SET-BASED COMPUTATIONS

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By
Nasser Noroozi
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ABSTRACT

The representation of uncertain information and inference with such information are some of the fundamental issues in uncertainty management. Conventional methods for uncertainty management usually use a single value with the exception of interval-based approaches. In interval-based approaches, an interval is used to represent the uncertain information. It is assumed that the true, possibly unknown, value lies in an interval. However, in order to use interval-based methods, there must exist an order relation on the set of data values.

The main objective of this thesis is to extend single-valued and interval-valued methods by introducing a framework of set-valued computations. In this model, uncertain information is described by a set without any further restrictions. Basic issues of set-based computations are investigated. Operations on set values are defined based on the corresponding point-based (i.e., single-value-based) operations on their members. The properties of set-based computations are examined in connection to the corresponding properties of the point-based computations. Within the proposed framework, a critical analysis of a number of existing set-based computation methods is presented. This provides further evidence supporting the proposed model. To a large extent, the present study may be regarded as a more explicit re-examination of methods that have been implicitly used in many studies, using a unified notion. The results of such an investigation will be useful in establishing a framework for more systematic study of set-based computations.

The set-based methods can be applied in a number of areas, such as interval-number algebra, interval-set algebra, interval-valued logic, interval-valued probabilistic reasoning, and set-based information systems. These applications are studied in this thesis.

Chapter 1

INTRODUCTION

One of the fundamental issues in information science is management of uncertain information. There are two basic problems involved: representation of uncertain information and reasoning using uncertain information.

Traditionally, probability theory is considered to be the only tool for uncertainty management. It can be applied in situations where the source of uncertainty is due to the presence of random variables. However, this is not the only source of uncertainty. There are situations where uncertainty stems from not clearly defined notions rather than randomness (Zadeh, 1965). In order to better manage uncertainty, different proposals have been made. In general, there are two different approaches to represent uncertain information, one is the use of a numeric structure, the other is the use of a non-numeric structure (Bhatnagar and Kanal, 1986). Among methods of the first approach (quantitative approach), the best known is Bayesian approach using probability functions (Neapolitean, 1990; Pearl, 1988; Shafer, 1976). The second approach (qualitative approach) is particularly useful for modeling uncertainty when numeric values are not readily available (Bhatnagar and Kanal, 1986; Fine, 1973; Luzeaux, 1991; Satoh, 1989). Examples of such approaches are fuzzy set theory (Zadeh, 1965), rough set theory (Pawlak, 1982), incidence calculus (Bundy, 1985), and interval-set algebra (Yao, 1993).

The notion of intervals is used in both two approaches. For example, in quantitative approach, to describe uncertainty one can construct an interval using a pair of belief and plausibility functions in which lies the true probability (Dubois and Prade, 1986; Halpern and Fagin, 1992; Shafer, 1976; Smets, 1988). For the second approach, the notion of intervals is an underlying concept. For instance, rough set theory, incidence calculus, and interval-set algebra all use intervals of poset (i.e., partially ordered set). It is obvious that these interval-based structures are based upon an order relation implicitly defined over their universes. The requirement of such an order relation restricts applications of these models. In this thesis, single-valued and interval-valued methods are extended by introducing a framework of set-valued computations.

Set-based computations can be motivated from two different point of views. One is the need for the representation of uncertain information when there is no order relation defined on the universe. The other is the need of a unified framework for the computations involved in inference processes using set-valued parameter or functions instead of single-valued ones. In many practical situations, it may be impossible to specify the exact values of certain parameters under consideration. For example, it may be difficult to measure the precise value of the temperature at the moment. One can therefore give only a lower bound and an upper bound to indicate its range, i.e., using interval for representing this information. There may also be situations where it is impossible to represent available information by an interval. For example, suppose a given person may speak English or French, which cannot be represented by an interval. It may be more reasonable to represent such information by a set. Intervals can be considered as sets of elements. The framework of set-based computations is therefore a natural generalization of interval-based computations.

The main objective of this thesis is to investigate some basic issues of set-based computations. The underlying assumptions of interval based models will be examined.

Possible extensions of computations in point-valued systems to set-valued ones will be studied. Operations and relations on set-valued parameters are defined based on the corresponding operations and relations available from single-valued parameters. These operations and relations are set-wise extensions of the original ones. Extended operations and relations are analyzed and their properties are given. It will be shown that the paradigm of set-based computations has been used implicitly in many studies.

The rest of this thesis is organized as follows. In Chapter 2, a brief review of basic notations and previous related works is provided. In Chapter 3, a framework of set-based computations is introduced and discussed. Extended operations and relations are defined and their properties are investigated. Set-based extension of a relational system is defined. Set-based extension of some basic structures in uncertainty management is investigated. In Chapter 4, interval-based computations, a special case of set-based computations, is discussed. In Chapter 5, set-based information systems and multi-valued logic are discussed as applications of set-based computations. Finally, Chapter 6 summaries the results of this study.

Chapter 2

PRELIMINARIES

This chapter presents a brief review of mathematical structures pertinent to the subsequent discussion.

2.1 Partially Ordered Set, Lattice and Boolean Algebra

Let P be a set and $\preceq$ a binary relation on P. The relation $\preceq$ is called a partial order (partial ordering) if and only if $\preceq$ is reflexive, antisymmetric and transitive (Trembley and Manohar, 1975). Let $\preceq$ be a partial order on P, the pair $\langle P, \preceq\rangle$ is called a partially ordered set or simply *poset*. By definition, not every pair of elements of P is related under a partial order. If relation $\preceq$ is defined for any two elements of P, the relation will be called a *total* order and a set with such a relation is called *totally* ordered set.

Let $\langle P, \preceq\rangle$ be a poset and let $A \subseteq P$. An element x in P is called an *upper bound* of A if for all $a \in A$, $a \preceq x$. Similarly, an element $x \in P$ is called a *lower bound* of A if for all $a \in A$, $x \preceq a$. An element $x \in P$ is called the *least upper bound (LUB)* of A, if x is an upper bound of A, and for any upper bound y of A, $x \preceq y$. Likewise, x is called the *greatest lower bound (GLB)* of A if x is a lower bound of A, and for any lower bound y of A, $y \preceq x$. An element $b \in P$ is said to cover another element $a \in P$

if $a \preceq b$ and there does not exist any element $c \in P$ such that $a \preceq c$ and $c \preceq b$, that is

$$b \;\; cover \;\; a \Longleftrightarrow (a \preceq b \wedge (a \preceq c \preceq b \Longrightarrow a = c \vee c = b)). \tag{2.1}$$

From the above discussion, it is obvious that for any given two elements a and b in P, if an element c covers both a and b then $c = LUB(a, b)$. Similarly, $c = GLB(a, b)$ if both a and b cover c.

A lattice is a partially ordered set $\langle L, \preceq \rangle$ in which every pair of elements $a, b \in L$ has a GLB and a LUB (Trembley and Manohar, 1975). GLB and LUB of $\{a, b\}$ are denoted by $a \otimes b$ and $a \oplus b$, i.e.,

$$\begin{aligned} GLB(a, b) &= a \otimes b, \\ LUB(a, b) &= a \oplus b. \end{aligned} \tag{2.2}$$

The operation $\otimes$ is called *meet* or *product* and the operation $\oplus$ is called *join* or *sum*. Using these symbols, a lattice is denoted by $(L, \otimes, \oplus)$.

The order relation $\preceq$ can be defined by operations $\otimes$ or $\oplus$. For example, the order relation $\preceq$ can be defined by operation $\oplus$ as a relation satisfying condition:

$$a \preceq b \Longleftrightarrow a \oplus b = b.$$

or equivalently, it can be defined using operation $\otimes$:

$$a \preceq b \Longleftrightarrow a \otimes b = a.$$

A lattice can also be defined axiomatically as a set L with two operations $\otimes$ and $\oplus$ satisfying the following axioms which are called axioms of lattice (Birkhoff, 1967; Trembley and Manohar, 1975):

for $a, b, c \in L$,

$$\text{Idempotent}: \quad a \oplus a = a, \quad a \otimes a = a;$$

$$\text{Commutativity}: \quad a \oplus b = b \oplus a, \quad a \otimes b = b \otimes a;$$

$$\text{Associativity}: \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c, \quad a \otimes (b \otimes c) = (a \otimes b) \otimes c;$$

$$\text{Absorption}: \quad a \otimes (a \oplus b) = a, \quad a \oplus (a \otimes b) = a.$$

A subset L' of L which is closed under these operations is a lattice itself. The lattice $\langle L', \otimes, \oplus \rangle$ is called a sublattice. A lattice is complete if every non-empty subset has a GLB and LUB. Every finite lattice is complete. For a complete lattice, GLB and LUB of the entire set are denoted by 0 and 1. For an element a in lattice L, if there exists an element $b \in L$ such that:

$$a \otimes b = 0, \qquad a \oplus b = 1,$$

b is called a complement of a. If every element of L has a complement, L is a complemented lattice. A lattice is distributive if operations $\otimes$ and $\oplus$ distribute over each other, namely,

$$\text{Distributivity}: \quad a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c), \qquad a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c).$$

As will be seen in Chapter 4, the notion of lattice will be used as a basis of interval-based computation models.

A complemented distributive lattice is called a *Boolean algebra* (Trembley and Manohar, 1975). A Boolean algebra is denoted by $\langle B, \otimes, \oplus, \sim, 1, 0 \rangle$, where the symbol $\sim$ denotes the complement operation, i.e., the complement of a is written as $\sim a$. A Boolean algebra can be defined axiomatically based on properties of its operations. For instance, it is possible to define a Boolean algebra as a set B with operations $\otimes$ and $\sim$ and a set of independent axioms about these operations. A partial order $\preceq$ on B can be defined as follows:

$$\begin{aligned} a \preceq b &\Leftrightarrow a \otimes b = a \\ &\Leftrightarrow a \oplus b = b \\ &\Leftrightarrow a \otimes \sim b = 0 \\ &\Leftrightarrow \sim b \preceq \sim a \\ &\Leftrightarrow \sim a \oplus b = 1. \end{aligned}$$

If $S \subseteq B$ and S contains elements 0 and 1, and closed under operations $\otimes, \oplus$ and $\sim$, $\langle S, \otimes, \oplus, \sim, 1, 0\rangle$ is called a *sub-Boolean algebra.*

Suppose S is a non-empty set. The pair $\langle 2^S, \subseteq\rangle$ is a poset, where 2^S is the power set of S. The system $\langle 2^S, \cap, \cup\rangle$ is a lattice, in which the join and meet operations are $\cup$ and $\cap$, respectively. Moreover, the system $\langle 2^S, \cap, \cup, \neg, 1, 0\rangle$ is a Boolean algebra, where the complement operation is $\neg$.

