Developing Understanding: Pre-service Elementary Teachers' Changing Conceptions of Mathematics

by

Wendy Margaret Stienstra

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Abstract

This descriptive case study explores how a conceptual understanding of fractions develops in pre-service elementary teachers enrolled in a reform-based, remedial mathematics skills course set at a middle school level, during their final year of an education program in a small Canadian university. I compared the learning trajectory of these adults to that of children and then developed a landscape of (re)learning fraction concepts within a constructivist-oriented environment.

Fourteen prospective teachers completed pre/posttest content exams and interviews that followed-up each of these test instruments, as well as two paired problem-solving interviews that used a modified think aloud protocol, and open ended questionnaires. All interviews were videotaped using two cameras, capturing both the participant(s) and a zoomed in view of their hands and written work. Data collection took place at four points in the school year between September and March, providing snapshots of the developmental progression of fraction understanding in the pre-service teachers over time. Analysis occurred at multiple layers: first with the individual pre-service teachers, then with three clusters of participants grouped according to their mathematical abilities, and finally with the group as a whole. The theoretical framework utilized a constructivist view of learning and a particular landscape of learning that highlights strategies, big ideas, and models.

The analysis revealed that the landscape of (re)learning fractions for adults is significantly different from that of children. Pre-service teachers moved from understanding fractions solely as procedures and worked to make sense of their fragmented knowledge around the intertwined concepts of fractions as relations, ratios, and operators. All participants demonstrated growth in their understanding of fractions; however, few participants developed a deep understanding of fractions as meaningful objects. The foundational concept of fractions as relations developed slowly over time and constrained the understanding of fractions as ratios and operators. Pre-service teachers experienced difficulties in understanding and identifying the appropriate referent unit, or fraction whole; this reflected the fragility of their understanding of fractions as relations. In order for pre-service teachers to understand fractions as meaningful objects and to shift to multiplicative thinking, modeling is a necessary but insufficient means.

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Chapter One: Introduction

Context of the Study

The current reform in mathematics education requires changes to our traditional understandings of what it means to do mathematics, of how one learns mathematics, and of what effective mathematics teaching looks like. Over the past twenty years the National Council of Teachers of Mathematics (NCTM) has provided both impetus and direction for these changes (NCTM, 1989, 1991, 1995, 2000). Canadian provinces have used these documents to revise their mathematics programs (e.g., Alberta Education, 2006; Ontario Ministry of Education, 2005). The focus in mathematics instruction is no longer the rote memorization of rules or procedures; instead, it is a focus on the development of deep understanding of mathematical concepts and relationships. Students are challenged to use their prior knowledge and reasoning skills to solve rich mathematical problems, set in meaningful contexts. They are encouraged to develop connections and relationships amongst the different aspects of their mathematical knowledge – rather than seeing mathematics as discrete pieces of information (Hiebert & Lefevre, 1986; NCTM, 1991). Finally, by hypothesizing, discussing, and justifying, students are prompted to develop personal autonomy and *make sense* of mathematics (Ball & Bass, 2003; Kamii, 2000).

The role of the teacher changes substantially in the reform mathematics classroom. Rather than being a dispenser of knowledge and answers, the teacher becomes a facilitator (Fosnot & Dolk, 2002). Hiebert et al. (1997) aptly describe the changing role of mathematics teachers who teach for understanding. First, teachers must choose rich mathematical tasks with multiple entry points that enable students to reflect on and communicate about mathematics. Next, teachers must know what to tell and what not to tell students (see also Chazan & Ball, 1999). Teachers need to be able to provide relevant information, such as mathematical conventions and alternative methods, and, they must be able to articulate the mathematical ideas embedded in students' methods. However, they also need to refrain from giving students the answers and thus depriving students of the opportunity to develop their own reasoning skills. Finally, teachers must be able to establish a classroom environment that fosters both discussion and risk taking. As students share their methods of solution and inevitably make mistakes, these mistakes must be taken as learning opportunities for the whole class rather than opportunities for ridicule; thus, teachers play a pivotal role in mathematics reform (Hiebert et al., 1997).

