$$

be a function of the $a_i(x)$'s of equation (4.4.1). Let

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (b_0(t) \equiv 1) ,$$

be the $P(u(x), v(x))$ transform of equation (4.4.1), where $u(x)$ and $v(x)$ are arbitrary non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^{n}[a, b]$. The function $V_3(a_i(x))$ is related to the function $V_3(b_i(t))$, which is given by equation (4.4.5), by

$$(4.4.6) \qquad V_3(b_i(t)) = (u(x))^{-3} V_3(a_i(x)) .$$

Lemma (4.4.1) guarantees that $V_3(b_i(t(x)))$ is non-vanishing for $x \in [a, b]$ if $V_3(a_i(x))$ is non-vanishing on $[a, b]$. We have used this fact in Theorem (1.4.2).

Recalling equation (1.4.12) of definition (1.4.4) we have that M is the set of all matrices of the form

$$\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)} = \begin{pmatrix} a_0(x) & a_1(x) & \cdot\ \cdot & a_n(x) \\ \frac{da_0(x)}{dx} & \frac{da_1(x)}{dx} & \cdot\ \cdot & \frac{da_n(x)}{dx} \\ \vdots & & & \vdots \\ \frac{d^n}{dx^n} a_0(x) & & & \frac{d^n}{dx^n} a_n(x) \end{pmatrix}.$$

We now make the following definition.

<u>Definition (4.4.1)</u>. Let $u(x)$ and $v(x)$ be arbitrary non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and

$v(x) \in C^n[a, b]$. Let R be a map from M to the set of all complex valued functions with domain $[a, b]$. Let $a_j(x)$, $j = 0,1,\ldots,n$, be the coefficients of equation (4.4.1) and let $b_j(t)$, $j = 0,1,\ldots,n$, be the coefficients of the $P(u(x), v(x))$ transform of equation (4.4.1). If there exists an integer j , such that for all $x \in [a, b]$ and for all $u(x)$ and $v(x)$ as defined above we have the identity

$$(4.4.7) \qquad R\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv (u(x))^j R\left(\left(\frac{d^i}{dt^i} b_j(t)\right)_{(n+1)\times(n+1)}\right) ,$$

where $\frac{dt}{dx} = u(x)$, then the function R is called a <u>relative invariant</u> of <u>weight</u> <u>j</u> of equation (4.4.1).

Remark. We have that a relative invariant of equation (4.4.1) is a function of the coefficients $a_i(x)$ of equation (4.4.1) and their derivatives with respect to x . This function has the property that for all $x \in [a, b]$ it has the same value as the product of the same function formed from the coefficients of any arbitrary equation which is P equivalent to equation (4.4.1), with an integral power of $u(x)$. Recall that $u(x)$ connects the independent variables x and t of equations (4.4.1) and (4.4.3) respectively, by $\frac{dt}{dx} = u(x)$.

Note that in the left hand side of the identity (4.4.7) the derivatives are with respect to x , while in the right hand side they are with respect to t .

Definition (4.4.2). A relative invariant of the differential equation (4.4.1) is called a fundamental relative invariant if it is independent of the order n of equation (4.4.1).

We formed equation (4.4.3) by letting $u(x)$ and $v(x)$ be arbitrary functions, hence the $b_i(t)$'s of equation (4.4.3) represent the coefficients of an arbitrary equation which is P equivalent to equation (4.4.1). In view of the two definitions just given, we see that the function $V_3(a_i(x))$ is a fundamental relative invariant of weight 3 of equation (4.4.1).

In lieu of giving the motivation for Lemma (4.4.1) the author refers the reader to the works [4], [6], [7], [14], [15], [18], [19], [20], [30], [31] and [41], which deal with finding relative invariants and other related problems. From these works it is evident that it is possible, in theory at least, to find $n - 3$ other relative invariants of equation (4.4.1), of weights 4 to n. Call these other relative invariants $V_j(a_i(x))$, $j = 4,\ldots,n$. By definition they have the property that

$$V_j(b_i(t)) = (u(x))^{-j} V_j(a_i(x)) \,, \qquad j = 4,\ldots,n \,,$$

where as usual the $b_i(t)$'s are the coefficients of an arbitrary $P(u(x), v(x))$ transform of equation (4.4.1). Note that derivatives in $V_j(b_i(t))$ are taken with respect to t, where as usual $\frac{dt}{dx} = u(x)$. The relative invariants $V_j(a_i(x))$, $j = 4,\ldots,n$, are not fundamental relative invariants, that is they depend on the order of the differential equation (4.4.1). All of the relative invariants $V_j(a_i(x))$, $j = 3,\ldots,n$, of a given differential equation (4.4.1) of order $n \geq 3$, are independent in the sense that if one of them is identically zero, this does not necessarily mean that any of the others are also identically zero. As we will see in the next chapter the relative invariants $V_j(a_i(x))$, $j = 3,\ldots,n$, are important since they can be used to define canonical transforms of equation (4.4.1).

Chapter 5

Canonical Transformations

(5.1) Introduction. By definition (1.4.6), any differential equation P equivalent to

$$(5.1.1)\qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 \ , \qquad (a_0(x) \equiv 1) \ ,$$

that has the property that each of its coefficients is an absolute invariant of equation (5.1.1), is called a canonical transform of equation (5.1.1).

In section (5.2) we find a canonical transform of equation (5.1.1). This canonical transform is defined in terms of the fundamental relative invariant $V_3(a_i(x))$, which we saw in section (4.4). In section (5.3) we find other canonical transforms of equation (5.1.1), that are defined in terms of the relative invariants $V_j(a_i(x))$, $j = 4,\dots,n$. The canonical transforms we give were first studied by Halphen [19].

In section (5.4) we prove the following theorem.

Let

$$V_3(a_i(x)) = -(a_1(x))^{(2)} + 3\left((a_2(x))^{(1)} - 2a_1(x)(a_1(x))^{(1)}\right)$$
$$- 2\left(a_3(x) - 3a_1(x)a_2(x) + 2(a_1(x))^3\right)$$

be a function of the $a_i(x)$'s of

$$(5.1.2)\qquad \sum_{k=0}^{3} \binom{3}{k} a_{3-k}(x)(y(x))^{(k)} = 0 \ , \qquad (a_0(x) \equiv 1) \ .$$

If $V_3(a_i(x)) \equiv 0$ on $[a, b]$ then the general solution of equation (5.1.2) is

$$y(x) = \exp\left(-\int a_1(x)dx\right)\left(\xi(x)\right)^2\left(c_1 + c_2\int\left(\xi(x)\right)^{-2}dx + c_3\left(\int\left(\xi(x)\right)^{-2}dx\right)^2\right) ,$$

where c_1 , c_2 and c_3 are arbitrary non-zero constants and $\xi(x)$ is any non-trivial solution of

$$\left(\xi(x)\right)^{(2)} + \frac{3}{4}\left(a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}\right)\xi(x) = 0 .$$

In section (5.5) we give explicit expressions for some of the absolute invariants of equation (5.1.1).

(5.2) The Fundamental Canonical Transform. Recall that the $P(u(x), v(x))$ transform of

$$(5.2.1) \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x) \frac{d^k}{dx^k} y(x) = 0 , \qquad (a_0(x) \equiv 1) ,$$

is

$$(5.2.2) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t) \frac{d^k}{dt^k} z(t) = 0 , \qquad (b_0(t) \equiv 1) ,$$

where

$$(5.2.3) \qquad b_{n-\ell}(t) = (u^n v)^{-1}(n-\ell)! \sum_{k=0}^{n-\ell} \sum_{j=0}^{n-\ell-k} \frac{a_k(x)\ell!}{k!j!(n-k-j)!} \phi(n-k-j,\ell;u)v^{(j)} ,$$

for $\ell = 0,1,\ldots,n$.

We now prove the following theorem that has already been referred to in section (1.4). The proof is similar to that of Theorems (4.2.1) and (4.3.1). In the proof we will as usual use the notation that subscripting by a variable indicates the variable that differentiation is with respect to, if it is other than x . For example ϕ_τ indicates that derivatives in Faà de Bruno's Formula, given by equation (1.3.7), are to be taken with respect to τ rather than x . The theorem says that the $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.1) is a canonical transform of equation (5.2.1), where $V_3(a_i(x))$ is the fundamental relative invariant we saw in section (4.4) and C is an arbitrary non-zero constant. Before proving the theorem we again make a comment about constants of integration. If, for example, $a_1(x) = x$ then the $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right) =$ $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int x\,dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.1) is defined by

$$\frac{dt}{dx} = \left(V_3(a_i(x))C^{-1}\right)^{1/3}$$

and

$$\begin{aligned} y(x) &= \exp\left(-\int x\,dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}} z(t) \\ &= \exp\left(-\frac{x^2}{2} - c\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}} z(t) \ , \end{aligned}$$

where c is a constant of integration. We always take the constant of integration c , that stems from integrating $a_1(x)$ with respect to x , to be zero. That is, in the above example we have

$$y(x) = \exp\left(-\frac{x^2}{2}\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}} z(t) .$$

Taking the constant of integration to be zero results in no loss of generality since we are only interested in finding one particular canonical transform.