2.2 Fuzzy Sets

The notion of fuzzy sets was introduced by Zadeh (1965). Let U be a set called the universe. A *fuzzy set* A in U is defined by a membership function $\mu_A : U \longrightarrow [0, 1]$. The function μ_A associates each element of U with a real number in the interval $[0, 1]$. The value μ_A represents the grade of *membership* of x in A. The concept of fuzzy sets is a generalization of ordinary set. If A is an ordinary (crisp) set, its membership function, μ_A, can take either 0 or 1. For example, the universe and the empty set can be represented by their membership functions as:

$$\mu_U(x) = 1, \qquad \mu_\emptyset(x) = 0.$$

Given two fuzzy sets A and B with membership functions μ_A and μ_B, their intersection $A \cap B$, union $A \cup B$, and complement $\neg A$ can be defined component-wise as follows:

$$\begin{aligned}\mu_{A\cap B}(x) &= \min(\mu_A(x), \mu_B(x)),\\ \mu_{A\cup B}(x) &= \max(\mu_A(x), \mu_B(x)),\\ \mu_{\neg A}(x) &= 1-\mu_A(x).\end{aligned} \tag{2.3}$$

Fuzzy sets inclusion is defined by:

$$A \subseteq B \Longleftrightarrow \forall x \in U, \mu_A(x) \preceq \mu_B(x). \tag{2.4}$$

For fuzzy sets A, B, the following properties hold:

$$\begin{aligned}&\text{Double negation :} && \neg(\neg A) = A,\\ &\text{De Morgan's law :} && \neg(A \cup B) = (\neg A) \cap (\neg B),\\ & && \neg(A \cap B) = (\neg A) \cup (\neg B).\end{aligned}$$

These properties are the same as those of ordinary sets. However, in general the following inequalities hold for fuzzy sets:

$$A \cup \neg A \neq X, \qquad A \cap \neg A \neq \emptyset.$$

Many different proposals have also been proposed for the definition of fuzzy set operations. For example, the *probabilistic-like* definition is given by:

$$\begin{aligned}\mu_{A\cap B}(x) &= \mu_A(x).\mu_B(x),\\ \mu_{A\cup B}(x) &= \mu_A(x)+\mu_B(x)) - \mu_A(x).\mu_B(x),\\ \mu_{\neg A}(x) &= 1-\mu_A(x).\end{aligned} \tag{2.5}$$

Using the min-max definition, it follows

$$\mu_{A\cap\neg A} = \min(\mu_A(x), 1-\mu_A(x)) \neq \mu_\emptyset(x), \quad \mu_A(x) \neq 0. \tag{2.6}$$

That is, in general $A \cap \neg A$ is not necessarily equal to $\emptyset$. By probabilistic-like definition,

$$\mu_{A \cap A}(x) = \mu_A(x).\mu_A(x) \neq \mu_A(x), \quad \mu_A(x) \neq 0, 1. \tag{2.7}$$

It implies that idempotent does not hold unless $\mu_A(x) = 0$ or $\mu_A(x) = 1$. Therefore, the above two definitions build up two different structures for fuzzy set systems. The min-max structure induces a lattice but not a Boolean algebra, while the probabilistic-like definition does not induce even a lattice.

In order to describe qualitatively a fuzzy set A, the concept of *core* and *support* are used. The core of a fuzzy set is defined by:

$$A_c = \{x \in X \mid \mu_A(x) = 1\}, \tag{2.8}$$

which consists of all those elements with membership 1. The support of a fuzzy set is denoted by:

$$A_s = \{x \in X \mid \mu_A(x) > 0\}, \tag{2.9}$$

which consists of all those elements with non-zero membership. For both max-min and probabilistic-like definitions, the following properties hold for A_c and A_s:

i). $A_c \subseteq A_s$,

ii). $(A \cap B)_c = A_c \cap B_c$, $(A \cup B)_c = A_c \cup B_c$,

iii). $(A \cap B)_s = A_s \cap B_s$, $(A \cup B)_s = A_s \cup B_s$.

2.3 Rough Sets

The notion of rough sets was introduced by Pawlak (1982). Let U be a set called the universe, and let R be an equivalence relation on U. The pair $Apr = (U, R)$ is called an approximation space. If $x, y \in U$ and $(x, y) \in R$, x and y are *indiscernible.* The empty set $\emptyset$ and the equivalence classes of relation R is called elementary sets (atoms) in Apr. Every finite union of elementary sets is called a composed set in A, or a composed set for short. The family of all composed sets, denoted by $Com(A)$, is closed under intersection, union and complement of sets. Thus, $Com(A)$ is a Boolean algebra.

Let A be a subset of U. The least composed set containing A is called the best upper approximation of A, written as $\overline{Apr}(A)$. The greatest composed set contained in A is called the best lower approximation of A, written as $\underline{Apr}(A)$. The pair $Apr(X) = (\underline{Apr}(A), \overline{Apr}(A))$ is called a rough set of A in Apr. For any two subsets $A, B \subseteq U$, the following properties hold:

$$
\begin{array}{ll}
\text{(R0)} & \underline{Apr}(A) \subseteq \overline{Apr}(A), \\
\text{(R1)} & \underline{Apr}(A \cap B) = \underline{Apr}(A) \cap \underline{Apr}(B), \\
 & \overline{Apr}(A \cap B) \subseteq \overline{Apr}(A) \cap \overline{Apr}(B), \\
\text{(R2)} & \underline{Apr}(A \cup B) \supseteq \underline{Apr}(A) \cup \underline{Apr}(B), \\
 & \overline{Apr}(A \cup B) = \overline{Apr}(A) \cup \overline{Apr}(B), \\
\text{(R3)} & \underline{Apr}(A - B) = \underline{Apr}(A) - \overline{Apr}(B), \\
 & \overline{Apr}(A - B) \subseteq \overline{Apr}(A) - \underline{Apr}(B), \\
\text{(R4)} & A \subseteq B \Longrightarrow \underline{Apr}(A) \subseteq \underline{Apr}(B), \\
 & A \subseteq B \Longrightarrow \overline{Apr}(A) \subseteq \overline{Apr}(B).
\end{array}
$$

It is interesting to note that they are different from some of properties of fuzzy sets.

Chapter 3

A MODEL OF SET-BASED COMPUTATIONS

This chapter presents a basic model of set-based computations using the notion of relational systems. A number of fundamental issues of set-based computations are addressed. Set-based operations and relations are defined by extending single-valued operations and relations. Properties of set-based operations and relations are examined.

3.1 Relational Systems

Different mathematical systems have their own special computation methods. For example, in numerical computations, on uses the system $(\Re, +, -, \times, /, \geq, =)$, where $\Re$ is the set of real numbers, $+, -, \times$ and $/$ are arithmetic operations, and $\geq$ and $=$ are binary relations on $\Re$. In logical inference, one uses the system $(V, \wedge, \vee, \neg)$, where V is a set of well formed formulas, and $\wedge$, $\vee$ and $\neg$ are operations on V. Abstracting from these examples, the concept of relational systems from measurement theory is adopted to describe various computational models (Roberts, 1979).

Definition 3.1 *A relational system is an ordered $(p+q+1)$-tuple,*

$$RS = (U, \circ_1, \ldots, \circ_p, R_1, \ldots, R_q), \tag{3.1}$$

where U is a nonempty set, $\circ_1, \ldots, \circ_p$ are operations on U, and $R_1, \ldots, R_q$ are relations (not necessarily binary) on U.

Although the definition of relational systems is very general, practical problems are usually involved with a small number of operations and relations. In this thesis, only unary, binary operations, and binary relations will be considered.

In a relational system, all operations and relations are defined on elements of U. In practice, it may not always be possible to represent a physical quantity using a single element of U. For example, it may be difficult to describe the temperature at the moment precisely using a single number. It may be more reasonable to say, for instance, that the temperature is between 19.5°C and 20.5°C. In these situations, it may be more appropriate to use subsets of U. To accommodate this set-based representation scheme, one may extend operations and relations on elements of U into operations and relations on subsets of U. This leads to extended relational systems. The former is referred to as point-based (single-valued) computations, and the latter as set-based (set-valued) computations.

3.2 Extended Relational Systems

Extended relational systems are obtained by extending operations and relations on U to subsets of U. It seems to be reasonable to require that extended relational systems should preserve as many characteristics as that of the original systems. In particular, when only singleton subsets of U are used, set-based computations must reduce to point-based computations.

3.2.1 Extended operations

An operation $\circ$ on U can be extended into an operation $\circ'$ on 2^U by applying it to members of subsets of U. For a unary operation, given a subset $A \subseteq U$, we can construct another subset $A' = \{\circ a \mid a \in A\}$. For a binary operation, given two subsets $A, B \in 2^U$, one can derive another subset $C \in 2^U$ by collecting all elements $a \circ b$, where $a \in A$ and $b \in B$. For clarity and simplicity, the empty set will not be considered.

Definition 3.2 *Suppose $\circ$ is a unary operation on U, and $A \neq \emptyset$ is an element of 2^U. An extended unary operation $\circ'$ on $2^U - \{\emptyset\}$ is defined by:*

$$\circ' A = \{\circ a \mid a \in A\}. \tag{3.2}$$

Definition 3.3 *Suppose $\circ$ is a binary operation on U, and $A, B \neq \emptyset$ are two elements of 2^U. An extended binary operation $\circ'$ on $2^U - \{\emptyset\}$ is defined by:*

$$A \ \circ' \ B = \{a \circ b \mid a \in A, b \in B\}. \tag{3.3}$$

If only singleton subsets of U are used, operation $\circ'$ reduces to $\circ$. Many properties of operations in the original system can be carried over by extended operations. The theorem given below shows that commutativity and associativity are preserved.

Theorem 3.1 *Suppose $\circ$ is a binary operation on U, and $\circ'$ on $2^U - \{\emptyset\}$ is the extended binary operation defined by equation (3.3). Then,*

(a). if $\circ$ is commutative, $\circ'$ is commutative,

(b). if $\circ$ is associative, $\circ'$ is associative.

Example 3.1 Consider the problem of constructing languages using strings out of a set of alphabets (Aho, Sethi and Ullman, 1988). An *alphabet* or a *character class* denotes any finite set of symbols. A string over some alphabet is a finite sequence of symbols drawn from that alphabet. The empty string, a string with no character, is included in this definition and is denoted by ϵ. A *language* is any set of strings over some fixed alphabet. An operation used to build string is concatenation. This concatenation operation also can be extended to languages. If L and M are two languages, the concatenation of L and M is denoted by LM which is also a language. In fact, $LM = \{st \mid \; s \in L, t \in M\}$. The operation of concatenating characters to make strings is extended to the concatenation operation in languages to concatenate any two languages. Since the concatenation of strings is associative, the concatenation of language is also associative.