Unfortunately, these changes are often difficult for teachers to implement. Many teachers are hampered by their own experiences of mathematics instruction that differed drastically from experiences provided by reform-based instruction (Battista, 1994). In addition, teachers often lack what Ma (1999) calls *profound understanding of fundamental mathematics*. In her comparison of Chinese and American elementary teachers, Ma argued that teachers need a solid conceptual understanding of mathematics. They need to be able to make connections between various mathematical concepts and procedures; they must be able to utilize and explain multiple perspectives and solution methods; they must be aware of the *big ideas* that are benchmarks for developing mathematical understanding; and finally, they must have a broad understanding of the cohesive nature of the elementary mathematics curriculum. We have a good deal of evidence that many elementary teachers often lack this in-depth conceptual understanding of the mathematics they are required to teach (Ball, Lubienski & Mewborn, 2001; Fennema & Franke, 1992; Mewborn, 2003). For example, the case studies of four grade 5 teachers presented by Putnum et al. (1992) reveal that these teachers not only had areas of weak content knowledge but also encountered instances in which they were unaware that mathematics was being misrepresented. While these teachers attempted to teach for understanding, at times the mathematical content became less important than the activity. The limitations of the teachers' mathematics knowledge thus weakened the students' mathematical experiences.

This lack of rich understanding of mathematics is, not unexpectedly, also evident in pre-service teachers (Brown, Cooney & Jones, 1990). For example, pre-service teachers tend to have an understanding of division that is limited to the partitive model (Ball, 1990a; Graeber, Tirosh & Glover, 1989; Simon & Schifter, 1993; Tirosh & Graeber, 1991). They may have strong procedural knowledge, but as was shown in Simon's (1993) study of 33 prospective elementary teachers, they were not able to explain correctly why the traditional division algorithm worked. Similarly, concepts such as multiplication, place value, fractions, and decimals pose great difficulty for pre-service teachers (Ball, 1990a; Graeber & Tirosh, 1988; Newton, 2008).

In order to break the cycle of inadequate mathematical teaching producing limited mathematical understanding which in turn leads to pre-service elementary teachers with inadequate mathematics knowledge, there have been calls to reexamine the education of prospective teachers (Mathematical Sciences Education Board, 1996; National Research Council, 2001). For example, the Conference Board of the Mathematical Sciences (2001) recommends that prospective elementary teachers take additional courses in mathematics to develop a deep understanding of the mathematics they need in order to teach. While, there is no universal consensus on what mathematics knowledge is required in order to teach (Ball et al., 2001), there is nonetheless contention that it should be deep and connect or, be as Ma (1999) suggests, a *profound understanding of fundamental mathematics*. Yet there is limited information on how that deep understanding develops in adults who have already progressed through the traditional mathematics education system with a limited understanding of foundational concepts.

Purpose of the Study

The purpose of this study is to investigate the development of an understanding of basic fraction concepts in primary/junior (K-6) prospective teachers attending a small Canadian university, as they participated in a remedial reform-based mathematics course set at the middle school level. Most of these students had not achieved a minimum of 75% on a mathematics content exam set at the Grade 6-7 level. As these adults (re)learn fraction concepts their learning trajectory is compared to that of children who are initially learning fractions in a reform-based learning environment. The progress of these pre-service teachers is described in terms of Fosnot and Dolk's (2002; 2007) big ideas, strategies, and models. The intent is to develop a landscape of conceptual understanding that reflects learning based on reform methods of mathematics instruction.

Research Questions

How does conceptual understanding of mathematics content knowledge, specifically in the strand of number sense in the area of fractions, develop in pre-service elementary teachers enrolled in a reform-based mathematics skills development course?

a) What changes occur in students' conceptual understanding of fractions?
b) What specific concepts cause students great difficulty as they (re)learn?
c) What are the typical stages of development of understanding and misunderstanding?