<u>Theorem (5.2.1)</u>. Let the order of equation (5.2.1) be 3 or greater. Assume that the function

$$V_3(a_i(x)) = -a_1^{(2)} + 3(a_2^{(1)} - 2a_1a_1^{(1)}) - 2(a_3 - 3a_1a_2 + 2a_1^3)$$

is non-vanishing on $[a, b]$. Moreover let C be an arbitrary non-zero constant. Under these assumptions the $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.1) is a canonical transform of equation (5.2.1).

<u>Proof</u>: The defining equations of the $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.1) are

$$\frac{dt}{dx} = \left(V_3(a_i(x))C^{-1}\right)^{1/3}$$

and

$$y(x) = \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}} \bar{y}(t) . \tag{5.2.4}$$

Note that we are taking the constant of integration, that stems from integrating $a_1(x)$ with respect to x , to be zero.

The $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3},\ \exp\left[-\int a_1(x)dx\right]\left(V\ (a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.1) is

$$(5.2.5)\qquad \sum_{k=0}^{n}\binom{n}{k} b_{n-k}(t)\,(\overline{y}(t))_t^{(k)} = 0\ , \qquad (b_0(t) \equiv 1)\ ,$$

where

$$(5.2.6)\qquad b_{n-\ell}(t) = \exp\left[\int a_1(x)dx\right]\left(V_3(a_i(x))C^{-1}\right)^{\frac{-1-n}{6}}(n-\ell)!$$

$$\cdot \sum_{k=0}^{n-\ell}\ \sum_{j=0}^{n-\ell-k} \frac{a_k(x)\ell!}{k!j!(n-k-j)!}\,\phi\left(n-k-j,\ell;\ \left(V_3(a_i(x))C^{-1}\right)^{1/3}\right)$$

$$\cdot \left(\exp\left[-\int a_1(x)dx\right]\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)^{(j)}$$

for $\ell = 0,1,\ldots,n$. Each $b_k(t)$ is a function of the $a_i(x)$'s and their derivatives with respect to x , hence in the notation of definition (1.4.4) we can write

$$(5.2.7)\qquad b_k(t) = B_k\left(\left(\frac{d^i}{dx^i}a_j(x)\right)_{(n+1)\times(n+1)}\right)\ , \qquad k = 0,1,\ldots,n\ .$$

To be done we need to show for $k = 0,1,\ldots,n$, that for all $x \in [a, b]$ the function $B_k\left(\left(\frac{d^i}{dx^i}a_j(x)\right)_{(n+1)\times(n+1)}\right)$ has the same value as the same function formed from the coefficients of any arbitrary equation which is P equivalent to equation (5.2.1).

Let $u(x)$ and $v(x)$ be arbitrary non-vanishing functions on $[a, b]$ such that $u(x) \in C^{n-1}[a, b]$ and $v(x) \in C^n[a, b]$. The $P(u(x), v(x))$ transform, defined by

$$\frac{d\tau}{dx} = u(x)$$

and

$$y(x) = v(x)z(\tau) ,$$

of equation (5.2.1) is

$$(5.2.8) \qquad \sum_{k=0}^{n} \binom{n}{k} c_{n-k}(\tau) \frac{d^k}{d\tau^k} z(\tau) = 0 , \qquad (c_0(\tau) \equiv 1) ,$$

where

$$(5.2.9) \qquad c_{n-\ell}(\tau) = (u^n v)^{-1} (n-\ell)! \sum_{k=0}^{n-\ell} \sum_{j=0}^{n-\ell-k} \frac{a_k(x)\ell!}{k!j!(n-k-j)!} \phi(n-k-j,\ell;u)v^{(j)} ,$$

for $\ell = 0,1,\ldots,n$. The coefficients $c_{n-\ell}(\tau)$ of equation (5.2.8) represent the coefficients of any arbitrary equation that is P equivalent to equation (5.2.1). To complete the proof we need to show that for all $x \in [a, b]$

$$(5.2.10) \qquad B_k\left(\left(\frac{d^i}{dx^i} a_j(x)\right)_{(n+1)\times(n+1)}\right) \equiv B_k\left(\left(\frac{d^i}{d\tau^i} c_j(\tau)\right)_{(n+1)\times(n+1)}\right) ,$$

$$k = 0,1,\ldots,n ,$$

where $\frac{d\tau}{dx} = u(x)$ and the function B_k is defined by equations (5.2.7) and (5.2.6). Note that derivatives and integrations are taken with respect to the independent variable of the function in question. Clearly the $B_k\left(\left(\frac{d^i}{d\tau^i} c_j(\tau)\right)_{(n+1)\times(n+1)}\right)$'s are just the coefficients of the $P\left(\left(V_3(c_i(\tau))C^{-1}\right)^{1/3},\ \exp\left(-\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.8), where

$$V_3(c_i(\tau)) = -(c_1(\tau))_\tau^{(2)} + 3\left((c_2(\tau))_\tau^{(1)} - 2c_1(\tau)(c_1(\tau))_\tau^{(1)}\right)$$
$$- 2\left(c_3(\tau) - 3c_1(\tau)c_2(\tau) + 2(c_1(\tau))^3\right) .$$

Letting the defining equations of this transform be

$$\frac{ds}{d\tau} = \left(V_3(c_i(\tau))C^{-1}\right)^{1/3}$$

and

$$z(\tau) = \exp\left(-\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{1-n}{6}} \bar{z}(s) ,$$

we find that the $P\left(\left(V_3(c_i(\tau))C^{-1}\right)^{1/3},\ \exp\left(-\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.2.8) is given by

$$(5.2.11) \qquad \sum_{k=0}^{n} \binom{n}{k} d_{n-k}(s) \frac{d^k}{ds^k} \bar{z}(s) = 0 , \qquad (d_0(s) \equiv 1) ,$$

where

$$(5.2.12) \quad d_{n-\ell}(s) = \exp\left(\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{-1-n}{6}} (n-\ell)!$$
$$\cdot \sum_{k=0}^{n-\ell} \sum_{j=0}^{n-\ell-k} \frac{c_k(\tau)\ell!}{k!j!(n-k-j)!}$$
$$\cdot \phi_\tau\left(n-k-j,\ \ell;\ \left(V_3(c_i(\tau))C^{-1}\right)^{1/3}\right)\left(\exp\left(-\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{1-n}{6}}\right)_\tau^{(j)}$$

for $\ell = 0,1,\ldots,n$.

Note that we are taking the constant of integration, that stems from integrating $c_1(\tau)$ with respect to τ, to be zero. It is important that this constant of integration be taken to be zero since we took the constant of integration when integrating $a_1(x)$ to be zero. By equations (5.2.9), (2.4.3) and (2.4.4) we have that

$$c_1(\tau) = \frac{a_1(x)}{u(x)} + \frac{n-1}{2}(u(x))^{-2}(u(x))^{(1)} + (u(x)v(x))^{-1}(v(x))^{(1)} .$$

Recalling that $\frac{d\tau}{dx} = u(x)$ we integrate $c_1(\tau)$ with respect to τ, taking the constant of integration to be zero, to obtain

$$\text{(5.2.13)} \qquad \int c_1(\tau)d\tau = \int c_1(\tau)u(x)dx$$

$$= \int a_1(x)dx + \frac{n-1}{2}\ell n\, u(x) + \ell n\, v(x) .$$

By equations (5.2.6), (5.2.7) and (5.2.12) we have as expected that

$$\text{(5.2.14)} \qquad B_k\left(\left(\frac{d^i}{d\tau^i}c_j(\tau)\right)_{(n+1)\times(n+1)}\right) = d_k(s) , \qquad k = 0,1,\ldots,n .$$

To show that the required identity (5.2.10) is true, it suffices to show that for all $x \in [a, b]$

$$b_k(t) = d_k(s) , \qquad k = 0,1,\ldots,n ,$$

(see equations (5.2.7) and (5.2.14)). That is, to be done we need only show that the differential equations (5.2.5) and (5.2.11) are the same. First we show that they have the same linearly independent solutions.