3.2.2 Extended relations

The extension of a binary relation R can be done in a similar manner as that of operations. However, this process is more complicated. For any two arbitrary sets $A, B \subseteq U$, relation may hold for all elements in A, B (i.e., $A \times B \subseteq R$), or it may hold only for few elements (i.e., $A \times B \not\subseteq R, (A \times B) \cap R \neq \emptyset$), or it does not hold for any element (i.e., $(A \times B) \cap R = \emptyset$). Accordingly, four distinct classes can be identified.

Definition 3.4 *Suppose R is a binary relation on U. Four types of set-wise extensions of R are defined by: for $A, B \in 2^U - \{\emptyset\}$,*

$$\text{I.} \quad A \; \overline{R}^* \; B \Longleftrightarrow (\exists a \in A)(\exists b \in B) \; a \; R \; b,$$

$$\text{II.} \quad A \; \underline{R}_* \; B \Longleftrightarrow (\forall a \in A)(\forall b \in B) \; a \; R \; b,$$

$$\text{III.} \quad A \; \underline{R}^* \; B \Longleftrightarrow (\forall a \in A)(\exists b \in B) \; a \; R \; b,$$

$$\text{IV.} \quad A \; \overline{R}_* \; B \Longleftrightarrow (\exists a \in A)(\forall b \in B) \; a \; R \; b.$$

It is easy to verify that for any $A, B \in 2^U$:

$$A \; {}_*R_* \; B \implies A \; {}^*R^* \; B,$$

$$A \; {}_*R_* \; B \implies A \; {}_*R^* \; B,$$

$$A \; {}_*R_* \; B \implies A \; {}^*R_* \; B,$$

$$A \; {}_*R^* \; B \implies A \; {}^*R^* \; B,$$

$$A \; {}^*R_* \; B \implies A \; {}^*R^* \; B.$$

That is,

$$ {}_*R_* \subseteq {}_*R^* \subseteq {}^*R^*, \qquad {}_*R_* \subseteq {}^*R_* \subseteq {}^*R^* \tag{3.4}$$

Using these extended relations, two additional relations can be defined:

$$\text{V.} \quad R_\cup = {}_*R^* \cup {}^*R_*$$

$$\text{VI.} \quad R_\cap = {}_*R^* \cap {}^*R_*$$

Relation $\subseteq$ (on the set of extended relations $\{{}^*R^*, {}_*R_*, {}_*R^*, {}^*R_*, R_\cup, R_\cap\}$) is a partial order relation. Equation (3.4) can be extended to:

$$ {}_*R_* \subseteq R_\cap \subseteq {}_*R^* \subseteq R_\cup \subseteq {}^*R^*,$$

$$ {}_*R_* \subseteq R_\cap \subseteq {}^*R_* \subseteq R_\cup \subseteq {}^*R^*. \tag{3.5}$$

In general, the above six extended relations are different. They form a lattice as shown in Figure 3.1. This lattice is denoted by $L(R)$.

Extended relations in $L(R)$ carry many properties of of R. Properties that cannot be carried over by ${}^*R^*$ cannot be carried over by other extended relations. The following theorem shows properties of R that can be carried over by extended relations.

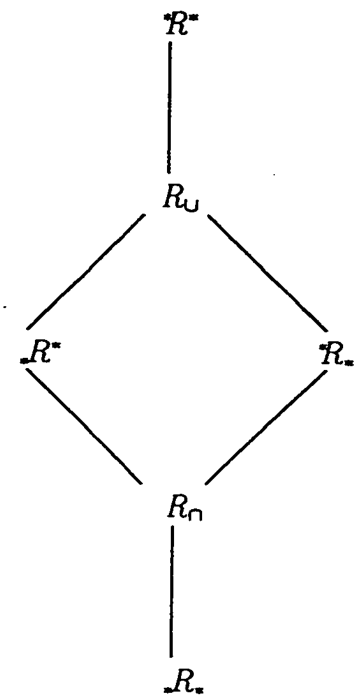

Figure 3.1: *Hasse* Diagram of Extended Relations

Theorem 3.2 *Suppose R is a binary relation on U, and $L(R)$ is a lattice of extended relations on $2^U - \{\emptyset\}$. Then,*

(a). if R is reflexive, ${}^{*}R^{*}$, ${}_{*}R^{*}$, and $R_{\cup}$ are reflexive,

(b). if R is symmetric, ${}_{*}R_{*}$ and ${}^{*}R^{*}$ are symmetric,

(c). if R is antisymmetric, only ${}_{*}R_{*}$ is antisymmetric,

(d). if R is transitive, ${}_{*}R_{*}$, ${}_{*}R^{*}$, ${}^{*}R_{*}$, and $R_{\cap}$ are transitive.

The proof is given in Section 3.3.

In real applications, it may be useful to define other kinds of extended relations. Nevertheless, extended relations defined above provide some general guidelines. For example, it is reasonable to assume that any extended relation should be bounded by the two extreme points of $L(R)$. In general, one may use two points in $L(R)$ as the bounds of an extended relation, namely, an interval $[R, R']$ such that $R, R' \in L(R)$ and $R \subseteq R'$.

3.2.3 Extended systems

Based on extended operations and relations, one can build an extended relational system from the original one.

Definition 3.5 *Let $RS = (U, \circ_1, \ldots, \circ_p, R_1, \ldots, R_q)$ be a relational system, and let $L(R_i)$ be the lattice of extended relations for R_i. The corresponding set-wise extended relational system, called set-based system, is:*

$$RS' = (\Gamma(2^U), \circ'_1, \ldots, \circ'_p, L(R_1), \ldots L(R_q)), \tag{3.6}$$

where $\Gamma(2^U) \subseteq 2^U$ is closed under extended operations $\circ'$.

Although each extended operation $\circ'$ is defined on 2^U, it is not necessary to use the entire set 2^U to construct a system for set-based computations. In many situations, one may find that it is more meaningful to use a subset of 2^U. For example, if U is an ordered set, one may consider the set of all closed intervals, which is only a subset of 2^U. Next chapter examines such systems and shows that other properties of $\circ$ may be carried over by $\circ'$. It should be noted that there are no constraints on the elements of U. Elements of U may in fact be sets themselves. Thus, the proposed framework provides a simple, yet powerful enough, model of set-based computations.

Extended relations R^* and R_* represent two extreme point of views. Informally speaking, they are the upper and lower limits for any plausible definition of extended relations. These two views play a very important role in many systems for uncertainty management. The notion of *possibility* and *necessity* in possibility theory, *upper approximation* and *lower approximation* in rough set theory, are related to such notions.

3.2.4 An example

To illustrate the basic concepts of set-based computations, a simple example is given below. Consider the following relational system,

$$RS = (U, \circ, R),$$

where $U = \{a, b, c\}$, a binary operation $\circ$ is given by:

$\circ$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	b

and a binary relation R is defined by:

R	a	b	c
a	1	0	1
b	0	1	0
c	1	0	1

An entry with value 1 in the relation table indicates that the relation holds between the elements in the corresponding row and column. For example, a is related to a and c, but not related to b. Obviously, operation $\circ$ is commutative and associative, and relation R is reflexive, symmetric and transitive, i.e., R is an equivalence relation.

From equation (3.3), extended operation $\circ'$ is given by:

$\circ'$	a	b	c	ab	ac	bc	abc
a	a	a	a	a	a	a	a
b	a	b	a	ab	a	ab	ab
c	a	a	b	a	ab	ab	ab
ab	a	ab	a	ab	a	ab	ab
ac	a	a	ab	a	ab	ab	ab
bc	a	ab	ab	ab	ab	ab	ab
abc	a	ab	ab	ab	ab	ab	ab

In the above table, a subset is represented by its members. For instance, ab stands for the subset $\{a, b\}$. It can be checked that extended operation $\circ'$ is also commutative and associative. According to Definition 3.4, four extended relations can be represented as follow:

$\mathcal{R}_*$	a	b	c	ab	ac	bc	abc
a	1	0	1	0	1	0	0
b	0	1	0	0	0	0	0
c	1	0	1	0	1	0	0
ab	0	0	0	0	0	0	0
ac	1	0	1	0	1	0	0
bc	0	0	0	0	0	0	0
abc	0	0	0	0	0	0	0

$\mathcal{R}^*$	a	b	c	ab	ac	bc	abc
a	1	0	1	1	1	1	1
b	0	1	0	1	0	1	1
c	1	0	1	1	1	1	1
ab	1	1	1	1	1	1	1
ac	1	0	1	1	1	1	1
bc	1	1	1	1	1	1	1
abc	1	1	1	1	1	1	1

$\mathcal{R}^*$	a	b	c	ab	ac	bc	abc
a	1	0	1	1	1	1	1
b	0	1	0	1	0	1	1
c	1	0	1	1	1	1	1
ab	0	0	0	1	0	1	1
ac	1	0	1	1	1	1	1
bc	0	0	0	1	0	1	1
abc	0	0	0	1	0	1	1

$\mathcal{R}_*$	a	b	c	ab	ac	bc	abc
a	1	0	1	0	1	0	0
b	0	1	0	0	0	0	0
c	1	0	1	0	1	0	0
ab	1	1	1	1	1	1	1
ac	1	0	1	0	1	0	0
bc	1	1	1	1	1	1	1
abc	1	1	1	1	1	1	1

The tables for the other two relations in $L(R)$, i.e., $\mathcal{R}_* \bigcup {}_*R^*$ and $\mathcal{R}_* \bigcap {}_*R^*$ can be obtained easily.

From this example, it is clear that if extended operation and relations are restricted to singleton subsets of U, the original operation and relation are obtained. Moreover, extended operation and relations carry over some properties of the original operation and relation as stated in Theorems 3.1 and 3.2.

3.3 Proof

Proof of Theorem 3.2: Suppose R is a relation on the universe U. The theorem is proved in four parts:

(a) If R is reflexive,

I) $\mathcal{R}^*$ is reflexive:

It is obvious that if $A \neq \emptyset$, there exists an element $a \in A$. By the reflexivity of R, it follows $a\ R\ a$. Thus, $A\ \mathcal{R}^*\ A$.

II) ${}_*R^*$ is reflexive:

Since every element a of A is related to itself (R is reflexive), it implies $A\ {}_*R^*\ A$.

III) $R_\cup$ is reflexive:

This follows from the fact that ${}_*R^*$ is reflexive.

