Intelligent States for Angular Momentum and Su(3) Observables

by

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Abstract

We generate expressions for all of the su(2)[‡] and su(3) intelligent states. To do so we combine well known coupling methods with unitary transformations; the construction is simple and efficient, and can be extended to generate intelligent states for any su(N) algebra. We also present a discussion of some of the properties of the su(2) and su(3) intelligent states.

[‡]The properties of the su(2) Lie algebra are analogous to quantum mechanical angular momentum properties.

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Contents

Chapter 1

Introduction

1.1 The Uncertainty Relation

In quantum mechanics, there is a lower bound on the uncertainty associated with the joint measurement of two non-commuting[‡], self-adjoint operators. This means that one cannot measure to arbitrary precision, with the same setup, the values of two non-commuting observables. The embodiment of this idea is Heisenberg's uncertainty relation [1];

$$\Delta q \Delta p \geq \frac{\hbar}{2}, \tag{1.1}$$

where p and q correspond, respectively, to the momentum and position of the particle, and $\hbar$ is Planck's constant, h, divided by 2π.

Uncertainty relations are a central feature of quantum physics. Thus, one way to better understand quantum mechanics is to further our understanding of uncertainty relations. In this thesis, an uncertainty relation more general than Eq.(1.1) will be used [2]:

$$\Delta\Omega\Delta\mathcal{O} \geqslant \tfrac{1}{2}|\langle[\hat{\Omega}, \hat{\mathcal{O}}]\rangle|, \tag{1.2}$$

where

$$\Delta\Omega = \sqrt{\langle\hat{\Omega}^2\rangle - \langle\hat{\Omega}\rangle^2} \tag{1.3}$$

is the standard deviation of the operator $\hat{\Omega}$, with $\langle\hat{\Omega}\rangle = \langle\psi|\hat{\Omega}|\psi\rangle$ the expectation value of the observable $\hat{\Omega}$ given the system is described by the state $|\psi\rangle$. This inequality, obtained by Robertson, is the correct generalization of the Heisenberg uncertainty relation to allow for arbitrary operators, though they must still be self-adjoint. It has been verified experimentally [3, 4, 5] and remains an effective tool, despite some objections [6] raised about the use of the standard deviation Eq.(1.3) as a good measure

[‡]See Appendix A for selected definitions. No additional terms will be indicated in the main text, but many terms and items that are not self evident will be defined in the appendix. This is done to preserve continuity of the main text.

of uncertainty in some specific cases. Eq.(1.2) does in fact reduce to Eq(1.1) by setting $\hat{\Omega} = \hat{q}$ and $\hat{\mathcal{O}} = \hat{p}$, and noting that the so-called commutator for $\hat{p}$ and $\hat{q}$ is

$$\begin{aligned} [\hat{p}, \hat{q}] &= \hat{p}\hat{q} - \hat{q}\hat{p} \\ &= -i\hbar \mathbb{1} \, . \end{aligned} \quad (1.4)$$

A derivation of Eq.(1.2) is included in Appendix B.1 for completeness.

The uncertainty described by $\Delta\Omega$ in Eq.(1.2) is not a statement about measurement techniques or equipment, but rather a feature built into quantum theory itself. The derivation of Eq.(1.2) is only based on the properties of the states and operators themselves. It does not account for the disturbance resulting from measurements made on the system, or the uncertainty associated with any limitations arising from the detectors or the experiment. It is worth noting that for this reason Eq.(1.2), as well as Eq.(1.1), has been the subject of debate [6, 7] with a wide range of suggestions on how to incorporate the measurement process into a more general expression for uncertainty. This, however, does not affect the results or motivation of this thesis.

The motivation for this thesis is the meaning of the lower bound in Eq.(1.2). The states that live at the lower bound of the uncertainty relation are called intelligent states [8], and they are states that satisfy the equation

$$\Delta\Omega\Delta\mathcal{O} = \tfrac{1}{2}|\langle[\hat{\Omega}, \hat{\mathcal{O}}]\rangle| \, . \quad (1.5)$$

The questions we are asking are "What are the states that live at this lower bound? Can they be constructed from existing technologies? What are some of their properties?". In other words, can we put together a recipe that uses known mathematical tools to generate these states, and if so, what are these states and how do they behave? Some answers to all three of these questions will be discussed in this thesis.

1.2 Coherent States

1.2.1 Coherent States as Harmonic Oscillator Minimum Uncertainty States

If we substitute $\hat{x}$ and $\hat{p}$ into Eq.(1.5), we are left with,

$$\Delta q \Delta p = \frac{\hbar}{2}. \quad (1.6)$$

Some of the states that satisfy this equation are already well known and well studied [9, 10]. These are harmonic oscillator coherent states. It is of interest, then, to relate the idea of coherent states as solutions to Eq.(1.6) to the general case of Eq.(1.5).

Harmonic oscillator coherent states were first introduced by Schrödinger [9] in 1926, though he did not refer to them by that name but simply as non-spreading wave packets. The label essentially captures the behaviour of coherent states; the shape of the probability distribution remains constant as time passes (*i.e.* it does not spread out like other states). It simply oscillates back and forth about a central point.

One of the reasons why coherent states are interesting is the classical-like behaviour they exhibit. For this reason we begin with a quick refresher on the classical harmonic oscillator.

Classically, simple harmonic motion is described by the differential equation:

$$\ddot{q} + m\omega^2 q = 0, \tag{1.7}$$

where $\dot{q} = \frac{dq}{dt}$. The oscillatory behaviour of the system is encapsulated in the solution to Eq.(1.7):

$$q = q_\circ \cos(\omega t + \phi)\,, \tag{1.8}$$

and

$$p = p_\circ \sin(\omega t + \phi)\,, \tag{1.9}$$

with $p = m\dot{q}$.

It is often convenient, however, to use the dimensionless variables [11];

$$\zeta(t) = \sqrt{\frac{m\omega}{2\hbar}}(q(t) + \frac{i}{m\omega}p(t))\,, \qquad \zeta^*(t) = \sqrt{\frac{m\omega}{2\hbar}}(q(t) - \frac{i}{m\omega}p(t))\,, \tag{1.10}$$

which evolve simply in time as

$$\dot{\zeta}(t) = -i\omega\zeta(t)\,. \tag{1.11}$$

The time dependent $\zeta(t)$ is then given by

$$\zeta(t) = \zeta_\circ e^{-i\omega t}\,, \qquad \zeta^*(t) = \zeta_\circ^* e^{i\omega t}\,, \tag{1.12}$$

where the constant

$$\zeta_\circ = \zeta(0) = -i\omega\sqrt{\frac{m}{2}}(q(0) + \frac{i}{m\omega}p(0))\,. \tag{1.13}$$

Thus a classical harmonic oscillator can be completely described by a vector in the complex plane with components given by the real and imaginary parts of $\zeta(t)$. The vector has a length of $|\zeta_\circ|$, and revolves through the angle $-\omega t$.

The quantum mechanical analogues to Eq.(1.10) are the so-called annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$;

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} + \frac{i}{m\omega}\hat{p})\,, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} - \frac{i}{m\omega}\hat{p})\,, \tag{1.14}$$

which have proven very useful in the study of the quantum mechanical harmonic oscillator, including the definition and studying of the harmonic oscillator coherent states. Their expectation values evolve in time the same way as their classical counterparts:

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -i\omega\langle\hat{a}(t)\rangle\,, \tag{1.15}$$

giving

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle \mathrm{e}^{-i\omega t}\,, \quad \langle \hat{a}^\dagger(t) \rangle = \langle \hat{a}^\dagger(0) \rangle \mathrm{e}^{i\omega t}\,. \tag{1.16}$$

They have the property that, in the so-called Fock basis;

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \tag{1.17}$$

with the number operator $\hat{N}$, which counts the number of elements in a basis state, defined as;

$$\hat{N}|n\rangle = \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle\,. \tag{1.18}$$

Their commutation relations are:

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{1}\,, \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0\,. \tag{1.19}$$

It was the work of Glauber [10], in 1963, that popularized the non-spreading wave packets, particularly to model laser output, and introduced the term harmonic oscillator coherent states. He showed that coherent states are an extremely useful set of states to use as a basis, despite the fact that they are non-orthogonal and overcomplete.

Glauber defined the harmonic oscillator coherent states in several separate, but equivalent, ways [10]. One of his definitions, and possibly the most well known, is that the set of coherent states are the eigenstates of the annihilation operator, $\hat{a}$, or more explicitly

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \tag{1.20}$$

where α is an arbitrary complex number. As it turns out Eq.(1.20) is a useful property to exploit when performing calculations, but it is not so useful as a definition that can be generalized. Since coherent states for other systems will be employed, we need an alternate definition.

In this thesis, a coherent state will be defined as a state obtained by an appropriate unitary transformation of a special state. For the harmonic oscillator the special state is the vacuum state, $|0\rangle$, which leads to the definition of the harmonic oscillator coherent state as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle\,, \tag{1.21}$$

where the operator $D(\alpha)$ is unitary (*i.e.* it satisfies $D(\alpha)D^\dagger(\alpha) = \mathbb{1}$). Physically $D(\alpha)$ is a translation operator, meaning that it causes a displacement,

$$D^{-1}(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha\,, \tag{1.22}$$

of the operator $\hat{a}$. Using Eq.(1.21) and Eq.(1.22), we have

$$\begin{aligned} \hat{a}|\alpha\rangle &= \hat{a}D(\alpha)|0\rangle \\ &= D(\alpha)[D^{-1}(\alpha)\hat{a}D(\alpha)]|0\rangle \\ &= D(\alpha)[\hat{a} + \alpha]|0\rangle \\ &= \alpha D(\alpha)|0\rangle \\ &= \alpha|\alpha\rangle\,, \end{aligned} \tag{1.23}$$

where $D^{-1}(\alpha)D(\alpha) = 1$ and $\hat{a}|0\rangle = 0$ have been used. Thus, Eq.(1.21) is equivalent to Eq.(1.20).

To see what this displacement means in terms of $\hat{q}$ and $\hat{p}$, we first need to determine how α is related to $\hat{q}$ and $\hat{p}$. We do this using the property Eq.(1.20) and the relations

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})\,, \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,. \tag{1.24}$$

From these we find that

$$\Re\mathrm{e}\,(\alpha) = \sqrt{\frac{m\omega}{2\hbar}}\langle\alpha|\hat{q}|\alpha\rangle\,, \quad \Im\mathrm{m}\,(\alpha) = \sqrt{\frac{1}{2m\omega\hbar}}\langle\alpha|\hat{p}|\alpha\rangle\,, \tag{1.25}$$

which gives us

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}(\langle\hat{q}\rangle + \frac{i}{m\omega}\langle\hat{p}\rangle)\,. \tag{1.26}$$

Putting this back into Eq.(1.22) gives us the new form:

$$\begin{aligned} D^{-1}(\alpha)\hat{a}D(\alpha) &= \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} + \frac{i}{m\omega}\hat{p}) + \sqrt{\frac{m\omega}{2\hbar}}(\langle\hat{q}\rangle + \frac{i}{m\omega}\langle\hat{p}\rangle) \\ &= \sqrt{\frac{m\omega}{2\hbar}}\left((\hat{q} + \langle\hat{q}\rangle) + \frac{i}{m\omega}(\hat{p} + \langle\hat{p}\rangle)\right)\,. \end{aligned} \tag{1.27}$$

It is now clear that the action of $D(\alpha)$ is to displace the states from their initial position, see Fig 1.1. Since $D(\alpha)$ acts on the vacuum, and the vacuum satisfies Eq.(1.6),

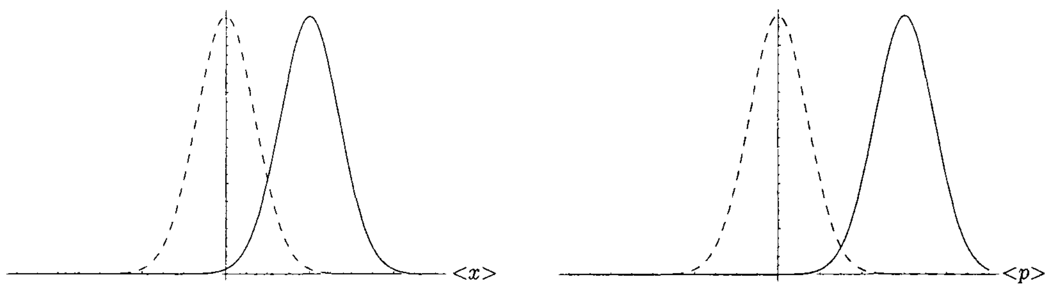

Figure 1.1: **Left:** The dashed curve shows the probability distribution in x for the ground state harmonic oscillator before the action of $D(\alpha)$ and the solid curve is the displaced distribution after. **Right:** Same as left, but for p.

the displaced vacuum states are also intelligent states. This can be shown by writing

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a}) \tag{1.28}$$

and

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})\,, \tag{1.29}$$

and using the property Eq.(1.20) to evaluate $\Delta q \Delta p$ in the state $|\alpha\rangle$.

The idea that the harmonic oscillator coherent states are related to intelligent states is extended to the construction of intelligent states in other systems.

1.2.2 Harmonic Oscillator Squeezing

The harmonic oscillator coherent states are a special set of states for a number of reasons, not simply for having minimum uncertainty. In fact, coherent states are a subset of a more general set of states called squeezed states. What makes the coherent states special in the bigger picture of squeezed states is that the standard deviations of the two observables $\hat{q}$ and $\hat{p}$, for the coherent states, are the same;

$$\Delta q = \Delta p = \frac{1}{\sqrt{2}}\,. \tag{1.30}$$

Here $\hat{p}$ and $\hat{q}$ are expressed in dimensionless form, as it is often more convenient.

The central idea of squeezing is to reshape the probability distributions so that $\Delta q \neq \Delta p$. There has been a lot of interest in squeezed states, from both the theoretical and experimental communities, since they were first introduced by Kennard [12] in 1927. The reason for the interest is the huge number of possibilities when one can significantly reduce the uncertainty associated with an observable (see [13, 14, 15] for reviews of the subject). Having said this, the squeezed states do not beat the uncertainty principle, they simply take advantage of the multiplicative property of the standard deviations on the left hand side of Eq(1.6). For instance, one can imagine a setup that does not give much knowledge on the position of a particle, *i.e.* $\Delta q = \frac{1}{\sqrt{2}}\mathrm{e}^{r}$ is very large. Given the constraints on the uncertainty principle one can then imagine being able to measure the momentum of that particle to a high precision, *i.e.* $\Delta p = \frac{1}{\sqrt{2}}\mathrm{e}^{-r}$ is very small. This is still within the bounds of the uncertainty principle, $\Delta q \Delta p = \frac{1}{2}$, so in effect, one can gain much knowledge of one aspect at the expense of knowledge of its conjugate.

It should be noted that harmonic oscillator squeezing only requires the ratio:

$$\frac{\Delta p}{\Delta q} \neq 1\,. \tag{1.31}$$

It will be shown later that the parameter α in the harmonic oscillator intelligent states functions to control the ratio $\frac{\Delta p}{\Delta q}$.

1.2.3 Coherent States for Angular Momentum and Other Systems

The idea of the coherent state as a displacement of a special state has been generalized to systems other than the harmonic oscillator. The reason that emphasis was put on the definition Eq.(1.21) is that when dealing with systems like angular momentum (the terms angular momentum algebra and su(2) algebra are synonymous), which has finite dimensional matrix representations, a problem is encountered if one tries to generalize the definition Eq.(1.20). For instance, when trying to find the eigenstates of the su(2) lowering operator, $\hat{L}_-$, one will find that it has only one non-zero eigenstate. This problem is encountered for all of the su(N) algebras.

It was Perelomov [16] who proposed the first useful generalization of the coherent states for arbitrary Lie algebras in a way that preserves/reproduces the properties found in the harmonic oscillator coherent states. These coherent states for other systems are essentially generalizations of Eq.(1.21). As well, Arecchi *et al* [17] studied the coherent states of two level atoms (which are SU(2) systems) and showed explicitly how the SU(2) coherent states can be constructed, using the SU(2) equivalent of the $D(\alpha)$ operator, via the generalization of Perelomov:

$$|\gamma,\beta\rangle = \mathrm{e}^{\zeta\hat{L}_+ - \zeta^*\hat{L}_-}|\ell,\ell\rangle, \tag{1.32}$$

where

$$\zeta = \frac{1}{2}\beta\mathrm{e}^{i\gamma} \tag{1.33}$$

depends on two real parameters β and γ. Only two parameters are necessary, even though the most general unitary SU(2) transformation is dependent on three, as will be seen later.

We would like to put Eq.(1.32) into a form that will be a bit more useful. The form of the transformation that will be most helpful is

$$|\gamma,\beta\rangle = R_z(\gamma)R_y(\beta)R_z(-\gamma)|\ell,\ell\rangle, \tag{1.34}$$

with $R_j(\varphi) = \mathrm{e}^{-i\varphi\hat{L}_j}$ denoting a rotation about the axis j through the angle φ. This form explicitly shows the unitary nature of the transformations. The forms Eq.(1.32) and Eq.(1.34) are equivalent (see Section 2.1.3), and describe the most general angular momentum coherent state. As is shown in [16], this definition of a coherent state can be extended to any su(N) algebra. The construction is straightforward: simply apply a unitary transformation to an extremal state and you have a coherent state. This will be useful in the construction of the su(3) intelligent states.

1.3 Motivation

In this thesis, we are studying the angular momentum and su(3) states that satisfy Eq.(1.5), and answering the questions stated in Section 1.1: "What are the states that live at this lower bound? Can they be constructed from existing technologies? What are some of their properties?". These states were originally studied by Aragone *et al* [8, 18], who also coined the terms intelligent states and intelligence.

Since Aragone *et al*, there have been others working to construct intelligent states for angular momentum systems. For instance Hillery and Mlodinow [19] devised a construction method that used a simple unitary transformation. However, not all of the angular momentum intelligent states can be produced in this way, and those which can are simply the coherent states.

One of the motivating factors for this method is that any finite unitary transformation (the transformations required to generate coherent states of SU(N)) can be realized experimentally using beam splitters and phase shifters [20, 21]. In particular Campos and Richard [22] have devised a setup to experimentally generate some of the su(2) intelligent states. The branch of solutions labeled by equation (9b) in [22] corresponds to the set of states generated using the method in this thesis if one takes ℓ_B, in Eq.(1.35), to be zero always. The result of their setup is to generate and verify a small subset of the angular momentum intelligent states. This subset corresponds to the intelligent states that can be generated using the method of [19].

The method of Rashid [23], on the other hand, solves for every angular momentum intelligent state using a non-unitary transformation. Though every state can be constructed in this way in a single shot, the method is not immediately transparent.

The first half of this thesis is based on published work [24], and is a reformulation of the method proposed by Milks and de Guise [25], who constructed intelligent states through the use of polynomial methods. While the method of [25] shows great promise, as all of the angular momentum intelligent states are found, the calculations can become tedious and do not clearly show the relation to coherent states. This thesis and [24] show more explicitly that the set of intelligent states is simply composed of coupled coherent states. It also serves to put the construction of these intelligent states into a clear light by showing that all of the $2\ell+1$ intelligent states, for a system with angular momentum ℓ, can be constructed in a simple way using existing and well known tools. These tools include coupling technology [26, 27] and unitary transformations.

For angular momentum, the method consists of coupling two separate systems, both of which has been subjected to a different unitary transformation,

$$[R(\theta)|\ell_A,\ell_A\rangle]\otimes[R(-\theta)|\ell_B,\ell_B\rangle], \tag{1.35}$$

where $R(\theta)$ is a unitary transformation and $|\ell,m\rangle$ is an angular momentum ket. These two separate systems are simply coherent states constructed in the manner of Eq.(1.34). Then, extracted from this using a non-unitary projection,

$$\hat{\Pi}^\ell=\sum_m|\ell,m\rangle\langle\ell,m|, \tag{1.36}$$

are the states which have good angular momentum. This projection can be likened to a measurement of ℓ.

The second half of this thesis is a generalization of the construction of the su(2) intelligent states to the su(3) intelligent states. Up until now, emphasis has been placed on constructing the intelligent states for the su(2) and su(1,1) algebras. This is due, in part, to the fact that it is possible to experimentally generate both SU(2) and SU(1,1) transformations reasonably easily. Also, only 3 generators are needed to completely describe each of the su(2) and su(1,1) algebras, while the su(3) algebra requires 8 generators. This fact alone compounds the difficulty of the problem, which made it impractical to solve, in general, for the complete set of intelligent states. It was felt that more insight as to the proper approach to use, could be gained by reducing the generality and solving the problem for a more restricted set of operators. As a result, the operators that were chosen are sufficiently complicated, so as to avoid a repetition of the su(2) problem, but are simple enough to allow a solution that is manageable. The overall method for the su(3) intelligent states follows the same procedure as the angular momentum intelligent states. The biggest difference is that, whereas the angular momentum case requires the coupling of two systems, as given in Eq.(1.35), the su(3) case requires the coupling of three systems. Note that it is possible to experimentally generate SU(3) transformations, and in principle SU(N) transformations, although the task is technically very difficult.

As well as generating the intelligent states for specific systems, the method presented herein was developed with interest in a construction of intelligent states that is generalizable to other Lie algebras without much modification. This was attempted in [25], but the complexity of the polynomial states grew quickly with the dimension of the problem. The method presented in this thesis largely circumvents that issue by exploiting Lie algebraic methods.

1.4 Intelligent States

1.4.1 Properties

The idea of intelligence and intelligent states is not a new one, and it has been studied from a number of different viewpoints [8, 18, 23, 22, 25, 28, 29, 30]. The definition of intelligence has not changed from that of [8]: it is simply any state for which the the inequality in Eq.(1.2) has been replaced by an equality. This definition comes with two constraints on the state $|\psi\rangle$. These constraints can be discerned from the derivation of Eq.(1.2), which is found in Appendix B.1. While the derivation given is not unique, see [31] for instance, the constraints themselves are properties of the uncertainty relation, and as such do not depend on the particular derivation used. The derivation presented in this thesis follows that of [32], and is used because it is straight forward and highlights the constraints clearly.

In order to get from Eq.(1.2) to Eq.(1.5), we begin by noting that

$$(\Delta\hat{\Omega})^2 = \langle\psi|(\hat{\Omega} - \langle\hat{\Omega}\rangle)^2|\psi\rangle. \tag{1.37}$$

Then, from Eq.(B.8)

$$(\Delta\hat{\Omega})^2(\Delta\hat{\mathcal{O}})^2 \geq |\langle\psi|\frac{1}{2}\{\hat{\Omega}-\langle\hat{\Omega}\rangle,\hat{\mathcal{O}}-\langle\hat{\mathcal{O}}\rangle\}+\frac{1}{2}[\hat{\Omega},\hat{\mathcal{O}}]|\psi\rangle|^2, \tag{1.38}$$

which can be written as

$$(\Delta\hat{\Omega})^2(\Delta\hat{\mathcal{O}})^2 \geq |\langle\psi|(\hat{\Omega}-\langle\hat{\Omega}\rangle)(\hat{\mathcal{O}}-\langle\hat{\mathcal{O}}\rangle)|\psi\rangle|^2. \tag{1.39}$$

Since the operators $\hat{\Omega}$ and $\hat{\mathcal{O}}$ are assumed self-adjoint, one can see that Eq.(1.39) is just a statement of the Schwartz inequality;

$$\langle\varphi|\varphi\rangle\langle\chi|\chi\rangle \geq |\langle\varphi|\chi\rangle|^2, \tag{1.40}$$

with

$$|\chi\rangle = (\hat{\mathcal{O}}-\langle\hat{\mathcal{O}}\rangle)|\psi\rangle, \quad |\varphi\rangle = (\hat{\Omega}-\langle\hat{\Omega}\rangle)|\psi\rangle. \tag{1.41}$$

Requiring the equality in Eq.(1.40), is equivalent to requiring the equality in Eq.(1.38). The condition for this is known, and is simply that the two vectors $|\chi\rangle$ and $|\varphi\rangle$ be collinear, or

$$|\varphi\rangle = i\alpha|\chi\rangle. \tag{1.42}$$

Using Eq.(1.41) gives

$$(\hat{\Omega}-\langle\hat{\Omega}\rangle)|\psi\rangle = i\alpha(\hat{\mathcal{O}}-\langle\hat{\mathcal{O}}\rangle)|\psi\rangle. \tag{1.43}$$

where α is an arbitrary complex number. We now have the equality in Eq.(1.38)

$$(\Delta\hat{\Omega})^2(\Delta\hat{\mathcal{O}})^2 = |\langle\psi|\frac{1}{2}\{\hat{\Omega}-\langle\hat{\Omega}\rangle,\hat{\mathcal{O}}-\langle\hat{\mathcal{O}}\rangle\}+\frac{1}{2}[\hat{\Omega},\hat{\mathcal{O}}]|\psi\rangle|^2. \tag{1.44}$$

At this point, the solutions to Eq.(1.43) are called the generalized intelligent states. This work, however, deals with states satisfying Eq.(1.5), so we require the anti-commutator term, $\langle\{\hat{\Omega}-\langle\hat{\Omega}\rangle,\hat{\mathcal{O}}-\langle\hat{\mathcal{O}}\rangle\}\rangle$, in Eq.(1.44) to be zero. The first condition, Eq.(1.43), combined with the additional requirement that $-\infty \leq \alpha \leq \infty$ is real, is sufficient to make this happen. Looking again at Eq.(1.43), and rearranging, results in an equation that depends only on the parameter α:

$$(\hat{\Omega}-i\alpha\hat{\mathcal{O}})|\psi\rangle = \lambda|\psi\rangle. \quad \begin{cases} \lambda = \langle\hat{\Omega}\rangle - i\alpha\langle\hat{\mathcal{O}}\rangle \\ \alpha\in\mathbb{R} \end{cases} \tag{1.45}$$

This is nothing more than an eigenvalue equation, the solutions of which are an intelligent states.

1.4.2 Example: $\hat{x}$ and $\hat{p}$

To illustrate the situation, consider the following example. If the operators $\hat{x} = x$ and $\hat{p} = \frac{1}{i}\frac{d}{dx}$, with $\hbar = 1$, are substituted into Eq.(1.45), the result is a differential equation of the form

$$(x - i\alpha(\frac{1}{i}\frac{d}{dx}))\psi(x) = (\langle\hat{x}\rangle - i\alpha\langle\hat{p}\rangle)\psi(x). \tag{1.46}$$

What is interesting is the solution to this equation,

$$\psi(x) = c\, e^{i\langle p\rangle x - \frac{1}{2\alpha}(x-\langle x\rangle)^2}, \tag{1.47}$$

is a harmonic oscillator ground state, with the parameter α controlling the ratio of $\Delta x/\Delta p$. For $\alpha = 1$, we have a coherent state; for $\alpha \neq 1$ the state is squeezed. This result is part of the motivation to consider coherent states as tools to construct intelligent states in other systems.

1.4.3 Example: Spin 1/2 Case

For an introductory example to the construction of su(2) intelligent states, we will work through the explicit solution for the spin-$\frac{1}{2}$ problem. To do this we consider the simplest realization of $\hat{L}_x - i\alpha\hat{L}_y$, using the basis states, $|\ell = \frac{1}{2}, m = \frac{1}{2}\rangle$ and $|\ell = \frac{1}{2}, m = -\frac{1}{2}\rangle$, for which

$$\hat{L}_z \mapsto \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \hat{L}_x \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \hat{L}_y \mapsto \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.48}$$

Equation (1.5) then takes the form

$$\Delta L_x \Delta L_y = \tfrac{1}{2}|\langle\hat{L}_z\rangle|\,. \tag{1.49}$$

From Eq.(1.48), we can obtain the 2×2 matrix

$$\hat{L}_x - i\alpha\hat{L}_y \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1-\alpha \\ 1+\alpha & 0 \end{pmatrix}. \tag{1.50}$$

The (unnormalized) eigenstates of Eq.(1.50), which are by definition intelligent states, are just

$$\begin{pmatrix} 1 \\ \frac{1+\alpha}{\sqrt{1-\alpha^2}} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\frac{1+\alpha}{\sqrt{1-\alpha^2}} \end{pmatrix}, \tag{1.51}$$

with respective eigenvalues

$$\lambda_+ = \lambda \equiv \tfrac{1}{2}\sqrt{1-\alpha^2}\,, \quad \lambda_- = -\lambda\,. \tag{1.52}$$

In order to simplify the notation it is convenient to introduce the quantity

$$\mu = \frac{1+\alpha}{\sqrt{1-\alpha^2}}, \tag{1.53}$$

which has the property that when $|\alpha| \leq 1$, μ is real, while when $|\alpha| > 1$, μ is purely imaginary. We can now write the normalized $\ell = \frac{1}{2}$ intelligent states in a cleaner manner, as

$$|\psi_-^{1/2}(\alpha)\rangle = \frac{1}{\sqrt{1+|\mu|^2}} \begin{pmatrix} 1 \\ -\mu \end{pmatrix} \tag{1.54}$$

$$|\psi_+^{1/2}(\mu)\rangle = \frac{1}{\sqrt{1+|\mu|^2}} \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \tag{1.55}$$

We can rewrite these, using the identification

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto |\tfrac{1}{2}, \tfrac{1}{2}\rangle, \tag{1.56}$$

to more explicitly show the composition in terms of angular momentum $\ell = \frac{1}{2}$:

$$|\psi_-^{1/2}(\alpha)\rangle = \frac{1}{\sqrt{1+|\mu|^2}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle - \frac{\mu}{\sqrt{1+|\mu|^2}}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \tag{1.57}$$

$$|\psi_+^{1/2}(\mu)\rangle = \frac{1}{\sqrt{1+|\mu|^2}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle + \frac{\mu}{\sqrt{1+|\mu|^2}}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle. \tag{1.58}$$

Recalling that if $|\alpha| < 1$, μ is real, we can write

$$|\psi_\pm^{1/2}(\alpha)\rangle = \begin{pmatrix} \cos {}^{\beta}\!/_2 & \mp \sin {}^{\beta}\!/_2 \\ \pm \sin {}^{\beta}\!/_2 & \cos {}^{\beta}\!/_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R_y(\pm\beta)|\tfrac{1}{2}, \tfrac{1}{2}\rangle, \tag{1.59}$$

with

$$R_y(\pm\beta) = \mathrm{e}^{\mp i\beta\hat{L}_y}, \tag{1.60}$$

$\hat{L}_y$ given in Eq.(1.48), and

$$\cos\tfrac{\beta}{2} = \frac{1}{\sqrt{1+|\mu|^2}}, \quad \sin\tfrac{\beta}{2} = \frac{\mu}{\sqrt{1+|\mu|^2}}. \tag{1.61}$$

We can clearly see now that Eq.(1.59) is equivalent to Eq.(1.34) with $\ell = \frac{1}{2}$ and $\gamma = 0$, thus showing how Eq.(1.59) is related to SU(2) coherent states.

On the other hand, when $|\alpha| \geq 1$, μ is purely imaginary, we have

$$|\psi_\pm^{1/2}(\beta)\rangle = R_x(\pm\beta)|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \mathrm{e}^{\mp i\beta\hat{L}_x}|\tfrac{1}{2}, \tfrac{1}{2}\rangle, \tag{1.62}$$

where, this time,

$$\cos\frac{\beta}{2} = \frac{1}{\sqrt{1+|\mu|^2}}, \quad i\sin\frac{\beta}{2} = \frac{\mu}{\sqrt{1+|\mu|^2}}. \tag{1.63}$$

It is less clear that this is a coherent state, but using Eq.(1.34) with $\ell = \frac{1}{2}$ and $\gamma = \frac{\pi}{2}$, you will recover Eq.(1.62). Thus the two $\ell = \frac{1}{2}$ angular momentum intelligent states are just angular momentum coherent states.