By equation (5.2.4) the n linearly independent solutions $\overline{y}_j(t)$, $j = 1,\dots,n$, of equation (5.2.5) are related to the n linearly independent solutions $y_j(x)$, $j = 1,\dots,n$, of equation (5.2.1) by

$$\overline{y}_j(t) = \exp\left(\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{n-1}{6}} y_j(x) , \qquad j=1,\dots,n . \tag{5.2.15}$$

We also have that $y(x) = v(x)z(\tau)$ and $z(\tau) = \exp\left(-\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{1-n}{6}} \overline{z}(s)$, hence

$$\overline{z}(s) = (v(x))^{-1}\exp\left(\int c_1(\tau)d\tau\right)\left(V_3(c_i(\tau))C^{-1}\right)^{\frac{n-1}{6}} y(x) . \tag{5.2.16}$$

From equation (5.2.13) it follows that

$$\exp\left(\int c_1(\tau)d\tau\right) = \exp\left(\int a_1(x)dx\right)(u(x))^{\frac{n-1}{2}} v(x) . \tag{5.2.17}$$

In section (4.4) we saw that $V_3(a_i(x))$ is related to $V_3(c_i(\tau))$ by

$$V_3(c_i(\tau)) = (u(x))^{-3}V_3(a_i(x)) . \tag{5.2.18}$$

Using equations (5.2.17) and (5.2.18) in equation (5.2.16) we obtain

$$\overline{z}(s) = \exp\left(\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{n-1}{6}} y(x) .$$

It follows that the n linearly independent solutions $\overline{z}_j(s)$, $j = 1,\dots,n$, of equation (5.2.11) are related to the n linearly independent solutions $y_j(x)$, $j = 1,\dots,n$, of equation (5.2.1) by

$$\bar{z}_j(s) = \exp\left(\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{n-1}{6}} y_j(x) \; , \qquad j=1,\ldots,n \; .$$

Comparing this equation with equation (5.2.15) we see that the differential equations (5.2.5) and (5.2.11) have the same n linearly independent solutions.

We now show that the independent variables t and s, of equations (5.2.5) and (5.2.11) respectively, are related by $\frac{ds}{dt} = 1$. Since $\frac{ds}{d\tau} = \left(V_3(c_i(\tau))C^{-1}\right)^{1/3}$ we can use equation (5.2.18) and $\frac{d\tau}{dx} = u(x)$ to obtain that

$$\begin{aligned} \frac{ds}{dx} &= \frac{d\tau}{dx}\frac{ds}{d\tau} \\ &= u(x)\left(V_3(a_i(x))C^{-1}(u(x))^{-3}\right)^{1/3} \\ &= \left(V_3(a_i(x))C^{-1}\right)^{1/3} \; . \end{aligned}$$

That is

$$\frac{ds}{dx} = \left(V_3(a_i(x))C^{-1}\right)^{1/3} \; .$$

Comparing this equation with $\frac{dt}{dx} = \left(V_3(a_i(x))C^{-1}\right)^{1/3}$, we see that $\frac{ds}{dt} = 1$, which is what we wanted to show.

Using exactly the same argument that we used in Theorem (4.3.1) we have for all $x \in [a, b]$ that

$$b_k(t(x)) = d_k(s(\tau(x))) \; , \qquad k = 0,1,\ldots,n \; .$$

Q.E.D.

Note that in a manner analogous to the direct proofs of Theorems (4.2.1) and (4.3.1) (see the remarks following those theorems), it is possible to prove Theorem (5.2.1) directly. This alternate proof, which makes use of the convolution Theorems (2.5.1) and (2.5.2), is extremely lengthy and it is omitted.

Halphen [19] omitted entirely his proof of Theorem (5.2.1), saying only that it was obvious.

Remark. The canonical transform, given by equation (5.2.5), of equation (5.2.1) depends on the fundamental relative invariant $V_3(a_i(x))$, hence it is referred to as the <u>fundamental canonical transform</u>.

(5.3) <u>Other Canonical Transforms</u>. We now assume that we can find, besides $V_3(a_i(x))$, the $n - 3$ other relative invariants $V_j(a_i(x))$, $j = 4,\ldots,n$, of equation (5.2.1). These functions have the property that

$$V_j(c_i(\tau)) = (u(x))^{-j} V_j(a_i(x)) ,$$

where $\frac{d\tau}{dx} = u(x)$ and the $c_i(\tau)$'s are the coefficients of an arbitrary $P(u(x), v(x))$ transform of equation (5.2.1). Note that derivatives in $V_j(c_i(\tau))$ are taken with respect to τ , where $\frac{d\tau}{dx} = u(x)$.

We have the following theorem.

Theorem (5.3.1). Let the order n of equation (5.2.1) be 3 or greater. Assume that $V_j(a_i(x))$ is the first non-vanishing relative invariant of equation (5.2.1) on $[a, b]$ where $3 \le j \le n$. Moreover let C be any non-zero constant. Under these assumptions the $P\left(\left(V_j(a_i(x))C^{-1}\right)^{1/j}, \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))C^{-1}\right)^{\frac{1-n}{2j}}\right)$ transform of equation (5.2.1) is a canonical transform of equation (5.2.1).

Proof: The proof follows exactly that of Theorem (5.2.1).

Q.E.D.

Remark. The case $j = 3$ of Theorem (5.3.1) is precisely Theorem (5.2.1).

Theorem (5.3.2). Let the order n of equation (5.2.1) be 3 or greater. Assume that $V_j(a_i(x))$ is the first non-vanishing relative invariant of equation (5.2.1) on $[a, b]$, where $3 \le j \le n$. Moreover let C be an arbitrary non-zero constant. Under these assumptions there exists a constant coefficient differential equation of the form

$$(5.3.1) \qquad \sum_{k=0}^{n} \binom{n}{k} c_{n-k} \frac{d^k}{dt^k} z(t) = 0 , \qquad (c_0 \equiv 1) ,$$

that is P equivalent to equation (5.2.1), if and only if the $P\left(\left(V_j(a_i(x))C^{-1}\right)^{1/j}, \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))C^{-1}\right)^{\frac{1-n}{2j}}\right)$ transform of equation (5.2.1) is a constant coefficient differential equation of the form of equation (5.3.1).

<u>Proof</u>: Since the $P\left(\left(V_j(a_i(x))C^{-1}\right)^{1/j}, \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))C^{-1}\right)^{\frac{1-n}{2j}}\right)$ transform of equation (5.2.1) is a canonical transform of equation (5.2.1) the proof follows that of Theorem (1.4.2).

Q.E.D.

Remark. The case $j = 3$ of Theorem (5.3.2) is precisely Theorem (1.4.2).

For an example consider the nth order Euler differential equation

$$\sum_{k=0}^{n} \binom{n}{k} x^{k-n} \frac{d^k}{dx^k} y(x) = 0 . \tag{5.3.2}$$

For this differential equation $a_i(x) = x^{-i}$, hence we find that $V_3(a_i(x)) = -2x^{-3}$. That is, $V_3(a_i(x))$ of equation (5.3.2) is non-vanishing for any real interval $[a, b]$, however the singularity $x = 0$ of $V_3(a_i(x))$ requires that we only consider intervals that exclude $x = 0$. Letting the arbitrary constant in Theorem (5.3.2) be $C = -2$, we find that $\left(V_3(a_i(x))(-2)^{-1}\right)^{1/3} = x^{-1}$, and $\exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))(-2)^{-1}\right)^{-1/3} = \exp(-\ell n\ x)x = 1$. Hence by Theorem (5.3.2) there exists a constant coefficient differential equation that is P equivalent to equation (5.3.2), if and only if the $P(x^{-1}, 1)$ transform of equation (5.3.2) is a constant coefficient differential equation. The defining equations of this transform are $\frac{dt}{dx} = \frac{1}{x}$ and $y(x) = z(t)$, from which $t = \ell n\ x$. It is well known that the transform defined by letting $t = \ell n\ x$ takes Euler differential equations to constant coefficient differential equations.

Theorem (5.3.2) suggests a method of handling the exceptional case of Theorem (1.4.2) where $V_3(a_i(x)) \equiv 0$ on $[a, b]$ and $n > 3$. When $V_3(a_i(x)) \equiv 0$ on $[a, b]$ we cannot use $V_3(a_i(x))$ to define a canonical transform of equation (5.2.1). We may however be able to find another canonical transform of equation (5.2.1) such that if equation (5.2.1) is P equivalent to a constant coefficient differential equation, then this canonical transform of equation (5.2.1) will be a constant coefficient differential equation. By Theorem (5.3.2) we see that we can determine such a canonical transform if we can find a relative invariant $V_j(a_i(x))$, $4 \le j \le n$, such that $V_j(a_i(x))$ does not vanish on $[a, b]$. As mentioned in section (4.4) reference material concerning the problem of finding these $V_j(a_i(x))$'s can be found in [4], [6], [7], [14], [15], [18], [19], [20], [30], [31] and [41].