(b) If R is symmetric,

I) $\mathcal{R}^*$ is symmetric:

$$\begin{aligned} A\ \mathcal{R}^*\ B &\iff \exists a \in A, \exists b \in B, a\ R\ b \\ &\iff \exists b \in B, \exists a \in A, b\ R\ a \\ &\iff B\ \mathcal{R}^*\ A. \end{aligned}$$

II) $\mathcal{R}_*$ is symmetric:

$$\begin{aligned} A\ \mathcal{R}_*\ B &\iff \forall a \in A, \forall b \in B, a\ R\ b \\ &\iff \forall b \in B, \forall a \in A, b\ R\ a \\ &\iff B\ \mathcal{R}_*\ A. \end{aligned}$$

(c) If R is antisymmetric, $\mathcal{R}_*$ is antisymmetric:

Suppose $\mathcal{R}_*$ is not antisymmetric. Then there are two nonempty sets $A, B \in 2^U$ such that $A\ \mathcal{R}_*\ B$, $B\ \mathcal{R}_*\ A$, and $A \neq B$. Since $A \neq B$ there are elements $a \in A$, $b \in B$ such that $a \neq b$. By $A\ \mathcal{R}_*\ B$ it can be inferred that $a\ R\ b$, and by $B\ \mathcal{R}_*\ A$ it can be inferred that $b\ R\ a$. The last two inferences and $a \neq b$ indicates that R is not antisymmetric. This is a contradiction. Therefore, $\mathcal{R}_*$ must be antisymmetric.

(d) If R is transitive,

I) $\mathcal{R}_*$ is transitive:

Suppose there are $A, B, C \in 2^U$ such that, $A\ \mathcal{R}_*\ B$ and $B\ \mathcal{R}_*\ C$. By definition,

$$\begin{aligned} A\ \mathcal{R}_*\ B &\implies \forall a \in A, \forall b \in B,\ \ a\ R\ b, \\ B\ \mathcal{R}_*\ C &\implies \forall b \in B, \forall c \in C,\ \ b\ R\ c. \end{aligned}$$

From the above equations, it can be inferred that for every $a \in A$, $c \in C$, $a \; R \; c$. In other words, $A \; {}_{*}R_{*} \; C$.

II) ${}_{*}R^{*}$ is transitive:

Suppose there are $A, B, C \in 2^U$ such that, $A \; {}_{*}R^{*} \; B$ and $B \; {}_{*}R^{*} \; C$. By definition,

$$A \; {}_{*}R^{*} \; B \implies \forall a \in A, \exists b \in B, \;\; a \; R \; b,$$

$$B \; {}_{*}R^{*} \; C \implies \forall b \in B, \exists c \in C, \;\; b \; R \; c.$$

From the above equations, it can be inferred, $\forall a \in A$, $\exists c \in C$ such that $a \; R \; c$ which means $A \; {}_{*}R_{*} \; C$.

III) ${}^{*}R_{*}$ is transitive:

Similar to the above.

IV) $R_\cap$ is transitive:

It follows from the fact that both ${}_{*}R^{*}$ and ${}^{*}R_{*}$ are transitive.

Chapter 4

INTERVAL-BASED COMPUTATIONS

Suppose $\langle U, \preceq \rangle$ is a poset, where $\preceq$ is a reflexive and antisymmetric relation. In this system, there is only one binary relation. By the results of Chapter 3, the set-wise extension of $(U, \preceq)$ is $(2^U, L(\preceq))$. However, none of the relations ${}^*\!\preceq^*$, ${}_*\!\preceq_*$, ${}^*\!\preceq_*$, ${}_*\!\preceq^*$, $\preceq_\cup$, and $\preceq_\cap$ is a partial order relation. This immediately follows from Theorem 3.2. That is, extension of poset is no longer a poset. Similar results can be obtained for lattice and Boolean algebra.

The set-based computations model does not produce interesting results when applied to poset, lattice and Boolean algebra. However, in many situations, it may be sufficient to consider only special kinds of subsets of the universe. It may be more natural to examine set-based computations when only intervals of ordered set are used. In fact, many well known systems, such as interval-number algebra, interval-set algebra, interval-valued logic, are special cases of set-based computations. Within the framework of set-based computations, this chapter examines a number of interval-based computation models.

All theorems developed in this chapter follow from definitions. Their proofs are therefore not included.

4.1 Interval-number Algebra

The interval-number algebra is a set-based extension of real number algebra (Moore. 1966). Let the real numbers algebra be represented by:

$$(\Re, +, -, \times, /, \leq, =). \tag{4.1}$$

An interval over real numbers is defined by a subset of real numbers between two real numbers called the lower bound and the upper bound. If $a_1, a_2 \in \Re$ and $a_1 \leq a_2$, $[a_1, a_2]$ is called a closed interval with a_1 as the lower bound and a_2 as the upper bound, namely,

$$[a_1, a_2] = \{x \in \Re \mid a_1 \leq x \leq a_2\}.$$

If an interval does not include the lower bound, i.e., $(a_1, a_2] = \{x \in \Re \mid a_1 < x \leq a_2\}$, the upper bound, i.e., $[a_1, a_2)$, or both, i.e., (a_1, a_2), it is called an open interval. In this chapter, the word interval will refer to closed interval unless specified otherwise. The set of all intervals in $\Re$ is denoted by $I(\Re)$. The notion of intervals is particularly useful for representing a value when only its range is known. For example, it has used extensively for error computations in numerical analysis.

An interval is a special subset of $\Re$. For cases where the lower and upper bounds are equal, i.e., degenerate interval $[a, a]$, singleton sets of $\Re$ are obtained. They are equivalent to the corresponding real numbers. Operations and relations defined for real numbers in ordinary algebra can be extended to intervals according to equations (3.2), (3.3), and (3.4). Let $+, -, \times$ and $/$ be arithmetic operations in $\Re$. If $\circ$ represents any of these operations, set-wise extension of $\circ$ is defined as:

$$I \circ' J = \{x \circ y \mid x \in I, y \in J\}, \qquad I, J \in I(\Re). \tag{4.2}$$

If $r \circ s$ is not defined for some $r \in I, s \in J$, $I \circ J$ is undefined. Particularly, the following formula for each operation can be defined:

$$
\begin{aligned}
[a_1, a_2] +' [b_1, b_2] &= \{a + b \mid a \in [a_1, a_2], b \in [b_1, b_2]\} \\
&= [a_1 + b_1, a_2 + b_2], \\
[a_1, a_2] -' [b_1, b_2] &= \{a - b \mid a \in [a_1, a_2], b \in [b_1, b_2]\} \\
&= [a_1 - b_2, a_2 - b_1], \\
[a_1, a_2] \times' [b_1, b_2] &= \{ab \mid a \in \ [a_1, a_2], b \in [b_1, b_2]\} \\
&= [min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)], \\
[a_1, a_2]/'[b_1, b_2] &= \{a/b \mid a \in [a_1, a_2], b \in [b_1, b_2]\} \\
&= [a_1, a_2] \times' [\frac{1}{b_1}, \frac{1}{b_2}] \\
&= [min(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}), max(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2})], \quad 0 \notin [b_1, b_2].
\end{aligned}
$$

If $0 \in [b_1, b_2]$, then $[a_1, a_2]/[b_1, b_2]$ is not defined.

In some applications of interval-number algebra, only intervals with positive bounds are used, i.e., intervals $[a_1, a_2]$ with $0 \leq a_1 \leq a_2$. In this case, the above mentioned formulas for operations $\times'$ and $/'$ can be simplified. For any two interval numbers $[a_1, a_2]$, $[b_1, b_2]$ such that $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$:

$$
\begin{aligned}
[a_1, a_2] \times' [b_1, b_2] &= [a_1b_1, a_2b_2], \\
[a_1, a_2]/'[b_1, b_2] &= [\frac{a_1}{b_2}, \frac{a_2}{b_1}].
\end{aligned}
$$

For these extended operations, properties such as commutativity and associativity hold as stated in Theorem 3.1. For the unary operation $-$ in real numbers, a unary operation $-'$ in $I(\Re)$ can be defined according to Definition 3.2.

The identity element for $+'$ is $[0, 0]$, that is,

$$\forall A \in I(\Re), \qquad [0, 0] +' A = A +' [0, 0] = A.$$

The identity element of $\times$ is $[1,1]$. In $\Re$, for each element a, there are complement elements of a with respect to operations $+$ and $\times$. Let $-a$ and a^{-1} denote the complements of a with respect to $+$ and $\times$. The following conditions must hold:

$$\begin{aligned} a+(-a) &= (-a)+a=0, \\ a \times a^{-1} &= a^{-1} \times a = a, \qquad \text{if } a \neq 0. \end{aligned}$$

Even though unary operation $-$ can be extended to $-'$ by $-'A = \{-a \mid a \in A\} = -[a_1, a_2] = [-a_2, -a_1]$, it is not a complement element of the interval $[a_1, a_2]$ regarding operation $+'$. The same can be said about operation $\times$. That is, there is no complement for $A = [a_1, a_2]$ regarding operations $+$ and $\times$ unless $a_1 = a_2$.

For relations $\leq$ and $=$, a set of extended relations can be defined using Definition 3.4. For $I = [a_1, a_2]$, $J = [b_1, b_2] \in I(\Re)$ and relation $\leq$, the four extended relations are given by:

$$\begin{aligned} I \;{}_*\!\leq_*\; J &\iff (\forall x \in I)(\forall y \in J)\; x \leq y, \\ I \;{}_*\!\leq^*\; J &\iff (\forall x \in I)(\exists y \in J)\; x \leq y, \\ I \;{}^*\!\leq_*\; J &\iff (\exists x \in I)(\forall y \in J)\; x \leq y, \\ I \;{}^*\!\leq^*\; J &\iff (\exists x \in I)(\exists y \in J)\; x \leq y. \end{aligned}$$

The other two possible extensions, $\leq_\cap$ and $\leq_\cup$, can be obtained easily by a combination of ${}_*\!\leq^*$ and ${}^*\!\leq_*$. It is easy to verify that the above relations can be simplified as follows:

$$\begin{aligned} I \;{}_*\!\leq_*\; J &\iff a_2 \leq b_1, \\ I \;{}_*\!\leq^*\; J &\iff a_2 \leq b_2, \\ I \;{}^*\!\leq_*\; J &\iff a_1 \leq b_1, \\ I \;{}^*\!\leq^*\; J &\iff a_1 \leq b_2. \end{aligned}$$

The extended relation for $=$ can be defined in a similar manner. According to Theorem 3.2, some properties of $\leq$ and $=$ can be carried over to extended relation. Moreover, additional properties may also be carried over because of the special properties of real intervals. Table 4.1 summarizes properties of extended relations in comparison with that of the original relations. The symbol x in the table indicates that the relation has the property, and the symbol - indicates that the relation does not have the property. The appropriateness of each extended relation depends on particular applications. For example, the interval equality can be defined using ${}_*{=}^*$, i.e.,

$$[a_1, a_2] = [b_1, b_2] \iff ([a_1, a_2] \ {}_*{=}^* \ [b_1, b_2]) \wedge ([b_1, b_2] \ {}_*{=}^* \ [a_1, a_2]).$$

	reflexive	symmetric	antisymmetric	transitive
$\leq$	x	-	x	x
${}_*{\leq}_*$	-	-	x	x
${}_*{\leq}^*$	x	-	-	x
${}^*{\leq}_*$	x	-	-	x
$\leq_\cap$	x	-	x	x
$\leq_\cup$	x	-	-	-
${}^*{\leq}^*$	x	-	-	-
$=$	x	x	x	x
${}_*{=}_*$	-	x	x	x
${}_*{=}^*$	x	-	x	x
${}^*{=}_*$	-	-	x	x
$=_\cap$	-	-	x	x
$=_\cup$	x	-	-	-
${}^*{=}^*$	x	x	-	-