1.4.4 Another Example: Spin 1 Case

Consider now an $\ell = 1$ angular momentum system. The angular momentum operators are represented by 3×3 matrices. We will explicitly work through the solution and find the three $\ell = 1$ intelligent states, and show how two of them are simply angular momentum coherent states. This will allow us to compare later to the ones constructed from coupling the $\ell = \frac{1}{2}$ intelligent states together. This will verify the claim that any su(2) intelligent state can be constructed from coupling sufficiently many copies of the $|\psi_+^{1/2}(\alpha)\rangle$ and $|\psi_-^{1/2}(\alpha)\rangle$ together.

To begin we will explicitly construct the 3×3 operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$. We need the relations

$$\begin{aligned} \hat{L}_x &= \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \\ \hat{L}_y &= \frac{1}{2i}(\hat{L}_+ - \hat{L}_-), \end{aligned} \tag{1.64}$$

along with, taking $\hbar = 1$,

$$\hat{L}_\pm|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle, \tag{1.65}$$

and

$$\hat{L}_z|\ell, m\rangle = m|\ell, m\rangle. \tag{1.66}$$

To construct $\hat{L}_z$, for example, we need to construct the matrix elements $\langle 1, m'|\hat{L}_z|1, m\rangle$ for all m' and m from 1 to -1. Using the identification;

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto |1,1\rangle, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto |1,0\rangle, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto |1,-1\rangle, \tag{1.67}$$

the three matrices are then

$$\hat{L}_x \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad \hat{L}_y \mapsto \frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix},$$

$$\hat{L}_z \mapsto \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{1.68}$$

We can now find the intelligent states by diagonalization, this time, of the 3×3 matrix

$$\hat{L}_x - i\alpha\hat{L}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-\alpha & 0 \\ 1+\alpha & 0 & 1-\alpha \\ 0 & 1+\alpha & 0 \end{pmatrix}, \tag{1.69}$$

and rapidly determine the three unnormalized intelligent states as:

$$|\psi^1_{--}\rangle = \begin{pmatrix} 1 \\ -\sqrt{2}\mu \\ \mu^2 \end{pmatrix}, \quad |\psi^1_{+-}\rangle = \begin{pmatrix} 1 \\ 0 \\ -\mu^2 \end{pmatrix}, \quad |\psi^1_{++}\rangle = \begin{pmatrix} 1 \\ \sqrt{2}\mu \\ \mu^2 \end{pmatrix}, \tag{1.70}$$

where μ is defined in Eq.(1.53) and the reason for the notation will become clear later.
We can write the normalized intelligent states as;

$$\begin{aligned}
|\psi^1_{--}\rangle &= \frac{1}{1+|\mu|^2}(|1,1\rangle - \sqrt{2}\mu|1,0\rangle + \mu^2|1,-1\rangle) \\
|\psi^1_{+-}\rangle &= \frac{1}{\sqrt{1+|\mu|^4}}(|1,1\rangle - \mu^2|1,-1\rangle) \\
|\psi^1_{++}\rangle &= \frac{1}{1+|\mu|^2}(|1,1\rangle + \sqrt{2}\mu|1,0\rangle + \mu^2|1,-1\rangle).
\end{aligned} \tag{1.71}$$

In the case where μ is real, we can use Eq.(1.61) and

$$\mu^2 = \frac{1+\cos\beta}{1-\cos\beta}, \tag{1.72}$$

to further rewrite the $\ell = 1$ intelligent states;

$$\begin{aligned}
|\psi^1_{--}\rangle &= \cos^2\tfrac{\beta}{2}|1,1\rangle - \sqrt{2}\cos\tfrac{\beta}{2}\sin\tfrac{\beta}{2}|1,0\rangle + \sin^2\tfrac{\beta}{2}|1,-1\rangle \\
|\psi^1_{+-}\rangle &= \frac{2}{\sqrt{3+\cos 2\beta}}\left(\cos^2\tfrac{\beta}{2}|1,1\rangle - \sin^2\tfrac{\beta}{2}|1,-1\rangle\right) \\
|\psi^1_{++}\rangle &= \cos^2\tfrac{\beta}{2}|1,1\rangle + \sqrt{2}\cos\tfrac{\beta}{2}\sin\tfrac{\beta}{2}|1,0\rangle + \sin^2\tfrac{\beta}{2}|1,-1\rangle.
\end{aligned} \tag{1.73}$$

Although it is not immediately clear, the states $|\psi^1_{--}\rangle$ and $|\psi^1_{++}\rangle$ are $\ell = 1$ coherent states. This can be shown quite easily by using $\hat{L}_y$ from Eq.(1.68) and

$$\hat{L}_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \hat{L}_y^3 = \hat{L}_y, \tag{1.74}$$

and expanding the rotation $R_y(\beta)$ using a Taylor series in β:

$$\mathrm{e}^{-i\beta\hat{L}_y} = \mathbb{1} - i\beta\hat{L}_y - \frac{\beta^2}{2}\hat{L}_y^2 + i\frac{\beta^3}{3!}\hat{L}_y + \ldots \tag{1.75}$$

If we then add zero in the form of $\hat{L}_y^2 - \hat{L}_y^2$ we can recompose the series in terms of $\cos\beta$ and $\sin\beta$:

$$\begin{aligned} e^{-i\beta\hat{L}_y} &= \mathbb{1} - \hat{L}_y^2 + \hat{L}_y^2(1 - \frac{\beta^2}{2} + \ldots) - i\hat{L}_y(\beta - \frac{\beta^3}{3!} + \ldots) \\ &= \mathbb{1} - \hat{L}_y^2 + \hat{L}_y^2\cos\beta - i\hat{L}_y\sin\beta. \end{aligned} \tag{1.76}$$

We can now use this to explicitly rotate the state, $|1,1\rangle$, using the form of Eq.(1.67):

$$\begin{aligned} e^{-i\beta\hat{L}_y}|1,1\rangle &= (\mathbb{1} - \hat{L}_y^2 + \hat{L}_y^2\cos\beta - i\hat{L}_y\sin\beta)|1,1\rangle \\ &= \frac{1}{2}\begin{pmatrix} 1+\cos\beta & -\sqrt{2}\sin\beta & 1-\cos\beta \\ \sqrt{2}\sin\beta & 2\cos\beta & -\sqrt{2}\sin\beta \\ 1-\cos\beta & \sqrt{2}\sin\beta & 1+\cos\beta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1+\cos\beta \\ \sqrt{2}\sin\beta \\ 1-\cos\beta \end{pmatrix} \end{aligned} \tag{1.77}$$

We can now break up this state into its separate angular momentum components:

$$\begin{aligned} e^{-i\beta\hat{L}_y}|1,1\rangle &= \frac{1+\cos\beta}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1-\cos\beta}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1+\cos\beta}{2}|1,1\rangle + \frac{\sin\beta}{\sqrt{2}}|1,0\rangle + \frac{1-\cos\beta}{2}|1,-1\rangle \end{aligned} \tag{1.78}$$

Finally, using the identities

$$\begin{aligned} 1+\cos\beta &= 2\cos^2\tfrac{\beta}{2}\,, \\ 1-\cos\beta &= 2\sin^2\tfrac{\beta}{2}\,, \\ \sin\beta &= 2\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\,, \end{aligned} \tag{1.79}$$

we have the angular momentum coherent state that is of the form $R_y(\beta)|\ell,\ell\rangle$;

$$\begin{aligned} R_y(\beta)|1,1\rangle &= e^{-i\beta\hat{L}_y}|1,1\rangle \\ &= \cos^2\tfrac{\beta}{2}|1,1\rangle + \sqrt{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}|1,0\rangle + \sin^2\tfrac{\beta}{2}|1,-1\rangle \\ &= |\psi^1_{++}\rangle\,. \end{aligned} \tag{1.80}$$

Also, if we pick $-\beta$ instead, we find that Eq.(1.80) becomes:

$$\begin{aligned} R_y(-\beta)|1,1\rangle &= e^{i\beta\hat{L}_y}|1,1\rangle \\ &= \cos^2\tfrac{\beta}{2}|1,1\rangle - \sqrt{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}|1,0\rangle + \sin^2\tfrac{\beta}{2}|1,-1\rangle \\ &= |\psi^1_{--}\rangle\,, \end{aligned} \tag{1.81}$$

which is simply the coherent state for the corresponding negative rotation. It is clear now that there is a relation between coherent states and intelligent states. What is interesting is that there are angular momentum intelligent states which are not simply angular momentum coherent states. The third $\ell = 1$ intelligent state, $|\psi^1_{+-}\rangle$, does not correspond to an $\ell = 1$ coherent state, but is in fact, the result of coupling the two $\ell = \frac{1}{2}$ coherent states $R_y(\beta)|\frac{1}{2}, \frac{1}{2}\rangle$ and $R_y(-\beta)|\frac{1}{2}, \frac{1}{2}\rangle$. This will be shown explicitly when the idea that all angular momentum intelligent states can be constructed using these two $\ell = \frac{1}{2}$ coherent states is investigated.

Chapter 2

su(2) Intelligent States

2.1 Some Background

2.1.1 The su(2) Algebra

The su(2) algebra is commonly constructed from the Pauli spin matrices, which are the traceless hermitian operators

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

The operation that is used to define the su(2) and all of the su(N) algebras is the commutator. The matrix commutator is the Lie bracket. The su(2) operators, Eq.(2.1), then satisfy the following commutation relations, where $\hat{L}_j = \frac{1}{2}\hat{\sigma}_j$:

$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hat{L}_y, \tag{2.2}$$

along with the Jacobi identity,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{2.3}$$

The commutator of any two arbitrary linear combinations of generators is another linear combination of generators. Thus the su(2) algebra is said to be closed under commutation. As an example, consider the 2×2 representation of Eq.(2.1). Using

$\hat{L}_j = \frac{1}{2}\hat{\sigma}_j$ one finds;

$$\begin{aligned} [\hat{L}_x, \hat{L}_y] &= \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x \\ &= \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hat{L}_z \end{aligned} \tag{2.4}$$

The elements of the su(2) algebra act linearly on composite states; consider the operator

$$\begin{aligned} \hat{L}_j &= \hat{L}_j \otimes 1\!\!1_2 \otimes \ldots \otimes 1\!\!1_N + 1\!\!1_1 \otimes \hat{L}_j \otimes \ldots \otimes 1\!\!1_N \\ &\quad + 1\!\!1_1 \otimes 1\!\!1_2 \otimes \ldots \otimes \hat{L}_j \\ &= \hat{L}_{1,j} + \hat{L}_{2,j} + \ldots + \hat{L}_{N,j}. \end{aligned} \tag{2.5}$$

The action of the operator $\hat{L}_j$ on a composite system is

$$\begin{aligned} &\hat{L}_j|\phi\rangle = (\hat{L}_{1,j} + \hat{L}_{2,j} + \ldots + \hat{L}_{N,j})[|\phi_A\rangle_1 \otimes |\phi_B\rangle_2 \otimes \ldots \otimes |\phi_V\rangle_N] \\ &= [\hat{L}_{1,j}|\phi_A\rangle_1] \otimes |\phi_B\rangle_2 \otimes \ldots \otimes |\phi_V\rangle_N + |\phi_A\rangle_1 \otimes [\hat{L}_{2,j}|\phi_B\rangle_2] \otimes \ldots \otimes |\phi_V\rangle_N \\ &\quad + |\phi_A\rangle_1 \otimes |\phi_B\rangle_2 \otimes \ldots \otimes [\hat{L}_{N,j}|\phi_V\rangle_N] \end{aligned} \tag{2.6}$$

The ability to combine elements of the algebra in a linear manner leads to the definition of the very useful su(2) ladder operators:

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y, \tag{2.7}$$

which have the following commutation relations:

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_z, \quad [\hat{L}_z, \hat{L}_\pm] = \pm\hat{L}_\pm. \tag{2.8}$$

Using the usual notation $|\ell, m\rangle$, where $|\ell, m\rangle$ is an eigenstate of both $\hat{L}_z$ and $\hat{L}^2$, the action of the ladder operators is

$$\hat{L}_\pm|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m \pm 1)}|\ell, m \pm 1\rangle. \tag{2.9}$$

The ladder operators allow one to move between states that have the same angular momentum, ℓ.

2.1.2 The SU(2) Group

The set of 2×2 unitary matrices with determinant 1 form a group called SU(2) (for Special (*i.e.* determinant 1) Unitary matrices in dimension 2), *i.e.* the 2×2 matrix T is an element of SU(2) if

$$T^\dagger = T^{-1} \tag{2.10}$$

and

$$\det(T) = +1\,. \tag{2.11}$$

The SU(2) group is also a Lie group; for the purpose of this thesis, this means every SU(2) element can be obtained by exponentiation of an element of the su(2) Lie algebra *i.e.* the operators of Eq.(2.1) or any arbitrary real linear combination thereof. Thus, any group element can be constructed as

$$T = \exp(i(\alpha_x\hat{\sigma}_x + \alpha_y\hat{\sigma}_y + \alpha_z\hat{\sigma}_z)/2). \tag{2.12}$$

The expression of Eq.(2.12) is not unique. In fact, we also have quite generally

$$T = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}, \tag{2.13}$$

with a and b complex numbers satisfying

$$aa^* + bb^* = 1. \tag{2.14}$$

In particular, if we write

$$a = |a|\, e^{-i\eta_a}\,, \quad b = |b|\, e^{-i\eta_b} \tag{2.15}$$

then we can define

$$|a| = \cos\tfrac{\beta}{2}\,, \quad |b| = \sin\tfrac{\beta}{2}, \tag{2.16}$$

and one can easily verify that the matrix T can be written as

$$\begin{aligned} T &= \begin{pmatrix} e^{-i\eta_a}\cos\frac{\beta}{2} & -e^{-i\eta_b}\sin\frac{\beta}{2} \\ e^{i\eta_b}\sin\frac{\beta}{2} & e^{i\eta_a}\cos\frac{\beta}{2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix} \\ &= R_z(\gamma)R_y(\beta)R_z(\vartheta), \end{aligned} \tag{2.17}$$

where $\eta_a = \frac{1}{2}(\gamma+\vartheta)$, $\eta_b = \frac{1}{2}(\gamma-\vartheta)$, and

$$R_z(\gamma) = e^{-i\gamma\hat{L}_z}\,, \qquad R_y(\beta) = e^{-i\beta\hat{L}_y}\,. \tag{2.18}$$

For an example, consider the element

$$R_x(\varphi) = \mathrm{e}^{-i\varphi\hat{L}_x} = \begin{pmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}. \tag{2.19}$$

It is unitary;

$$\begin{aligned} R_x(\varphi)(R_x(\varphi))^\dagger &= \begin{pmatrix} \cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2} & -i(\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}-\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}) \\ -i(\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}-\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}) & \cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= R_x(\varphi)(R_x(\varphi))^{-1}, \end{aligned} \tag{2.20}$$

so that

$$(R_x(\varphi))^\dagger = R_x(-\varphi) = (R_x(\varphi))^{-1}, \tag{2.21}$$

and has a determinant of +1;

$$\begin{aligned} \det(R_x(\varphi)) &= \begin{vmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{vmatrix} \\ &= \cos^2\frac{\varphi}{2} - i^2\sin^2\frac{\varphi}{2} \\ &= \cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2} \\ &= 1. \end{aligned} \tag{2.22}$$

The SU(2) group is a group of order ∞ because the group elements, Eq.(2.17), are defined in terms of continuous real parameters. Therefore, every unique choice of these yields a unique group element.

Every group has an operation that specifies how the elements are combined. The group operation for the Lie groups is multiplication. To combine group elements, one needs simply to multiply them together. The result of such a multiplication is itself an element of the SU(2) group. Take the following product for example:

$$\begin{aligned} R_y(\beta)R_z(\gamma) &= \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\gamma/2} & 0 \\ 0 & \mathrm{e}^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-i\gamma/2}\cos\frac{\beta}{2} & -\mathrm{e}^{i\gamma/2}\sin\frac{\beta}{2} \\ \mathrm{e}^{-i\gamma/2}\sin\frac{\beta}{2} & \mathrm{e}^{i\gamma/2}\cos\frac{\beta}{2} \end{pmatrix}. \end{aligned} \tag{2.23}$$

It is not immediately clear that Eq.(2.23) is an element of the SU(2) group. However, we can easily show that it possesses the properties of an SU(2) element. Using the fact that $R_y(\beta)$ and $R_z(\gamma)$ are SU(2) group elements, and thus both have determinant equal to 1, along with the multiplicative property of determinants [33],

$$\det(\hat{A}\hat{B}) = \det(\hat{A})\det(\hat{B}), \tag{2.24}$$

we can see that the product $R_y(\beta)R_z(\gamma)$ also has determinant 1:

$$\begin{aligned} \det(R_y(\beta)R_z(\gamma)) &= \det(R_y(\beta))\det(R_z(\gamma)) \\ &= (1)(1) = 1. \end{aligned} \tag{2.25}$$

Evaluating the product $R_y(\beta)R_z(\gamma)[R_y(\beta)R_z(\gamma)]^\dagger$ using equivalent forms of the relation Eq.(2.21) for $R_y(\beta)$ and $R_z(\gamma)$, one quickly finds

$$\begin{aligned} R_y(\beta)R_z(\gamma)[R_y(\beta)R_z(\gamma)]^\dagger &= R_y(\beta)R_z(\gamma)(R_z(\gamma))^\dagger(R_y(\beta))^\dagger \\ &= R_y(\beta)R_z(\gamma)(R_z(\gamma))^{-1}(R_y(\beta))^{-1} \\ &= R_y(\beta)\,\mathbb{1}\,(R_y(\beta))^{-1} \\ &= \mathbb{1}\,R_y(\beta)(R_y(\beta))^{-1} \\ &= \mathbb{1}, \end{aligned} \tag{2.26}$$

implying that

$$[R_y(\beta)R_z(\gamma)]^\dagger = [R_y(\beta)R_z(\gamma)]^{-1}. \tag{2.27}$$

The product $R_y(\beta)R_z(\gamma)$ is thus an element of the SU(2) group. Determining which element, however, is not a trivial endeavour. Since determining this is not central to this thesis, it will be omitted with a note that the details of the calculation can be found in [26].

When one has a composite state, $|\phi\rangle = |\phi_A\rangle_1 \otimes |\phi_B\rangle_2 \otimes \ldots \otimes |\phi_V\rangle_N$, the composite group element is constructed as the product of the group elements for each system:

$$R_j(\theta) = R_j^1(\theta) \otimes R_j^2(\theta) \otimes \ldots \otimes R_j^N(\theta). \tag{2.28}$$

The action of this composite group element on the composite system is simply for each individual rotation to act on its appropriate system and no others:

$$R_j(\theta)|\phi\rangle = [R_j^1(\theta)|\phi_A\rangle_1] \otimes [R_j^2(\theta)|\phi_B\rangle_2] \otimes \ldots \otimes [R_j^N(\theta)|\phi_V\rangle_N]. \tag{2.29}$$

Finally we note that it is possible to find matrices which satisfy Eq.(2.2) but are of dimension larger than 2. For instance, the three-dimensional matrix representation of angular momentum operators for states of angular momentum $\ell = 1$:

$$\hat{L}_x \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \hat{L}_y \mapsto \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \hat{L}_z \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.30}$$

The exponentiation of these matrices produces a 3×3 representation of the corresponding group element. One important feature of this representation is that, if the 2×2 unitary matrices T_1 and T_2:

$$T_1 = \exp(i(\alpha_x\hat{\sigma}_x + \alpha_y\hat{\sigma}_y + \alpha_z\hat{\sigma}_z)/2) \tag{2.31}$$

$$T_2 = \exp(i(\beta_x\hat{\sigma}_x + \beta_y\hat{\sigma}_y + \beta_z\hat{\sigma}_z)/2), \tag{2.32}$$

combine as

$$T_1 \cdot T_2 = T_3, \tag{2.33}$$

with

$$T_3 = \exp(i(\gamma_x\hat{\sigma}_x + \gamma_y\hat{\sigma}_y + \gamma_z\hat{\sigma}_z)/2) \tag{2.34}$$

and $\gamma_x, \gamma_y, \gamma_z$ some complicated function of α_x, β_x etc, then the corresponding 3×3 matrices

$$\widetilde{T}_1 = \exp(i(\alpha_x \hat{L}_x + \alpha_y \hat{L}_y + \alpha_z \hat{L}_z)) \tag{2.35}$$
$$\widetilde{T}_2 = \exp(i(\beta_x \hat{L}_x + \beta_y \hat{L}_y + \beta_z \hat{L}_z)), \tag{2.36}$$

combine in an identical way:

$$\widetilde{T}_1 \cdot \widetilde{T}_2 = \widetilde{T}_3 \tag{2.37}$$
$$\widetilde{T}_3 = \exp(i(\gamma_x \hat{L}_x + \gamma_y \hat{L}_y + \gamma_z \hat{L}_z)). \tag{2.38}$$

2.1.3 Angular Momentum Coherent States

In section 1.2.3, the idea that we can extend the construction of coherent states to systems other than the harmonic oscillator was introduced, and the form of the angular momentum coherent state was given without much explanation. This section is devoted to showing how the construction of the coherent state can be realized in angular momentum systems, using a definition analogous to Eq.(1.21).

It is useful to note that the operator $\hat{L}_z$ is a diagonal operator in the basis spanned by $\{|\ell, m\rangle,\ m = -\ell,\ -\ell+1,\ \ldots,\ \ell-1,\ \ell\}$. We then rewrite the rotation Eq.(2.17) as

$$e^{i\gamma\hat{L}_z} e^{i\beta\hat{L}_y} e^{-i\gamma\hat{L}_z} e^{i(\vartheta+\gamma)\hat{L}_z}, \tag{2.39}$$

so that the transformation of the special state $|\ell, \ell\rangle$, which is simply an extremal state, becomes

$$e^{i\gamma\hat{L}_z} e^{i\beta\hat{L}_y} e^{-i\gamma\hat{L}_z} e^{i(\vartheta+\gamma)\hat{L}_z} |\ell, \ell\rangle = e^{i(\vartheta+\gamma)\ell} (e^{i\gamma\hat{L}_z} e^{i\beta\hat{L}_y} e^{-i\gamma\hat{L}_z} |\ell, \ell\rangle). \tag{2.40}$$

Due to the fact that the rightmost rotation, $e^{i(\vartheta+\gamma)\hat{L}_z}$, is about the z-axis, it simply produces a global phase when acting on $|\ell, \ell\rangle$. Because this is common to the entire state, it does not affect any property of this state and therefore ϑ can be chosen so that the phase is zero. This gives us a general form for the SU(2) coherent states in terms of two parameters:

$$|\beta, \gamma\rangle = R_z(\gamma) R_y(\beta) R_z(-\gamma) |\ell, \ell\rangle. \tag{2.41}$$

Now that we have the general form, we can show that it is equivalent to the form of Eq.(1.32).

We begin by using $e^{-A}e^{A} = \mathbb{1}$ to form the relation

$$\begin{aligned}
e^{A}e^{B}e^{-A} &= e^{A}\left(\mathbb{1} + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \ldots\right)e^{-A} \\
&= \mathbb{1} + e^{A}Be^{-A} + \frac{1}{2}(e^{A}Be^{-A})(e^{A}Be^{-A}) \\
&\quad + \frac{1}{3!}(e^{A}Be^{-A})(e^{A}Be^{-A})(e^{A}Be^{-A}) + \ldots \\
&= \mathbb{1} + e^{A}Be^{-A} + \frac{1}{2}(e^{A}Be^{-A})^2 + \frac{1}{3!}(e^{A}Be^{-A})^3 + \ldots \\
&= \exp(e^{A}Be^{-A}).
\end{aligned} \tag{2.42}$$

This allows us to write

$$\begin{aligned}
R_z(\gamma)R_y(\beta)R_z(-\gamma)|\ell,\ell\rangle &= e^{i\gamma\hat{L}_z}e^{i\beta\hat{L}_y}e^{-i\gamma\hat{L}_z}|\ell,\ell\rangle \\
&= \exp(i\beta(e^{i\gamma\hat{L}_z}\hat{L}_y e^{-i\gamma\hat{L}_z}))|\ell,\ell\rangle.
\end{aligned} \tag{2.43}$$

Noting that

$$\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) \tag{2.44}$$

and using the identity,

$$\begin{aligned}
e^{i\gamma\hat{L}_z}\hat{L}_\pm e^{-i\gamma\hat{L}_z} &= \hat{L}_\pm + i\gamma[\hat{L}_z,\hat{L}_\pm] - \gamma^2[\hat{L}_z,[\hat{L}_z,\hat{L}_\pm]] - i\gamma^3[\hat{L}_z,[\hat{L}_z,[\hat{L}_z,\hat{L}_\pm]]] + \ldots \\
&= \hat{L}_\pm \pm i\gamma\hat{L}_\pm - \frac{\gamma^2}{2}\hat{L}_\pm \mp \frac{i\gamma^3}{3!}\hat{L}_\pm + \ldots \\
&= \left(\mathbb{1} \pm i\gamma - \frac{\gamma^2}{2} \mp \frac{i\gamma^3}{3!} + \ldots\right)\hat{L}_\pm \\
&= e^{\pm i\gamma}\hat{L}_\pm,
\end{aligned} \tag{2.45}$$

we can write,

$$\begin{aligned}
\exp(e^{i\gamma\hat{L}_z}i\beta\hat{L}_y e^{-i\gamma\hat{L}_z})|\ell,\ell\rangle &= \exp(i\beta e^{i\gamma\hat{L}_z}\frac{1}{2i}(\hat{L}_+ - \hat{L}_-)e^{-i\gamma\hat{L}_z})|\ell,\ell\rangle \\
&= \exp(\frac{1}{2}\beta e^{i\gamma\hat{L}_z}(\hat{L}_+ - \hat{L}_-)e^{-i\gamma\hat{L}_z})|\ell,\ell\rangle \\
&= \exp(\frac{1}{2}\beta(e^{i\gamma}\hat{L}_+ - e^{-i\gamma}\hat{L}_-))|\ell,\ell\rangle.
\end{aligned} \tag{2.46}$$

All we have to do now is introduce the complex variable $\zeta = \frac{1}{2}\beta e^{i\gamma}$ and we have

$$|\beta,\gamma\rangle = e^{\zeta\hat{L}_+ - \zeta^*\hat{L}_-}|\ell,\ell\rangle, \tag{2.47}$$

which is exactly what we are looking for, showing the two forms Eq.(1.32) and Eq.(1.34) are equivalent. It is important to note that, since β and γ are arbitrary, every unique pair (β,γ) yields a unique coherent state.

We will be using the form of Eq.(1.34) since it is more convenient for the purpose of this thesis.

Since the states of Eq.(2.47) are coherent states they have specific properties like non-orthogonality and overcompleteness. See Appendix B.2 for the proof that these states do possess the required properties.

2.2 The SU(2) Building Blocks

Now that it is clear that angular momentum coherent states can be considered as arising from a unitary transformation of an extremal state, we will see how they can be combined together and built up to systems with higher values of angular momentum ℓ. In Section 1.4.3 it was shown how the solution to the simplest case, the spin-$\frac{1}{2}$ case, is a pair of $\ell = \frac{1}{2}$ angular momentum coherent states. Section 1.4.4 saw the explicit solution to the $\ell = 1$ problem using the 3×3 matrix representation. The results are two $\ell = 1$ angular momentum coherent states, plus one state that is not a coherent state. As is suggested by these results, the building blocks for angular momentum intelligent states are angular momentum coherent states. Let us, then, take a closer look at the angular momentum coherent state building blocks.

To construct coherent states we need an extremal state, *i.e.* what is known as a state of highest weight. States that are of highest weight are defined as those that return zero under the action of raising operators. The action of the raising operator for the su(2) algebra is given by

$$\hat{L}_+|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m+1)}|\ell, m+1\rangle. \tag{2.48}$$

For the right hand side of Eq.(2.48) to be zero we need

$$\ell(\ell+1) - m(m+1) = 0, \tag{2.49}$$

which leads to the condition

$$m = \ell. \tag{2.50}$$

Thus the su(2) state that has the highest weight is the state $|\ell, \ell\rangle$, so that

$$\hat{L}_+|\ell, \ell\rangle = 0. \tag{2.51}$$

If we consider the composition of two systems A and B, we can construct the composite operators for the system as

$$\begin{aligned} \hat{L}_{A,x} &\equiv \hat{L}_x \otimes 1\!\!1_B\ , \quad \hat{L}_{B,x} \equiv 1\!\!1_A \otimes \hat{L}_x\ , & (2.52)\\ \hat{L}_x &= \hat{L}_{A,x} + \hat{L}_{B,x}\ , & (2.53)\end{aligned}$$

where $1\!\!1_A$ and $1\!\!1_B$ are unit operators in their respective subspaces. Eq.(2.52) simply means that $\hat{L}_{A,x}$ acts on the first (or "A") subsystem only, leaving the second (or "B") subsystem alone, and similarly for $\hat{L}_{B,x}$. The operators

$$\hat{L}_y = \hat{L}_{A,y} + \hat{L}_{B,y}, \tag{2.54}$$
$$\hat{L}_z = \hat{L}_{A,z} + \hat{L}_{B,z} \tag{2.55}$$

are defined in a similar manner, with $[\hat{L}_x, \hat{L}_y] = i\hat{L}_z$. The raising operator then becomes

$$\hat{L}_+ = \hat{L}_{A,+} + \hat{L}_{B,+}. \tag{2.56}$$

This allows us to construct a highest weight state from two spin-$\frac{1}{2}$ states. To show this we begin by coupling two spin-$\frac{1}{2}$ highest weight states and act on the coupled states with $\hat{L}_+$. Using Eq.(2.51) one quickly finds

$$\begin{aligned}\hat{L}_+[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_B] &= [\hat{L}_{A,+}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A] \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_B + |\tfrac{1}{2},\tfrac{1}{2}\rangle_A \otimes [\hat{L}_{B,+}|\tfrac{1}{2},\tfrac{1}{2}\rangle_B] \\ &= 0,\end{aligned} \tag{2.57}$$

which means that the state‡ $|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle$ is a highest weight state. To find the value of ℓ we simply need to use the fact that

$$\hat{L}_z|\ell, m\rangle = m|\ell, m\rangle \tag{2.58}$$

along with the condition of Eq.(2.50) for highest weight states:

$$\begin{aligned}\hat{L}_z[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B] &= \left[\hat{L}_{A,z}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A\right]|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + |\tfrac{1}{2},\tfrac{1}{2}\rangle_A\left[\hat{L}_{B,z}|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] \\ &= \tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + \tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B \\ &= (1)|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B.\end{aligned} \tag{2.59}$$

Thus, the result of coupling two highest weight spin-$\frac{1}{2}$ states is a state that is a highest weight state and has angular momentum $\ell = 1$, or

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,1\rangle. \tag{2.60}$$

We can continue in this way and construct

$$|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle. \tag{2.61}$$

To get to the state $|\ell,\ell\rangle$, we need to simply couple 2ℓ spin-$\frac{1}{2}$ states together:

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B \dots |\tfrac{1}{2},\tfrac{1}{2}\rangle_{2\ell} = |\ell,\ell\rangle. \tag{2.62}$$

Once the required state has been generated via Eq.(2.62), the appropriate transformation can be applied to generate the coherent state:

$$\begin{aligned}|\beta,\gamma\rangle &= R_z(\gamma)R_y(\beta)R_z(-\gamma)\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B \dots |\tfrac{1}{2},\tfrac{1}{2}\rangle_{2\ell}\right] \\ &= R_z(\gamma)R_y(\beta)R_z(-\gamma)|\ell,\ell\rangle.\end{aligned} \tag{2.63}$$

‡For the remainder of this thesis the $\otimes$ will mostly be omitted when coupling states, unless a distinction is necessary, so that $|\varphi_1\rangle_A \otimes |\varphi_2\rangle_B \equiv |\varphi_1\rangle_A|\varphi_2\rangle_B$.