(5.4) <u>The Third Order Exceptional Case</u>. In this section we consider the third order exceptional case to Theorem's (1.4.2) and (5.3.2). That is, we consider the differential equation

$$(5.4.1) \qquad \sum_{k=0}^{3} \binom{3}{k} a_{3-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

where $V_3(a_i(x)) \equiv 0$ on $[a, b]$.

<u>Theorem (5.4.1)</u>. Let $a_i(x)$, $i = 0,1,2,3$, be the coefficients of equation (5.4.1) and let c_1 , c_2 and c_3 be arbitrary non-zero constants. If the function

$$(5.4.2) \qquad V_3(a_i(x)) = -(a_1(x))^{(2)} + 3\left((a_2(x))^{(1)} - 2a_1(x)(a_1(x))^{(1)}\right)$$

$$- 2\left(a_3(x) - 3a_1(x)a_2(x) + 2(a_1(x))^3\right)$$

is identically zero on $[a, b]$, then the general solution of equation (5.4.1) is

$$(5.4.3) \qquad y(x) = \exp\left(-\int a_1(x)dx\right)\left(\xi(x)\right)^2\left(c_1 + c_2\int\left(\xi(x)\right)^{-2}dx + c_3\left(\int\left(\xi(x)\right)^{-2}dx\right)^2\right)$$

where $\xi(x)$ is any non-trivial solution of the differential equation

$$(5.4.4) \qquad \left(\xi(x)\right)^{(2)} + \frac{3}{4}\left(a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}\right)\xi(x) = 0 .$$

<u>Proof</u>: In section (3.3) we saw that the $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (5.4.1), defined by

$$y(x) = \exp\left(-\int a_1(x)dx\right)\bar{y}(x) ,$$

is

$$(5.4.5) \qquad \sum_{k=0}^{3}\binom{3}{k} b_{3-k}(x)(\bar{y}(x))^{(k)} = 0 , \qquad (b_0(x) \equiv 1) ,$$

where

$$b_1(x) = 0 ,$$

$$(5.4.6) \qquad b_2(x) = a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}$$

and

(5.4.7) $$b_3(x) = a_3(x) - 3a_1(x)a_2(x) + 2(a_1(x))^3 - (a_1(x))^{(2)} .$$

If we let $t = x$ and

$$y(x) = \exp\left(-\int a_1(x)dx\right)\overline{y}(t) = \exp\left(-\int a_1(x)dx\right)\overline{y}(x) ,$$

we see that the $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (5.4.1) is the same as the $P\left(1, \exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (5.4.1) (see Lemma (3.3.1)). Since $t = x$ it follows that $\frac{dt}{dx} = u(x) = 1$, hence equation (4.4.6) of Lemma (4.4.1) gives that

$$\begin{aligned} V_3(a_i(x)) &= V_3(b_i(x)) \\ &= -(b_1(x))^{(2)} + 3\left((b_2(x))^{(1)} - 2b_1(x)(b_1(x))^{(1)}\right) \\ &\quad - 2\left(b_3(x) - 3b_1(x)\ {}_2(x) + 2(b_1(x))^3\right) . \end{aligned}$$

Using the fact that $b_1(x) = 0$, this reduces to

(5.4.8) $$V_3(a_i(x)) = V\ (b_i(x)) = 3(b_2(x))^{(1)} - 2b_3(x) .$$

Equation (5.4.8) can be verified directly using equations (5.4.2), (5.4.6) and (5.4.7). By hypothesis $V_3(a_i(x)) \equiv 0$ on $[a, b]$, hence equation (5.4.8) gives that

(5.4.9) $$3(b_2(x))^{(1)} - 2b_3(x) = 0 , \qquad (x \in [a, b]) .$$

By the corollary to Theorem (3.7.2) the $P\left(\left(\xi(x)\right)^{-2}, \left(\xi(x)\right)^{2}\right)$ transform of equation (5.4.5), defined by

$$\frac{dt}{dx} = \left(\xi(x)\right)^{-2} \tag{5.4.10}$$

and

$$\bar{y}(x) = \left(\xi(x)\right)^{2} z(t) , \tag{5.4.11}$$

is

$$(z(t))_t^{(3)} - \frac{1}{2}\left(\xi(x)\right)^{6}\left(3(b_2(x))^{(1)} - 2b_3(x)\right) z(t) = 0 , \tag{5.4.12}$$

where $\xi(x)$ is any solution of

$$\left(\xi(x)\right)^{(2)} + \frac{3}{4} b_2(x)\xi(x) = 0 , \tag{5.4.13}$$

that does not vanish on $[a, b]$. By equation (5.4.9) we see that equation (5.4.12) is

$$(z(t))_t^{(3)} = 0 .$$

The general solution of this differential equation is

$$z(t) = c_1 + c_2 t + c_3 t^2 , \tag{5.4.14}$$

where c_1 , c_2 and c_3 are arbitrary non-zero constants. Using equations (5.4.10), (5.4.11) and (5.4.14) it follows that the general solution of equation (5.4.5) is

$$\bar{y}(x) = \left(\xi(x)\right)^2(c_1 + c_2 t + c_3 t^2)$$
$$= \left(\xi(x)\right)^2\left(c_1 + c_2\int\left(\xi(x)\right)^{-2}dx + c_3\left(\int\left(\xi(x)\right)^{-2}dx\right)^2\right) .$$

Since $y(x) = \exp\left(-\int a_1(x)dx\right)\bar{y}(x)$ it follows that the general solution of equation (5.4.1) is given by equation (5.4.3). Using equation (5.4.6) we see that equation (5.4.13) is equation (5.4.4).

Q.E.D.

Remark. Theorem (5.4.1) was known to Laguerre [25], Brioschi (see [6] and [7]), Halphen (see [19] and [20]) and Wilczynski [41]. It has also been proven independently by Kostenko [24].

It can be shown (see [7] and [41]) that if all the relative invariants $V_j(a_i(x))$, $j = 3,4,\ldots,n$, of

(5.4.15) $$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

are identically zero on $[a, b]$, then the general solution of equation (5.4.15) is

$$y(x) = \exp\left(-\int a_1(x)dx\right)\left(\xi(x)\right)^{n-1} \sum_{i=1}^{n} c_i\left(\int\left(\xi(x)\right)^{-2}dx\right)^{i-1} ,$$

where $\xi(x)$ is any non-trivial solution of

$$\left(\xi(x)\right)^{(2)} + \frac{3}{n+1}\left(a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}\right)\xi(x) = 0 ,$$

and c_i , $i = 1,\ldots,n$, are arbitrary non-zero constants.

We now illustrate Theorem (5.4.1) with an example. Consider the differential equation

$$\sum_{k=0}^{3} \binom{3}{k} x^{3-k}(y(x))^{(k)} = 0 , \qquad (x \in [a, b]) . \tag{5.4.16}$$

For equation (5.4.16) we have that $a_i(x) = x^i$, $i = 0,1,2,3$. We easily find that $V_3(a_i(x)) = 0$ for $x \in [a, b]$, where $[a, b]$ is any real interval. We also have that

$$a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)} = -1 .$$

By Theorem (5.4.1) the general solution of equation (5.4.16) is

$$y(x) = \exp\left(-\int x\,dx\right)\left(\xi(x)\right)^2\left(c_1 + c_2\int\left(\xi(x)\right)^{-2}dx + c_3\left(\int\left(\xi(x)\right)^{-2}\right)^2\right) ,$$

where $\xi(x)$ is any non-trivial solution of

$$\left(\xi(x)\right)^{(2)} - \frac{3}{4}\xi(x) = 0 .$$

A non-trivial solution of this equation is

$$\xi(x) = \exp\left(\frac{\sqrt{3}}{2}x\right) ,$$

hence the general solution of equation (5.4.16) is

$$\begin{aligned} y(x) &= \exp\left(-\frac{x^2}{2}\right)\exp(\sqrt{3}\,x)\left(c_1 + c_2\int\exp(-\sqrt{3}\,x)dx + c_3\left(\int\exp(-\sqrt{3}\,x)dx\right)^2\right) \\ &= \exp\left(\frac{2\sqrt{3}\,x - x^2}{2}\right)\left(c_1 + c_2\,\frac{\exp(-\sqrt{3}\,x)}{-\sqrt{3}} + c_3\,\frac{\exp(-2\sqrt{3}\,x}{3}\right) . \end{aligned}$$

That is, the general solution of equation (5.4.16) is of the form

$$(5.4.17) \qquad y(x) = \exp\left(-\frac{x^2}{2}\right)\left(c_1 \exp(\sqrt{3}\,x) + \bar{c}_2 + \bar{c}_3 \exp(-\sqrt{3}\,x)\right) ,$$

where c_1 , $\bar{c}_2$ and $\bar{c}_3$ are arbitrary non-zero constants.