Table 4.1: Properties of Extended Relations on Closed Real Intervals

4.2 Interval-set Algebra

Interval-set algebra is a special case of set-based computations (Yao, 1993). Let U be a finite nonempty set called the universe, and let 2^U be its power set. Suppose $A_1, A_2 \in 2^U$ and $A_1 \subseteq A_2$, a closed *interval set* is defined by the set of all elements of 2^U which are supersets of A_1, and subsets of A_2, and represented by $[A_1, A_2]$, i.e.,

$$\mathcal{A} = [A_1, A_2] = \{X \in 2^U \mid A_1 \subseteq X \subseteq A_2\}. \tag{4.3}$$

The set A_1 is called the lower bound, and A_2 the upper bound, of the interval set. An interval set is a subset of 2^U bounded by two elements of 2^U. Let $I(2^U)$ denote the set of all closed interval sets. According to Definitions 3.2 and 3.3, these binary operations may be extended to interval sets. If $\mathcal{A} = [A_1, A_2]$ and $\mathcal{B} = [B_1, B_2]$ are two arbitrary interval sets from $I(2^U)$, these extended operations are defined by:

$$\begin{aligned}
\mathcal{A} \sqcap \mathcal{B} &= \{X \cap Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}, \\
\mathcal{A} \sqcup \mathcal{B} &= \{X \cup Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}, \\
\mathcal{A} \setminus \mathcal{B} &= \{X - Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}.
\end{aligned} \tag{4.4}$$

The above operations are closed on $I(2^U)$, i.e., $\mathcal{A} \sqcap \mathcal{B}$, $\mathcal{A} \sqcup \mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ are interval sets. As in interval-number algebra, the result of the above operations can be computed directly using:

$$\begin{aligned}
\mathcal{A} \sqcap \mathcal{B} &= [A_1 \cap B_1, A_2 \cap B_2], \\
\mathcal{A} \sqcup \mathcal{B} &= [A_1 \cup B_1, A_2 \cup B_2], \\
\mathcal{A} \setminus \mathcal{B} &= [A_1 - B_2, A_2 - B_1].
\end{aligned} \tag{4.5}$$

In 2^U, the complement of an element A is denoted by A^c. Using Definition 3.2, a psudo-complement of $[A_1, A_2]$, denoted by $\neg[A_1, A_2]$, is defined by

$$\neg[A_1, A_2] = \{A^c \mid A_1 \subseteq A \subseteq A_2\} = [U, U] \setminus [A_1, A_2]$$

$$= \quad [U - A_2, U - A_1] = [A_2{}^c, A_1{}^c]$$

Degenerated interval set $[A, A]$ is equivalent to A itself. The proposed operations $\sqcap$, $\sqcup$, $\setminus$, and $\neg$ for degenerated intervals reduce to the usual set-theoretic operations. According to Theorem 3.1, commutativity and associativity hold for interval-set union ($\sqcup$) and intersection ($\sqcap$). Idempotent, absorption and Demorgan's law hold as well. The double negation law holds for interval-set psudo-complement.

According to Definition 3.4, relation $\subseteq$ on 2^U can be extended to relations on interval sets. For any two interval sets $\mathcal{A} = [A_1, A_2]$ and $\mathcal{A} = [B_1, B_2]$, the following extensions for the inclusion relation between interval sets can be defined:

$$\begin{array}{lcccl}
\mathcal{A} \,{}_{*}\!\subseteq_{*} \mathcal{B} & \Longleftrightarrow & (\forall A \in \mathcal{A})(\forall B \in \mathcal{B}) \;\; A \subseteq B & \Longleftrightarrow & A_2 \subseteq B_1, \\
\mathcal{A} \,{}_{*}\!\subseteq^{*} \mathcal{B} & \Longleftrightarrow & (\forall A \in \mathcal{A})(\exists B \in \mathcal{B}) \;\; A \subseteq B & \Longleftrightarrow & A_2 \subseteq B_2, \\
\mathcal{A} \,{}^{*}\!\subseteq_{*} \mathcal{B} & \Longleftrightarrow & (\exists A \in \mathcal{A})(\forall B \in \mathcal{B}) \;\; A \subseteq B & \Longleftrightarrow & A_1 \subseteq B_1, \\
\mathcal{A} \,{}^{*}\!\subseteq^{*} \mathcal{B} & \Longleftrightarrow & (\exists A \in \mathcal{A})(B \in \mathcal{B}) \;\; A \subseteq B & \Longleftrightarrow & A_1 \subseteq B_2, \\
\mathcal{A} \subseteq_{\cup} \mathcal{B} & \Longleftrightarrow & (\mathcal{A} \,{}^{*}\!\subseteq_{*} \mathcal{B}) \vee (\mathcal{A} \,{}_{*}\!\subseteq^{*} \mathcal{B}) & \Longleftrightarrow & (A_1 \subseteq B_1) \vee (A_2 \subseteq B_2), \\
\mathcal{A} \subseteq_{\cap} \mathcal{B} & \Longleftrightarrow & (\mathcal{A} \,{}^{*}\!\subseteq_{*} \mathcal{B}) \wedge (\mathcal{A} \,{}_{*}\!\subseteq^{*} \mathcal{B}) & \Longleftrightarrow & (A_1 \subseteq B_1) \wedge (A_2 \subseteq B_2).
\end{array}$$

It can be verified that according to Theorem 3.2 relation ${}_{*}\!\subseteq_{*}$ is antisymmetric and transitive but not reflexive and not symmetric. In fact, properties of these extended relations are similar to those in Table 4.1.

The set $I(2^U)$ is closed under operations $\sqcup$ and $\sqcap$ and according to properties of these operations, the following theorem can be asserted.

Theorem 4.1 *Suppose $\sqcup$ and $\sqcap$ are interval-set intersection and union. Then, $(2^U, \sqcap, \sqcup)$ is a complete distributive lattice.*

Like the inclusion relation $\subseteq$ for ordinary sets, the inclusion relation on $I(2^U)$ can be defined as follow:

Definition 4.1 *For any two interval set $\mathcal{A} = [A_1, A_2]$, $\mathcal{B} = [B_1, B_2]$, $\mathcal{A} \sqsubseteq \mathcal{B} \Longleftrightarrow (A_1 \subseteq B_1) \wedge (A_2 \subseteq B_2)$*

Based on this definition, for two interval sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} = \mathcal{B}$ if and only if $(\mathcal{A} \sqsubseteq \mathcal{B}) \wedge (\mathcal{B} \sqsubseteq \mathcal{A})$.

The following theorem shows that $\sqsubseteq$ is exactly the relation $\subseteq_{\cap}$ in extended relational systems.

Theorem 4.2 *The inclusion of interval sets, i.e.,* $\sqsubseteq$*, is equivalent to* $\subseteq_{\cap}$.

The relation $\sqsubseteq$ is a partial ordering and the lattice $\langle I(2^U), \sqsubseteq\rangle$ is the same lattice as $\langle I(2^U), \sqcap, \sqcup\rangle$. It should be noted that the lattice $\langle I(2^U), \sqcup, \sqcap\rangle$ is complete but not complemented. That is, $[U, U]$ and $[\emptyset, \emptyset]$ are the upper and the lower bound for the lattice, but for a given interval set such as $\mathcal{A}$, $\mathcal{A} \sqcap \neg\mathcal{A}$ is not necessarily equal to $[\emptyset, \emptyset]$, $\mathcal{A} \sqcup \neg\mathcal{A}$ is not necessarily equal to $[U, U]$, and $\mathcal{A} \setminus \mathcal{A}$ is not necessarily equal to $[\emptyset, \emptyset]$.

4.3 Interval Lattice

Let L be a lattice with operations $\otimes$ and $\oplus$. If L is a Boolean lattice, the symbol $\sim$ denotes the complement operation. Given two elements $a_1, a_2 \in L$ with $a_1 \preceq a_2$, an interval $[a_1, a_2]$ is defined by the set:

$$[a_1, a_2] = \{x \in L \mid a_1 \preceq x \preceq a_2\}. \tag{4.6}$$

That is, $[a_1, a_2]$ consists of these elements of L that are bounded by a_1 and a_2. It is a sublattice of L. An element $a \in L$ may be represented as a degenerate interval of the form $[a, a]$. Let $I(L)$ denote the set of all intervals formed from L. Using Definitions 3.3, 3.2, operations $\otimes$, $\oplus$ and $\sim$ can be extended to the elements of $I(L)$ as follows:

$$\begin{aligned}
[a_1, a_2] \otimes' [b_1, b_2] &= \{x \otimes y \mid a_1 \preceq x \preceq a_2,\ \ b_1 \preceq y \preceq b_2\}, \\
[a_1, a_2] \oplus' [b_1, b_2] &= \{x \oplus y \mid a_1 \preceq x \preceq a_2,\ \ b_1 \preceq y \preceq b_2\}, \\
\sim' [a_1, a_2] &= \{\sim x \mid a_1 \preceq x \preceq a_2\}.
\end{aligned} \tag{4.7}$$

Suppose L is a Boolean lattice, then extended operations on $I(L)$ are closed and can be computed by:

$$\begin{aligned} [a_1, a_2] \otimes' [b_1, b_2] &= [a_1 \otimes b_1, a_2 \otimes b_2], \\ [a_1, a_2] \oplus' [b_1, b_2] &= [a_1 \oplus b_1, a_2 \oplus b_2], \\ \sim' [a_1, a_2] &= [\sim a_2, \sim a_1]. \end{aligned} \tag{4.8}$$

The set $I(L)$ with the above operations forms a lattice. To differentiate it from the original lattice, $I(L)$ is referred to as an interval lattice. Many properties of $\otimes$, $\oplus$ and $\sim$ on L are carried over by their corresponding operations on the interval lattice. For example, if L is a complete distributive lattice, then $I(L)$ is a complete distributive lattice. However, if L is a Boolean lattice, $I(L)$ is not a Boolean lattice but a complete distributive lattice. The relation $\preceq_\cap$ can be considered as the partial ordering in the lattice $I(L)$.

Chapter 5

APPLICATIONS

To demonstrate the usefulness of the proposed framework, this chapter presents a number of its applications.