Alternatively, using

$$\hat{L}_j = \sum_k \hat{L}_{k,j}\,, \quad \begin{cases} j = x,\, y,\, z, \\ k = A,\, B,\, \ldots \end{cases} \tag{2.64}$$

one can start with 2ℓ copies of the coherent state $R_z(\gamma)R_y(\beta)R_z(-\gamma)|\frac{1}{2},\frac{1}{2}\rangle$, and couple these directly together:

$$\begin{aligned} |\beta,\gamma\rangle &= [R_z^A(\gamma)R_y^A(\beta)R_z^A(-\gamma)|\tfrac{1}{2},\tfrac{1}{2}\rangle_A] \otimes [R_z^B(\gamma)R_y^B(\beta)R_z^B(-\gamma)|\tfrac{1}{2},\tfrac{1}{2}\rangle_B] \otimes \ldots \\ &\quad \otimes \left[R_z^{2\ell}(\gamma)R_y^{2\ell}(\beta)R_z^{2\ell}(-\gamma)|\tfrac{1}{2},\tfrac{1}{2}\rangle_{2\ell}\right] \\ &= R_z(\gamma)R_y(\beta)R_z(-\gamma)\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B \ldots |\tfrac{1}{2},\tfrac{1}{2}\rangle_{2\ell}\right] \\ &= R_z(\gamma)R_y(\beta)R_z(-\gamma)|\ell,\ell\rangle, \end{aligned} \tag{2.65}$$

where

$$R_j(\varphi) = \exp(i\varphi \sum_k^{2\ell} \hat{L}_{k,j}), \quad j = x,\, y,\, z. \tag{2.66}$$

This is only true for states that are generated as a result of the same rotations. If you couple two states that are generated with unlike rotations then this method cannot be applied.

The states $R_z(\gamma)R_y(\pm\beta)R_z(-\gamma)|\frac{1}{2},\frac{1}{2}\rangle$, with $\gamma = 0,\ \pi/2$, are precisely the solutions to the spin-$\frac{1}{2}$ intelligent state problem. The ability of the highest weight states to be easily built up is what allows us to straightforwardly construct the angular momentum intelligent states of $\ell > \frac{1}{2}$ using the spin-$\frac{1}{2}$ states as a set of building blocks.

2.3 A General Construction

2.3.1 Combining Intelligent States

To explore the idea of combining simpler intelligent states to construct more complicated ones, we consider once again a composite system made from two independent subsystems, denoted by the subscripts A and B. Let $|\chi(\alpha)\rangle_A$ and $|\phi(\alpha)\rangle_B$ be states of subsystems A and B, respectively, with the property that

$$(\hat{L}_{A,x} - i\alpha\hat{L}_{A,y})|\chi(\alpha)\rangle_A = \lambda_A|\chi(\alpha)\rangle_A \tag{2.67}$$

$$(\hat{L}_{B,x} - i\alpha\hat{L}_{B,y})|\phi(\alpha)\rangle_B = \nu_B|\phi(\alpha)\rangle_B\,, \tag{2.68}$$

i.e. $|\chi(\alpha)\rangle_A$ and $|\phi(\alpha)\rangle_B$ are intelligent in their respective subsystems. Then,

$$|\psi(\alpha)\rangle = |\chi(\alpha)\rangle_A \otimes |\phi(\alpha)\rangle_B \equiv |\chi(\alpha)\rangle_A\, |\phi(\alpha)\rangle_B \tag{2.69}$$

is intelligent since

$$\begin{aligned}(\hat{L}_x - i\alpha\hat{L}_y)|\psi(\alpha)\rangle &= \left[(\hat{L}_{A,x} - i\alpha\hat{L}_{A,y})|\chi(\alpha)\rangle_A\right]|\phi(\alpha)\rangle_B \\ &\quad +|\chi(\alpha)\rangle_A\left[(\hat{L}_{B,x} - i\alpha\hat{L}_{B,y})|\phi(\alpha)\rangle_B\right], \\ &= (\lambda_A + \nu_B)|\chi(\alpha)\rangle_A|\phi(\alpha)\rangle_B. \end{aligned} \tag{2.70}$$

In other words, the direct product of two intelligent states is also intelligent, provided that one thinks of the resulting state as a composite state constructed from two separate systems. This simple result is quite powerful as it indicates that intelligent states can be "built-up" by putting together other intelligent states.

Quite clearly, the task now at hand is to find the simplest intelligent states and use them as building blocks to construct more complicated ones.

2.3.2 Spin 1 System

Reconstruction Method

We can use $|\psi_{\pm}^{1/2}(\mu)\rangle$ and Eq.(2.70) to construct $\ell = 1$ intelligent states as follows. Consider first the composite state

$$|\phi_{--}(\mu)\rangle = |\psi_{-}^{1/2}(\mu)\rangle_A \otimes |\psi_{-}^{1/2}(\mu)\rangle_B \equiv |\psi_{-}^{1/2}(\mu)\rangle_A\,|\psi_{-}^{1/2}(\mu)\rangle_B\,. \tag{2.71}$$

Using Eq.(2.53) and Eq.(2.54), one easily verifies that $|\phi_{--}(\mu)\rangle$ is intelligent, with eigenvalue $\lambda_{--} = -2\lambda$. Using Eq.(1.57) and distributing the product, we obtain

$$\begin{aligned}|\phi_{--}(\mu)\rangle = \frac{1}{1+|\mu|^2}&\left[|\tfrac12,\tfrac12\rangle_A|\tfrac12,\tfrac12\rangle_B - \mu\left(|\tfrac12,\tfrac12\rangle_A|\tfrac12,-\tfrac12\rangle_B + |\tfrac12,-\tfrac12\rangle_A|\tfrac12,\tfrac12\rangle_B\right)\right. \\ &\left. + \mu^2\,|\tfrac12,-\tfrac12\rangle_A|\tfrac12,-\tfrac12\rangle_B\right]. \end{aligned} \tag{2.72}$$

If we simply note that

$$\begin{aligned}|\tfrac12,\tfrac12\rangle_A|\tfrac12,\tfrac12\rangle_B &= |\ell=1, m=1\rangle\,, \\ \tfrac{1}{\sqrt2}\left[|\tfrac12,\tfrac12\rangle_A|\tfrac12,-\tfrac12\rangle_B + |\tfrac12,-\tfrac12\rangle_A|\tfrac12,\tfrac12\rangle_B\right] &= |1,0\rangle\,, \\ |\tfrac12,-\tfrac12\rangle_A|\tfrac12,-\tfrac12\rangle_B &= |1,-1\rangle\,, \end{aligned} \tag{2.73}$$

we can rewrite, assuming $|\alpha| < 1$,

$$|\phi_{--}(\mu)\rangle = \cos^2\tfrac{\beta}{2}\,|1,1\rangle - \sqrt2\cos\tfrac{\beta}{2}\,\sin\tfrac{\beta}{2}\,|1,0\rangle + \sin^2\tfrac{\beta}{2}\,|1,-1\rangle\,. \tag{2.74}$$

In a similar manner,

$$|\phi_{++}(\mu)\rangle = \cos^2\tfrac{\beta}{2}\,|1,1\rangle + \sqrt2\cos\tfrac{\beta}{2}\,\sin\tfrac{\beta}{2}\,|1,0\rangle + \sin^2\tfrac{\beta}{2}\,|1,-1\rangle\,, \tag{2.75}$$

is also intelligent, with eigenvalue $\lambda_{++} = 2\lambda$. These two states are identical to the states $|\psi^1_{--}\rangle$ and $|\psi^1_{++}\rangle$ of Eq.(1.73) that were found explicitly using the 3×3 representation for the $\ell = 1$ case.

The state $|\phi_{+-}(\mu)\rangle$ is also intelligent but is a linear combination of $\ell = 1$ and $\ell = 0$ states. More specifically, write

$$|\phi_{+-}(\mu)\rangle = \frac{1}{1+|\mu|^2}\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B - \mu\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right) \right.$$
$$\left. - \mu^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B\right] . \tag{2.76}$$

Recalling that (see Appendix B.3.1)

$$\frac{1}{\sqrt{2}}\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] = |0,0\rangle, \tag{2.77}$$

has angular momentum $\ell = 0$, it must be projected out if we are to remain in the $\ell = 1$ subspace. Thus, the third intelligent state with $\ell = 1$ is proportional to

$$|\psi^1_{+-}(\mu)\rangle \propto \frac{1}{1+|\mu|^2}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B - \mu^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B\right). \tag{2.78}$$

Having removed the $\ell = 0$ state, the correctly normalized state is

$$|\psi^1_{+-}(\mu)\rangle = \sqrt{\frac{2}{3+\cos 2\beta}}\left(\cos^2\tfrac{\beta}{2}\,|1,1\rangle - \sin^2\tfrac{\beta}{2}\,|1,-1\rangle\right). \tag{2.79}$$

This is the third $\ell = 1$ intelligent state found in Sec. 1.4.4. The eigenvalue associated with this intelligent state is $\lambda_{+-} = \lambda_+ + \lambda_- = \lambda - \lambda = 0$. From this it can be seen that coupling the $\ell = \frac{1}{2}$ intelligent states can be used as a means of constructing angular momentum intelligent states for higher values of ℓ. The benefit of this method is that it bypasses the need to solve for the eigenvectors of a matrix that has dimension $2\ell+1$.

Coupling Method

The previous paragraphs illustrate how to construct intelligent states from longer and longer strings of $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$. The limiting factor is the ability to quickly recognize, in a sum of products of $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$, which linear combinations belong to which ℓ subspace.

The case $|\alpha| < 1$

Assuming $|\alpha| < 1$ for the purpose of discussion, a good deal of progress can be made by coming back to Eq.(1.59) and observing that Eq.(2.71) and Eq.(2.72) can be rewritten as

$$\begin{aligned} |\psi^1_{--}(\beta)\rangle &= [R^A_y(-\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_A] \otimes [R^B_y(-\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_B]\,, \\ &= R_y(-\beta)[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B]\,, \\ &= R_y(-\beta)\,|1,1\rangle\,, \qquad (2.80) \\ |\psi^1_{++}(\beta)\rangle &= [R^A_y(\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_A] \otimes [R^B_y(\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_B]\,, \\ &= R_y(\beta)\,|1,1\rangle\,, \qquad (2.81) \end{aligned}$$

where the rotations $R_y(\mp\beta)$ of Eq.(2.80) and Eq.(2.81) are composite rotations generated by the collective operator

$$\begin{aligned} \hat{L}_y &= \hat{L}_y \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \hat{L}_y \\ &\equiv \hat{L}_{A,y} + \hat{L}_{B,y}\,. \qquad (2.82) \end{aligned}$$

In the parametrization of Eq.(1.34), Eq.(2.80) and Eq.(2.81) are SU(2) coherent states with $\gamma = 0$. They are known to be intelligent from Eq.(1.73), and belong to the $\ell = 1$ subspace. Their respective angular momentum expansion is simply given by

$$\begin{aligned} |\psi^1_{\pm\pm}(\beta)\rangle &= \sum_m |1,m\rangle\langle 1,m|\,R_y(\pm\beta)|1,1\rangle\,, \\ &= \sum_m |1,m\rangle\, d^1_{m,1}(\pm\beta)\,, \qquad (2.83) \end{aligned}$$

where

$$d^\ell_{m,m'}(\beta) \equiv \langle \ell, m|\,R_y(\beta)|\ell,m'\rangle \qquad (2.84)$$

is the reduced Wigner function [26], described in Appendix B.4.

The situation is slightly more complicated for

$$|\phi_{-+}(\beta)\rangle = [R_y(-\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_A] \otimes [R_y(\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_B]\,. \qquad (2.85)$$

This state is also a product of two SU(2) coherent states. However, because each state has been subject to a different rotation, it is not possible to "factor out" a collective rotation. Nevertheless, one can separately expand

$$|\phi_{-+}(\beta)\rangle = \left[\sum_{m_A} |\tfrac{1}{2},m_A\rangle\; d^{1/2}_{m_A,1/2}(-\beta)\right] \otimes \left[\sum_{m_B} |\tfrac{1}{2},m_B\rangle\, d^{1/2}_{m_B,1/2}(\beta)\right]\,, \qquad (2.86)$$

and collect the terms, since the summations are independent,

$$|\phi_{-+}(\beta)\rangle = \sum_{m_A,m_B} |\tfrac{1}{2},m_A\rangle|\tfrac{1}{2},m_B\rangle\, d^{1/2}_{m_A,1/2}(-\beta)\, d^{1/2}_{m_B,1/2}(\beta)\,. \qquad (2.87)$$

Then project into the $\ell = 1$ subspace by specializing the projector

$$\hat{\Pi}^\ell = \sum_{m=-\ell}^{\ell} |\ell,m\rangle\langle\ell,m| \qquad (2.88)$$

to $\ell = 1$ so as to obtain

$$|\psi^1_{-+}(\beta)\rangle \propto \sum_m |1,m\rangle\, \kappa^{1m}_{1/2,1/2}(\beta)\,, \tag{2.89}$$

where

$$\kappa^{\ell m}_{\ell_A,\ell_B}(\beta) = \sum_{m_A(m_B)} \left\langle \begin{matrix} \ell_A & \ell_B \\ m_A & m_B \end{matrix} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \times d^{\ell_A}_{m_A,\ell_A}(-\beta)\, d^{\ell_B}_{m_B,\ell_B}(\beta)\,, \tag{2.90}$$

and $\left\langle \begin{matrix} \ell_A & \ell_B \\ m_A & m_B \end{matrix} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle$ is an su(2) Clebsch-Gordan coefficient, defined in Appendix B.3.1.

To show that the state of Eq.(2.89) is intelligent, we note that the operator

$$\hat{\Pi}^{\ell=1} = \sum_{m=-1}^{1} |1,m\rangle\langle 1,m| \tag{2.91}$$

acts as the unit operator on any state completely in the $\ell = 1$ subspace, and annihilates any state with no part in this subspace. Hence, the collective $\hat{L}_y$ operator of Eq.(2.82) and its $\hat{L}_x$ counterpart

$$\begin{aligned} \hat{L}_x &= \hat{L}_x \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \hat{L}_x \\ &\equiv \hat{L}_{A,x} + \hat{L}_{B,x} \end{aligned} \tag{2.92}$$

must commute with the projection $\hat{\Pi}^{\ell=1}$ of Eq.(2.88) since neither $\hat{L}_y$ nor $\hat{L}_x$ can change ℓ. Thus,

$$\left(\hat{L}_y - i\alpha\hat{L}_x\right)|\psi^1_{-+}(\beta)\rangle = \hat{\Pi}^{\ell=1}\left(\hat{L}_y - i\alpha\hat{L}_x\right)|\phi_{-+}(\beta)\rangle\,, \tag{2.93}$$

$$= (\lambda_+ + \lambda_-)|\psi^1_{-+}(\beta)\rangle\,. \tag{2.94}$$

The projection does not preserve the norm so, as indicated before, $|\psi^1_{-+}(\beta)\rangle$ must be normalized after the projection.

Since $|\psi^{1/2}_{\pm}(\beta)\rangle$ is coherent, we see that $|\psi^1_{-+}(\beta)\rangle$ is the result of coupling two SU(2) coherent states, *i.e.* $|\psi^1_{-+}(\beta)\rangle$ is a coupled SU(2) coherent state.

The eigenvalue problem in the $\ell = 1$ subspace has only three linearly independent solutions. Hence, those solutions must be the three states of Eq.(2.81), Eq.(2.80) and Eq.(2.89), as they are linearly independent except when $\beta = 0, \pi$, which implies $\alpha = \pm 1$. For $\alpha = \pm 1$, the operator $\hat{L}_x - i\alpha\hat{L}_y$ collapses to the operators $\hat{L}_{\mp}$;

$$\hat{L}_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{L}_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.95}$$

each of which only have a single eigenstate.

The case $|\alpha| \geq 1$.

In this case, we have, from Eq.(1.62), a rotation about the x-axis. This simply introduces a phase via

$$\begin{aligned}\langle \ell, m|\, R_x(\beta)|\ell,\ell\rangle & \\ &= \langle \ell, m|\, R_z(-{}^{\pi}\!/_2)\, R_y(\beta)\, R_z({}^{\pi}\!/_2)\, |\ell,\ell\rangle\,, & (2.96)\\ &= \mathrm{e}^{-i\pi(\ell-m)/2} d^{\ell}_{m,\ell}(\beta)\,, & (2.97)\end{aligned}$$

so that, for instance,

$$|\psi^1_{-+}(\beta)\rangle \propto \sum_m |1,m\rangle\, \mathrm{e}^{-i\pi(1-m)/2}\, \kappa^{1m}_{{}^1\!/_2,{}^1\!/_2}(\beta)\,, \tag{2.98}$$

is intelligent by the same argument given for the $|\alpha| < 1$ case.

2.3.3 Spin 5/2 system

As a final example, consider how we can use the states $|\psi^{1/2}_{\pm}(\beta)\rangle$ of Eq.(1.57) and Eq.(1.58) to construct $\ell = {}^5\!/_2$ intelligent states. Start with the product

$$\begin{aligned}|\phi_{+++--}(\beta)\rangle &= \left[|\psi^{1/2}_+(\beta)\rangle_1\, |\psi^{1/2}_+(\beta)\rangle_2\, |\psi^{1/2}_+(\beta)\rangle_3\right] \\ &\quad \otimes \left[|\psi^{1/2}_-(\beta)\rangle_4\, |\psi^{1/2}_-(\beta)\rangle_5\right]\,. \end{aligned} \tag{2.99}$$

If we expand every $|\psi^{1/2}_+(\beta)\rangle_i$, where the index i labels one of five spin-$\frac{1}{2}$ subsystems, and distribute the product, the first term of the resulting expression is given by

$$|\ell = {}^5\!/_2\,, m = {}^5\!/_2\rangle = \cos^5\tfrac{\beta}{2}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_1|\tfrac{1}{2},\tfrac{1}{2}\rangle_2|\tfrac{1}{2},\tfrac{1}{2}\rangle_3|\tfrac{1}{2},\tfrac{1}{2}\rangle_4|\tfrac{1}{2},\tfrac{1}{2}\rangle_5\right)\,. \tag{2.100}$$

This term is fully symmetric under permutation.

Let us use the shorthands

$$\begin{aligned}\hat{L}_{1,x} &= \hat{L}_x \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \mathbb{1}_5\,,\\ \hat{L}_{2,x} &= \mathbb{1}_1 \otimes \hat{L}_x \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \mathbb{1}_5\end{aligned} \tag{2.101}$$

etc., so that each $\hat{L}_{x,i}$ acts only on the i'th subspace (of dimension 2). Let

$$\begin{aligned}\hat{L}_{A,x} &= \hat{L}_{1,x} + \hat{L}_{2,x} + \hat{L}_{3,x}\,,\\ \hat{L}_{B,x} &= \hat{L}_{4,x} + \hat{L}_{5,x}\,,\end{aligned} \tag{2.102}$$

and define

$$\hat{L}_x = \hat{L}_{A,x} + \hat{L}_{B,x}\,. \tag{2.103}$$

The collective operators $\hat{L}_y$ and $\hat{L}_z$ are defined similarly, as are $\hat{L}_\pm$:

$$\hat{L}_\pm = \hat{L}_x \pm i\,\hat{L}_y\,. \tag{2.104}$$

Because the collective operators are fully symmetric under permutation of any two subspace index i in Eq.(2.103), and act on the symmetric state $|\frac{5}{2},\frac{5}{2}\rangle$, every state of angular momentum $\ell = {}^5\!/_2$ will be symmetric under permutation. Thus, the order in which the $|\frac{1}{2},\frac{1}{2}\rangle$'s or $|\frac{1}{2},-\frac{1}{2}\rangle$'s occur is unimportant.

The case $|\alpha| < 1$

With $|\alpha| < 1$, every $|\psi_\pm^{1/2}(\beta)\rangle$ is obtained by rotation about the y–axis. Thus, we can write

$$|\phi_{+++--}(\beta)\rangle = \left[R_y^A(\beta)|\tfrac{3}{2},\tfrac{3}{2}\rangle_A\right]\left[R_y^B(-\beta)|1,1\rangle_B\right]\,, \tag{2.105}$$

where we have directly coupled

$$\begin{aligned}
&\left[R_y(\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_1\right]\otimes\left[R_y(\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_2\right]\otimes\left[R_y(\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_3\right]\\
&\qquad = R_y^A(\beta)\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_1|\tfrac{1}{2},\tfrac{1}{2}\rangle_2|\tfrac{1}{2},\tfrac{1}{2}\rangle_3\right]\,,\\
&\qquad = R_y^A(\beta)|\tfrac{3}{2},\tfrac{3}{2}\rangle_A\,,
\end{aligned} \tag{2.106}$$

and

$$\left[R_y(-\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_4\right]\otimes\left[R_y(-\beta)|\tfrac{1}{2},\tfrac{1}{2}\rangle_5\right] = R_y^B(-\beta)|1,1\rangle_B\,. \tag{2.107}$$

Here, the rotation operator $R_y^A(\beta) = \mathrm{e}^{-i\beta\hat{L}_{A,y}}$ while $R_y^B(-\beta) = \mathrm{e}^{i\beta\hat{L}_{B,y}}$. Note that the states of Eq.(2.106) and Eq.(2.107) are both angular momentum coherent states.

Eq.(2.105) can now be expanded as

$$\sum_{m_A,m_B} |\tfrac{3}{2},m_A\rangle_A\,|1,m_B\rangle_B\, d^{3/2}_{m_A,3/2}(\beta)\, d^1_{m_B,1}(-\beta)\,. \tag{2.108}$$

To project into the $\ell = {}^5\!/_2$ subspace, we specialize the projector Eq.(1.36) to $\ell = {}^5\!/_2$,

$$\hat{\Pi}^{5/2} = \sum_{m=-5/2}^{5/2} |\tfrac{5}{2},m\rangle\langle\tfrac{5}{2},m|, \tag{2.109}$$

so as to obtain

$$|\psi^{5/2}_{+++--}(\beta)\rangle \propto \sum_m |{}^5\!/_2,m\rangle\,\kappa^{5/2,m}_{3/2,1}(\beta)\,, \tag{2.110}$$

where

$$\kappa^{\ell m}_{\ell_A,\ell_B}(\beta) = \sum_{m_A(m_B)} \left\langle \begin{matrix}\ell_A & \ell_B\\ m_A & m_B\end{matrix} \middle| \begin{matrix}\ell\\ m\end{matrix}\right\rangle \times d^{\ell_A}_{m_A,\ell_A}(-\beta)\, d^{\ell_B}_{m_B,\ell_B}(\beta)\,, \tag{2.111}$$

and $\left\langle \begin{matrix} \ell_A & \ell_B \\ m_A & m_B \end{matrix} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle$ is an su(2) Clebsch-Gordan coefficient.

A better, more compact notation for $|\psi^{5/2}_{+++--}\rangle$ is

$$|\psi^{5/2}_{+++--}\rangle \equiv |\psi^{5/2}_{3/2,1}(\beta)\rangle\,. \tag{2.112}$$

This emphasizes that only the total number of $|\frac{1}{2},\frac{1}{2}\rangle_i$ states and the total number of $|\frac{1}{2},-\frac{1}{2}\rangle_j$ states are relevant for the construction of an intelligent state of angular momentum $\ell = \ell_A+\ell_B$. The state $|\psi^{5/2}_{++-+-}(\beta)\rangle$, for instance, can differ from $|\psi^{5/2}_{+++--}(\beta)\rangle$ by at most a phase.

To show that the state of Eq.(2.112) is intelligent, we note once more that the operator $\hat{\Pi}^{5/2}$ of Eq.(2.88) acts as the unit operator on any state completely in the $\ell = 5/2$ subspace, and annihilates any state with no part in this subspace. Hence, the collective $\hat{L}_y = \hat{L}_{A,y} + \hat{L}_{B,y}$ operator and its $\hat{L}_x$ counterpart must commute with the projection $\hat{\Pi}^{5/2}$ of Eq.(2.88) since neither $\hat{L}_y$ nor $\hat{L}_x$ can change ℓ. Thus,

$$\begin{aligned}\left(\hat{L}_y - i\alpha\hat{L}_x\right)|\psi^{5/2}_{3/2,1}(\beta)\rangle &= \hat{\Pi}^{5/2}\left(\hat{L}_y - i\alpha\hat{L}_x\right)|\phi_{3/2,1}(\beta)\rangle\,, &(2.113)\\ &= (3\lambda_+ + 2\lambda_-)|\psi^{5/2}_{3/2,1}(\beta)\rangle\,. &(2.114)\end{aligned}$$

The projection does not preserve the norm so $|\psi^{5/2}_{3/2,1}(\beta)\rangle$ must be normalized after the projection.

Since $R^A_y(\beta)|\frac{3}{2},\frac{3}{2}\rangle_A$ and $R^B_y(-\beta)|1,1\rangle_B$ are coherent, we see that $|\psi^{5/2}_{3/2,1}(\beta)\rangle$ is the result of coupling two su(2) coherent states.

The case $|\alpha| \geq 1$.

In this case, we note that

$$\begin{aligned}&\langle \ell, m|\, R_x(\beta)|\ell,\ell\rangle \\ &\quad= \langle \ell, m|\, R_z(-\pi/2)\, R_y(\beta)\, R_z(\pi/2)\,|\ell,\ell\rangle\,, &(2.115)\\ &\quad= \mathrm{e}^{-i\pi(\ell-m)/2} d^{\ell}_{m,\ell}(\beta)\,, &(2.116)\end{aligned}$$

so that, for instance,

$$|\psi^{5/2}_{3/2,1}(\beta)\rangle \propto \sum_m |\tfrac{5}{2}, m\rangle\, \mathrm{e}^{-i\pi(\frac{5}{2}-m)/2}\, \kappa^{5/2m}_{3/2,1}(\beta)\,, \tag{2.117}$$

is intelligent by the same argument given for the $|\alpha| < 1$ case.

2.3.4 A General Expression

More generally, it is now clear that if we start with $2\ell_A$ copies of $|\psi_+^{1/2}(\beta)\rangle$ and $2\ell_B$ copies of $|\psi_-^{1/2}(\beta)\rangle$, we can write for $\alpha \leq 1$

$$|\phi_{\ell_A,\ell_B}\rangle = \left[R_y^A(\beta)|\ell_A,\ell_A\rangle\right] \otimes \left[R_y^B(-\beta)|\ell_B,\ell_B\rangle\right] . \tag{2.118}$$

We must treat the two rotations $R_y^A(\beta)$ and $R_y^B(-\beta)$ separately. To do this we take advantage of the fact that

$$\begin{aligned} \mathbb{1}_A &= \sum_{m_A=-\ell_A}^{\ell_A} |\ell_A, m_A\rangle\langle\ell_A, m_A| \\ \mathbb{1}_B &= \sum_{m_B=-\ell_B}^{\ell_B} |\ell_B, m_B\rangle\langle\ell_B, m_B| . \end{aligned} \tag{2.119}$$

Since $\mathbb{1}_A$ and $\mathbb{1}_B$ are unit operators we can insert them where appropriate in Eq.(2.118) without changing it. This yields

$$\begin{aligned} |\phi_{\ell_A,\ell_B}\rangle &= \sum_{m_A}\left[|\ell_A,m_A\rangle\langle\ell_A,m_A|R_y^A(\beta)|\ell_A,\ell_A\rangle\right] \\ &\qquad \otimes \sum_{m_B}\left[|\ell_B,m_B\rangle\langle\ell_B,m_B|R_y^B(-\beta)|\ell_B,\ell_B\rangle\right] . \end{aligned} \tag{2.120}$$

Using the definition of the reduced Wigner function, Eq.(2.84), and the fact that the two summations are independent, we arrive at

$$|\phi_{\ell_A,\ell_B}\rangle = \sum_{m_A,m_B} |\ell_A,m_A\rangle|\ell_B,m_B\rangle\, d^{\ell_A}_{m_A,\ell_A}(\beta)\, d^{\ell_B}_{m_B,\ell_B}(-\beta). \tag{2.121}$$

As is shown in Appendix B.3.1, the coupling $|\ell_A,m_A\rangle|\ell_B,m_B\rangle$ will not only result in states with $\ell = \ell_A + \ell_B$, but in a number of different states with total $\ell_i < \ell_A + \ell_B$. We are interested in only the states with total $\ell = \ell_A + \ell_B$, and thus we need to throw away other angular momenta. To do this we use the projection operator of Eq.(1.36). After projecting out the unwanted states we have an intelligent state of angular momentum $\ell = \ell_A + \ell_B$ as

$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle \propto \sum_m |\ell,m\rangle\, \kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) , \tag{2.122}$$

with $m_B = m - m_A$, since the Clebsch-Gordan coefficients are zero otherwise, and

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = \sum_{m_A,(m_B)} \left\langle \begin{matrix} \ell_A & \ell_B \\ m_A & m_B \end{matrix} \middle| \begin{matrix} \ell \\ m \end{matrix} \right\rangle d^{\ell_A}_{m_A,\ell_A}(\beta)\, d^{\ell_B}_{m_B,\ell_B}(-\beta). \tag{2.123}$$

Eq.(2.118) and Eq.(2.122) show explicitly how su(2) intelligent states with angular momentum ℓ can be constructed by appropriately coupling SU(2) coherent states. The state of Eq.(2.118) is explicitly intelligent and remains intelligent under projection by $\hat{\Pi}^{\ell}$ of Eq.(1.36), thus yielding Eq.(2.122).

The expression for the coefficient $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$ can be simplified significantly. To do this, we begin by noting that [26]

$$\begin{aligned} d^{\ell_B}_{m_B,\ell_B}(-\beta) &= (-1)^{m_B-\ell_B}\, d^{\ell_B}_{m_B,\ell_B}(\beta)\,, &(2.124)\\ d^{\ell_A}_{m_A,\ell_A}(\beta)\, d^{\ell_B}_{m_B,\ell_B}(\beta) &= \left\langle \begin{matrix}\ell_A & \ell_B\\ m_A & m_B\end{matrix} \middle| \begin{matrix}\ell\\ m\end{matrix}\right\rangle \times d^{\ell}_{m,\ell}(\beta)\,, &(2.125)\end{aligned}$$

where $\ell = \ell_A + \ell_B$ and $\left\langle \begin{matrix}\ell_A & \ell_B\\ \ell_A & \ell_B\end{matrix} \middle| \begin{matrix}\ell_A+\ell_B\\ \ell_A+\ell_B\end{matrix}\right\rangle = 1$ have been used. Thus,

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = d^{\ell}_{m,\ell}(\beta) \times \left[\sum_{m_A(m_B)} (-1)^{m_B-\ell_B} \left\langle \begin{matrix}\ell_A & \ell_B\\ m_A & m_B\end{matrix} \middle| \begin{matrix}\ell\\ m\end{matrix}\right\rangle^2\right]. \tag{2.126}$$

A little more mileage can be done because Clebsch-Gordan coefficients for which $\ell = \ell_A + \ell_B$ have known expressions [26]. Using this and the condition $m = m_A + m_B$, we obtain

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = \frac{d^{\ell}_{m,\ell}(\beta)}{\binom{2\ell}{\ell-m}} \left[\sum_{n=0}^{2\ell_B} (-1)^n \binom{2\ell_A}{\ell-m-n}\binom{2\ell_B}{n}\right]. \tag{2.127}$$

The coefficient in the bracket can be identified with the coefficient of $x^{\ell-m}$ in the expansion of $(1+x)^{2\ell_A}(1-x)^{2\ell_B}$. This is similar to the states that were used in [25]. Finally [26],

$$\sum_{n=0}^{2\ell_B} (-1)^n \binom{2\ell_A}{\ell-m-n}\binom{2\ell_B}{n} = 2^{\ell}\sqrt{\frac{(2\ell_B)!\,(2\ell_A)!}{(\ell+m)!\,(\ell-m)!}}\, d^{\ell}_{\ell_B-\ell_A,m}\left(\tfrac{\pi}{2}\right). \tag{2.128}$$

Inserting this into $\kappa^{\ell,m}_{\ell_A\ell_B}(\beta)$ gives

$$\kappa^{\ell,m}_{\ell_A\ell_B}(\beta) = 2^{\ell}\frac{\sqrt{(2\ell_B)!\,(2\ell_A)!\,(\ell+m)!\,(\ell-m)!}}{(2\ell)!}\, d^{\ell}_{\ell_B-\ell_A,m}\left(\tfrac{\pi}{2}\right)\, d^{\ell}_{m,\ell}(\beta). \tag{2.129}$$

Note that the appearance of a rotation by $\pi/2$ about the $\hat{y}$ axis:

$$\begin{aligned} d^{\ell}_{m,\ell_B-\ell_A}\left(\tfrac{\pi}{2}\right) &= d^{\ell}_{\ell_B-\ell_A,m}\left(-\tfrac{\pi}{2}\right)\\ &= \langle \ell, \ell_B-\ell_A| e^{-i\frac{\pi}{2}\hat{L}_y}|\ell, m\rangle\,, &(2.130)\end{aligned}$$

is reminiscent of an expression found in [23].