Note that the general solution of equation (5.4.16), given by equation (5.4.17), could also have been found by applying Theorem (4.2.2) to equation (5.4.16). That is, the $S\left(\exp\left(-\int a_1(x)dx\right)\right) = S\left(\exp\left(-\frac{x^2}{2}\right)\right)$ transform of equation (5.4.16) is a constant coefficient differential equation.

(5.5) Some Absolute Invariants. Let us assume that C is an arbitrary non-zero constant and that the fundamental relative invariant

$$(5.5.1) \qquad V_3(a_i(x)) = -a_1^{(2)} + 3(a_2^{(1)} - 2a_1a_1^{(1)}) - 2(a_3 - 3a_1a_2 + 2a_1^3) ,$$

of

$$(5.5.2) \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

is non-vanishing on $[a, b]$. In section (5.2) we saw that the $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)$ transform of equation (5.5.2) is a canonical transform of equation (5.5.2). It is given by

$$(5.5.3) \qquad \sum_{k=0}^{n} \binom{n}{k} b_{n-k}(t)(z(t))_t^{(k)} = 0 , \qquad (b_0(t) \equiv 1) ,$$

where

$$(5.5.4) \qquad b_{n-s}(t) = (n-s)!\left(V_3(a_i(x))C^{-1}\right)^{\frac{-1-n}{6}} \exp\left(\int a_1(x)dx\right)$$

$$\cdot \sum_{j=0}^{n-s} \sum_{k=0}^{n-s-j} \frac{a_j(x)s!}{j!k!(n-j-k)!}$$

$$\cdot \phi\left(n-j-k,\ s;\ \left(V_3(a_i(x))C^{-1}\right)^{1/3}\right)\left(\exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{\frac{1-n}{6}}\right)^{(k)}$$

$s = 0,1,\ldots,n$. Recall that t and x are related by

$$(5.5.5) \qquad \frac{dt}{dx} = u(x) = \left(V_3(a_i(x))C^{-1}\right)^{1/3} .$$

By definition (1.4.6), of canonical transform, we have that each $b_k(t)$ given by equation (5.5.4) is an absolute invariant of equation (5.5.2). We now give explicit expressions, in terms of the $a_i(x)$'s of equation (5.5.2), for the absolute invariants $b_i(t)$, $i = 0,1,2,3$.

Making use of equations (5.5.4), (2.4.3) and (2.4.4) we obtain

$$(5.5.6) \qquad b_0(t) \equiv 1$$

and

$$(5.5.7) \qquad b_1(t) \equiv 0 .$$

We have that $b_0(t)$ and $b_1(t)$ are constants, hence they are trivially absolute invariants of equation (5.5.2). Using equations (5.5.4), (2.4.3), (2.4.4) and (2.4.5) we obtain

$$(5.5.8)\qquad b_2(t) = C^{2/3}\left(27\left(V_3(a_i(x))\right)^{8/3}\right)^{-1}\left[27\left(V_3(a_i(x))\right)^2(a_2 - a_1^2 - a_1^{(1)})\right.$$

$$\left. + \frac{7(n+1)}{4}\left(\left(V_3(a_i(x))\right)^{(1)}\right)^2 - \frac{3(n+1)}{2} V_3(a_i(x))\left(V_3(a_i(x))\right)^{(2)}\right] .$$

Recalling that $V_3(a_i(x))$ is a fundamental relative invariant of weight 3 (see equations (4.4.6) and (5.5.5)), we have that

$$V_3(b_i(t)) = \left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}\right)^{-3} V_3(a_i(x)) = C ,$$

where

$$V_3(b_i(t)) = -(b_1(t))_t^{(2)} + 3\left((b_2(t))_t^{(1)} - 2b_1(t)(b_1(t))^{(1)}\right)$$

$$- 2\left(b_3(t) - 3b_1(t)b_2(t) + 2(b_1(t))^3\right) .$$

By the identity (5.5.7) $b_1(t(x)) = 0$ for $x \in [a, b]$, hence

$$V_3(b_i(t)) = 3(b_2(t))_t^{(1)} - 2b_3(t) = C .$$

We have that

$$\begin{aligned} b_3(t) &= -\frac{1}{2}\left(C - 3(b_2(t))_t^{(1)}\right) \\ &= -\frac{1}{2}\left(C - 3\,\frac{dx}{dt}\,(b_2(t))^{(1)}\right) \\ &= -\frac{1}{2}\left(C - 3\left(V_3(a_i(x))C^{-1}\right)^{-1/3}(b_2(t))^{(1)}\right) , \end{aligned}$$

where we have made use of equation (5.5.5).

That is

$$(5.5.9)\qquad b_3(t) = -\frac{1}{2}\left(C - 3\left(V_3(a_i(x))C^{-1}\right)^{-1/3}(b_2(t))^{(1)}\right) ,$$

where $b_2(t)$ is given by equation (5.5.8).

Note that although it is much more difficult, we could have found equation (5.5.9) directly from equation (5.5.4) on making use of equations (2.4.3) to (2.4.6).

Equations (5.5.8) and (5.5.9) illustrate the complicated structure that absolute invariants have.

Halphen [19] gave the absolute invariants $b_2(t)$ and $b_3(t)$ for the case $C = 1$.

Chapter 6

Applications

(6.1) Introduction. In section (6.2) we find the solutions to the differential equation

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

when it is S equivalent, T equivalent and/or P equivalent to a constant coefficient differential equation. Section (6.3) contains a detailed look at 3rd order linear differential equations that are P equivalent to constant coefficient differential equations. In section (6.4) we give some examples of nth order differential equations that are P equivalent to constant coefficient differential equations.

(6.2) Solutions of Differential Equations that are Equivalent to Constant Coefficient Differential Equations.

We now find the solutions of the differential equation

$$\text{(6.2.1)} \qquad \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)(y(x))^{(k)} = 0 , \qquad (a_0(x) \equiv 1) ,$$

when it is S equivalent, T equivalent and/or P equivalent to constant coefficient differential equations.

By Theorem (4.2.2), there exists a constant coefficient differential equation

$$\text{(6.2.2)} \qquad \sum_{k=0}^{n} \binom{n}{k} c_{n-k} \frac{d^k}{dx^k} y(x) = 0 , \qquad (c_0 = 1) ,$$

that is S equivalent to equation (6.2.1), if and only if the $S\left(\exp\left(-\int a_1(x)dx\right)\right)$ transform of equation (6.2.1) is a constant coefficient differential equation of the form of equation (6.2.2). We have the following theorem.

Theorem (6.2.1). Let $a_1(x)$ of equation (6.2.1) be non-vanishing on $[a, b]$. If there exists a constant coefficient differential equation that is S equivalent to equation (6.2.1), then the general solution of equation (6.2.1) is

$$(6.2.3) \qquad y(x) = \sum_{k=1}^{m} \sum_{\ell=1}^{r_k} \beta_{k\ell} \exp\left(-\int a_1(x)dx\right) x^{\ell-1} \exp(\lambda_k x) \quad ,$$

where $\sum_{k=1}^{m} r_k = n$ and the $\beta_{k\ell}$'s are arbitrary non-zero constants. The r_k's are the multiplicities of the roots λ_k of the characteristic equation of equation (6.2.2).

Proof: The theorem follows immediately from Theorem (4.2.2) since the solutions $y_i(x)$, $i = 1,\ldots,n$, of equation (6.2.1) are related to the solutions $\overline{y}_i(x)$, $i = 1,\ldots,n$, of equation (6.2.2) by

$$\begin{aligned} y_i(x) &= \exp\left(-\int a_1(x)dx\right) \overline{y}_i(x) \quad , \\ &= \exp\left(-\int a_1(x)dx\right) \exp(\lambda_i x) \quad , \qquad i = 1,\ldots,n \; . \end{aligned}$$

Note that if the characteristic equation of equation (6.2.2) has no multiple roots, then $r_k = 1$, $k = 1,\ldots,n$, and the term $x^{\ell-1}$

in equation (6.2.3) is unity. We are done.

By Theorem (3.4.1), there exists a constant coefficient differential equation

$$(6.2.4) \qquad \sum_{k=0}^{n} \binom{n}{k} c_{n-k} \frac{d^k}{dt^k} z(t) = 0 \ , \qquad (c_0 = 1) \ ,$$

that is T equivalent to equation (6.2.1), if and only if the $T\left[(a_n(x))^{1/n}\right]$ transform of equation (6.2.1) is a constant coefficient differential equation of the form of equation (6.2.4). We have the following theorem which was proven in [5] by Breuer and Gottlieb.