5.1 Set-based Information Systems

The study of information systems is one of the most desired area for set-based computations. Following Lipski (1981), Orlowska and Pawlak (1984), Pawlak (1981), and Vakarelov (1991), A set-based information system is defined to be a quadruple,

$$S = (O, A, \{V_a \mid a \in A\}, \{f_a \mid a \in A\}), \tag{5.1}$$

where

O is a nonempty set of objects,

A is a nonempty set of attributes,

V_a is a nonempty set of values of $a \in A$,

$f_a : O \times A \longrightarrow 2^{V_a}$ is an information function.

If all information functions map an object only to singleton sets of attribute values, we obtain a degenerate set-based information system commonly used in the rough-set model (Pawlak, 1982). The notion of information systems provides a convenient tool for the representation of objects in terms of their attribute values. By definition, a database system is an information system. More examples can be found in (Pawlak 1981).

Set-based computations introduced in Chapter 3 can be easily applied in set-based information systems. The following information system is used to demonstrate the main idea.

	AGE	HEIGHT	LANGUAGE
o_1	{35}	{tall}	{English, French}
o_2	[30, 35]	{medium}	{French}
o_3	{20}	[medium, tall]	{English}
o_4	[60, 61]	{short}	{English, French}
o_5	{54}	[short, tall]	{English}

The set-based representation in this example may arise in several ways. The available information may be insufficient to determine the exact value of an attribute. For instance, based on the given information, one may only infer that the age of o_2 is between 30 and 35. In the worst case, if one is totally ignorant of the value of an attribute a, the entire set V_a may be used to represent such an unknown value (Grzymala-Busse 1991). Any possible attribute value may in fact be the actual value of the attribute. The assignment of [short, tall] to the attribute HEIGHT of o_5 reflects such a situation. It is possible that an attribute takes a subset of V_a as its value. For example, o_1 speaks both English and French. An expert may feel that the prefixed grades in the system is not fine enough, and would rather use an interval formed by two adjacent values, say [medium, tall], as an additional value. From above discussion, it is obvious that the flexibility of set-based representation leads to a richer and more complicated semantics of set-based information systems. Applications of set-based computations depend on a well defined semantics of an information system.

Suppose all attributes take a single value and a set-based information system is used to represent uncertainty in specifying the actual value. One can immediately apply extended relations introduced in Chapter 3 to carry out the retrieval process, one of the basic operations in information system. Since attribute values for an object regarding a certain attribute is represented by a set, in general there are six different ways of evaluating a query with respect to an object. In this thesis, uncertainty in the query itself is not considered. It is assumed that the query is a single value (a singleton set), therefore, only two of those possible equality relations are meaningful. In this case, there are two possible retrieved sets, i.e., the sets ${}_*Ret_*$ and ${}^*Ret^*$. Consider a query

$$q_1 : \quad \text{LANGUAGE} \;=\; \text{English}.$$

It produces the following two sets:

$$
\begin{aligned}
{}_*Ret_*(q_1) &= \{o_3, o_5\}, \\
{}^*Ret^*(q_2) &= \{o_1, o_3, o_4, o_5\}.
\end{aligned}
$$

Elements of ${}_*Ret_*$ definitely satisfy the query, whereas elements of ${}^*Ret^* - {}_*Ret_*$ may satisfy the query. The pair $({}_*Ret_*, {}^*Ret^*)$ defines an interval set $[{}_*Ret_*, {}^*Ret^*]$, indicating the range of the set of objects that actually satisfies the query. They may also be interpreted as lower and upper approximations in the context rough-set model. For an ordered set of attribute values, in addition to comparison of equality, comparisons can also be made using an order relation. For a given ordered relation $\succeq$ defined on V_a, it induces two extended relation ${}_*\succeq_*$ and ${}_*\succeq^*$. Accordingly, two sets ${}_*Ret_*$ and ${}^*Ret^*$ will be produced. For example, for the query,

$$q_2 : \quad \text{AGE} \;\geq\; 34,$$

we have:

$$\underline{Ret}_*(q_2) = \{o_1, o_4, o_5\},$$

$$\overline{Ret}^*(q_2) = \{o_1, o_2, o_4, o_5\}.$$

They are a pair of lower and upper approximations.

By combining queries q_1 and q_2, two composite queries are obtained:

$$\begin{array}{ll} q_3: & q_1 \text{ and } q_2 \\ & (\text{LANGUAGE} = \text{English}) \text{ and } (\text{AGE} \geq 34), \\ q_4: & q_1 \text{ or } q_2 \\ & (\text{LANGUAGE} = \text{English}) \text{ or } (\text{AGE} \geq 34). \end{array}$$

Using these queries, one can derive the following two pairs of retrieved sets:

$$\underline{Ret}_*(q_1 \text{ and } q_2) = \{o_5\},$$

$$\overline{Ret}^*(q_1 \text{ and } q_2) = \{o_1, o_4, o_5\};$$

$$\underline{Ret}_*(q_1 \text{ or } q_2) = \{o_1, o_3, o_4, o_5\},$$

$$\overline{Ret}^*(q_1 \text{ or } q_2) = \{o_1, o_2, o_3, o_4, o_5\}.$$

Obviously, the following properties hold:

$$\begin{aligned} \underline{Ret}_*(q_1 \text{ and } q_2) &= \underline{Ret}_*(q_1) \cap_* \underline{Ret}_*(q_1), \\ \overline{Ret}^*(q_1 \text{ and } q_2) &= \overline{Ret}^*(q_1) \cap^* \overline{Ret}^*(q_1), \\ \underline{Ret}_*(q_1 \text{ or } q_2) &= \underline{Ret}_*(q_1) \cup_* \underline{Ret}_*(q_1), \\ \overline{Ret}^*(q_1 \text{ or } q_2) &= \overline{Ret}^*(q_1) \cup^* \overline{Ret}^*(q_1). \end{aligned} \tag{5.2}$$

They correspond to operations of the interval-set algebra. These rules may be considered as a generalization of the rules used in database systems. They may be used in a retrieval process of a set-based information system. However, it should be noted that these rules may not generat the tightest bounds.

The proposed two operations in a set-based information system are essentially the same as the modal operators proposed by Lipski in the study of incomplete databases (Lipski, 1981). If a different semantic interpretation of a set-based information system is used, one may introduce other types extended operations (Beaubouef and Petry, 1994; Lipski, 1981). For example, Vakarelov (1991) used set equality to define an indiscernibility relation, and "=" to define a similarity relation. Our analysis may be extended to the case where a query itself uses a nonsingleton set.

5.2 Multi-valued Logic

Multi-valued logic is a generalization of two-valued logic in which the truth values may be taken from a lattice L (Edmonds, 1980). It is assumed that well formed formulas are defined to be exactly the same as those in two-valued propositional logic except elements of L are used. Let $v(\phi) \in L$ denote the truth value of a proposition or formula ϕ. A possible way for evaluation of logical conjunction, disjunction, and negation is to use the greatest lower bound ($\otimes$), the least upper bound ($\oplus$), and the complement ($\sim$). In other words, they may be evaluated using the following rules:

$$
\begin{aligned}
v(\phi \wedge \psi) &= v(\phi) \otimes v(\psi), \\
v(\phi \vee \psi) &= v(\phi) \oplus v(\psi), \\
v(\neg\phi) &= \sim v(\phi), \\
v(\phi \rightarrow \psi) &= \sim v(\phi) \oplus v(\psi), \\
v(\phi \leftrightarrow \psi) &= (\sim v(\phi) \oplus v(\psi)) \otimes (v(\phi) \oplus \sim v(\psi))).
\end{aligned} \tag{5.3}
$$

Where the negation is defined only if L is a complemented lattice. The adoption of a lattice for the definition of a many-valued logic implies that logical connectives $\wedge$, $\vee$, and $\neg$ must have the same properties as that of $\otimes$, $\oplus$, and $\sim$. For example, if a lattice L is a distributive lattice, logical connectives must be distributive. Conversely, if logical connectives are distributive, one must choose a distributive lattice to represent the truth values.

In the following discussion, L is assumed to be a Boolean algebra. In an interval-based logic system, we assume that an interval $[v_*(\phi), v^*(\phi)]$ may be assigned to each proposition to indicate the range within which lies the truth value. Any element between $v_*(\phi)$ and $v^*(\phi)$ may be the actual truth value. Within the framework of set-based computations introduced in Chapter 3, the following rules can be used for logical conjunction, disjunction, negation, implication and equivalence:

$$
\begin{aligned}
[v_*(\phi \wedge \psi), v^*(\phi \wedge \psi)] &= [v_*(\phi) \otimes v_*(\psi), v^*(\phi) \otimes v^*(\psi)], \\
[v_*(\phi \vee \psi), v^*(\phi \vee \psi)] &= [v_*(\phi) \oplus v_*(\psi), v^*(\phi) \oplus v^*(\psi)], \\
[v_*(\neg\phi), v^*(\neg\phi)] &= [\sim v^*(\phi), \sim v_*(\phi)], \\
[v_*(\phi \rightarrow \psi), v^*(\phi \rightarrow \psi)] &= [\sim v^*(\phi) \oplus v_*(\phi), \sim v_*(\phi) \oplus v^*(\phi)], \\
[v_*(\phi \leftrightarrow \psi), v^*(\phi \leftrightarrow \psi)] &= [(\sim v^*(\phi) \oplus v_*(\psi)) \otimes (v_*(\phi) \oplus \sim v^*(\psi)),
\end{aligned}
$$

$$(\sim v_*(\phi) \oplus v^*(\psi)) \otimes (v^*(\phi) \oplus \sim v_*(\psi))]. \tag{5.4}$$

These rules are extensions of rules defined by equation (5.3) that uses a single element of L as the truth value of a proposition. The assignment of interval truth values suggests that the interval lattice $I(L)$, or a sublattice of $I(L)$, should be used. To reflect this property, such a logic is referred to as an interval-valued logic.