Introducing the normalization factor

$$\mathcal{N}^{\ell}_{\ell_A,\ell_B}(\beta) = \frac{1}{\sqrt{\sum_m |\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)|^2}}\,, \tag{2.131}$$

we obtain the final expression for our intelligent state as

$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle = \mathcal{N}^{\ell}_{\ell_A,\ell_B}(\beta)\sum_m |\ell,m\rangle\,\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)\,. \tag{2.132}$$

The construction for the intelligent states with $\alpha > 1$ follows the same steps. The only differences are

$$\cos\beta = -\frac{1}{\alpha}\,, \tag{2.133}$$

and the rotations are about the x-axis rather than the y-axis:

$$|\psi_{\ell_A,\ell_B}\rangle = \left[R^A_x(\beta)|\ell_A,\ell_A\rangle_A\right]\otimes\left[R^B_x(-\beta)|\ell_B,\ell_B\rangle_B\right]. \tag{2.134}$$

Remembering that we can write

$$\begin{aligned}\langle\ell,m|R_x(\beta)|\ell,\ell\rangle &= \langle\ell,m|R_z(-\pi/2)R_y(\beta)R_z(\pi/2)|\ell,\ell\rangle\\ &= \mathrm{e}^{-i\pi(\ell-m)/2}\,d^{\ell}_{m,\ell}(\beta),\end{aligned} \tag{2.135}$$

leads to the expression

$$|\psi_{\ell_A,\ell_B}\rangle \sum_{m_A,m_B} |\ell_A,m_A\rangle|\ell_B,m_B\rangle\,\mathrm{e}^{-i\pi(\ell-m)/2}\,d^{\ell_A}_{m_A,\ell_A}(\beta)\,d^{\ell_B}_{m_B,\ell_B}(-\beta) \tag{2.136}$$

with $\ell = \ell_A + \ell_B$, and $m = m_A + m_B$. The final expression after projecting onto the $\ell = \ell_A + \ell_B$ subspace and normalizing is

$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle = \mathcal{N}^{\ell}_{\ell_A,\ell_B}(\beta)\sum_m |\ell,m\rangle\,\mathrm{e}^{-i\pi(\ell-m)/2}\,\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)\,, \tag{2.137}$$

with $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$ and $\mathcal{N}^{\ell}_{\ell_A,\ell_B}(\beta)$ as defined above.

Although we have limited ourselves to expressions where $\ell = \ell_A + \ell_B$, the factor $\ell_B - \ell_A$ in Eq.(2.129) suggests that, up to a phase, it is only the difference between angular momenta that is here relevant. More precisely, if one considers $\ell'_A = \ell_A + j$, $\ell'_B = \ell_B + k$, then the tensor product $\ell'_A \otimes \ell'_B$ will contain a subspace of angular momentum ℓ. The coupled states in this subspace are also intelligent, but are simply proportional to the states obtained by coupling $\ell_A \otimes \ell_B$. In other words, no new state is found by considering cases other than $\ell = \ell_A + \ell_B$.

Finally, we note that the eigenvalue problem in the $\ell = \ell_A + \ell_B$ subspace has at most $2\ell + 1$ independent eigenvectors. Using Eq.(2.137), it is clear that (except when $\beta = 0$ or π) we can construct exactly the right number linearly independent states of the form by selecting in turn (ℓ_A, ℓ_B) to be $(\ell, 0), (\ell - 1/2, 1/2), \ldots, (0, \ell)$.

Hence, all $2\ell + 1$ intelligent states are coupled su(2) coherent states.

2.4 Selected Results

2.4.1 Expectations and standard deviations

The intelligent state of Eq.(2.137) is an eigenstate of $\hat{L}_x - i\alpha\hat{L}_y$ with eigenvalue

$$\lambda_{\ell_A,\ell_B} = \lambda\,(2\ell_A - 2\ell_B)\,. \tag{2.138}$$

If we assume $|\alpha| \leq 1$, then λ is real. Combining $\lambda = \frac{1}{2}\sqrt{1-\alpha^2}$ with $\cos\beta = -\alpha$ yields the result

$$\lambda_{\ell_A,\ell_B} = (\ell_A - \ell_B)\sin\beta\,. \tag{2.139}$$

Since α, $\langle\,\hat{L}_x\rangle$ and $\langle\,\hat{L}_y\rangle$ are real, this can be compared with $\lambda_{\ell_A,\ell_B} = \langle\,\hat{L}_x\rangle - i\alpha\langle\,\hat{L}_y\rangle$ to give

$$\langle\,\hat{L}_x\rangle = \tfrac{1}{2}\,(\ell_B - \ell_A)\sin\beta\,, \qquad \langle\,\hat{L}_y\rangle = 0\,. \tag{2.140}$$

If, on the other hand, $|\alpha| \geq 1$, we have

$$\langle\,\hat{L}_x\rangle = 0\,, \qquad \langle\,\hat{L}_y\rangle = -\tfrac{1}{2}\,(\ell_B - \ell_A)\sin\beta\,. \tag{2.141}$$

Furthermore, using Eq.(1.37) and Eq.(1.43) with the fact that

$$[\hat{L}_x - \langle\hat{L}_x\rangle, \hat{L}_y - \langle\hat{L}_y\rangle] = [\hat{L}_x, \hat{L}_y] = i\hat{L}_z, \tag{2.142}$$

one finds that the intelligent states generally satisfy

$$(\Delta L_y)^2 = -\frac{1}{2\alpha}\,\langle\,\hat{L}_z\rangle\;, \qquad (\Delta L_x)^2 = -\frac{1}{2}\alpha\,\langle\,\hat{L}_z\rangle\,. \tag{2.143}$$

To show this, simply expand the left hand side of Eq.(2.142) and compute the expectation value;

$$\langle(\hat{L}_x - \langle\hat{L}_x\rangle)(\hat{L}_y - \langle\hat{L}_y\rangle)\rangle - \langle(\hat{L}_y - \langle\hat{L}_y\rangle)(\hat{L}_x - \langle\hat{L}_x\rangle)\rangle = i\langle\hat{L}_z\rangle. \tag{2.144}$$

Using, now, the condition of Eq.(1.43) gives

$$-i\alpha\langle(\hat{L}_y - \langle\hat{L}_y\rangle)^2\rangle - i\alpha\langle(\hat{L}_y - \langle\hat{L}_y\rangle)^2\rangle = i\langle\hat{L}_z\rangle, \tag{2.145}$$

into which, inserting the relation of Eq.(1.37) yields

$$(\Delta L_y)^2 = -\frac{1}{2\alpha}\langle\hat{L}_z\rangle. \tag{2.146}$$

The relation for ΔL_x can easily be found from Eq.(2.146) using Eq.(1.43). This allows computation of all pertinent quantities from $\langle\,\hat{L}_z\rangle$, which is simply given by

$$\langle\,\hat{L}_z\rangle = \left(\mathcal{N}^{\ell}_{\ell_A\ell_B}(\beta)\right)^2\left(\sum_m m\,|\,\kappa^{\ell m}_{\ell_A,\ell_B}(\beta)\,|^2\right)\,. \tag{2.147}$$

2.4.2 Numerical results

If we look again at the equation that we have solved;

$$(\hat{L}_x - i\alpha\hat{L}_y)|\psi\rangle = \lambda|\psi\rangle, \tag{2.148}$$

it is possible to predict some general trends that the intelligent states should satisfy. First of all note that for finite dimensional representations, if the state $|\psi\rangle$ is an eigenstate of either $\hat{L}_x$ or $\hat{L}_y$ the uncertainty relation will reduce to $0 = 0$. To show this, assume that the state $|\psi_x\rangle$ is an eigenstate of $\hat{L}_x$ with eigenvalue λ_x. Putting this into the uncertainty relation yields

$$\begin{aligned}
\left(\sqrt{\langle\hat{L}_x^2\rangle - \langle\hat{L}_x\rangle^2}\right)\Delta L_y &\geq \tfrac{1}{2}|\langle\psi_x|(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)|\psi_x\rangle| \\
\left(\sqrt{\lambda_x\langle\hat{L}_x\rangle - \lambda_x^2}\right)\Delta L_y &\geq \tfrac{1}{2}|\langle\psi_x|(\lambda_x\hat{L}_y - \hat{L}_y\lambda_x)|\psi_x\rangle| \\
\left(\sqrt{\lambda_x^2 - \lambda_x^2}\right)\Delta L_y &\geq \tfrac{1}{2}|\lambda_x(\langle\hat{L}_y\rangle - \langle\hat{L}_y\rangle)| \\
0 &= 0.
\end{aligned} \tag{2.149}$$

Thus, we would expect that the uncertainty for the state $|\psi\rangle$ will be zero if $|\psi\rangle$ is an eigenstate of either $\hat{L}_x$ or $\hat{L}_y$. By looking at Eq.(2.148), it is easy to see that if $\alpha = 0$ then $|\psi\rangle$ is simply an eigenstate of $\hat{L}_x$. Alternatively, as $\alpha \to \infty$ the $\hat{L}_y$ term dominates and $|\psi\rangle$ is approaching an eigenstate of $\hat{L}_y$. We expect, then, that a plot of the uncertainty *vs.* α for any intelligent state will reflect this. Looking at Fig. 2.1 shows that these two features are present. The plot is clearly zero at $\alpha = 0$ and, for $|\alpha| \geq 1$, the uncertainty decays toward zero for increasing values of $|\alpha|$. Another interesting choice for α is $\alpha = \pm 1$. Choosing $\alpha = -1$ in Eq.(2.148) is equivalent to solving for the eigenstates of the operator $\hat{L}_+$. However, if one tries to solve for these states, one quickly finds that there is only one that is non-zero. This implies that the solutions to Eq.(2.148) collapse into a single solution at this point. The same is true if the value $\alpha = 1$, which produces $\hat{L}_-$, is chosen. As well, it is precisely at the values $\alpha = \pm 1$ where the solutions change from a rotation about the y-axis, for $|\alpha| < 1$, to a rotation about the x-axis. It is not surprising, then, that there is a discontinuity in the plots at this point.

Figure 2.2 compares the uncertainty curves for three of the $\ell = 5/2$ intelligent states. They are completely symmetric about $\alpha = 0$ so only $\alpha > 0$ is shown. As well, if the values of ℓ_A and ℓ_B are interchanged, the curve remains unchanged. It is clear that they all possess the expected features. In general, for the su(2) intelligent states of a given ℓ, the difference between ℓ_A and ℓ_B affects the height of the curve for a given value of α. The smaller the value of $|\ell_A - \ell_B|$, the higher the uncertainty.

Figures. 2.3 and 2.4 illustrate typical results. The figures give the ratio of the uncertainty products $(\Delta L_x \Delta L_y)_I$ of intelligent states to the coherent state $(\Delta L_x \Delta L_y)_c$,

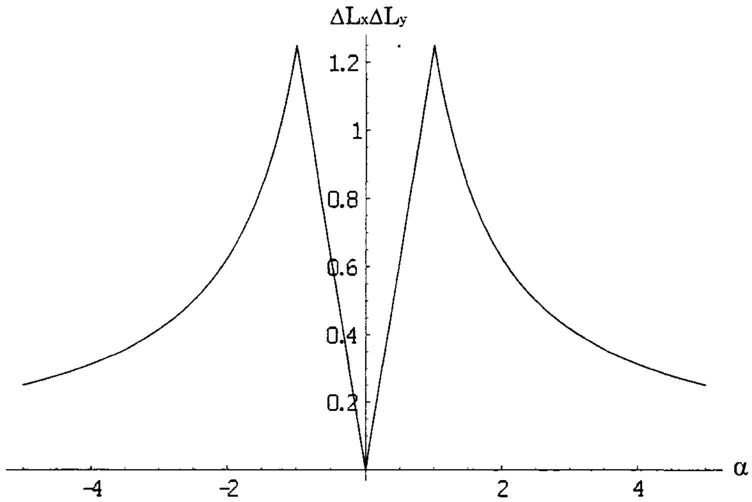

Figure 2.1: A plot of the uncertainty, $\Delta L_x \Delta L_y = \frac{1}{2}|\langle L_z \rangle|$ for the state $|\psi^{5/2}_{5/2,0}\rangle$. The uncertainty curve for every su(2) intelligent state has the same general shape, though they vary in height, depending on total ℓ, and sharpness, which is governed by the ratio $\frac{\ell_A - \ell_B}{\ell}$.

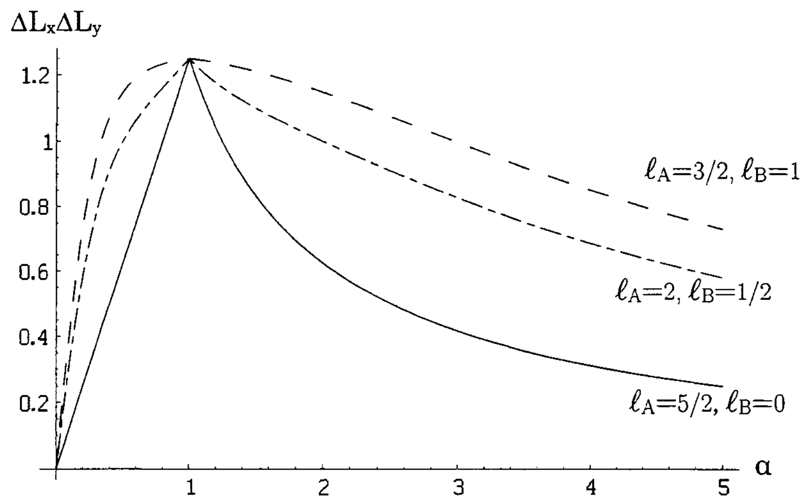

Figure 2.2: A plot of the uncertainty curves for three of the $\ell = 5/2$ intelligent states.

for which $\ell_A = \ell$. These ratios are just the ratios of $\langle \hat{L}_z \rangle$. For the coherent state, using the relation

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \tag{2.150}$$

one can easily find, for $|\alpha| < 1$,

$$\begin{aligned} \langle \hat{L}_z \rangle_c &= \langle \ell, \ell | R_y(-\beta) \hat{L}_z R_y(\beta) | \ell, \ell \rangle \\ &= \langle \ell, \ell | (\hat{L}_z + i\beta[\hat{L}_y, \hat{L}_z] - \frac{\beta^2}{2}[\hat{L}_y, [\hat{L}_y, \hat{L}_z]] - i\frac{\beta^3}{3!}[\hat{L}_y, [\hat{L}_y, [\hat{L}_y, \hat{L}_z]]] + \dots) | \ell, \ell \rangle \\ &= \langle \ell, \ell | (\hat{L}_z - \beta \hat{L}_x - \frac{\beta^2}{2}\hat{L}_z + \frac{\beta^3}{3!}\hat{L}_x + \dots) | \ell, \ell \rangle \\ &= \langle \ell, \ell | (\hat{L}_z \cos\beta - \hat{L}_x \sin\beta) | \ell, \ell \rangle \\ &= \langle \ell, \ell | \hat{L}_z | \ell, \ell \rangle \cos\beta + 0 \\ &= \ell \cos\beta \,. \end{aligned} \tag{2.151}$$

In Fig. 2.3, the ratios for intelligent states of angular momentum $\ell = {}^5\!/_2$ with

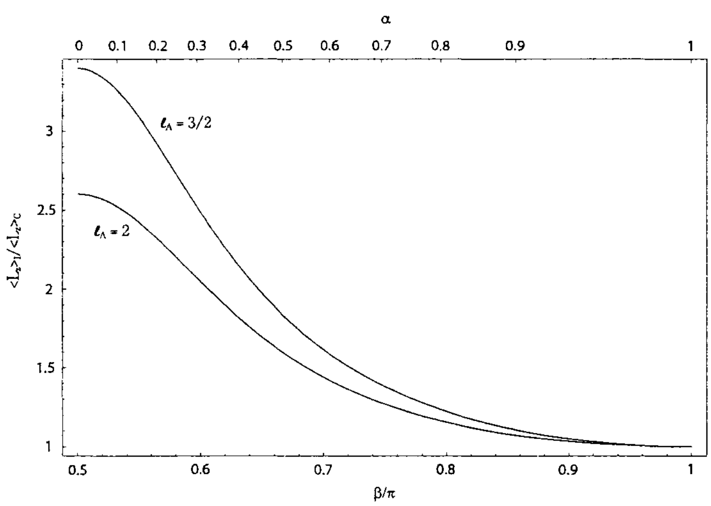

Figure 2.3: The ratio $| \langle \hat{L}_z \rangle |_I / | \langle \hat{L}_z \rangle |_c$ as a function of ${}^\beta\!/_\pi$ or α for $\ell = {}^5\!/_2$ and various values of ℓ_A and ℓ_B so that $\ell_A + \ell_B = {}^5\!/_2$. SOURCE: Reproduced with permission from B. R. Lavoie and H. de Guise *Su(2) intelligent states as coupled su(2) coherent states,* J. Phys. A: Math. Theor. **40** (2007) 2825–2837. © IOP Publishing.

$(\ell_A = 2, \ell_B = {}^1\!/_2)$ and $(\ell_A = {}^3\!/_2, \ell_B = 1)$ are given. The results are unchanged if one switches ℓ_A and ℓ_B. The curves $\alpha < 0$ are identical to those for $\alpha > 0$. Furthermore, the results with $|\alpha| > 1$ can be obtained from those with $|\alpha| < 1$ by the transformation

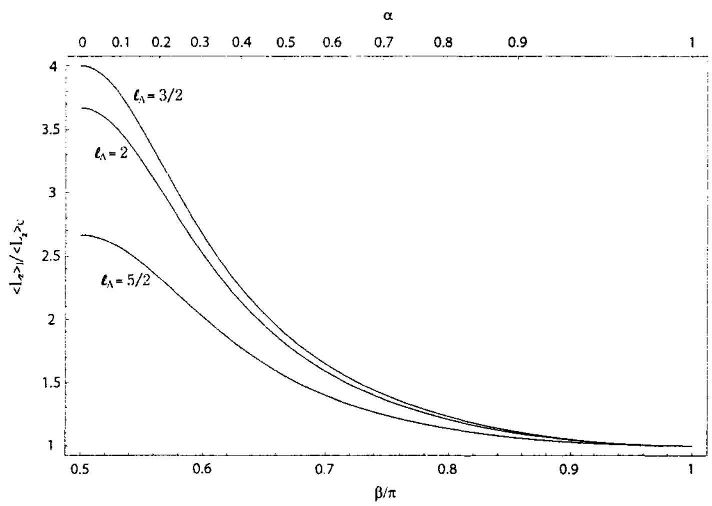

Figure 2.4: The ratio $|\langle \hat{L}_z \rangle|_I / |\langle \hat{L}_z \rangle|_c$ as a function of $^{\beta}/_{\pi}$ or α for $\ell = 3$ and various values of ℓ_A and ℓ_B so that $\ell_A + \ell_B = 3$. SOURCE: Reproduced with permission from B. R. Lavoie and H. de Guise *Su(2) intelligent states as coupled su(2) coherent states,* J. Phys. A: Math. Theor. **40** (2007) 2825–2837. © IOP Publishing.

$\alpha \to {}^1/_{\alpha}$, so the range $0 \le \alpha \le 1$ captures all qualitative features of the curves. Figure 2.4 is similar to 2.3, except that $\ell = 3$. The symmetries of Fig.2.3 are also present in Fig. 2.4.

One immediately observes that the uncertainty products for intelligent states (with $\ell_A \neq \ell$) is always greater than the corresponding product for the coherent state (with $\ell_A = \ell$). Insofar as the product $\Delta L_x \Delta L_y$ goes, the "worst" intelligent state is the one for which ℓ_A and ℓ_B are as close as possible. We have not been able to prove this analytically because Eq.(2.147) for $\langle \hat{L}_z \rangle$ is difficult to manipulate. However, we have verified that this observation holds over a wide range of values of ℓ. Other curves illustrating this behavior can be found in [25].

It is not difficult to show that the maximum of the product $\Delta L_x \Delta L_y$ is simply $\frac{1}{2}\ell$. Indeed, by Eq.(1.49), it is clear that the product is maximal when $|\langle \hat{L}_z \rangle|$ is maximal. This maximum is reached for the states $|\ell, \pm\ell\rangle$. From Eq.(2.118) and Eq.(2.122), it immediately follows that this will occur when $\beta = 0$ or $\beta = \pi$. This implies from

$$\cos\tfrac{\beta}{2} = \frac{1}{\sqrt{1+|\mu|^2}}, \quad \sin\tfrac{\beta}{2} = \frac{\mu}{\sqrt{1+|\mu|^2}}, \tag{2.152}$$

that $\mu = 0$ or $\mu = \infty$ which in turn, by Eq.(1.53), implies $\alpha = \pm 1$.

As $\alpha \to \pm 1$, all intelligent states converge to a single state. When $\alpha = \pm 1$ precisely, the operator $\hat{L}_x - i\alpha \hat{L}_y$ becomes the nilpotent $\hat{L}_+$ or $\hat{L}_-$ respectively, both of which have only one non-zero eigenvector.

Finally, Fig. 2.5 shows the population of various m substates in the intelligent state

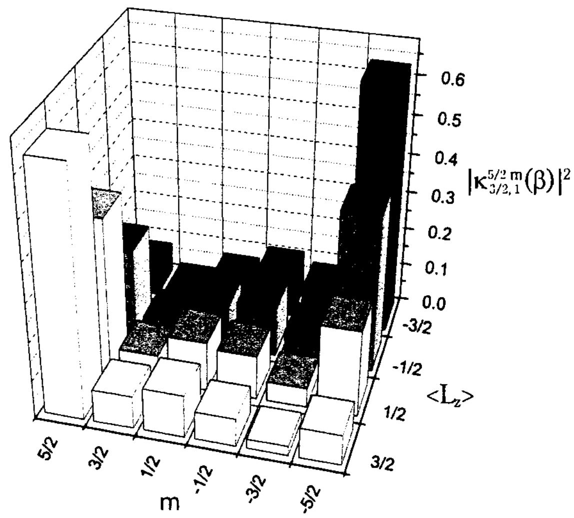

Figure 2.5: The populations of m substates $|\kappa^{5/2\,m}_{3/2,1}(\beta)|^2$ for different values of m and $\ell = {}^5\!/_2$. The values of β were selected so that $\langle \hat{L}_z \rangle = \pm{}^3\!/_2, \pm{}^1\!/_2$. SOURCE: Reproduced with permission from B. R. Lavoie and H. de Guise *Su(2) intelligent states as coupled su(2) coherent states,* J. Phys. A: Math. Theor. **40** (2007) 2825–2837. © IOP Publishing.

$|\psi^{5/2}_{3/2,1}(\beta)\rangle$. For clarity, we have restricted the calculations to angles β chosen so that $\langle \hat{L}_z \rangle = \pm{}^3\!/_2, \pm{}^1\!/_2$. This figure illustrates a very general symmetry: $|\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)|^2 = |\kappa^{\ell,-m}_{\ell_A,\ell_B}(-\beta)|^2$. This can be traced back to symmetries of the d-functions entering in the construction of the $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$ coefficients.

Chapter 3

su(3) Intelligent States

3.1 Some background

3.1.1 The SU(3) Group and the su(3) Algebra

The SU(3) group, like the SU(2) group, is a Lie group. There is an SU(3) element for every 3×3 unitary matrix with determinant 1. The group elements are constructed by exponentiating a set of 8 matrices called su(3) generators. In their simplest form, the generators are the set of 3×3 traceless hermitian matrices. A convenient basis for the generators is

$$\begin{aligned} \hat{C}_{ij} &= \hat{a}_i^\dagger \hat{a}_j, \quad i \neq j = 1,2,3, \\ \hat{h}_1 &= \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1\,, \\ \hat{h}_2 &= \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_2^\dagger \hat{a}_2\,. \end{aligned} \tag{3.1}$$

Using the usual commutation relation for harmonic oscillator creation and destruction operators, we find the abstract commutation relations

$$[\hat{C}_{ij}, \hat{C}_{k\ell}] = \hat{C}_{i\ell}\delta_{jk} - \hat{C}_{kj}\delta_{i\ell}. \tag{3.2}$$

As 3×3 matrices, the operator $\hat{C}_{ij}$ can be obtained using the harmonic oscillator basis states $|100\rangle, |010\rangle, |001\rangle$, where $|n_1, n_2, n_3\rangle$ is the harmonic oscillator state containing n_1 quanta in direction 1, n_2 quanta in direction 2 and n_3 quanta in direction 3. Thus, for instance,

$$\hat{C}_{12} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{C}_{21} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

so that

$$\hat{C}_{12} + \hat{C}_{21} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

$$\frac{1}{i}\left(\hat{C}_{12} - \hat{C}_{21}\right) \rightarrow \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

are traceless and hermitian. The operators $\hat{C}_{12}$ and $\hat{C}_{21}$ are ladder operators in su(3) in the same way that $\hat{L}_+$ and $\hat{L}_-$ are ladder operators in angular momentum theory. The major difference is that, whereas there are only two ladder operators in angular momentum systems, there are, in addition to $\hat{C}_{12}$ and $\hat{C}_{21}$, four other ladder operators: $\hat{C}_{13}, \hat{C}_{31}, \hat{C}_{23}$ and $\hat{C}_{32}$.

We are not restricted to a 3×3 matrix representation. Using the six $\Lambda = 2$ states $\{|200\rangle, |110\rangle, |101\rangle, |020\rangle, |011\rangle, |002\rangle\}$, it is clear that we can obtain a 6×6 matrix for each $\hat{C}_{ij}$ and $\hat{h}_i$. Upon exponentiation, this will result in a group element represented by a 6×6 matrix. Such a construction holds for any positive integer Λ, resulting in a representation of dimension $\frac{1}{2}(\Lambda+1)(\Lambda+2)$.

The laddering action of each $\hat{C}_{ij}$, and the commutation between any two elements, is best illustrated using a weight and a root diagram. The root diagram for su(3) is given in Fig.(3.1).

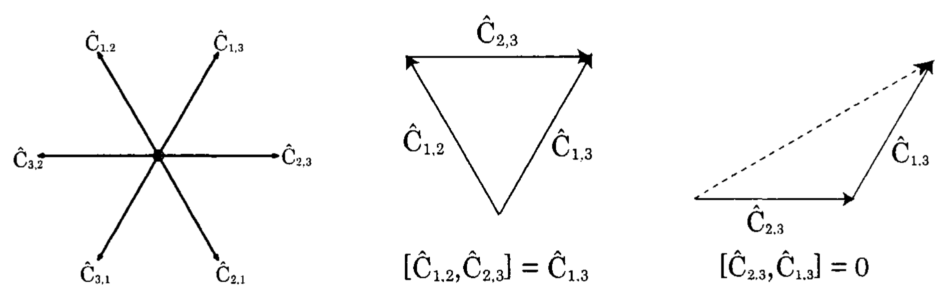

Figure 3.1: Representation of the su(3) ladder operators as vectors in the su(3) weight space. The six ladder operators $\hat{C}_{i,j}$ are shown explicitly, on the left, and the two operators $\hat{h}_1$ and $\hat{h}_2$ lie at the centre. The two vector additions on the right show the commutation relations, though the proportionality factors cannot be determined in this way.

The commutator $[\hat{C}_{ij}, \hat{C}_{k\ell}]$ is proportional to the vector sum of the roots associated to $\hat{C}_{ij}$ and $\hat{C}_{k\ell}$. If the resulting vector sum is the root associated to $\hat{C}_{ab}$, then $[\hat{C}_{ij}, \hat{C}_{k\ell}]$ is proportional to $\hat{C}_{ab}$.

The vector on the root diagram associated with $\hat{C}_{ij}$ is proportional to the raising action of $\hat{C}_{ij}$ on a basis state $|n_1 n_2 n_3\rangle$. This is illustrated in Fig.(3.2), which also

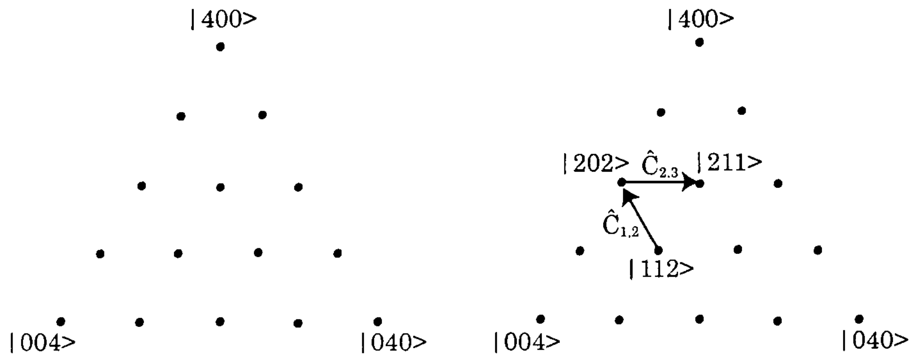

Figure 3.2: **Left:** Graphical representation of the geometry of the su(3) weight space. This is the weight diagram for $\Lambda = 4$. It is clear that the extremal weighted states are at the vertices of the triangle. **Right:** Showing how the ladder operators act in the weight space.

illustrates the geometry of the su(3) weights: harmonic oscillator states of the type $|n_1 n_2 n_3\rangle$ form a finite triangular lattice.

3.2 The SU(3) Building Blocks

3.2.1 SU(3) Coherent States

The definition of a coherent state in this thesis is a state obtained by a unitary transformation of a special state [16]. For SU(3), we choose the special state to be the state $|00\Lambda\rangle$. This choice is due to that fact that we define this state to be the highest weight SU(3) state (the details are covered in Appendix B.3.2). To construct a coherent state one needs, simply, to apply a unitary transformation:

$$|\mathcal{U}\rangle = \hat{U}|00\Lambda\rangle\,. \tag{3.6}$$

In general, the SU(3) element $\hat{U}$ depends on 8 parameters. It can be shown [34] that it is possible to factor the SU(3) transformation into the product of three block SU(2) transformations:

$$\hat{U} = T_{2,3}(\alpha_1, \beta_1, \gamma_1)T_{1,2}(\alpha_2, \beta_2, \alpha_2)T_{2,3}(\alpha_3, \beta_3, \gamma_3)\,, \tag{3.7}$$

where the sub-indices on the $T_{i,j}$ s indicate which subspace the operator acts in, and the $T_{i,j}$ s are:

$$T_{i,j}(\alpha, \beta, \gamma) = R_{(z)i,j}(\alpha)R_{(y)i,j}(\beta)R_{(z)i,j}(\gamma)\,. \tag{3.8}$$

Showing that the resulting coherent states are overcomplete and non-orthogonal is a non-trivial exercise. As these properties are not central to this work, the proofs will be omitted.

3.2.2 The Simplest su(3) Intelligent States

As with the angular momentum intelligent states, the building blocks for the su(3) intelligent states are the solutions to the simplest problem, which is $\Lambda = 1$. In this case it happens to be a 3×3 problem. Also, like the angular momentum case, the building blocks will be seen to be coherent states.