<u>Theorem (6.2.2)</u>. Let $a_n(x)$ of equation (6.2.1) be non-vanishing on $[a, b]$. If there exists a constant coefficient differential equation that is T equivalent to equation (6.2.1), then the general solution of equation (6.2.1) is

$$(6.2.5) \qquad y(x) = \sum_{k=1}^{m} \sum_{\ell=1}^{r_k} \beta_{k\ell} \left(\int (a_n(x))^{1/n} dx\right)^{\ell-1} \exp\left(\lambda_k \int (a_n(x))^{1/n} dx\right) \ ,$$

where $\sum_{k=1}^{m} r_k = n$ and the $\beta_{k\ell}$'s are arbitrary non-zero constants. The r_k's are the multiplicities of the roots λ_k of the characteristic equation of equation (6.2.4).

<u>Proof</u>: The theorem follows immediately from Theorem (3.4.1) since the solutions $y_i(x)$, $i = 1,\ldots,n$, of equation (6.2.1) are related to the solution $z_i(t)$, $i = 1,\ldots,n$, of equation (6.2.4) by

$$\begin{aligned} y_i(x) &= z_i(t) \\ &= \exp\left(\lambda_i t\right) \\ &= \exp\left(\lambda_i \int (a_n(x))^{1/n} dx\right) , \qquad i = 1,\dots,n . \end{aligned}$$

Note that if the characteristic equation of equation (6.2.4) has no multiple roots, then $r_k = 1$, $k = 1,\dots,n$, and the term $\left(\int (a_n(x))^{1/n} dx\right)^{\ell-1}$ in equation (6.2.5) is unity. We are done.

Let $V_j(a_i(x))$ be the first non-vanishing relative invariant of equation (6.2.1) on $[a, b]$, where $3 \le j \le n$. By Theorem (5.3.2), there exists a constant coefficient differential equation

$$(6.2.6) \qquad \sum_{k=0}^{n} \binom{n}{k} c_{n-k} \frac{d^k}{dt^k} z(t) = 0 , \qquad (c_0 = 1) ,$$

that is P equivalent to equation (6.2.1), if and only if the $P\left(\left(V_j(a_i(x))\right)^{1/j}, \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))\right)^{\frac{1-n}{2j}}\right)$ transform of equation (6.2.1) is a constant coefficient differential equation of the form of equation (6.2.6). Note that we have taken the arbitrary constant C , of Theorem (5.3.2), to be 1. We have the following theorem (see [3], p. 4).

<u>Theorem (6.2.3)</u>. Let the order of equation (6.2.1) be 3 or greater and assume that $V_j(a_i(x))$ is the first non-vanishing relative invariant of equation (6.2.1) on $[a, b]$, where $3 \le j \le n$. If there exists a constant coefficient differential equation that is P equivalent to equation (6.2.1) then the general solution of equation (6.2.1) is

$$(6.2.7) \qquad y(x) = \sum_{k=1}^{m} \sum_{\ell=1}^{r_k} \beta_{k\ell} \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))\right)^{\frac{1-n}{2j}}$$

$$\cdot \left(\int \left(V_j(a_i(x))\right)^{1/j}dx\right)^{\ell-1} \exp\left(\lambda_k \int \left(V_j(a_i(x))\right)^{1/j}dx\right),$$

where $\sum_{k=1}^{m} r_k = n$ and the $\beta_{k\ell}$'s are arbitrary non-zero constants. The r_k's are the multiplicities of the roots λ_k of the characteristic equation of equation (6.2.6).

Proof: The theorem follows immediately from Theorem (5.3.2) since the solutions $y_i(x)$, $i = 1,\ldots,n$, of equation (6.2.1) are related to the solutions $z_i(t)$, $i = 1,\ldots,n$, of equation (6.2.6) by

$$\begin{aligned} y_i(x) &= \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))\right)^{\frac{1-n}{2j}} z_i(t) \\ &= \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))\right)^{\frac{1-n}{2j}} \exp(\lambda_i t) \\ &= \exp\left(-\int a_1(x)dx\right)\left(V_j(a_i(x))\right)^{\frac{1-n}{2j}} \exp\left(\lambda_i \int \left(V_j(a_i(x))\right)^{1/j}dx\right). \end{aligned}$$

Note that if the characteristic equation of equation (6.2.6) has no multiple roots, the term $\left(\int \left(V_j(a_i(x))\right)^{1/j}dx\right)^{\ell-1}$ in equation (6.2.7) is unity. We are done.

(6.3) Third Order Differential Equations that are Equivalent to Constant Coefficient Differential Equations.

In this section we consider the third order differential equation

$$(6.3.1) \qquad \sum_{k=0}^{3} \binom{3}{k} a_{3-k}(x)(y(x))^{(k)} = 0, \qquad (a_0(x) \equiv 1).$$

<u>Theorem (6.3.1)</u>. Let the relative invariant

$V_3(a_i(x)) = -(a_1(x))^{(2)} + 3\left((a_2(x))^{(1)} - 2a_1(x)(a_1(x))^{(1)}\right)$

$- 2\left(a_3(x) - 3a_1(x)a_2(x) + 2(a_1(x))^3\right)$, of equation (6.3.1), be non-vanishing on $[a, b]$. There exists a constant coefficient differential equation that is P equivalent to equation (6.3.1), if and only if

$$(6.3.2) \qquad 27\left(V_3(a_i(x))\right)^2\left(a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}\right)$$

$$+ 7\left(\left(\left(V_3(a_i(x))\right)^{(1)}\right)^2\right) - 6V_3(a_i(x))\left(V_3(a_i(x))\right)^{(2)}$$

$$= c27\left(V_3(a_i(x))\right)^{8/3} ,$$

where c is some constant.

<u>Proof:</u> By Theorem (5.3.2), there exists a constant coefficient differential equation that is P equivalent to equation (6.3.1), if and only if the $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \left(\exp -\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{-1/3}\right)$ transform of equation (6.3.1) is a constant coefficient differential equation, where C is an arbitrary non-zero constant. This $P\left(\left(V_3(a_i(x))C^{-1}\right)^{1/3}, \exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))C^{-1}\right)^{-1/3}\right)$ transform of equation (6.3.1) is given by (see equation (5.5.3) to (5.5.9) with $n = 3$)

$$(6.3.3) \qquad \sum_{k=0}^{3} \binom{3}{k} b_{3-k}(t)(z(t))_t^{(k)} = 0 ,$$

where

$$b_0(t) \equiv 1 ,$$

$$b_1(t) \equiv 0 ,$$

(6.3.4) $$b_2(t) = C^{2/3}\left(27\left(V_3(a_i(x))\right)^{8/3}\right)^{-1}$$
$$\cdot\left[27\left(V_3(a_i(x))\right)^2\left(a_2(x) - (a_1(x))^2 - (a_1(x))^{(1)}\right)\right.$$
$$\left. + 7\left(\left(V_3(a_i(x))\right)^{(1)}\right)^2 - 6V_3(a_i(x))\left(V_3(a_i(x)\right)^{(2)}\right]$$

and

$$b_3(t) = -\frac{1}{2}\left(C - 3\left(V_3(a_i(x))C^{-1}\right)^{-1/3}(b_2(t))^{(1)}\right) .$$

Clearly $b_3(t)$ reduces to the constant $-\frac{C}{2}$ when $b_2(t)$ is a constant, hence we see that equation (6.3.3) is a constant coefficient differential equation if and only if $b_2(t)$ is a constant. The condition given by equation (6.3.2) now follows immediately from equation (6.3.4).

Q.E.D.

We now consider some examples, taken from [23], of third order differential equations that are P equivalent to constant coefficient differential equations.

The differential equation

(6.3.5) $$(y(x))^{(3)} - x^{-6}y(x) = 0 ,$$

has $a_1(x) \equiv a_2(x) \equiv 0$, $a_3(x) = -x^{-6}$ and $V_3(a_i(x)) = 2x^{-6}$. Letting $C = 2$ in Theorem (5.3.2), we find that equation (6.3.5) is P equivalent to the constant coefficient differential equation

$$(z(t))_t^{(3)} - z(t) = 0 . \tag{6.3.6}$$

Equation (6.3.6) has the characteristic equation

$$\lambda^3 - 1 = 0 . \tag{6.3.7}$$

Letting λ_k , $k = 1,2,3$ be the distinct roots of equation (6.3.7), the general solution of equation (6.3.5) is given by (see Theorem (6.2.3))

$$y(x) = \sum_{k=1}^{3} \beta_k \, x^2 \exp(-\lambda_k x^{-1}) .$$

This solution is valid on any interval $[a, b]$ not including $x = 0$.