The interval-valued system examined above is related to a number of systems. If the lattice $([0,1], \max, \min)$ is used, the interval-valued fuzzy logic system investigated by Kenevan and Neapolitan (1992) is obtained. All their inference rules have a counterpart in our framework. If three truth values T (true), F (false) and I (unknown or undetermined) are used (Rescher, 1969), one can draw the corresponding between such a three-valued logic and interval-valued logic (Yao and Li, 1993). This can be simply done by interpreting the truth value I as either an interval $[F, T]$ or as its equivalent set representation $\{F, T\}$, where it is assumed that $T \succ F$. Consider a three valued logic where $I(L) = \{\{F\}, \{F, T\}, \{T\}\}$. The sets $\{F\}$ and $\{T\}$ indicate that the proposition is *false* and *true*, respectively. The set $\{F, T\}$ indicates that the proposition is undetermined due to a lack of sufficient information or other uncertainty involved. Such a three valued logic is characterized by the following truth tables:

ϕ	$\neg\phi$
$\{T\}$	$\{F\}$
$\{F\}$	$\{T\}$
$\{F, T\}$	$\{F, T\}$

	$\phi \wedge \psi$			$\phi \vee \psi$		
ϕ \ ψ	$\{T\}$	$\{F,T\}$	$\{F\}$	$\{T\}$	$\{F,T\}$	$\{F\}$
$\{T\}$	$\{T\}$	$\{F,T\}$	$\{F\}$	$\{T\}$	$\{T\}$	$\{T\}$
$\{F,T\}$	$\{F,T\}$	$\{F,T\}$	$\{F\}$	$\{T\}$	$\{F,T\}$	$\{F,T\}$
$\{F\}$	$\{F\}$	$\{F\}$	$\{F\}$	$\{T\}$	$\{F,T\}$	$\{F\}$

	$\phi \rightarrow \psi$			$\phi \leftrightarrow \psi$		
ϕ \ ψ	$\{T\}$	$\{F,T\}$	$\{F\}$	$\{T\}$	$\{F,T\}$	$\{F\}$
$\{T\}$	$\{T\}$	$\{F,T\}$	$\{F\}$	$\{T\}$	$\{F,T\}$	$\{F\}$
$\{F,T\}$	$\{T\}$	$\{F,T\}$	$\{F,T\}$	$\{F,T\}$	$\{F,T\}$	$\{F,T\}$
$\{F\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{F\}$	$\{F,T\}$	$\{T\}$

They coincide with that of Kleene's three valued logic (Kleene, 1952).

5.3 Interval-valued Probabilistic Reasoning

In interval-valued probabilistic reasoning, the probability of a proposition is presented by an interval instead of a single value. In situations where it is difficult or impractical, if not impossible, to assign a single value for the probability of certain propositions, intervals may be used to indicate the true probability. Uncertainty of propositions is characterized by a family of probability functions bounded by such probability intervals. In general, there are two complementary interval-valued probabilistic reasoning approaches (Yao, 1993). One is a non-numeric approach proposed by Bundy (1985, 1986) and the other is a numeric approach introduced by Quinlan (1983). The results obtained in set-based computation can be applied in both approaches. Both methods can be studied using the unified notion of set-based computations introduced in this study. The following two subsections, drawn extensively from Yao (1994), illustrate this conclusion.

5.3.1 Incidence calculus

Incidence calculus provides a possible world semantics of two-valued propositional logic. Let Φ be a finite and non-empty set of propositions of interest. A propositional language formed from Φ is denoted by $L(\Phi)$. It is the smallest set containing the truth values, and closed under logic connectives (conjunction $\wedge$, disjunction $\vee$, negation $\neg$, and implication $\rightarrow$). Let W be a non-empty set of possible worlds. It represents the states or situations of the system being modeled. Each possible world can be considered as a partial interpretation of some logical formulas in the propositional language $L(\Phi)$. With respect to a possible world $w \in W$, a proposition is either *true* or *false*.

In incidence calculus, instead of using a numeric value, a subset $i(\phi) \subseteq W$ is assigned to a given proposition $\phi \in L(\Phi)$ to indicate that ϕ is true for all $w \in i(\phi)$, and ϕ is false for all $w \notin i(\phi)$. The set $i(\phi)$ is referred to as the *incidence set* of ϕ.

In practice, it may be difficult to specify precisely the incidence set of a proposition. One may be able to provide only *lower* and *upper* bounds of the incidence sets of certain propositions. Depending on the available information about a proposition, one may choose a pair of lower and upper bounds for the incidence set of ϕ such as $i_*(\phi)$ and $i^*(\phi)$. They specify an interval set $[i_*(\phi), i^*(\phi)]$ within which lies the true incidence set of the proposition (Yao, 1993). Obviously these two bounds can be described as the range of two mappings:

$$i_* : L(\Phi) \longrightarrow 2^W \text{ and } i^* : L(\Phi) \longrightarrow 2^W.$$

In the absence of any information about the proposition the obvious lower and upper bounds for the incidence set will be $\emptyset$ and W, respectively. A set of lower and upper bounds is said to be *consistent* if there exists an incidence structure i such that for all $\phi \in L(\Phi)$,

$$i_*(\phi) \subseteq i(\phi) \subseteq i^*(\phi). \tag{5.5}$$

Accordingly, if an incidence structure i satisfies the above equation, we say that i is *bounded* by the pair (i_*, i^*) (Wong, Wang and Yao, 1992). Incidence bounds can be sharpened using properties of rough sets. In practice, the following rules can be used:

(I1) $\quad i_*(\phi) \longleftarrow i_*(\phi) \cup (W - i^*(\neg\phi))$,

(I2) $\quad i^*(\phi) \longleftarrow i^*(\phi) \cap (W - i_*(\neg\phi))$,

(I3) $\quad i_*(\phi \wedge \psi) \longleftarrow i_*(\phi \wedge \psi) \cup (i_*(\phi) \cap i_*(\psi))$,

(I4) $\quad i^*(\phi \wedge \psi) \longleftarrow i^*(\phi \wedge \psi) \cap (i^*(\phi) \cap i^*(\psi))$,

(I5) $\quad i_*(\phi) \longleftarrow i_*(\phi) \cup (i_*(\phi \wedge \psi)$,

(I36 $\quad i^*(\phi) \longleftarrow i_*(\phi) \cap (i^*(\phi \wedge \psi) \cup (W - i_*(\psi)))$.

Probabilistic reasoning with incidence calculus is carried out using a probability function on W. Let P_W denote a probability function defined on W. The probability of a proposition ϕ is defined using its incidence set by:

$$P(\phi) = P_W(i(\phi)). \tag{5.6}$$

If only lower and upper bounds of the incidence sets are given, the corresponding lower and upper probabilities of a propositions are defined by:

$$P_*(\phi) = P_W(i_*(\phi)),$$

$$P^*(\phi) = P_W(i^*(\phi)).$$

Wong, Wang and Yao (1992) have shown that the lower and upper probabilities define a pair of belief and plausibility functions (Shafer, 1990). Thus, interval-valued probabilistic reasoning using incidence calculus is similar to evidential reasoning using belief functions (Correa da Silva and Bundy, 1990; Wong, Wang and Yao, 1992). However, it is also important to point out that the interval-valued probabilistic interpretation of belief and plausibility functions is only one of several views.

5.3.2 Numeric probabilistic reasoning

Suppose a pair of lower and upper probabilities $P^*(\phi)$ and $P_*(\phi)$ is associated with a proposition ϕ to indicate its bounds, i.e., $P(\phi) \in [P_*(\phi),\ P^*(\phi)]$. The pair $[P_*(\phi), P^*(\phi)]$ is referred to as interval-valued probability of ϕ and $P(\phi)$ as point-valued probability of ϕ. If one is totally ignorant of the probability of a proposition, the trivial bounds $[0, 1]$ can be used. The lower and upper bounds can be described by two mappings:

$$P_* : L(\Phi) \longrightarrow [0,1] \text{ and } P^* : L(\Phi) \longrightarrow [0,1].$$

A pair of probability bounds (P_*, P^*) is said to be consistent if there exist a probability function P such that for all $\phi \in L(\Phi)$,.

$$P_*(\phi) \leq P(\phi) \leq P^*(\phi). \tag{5.7}$$

A consistent pair of lower and upper bounds (P_*, P^*) can be interpreted as constraints on probability functions. They characterize the maximal family of probability functions:

$$\mathcal{P} = \{P \mid P_*(\phi) \leq P(\phi) \leq P^*(\phi) \text{ for every } \phi \in L(\Phi)\}.$$

This set can be equivalently defined by the pair of tightest bounds:

$$P_{0*}(\phi) = \inf_{P\in\mathcal{P}} P(\phi),$$

$$P_0^*(\phi) = \sup_{P\in\mathcal{P}} P(\phi).$$

From the definition of probability function, for any two propositions ϕ and ψ, we have the following equation:

$$P(\phi \wedge \psi) = P(\phi) + P(\psi) - P(\phi \vee \psi) \tag{5.8}$$

If interval-valued probabilities are given for propositions ϕ and ψ, we can find the bounds for $P(\phi \wedge \psi)$ by extending the above formula. The right hand side of equation (5.8) can be expressed as:

$$[P_*(\phi), P^*(\phi)] + [P_*(\psi), P^*(\psi)] + [P_*(\phi \vee \psi), P^*(\phi \vee \psi)]. \tag{5.9}$$

Operations $+$ and $-$ in the above equation are interpreted as interval-number arithmetic operations introduced by Moore (1966). The value of equation (5.9) can be simplified into:

$$[P_*(\phi) + P_*(\psi) - P^*(\phi \vee \psi), P^*(\phi) + P^*(\psi) - P_*(\phi \vee \psi)] \tag{5.10}$$

Which gives the bounds of the probability $P(\phi \wedge \psi)$.

Quinlan (1983) has proposed a set of inference axioms. Yao (1994) refined these inference rules using the results from interval-number algebra, as discussed above. A subset of inference axioms related to the primitive connectives $\neg$ and $\wedge$ is summarized as follow :

(P1) $\quad P_*(\phi) \longleftarrow \max\{P_*(\Phi), 1 - P^*(\neg\phi)\};$

(P2) $\quad P^*(\phi) \longleftarrow \min\{P^*(\Phi), 1 - P_*(\neg\phi)\};$

(P3) $\quad P_*(\phi \wedge \psi) \longleftarrow \max\{P_*(\Phi \wedge \psi), P_*(\phi) + P_*(\psi) - P^*(\phi \vee \psi)\};$

(P4) $\quad P^*(\phi \wedge \psi) \longleftarrow \min\{P^*(\Phi \wedge \psi), P^*(\phi), P^*(\psi), P^*(\phi) + P^*(\psi) - P_*(\phi \vee \psi)\};$

(P5) $\quad P_*(\phi) \longleftarrow \max\{P_*(\Phi), P_*(\phi \wedge \psi), P_*(\phi \wedge \psi) + P_*(\phi \vee \psi) - P^*(\psi)\};$

(P6) $\quad P^*(\phi) \longleftarrow \min\{P^*(\Phi), P^*(\phi \wedge \psi) + P^*(\phi \vee \psi) - P_*(\psi)\}.$

The application of inference rules (P1)-(P6) will increase lower bounds and decrease upper bounds.