The su(3) algebra contains eight elements, but the uncertainty relation, and thus the equation for intelligence, depends only on two hermitian operators plus a third resulting the commutator of the initial two. An attempt to solve the problem in general using two operators that are arbitrary linear combinations of the 8 generators proved to be unmanageable. Since the uncertainty relation, and thus the equation for intelligence, depends only on two operators, we decided instead to choose:

$$\hat{\mathcal{A}}' = \frac{2\pi}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{\mathcal{B}}' = \frac{2\pi i}{3\sqrt{3}}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \tag{3.9}$$

with the commutation relation,

$$\hat{\mathcal{C}}' = -i[\hat{\mathcal{A}}', \hat{\mathcal{B}}'] = \frac{4\pi^2}{9\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix}. \tag{3.10}$$

$\hat{\mathcal{A}}'$ and $\hat{\mathcal{B}}'$ are sufficiently simple, yet sufficiently general to capture features of the su(3) problem absent from the angular momentum case.

The physical motivation behind is this choice is an analogy with the 2×2 angular momentum operators of Eq.(1.48). If $|\psi_i^x\rangle$ and $|\phi_j^y\rangle$ denote any 2-dimensional eigenvector of $\hat{L}_x$ and $\hat{L}_y$, respectively, then these eigenvectors satisfy the so-called complementarity condition, which in two dimensions reads:

$$|\langle\psi_i^x \,|\, \phi_j^y\rangle|^2 = \frac{1}{2}. \tag{3.11}$$

Our choice of su(3) observables is such that, if $|\Psi_i^{\mathcal{A}'}\rangle$ and $|\Phi_j^{\mathcal{B}'}\rangle$ denote any 3-dimensional eigenvector of $\hat{\mathcal{A}}'$ and $\hat{\mathcal{B}}'$, respectively, then these eigenvectors satisfy the complementary condition in three dimensions:

$$|\langle\Psi_i^{\mathcal{A}'} \,|\, \Phi_j^{\mathcal{B}'}\rangle|^2 = \frac{1}{3}. \tag{3.12}$$

For simplicity, and to follow as closely as possible the results on angular momentum, we go to a basis where $\hat{\mathcal{C}}' = -i[\hat{\mathcal{A}}', \hat{\mathcal{B}}']$ is diagonal. This will make calculations easier later. To achieve this we use the unitary matrix

$$\hat{\mathcal{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \frac{1-\sqrt{3}}{\sqrt{3-\sqrt{3}}} & \frac{1+\sqrt{3}}{\sqrt{3+\sqrt{3}}} \\ -1 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \\ 1 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \end{pmatrix}, \tag{3.13}$$

to make the transformations:

$$\hat{\mathcal{A}} = \hat{T}^{-1}\hat{\mathcal{A}}'\hat{T} = -\frac{2\pi}{3}\begin{pmatrix} 0 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \\ \frac{1}{\sqrt{3-\sqrt{3}}} & 0 & 0 \\ \frac{1}{\sqrt{3+\sqrt{3}}} & 0 & 0 \end{pmatrix},$$

$$\hat{\mathcal{B}} = \hat{T}^{-1}\hat{\mathcal{B}}'\hat{T} = \frac{2\pi i}{3}\begin{pmatrix} 0 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{-1}{\sqrt{3+\sqrt{3}}} \\ \frac{-1}{\sqrt{3-\sqrt{3}}} & 0 & 0 \\ \frac{1}{\sqrt{3+\sqrt{3}}} & 0 & 0 \end{pmatrix}, \tag{3.14}$$

$$\hat{\mathcal{C}} = \hat{T}^{-1}\hat{\mathcal{C}}'\hat{T} = \frac{4\pi^2}{9\sqrt{3}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1-\sqrt{3} & 0 \\ 0 & 0 & -1+\sqrt{3} \end{pmatrix}.$$

This unitary transformation does not affect the complementary condition used to define our su(3) observables. Note that the commutation relation, Eq.(3.10), holds for the unprimed operators as well since, using $\hat{T}^{-1}\hat{T} = \mathbb{1}$:

$$\begin{aligned} \hat{\mathcal{C}} &= \hat{T}^{-1}\hat{\mathcal{C}}'\hat{T} \\ &= \hat{T}^{-1}(-i[\hat{\mathcal{A}}',\hat{\mathcal{B}}'])\hat{T} \\ &= -i\hat{T}^{-1}(\hat{\mathcal{A}}'\hat{\mathcal{B}}' - \hat{\mathcal{B}}'\hat{\mathcal{A}}')\hat{T} \\ &= -i\hat{T}^{-1}(\hat{T}\hat{\mathcal{A}}\hat{T}^{-1}\hat{T}\hat{\mathcal{B}}\hat{T}^{-1} - \hat{T}\hat{\mathcal{B}}\hat{T}^{-1}\hat{T}\hat{\mathcal{A}}\hat{T}^{-1})\hat{T} \\ &= -i\hat{T}^{-1}(\hat{T}\hat{\mathcal{A}}\hat{\mathcal{B}}\hat{T}^{-1} - \hat{T}\hat{\mathcal{B}}\hat{\mathcal{A}}\hat{T}^{-1})\hat{T} \\ &= -i\hat{T}^{-1}\hat{T}(\hat{\mathcal{A}}\hat{\mathcal{B}} - \hat{\mathcal{B}}\hat{\mathcal{A}})\hat{T}^{-1}\hat{T} \\ &= -i(\hat{\mathcal{A}}\hat{\mathcal{B}} - \hat{\mathcal{B}}\hat{\mathcal{A}}) \\ \hat{\mathcal{C}} &= -i[\hat{\mathcal{A}},\hat{\mathcal{B}}]. \end{aligned} \tag{3.15}$$

The equality that defines intelligence, Eq.(1.5), then reads

$$\Delta\mathcal{A}\Delta\mathcal{B} = \tfrac{1}{2}|\langle\hat{\mathcal{C}}\rangle|. \tag{3.16}$$

Eq.(1.45) now becomes a 3×3 eigenvalue problem for the operator $\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}}$, with eigenvalue $\lambda = \langle\hat{\mathcal{A}}\rangle - i\alpha\langle\hat{\mathcal{B}}\rangle$. Solving yields three intelligent states, which are functions of the parameter α. Introducing once more $\mu = \frac{1+\alpha}{\sqrt{1-\alpha^2}}$, we have

$$\begin{aligned}
|\psi_1^1\rangle &= \mathcal{N}_1 \begin{pmatrix} 0 \\ \frac{1-\sqrt{3}}{\sqrt{2}}\mu^2 \\ 1 \end{pmatrix}, \quad \mathcal{N}_1 = \frac{1}{\sqrt{1+(2-\sqrt{3})\,|\mu|^4}} \\
|\psi_2^1\rangle &= \mathcal{N}_2 \begin{pmatrix} \sqrt{3+\sqrt{3}}\mu \\ \sqrt{2+\sqrt{3}}\mu^2 \\ 1 \end{pmatrix}, \quad \mathcal{N}_2 = \frac{1}{\sqrt{(1+|\mu|^2)\,(1+(2+\sqrt{3})\,|\mu|^2)}} \\
|\psi_3^1\rangle &= \mathcal{N}_3 \begin{pmatrix} -\sqrt{3+\sqrt{3}}\mu \\ \sqrt{2+\sqrt{3}}\mu^2 \\ 1 \end{pmatrix}, \quad \mathcal{N}_3 = \frac{1}{\sqrt{(1+|\mu|^2)\,(1+(2+\sqrt{3})\,|\mu|^2)}}.
\end{aligned} \tag{3.17}$$

Notice, for completeness, that if we had used the primed variables to calculate the intelligent states we would have the equation:

$$\hat{\mathcal{A}}' - i\alpha\hat{\mathcal{B}}'|\psi'\rangle = \lambda|\psi'\rangle,$$

where $|\psi'\rangle$ is completely arbitrary for now. Writing it in terms of the unprimed operators, we get

$$\hat{T}(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}})\hat{T}^{-1}|\psi'\rangle = \lambda|\psi'\rangle,$$

and left multiplying by $\hat{T}^{-1}$ gives

$$(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}})[\hat{T}^{-1}|\psi'\rangle] = \lambda[\hat{T}^{-1}|\psi'\rangle]. \tag{3.18}$$

Since we already know that for $\Lambda = 1$ the eigenstates of $(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}})$ are the states $|\psi_i^1\rangle$, it follows that

$$\hat{T}^{-1}|\psi_i'^1\rangle = |\psi_i^1\rangle \tag{3.19}$$

or

$$|\psi_i'^1\rangle = \hat{T}|\psi_i^1\rangle, \tag{3.20}$$

which makes it clear that the solutions in the primed basis would simply be related to our solutions by the transformation Eq.(3.13).

We wish now, as we did with the su(2) states, to construct these simple states using only unitary transformations. We begin by noting that all of the states in Eq.(3.17) have a '1' in the third row. We take advantage of this by using a common state from which we can construct our three intelligent states:

$$|\psi_i^1\rangle = \hat{U}_i \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad i = 1, 2, 3. \tag{3.21}$$

The general form for a unitary 3×3 matrix, which depends on 8 separate parameters, is not simple. It was found that the most convenient way to proceed is to write the required SU(3) transformation as a product of three block 2×2 unitary transformations (see Appendix B.5 for a proof of this decomposition):

$$\hat{U}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_i^* & b_i \\ 0 & -b_i^* & a_i \end{pmatrix} \begin{pmatrix} c_i^* & 0 & d_i \\ 0 & 1 & 0 \\ -d_i^* & 0 & c_i \end{pmatrix} \begin{pmatrix} f_i^* & g_i & 0 \\ -g_i^* & f_i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.22}$$

This particular factorization is chosen because the first rotation (rightmost in the factorization) will not affect the state in Eq.(3.21). However any factorization could have been used. To determine the entries in the transformations, we simply determine which values solve

$$\hat{U}_i^{-1}|\psi_i^1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.23}$$

Let us consider $|\psi_2^1\rangle$ for the purpose of example. We begin by operating on $|\psi_2^1\rangle$ with the inverse of $\hat{U}_2$;

$$\hat{U}_2^{-1}|\psi_2^1(\alpha)\rangle = \mathcal{N}_2 \begin{pmatrix} c_2 & 0 & -d_2 \\ 0 & 1 & 0 \\ d_2^* & 0 & c_2^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & -b_2 \\ 0 & b_2^* & a_2^* \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt{3}}\mu \\ \sqrt{2+\sqrt{3}}\mu^2 \\ 1 \end{pmatrix}. \tag{3.24}$$

The first step is to find a_2 and b_2 so as to introduce a zero as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & -b_2 \\ 0 & b_2^* & a_2^* \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt{3}}\mu \\ \sqrt{2+\sqrt{3}}\mu^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3+\sqrt{3}}\mu \\ 0 \\ b_2^*\sqrt{2+\sqrt{3}}\mu^2 + a_2^* \end{pmatrix}. \tag{3.25}$$

From the condition $|a_2|^2 + |b_2|^2 = 1$ and $a_2\sqrt{2+\sqrt{3}}\mu^2 - b_2 = 0$, a_2 and b_2 can be completely determined, up to an overall phase. Now we can apply a final rotation and determine $\hat{U}_2$:

$$\mathcal{N}_2 \begin{pmatrix} c_2 & 0 & -d_2 \\ 0 & 1 & 0 \\ d_2^* & 0 & c_2^* \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt{3}}\mu \\ 0 \\ b_2^*\sqrt{2+\sqrt{3}}\mu^2 + a_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.26}$$

Again, we can completely determine c_2 and d_2 in a similar manner. There is no need to apply a third rotation: since vector length is preserved and the state is normalized, the 1 appears for free. The three unitary transformations are found by taking the inverse of $\hat{U}_i^{-1}$. For instance

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1^* & b_1 \\ 0 & -b_1^* & a_1 \end{pmatrix}, \quad \begin{cases} a_1 = \frac{1}{\sqrt{1+(2-\sqrt{3})|\mu|^4}} \\ b_1 = \frac{(1-\sqrt{3})\mu^2}{\sqrt{2}\sqrt{1+(2-\sqrt{3})|\mu|^4}} \end{cases}. \tag{3.27}$$

$\hat{U}_2$ and $\hat{U}_3$ are closely related:

$$\hat{U}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2^* & b_2 \\ 0 & -b_2^* & a_2 \end{pmatrix} \begin{pmatrix} c_2^* & 0 & d_2 \\ 0 & 1 & 0 \\ -d_2^* & 0 & c_2 \end{pmatrix}$$
$$\hat{U}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2^* & b_2 \\ 0 & -b_2^* & a_2 \end{pmatrix} \begin{pmatrix} c_2^* & 0 & -d_2 \\ 0 & 1 & 0 \\ d_2^* & 0 & c_2 \end{pmatrix}, \quad \begin{cases} a_2 = \frac{1}{\sqrt{1+(2+\sqrt{3})|\mu|^4}} \\ b_2 = \frac{\sqrt{2+\sqrt{3}}\,\mu^2}{\sqrt{1+(2+\sqrt{3})|\mu|^4}} \\ c_2 = \frac{\sqrt{1+(2+\sqrt{3})|\mu|^4}}{\sqrt{(1+|\mu|^2)(1+(2+\sqrt{3})|\mu|^2)}} \\ d_2 = \frac{\sqrt{3+\sqrt{3}}\,\mu}{\sqrt{(1+|\mu|^2)(1+(2+\sqrt{3})|\mu|^2)}}. \end{cases} \tag{3.28}$$

Now that we have our three states in terms of unitary rotations, we will introduce an identification similar to Eq.(1.56):

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto |001\rangle, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto |010\rangle, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto |100\rangle. \tag{3.29}$$

We are now able to rewrite Eq.(1.45) in a more useful manner:

$$\left(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}}\right) \hat{U}_i(\alpha)|001\rangle = \lambda \hat{U}_i(\alpha)|001\rangle, \tag{3.30}$$

and our three building block solutions as

$$|\psi_i^1\rangle = \hat{U}_i(\alpha)|001\rangle, \quad i = 1, 2, 3. \tag{3.31}$$

Now that we have the simple building blocks, we need to work out how they combine together. As with the SU(2) states, the coupling of two identical extremal states yields an extremal state with unit probability. Thus, for instance,

$$\begin{aligned} \hat{C}_{3,2}\left[|001\rangle_A|001\rangle_B\right] &= \left(\hat{C}_{3,2}^A + \hat{C}_{3,2}^B\right)\left[|001\rangle_A|001\rangle_B\right] \\ &= \left[\hat{C}_{3,2}^A|001\rangle_A\right]|001\rangle_B + |001\rangle_A\left[\hat{C}_{3,2}^B|001\rangle_B\right] \\ &= 0 = \hat{C}_{3,2}|002\rangle, \end{aligned} \tag{3.32}$$

where the operator $\hat{C}_{3,2}$ is defined by

$$\begin{aligned} \hat{C}_{3,2} &= \hat{C}_{3,2} \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \hat{C}_{3,2} \\ &= \hat{C}_{3,2}^A + \hat{C}_{3,2}^B. \end{aligned} \tag{3.33}$$

The same is true for the other two raising operators $\hat{C}_{3,1}$ and $\hat{C}_{2,1}$ (see Appendix B.3.2 for a discussion of su(3) raising and lowering operators). Using this we can construct

$$|001\rangle|001\rangle = |002\rangle, \tag{3.34}$$

and then coupling this to another extremal state will continue the process:

$$|002\rangle|001\rangle = |003\rangle. \tag{3.35}$$

To get to the state $|00\Lambda\rangle$, we simply need to couple Λ states together:

$$|001\rangle_A|001\rangle_B \dots |001\rangle_\Lambda = |00\Lambda\rangle. \tag{3.36}$$

From this extremal state we can obtain a coherent state via the $U_i(\alpha)$ operators. As well, coupling together Λ identical coherent states of the form $\hat{U}_i(\alpha)|001\rangle$ will produce a coherent state:

$$\begin{aligned}
&\left[\hat{U}_i^A(\alpha)|001\rangle_A\right] \otimes \left[\hat{U}_i^B(\alpha)|001\rangle_B\right] \otimes \dots \otimes \left[\hat{U}_i^\Lambda(\alpha)|001\rangle_\Lambda\right] \\
&\quad = \hat{U}_i(\alpha)\left[|001\rangle_A|001\rangle_B \dots |001\rangle_\Lambda\right] \\
&\quad = \hat{U}_i(\alpha)|00\Lambda\rangle \\
&\quad = |\psi_i^\Lambda(\alpha)\rangle\,,
\end{aligned} \tag{3.37}$$

where

$$\hat{U}_i(\alpha) = \hat{U}_i^A(\alpha) \otimes \hat{U}_i^B(\alpha) \otimes \dots \otimes \hat{U}_i^\Lambda(\alpha). \tag{3.38}$$

As with the SU(2) states, we take advantage of the ease of building up the SU(3) coherent states to construct the su(3) intelligent states.

Since we have factored the $\hat{U}_i(\alpha)$s into block su(2) rotations, $\hat{U}_i(\alpha) = R_{2,3}(\theta_i)R_{1,3}(\phi_i)$, we can write the intelligent states as $R_{2,3}(\theta_i)R_{1,3}(\phi_i)|00\Lambda\rangle$ and note that, for $|\alpha| \leq 1$, we can define

$$b_i = -\sin\tfrac{\theta_i}{2}\,, \qquad d_i = -\sin\tfrac{\phi_i}{2}. \tag{3.39}$$

We now have our three basic intelligent states in the form

$$\begin{aligned}
|\psi_1^1(\alpha)\rangle &= R_{2,3}(\theta_1)|001\rangle \\
|\psi_2^1(\alpha)\rangle &= R_{2,3}(\theta_2)R_{1,3}(\phi_2)|001\rangle \\
|\psi_3^1(\alpha)\rangle &= R_{2,3}(\theta_2)R_{1,3}(-\phi_2)|001\rangle.
\end{aligned} \tag{3.40}$$

Comparing these with Eq.(3.6) shows that they are coherent states.

3.3 A General Construction

3.3.1 An Example: The Six-Dimensional Case

As an illustrative example in a higher dimension, we will consider the case where the total number of quantum elements in the system is $n_1 + n_2 + n_3 = \Lambda = 2$. This translates into a 6×6 matrix representation. We need to solve for the eigenvalues and eigenstates of the matrix

$$\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}} = -\frac{2\pi}{3}\begin{pmatrix} 0 & (1-\alpha)\eta_- & 0 & (1+\alpha)\eta_+ & 0 & 0 \\ (1+\alpha)\eta_- & 0 & (1-\alpha)\eta_- & 0 & \frac{(1+\alpha)\eta_+}{\sqrt{2}} & 0 \\ 0 & (1+\alpha)\eta_- & 0 & 0 & 0 & 0 \\ (1-\alpha)\eta_+ & 0 & 0 & 0 & \frac{(1-\alpha)\eta_-}{\sqrt{2}} & (1+\alpha)\eta_+ \\ 0 & \frac{(1-\alpha)\eta_+}{\sqrt{2}} & 0 & \frac{(1+\alpha)\eta_-}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & (1-\alpha)\eta_+ & 0 & 0 \end{pmatrix}.$$

with $\eta_+ = \sqrt{\frac{2}{3+\sqrt{3}}}$, and $\eta_- = \sqrt{\frac{2}{3-\sqrt{3}}}$. Clearly, the complexity of the problem grows rapidly with Λ, which is where the power of the coupling method can be best seen.

The SU(3) building blocks are the solution to the $\Lambda = 1$ problem, given by Eq.(3.40). By repeating, for $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, the steps leading to Eq.(2.70), we know that if we couple two $\Lambda = 1$ solutions together, we get another solution. There are a total of nine possible products of the three basic building blocks:

$$\begin{aligned} &|\psi_1^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \ |\psi_1^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B, \ |\psi_1^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B, \\ &|\psi_2^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \ |\psi_2^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B, \ |\psi_2^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B, \\ &|\psi_3^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \ |\psi_3^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B, \ |\psi_3^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B. \end{aligned} \tag{3.41}$$

From this, we can extract six states symmetric under the exchange of A and B :

$$\begin{aligned} &|\psi_1^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \ |\psi_2^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B, \ |\psi_3^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B \\ &|\psi_1^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B + |\psi_2^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \\ &|\psi_1^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B + |\psi_3^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \\ &|\psi_2^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B + |\psi_3^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B \end{aligned} \tag{3.42}$$

and three states antisymmetric under exchange of A and B :

$$\begin{aligned} &|\psi_1^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B - |\psi_2^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \\ &|\psi_1^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B - |\psi_3^1(\alpha)\rangle_A |\psi_1^1(\alpha)\rangle_B, \\ &|\psi_2^1(\alpha)\rangle_A |\psi_3^1(\alpha)\rangle_B - |\psi_3^1(\alpha)\rangle_A |\psi_2^1(\alpha)\rangle_B. \end{aligned} \tag{3.43}$$

The situation is reminiscent of angular momentum systems, where coupling the two spin-$\frac{1}{2}$ particles yields symmetric $\ell = 1$ states and an antisymmetric $\ell = 0$ state. This is symbolically written as

$$\frac{1}{2} \otimes \frac{1}{2} \rightarrow 1 \oplus 0. \tag{3.44}$$

The rules for combining su(3) states are similar. Using the notation of Appendix B.3.2, the symmetric product of two states in $(0,1)$ is a state in $(0,2)$ while the anti-symmetric product is a state in $(1,0)$, or

$$(0,1)\otimes(0,1)\rightarrow(0,2)\oplus(1,0). \tag{3.45}$$

While the rules for combining two general su(3) states are quite complicated, one can show quite generally that,

$$(0,q_1)\otimes(0,q_2)\rightarrow(0,q_1+q_2)\oplus\ldots, \tag{3.46}$$

which confirms the statement that the product of two highest weights is also a highest weight.

Much like we did for angular momentum, where we restricted, using a projection operator, the total angular momentum to satisfy $\ell=\ell_A+\ell_B,$ we will, in this thesis, use a projection operator to restrict the triple product $(0,q_1)\otimes(0,q_2)\otimes(0,q_3)$ to states in $(0,q_1+q_2+q_3)$. In terms of harmonic oscillator states, this condition reads $\Lambda=\Lambda_1+\Lambda_2+\Lambda_3$.

Before projection into the $(0,q_1+q_2+q_3)$ subspace, our states will be denoted as

$$|\phi_{\Lambda_1,\Lambda_2,\Lambda_3}\rangle=|\psi_1^{\Lambda_1}(\alpha)\rangle_A\otimes|\psi_2^{\Lambda_2}(\alpha)\rangle_B\otimes|\psi_3^{\Lambda_3}(\alpha)\rangle_C. \tag{3.47}$$

We now proceed with the example of the coupling

$$\begin{aligned}|\phi_{1,1,0}\rangle &= \left[|\psi_1^1(\alpha)\rangle_A\right]\otimes\left[|\psi_2^1(\alpha)\rangle_B\right]\\ &= \left[R^A_{2,3}(\theta_1)|001\rangle_A\right]\otimes\left[R^B_{2,3}(\theta_2)R^B_{1,3}(\phi_2)|001\rangle_B\right].\end{aligned} \tag{3.48}$$

The Case $|\alpha|\le 1$

For $|\alpha|<1$, the operators $R_{i,j}(\vartheta)$ correspond to rotations about the y-axis, and can be written as $R_{(y)i,j}(\vartheta)$. As we will be using this case to do most of the calculations, we will suppress the y subscript unless a distinction is necessary.

We can use the $\Lambda=1$ unit operator,

$$\mathbb{1}=\sum_{n=0}^{1}\sum_{m=1}^{n}|m,n-m,1-n\rangle\langle m,n-m,1-n|\,, \tag{3.49}$$

to deal with the $R_{i,j}(\vartheta)$s. Since each rotation only happens in an SU(2) subspace, we can make the identification of the type

$$|n_1,n_2,n_3\rangle\mapsto|\ell_{23},m_{23}\rangle \tag{3.50}$$

via,

$$\ell_{23}=\frac{1}{2}(n_2+n_3),\quad m_{23}=\frac{1}{2}(n_2-n_3). \tag{3.51}$$

This particular identification would be valid for $R_{2,3}(\vartheta)$ as this transformation does not affect n_1. This allows us to employ the same Wigner d-function as we did for the su(2) states. Starting with the "A" subspace we quickly get

$$\sum_{n_A=0}^{1} |0, n_A, 1-n_A\rangle_A \langle 0, n_A, 1-n_A| R^A_{2,3}(\theta_1)|001\rangle_A$$
$$= \sum_{n_A} |0, n_A, 1-n_A\rangle_A \, d^{\frac{1}{2}}_{\frac{1}{2}(2n_A-1), -\frac{1}{2}}(\theta_1). \tag{3.52}$$

There are two rotations in the "B" subspace, and they must be dealt with separately. Inserting the unit once gives:

$$R^B_{2,3}(\theta_2) \sum_{\nu_B=0}^{1} |\nu_B, 0, 1-\nu_B\rangle_B \langle \nu_B, 0, 1-\nu_B| R^B_{1,3}(\phi_2)|001\rangle_B$$
$$= R^B_{2,3}(\theta_2) \sum_{\nu_B} |\nu_B, 0, 1-\nu_B\rangle_B \, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1), -\frac{1}{2}}(\phi_2)$$
$$= \sum_{\nu_B} R^B_{2,3}(\theta_2)|\nu_B, 0, 1-\nu_B\rangle_B \, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1), -\frac{1}{2}}(\phi_2). \tag{3.53}$$

A second application of the unit yields:

$$\sum_{\nu_B} \sum_{n_B=0}^{1-\nu_B} |\nu_B, n_B, 1-\nu_B-n_B\rangle_B \, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1), -\frac{1}{2}}(\phi_2)$$
$$\times {}_B\langle \nu_B, n_B, 1-\nu_B-n_B| R^B_{2,3}(\theta_2)|\nu_B, 0, 1-\nu_B\rangle_B$$
$$= \sum_{\nu_B} \sum_{n_B} |\nu_B, n_B, 1-\nu_B-n_B\rangle_B$$
$$\times d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1), -\frac{1}{2}}(\phi_2) \, d^{\frac{1}{2}(1-\nu_B)}_{\frac{1}{2}(2n_B+\nu_B-1), \frac{1}{2}(\nu_B-1)}(\theta_2). \tag{3.54}$$

Putting these back together gives

$$|\phi_{1,1,0}\rangle = \sum_{n_A, \nu_B, n_B} |0, n_A, 1-n_A\rangle_A |\nu_B, n_B, 1-\nu_B-n_B\rangle_B$$
$$\times d^{\frac{1}{2}}_{\frac{1}{2}(2n_A-1), -\frac{1}{2}}(\theta_1) \, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1), -\frac{1}{2}}(\phi_2) \, d^{\frac{1}{2}(1-\nu_B)}_{\frac{1}{2}(2n_B+\nu_B-1), \frac{1}{2}(\nu_B-1)}(\theta_2). \tag{3.55}$$

All that remains to do is couple the two kets together and we will have all of the $\Lambda = 2$ intelligent states (see Appendix B.3.2 for a discussion on su(3) coupling). When we are coupling the states, we only want to keep the state with $\Lambda = \Lambda_A + \Lambda_B = 2$. The state we will keep is

$$|\nu_B, n_A + n_B, 2 - n_A - \nu_B - n_B\rangle, \tag{3.56}$$

with coupling coefficient [27]

$$\left[\frac{(2-n_A-n_B-\nu_B)!\,(n_A+n_B)!}{2!}\right]^{\frac{1}{2}} . \tag{3.57}$$

We now have our unnormalized $\Lambda = 2$ intelligent state as

$$\begin{aligned}|\psi^2_{1,1,0}(\alpha)\rangle &= \sum_{n_A,\nu_B,n_B} |\nu_B, n_A+n_B, 2-n_A-\nu_B-n_B\rangle \\ &\times\sqrt{\frac{(2-n_A-n_B-\nu_B)!\,(n_A+n_B)!}{2!}} \\ &\times d^{\frac{1}{2}}_{\frac{1}{2}(2n_A-1),-\frac{1}{2}}(\theta_1)\, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1),-\frac{1}{2}}(\phi_2)\, d^{\frac{1}{2}(1-\nu_B)}_{\frac{1}{2}(2n_B+\nu_B-1),\frac{1}{2}(\nu_B-1)}(\theta_2),\end{aligned} \tag{3.58}$$

where the relation between the angles and α is given by Eq.(3.39).

The Case $|\alpha| > 1$

To approach the $|\alpha| > 1$ problem, we note that the only change that occurs is in the value of μ. For the previous case μ was a real number. In this case, however, μ becomes purely imaginary. This does not affect the process by which we construct the intelligent states; it simply changes the form of one of the rotations. Looking at Eq.(3.27) and Eq.(3.28), one can see that there are sign changes, but that is all for the $R_{2,3}$ rotations. The $R_{1,3}$ rotation, on the other hand, changes in a way that we have seen before. Two of its matrix elements become imaginary, transforming the rotation into one about the x-axis. We then can use the relation

$$R_x(\vartheta) = R_z(-{}^{\pi}\!/2)R_y(\vartheta)R_z({}^{\pi}\!/2) \tag{3.59}$$

to evaluate the effect of this change. The only place that this rotation appears is in Eq.(3.53). Using the definition of the Wigner D-function [26] to evaluate the expectation value yields

$$\begin{aligned}&{}_B\langle\nu_B, 0, 1-\nu_B|R^B_{(x)1,3}(\phi_2)|001\rangle_B \\ &={}_B\,\langle\nu_B, 0, 1-\nu_B|R^B_{(z)1,3}(-{}^{\pi}\!/2)R^B_{(y)1,3}(\phi_2)R^B_{(z)1,3}({}^{\pi}\!/2)|001\rangle_B \\ &= D^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1),-\frac{1}{2}}(-{}^{\pi}\!/2,\ \phi_2,\ {}^{\pi}\!/2) \\ &= \mathrm{e}^{i\pi(\nu_B-\frac{1}{2})/2}\, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1),-\frac{1}{2}}(\phi_2)\,\mathrm{e}^{i\pi/4} \\ &= \mathrm{e}^{i\pi\nu_B/2}\, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1),-\frac{1}{2}}(\phi_2)\end{aligned} \tag{3.60}$$

The unnormalized intelligent state for $|\alpha| > 1$ is then

$$
\begin{aligned}
|\psi^2_{1,1,0}(\alpha)\rangle = & \sum_{n_A,\nu_B,n_B} |\nu_B, n_A + n_B, 2 - n_A - \nu_B - n_B\rangle \\
& \times e^{i\pi\nu_B/2}\sqrt{\frac{(2-n_A-n_B-\nu_B)!\,(n_A+n_B)!}{2!}} \\
& \times d^{\frac{1}{2}}_{\frac{1}{2}(2n_A-1),-\frac{1}{2}}(\theta_1)\, d^{\frac{1}{2}}_{\frac{1}{2}(2\nu_B-1),-\frac{1}{2}}(\phi_2)\, d^{\frac{1}{2}(1-\nu_B)}_{\frac{1}{2}(2n_B+\nu_B-1),\frac{1}{2}(\nu_B-1)}(\theta_2)\,. \qquad (3.61)
\end{aligned}
$$

3.3.2 The General Expression

To develop the general construction for the su(3) intelligent states, we begin by noting that the three solutions to the $\Lambda = 1$ or 3×3 case are coherent states. These three simple intelligent states will serve as the building blocks with which we can construct all of the intelligent states for any given $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$. We will assume $\alpha < 1$ for the purpose of illustration. To construct the general expression we begin with Eq.(3.37) and couple Λ_1 copies of the state $|\psi^1_1\rangle$, Λ_2 copies of $|\psi^1_2\rangle$, and Λ_3 copies of $|\psi^1_3\rangle$:

$$
\begin{aligned}
|\phi_{\Lambda_1,\Lambda_2,\Lambda_3}\rangle &= [\hat{U}_1(\alpha)|00\Lambda_1\rangle] \otimes [\hat{U}_2(\alpha)|00\Lambda_2\rangle] \otimes [\hat{U}_3(\alpha)|00\Lambda_3\rangle] \\
&= [R_{2,3}(\theta_1)|00\Lambda_1\rangle] \otimes [R_{2,3}(\theta_2)R_{1,3}(\phi_2)|00\Lambda_2\rangle] \\
&\qquad \otimes [R_{2,3}(\theta_2)R_{1,3}(-\phi_2)|00\Lambda_3\rangle] \\
&= [R_{2,3}(\theta_1)|00\Lambda_1\rangle] \otimes R_{2,3}(\theta_2)\,[R_{1,3}(\phi_2)|00\Lambda_2\rangle \otimes R_{1,3}(-\phi_2)|00\Lambda_3\rangle]\,.
\end{aligned} \qquad (3.62)
$$

We are now left to write Eq.(3.62) in a way that is more transparent. To do this we must project all of the coherent states onto the same basis, using the unit operator:

$$
\mathbb{1}_j = \sum_{n_j=0}^{\Lambda_j}\sum_{m_j=0}^{n_j} |m_j, n_j - m_j, \Lambda_j - n_j\rangle\langle m_j, n_j - m_j, \Lambda_j - n_j|, \qquad (3.63)
$$

where j indicates the subspace in which the operator acts. Looking at the state $R_{2,3}(\theta_1)|00\Lambda_1\rangle$, we note that the transformation $R_{2,3}(\theta_1)$ only mixes the 2nd and 3rd slots of $|00\Lambda_1\rangle$ and leaves the first as zero. This means that the projection will take the form

$$
R_{2,3}(\theta_1)|00\Lambda_1\rangle = \sum_{n_1=0}^{\Lambda_1} |0, n_1, \Lambda_1 - n_1\rangle\langle 0, n_1, \Lambda_1 - n_1|R_{2,3}(\theta_1)|00\Lambda_1\rangle. \qquad (3.64)
$$

Again, each rotation only happens in an SU(2) subspace, and we make, again, the identification

$$
|n_1, n_2, n_3\rangle \mapsto |\ell_{23}, m_{23}\rangle \qquad (3.65)
$$

via,

$$\ell_{23} = \frac{1}{2}(n_2 + n_3), \quad m_{23} = \frac{1}{2}(n_2 - n_3), \tag{3.66}$$

so as to employ the SU(2) Wigner d-function.