Consider the differential equation

$$(y(x))^{(3)} + 3\,\frac{ax^{2\nu}+1-\nu^2}{3x^2}\,y^{(1)} + \frac{bx^{3\nu}+a(\nu-1)x^{2\nu}+\nu^2-1}{x^3}\,y(x) = 0 , \tag{6.3.8}$$

where a, b and ν are constants and $b \neq 0$. For this equation $a_1(x) \equiv 0$, $a_2(x) = \dfrac{ax^{2\nu}+1-\nu^2}{3x^2}$, $a_3(x) = \dfrac{bx^{3\nu}+a(\nu-1)x^{2\nu}+\nu^2-1}{x^3}$ and $V_3(a_i(x)) = -2bx^{3\nu-3}$. Letting $C = -2b$ in Theorem (5.3.2), we find that equation (6.3.8) is P equivalent to the constant coefficient differential equation

$$(z(t))_t^{(3)} + a(z(t))_t^{(1)} + bz(t) = 0 . \tag{6.3.9}$$

Equation (6.3.9) has the characteristic equation

$$\lambda^3 + a\lambda + b = 0 . \tag{6.3.10}$$

By Theorem (6.2.3), the general solution of equation (6.3.8) is

$$y(x) = \sum_{k=1}^{m} \sum_{\ell=1}^{r_k} \beta_{k\ell} x^{1-\nu} \left(\frac{x^\nu}{\nu}\right)^{\ell-1} \exp\left(\lambda_k \frac{x^\nu}{\nu}\right) ,$$

where $\sum_{k=1}^{m} r_k = 3$ and the λ_k's are the roots, of multiplicities r_k, of equation (6.3.10). This solution is valid for any interval $[a, b]$ not including $x = 0$.

Consider the differential equation

$$(y(x))^{(3)} + 3\frac{2}{x}(y(x))^{(2)} + 3\frac{2}{x^2}(y(x))^{(1)} + ay(x) = 0 , \tag{6.3.11}$$

where a is a non-zero constant. For this equation $a_1(x) = \frac{2}{x}$, $a_2(x) = \frac{2}{x^2}$, $a_3(x) = a$ and $V_3(a_i(x)) = -2a$. Letting $C = -2a$ in Theorem (5.3.2), we find that equation (6.3.11) is P equivalent to the constant coefficient differential equation

$$(z(t))_t^{(3)} + az(t) = 0 . \tag{6.3.12}$$

Equation (6.3.12) has the characteristic equation

$$\lambda^3 + a = 0 . \tag{6.3.13}$$

Letting λ_k, $k = 1,2,3$, be the distinct roots of equation (6.3.13), the general solution of equation (6.3.11) (see Theorem (6.2.3)) is

$$y(x) = \sum_{k=1}^{3} \beta_k x^{-2} \exp(\lambda_k x) .$$

This solution is valid for any interval $[a, b]$ not including $x = 0$.

The following two examples illustrate the fact that a given differential equation may be P equivalent to more than one constant coefficient differential equation.

The $P(x^{-1}, x)$ transform of

$$(y(x))^{(3)} - \frac{1}{2} x^{-3} y(x) = 0 , \tag{6.3.14}$$

defined by

$$\frac{dt}{dx} = x^{-1} \tag{6.3.15}$$

and

$$y(x) = xz(t) , \tag{6.3.16}$$

is

$$(z(t))_t^{(3)} - (z(t))_t^{(1)} - \frac{1}{2} z(t) = 0 . \tag{6.3.17}$$

The $T(x^{-1})$ transform of equation (6.3.14), defined by

$$\frac{dt}{dx} = x^{-1} \tag{6.3.18}$$

and

$$y(x) = \xi(t) , \tag{6.3.19}$$

is

$$(\xi(t))_t^{(3)} - 3(\xi(t))_t^{(2)} + 2(\xi(t))_t^{(1)} - \frac{1}{2} \xi(t) = 0 . \tag{6.3.20}$$

By Lemma (3.4.2) equation (6.3.20) is the same as the $P(x^{-1}, 1)$ transform of equation (6.3.14), hence from equations (6.3.17) and (6.3.20) we see that equation (6.3.14) is P equivalent to more than one constant coefficient differential equation. From equations (6.3.15), (6.3.16) and (6.3.17) we find that the solutions of equation (6.3.14) are

$$\begin{aligned} y(x) &= xz(t) \\ &= x \exp(\lambda_k t) \\ &= x \exp(\lambda_k \ell n\ x) \\ &= x^{\lambda_k + 1}, \end{aligned}$$

where λ_k , $k = 1,2,3$, is a root of

(6.3.21) $$\lambda^3 - \lambda - \frac{1}{2} = 0 .$$

That is

(6.3.22) $$y(x) = x^{\lambda_k+1}, \qquad k = 1,2,3 .$$

From equations (6.3.18), (6.3.19) and (6.3.20) we find that the solutions of equation (6.3.14) are also given by

$$\begin{aligned} y(x) &= \xi(t) \\ &= \exp(r_k t) \\ &= \exp(r_k \ell n\ x) \\ &= x^{r_k} , \end{aligned}$$

where r_k , $k = 1,2,3$, is a root of

(6.3.23) $$r^3 - 3r^2 + 2r - \frac{1}{2} = 0 .$$

That is

(6.3.24) $$y(x) = x^{r_k} , \qquad k = 1,2,3 .$$

From equations (6.3.22) and (6.3.24) we expect that $r_k = \lambda_k + 1$, which is easy to verify. That is, equation (6.2.23) becomes equation (6.3.21) where r is replaced by $\lambda + 1$.

The $P\left(\exp(-\sqrt{3}\ x),\ \exp\left(\frac{2\sqrt{3}\ x - x^2}{2}\right)\right)$ transform of

(6.3.25) $$(y(x))^{(3)} + 3x(y(x))^{(2)} + 3x^2(y(x))^{(1)} + x^3 y(x) = 0 ,$$

defined by

(6.3.26) $$\frac{dt}{dx} = \exp(-\sqrt{3}\ x)$$

and

(6.3.27) $$y(x) = \exp\left(\frac{2\sqrt{3}\ x - x^2}{2}\right) z(t) ,$$

(see section (5.4)) is

$$(z(t))_t^{(3)} = 0 \tag{6.3.28}$$

The $S\left(\exp\left(-\frac{x^2}{2}\right)\right)$ transform of equation (6.3.25), defined by

$$y(x) = \exp\left(-\frac{x^2}{2}\right)\bar{y}(x) , \tag{6.3.29}$$

is

$$(\bar{y}(x))^{(3)} - 3(\bar{y}(x))^{(1)} = 0 . \tag{6.3.30}$$

By Lemma (3.3.1) equation (6.3.30) is the same as the $P\left(1, \exp\left(-\frac{x^2}{2}\right)\right)$ transform of equation (6.3.25), hence from equations (6.3.28) and (6.3.30) we see that equation (6.3.25) is P equivalent to more than one constant coefficient differential equation. From equations (6.3.26), (6.3.27) and (6.3.28) we find that the solutions of equation (6.3.25) are

$$\begin{aligned} y(x) &= \exp\left(\frac{2\sqrt{3}\,x - x^2}{2}\right) z(t) \\ &= \exp\left(\frac{2\sqrt{3}\,x - x^2}{2}\right) t^j \\ &= \exp\left(\frac{2\sqrt{3}\,x - x^2}{2}\right)\left(\frac{\exp(-\sqrt{3}\,x)}{-\sqrt{3}}\right)^j , \quad j = 0,1,2 . \end{aligned}$$

The factor $(-\sqrt{3})^{-j}$ can be replaced by one without loss of generality, hence

$$y(x) = \exp\left(\frac{2\sqrt{3}\,x - x^2}{2}\right)\left(\exp(-\sqrt{3}\,x)\right)^j , \quad j = 0,1,2 . \tag{6.3.31}$$

From equations (6.3.29) and (6.3.30) we find that the solutions of equation (6.3.25) are also given by

$$\begin{aligned} y(x) &= \exp\left(-\frac{x^2}{2}\right)\bar{y}(x) \\ &= \exp\left(-\frac{x^2}{2}\right)\exp(\lambda_k x) \\ &= \exp\left(\frac{2\lambda_k x - x^2}{2}\right) \end{aligned}$$

where λ_k , $k = 1,2,3$, is a root of

$$\lambda^3 - 3\lambda = 0 \ . \tag{6.3.32}$$

Equation (6.3.32) has the 3 distinct roots $\lambda_1 = \sqrt{3}$, $\lambda_2 = 0$ and $\lambda_3 = -\sqrt{3}$, hence

$$y(x) = \exp\left(\frac{2\lambda_k x - x^2}{2}\right), \tag{6.3.33}$$

where $\lambda_1 = \sqrt{3}$, $\lambda_2 = 0$ and $\lambda_3 = -\sqrt{3}$. Comparing equation (6.3.33) with equation (6.3.31) we see that we have obtained the same solutions of equation (6.3.25) by transforming it to two different constant coefficient differential equations.