5.4 Interval-valued Fuzzy Reasoning

Fuzzy logic is based on the definition of fuzzy sets. The basic assumption is that for a given proposition the truth value can fall in the interval $[0, 1]$, unlike the classic logic in which the only two possible truth values are *true* and *false*. In fuzzy logic, the truth value 0 and 1 can be regarded as absolute false or absolute true, respectively. Based on fuzzy set operations, logic connectives, $\wedge$, $\vee$ and $\sim$ can be defined. Suppose the fuzzy truth value of propositions ϕ and ψ are represented by $f(\phi)$ and $f(\psi)$. Using equation (2.3), fuzzy logic connectives may be defined as:

$$\begin{aligned} f(\phi \wedge \psi) &= \min(f(\phi), f(\psi)), \\ f(\phi \vee \psi) &= \max(f(\phi), f(\psi)), \\ f(\sim \phi) &= 1 - f(\phi). \end{aligned} \tag{5.11}$$

Alternatively, according to equation (2.5), it may also be defined as:

$$\begin{aligned} f(\phi \wedge \psi) &= f(\phi).f(\psi)), \\ f(\phi \vee \psi) &= f(\phi) + f(\psi)) - f(\phi).f(\psi), \\ f(\sim \phi) &= 1 - f(\phi). \end{aligned} \tag{5.12}$$

These definitions can be extended to interval-based framework.

In situations where it is impossible to have precise value for the truth value of a given proposition, it might be possible to have a set of plausible truth values, or, as a special case, an interval for the truth value of the proposition. The idea of extending a single-valued fuzzy logic to an interval-valued fuzzy logic is specially useful to establish a proof theory (Kenevan and Neapolitan, 1992). Suppose that for propositions ϕ and ψ the fuzzy interval-valued truth values are represented by:

$$f(\phi) = [\phi_0,\ \phi_1], \quad 0 \leq \phi_0 \leq \phi_1 \leq 1,$$

$$f(\psi) = [\psi_0,\ \psi_1], \quad 0 \leq \psi_0 \leq \psi_1 \leq 1.$$

The definition for the first two extended fuzzy logic connectives based on the equation (5.11) will be:

$$\begin{aligned} f(\phi \wedge \psi) &= \min([\phi_0,\ \phi_1], [\psi_0,\ \psi_1]), \\ f(\phi \vee \psi) &= \max([\phi_0,\ \phi_1], [\psi_0,\ \psi_1]). \end{aligned} \tag{5.13}$$

According to definition for interval-extended operations in interval-number algebra (Moore, 1966), it follows:

$$\begin{aligned} \min([\phi_0,\ \phi_1], [\psi_0,\ \psi_1]) &= \{\min(x, y) \mid \ x \in [\phi_0,\ \phi_1],\ y \in [\psi_0,\ \psi_1]\} \\ &= [\min(\phi_0, \psi_0), \min(\phi_1, \psi_1)], \\ \max([\phi_0,\ \phi_1], [\psi_0,\ \psi_1]) &= \{\max(x, y) \mid \ x \in [\phi_0,\ \phi_1],\ y \in [\psi_0,\ \psi_1]\} \\ &= [\max(\phi_0, \psi_0),\ \max(\phi_1, \psi_1)]. \end{aligned}$$

Therefore,

$$f(\phi \wedge \psi) = [\min(\phi_0, \psi_0),\ \min(\phi_1, \psi_1)],$$

$$f(\phi \vee \psi) = [\max(\phi_0, \psi_0),\ \max(\phi_1, \psi_1)].$$

The definition for the negation would be:

$$f(\sim \phi) = [1, 1] - f(\phi) = [1 - \phi_1,\ 1 - \phi_0]$$

Now suppose equation (5.12) is used, then:

$$\begin{aligned} f(\phi \wedge \psi) &= \{x.y \mid x \in [\phi_0,\ \phi_1],\ y \in [\psi_0,\ \psi_1]\}, \\ f(\phi \vee \psi) &= \{x + y - x.y \mid x \in [\phi_0,\ \phi_1],\ y \in [\psi_0,\ \psi_1]\}. \end{aligned}$$

Since $\phi_0, \phi_1, \psi_0, \psi_1 \in [0, 1]$, then:

$$\{x.y \mid x \in [\phi_0,\ \phi_1],\ y \in [\psi_0,\ \psi_1]\} = [\phi_0.\psi_0,\ \phi_1.\psi_1],$$

$$\{x + y - x.y \mid x \in [\phi_0,\ \phi_1],\ y \in [\psi_0,\ \psi_1]\} = [\phi_0 + \psi_0 - \phi_0.\psi_0,\ \phi_1 + \psi_1 - \phi_1.\psi_1]$$

Therefore, in this case, the definition for the above operations will be:

$$\begin{aligned} f(\phi \wedge \psi) &= [\phi_0.\psi_0,\ \phi_1.\psi_1], \\ f(\phi \wedge \psi) &= [\phi_0 + \psi_0 - \phi_0.\psi_0,\ \phi_1 + \psi_1 - \phi_1.\psi_1] \end{aligned}$$

A more complete study of interval fuzzy reasoning using t-norms and conorms can be found in Wang and Yao (1995). The max-min and probabilistic-like definitions are two special cases.

Chapter 6

CONCLUSION

In this thesis, a framework of set-based computations is developed as a more flexible model for representing and inference in situations with vague or incomplete information. This model is particularly useful when it is difficult to obtain a precise value of certain parameter, or where set-valued attributes play an important role.

Conventionally, mathematical systems for computations, consist of a set of elements called the universe, and some operations and relations defined on the elements of the universe. These systems can be formally defined in terms of relational systems. The major issue discussed in this thesis is the extension of such relational systems. A relational system with the universe U is extended to another relational system with the universe 2^U. The operations and relations are extended so that they preserve most of the characteristics of the original ones. In fact, they are defined component-wise based on the corresponding operations or relation in the original system.

The process of extending operations is rather simple and straightforward. The extension of relations is more complicated. Given two subsets of the universe, there may be different number of pairs related to each other. A number of different grades of the relationship are considered. All the possible grades of relations between the two subsets are classified in six different cases. Properties that are transferable from the original operations and relations to the extended ones are considered and analyzed.

The internal structures of the extended relations and their relations to each other are discussed.

Interval computations may be considered as a special case of set-based computations. From this aspect, some of the interval-based models such as interval-number algebra, interval-set algebra, and interval lattice are re-examined. It should be pointed out that there must exist an order relation in order to carry out interval-based computations. In contrast, the set-based computations model does not have this restriction.

To show the applicability of the proposed set-based framework, set-based computations techniques are used for solving problems in set-based information systems, multi-valued logic, interval-valued probabilistic reasoning and interval-valued fuzzy reasoning. In these applications, some boundaries are given for the inference rules when the set-based parameters are involved.

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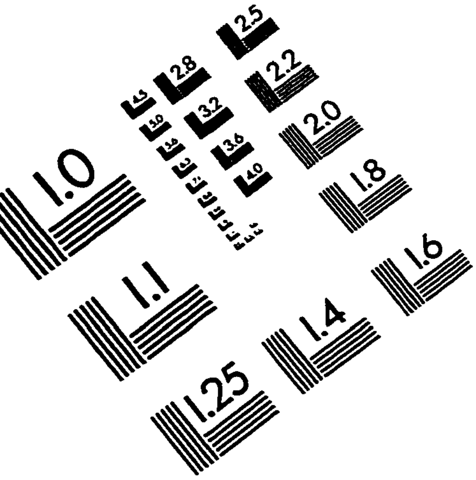
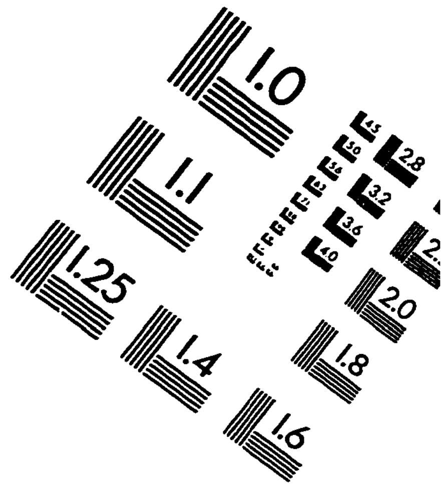
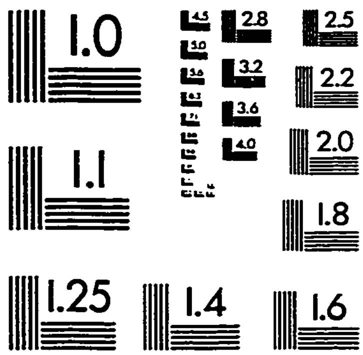
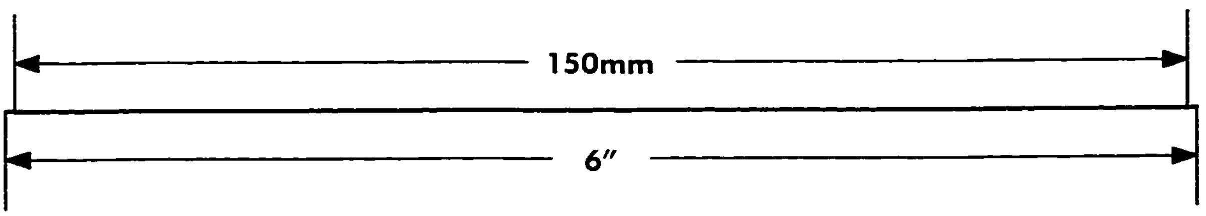
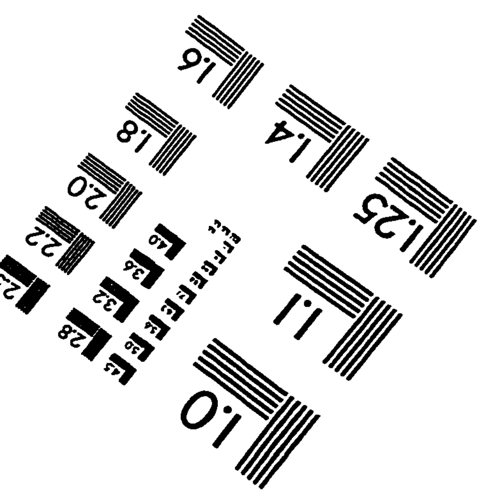
IMAGE EVALUATION TEST TARGET (QA-3)

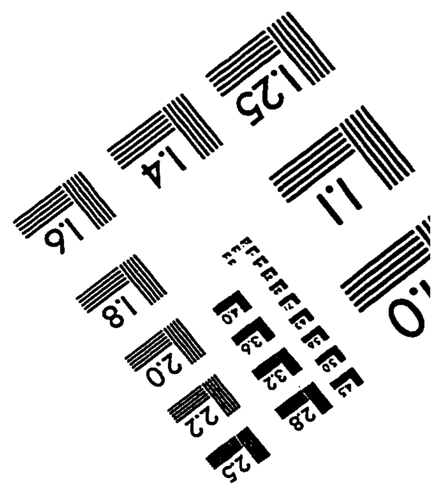












APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989