Beginning with the state $R_{2,3}(\theta_1)|00\Lambda_1\rangle$, we write

$$\begin{aligned} R_{2,3}(\theta_1)|00\Lambda_1\rangle &= \sum_{n_1=0}^{\Lambda_1} |0, n_1, \Lambda_1 - n_1\rangle\langle 0, n_1, \Lambda_1 - n_1|R_{2,3}(\theta_1)|00\Lambda_1\rangle \\ &= \sum_{n_1=0}^{\Lambda_1} |0, n_1, \Lambda_1 - n_1\rangle\, d^{\frac{1}{2}\Lambda_1}_{\{2n_1,\Lambda_1\},\{0,\Lambda_1\}}(\theta_1), \end{aligned} \tag{3.67}$$

where the pairs

$$\{a, b\} \equiv \frac{1}{2}(a - b) \tag{3.68}$$

in the d-functions.

Likewise,

$$\begin{aligned} R_{1,3}(\phi_2)|00\Lambda_2\rangle &= \sum_{n_2=0}^{\Lambda_2} |n_2, 0, \Lambda_2 - n_2\rangle\langle n_2, 0, \Lambda_2 - n_2|R_{1,3}(\phi_2)|00\Lambda_2\rangle \\ &= \sum_{n_2=0}^{\Lambda_2} |n_2, 0, \Lambda_2 - n_2\rangle\, d^{\frac{1}{2}\Lambda_2}_{\{2n_2,\Lambda_2\},\{0,\Lambda_2\}}(\phi_2), \end{aligned} \tag{3.69}$$

and

$$\begin{aligned} R_{1,3}(-\phi_2)|00\Lambda_3\rangle &= \sum_{n_3=0}^{\Lambda_3} |n_3, 0, \Lambda_3 - n_3\rangle\langle n_3, 0, \Lambda_3 - n_3|R_{1,3}(-\phi_2)|00\Lambda_3\rangle \\ &= \sum_{n_3=0}^{\Lambda_3} |n_3, 0, \Lambda_3 - n_3\rangle\, d^{\frac{1}{2}\Lambda_3}_{\{2n_3,\Lambda_3\},\{0,\Lambda_3\}}(-\phi_2). \end{aligned} \tag{3.70}$$

Putting the pieces back together yields

$$\begin{aligned} |\psi^{\Lambda}_{\Lambda_1,\Lambda_2,\Lambda_3}(\alpha)\rangle = \sum_{n_1=0}^{\Lambda_1} |0, n_1, \Lambda_1 - n_1\rangle\, d^{\frac{1}{2}\Lambda_1}_{(2n_1,\Lambda_1),(0,\Lambda_1)}(\theta_1) \\ \otimes R_{2,3}(\theta_2)\left[\sum_{n_2=0}^{\Lambda_2} |n_2, 0, \Lambda_2 - n_2\rangle\, d^{\frac{1}{2}\Lambda_2}_{\{2n_2,\Lambda_2\},\{0,\Lambda_2\}}(\phi_2)\right. \\ \left.\times \sum_{n_3=0}^{\Lambda_3} |n_3, 0, \Lambda_3 - n_3\rangle\, d^{\frac{1}{2}\Lambda_3}_{\{2n_3,\Lambda_3\},\{0,\Lambda_3\}}(-\phi_2)\right]. \end{aligned} \tag{3.71}$$

Since the d-functions of the form $d^j_{m,-j}(\beta)$ have a simple expression [26],

$$d^j_{m,-j}(\beta) = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\left(\cos\frac{\beta}{2}\right)^{j-m}\left(-\sin\frac{\beta}{2}\right)^{j+m}. \tag{3.72}$$

we can use that to combine the two d-functions with argument ϕ_2,

$$\begin{aligned} d^{\frac{1}{2}\Lambda_2}_{\{2n_2,\Lambda_2\},\{0,\Lambda_2\}}(\phi_2)\, d^{\frac{1}{2}\Lambda_3}_{\{2n_3,\Lambda_3\},\{0,\Lambda_3\}}(-\phi_2) = {} & (-1)^{n_3}\sqrt{\frac{\Lambda_2!\Lambda_3!(\Lambda_{23}-n_{23})!n_{23}!}{\Lambda_{23}!(\Lambda_2-n_2)!(\Lambda_3-n_3)!n_2!n_3!}} \\ & \times d^{\frac{1}{2}\Lambda_{23}}_{\{2n_{23},\Lambda_{23}\},\{0,\Lambda_{23}\}}(\phi_2)\,, \end{aligned} \tag{3.73}$$

where we have used

$$\begin{aligned} n_{ijk} &\equiv n_i + n_j + n_k\,, \\ n_{ij} &\equiv n_i + n_j\,, \\ \Lambda_{ij} &\equiv \Lambda_i + \Lambda_j\,. \end{aligned} \tag{3.74}$$

This leaves us with

$$\begin{aligned} |\psi^{\Lambda}_{\Lambda_1,\Lambda_2,\Lambda_3}(\alpha)\rangle = {} & \sum_{n_1,n_2,n_3}(-1)^{n_3}\sqrt{\frac{\Lambda_2!\Lambda_3!(\Lambda_{23}-n_{23})!n_{23}!}{\Lambda_{23}!(\Lambda_2-n_2)!(\Lambda_3-n_3)!n_2!n_3!}} \\ & \times|0,n_1,\Lambda_1-n_1\rangle\, d^{\frac{1}{2}\Lambda_1}_{\{2n_1,\Lambda_1\},\{0,\Lambda_1\}}(\theta_1)\, d^{\frac{1}{2}\Lambda_{23}}_{\{2n_{23},\Lambda_{23}\},\{0,\Lambda_{23}\}}(\phi_2) \\ & \times R_{2,3}(\theta_2)\left[|n_2,0,\Lambda_2-n_2\rangle|n_3,0,\Lambda_3-n_3\rangle\right]. \end{aligned} \tag{3.75}$$

Before we can perform the last rotation, $R_{2,3}(\theta_2)$, we need to couple the two kets so that we can rotate a single state. We couple su(3) states in a manner that is covered in Appendix B.3.2. Since we are only interested in states that satisfy $\Lambda_{23} = \Lambda_2 + \Lambda_3$, we note that the only relevant su(3) Clebsch-Gordan coefficient is

$$\begin{aligned} & \langle n_{23},0,\Lambda_{23}-n_{23}\,|\,n_2,0,\Lambda_2-n_2\,;n_3,0,\Lambda_3-n_3\rangle \\ & \quad = \sqrt{\frac{\Lambda_2!\Lambda_3!(\Lambda_{23}-n_{23})!n_{23}!}{\Lambda_{23}!(\Lambda_2-n_2)!(\Lambda_3-n_3)!n_2!n_3!}}. \end{aligned} \tag{3.76}$$

We can now apply the $R_{2,3}(\theta_2)$ rotation:

$$\begin{aligned} R_{2,3}(\theta_2)|n_{23},0,\Lambda_{23}-n_{23}\rangle = {} & \sum_{\nu=0}^{\Lambda_{23}-n_{23}}|n_{23},\nu,\Lambda_{23}-n_{23}-\nu\rangle \\ & \times d^{\frac{1}{2}(\Lambda_{23}-n_{23})}_{\{2\nu+n_{23},\Lambda_{23}\},\{n_{23},\Lambda_{23}\}}(\theta_2). \end{aligned} \tag{3.77}$$

In coupling the last two remaining states, we use the Clebsch

$$\begin{aligned} & \langle n_{23},n_1+\nu,\Lambda-n_{123}-\nu\,|\,0,n_1,\Lambda_1-n_1\,;n_{23},\nu,\Lambda_{23}-n_{23}-\nu\rangle \\ & \quad = \sqrt{\frac{\Lambda_1!(\Lambda_{23})!(\Lambda-n_{123}-\nu)!(n_1+\nu)!}{(\Lambda)!(\Lambda_1-n_1)!(\Lambda_{23}-n_{23}-\nu)!n_1!\nu!}}. \end{aligned} \tag{3.78}$$

The final expression for the unnormalized intelligent states is

$$|\psi^{\Lambda}_{\Lambda_1,\Lambda_2,\Lambda_3}(\alpha)\rangle = \sum_{n_1,n_2,n_3,\nu} |n_{23}, n_1+\nu, \Lambda - n_{123} - \nu\rangle K^{\Lambda,\Lambda_1,\Lambda_2,\Lambda_3}_{n_1,n_2,n_3,\nu}(\theta_1,\theta_2,\phi_2), \quad (3.79)$$

with

$$\begin{aligned} K^{\Lambda,\Lambda_1,\Lambda_2,\Lambda_3}_{n_1,n_2,n_3,\nu}(\theta_1,\theta_2,\phi_2) &= (-1)^{n_3}\frac{\Lambda_2!\Lambda_3!(\Lambda_{23}-n_{23})!n_{23}!}{(\Lambda_2-n_2)!(\Lambda_3-n_3)!n_2!n_3!} \\ &\times\sqrt{\frac{\Lambda_1!(\Lambda-n_{123}-\nu)!(n_1+\nu)!}{\Lambda!\Lambda_{23}!(\Lambda_1-n_1)!(\Lambda_{23}-n_{23}-\nu)!n_1!\nu!}} \\ &\times d^{\frac{1}{2}\Lambda_1}_{\{2n_1,\Lambda_1\},\{0,\Lambda_1\}}(\theta_1)\, d^{\frac{1}{2}\Lambda_{23}}_{\{2n_{23},\Lambda_{23}\},\{0,\Lambda_{23}\}}(\phi_2)\, d^{\frac{1}{2}(\Lambda_{23}-n_{23})}_{\{2\nu+n_{23},\Lambda_{23}\},\{n_{23},\Lambda_{23}\}}(\theta_2) \end{aligned} \quad (3.80)$$

This may seem to be a complicated expression. However, consider the case for $\Lambda = 3$. That could be the result of simply coupling 1 copy of each of the three basic intelligent states. This would amount to diagonalizing a 10×10 matrix. If you want to construct a $\Lambda = 4$ state you would have a 15×15 matrix to diagonalize; and in general, the size of the matrix scales with Λ as $\frac{1}{2}(\Lambda+2)(\Lambda+1)$. The relative complexity of the coupling method is clearly less with increasing Λ.

To construct the expression for $|\alpha| > 1$ we need to look at the form of μ,

$$\mu = \frac{1+\alpha}{\sqrt{1-\alpha^2}}. \quad (3.81)$$

When $|\alpha| < 1$ then μ is real, but when $|\alpha| > 1$ μ is completely imaginary. This causes a small change in the forms of the $\hat{U}_i(\alpha)$s. In particular, the $R_{(y)1,3}(\phi_i)$s become rotations about the x-axis, $R_{(x)1,3}(\phi_i)$. This simply implies that we must factor it further:

$$\begin{aligned} R_{(x)1,3}(\phi_i) &= R_{(z)1,3}(-\tfrac{\pi}{2})R_{(y)1,3}(\phi_i)R_{(z)1,3}(\tfrac{\pi}{2}) \\ \langle \ell, m|R_{(x)1,3}(\phi_i)|\ell, m'\rangle &= D^{\ell}_{m,m'}(-\tfrac{\pi}{2},\phi_i,\tfrac{\pi}{2}) \\ &= \mathrm{e}^{im\pi/2}\, d^{\ell}_{m,m'}(\phi_i)\mathrm{e}^{-im'\pi/2}. \end{aligned} \quad (3.82)$$

The only difference, then, between the states for $|\alpha| < 0$ and those for $|\alpha| > 0$ is the inclusion of a factor of $\mathrm{e}^{i\pi(n_2+n_3)/2}$ in the latter:

$$\begin{aligned} |\psi^{\Lambda}_{\Lambda_1,\Lambda_2,\Lambda_3}(\alpha)\rangle = \sum_{n_1,n_2,n_3,\nu} & |n_{23}, n_1+\nu, \Lambda - n_{123} - \nu\rangle \\ & \times \mathrm{e}^{i\pi n_{23}/2} K^{\Lambda,\Lambda_1,\Lambda_2,\Lambda_3}_{n_1,n_2,n_3,\nu}(\theta_1,\theta_2,\phi_2)\,. \end{aligned} \quad (3.83)$$

3.3.3 Selected Results

The su(3) intelligent states are the solutions to the eigenvalue equation

$$(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}})|\psi\rangle = \lambda|\psi\rangle\,, \quad (3.84)$$

and they have the eigenvalues

$$\lambda = \frac{2\pi}{3}\sqrt{1-\alpha^2}(\Lambda_3 - \Lambda_2)\,. \tag{3.85}$$

One can also show that the following relations hold:

$$(\Delta\mathcal{A})^2 = -\frac{1}{2}\alpha\langle\hat{\mathcal{C}}\rangle, \qquad (\Delta\mathcal{B})^2 = -\frac{1}{2\alpha}\langle\hat{\mathcal{C}}\rangle\,. \tag{3.86}$$

This is done in a manner similar to the su(2) states.

The uncertainty curves for the su(3) intelligent states display the expected behaviour that was discussed in Section 2.4.2, *i.e.* the uncertainty is zero at $\alpha = 0, \pm\infty$, and there are discontinuities at $\alpha = \pm 1$. The similarities seem to stop there, however. We have not been able to determine a trend, similar to the one shown in Fig 2.2, where the uncertainty, overall, is higher or lower for states of a given Λ with different values of Λ_1, Λ_2, Λ_3.

Figures 3.3 and 3.4 illustrate typical uncertainty curves for the su(3) states. The

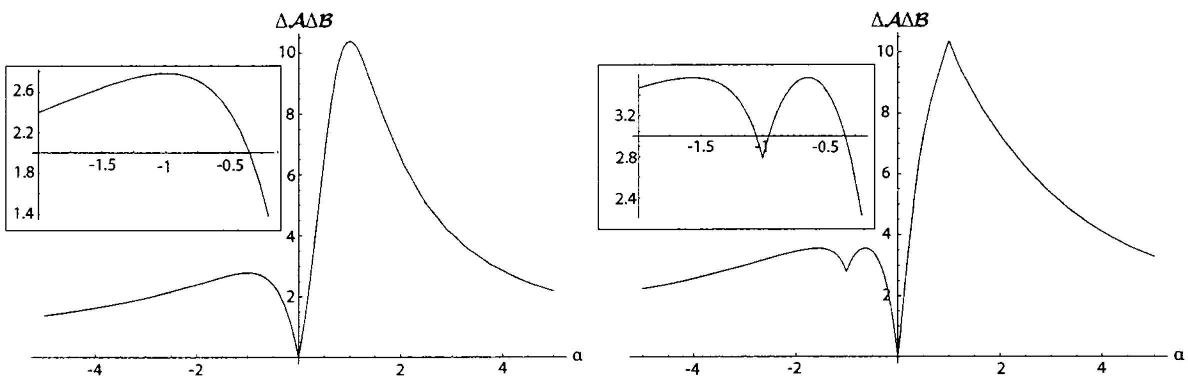

Figure 3.3: Two plots of $\Delta\mathcal{A}\Delta\mathcal{B}$ for $\Lambda = 3$. The inset is an expanded view around $\alpha = -1$. **Left:** $\Lambda_1 = 3$, $\Lambda_2 = 0$, $\Lambda_3 = 0$, **Right:** $\Lambda_1 = 1$, $\Lambda_2 = 2$, $\Lambda_3 = 0$.

section of these curves for $\alpha > 0$ are reminiscent of the su(2) curves; for $\alpha < 0$ however, there are clear differences.

The most striking feature of the su(3) graphs is the difference in amplitude between positive and negative α. The overall uncertainty for negative α is significantly less than that for positive α for every graph produced up to this point. However, the height of the graph at $\alpha = -1$ can easily be determined. From the definition of μ, Eq.(3.81), it can be shown that

$$\lim_{\alpha\to -1}\frac{1+\alpha}{\sqrt{1-\alpha^2}} = 0. \tag{3.87}$$

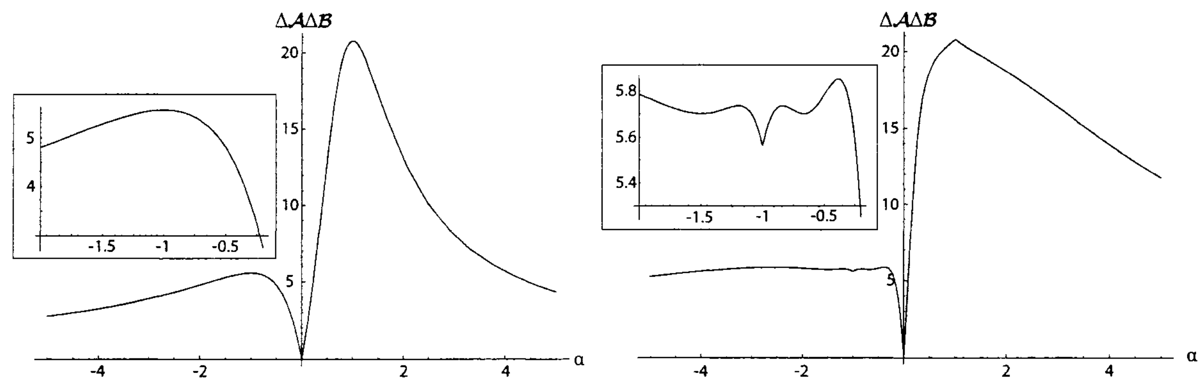

Figure 3.4: Two plots of $\Delta\mathcal{A}\Delta\mathcal{B}$ for $\Lambda = 6$. The inset is an expanded view around $\alpha = -1$. **Left:** $\Lambda_1 = 6$, $\Lambda_2 = 0$, $\Lambda_3 = 0$, **Right:** $\Lambda_1 = 2$, $\Lambda_2 = 3$, $\Lambda_3 = 1$.

Using this, and looking at the transformations of Eq.(3.27) and Eq.(3.28), it becomes clear that at $\alpha = -1$ the transformations all become unit transformations and the uncertainty is then simply

$$\begin{aligned}
\Delta\mathcal{A}\Delta\mathcal{B} &= \tfrac{1}{2}|\langle\hat{C}\rangle| \\
&= \tfrac{1}{2}|\langle\psi^{\Lambda}_{\Lambda_1,\Lambda_2,\Lambda_3}(-1)|\hat{C}|\psi^{\Lambda}_{\Lambda_1,\Lambda_2,\Lambda_3}(-1)\rangle| \\
&= \tfrac{1}{2}|\langle 00\Lambda|\hat{C}|00\Lambda\rangle| \\
&= \frac{2\pi^2}{9\sqrt{3}}|\langle 00\Lambda|(2\hat{a}_1\hat{a}_1^\dagger - (1+\sqrt{3})\hat{a}_2\hat{a}_2^\dagger + (\sqrt{3}-1)\hat{a}_3\hat{a}_3^\dagger)|00\Lambda\rangle| \\
&= \frac{2\pi^2(\sqrt{3}-1)\Lambda}{9\sqrt{3}}. \qquad (3.88)
\end{aligned}$$

Thus, for any Λ the uncertainty is easily determined for $\alpha = -1$.

Due to the complexity of the expression for the su(3) intelligent states, Eq.(3.79), it was not feasible to begin a detailed exploration of the properties of these states within the context of this thesis. However, it is worth noting that there are clear differences between the su(2) and su(3) intelligent states so they are worth investigating in their own right.

Chapter 4

Conclusions

In this thesis, we asked the questions "What are the states which always satisfy the equality in the uncertainty relation?", "Can we construct them, using known mathematical tools, in a useful and straight-forward manner?", and "Will this construction be easily generalizable to other systems?" In particular, we studied these states for angular momentum and the su(3) algebra.

The answer to the first question is simply: These states are the solutions to Eq.(1.45) and are called intelligent states. Equation(1.45), however, is an eigenvalue equation. This proves to be problematic as the dimension of the problem, even in the su(2) algebra, grows. The challenge, then, was to find a better way to get intelligent states than solving a large eigenvalue problem.

"Can we construct them, using known mathematical tools, in a useful and straight-forward manner?" Yes. The solutions to the simplest cases, the angular momentum spin-$\frac{1}{2}$ and the $\Lambda = 1$ su(3) cases, provide the insight as to what is the composition of intelligent states. More complex systems however, those with higher values of ℓ and Λ, have solutions which are not simple coherent states. What these more complex intelligent states are composed of was not well understood. It has been shown in this thesis that these more complex intelligent states can be thought of as coupled coherent states. As such, they can be broken down into smaller pieces and recombined in a simple way. In fact, the solution presented in this thesis requires only mathematical tools that are well studied and often tabulated.

In this way, every intelligent state can be constructed by coupling the appropriate coherent states.

The method of constructing the intelligent states consists mainly of applying unitary rotations to appropriate systems, then coupling them together. Once this is done, one simply needs to project onto the appropriate subspace to recover the intelligent states within this subspace. The subspace one needs to project onto, for angular momentum, is simply the space in which $\ell = \ell_A + \ell_B$. This is because, any state found such that $\ell < \ell_A + \ell_B$ can be reconstructed in a simpler manner through a better choice of ℓ_A and ℓ_B. It is not known if this holds in general for the su(3) case, but it is believed that it does, because the method presented here always produces a complete set of intelligent states within a subspace of specified dimension.

Although only the su(2) and su(3) intelligent states were discussed, the method for generating the intelligent states is the same for all of the su(N) algebras, which answers the third question. Thus in principle, one could generate an expression for the intelligent states as coupled coherent states in any su(N) algebra through this method.

Since the method of coupled coherent states relies nearly entirely, apart from the single projection, on unitary transformations, there is the possibility that these states can be experimentally produced. This is because methods for experimentally generating SU(2), and in principle any SU(N), transformations are known. Regrettably, it is not clear how the projection to the appropriate subspace can be experimentally realized.

In conclusion, we have answered the questions which had been posed. We know now that intelligent states are simply coupled coherent states. We have been able to construct them, for two separate systems, from mathematical tools that are readily available. This construction is simpler than explicitly solving the eigenvalue problem, especially for large systems, or the recursion method originally proposed by Aragone. As well, it provides a clearer window into some of the properties of these states. Finally, we have a proof of concept that the intelligent states for any su(N) algebra can be constructed in this manner.

Appendix A

Glossary of Terms

A.1 Some Relevant Definitions

Baker-Campbell-Hausdorff (BCH) theorem: The BCH theorem is a method of unraveling the product of $e^A e^B$ when A and B do not commute. It is usually an infinite series, of which the first few terms are

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+\frac{1}{12}([\hat{A},[\hat{A},\hat{B}]]+[[\hat{A},\hat{B}],\hat{B}])+\ldots}. \tag{A.1}$$

There is another important result called the Baker-Campbell-Hausdorff lemma;

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \ldots \tag{A.2}$$

This can be shown by expanding the exponentials and regrouping the resulting series.

Commuting: Two operators are said to commute when the so-called commutator,

$$[\hat{\mathcal{O}},\hat{\Omega}] = \hat{\mathcal{O}}\hat{\Omega} - \hat{\Omega}\hat{\mathcal{O}}, \tag{A.3}$$

is zero, otherwise they are non-commuting.

Group: A group, G, is a set of elements, g_i, (*i.e.* numbers, matrices, even physical rotations), along with an operation, $*$ (addition, multiplication, *etc*) called the group operation, that satisfy four axioms:

1. *Closure:* The result of the group operation between two group elements must be an element of the group, or $g_i,\ g_j \in G,\ g_i * g_j = g_k \in G$.
2. *Associativeness:* $g_i * (g_j * g_k) = (g_i * g_j) * g_k$.
3. *Unique identity:* The group must contain an element, called the identity, such that the result of the group operation between this element and any element, g_i, of the group, including the identity itself, returns the element g_i. The identity is sometimes labeled as e.

4. *Unique inverse:* For every element, g_i, of the group, there must correspond a single element g_k, also in the group, such that the operation of these two elements produces the identity; $g_i * g_k = g_i * g_i^{-1} = e$.

Group, Order of: The number of elements in a group is called the order of a group. Thus, a group with $N < \infty$ elements is said to be of finite order, and of order N. If the group has an infinite number of elements, whether they are discrete or continuous, it is said to be of infinite order. This means that the Lie groups, even though they have a finite number of generators, are of infinite order because the elements depend on real continuous parameters.

Inner product: An inner product, sometimes called a scalar product, is the generalization of the dot product of Euclidean spaces to an arbitrary space. In particular, for linear spaces the inner product is defined as

$$\langle x|y\rangle = \sum_{j=1}^{n} x_j^* y_j, \tag{A.4}$$

with $\langle x| = |x\rangle^\dagger$, and for functions,

$$\langle f|g\rangle = \int_a^b f^*(r)g(r)d\mu_r\,, \tag{A.5}$$

where $d\mu_r$ is a suitable measure.

Irreducible representation: A representation is called reducible if a similarity transformation will bring each matrix of the representation into the block diagonal form:

$$\Gamma(T_i) = \begin{pmatrix} A(T_i) & 0 \\ 0 & B(T_i) \end{pmatrix}, \tag{A.6}$$

where $A(T_i)$ and $B(T_i)$ are matrices that depend on T_i. The set of transformed matrices $\Gamma(T_i)$ must still possess the required properties of a group. The matrices that compose the blocks are themselves a representation, and as such, may or may not be reducible. If no transformation will produce Eq.(A.6), then the representation is called an irreducible representation or irrep for short.

Lie algebra: A Lie algebra $\mathcal{L}$ is a set of elements $\{a, b, c, \ldots\}$, usually represented as square matrices, that form an antisymmetric Lie product $[a, b] = -[b, a]$. For square matrices, the Lie product is commonly called a commutator and is defined in Eq.(A.3). Furthermore, this set of elements must possess certain properties if it is to be called a Lie algebra.

1. The Lie product of two elements must be, itself, an element of the Lie algebra: $[a, b] \in \mathcal{L}$.
2. Given that the elements a and b are in the Lie algebra, then so too is the linear combination $\alpha a + \beta b$, where α and β are arbitrary real numbers: $\alpha a + \beta b \in \mathcal{L}$.

3. The Lie product is linear: $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]$.
4. The previous property, along with $[a, b] = -[b, a]$, leads to the following: $[a, \beta b + \gamma c] = \beta[a, b] + \gamma[a, c]$ and $[c, c] = 0$.
5. The Jacobi identity $[a, [b, c]] + [b, [a, c]] + [c[a, b]] = 0$ must be satisfied.

Any set of elements satisfying these properties constitutes a Lie algebra.

Lie group: A Lie group is the set of elements formed from the exponentiation of the elements of the corresponding Lie algebra: $e^{-i\varphi a}$, where $a \in \mathcal{L}$. Lie groups are defined in terms of real continuous parameters, and thus are of order ∞, since any unique choice of the parameters yields a unique group element. They are, however, representable by finite dimensional matrices.

Orthogonality: Two vectors are orthogonal if their inner product is equal to zero, otherwise they are non-orthogonal.

Overcompleteness: An overcomplete system is one that contains more states than is necessary to decompose an arbitrary vector into its basis components. As a simple example, consider using a basis of 3 vectors, each at 120° to the others, to span the 2-dimensional plane. The vectors are clearly non-orthogonal, and one can completely describe any arbitrary vector using only two of the three basis vectors, making it, in essence, an overcomplete set.

Representation: A representation is an explicit form of the abstract combination rules that define a group or algebra. For instance, the representation of a group by square matrices preserves the multiplicative properties of the group, while that of a Lie algebra will preserve the Lie product as the commutator.

Self-adjoint: For an operator $\hat{O}$ to be self-adjoint, it must be equal to its adjoint $\hat{O}^\dagger$, that is [31]:

1. It must satisfy the equation

$$\langle\phi|\hat{O}|\psi\rangle = \langle\psi|\hat{O}|\phi\rangle^*. \tag{A.7}$$

2. The set of vectors, $|\phi_i\rangle$, on which $\hat{O}|\phi_i\rangle$ is well defined, must be the same for $\hat{O}^\dagger$.

Operators that only satisfy the first condition, Eq.(A.7), and not the second, are called hermitian. If an operator acting in a finite dimensional vector space (representable by a matrix of finite dimensions) is hermitian, it is automatically self-adjoint. This is the case with all operators relevant to this thesis so there will be no further discussion on the subject here. For more detail on the distinction between hermitian and self-adjoint see [31], for instance.

su(N) and SU(N): The su(N) algebra is a Lie algebra, defined by $n \times n$ hermitian traceless matrices. The SU(N) group is the corresponding Lie group, constructed by exponentiating the elements of the algebra.

Trace: The trace of a matrix it the sum of its diagonal elements. It is interesting to note that the trace of a matrix is also preserved under a unitary transformation. Consider a diagonal matrix; the diagonal elements are simply the eigenvalues, and the trace is the sum of the eigenvalues. Since a unitary transformation preserves the eigenvalues of a matrix, the trace of a matrix is also preserved under a unitary transformation.