(6.4) <u>Nth Order Differential Equations that are Equivalent to Constant Coefficient Differential Equations</u>.

We now give some examples of nth order differential equations,

$$\sum_{k=0}^{n} \binom{n}{k} a_{n-k}(x)\,(y(x))^{(k)} = 0 \ , \qquad (a_0(x) \equiv 1) \ ,$$

that are P equivalent to constant coefficient differential equations.

As noted in section (5.3) it is well known that the $P(x^{-1}, 1)$ transform of the Euler differential equation (5.3.2),

$$\sum_{k=0}^{n} \binom{n}{k} x^{k-n}(y(x))^{(k)} = 0 , \tag{6.4.1}$$

is a constant coefficient differential equation. The solutions of equation (6.4.1) are of the form $y(x) = x^{\lambda}$ where λ is a constant.

We now consider the differential equation

$$(y(x))^{(n)} + \sum_{k=0}^{n-1} \binom{n}{k}(n - k)!\,(1 + \frac{1}{x})(y(x))^{(k)} = 0 . \tag{6.4.2}$$

This equation has $a_i(x) = i!\,(1 + \frac{1}{x})$, $i = 1,2,3$, hence $V_3(a_i(x))$ is easily found to be -4. Letting the arbitrary constant in Theorem (5.3.2) be $C = -4$, we find that $\left(V_3(a_i(x))(-4)^{-1}\right)^{1/3} = 1$ and

$$\begin{aligned}\exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))(-4)^{-1}\right)^{\frac{1-n}{6}} &= \exp\left(-\int a_1(x)\right) \\ &= \exp\left(-\int\left(1 + \frac{1}{x}\right)dx\right) \\ &= \frac{\exp(-x)}{x} .\end{aligned}$$

By Theorem (5.3.2), there exists a constant coefficient differential equation that is P equivalent to equation (6.4.2), if and only if the $P\left(1, \frac{\exp(-x)}{x}\right)$ transform of equation (6.4.2) is a constant coefficient differential equation. Recalling that $P\left(1, \frac{\exp(-x)}{x}\right) = S\left(\frac{\exp(-x)}{x}\right)$ (see Lemma (3.3.1)), we find that the

$P\left(1, \frac{\exp(-x)}{x}\right)$ transform of equation (6.4.2), defined by $y(x) = \frac{\exp(-x)}{x}\,\overline{y}(x)$, is a constant coefficient differential equation. That is, $\overline{y}(x)$ is of the form $\exp(\lambda x)$, hence the solutions of equation (6.4.2) are of the form $y(x) = \frac{\exp((\lambda-1)x)}{x}$, where λ is a constant.

We now consider the differential equation

$$\text{(6.4.3)} \qquad \sum_{k=0}^{n} \binom{n}{k} (1 + x)^k x^{n-k} (y(x))^{(k)} = 0 .$$

The coefficient of $(y(x))^{(n)}$ in equation (6.4.3) is $(1 + x)^n$, hence we must divide equation (6.4.3) through by $(1 + x)^n$ to normalize its leading coefficient to 1. Carrying out this division we find that equation (6.4.3) is equivalent to

$$\text{(6.4.4)} \qquad \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{1 + x}\right)^{n-k} (y(x))^{(k)} = 0 ,$$

which has $a_i(x) = \left(\frac{x}{1 + x}\right)^i$. We easily find that $V_3(a_i(x))$ of equation (6.4.4) is $\frac{2}{(1 + x)^3}$. Letting the arbitrary constant C in Theorem (5.3.2) be $C = 2$, we find that $\left(V_3(a_i(x))2^{-1}\right)^{1/3} = (1+x)^{-1}$ and

$$\begin{aligned}
\exp\left(-\int a_1(x)dx\right)\left(V_3(a_i(x))2^{-1}\right)^{\frac{1-n}{6}} &= \exp\left(-\int \frac{x}{1+x}\, dx\right)(1 + x)^{\frac{n-1}{2}} \\
&= \exp\left(-1 - x + \ln(1 + x)\right)(1 + x)^{\frac{n-1}{2}} \\
&= \exp(-(1 + x))(1 + x)^{\frac{n+1}{2}} .
\end{aligned}$$

By Theorem (5.3.2), there exists a constant coefficient differential equation that is P equivalent to equation (6.4.4), if and only if the $P\left((1+x)^{-1}, \exp(-(1+x))(1+x)^{\frac{n+1}{2}}\right)$ transform of equation (6.4.4) is a constant coefficient differential equation. Assuming that the $P\left((1+x)^{-1}, \exp(-(1+x))(1+x)^{\frac{n+1}{2}}\right)$ transform of equation (6.4.4) is a constant coefficient differential equation, with dependent variable $z(t)$, we must have that

$$y(x) = \exp(-(1+x))(1+x)^{\frac{n+1}{2}} z(t) .$$

By assumption $z(t)$ is the solution of a constant coefficient differential equation, where $t = \int (1+x)^{-1} dx$, hence we have that $z(t)$ is of the form

$$\begin{aligned} z(t) &= \exp(\lambda t) \\ &= \exp\left(\lambda \int (1+x)^{-1} dx\right) \\ &= (1+x)^{\lambda} , \end{aligned}$$

where λ is a constant. We must have that

$$\begin{aligned} y(x) &= \exp(-(1+x))(1+x)^{\frac{n+1}{2}}(1+x)^{\lambda} \\ &= \exp(-1)\exp(-x)(1+x)^{\frac{n+1+2\lambda}{2}} . \end{aligned}$$

Since the factor $\exp(-1)$ is a constant, it can be replaced by one, hence we see that $y(x)$ is of the form

$$y(x) = \exp(-x)(1+x)^r , \tag{6.4.5}$$

if equation (6.4.4) is P equivalent to a constant coefficient differential equation. The solutions $y(x)$, of equations (6.4.3) and (6.4.4), are known to be of the form given by equation (6.4.5) (see Allaway [1]), where r is a root of the Poisson-Charlier polynomial $\{c_n(r;\ 1)\}$ defined by (see Szegö [40, p. 35])

$$\exp(-x)(1+x)^r = \sum_{\ell=0}^{\infty} c_\ell(r;\ 1)\,\frac{x^\ell}{\ell!} .$$

Clearly our assumption, that the $P\left((1+x)^{-1},\ \exp\left(-(1+x)\right)(1+x)^{\frac{n+1}{2}}\right)$ transform of equation (6.4.4) is a constant coefficient differential equation, was justified. Note that to actually effect this transformation of equation (6.4.4), for arbitrarily high n , is not practical because of the computations that would be involved. That is, although Theorem (5.4.2) is true for arbitrarily high n it is not practical, in general, to try and effect $P(u(x),\ v(x))$ transforms of differential equations of orders greater than 3 or 4.

Appendix

Interchange of Summation Formulas

Rainville [36], p. 57, shows that

$$\text{(A.1.1)} \qquad \sum_{k=0}^{\infty} \sum_{j=0}^{k} P(k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} P(k + j, j) ,$$

where P is a function of k and j .

We easily obtain the following lemma from equation (A.1.1).

Lemma (A.1.1). Let n be a positive integer and let P be a function of k and j , then

$$\text{(A.1.2)} \qquad \sum_{k=0}^{n} \sum_{j=0}^{k} P(k, j) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} P(k + j, j) .$$

Proof: Letting

$$\lambda_k = \begin{cases} 1 & k \le n \\ 0 & k > n \end{cases} ,$$

we have

$$\sum_{k=0}^{n} \sum_{j=0}^{k} P(k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \lambda_k P(k, j) .$$

By equation (A.1.1) this becomes

$$\sum_{k=0}^{n} \sum_{j=0}^{k} P(k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \lambda_{k+j} P(k + j, j) ,$$

where

$$\text{(A.1.3)} \qquad \lambda_{k+j} = \begin{cases} 1 & k + j \leq n \\ 0 & k + j > n \end{cases}$$

It is easy to see that the infinite k and j summations above can be interchanged, hence

$$\sum_{k=0}^{n} \sum_{j=0}^{k} P(k, j) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_{k+j} P(k + j, j) .$$

By equation (A.1.3) this becomes

$$\sum_{k=0}^{n} \sum_{j=0}^{k} P(k, j) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} P(k + j, j) .$$

Q.E.D.

Corollary.

$$\text{(A.1.4)} \qquad \sum_{k=0}^{n} \sum_{j=0}^{k} P(k, j) = \sum_{j=0}^{n} \sum_{k=j}^{n} P(k, j) .$$

Proof: Raising the k index of sumation in the right hand side of equation (A.1.2) by j we immediately obtain equation (A.1.4).

Q.E.D.

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