Unitary: A group element, such as a matrix, is said to be unitary if its adjoint is also its inverse,

$$U^{\dagger}U = U^{-1}U = \mathbb{1}\,. \tag{A.8}$$

Appendix B

Mathematical and Technical Discussions

B.1 The Robertson Uncertainty Relation

A derivation of Eq.(1.2) is in order, since some of the elements of the derivation will be modified to develop the equation for intelligence. The derivation will follow that presented in [32] since the notation used therein is more contemporary. We begin by defining the shifted operators

$$\hat{\Omega}' = \hat{\Omega} - \langle \hat{\Omega} \rangle \,, \qquad \hat{\mathcal{O}}' = \hat{\mathcal{O}} - \langle \hat{\mathcal{O}} \rangle, \tag{B.1}$$

where

$$\hat{\Omega}^\dagger = \hat{\Omega}, \quad \hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}} \tag{B.2}$$

are self-adjoint. Note that the commutator remains unchanged, $[\hat{\Omega}', \hat{\mathcal{O}}'] = [\hat{\Omega}, \hat{\mathcal{O}}]$, but we now have $\langle \hat{\Omega}' \rangle = \langle \hat{\mathcal{O}}' \rangle = 0$, which gives us

$$(\Delta\hat{\Omega}')^2 = (\Delta\hat{\Omega})^2 = \langle (\hat{\Omega}')^2 \rangle \,, \qquad (\Delta\hat{\mathcal{O}}')^2 = (\Delta\hat{\mathcal{O}})^2 = \langle (\hat{\mathcal{O}}')^2 \rangle. \tag{B.3}$$

This allows us to form the product

$$(\Delta\hat{\Omega})^2 (\Delta\hat{\mathcal{O}})^2 = \langle \psi | (\hat{\Omega}')^2 | \psi \rangle \langle \psi | (\hat{\mathcal{O}}')^2 | \psi \rangle \,, \tag{B.4}$$

and use the Schwartz inequality

$$\langle \varphi | \varphi \rangle \langle \chi | \chi \rangle \geq | \langle \varphi | \chi \rangle |^2 \tag{B.5}$$

if we make the identifications $\hat{\Omega}' |\psi\rangle = |\varphi\rangle$ and $\hat{\mathcal{O}}' |\psi\rangle = |\chi\rangle$. Using the fact that $\hat{\Omega}'$ and $\hat{\mathcal{O}}'$ are self-adjoint: $\hat{\Omega}' = (\hat{\Omega}')^\dagger$ and $\hat{\mathcal{O}}' = (\hat{\mathcal{O}}')^\dagger$, it is clear that Eq.(B.4) and Eq.(B.5) can be combined to give

$$(\Delta\hat{\Omega})^2 (\Delta\hat{\mathcal{O}})^2 \geq | \langle \psi | \hat{\Omega}' \hat{\mathcal{O}}' | \psi \rangle |^2. \tag{B.6}$$

It is always possible to write:

$$\begin{aligned}\hat{\Omega}'\hat{O}' &= \tfrac{1}{2}(\hat{\Omega}'\hat{O}' + \hat{O}'\hat{\Omega}') + \tfrac{1}{2}(\hat{\Omega}'\hat{O}' - \hat{O}'\hat{\Omega}') \\ &= \tfrac{1}{2}\{\hat{\Omega}', \hat{O}'\} + \tfrac{1}{2}[\hat{\Omega}', \hat{O}'] \\ &= \tfrac{1}{2}\{\hat{\Omega}', \hat{O}'\} + \tfrac{1}{2}[\hat{\Omega}, \hat{O}],\end{aligned} \tag{B.7}$$

which allows us to rewrite Eq.(B.6) as

$$(\Delta\hat{\Omega})^2(\Delta\hat{O})^2 \geq |\langle\psi|\tfrac{1}{2}\{\hat{\Omega}', \hat{O}'\} + \tfrac{1}{2}[\hat{\Omega}, \hat{O}]|\psi\rangle|^2. \tag{B.8}$$

Using Eq.(B.2) we can see that $\langle[\hat{\Omega}, \hat{O}]\rangle$ is purely imaginary by taking the hermitian conjugate

$$\begin{aligned}[\hat{\Omega}, \hat{O}]^\dagger &= (\hat{\Omega}\hat{O} - \hat{O}\hat{\Omega})^\dagger \\ &= \hat{O}^\dagger\hat{\Omega}^\dagger - \hat{\Omega}^\dagger\hat{O}^\dagger \\ &= -\hat{\Omega}\hat{O} + \hat{O}\hat{\Omega} \\ &= -[\hat{\Omega}, \hat{O}].\end{aligned} \tag{B.9}$$

If we write $[\hat{\Omega}, \hat{O}] = i\hat{C}_-$, where $\hat{C}_-$ is hermitian, then

$$(i\hat{C}_-)^\dagger = -i\hat{C}_-^\dagger = -i\hat{C}_-, \tag{B.10}$$

which leads to

$$\langle[\hat{\Omega}, \hat{O}]\rangle = i\langle\hat{C}_-\rangle = ic_-. \tag{B.11}$$

The result of Eq.(B.11) must be purely imaginary, since the expectation value of a self-adjoint operator is always real. Similarly, if we take the hermitian conjugate:

$$\begin{aligned}\{\hat{\Omega}', \hat{O}'\}^\dagger &= (\hat{\Omega}'\hat{O}' + \hat{O}'\hat{\Omega}')^\dagger \\ &= (\hat{O}')^\dagger(\hat{\Omega}')^\dagger + (\hat{\Omega}')^\dagger(\hat{O}')^\dagger \\ &= \hat{\Omega}'\hat{O}' + \hat{O}'\hat{\Omega}' \\ &= \{\hat{\Omega}', \hat{O}'\},\end{aligned} \tag{B.12}$$

we find that it is already a self-adjoint operator. Thus we write $\{\hat{\Omega}', \hat{O}'\} = \hat{C}_+$, with $\hat{C}_+$ a self-adjoint operator to get

$$\langle\{\hat{\Omega}', \hat{O}'\}\rangle = \langle\hat{C}_+\rangle = c_+, \tag{B.13}$$

which is again a real number.

Taking the right hand side of Eq.(B.8) we write it as

$$\begin{aligned}|\langle\psi|\tfrac{1}{2}\{\hat{\Omega}', \hat{O}'\} + \tfrac{1}{2}[\hat{\Omega}, \hat{O}]|\psi\rangle|^2 &= |\langle\psi|\tfrac{1}{2}\hat{C}_+ + \tfrac{1}{2}i\hat{C}_-|\psi\rangle|^2 \\ &= |\tfrac{1}{2}\langle\hat{C}_+\rangle + \tfrac{1}{2}i\langle\hat{C}_-\rangle|^2 \\ &= |\tfrac{1}{2}c_+ + \tfrac{1}{2}ic_-|^2 \\ &= \tfrac{1}{4}(c_+^2 + c_-^2) \\ &\geq \tfrac{1}{4}c_-^2.\end{aligned} \tag{B.14}$$

Using

$$c_-^2 = |i\langle\hat{C}_-\rangle|^2 = |\langle[\hat{\Omega},\hat{\mathcal{O}}]\rangle|^2, \tag{B.15}$$

with Eq.(B.14) and Eq.(B.8), one can quickly see that

$$(\Delta\hat{\Omega})^2(\Delta\hat{\mathcal{O}})^2 \geq \tfrac{1}{4}|\langle[\hat{\Omega},\hat{\mathcal{O}}]\rangle|^2. \tag{B.16}$$

Finally, taking the square root of Eq.(B.16) yields the familiar relation

$$(\Delta\hat{\Omega})(\Delta\hat{\mathcal{O}}) \geq \tfrac{1}{2}|\langle[\hat{\Omega},\hat{\mathcal{O}}]\rangle|. \tag{B.17}$$

B.2 Angular Momentum Coherent States

We prove that the coherent states defined in Section 2.1.3 span a non-orthogonal basis as well as an overcomplete one. It is more convenient to show that the states in Eq.(2.41) possess both of these properties if we express them in terms of the basis states $|\ell, m\rangle$. To do this, we exploit the fact that the commutation relations in Eqs.(2.2) and (2.8) are independent of the angular momentum of the system. We will use the $\ell = \frac{1}{2}$, or 2×2, system to obtain the result. The operators are given in Eq.(1.48), with the ladder operators as,

$$\hat{L}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{L}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{B.18}$$

With this we can write:

$$\begin{aligned} e^{i\gamma\hat{L}_z}e^{i\beta\hat{L}_y}e^{-i\gamma\hat{L}_z} &= \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}\begin{pmatrix} \cos\frac{1}{2}\beta & -\sin\frac{1}{2}\beta \\ \sin\frac{1}{2}\beta & \cos\frac{1}{2}\beta \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{1}{2}\beta & -e^{i\gamma}\sin\frac{1}{2}\beta \\ e^{-i\gamma}\sin\frac{1}{2}\beta & \cos\frac{1}{2}\beta \end{pmatrix}. \end{aligned} \tag{B.19}$$

We now compare this to the form

$$\begin{aligned} e^{\xi\hat{L}_-}e^{\chi\hat{L}_z}e^{-\xi^*\hat{L}_+} &= \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}\begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix}\begin{pmatrix} 1 & -\xi^* \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\chi/2} & -\xi^* e^{\chi/2} \\ \xi e^{\chi/2} & e^{-\chi/2} - |\xi|^2 e^{\chi/2} \end{pmatrix}, \end{aligned} \tag{B.20}$$

which gives

$$e^{\chi/2} = \cos\tfrac{1}{2}\beta, \tag{B.21}$$

$$\begin{aligned} -\xi^* e^{\chi/2} &= -\xi^*\cos\tfrac{1}{2}\beta \\ \xi &= e^{-i\gamma}\tan\tfrac{1}{2}\beta. \end{aligned} \tag{B.22}$$

Now we write the coherent state as

$$|\beta,\gamma\rangle = \mathrm{e}^{\xi\hat{L}_-}\mathrm{e}^{\chi\hat{L}_z}\mathrm{e}^{-\xi^*\hat{L}_+}|\ell,\ell\rangle. \tag{B.23}$$

This is helpful since we know that

$$\hat{L}_+|\ell,\ell\rangle = 0, \tag{B.24}$$

so the effect of the exponential operator in $\hat{L}_+$ is

$$\begin{aligned} \mathrm{e}^{-\xi^*\hat{L}_+}|\ell,\ell\rangle &= \left[\mathbb{1} - \xi^*\hat{L}_+ + \frac{(\xi^*)^2}{2}\hat{L}_+^2 + \ldots\right]|\ell,\ell\rangle \\ &= \mathbb{1}|\ell,\ell\rangle - \xi^*\hat{L}_+|\ell,\ell\rangle + \frac{(\xi^*)^2}{2}\hat{L}_+^2|\ell,\ell\rangle + \ldots \\ &= |\ell,\ell\rangle. \end{aligned} \tag{B.25}$$

This simplifies Eq.(B.23) to

$$\begin{aligned} |\beta,\gamma\rangle &= \mathrm{e}^{\xi\hat{L}_-}\mathrm{e}^{\chi\hat{L}_z}|\ell,\ell\rangle \\ &= \mathrm{e}^{\chi\ell}\mathrm{e}^{\xi\hat{L}_-}|\ell,\ell\rangle \\ &= (\cos\tfrac{1}{2}\beta)^{2\ell}\left[\sum_{k=0}^{2\ell}\frac{\xi^k}{k!}\hat{L}_-^k|\ell,\ell\rangle\right]. \end{aligned} \tag{B.26}$$

Now, to find $\hat{L}_-^k|\ell,\ell\rangle$,

$$\begin{aligned} \hat{L}_-|\ell,\ell\rangle &= \sqrt{2\ell}|\ell,\ell-1\rangle \\ \hat{L}_-^2|\ell,\ell\rangle &= \sqrt{(2\ell)(2\ell-1)2}|\ell,\ell-2\rangle \\ &= \sqrt{\frac{(2\ell)!2}{(2\ell-2)!}}|\ell,\ell-2\rangle \\ \hat{L}_-^3|\ell,\ell\rangle &= \sqrt{(2\ell)(2\ell-1)(2\ell-2)(2)(3)}|\ell,\ell-3\rangle \\ &= \sqrt{\frac{(2\ell)!3!}{(2\ell-3)!}}|\ell,\ell-3\rangle \\ \hat{L}_-^k|\ell,\ell\rangle &= \sqrt{\frac{(2\ell)!k!}{(2\ell-k)!}}|\ell,\ell-k\rangle. \end{aligned} \tag{B.27}$$

Putting this back into Eq.(B.26) gives

$$\begin{aligned} |\beta,\gamma\rangle &= (\cos\tfrac{1}{2}\beta)^{2\ell}\left[\sum_{k=0}^{2\ell}\frac{\xi^k}{k!}\sqrt{\frac{(2\ell)!k!}{(2\ell-k)!}}|\ell,\ell-k\rangle\right] \\ &= \sqrt{\frac{1}{(1+|\xi|^2)^{2\ell}}}\left[\sum_{k=0}^{2\ell}\xi^k\sqrt{\frac{(2\ell)!}{k!(2\ell-k)!}}|\ell,\ell-k\rangle\right]. \end{aligned} \tag{B.28}$$

If we now compute the inner product, $\langle\beta_1,\gamma_1\,|\,\beta_2,\gamma_2\rangle$, we can show that two angular momentum coherent states are generally non-orthogonal:

$$\langle\beta_1,\gamma_1\,|\,\beta_2,\gamma_2\rangle = \left(1+|\xi_1|^2\right)^{-\ell}\left(1+|\xi_2|^2\right)^{-\ell} \times \sum_{k_1=0}^{2\ell}\sum_{k_2=0}^{2\ell}(\xi_1^*)^{k_1}\xi_2^{k_2}\sqrt{\frac{(2\ell)!}{k_1!(2\ell-k_1)!}}\sqrt{\frac{(2\ell)!}{k_2!(2\ell-k_2)!}}\langle\ell,\ell-k_1\,|\,\ell,\ell-k_2\rangle. \tag{B.29}$$

Note that the inner product

$$\langle\ell,\ell-k_1\,|\,\ell,\ell-k_2\rangle = \delta_{k_1,k_2} \tag{B.30}$$

which produces the condition

$$k_1 = k_2 = k. \tag{B.31}$$

This gives

$$\langle\beta_1,\gamma_1\,|\,\beta_2,\gamma_2\rangle = (1+|\xi_1|^2)^{-\ell}(1+|\xi_2|^2)^{-\ell}\sum_{k=0}^{2\ell}(\xi_1^*\xi_2)^k\frac{(2\ell)!}{k!(2\ell-k)!}. \tag{B.32}$$

Taking into account that

$$\frac{(2\ell)!}{k!(2\ell-k)!} = \binom{2\ell}{k}, \tag{B.33}$$

and that the binomial series takes the form

$$(x+a)^n = \sum_{k=0}^{n}\binom{n}{k}x^k a^{n-k}, \tag{B.34}$$

we can simplify the sum in Eq.(B.32):

$$\langle\beta_1,\gamma_1\,|\,\beta_2,\gamma_2\rangle = \frac{(1+\xi_1^*\xi_2)^{2\ell}}{(1+|\xi_1|^2)^{\ell}(1+|\xi_2|^2)^{\ell}}, \tag{B.35}$$

which is clearly non-zero, indicating that these states are non-orthogonal unless $\xi_1^*\xi_2 = -1$. This special case applies only to coherent states that lie antipodal on the sphere with points located at angles γ and β. For instance, two coherent states, one at the north pole and the other at the south pole are orthogonal, and likewise any two states that lie opposite on the sphere.

To test for overcompleteness, we need to modify Eq.(B.28) slightly by picking

$$m = \ell - k. \tag{B.36}$$

To rewrite the sum for m, we set $k = 0$ and $k = 2\ell$ to get the limits of ℓ and $-\ell$ respectively, and Eq.(B.28) becomes

$$|\beta,\gamma\rangle = \sqrt{\frac{1}{(1+|\xi|^2)^{2\ell}}}\left[\sum_{m=-\ell}^{\ell}\xi^{\ell-m}\sqrt{\frac{(2\ell)!}{(\ell-m)!(\ell+m)!}}|\ell,m\rangle\right]. \tag{B.37}$$

We will need to rewrite this in terms of β and γ,

$$\begin{aligned} |\beta,\gamma\rangle &= (\cos\tfrac{1}{2}\beta)^{2\ell}\sum_{m=-\ell}^{\ell}\left(e^{-i\gamma}\tan\tfrac{1}{2}\beta\right)^{\ell-m}\sqrt{\frac{(2\ell)!}{(\ell-m)!(\ell+m)!}}|\ell,m\rangle \\ &= \sum_{m=-\ell}^{\ell} e^{-i(\ell-m)\gamma}(\cos\tfrac{1}{2}\beta)^{\ell+m}(\sin\tfrac{1}{2}\beta)^{\ell-m}\sqrt{\tfrac{(2\ell)!}{(\ell-m)!(\ell+m)!}}|\ell,m\rangle. \end{aligned} \quad \text{(B.38)}$$

We can see easily that the usual basis states $|\ell,m\rangle$ are complete, but not overcomplete, since the operator

$$\sum_m |\ell,m\rangle\langle\ell,m| = \mathbb{1} \quad \text{(B.39)}$$

is the unit operator. This can be seen for any state $|\ell,m'\rangle$:

$$\begin{aligned} \left[\sum_m |\ell,m\rangle\langle\ell,m|\right]|\ell,m'\rangle &= |\ell,-\ell\rangle\langle\ell,-\ell\,|\,\ell,m'\rangle + |\ell,-\ell+1\rangle\langle\ell,-\ell+1\,|\,\ell,m'\rangle \\ &\quad + \ldots + |\ell,m'\rangle\langle\ell,m'\,|\,\ell,m'\rangle + \ldots \\ &= 0+0+\ldots+|\ell,m'\rangle+\ldots \\ &= |\ell,m'\rangle\,, \end{aligned} \quad \text{(B.40)}$$

so the operator of Eq.(B.39) simply returns that state back again. This means that the decomposition of states into the basis $|\ell,m\rangle$ is possible for any state of angular momentum ℓ as well as being unique.

For the angular momentum coherent states this is not the case. To resolve the unit from the coherent states, we must integrate over the parameters β and γ. Starting from Eq.(B.38), we integrate over the entire range:

$$\begin{aligned} &\int_0^{\pi}\int_0^{2\pi}\sin\beta\,d\gamma d\beta|\beta,\gamma\rangle\langle\beta,\gamma| = \\ &\quad\int_0^{\pi}\int_0^{2\pi}\sin\beta\,d\gamma d\beta\left[\binom{2\ell}{\ell+m}^{\frac{1}{2}}\binom{2\ell}{\ell+m'}^{\frac{1}{2}}(\cos\tfrac{1}{2}\beta)^{2\ell+m+m'}\right. \\ &\quad\quad\left.\times(\sin\tfrac{1}{2}\beta)^{2\ell-m-m'}e^{i(m-m')\gamma}|\ell,m\rangle\langle\ell,m'|\right]. \end{aligned} \quad \text{(B.41)}$$

These two integrals can be separated and evaluated individually. It is beneficial to evaluate the integral in γ first, since

$$\int_0^{2\pi} d\gamma\, e^{i(m-m')\gamma} = 2\pi\delta_{m'm} \quad \text{(B.42)}$$

when m' and m are both integers or half-integers, leaving us with

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi} & \sin\beta\, d\gamma d\beta |\beta,\gamma\rangle\langle\beta,\gamma| = \\ & 2\pi\int_0^{\pi}\sin\beta\, d\beta\binom{2\ell}{\ell+m}(\cos\tfrac{1}{2}\beta)^{2(\ell+m)}(\sin\tfrac{1}{2}\beta)^{2(\ell-m)}|\ell,m\rangle\langle\ell,m| \\ &= \frac{4\pi}{2\ell+1}\sum_m |\ell,m\rangle\langle\ell,m|\,. \end{aligned} \tag{B.43}$$

Thus we have

$$\frac{2\ell+1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sin\beta\, d\gamma d\beta|\beta,\gamma\rangle\langle\beta,\gamma| = \mathbb{1}\,. \tag{B.44}$$

The fact that integration is required to resolve the unit from the coherent states is the reason that we can say they form an over complete basis. Whereas the operator of Eq.(B.39) acts in a $(2\ell+1)$-dimensional space and contains a sum of $2\ell+1$ projectors, the (continuous) sum of Eq.(B.44) contains an infinite number of coherent states.

B.3 State Coupling Methods

The purpose of this section is to outline the calculations involved in the coupling of two systems. The application of the method yields the so-called Clebsch-Gordan coefficients. They come about when one looks at the possible outcome of coupling two single systems together, or alternatively, when one wishes to decompose a single state into two coupled states.

B.3.1 Clebsch-Gordan Technology

We will show how to construct a single state of given angular momentum, $|\ell, m\rangle$, by bringing together two systems of angular momentum ℓ_A and ℓ_B, such that [11]

$$m = m_A + m_B, \quad |\ell_A - \ell_B| \le \ell \le \ell_A + \ell_B\,. \tag{B.45}$$

We write the final expression in the form

$$|\ell,m\rangle = \sum_{m_A(m_B)}\left\langle \begin{matrix}\ell_A & \ell_B\\ m_A & m_B\end{matrix}\,\middle|\, \begin{matrix}\ell\\ m\end{matrix}\right\rangle |\ell_A, m_A\rangle|\ell_B, m_B\rangle\,, \tag{B.46}$$

where $\left\langle \begin{matrix}\ell_A & \ell_B\\ m_A & m_B\end{matrix}\,\middle|\, \begin{matrix}\ell\\ m\end{matrix}\right\rangle$ is called a Clebsch-Gordan coefficient. The sum is taken for all values of m_A and m_B, such that $m = m_A + m_B$, and, for any system, $m_i = -\ell_i, -\ell_i+1 \ldots \ell_i - 1, \ell_i$.

We define the operators that act on the system in the same way as Eq.(2.52) and Eq.(2.53);

$$\hat{L}_{i,A} \equiv \hat{L}_i \otimes \mathbb{1}_B\,, \quad \hat{L}_{i,B} \equiv \mathbb{1}_A \otimes \hat{L}_i\,, \tag{B.47}$$

leading to

$$\begin{aligned}\hat{L}_z &= \hat{L}_{z,A} + \hat{L}_{z,B} \\ \hat{L}_+ &= \hat{L}_{+,A} + \hat{L}_{+,B} \\ \hat{L}_- &= \hat{L}_{-,A} + \hat{L}_{-,B}\,.\end{aligned} \tag{B.48}$$

The operators $\hat{L}_+$ and $\hat{L}_-$ are the angular momentum ladder operators. They change the value of m when they act on a ket:

$$\hat{L}_\pm|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle\,. \tag{B.49}$$

The easiest way to get acquainted with Clebsch-Gordan technology is through an example. Consider the construction of the $\ell = 1$ states from two $\ell = \frac{1}{2}$ states. To begin, we want to find an extremal state, one that has either the maximum or minimum allowed value of m. For this example, we will look for the total angular momentum to be

$$\ell = \ell_A + \ell_B = \frac{1}{2} + \frac{1}{2} = 1\,. \tag{B.50}$$

We know that the highest value that can be returned by the operator $\hat{L}_z$ is $m = \ell = 1$. So if we couple the systems A and B together, in a manner that the action of $\hat{L}_z$ returns the value $m = 1$ we will know that our state is the state $|1,1\rangle$. In mathematical terminology, this state is said to have the highest weight because the result of $\hat{L}_+$ acting on it returns zero. We start with this state because there is only one way to couple the two systems together to give this state. Alternatively, one could begin with the lowest weight state $|\ell, -\ell\rangle$.

Starting with the state $|\frac{1}{2},\frac{1}{2}\rangle_A|\frac{1}{2},\frac{1}{2}\rangle_B$ we find:

$$\begin{aligned}\hat{L}_z\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] &= \left(\hat{L}_{z,A} + \hat{L}_{z,B}\right)\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] \\ &= \left[\hat{L}_{z,A}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A\right]|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + |\tfrac{1}{2},\tfrac{1}{2}\rangle_A\left[\hat{L}_{z,B}|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] \\ &= \frac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + \frac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B \\ &= (1)|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\,.\end{aligned} \tag{B.51}$$

This means that we have found our highest weight state, and can write the equality

$$|1,1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\,. \tag{B.52}$$

We also have, nearly for free, the first Clebsch-Gordan coefficient:

$$\left\langle \begin{matrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{matrix} \middle| \begin{matrix}1 \\ 1\end{matrix}\right\rangle = 1\,. \tag{B.53}$$

Now that we have our highest weight state, we can work our way down to the other states with $\ell = 1$. Had we started at $|1,-1\rangle$ we would simply work our way up. We do this through repeated use of the lowering operator $\hat{L}_-$. The reason we cannot just couple the two systems together to give us the state for $m = \ell - 1 = 0$, is that there are two ways to do this, namely $|\frac{1}{2},\frac{1}{2}\rangle_A|\frac{1}{2},-\frac{1}{2}\rangle_B$ and $|\frac{1}{2},-\frac{1}{2}\rangle_A|\frac{1}{2},\frac{1}{2}\rangle_B$, both have $m = 0$. We need to know what fraction of each to include in our construction of the $|1,0\rangle$ state. This doesn't seem too difficult for this simple case, but for larger ℓ there can be a number of ways to obtain a given m from the various states in the two systems.

The next state is found by applying $\hat{L}_-$ to the state $|1,1\rangle$ in the following way:

$$\begin{aligned}
\hat{L}_-|1,1\rangle &= \hat{L}_-\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] \\
\sqrt{2}|1,0\rangle &= \left[\hat{L}_{-,A}|\tfrac{1}{2},\tfrac{1}{2}\rangle_A\right]|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + |\tfrac{1}{2},\tfrac{1}{2}\rangle_A\left[\hat{L}_{-,A}|\tfrac{1}{2},\tfrac{1}{2}\rangle_B\right] \\
&= |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + |\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B \\
|1,0\rangle &= \sqrt{\tfrac{1}{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B + |\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B\right) .
\end{aligned} \tag{B.54}$$

Looking at Eq.(B.54), and comparing to Eq.(B.46), gives us immediately the Clebsch-Gordan coefficients:

$$\left\langle \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = \sqrt{\frac{1}{2}}, \qquad \left\langle \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = \sqrt{\frac{1}{2}}. \tag{B.55}$$

Continuing on we apply $\hat{L}_-$ to Eq.(B.54) to get to the next state:

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B, \tag{B.56}$$

which is the lowest weight state, since both sides return zero when we try to ramp down one more time.

We are not finished yet, however. There are four possible ways to combine the two systems A and B:

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B, \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B, \tag{B.57}$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B, \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B, \tag{B.58}$$

but we have only generated three states so far, and only three states that have $\ell = 1$ exist, one for each value of m. What about the fourth state? If we look again at the conditions on ℓ, Eq.(B.45), it becomes clear that $\ell = 1$ is not the only possibility. We can't choose $\ell = \frac{1}{2}$ because our possible values of m are integers and therefore so must be our values of ℓ. The only value we are left with is $\ell = 0$. We can use the requirement

$$\langle \ell', m' \,|\, \ell, m\rangle = \delta_{\ell',\ell}\delta_{m',m}, \tag{B.59}$$

where $\delta_{i,j}$ is the Kronecker δ and defined as:

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{B.60}$$

so that the state we choose to be an $\ell = 0$ state must be orthogonal to all of the $\ell = 1$ states which we have just constructed, *i.e.*

$$\langle 0, m_0 \,|\, 1, m_1 \rangle = 0\,, \tag{B.61}$$

for any values of m_0 and m_1. A simple choice is the state

$$|\varphi\rangle = \sqrt{\tfrac{1}{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B - |\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B\right)\,, \tag{B.62}$$

which is orthogonal to our three other states. A quick check with $\hat{L}_+$ shows that it is the highest weight state, since $\hat{L}_+|\varphi\rangle = 0$. It also happens to be the lowest weight state since $\hat{L}_-|\varphi\rangle = 0$, which means it is the only state for $\ell = 0$. This means that

$$|0,0\rangle = \sqrt{\tfrac{1}{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle_B - |\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle_B\right)\,, \tag{B.63}$$

gives us our fourth state and the final two coefficients:

$$\left\langle \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = \sqrt{\frac{1}{2}}\,, \quad \left\langle \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = -\sqrt{\frac{1}{2}}\,. \tag{B.64}$$

The construction of the $\ell = 0$ state illustrates that it is possible to arrive at states with a range of different values of ℓ, not simply those that have $\ell = \ell_A + \ell_B$.

We have now constructed all of the Clebsch-Gordan coefficients for the coupling of two states with angular momentum $\ell = \frac{1}{2}$. This same process can be repeated for any possible coupling of two systems. The number of states, including the number of orthogonal states that must be produced, increases with the total angular momentum, but always the process remains the same; find the highest weight state, determine its total angular momentum, and work your way down to the lowest weight state, keeping track of the coefficients as you go. In this way you can, in principle, generate the Clebsch-Gordan coefficient for any coupling. This is usually unnecessary, as the Clebsch-Gordan coefficients are tabulated in numerous places, see [26] for instance, or the program Mathematica© contains a built-in function that will generate the Clebsch-Gordan coefficient, given the relevant information.

B.3.2 SU(3) Coupling

The coupling of SU(3) systems is approached in the same manner as angular momentum systems. The motivation is to be able to properly express, via SU(3) Clebsch-Gordan coefficients, a state in terms of two coupled su(3) states, or to predict the outcome of coupling two states together.

We define a basis for the su(3) operators in terms of the harmonic oscillator creation and annihilation operators;

$$\begin{aligned} \hat{C}_{ij} &= \hat{a}_i^\dagger \hat{a}_j\,, \quad i \neq j = 1,\, 2,\, 3\,, \\ \hat{h}_1 &= \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1\,, \qquad \text{(B.65)} \\ \hat{h}_2 &= \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_2^\dagger \hat{a}_2\,, \qquad \text{(B.66)} \end{aligned}$$

Because the su(3) algebra is more complicated than the su(2) algebra, we need more than two numbers to completely define the state (with angular momentum we needed only two numbers: ℓ to determine the total angular momentum and m to fully determine the state). In general, su(3) states are fully labeled by five numbers. Two of them, p and q, play a role similar to the role of ℓ in angular momentum. Two others, h_1 and h_2, are eigenvalues of diagonal operators and play a role similar to m in angular momentum. Unlike angular momentum, where the eigenvalue m can occur at most once in the set of states labeled by ℓ, the pair (h_1, h_2) may occur more than once in a set of states labeled by (p, q). Thus, a fifth label, sometimes written as α, is needed to distinguish the repeated occurrences of (h_1, h_2). An su(3) state, then, should be labeled as:

$$|(p,q)\alpha, h_1, h_2\rangle \ . \tag{B.67}$$

However, in this thesis, and in this example, we will be considering su(3) states of the type $(0, q)$ and $(p, 0)$, where this extra label α is not required and will be generally omitted.

The four numbers p, q, h_1, and h_2 are all found through the action of the two operators $\hat{h}_1$ and $\hat{h}_2$. As with angular momentum coupling, to find all of the possible couplings we must find orthogonal states to those we generate as we work our way down the ladder. The orthogonal su(3) states, unlike the orthogonal su(2) states, do not only exist in a space with a different dimensionality, but also with a different geometry. The su(3) states that can be labeled by $|n_1 n_2 n_3\rangle$ have an upright triangular geometry like that shown in Fig.(3.2) or Fig.(B.1). The orthogonal su(3) states to those generated via the ladder operators, can also have a hexagonal geometry or an inverted triangular geometry. The states that have these other two geometries cannot be fully labeled by the $|n_1 n_2 n_3\rangle$ notation, which we will call the Λ-notation. In this case we use the so-called (p, q)-notation of Eq.(B.67). The states in the (p, q)-notation that also have a Λ-notation are those for which $p = 0$:

$$|n_1, n_2, n_3\rangle \mapsto |(0, \Lambda)\, n_2 - n_1, n_3 - n_2\rangle \, . \tag{B.68}$$

As with the SU(2) case we need an extremal state to start from. We will choose the state killed by $\hat{C}_{32}$, $\hat{C}_{31}$, and $\hat{C}_{21}$, as the highest weight state. Thus, we declare $\{\hat{C}_{32}, \hat{C}_{31}, \hat{C}_{21}\}$ to be raising operators and find (p, q) as the labels h_1, h_2 of the state that is killed by all raising operators. (Note: The choice of raising operators is not unique, but must satisfy some technical criteria obtained from the theory of Lie algebra. Our choice of raising operators satisfies these criteria.)

To see that this will be a highest weight state, compare the action of the operators in Fig.(3.1) acting in the weight diagram of Fig.(B.1). The eigenvalues of the two operators $\hat{h}_1$ and $\hat{h}_2$ acting on the extremal state determine the two values p and q respectively. The condition, then, for the state to be of highest weight is

$$\hat{C}_{32}|(p,q)\, h_1, h_2\rangle = \hat{C}_{31}|(p,q)\, h_1, h_2\rangle = \hat{C}_{21}|(p,q)\, h_1, h_2\rangle = 0 \, . \tag{B.69}$$

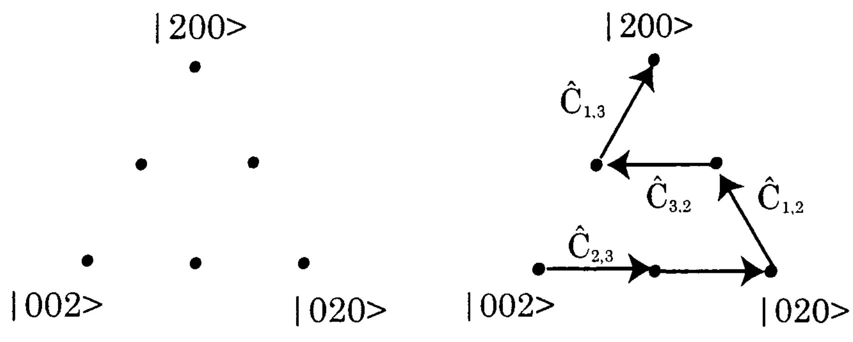

Figure B.1: **Left:** The weight diagram for $\Lambda = 2$. The highest weight state is $|002\rangle = |(0,2)\,0,2\rangle$ in the lower left corner. **Right:** The path we take to generate each state of the form $|(0,2)\,n_2 - n_1, n_3 - n_2\rangle$.

To determine the form of the highest weight states recall that the annihilation operator $\hat{a}_j$ will only produce a zero if there is a zero present in the j'th position of the state; for example, using the Λ-notation

$$\begin{aligned} \hat{a}_3|00\Lambda\rangle &= \sqrt{\Lambda}|00\Lambda-1\rangle \neq 0 \\ \hat{a}_2|00\Lambda\rangle &= \sqrt{0}|00\Lambda\rangle = 0 \\ \hat{a}_1|00\Lambda\rangle &= \sqrt{0}|00\Lambda\rangle = 0\,. \end{aligned} \tag{B.70}$$

Thus, we have the family of states $|00\Lambda\rangle$ as extremal states.

As an example, consider the coupling of two $\Lambda = 1$ states. There are 9 possible ways to couple two $\Lambda = 1$ states:

$$\begin{aligned} &|001\rangle_A|001\rangle_B\,, \quad |001\rangle_A|010\rangle_B\,, \quad |001\rangle_A|100\rangle_B\,, \\ &|010\rangle_A|001\rangle_B\,, \quad |010\rangle_A|010\rangle_B\,, \quad |010\rangle_A|100\rangle_B\,, \\ &|100\rangle_A|001\rangle_B\,, \quad |100\rangle_A|010\rangle_B\,, \quad |100\rangle_A|100\rangle_B\,. \end{aligned} \tag{B.71}$$

The composite operators are defined as before so that, for example

$$\hat{C}^A_{ij} = \hat{C}_{ij} \otimes 1\!\!1_B\,, \quad \hat{C}^B_{ij} = 1\!\!1_A \otimes \hat{C}_{ij}\,, \tag{B.72}$$

to give

$$\hat{C}_{ij} = \hat{C}^A_{ij} + \hat{C}^B_{ij}\,. \tag{B.73}$$

We will begin with the state $|001\rangle_A|001\rangle_B$. This is a highest weight state, since it satisfies Eq.(B.69); for example

$$\begin{aligned} \hat{C}_{21}\left[|001\rangle_A|001\rangle_B\right] &= (\hat{C}^A_{21} + \hat{C}^B_{21})\left[|001\rangle_A|001\rangle_B\right] \\ &= \left[\hat{C}^A_{21}|001\rangle_A\right]|001\rangle_B + |001\rangle_A\left[\hat{C}^B_{21}|001\rangle_B\right] \\ &= 0\,. \end{aligned} \tag{B.74}$$

From the eigenvalues h_1, h_2 and the fact that $\Lambda = 2$, one can determine the resulting state:

$$\begin{aligned}\hat{h}_2\left[|001\rangle_A|001\rangle_B\right] &= \left[\hat{h}_2^A|001\rangle_A\right]|001\rangle_B + |001\rangle_A\left[\hat{h}_2^B|001\rangle_B\right] \\ &= \left[(\hat{C}_{33}^A - \hat{C}_{22}^A)|001\rangle_A\right]|001\rangle_B + |001\rangle_A\left[(\hat{C}_{33}^B - \hat{C}_{22}^B)|001\rangle_B\right] \\ &= \left[(1)|001\rangle_A\right]|001\rangle_B + |001\rangle_A\left[(1)|001\rangle_B\right] \\ &= 2|001\rangle_A|001\rangle_B\,, \end{aligned} \tag{B.75}$$

giving $h_2 = 2$. In a similar manner, one finds that $h_1 = 0$ for this state. We can now write our coupled state as a single state with $\Lambda = 2$; using both notations we find

$$|001\rangle_A|001\rangle_B = |002\rangle = |(0,2)\,0,2\rangle\,. \tag{B.76}$$

Analogously, one can quickly determine that

$$|001\rangle = |(0,1)\,0,1\rangle\,. \tag{B.77}$$

We now have the first su(3) Clebsch-Gordan coefficient:

$$\left\langle \begin{matrix}(0,1) & (0,1)\\ 0,1 & 0,1\end{matrix} \middle| \begin{matrix}(0,2)\\ 0,2\end{matrix}\right\rangle = 1\,. \tag{B.78}$$

To determine the rest of the Clebsch-Gordan coefficients, we must choose one of the three remaining ladder operators and begin "mapping" the weight diagram. We will begin with $\hat{C}_{23}$ and continue until we reach a state that is killed by it. Using the Λ notation for the purpose of calculation, we find

$$\begin{aligned}\hat{C}_{23}|002\rangle &= \hat{C}_{23}(|001\rangle_A|001\rangle_B) \\ \sqrt{2}|011\rangle &= |010\rangle_A|001\rangle_B + |001\rangle_A|010\rangle_B \\ |011\rangle &= \frac{1}{\sqrt{2}}\left(|010\rangle_A|001\rangle_B + |001\rangle_A|010\rangle_B\right)\,,\end{aligned} \tag{B.79}$$

and in the (p,q)-notation it looks like:

$$|(0,2)\,1,0\rangle = \frac{1}{\sqrt{2}}\left(|(0,1)\,1,-1\rangle_A\,|(0,1)\,0,1\rangle_B + |(0,1)\,0,1\rangle_A\,|(0,1)\,1,-1\rangle_B\right)\,. \tag{B.80}$$

This gives us two more coefficients:

$$\left\langle \begin{matrix}(0,1) & (0,1)\\ 1,-1 & 0,1\end{matrix} \middle| \begin{matrix}(0,2)\\ 1,0\end{matrix}\right\rangle = \left\langle \begin{matrix}(0,1) & (0,1)\\ 0,1 & 1,-1\end{matrix} \middle| \begin{matrix}(0,2)\\ 1,0\end{matrix}\right\rangle = \frac{1}{\sqrt{2}}\,. \tag{B.81}$$

Continuing the process, we act with $\hat{C}_{23}$ on our new state, again using the Λ-notation to make the operation explicit:

$$\begin{aligned} \hat{C}_{23}|011\rangle &= \frac{1}{\sqrt{2}}\hat{C}_{23}(|010\rangle_A|001\rangle_B + |001\rangle_A|010\rangle_B) \\ \sqrt{2}|020\rangle &= \frac{1}{\sqrt{2}}(|010\rangle_A|010\rangle_B + |010\rangle_A|010\rangle_B \\ |020\rangle &= |010\rangle_A|010\rangle_B \\ |(0,2)\,2,-2\rangle &= |(0,1)\,1,-1\rangle_A|(0,1)\,1,-1\rangle_B\,. \end{aligned} \tag{B.82}$$

We have one more Clebsch-Gordan coefficient:

$$\left\langle \begin{matrix} (0,1) & (0,1) \\ 1,-1 & 1,-1 \end{matrix} \middle| \begin{matrix} (0,2) \\ 2,-2 \end{matrix} \right\rangle = 1\,. \tag{B.83}$$

From Eq.(B.82) we can see that another application of $\hat{C}_{23}$ will produce a zero, meaning we have reached the end of a string of weights. Thus, we must change operators so as to generate another string. By inspecting Eq.(B.82), it is evident that $\hat{C}_{13}$ will also produce zero, so we will proceed by using $\hat{C}_{12}$:

$$\begin{aligned} \hat{C}_{12}|020\rangle &= \hat{C}_{12}(|010\rangle_A|010\rangle_B) \\ \sqrt{2}|110\rangle &= |100\rangle_A|010\rangle_B + |010\rangle_A|100\rangle_B \\ |110\rangle &= \frac{1}{\sqrt{2}}(|100\rangle_A|010\rangle_B + |010\rangle_A|100\rangle_B) \\ |(0,2)\,0,-1\rangle &= \frac{1}{\sqrt{2}}(|(0,1)\,-1,0\rangle_A|(0,1)\,1,-1\rangle_B \\ &\qquad +|(0,1)\,1,-1\rangle_A|(0,1)\,-1,0\rangle_B)\,. \end{aligned} \tag{B.84}$$

The Clebsch-Gordan coefficients will no longer be explicitly written, since they can be taken directly from the states.

Now that we have a state not in the initial string, we will begin using $\hat{C}_{32}$ to follow this new string to the end. As a visual aid, consider again Fig.(B.1). Starting from the bottom left corner, we can see how the weight space is beginning to take shape. We have the entire bottom string, and the last state we found is the rightmost on the second string.

Continuing, we act on Eq.(B.84) with $\hat{C}_{32}$:

$$\begin{aligned} \hat{C}_{32}|110\rangle &= \frac{1}{\sqrt{2}}\hat{C}_{32}(|100\rangle_A|010\rangle_B + |010\rangle_A|100\rangle_B) \\ |101\rangle &= \frac{1}{\sqrt{2}}(|100\rangle_A|001\rangle_B + |001\rangle_A|100\rangle_B) \\ |(0,2)\,-1,1\rangle &= \frac{1}{\sqrt{2}}(|(0,1)\,-1,0\rangle_A|(0,1)\,0,1\rangle_B \\ &\qquad +|(0,1)\,0,1\rangle_A|(0,1)\,-1,0\rangle_B)\,, \end{aligned} \tag{B.85}$$

and we have the next state in the second string. We cannot go any further with $\hat{C}_{32}$, since it will produce a zero now, which means that we have already found all of the states in the second string. The only direction left to go now is $\hat{C}_{13}$, since going back, with $\hat{C}_{21}$ for example, will bring us to a state that we have found already. We find, then, for the next state:

$$\begin{aligned}
\hat{C}_{13}|101\rangle &= \frac{1}{\sqrt{2}}\hat{C}_{13}(|100\rangle_A|001\rangle_B + |001\rangle_A|100\rangle_B) \\
\sqrt{2}|200\rangle &= \frac{1}{\sqrt{2}}(|100\rangle_A|100\rangle_B + |100\rangle_A|100\rangle_B) \\
|200\rangle &= |100\rangle_A|100\rangle_B \\
|(0,2)\ -2,0\rangle &= |(0,1)\ -1,0\rangle_A|(0,1)\ -1,0\rangle_B\,.
\end{aligned} \tag{B.86}$$

This state is killed by all but two of the ladder operators, and those two that do not return a zero will bring us back to a state that we have already found. This means that we have six out of nine states, but cannot generate any new ones through the use of the ladder operators. To continue, we need to pick a state that is normalized and orthogonal to the six we already have. One choice is the state

$$\frac{1}{\sqrt{2}}(|010\rangle_A|001\rangle_B - |001\rangle_A|010\rangle_B)\,. \tag{B.87}$$

To verify that it is a highest weight state, we apply the condition of Eq.(B.69), beginning with the operator $\hat{C}_{32}$:

$$\begin{aligned}
\frac{1}{\sqrt{2}}\hat{C}_{32}(|010\rangle_A|001\rangle_B - |001\rangle_A|010\rangle_B) &= \frac{1}{\sqrt{2}}(|001\rangle_A|001\rangle_B - |001\rangle_A|001\rangle_B) \\
&= 0\,.
\end{aligned} \tag{B.88}$$

It is killed by $\hat{C}_{32}$, and a quick inspection reveals that it is also killed by $\hat{C}_{31}$ and $\hat{C}_{21}$, meaning that it is a highest weight state. To determine which state it is we use the two operators $\hat{h}_1$ and $\hat{h}_2$:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\hat{h}_1(|010\rangle_A|001\rangle_B - |001\rangle_A|010\rangle_B) \\
&\quad = \frac{1}{\sqrt{2}}\Big(\left[\hat{h}_1^A|010\rangle_A\right]|001\rangle_B + |010\rangle_A\left[\hat{h}_1^B|001\rangle_B\right] \\
&\qquad\quad - \left[\hat{h}_1^A|001\rangle_A\right]|010\rangle_B - |001\rangle_A\left[\hat{h}_1^B|010\rangle_B\right]\Big) \\
&\quad = (1)\frac{1}{\sqrt{2}}(|010\rangle_A|001\rangle_B - |001\rangle_A|010\rangle_B)\,,
\end{aligned} \tag{B.89}$$

giving $h_1 = 1$. In a similar manner we find that $h_2 = 0$, which means that we have the state

$$|(1,0)\,1,0\rangle = \frac{1}{\sqrt{2}}(|010\rangle_A|001\rangle_B - |001\rangle_A|010\rangle_B)\,, \tag{B.90}$$

as an extremal one.

Notice that this state cannot be represented by a single state in the Λ-notation, since we have already shown that the state $|011\rangle = |(0,2)\,1,1\rangle$ is produced by a sum of the coupled states. We will also find that the three remaining states form an inverted triangle in the weight space, as shown in Fig.(B.2).

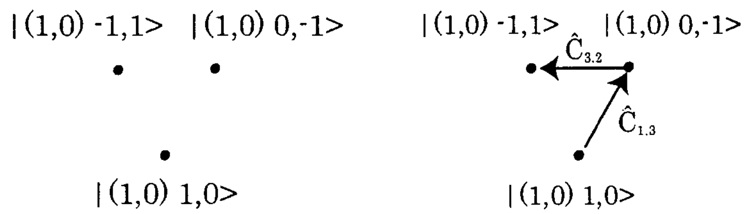

Figure B.2: The same as Fig(B.1), except this shows the set of states $|(1,0)\,h_1, h_2\rangle$.

To find the next state, we continue using the ladder operators. The only two that will not produce a zero are $\hat{C}_{13}$ and $\hat{C}_{12}$. We will start with $\hat{C}_{13}$, only because we have to choose one or the other. Acting on the new state yields

$$\begin{aligned}
\hat{C}_{13}|(1,0)\,1,0\rangle &= \frac{1}{\sqrt{2}}\hat{C}_{13}(|010\rangle_A|001\rangle_B - |001\rangle_A|010\rangle_B) \\
&= \frac{1}{\sqrt{2}}(|010\rangle_A|100\rangle_B - |100\rangle_A|010\rangle_B) \\
|(1,0)\,0,-1\rangle &= \frac{1}{\sqrt{2}}(|(0,1)\,1,-1\rangle_A|(0,1)\,-1,0\rangle_B \\
&\quad -|(0,1)\,-1,0\rangle_A|(0,1)\,1,-1\rangle_B)\,, \qquad \text{(B.91)}
\end{aligned}$$

and the values of h_1 and h_2 can be determined from $h_j = h_j^A + h_j^B$ and the Λ-notation representation of the right hand side.

The next state can be found using $\hat{C}_{32}$, which completes the inverted triangle:

$$\begin{aligned}
\hat{C}_{32}|(1,0)\,0,-1\rangle &= \frac{1}{\sqrt{2}}\hat{C}_{32}(|010\rangle_A|100\rangle_B - |100\rangle_A|010\rangle_B) \\
&= \frac{1}{\sqrt{2}}(|001\rangle_A|100\rangle_B - |100\rangle_A|001\rangle_B) \\
|(1,0)\,-1,1\rangle &= \frac{1}{\sqrt{2}}(|(0,1)\,0,1\rangle_A|(0,1)\,-1,0\rangle_B \\
&\quad -|(0,1)\,-1,0\rangle_A|(0,1)\,0,1\rangle_B)\,. \qquad \text{(B.92)}
\end{aligned}$$

We have now found all of the 9 possible combinations. This process can be repeated for any two coupled states. It is tedious, however, especially if Λ is large, and there are better ways of determining the su(3) Clebsch-Gordan coefficients [27].

Fortunately, the work contained in this thesis relies only on those states which can be represented in the Λ-notation. To avoid overcomplicating the issue, the remainder of this discussion will be limited to only those aspects of SU(3) coupling which are relevant to this work. For a full discussion of SU(3) coupling methods, the reader is directed to [35, 27].

It can be shown [27] that the su(3) Clebsch-Gordan coefficients for the Λ-states, with $\Lambda = n_1 + n_2 + n_3$, $M = m_1 + m_2 + m_3$, $V = \nu_1 + \nu_2 + \nu_3$, and $\Lambda = M + V$, can be factored into the form

$$\langle n_1 n_2 n_3 | m_1 m_2 m_3; \nu_1 \nu_2 \nu_3 \rangle = \left\langle \begin{matrix} \frac{1}{2}(M-m_3) & \frac{1}{2}(V-\nu_3) \\ \frac{1}{2}(m_2-m_1) & \frac{1}{2}(\nu_2-\nu_1) \end{matrix} \middle| \begin{matrix} \frac{1}{2}(\Lambda-n_3) \\ \frac{1}{2}(n_2-n_1) \end{matrix} \right\rangle \times \left\langle \begin{matrix} 0, M \\ m_3, \frac{1}{2}(M-m_3) \end{matrix} ; \begin{matrix} 0, V \\ \nu_3, \frac{1}{2}(V-\nu_3) \end{matrix} \middle\| \begin{matrix} 0, \Lambda \\ n_3, \frac{1}{2}(\Lambda-n_3) \end{matrix} \right\rangle . \tag{B.93}$$

The first term on the right hand side of Eq.(B.93) is a usual su(2) Clebsch-Gordan coefficient, which is covered in the previous section. The second term is called an isoscalar factor or su(2)-reduced Clebsch-Gordan coefficient. These have the form [27]

$$\left\langle \begin{matrix} 0, M \\ m_3, \frac{1}{2}(M-m_3) \end{matrix} ; \begin{matrix} 0, V \\ \nu_3, \frac{1}{2}(V-\nu_3) \end{matrix} \middle\| \begin{matrix} 0, \Lambda \\ n_3, \frac{1}{2}(\Lambda-n_3) \end{matrix} \right\rangle = \left[\frac{(\Lambda - n_3)!\, M!\, V!\, n_3!}{\Lambda!\, (M - m_3)!\, (V - \nu_3)!\, m_3!\, \nu_3!} \right]^{\frac{1}{2}} . \tag{B.94}$$

B.4 The Wigner d-function

The general SU(2) transformation $R_z(\gamma)R_y(\beta)R_z(\vartheta)$ acting on the state $|\ell, m\rangle$ transforms it into a new state, which depends on the parameters γ, β, ϑ. This new state can be represented by a superposition of the basis states $|\ell, m\rangle$, $m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$. The expansion coefficients can be written as a $(2\ell + 1) \times (2\ell + 1)$ matrix. As an example, consider the transformation in the second line of Eq.(1.77). It is nothing more than the matrix representation of $D^1_{m,m'}(0, \beta, 0)$. Also, the matrix T in Eq.(2.17) shows different ways of writing $D^{\frac{1}{2}}_{m,m'}(\gamma, \beta, \vartheta)$. The expansion coefficients are not constants, but functions of the angles γ, β, ϑ. These coefficients are called the Wigner D-functions [26] and are computed as:

$$D^{\ell}_{m',m}(\gamma, \beta, \vartheta) = \langle \ell, m' | R_z(\gamma) R_y(\beta) R_z(\vartheta) | \ell, m \rangle . \tag{B.95}$$

If we evaluate the effect of the z rotations on the right hand side of Eq.(B.95) and use the fact that $\langle \ell, m'|\mathrm{e}^{-i\gamma\hat{L}_z} = \left(\mathrm{e}^{i\gamma\hat{L}_z}|\ell, m'\rangle\right)^\dagger$:

$$\begin{aligned} D^{\ell}_{m',m}(\gamma,\beta,\vartheta) &= \langle \ell, m'|\mathrm{e}^{-i\gamma\hat{L}_z} R_y(\beta)\mathrm{e}^{-i\vartheta\hat{L}_z}|\ell, m\rangle \\ &= \langle \ell, m'|\mathrm{e}^{-i\gamma m'} R_y(\beta)\mathrm{e}^{-i\vartheta m}|\ell, m\rangle \\ &= \mathrm{e}^{-i\gamma m'}\langle \ell, m'|R_y(\beta)|\ell, m\rangle \mathrm{e}^{-i\vartheta m} \\ &= \mathrm{e}^{-i\gamma m'} d^{\ell}_{m',m}(\beta)\mathrm{e}^{-i\vartheta m}. \end{aligned} \tag{B.96}$$

We now have the Wigner d-function, or reduced Wigner function, defined as

$$d^{\ell}_{m',m}(\beta) = \langle \ell, m'|R_y(\beta)|\ell, m\rangle. \tag{B.97}$$

It is this form that will be used throughout this thesis.

One way of calculating the d-functions is through a recursion relation [36]:

$$\begin{aligned} \sqrt{(\ell - m')(\ell + m' + 1)}\, d^{\ell}_{m,m'+1}(\beta) + \sqrt{(\ell + m')(\ell - m' + 1)}\, d^{\ell}_{m,m'-1}(\beta) \\ = 2\csc\beta\,(m'\cos\beta - m)\, d^{\ell}_{m,m'}(\beta)\,. \end{aligned} \tag{B.98}$$

To start the recursion, recall from Eq.(B.20) that the transformation $R_y(\beta)$ can be factored as

$$R_y(\beta) = \mathrm{e}^{-i\beta\hat{L}_y} = \mathrm{e}^{\xi\hat{L}_-}\mathrm{e}^{\chi\hat{L}_z}\mathrm{e}^{\xi^*\hat{L}_+}\,, \tag{B.99}$$

where $\mathrm{e}^{\chi/2} = \cos\frac{\beta}{2}$ and $\xi = \tan\frac{\beta}{2}$. It is then straight forward to find the value of $d^{\ell}_{m,\ell}(\beta)$. Simply use the fact that

$$\mathrm{e}^{\varphi\hat{L}_+}|\ell,\ell\rangle = |\ell,\ell\rangle \tag{B.100}$$

to compute the matrix element for $d^{\ell}_{m,\ell}(\beta)$ as a starting point:

$$\begin{aligned} \langle \ell, m|\mathrm{e}^{\xi\hat{L}_-}\mathrm{e}^{\chi\hat{L}_z}\mathrm{e}^{\xi^*\hat{L}_+}|\ell\ell\rangle &= \mathrm{e}^{\chi\ell}\langle \ell, m|\mathrm{e}^{\xi\hat{L}_-}|\ell,\ell\rangle \\ &= \mathrm{e}^{\chi\ell}\langle \ell, m|(\mathbb{1} + \xi\hat{L}_- + \frac{\xi^2}{2}\hat{L}_-^2 + \ldots)|\ell,\ell\rangle \\ &= \frac{\xi^{\ell-m}\mathrm{e}^{\chi\ell}}{(\ell-m)!}\langle \ell, m|\hat{L}_-^{\ell-m}|\ell,\ell\rangle \\ &= \frac{(\tan\frac{\beta}{2})^{\ell-m}(\cos\frac{\beta}{2})^{2\ell}}{(\ell-m)!}\langle \ell, m|\sqrt{\frac{(2\ell)!\,(\ell-m)!}{(\ell+m)!}}|\ell,\ell\rangle \\ d^{\ell}_{m,\ell}(\beta) &= \sqrt{\frac{(2\ell)!}{(\ell+m)!\,(\ell-m)!}}(\cos\tfrac{\beta}{2})^{\ell+m}(\sin\tfrac{\beta}{2})^{\ell-m}. \end{aligned} \tag{B.101}$$

Now all one needs to do is apply Eq.(B.98) enough times to recover the appropriate d-function.

B.5 Factorization of a Unitary SU(3) Transformation

An SU(3) matrix is defined as a complex 3×3 matrix of determinant $+1$, such that the inverse of the matrix is the transpose complex conjugate of the matrix itself. These SU(3) matrices act on complex 3-dimensional vectors. The following section demonstrates that any arbitrary unitary SU(3) transformation U can be decomposed into the product of three unitary block SU(2) transformations $R_{i,j}$. First, we need to define a block SU(2) transformation. A block SU(2) transformation is one that acts in a subspace in a way that mimics an SU(2) transformation on that subspace. For an example, consider the three 3×3 block SU(2) transformations:

$$\begin{aligned} R_{1,2} &= \begin{pmatrix} a & b & 0 \\ -b^* & a^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ R_{2,3} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b^* & a^* \end{pmatrix} \\ R_{1,3} &= \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b^* & 0 & a^* \end{pmatrix}, \end{aligned} \tag{B.102}$$

with $|a|^2 + |b|^2 = 1$. Each of these $R_{i,j}$ transformations only affect the elements in the i, j subspace, leaving each of the other elements unchanged. To further illustrate the idea, consider a 5×5 block SU(2) transformation acting inside a space of dimension 5.

$$R_{1,3} \mapsto \begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -b^* & 0 & a^* & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.103}$$

This transformation is simply a generalization of Eq.(B.102) to the corresponding 5×5 transformation. It still only acts on the $1, 3$ subspace, leaving $2, 4, 5$ unchanged.

We will show that any 3×3 SU(3) transformation can be decomposed into a product of three block SU(2) transformations of the form Eq.(B.102). The result is then generalizable to SU(3) transformations acting in spaces of arbitrary dimensions, since it relies solely on operator relations. There are a number of possible combinations of Eq.(B.102) that work. However we are interested in obtaining the specific decomposition $U = R_{2,3}R_{1,3}R_{1,2}$. This method is an adaptation of a proof given by Rowe *et al* [34].

To show that the decomposition $U = R_{2,3}R_{1,3}R_{1,2}$ is possible, we start with a completely general unitary SU(3) element, assuming only that the final column is of the form:

$$\begin{pmatrix} * & * & x \\ * & * & y \\ * & * & z \end{pmatrix}, \tag{B.104}$$

where the $*$s are arbitrary entries (not necessarily the same) and the three entries x, y, and z are complex numbers. Note that since Eq.(B.104) is unitary, the three labeled entries must satisfy $|x|^2 + |y|^2 + |z|^2 = 1$.

Noting that the product of any two unitary transformations is also a unitary transformation, the first step is to make the following $R_{2,3}$ multiplication on Eq.(B.104);

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & Y & -Z \\ 0 & Z^* & Y^* \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & y \\ * & * & z \end{pmatrix} = \begin{pmatrix} * & * & a \\ * & * & 0 \\ * & * & b \end{pmatrix}. \tag{B.105}$$

This gives:

$$x = a \tag{B.106}$$

$$Yy - Zz = 0 \tag{B.107}$$

$$Z^*y + Y^*z = b. \tag{B.108}$$

We need now to consider separately the cases $z = 0$ and $z \neq 0$. We will begin with the latter.

For the case $z \neq 0$:
Since $Yy = Zz$ and $|Y|^2 + |Z|^2 = 1$, we can solve for Y (note that $z \neq 0$ implies $|x| \neq 1$);

$$\begin{aligned} |Y|^2|y|^2 &= |Z|^2|z|^2 \\ |Y|^2|y|^2 &= \left(1 - |Y|^2\right)|z|^2 \\ |Y|^2 &= \frac{|z|^2}{|y|^2 + |z|^2} \\ |Y|^2 &= \frac{|z|^2}{1 - |x|^2} \\ Y &= \frac{z}{\sqrt{1 - |x|^2}}. \end{aligned} \tag{B.109}$$

This implies that

$$Z = \frac{y}{\sqrt{1 - |x|^2}} \tag{B.110}$$

$$b = \sqrt{1 - |x|^2}. \tag{B.111}$$

At this point, if $x = 0$, we already have the factored form we are looking for and we are done. If on the other hand $x \neq 0$, then our next transformation must be of the type $R_{1,3}$ and, acting on the right hand side of Eq.(B.105), must be of the form

$$\begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B^* & 0 & A^* \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & 0 \\ * & * & \sqrt{1-|x|^2} \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.112}$$

This gives the equations:

$$Ax - B\sqrt{1-|x|^2} = 0 \tag{B.113}$$
$$B^*x + A^*\sqrt{1-|x|^2} = 1. \tag{B.114}$$

Combining them we get

$$\begin{aligned} Bx^* + \frac{B}{x}\left(1-|x|^2\right) &= 1 \\ B|x|^2 + B\left(1-|x|^2\right) &= x \\ B\left(|x|^2 + 1 - |x|^2\right) &= x \\ B &= x\,, \end{aligned} \tag{B.115}$$

and

$$\begin{aligned} Ax &= x\sqrt{1-|x|^2} \\ A &= \sqrt{1-|x|^2}. \end{aligned} \tag{B.116}$$

Since we have forced the (3,3) entry to be 1 in Eq.(B.117), and remembering these are all unitary, the zeros in the third row follow automatically. Our transformations now have the form

$$\begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B^* & 0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & Y & -Z \\ 0 & Z^* & Y^* \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & y \\ * & * & z \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.117}$$

Using the fact that any unitary transformation $U^\dagger U = \mathbb{1}$, we can invert the equation, giving

$$\begin{pmatrix} * & * & x \\ * & * & y \\ * & * & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Y^* & Z \\ 0 & -Z^* & Y \end{pmatrix} \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ -B^* & 0 & A \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.118}$$

which is the in the form $R_{2,3}R_{1,3}R_{1,2}$ that we are looking for.

For the $z = 0$ case, we have the following;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & Y & -Z \\ 0 & Z^* & Y^* \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & y \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & a \\ * & * & 0 \\ * & * & b \end{pmatrix}. \tag{B.119}$$

This immediately gives $Y = 0$ and thus we can pick $Z = \frac{y}{|y|}$ (if $|y| = 0$, then we simply pick $Z = 1$).

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -Z \\ 0 & Z^* & 0 \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & y \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & x \\ * & * & 0 \\ * & * & |y| \end{pmatrix}. \quad \text{(B.120)}$$

This produces $b = |y|$. The next transformation,

$$\begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B^* & 0 & A^* \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & 0 \\ * & * & |y| \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(B.121)}$$

implies that $A = |y|$ and $B = x$, giving

$$\begin{pmatrix} |y| & 0 & -x \\ 0 & 1 & 0 \\ x^* & 0 & |y| \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & 0 \\ * & * & |y| \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(B.122)}$$

or

$$\begin{pmatrix} |y| & 0 & -x \\ 0 & 1 & 0 \\ x^* & 0 & |y| \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -Z \\ 0 & Z^* & 0 \end{pmatrix} \begin{pmatrix} * & * & x \\ * & * & y \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{(B.123)}$$

Inverting we get the factorization;

$$\begin{pmatrix} * & * & x \\ * & * & y \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & Z \\ 0 & -Z^* & 0 \end{pmatrix} \begin{pmatrix} |y| & 0 & x \\ 0 & 1 & 0 \\ -x^* & 0 & |y| \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(B.124)}$$

which is identical to Eq.(B.118) if you take $z = 0$.

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