Significance of the Study

This study has direct significance for both teacher educators and the Canadian mathematics education system. The empirical evidence regarding the mathematical knowledge of pre-service teachers can inform decisions about the restructuring of mathematics education programs. Identifying areas of content knowledge weakness or misunderstandings in elementary education students gives us insight into the results of the typical mathematics education system. The continuum on conceptual understanding will provide mathematics teacher educators with insight into areas of difficulty and potential paths for a deeper understanding of mathematics in their students. This information can also be used to restructure mathematics methods courses to ensure that prospective teachers have a profound understanding of these fundamental areas of mathematics and that they are able to recognize potential areas of difficulty their future students may have as they encounter these mathematics concepts.

Organization of the Dissertation

This chapter introduces the central focus of the study: the development of conceptual understanding of fraction content knowledge in pre-service teachers. Chapter Two examines a review of the related literature, outlining the nature of reform based mathematics education, the limitations of pre-service teachers' mathematical knowledge, and key elements of fraction knowledge. Chapter Three outlines the case study research methodology, indicating the design used to determine the conceptual development over time. Chapter Four introduces the participants and their initial perceptions of mathematics. Chapters Five to Eight report on the development of the 14 participants' mathematical understanding over the four test instruments. Results are grouped using three distinct clusters of participants. Analysis of the conceptual development is discussed using the theoretical framework. Chapter Nine expands the discussion by answering the main research questions that frame this study, concluding with implications for future research.

Chapter Two: Literature Review

The purpose of this study is to investigate how an understanding of basic mathematical concepts, especially that of fractions, develops in pre-service elementary teachers as they participate in a reform-based remedial mathematics course set at the middle school level. The goal is to create a landscape of fraction learning that reflects a continuum of conceptual understanding as it develops in the context of a reform-oriented learning environment. Thus, the areas covered in this chapter will include the need for reform in mathematics education, the nature of mathematical understanding, relevant learning theories and their implications for instruction, mathematical knowledge for teachers, and the limitations of pre-service teachers' mathematical knowledge.

Reform in Mathematics Education

The current reform movement in mathematics education, which began in the mid 1980s, represents a significant shift in our understanding of mathematics teaching and learning. Mathematics is seen as a dynamic body of knowledge rather than isolated bodies of facts and procedures (National Research Council, 1989; Thompson, 1992). This viewpoint, also held by many mathematicians, maintains that *knowing* mathematics is *making* mathematics (Thompson, 1992, p. 128). Thus the reform movement encourages the use of problem solving to promote thinking and reasoning, contrary to the "new math" of the 1960s, which focused on fundamental principles of logical deduction and formal notation, and the "back to basics" movement of the 1970s, which emphasized rote memorization (Barb & Quinn, 1997).

Lack of Understanding

This shift was initiated when national and international large scale testing documented the failure of the then current teaching methods in the United States (Battista, 1999; Geist, 2000; O'Brien, 1999). The performance of American students was disappointing in relation to other nations that participated in both the Second International Mathematics Study (SIMS) in 1982 and repeated in the Third International Mathematics and Science Study (TIMSS) in 1995. Students were unable to make sense of mathematics. They were not able to use what they had learned in new situations. For example, in the mid 1980s, the following problem was posed to 45,000 13-year-olds in the third mathematical assessment of the National Assessment of Educational

Progress (NAEP): "*An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?*" Two-thirds of the students used the long division algorithm appropriately, but only 23% correctly solved the problem (O'Brien, 1999). Many of the students ignored the remainder or rounded down and gave an answer of 31 buses, or they responded that "31 remainder 12 buses" were needed. They were not able to use their procedural knowledge of mathematics to make sense of the context of the problem.

Another example from the second mathematics assessment of the NAEP study further exemplifies this lack of understanding. Students were asked to: "*Estimate the answer to 12/13 + 7/8. You will not have time to solve the problem using paper and pencil*." Only 24% of 13-year-olds and 37% of 17-year-olds were able to answer the question correctly (Carpenter & Corbitt, 1981). The majority of participants were not able to recognize that each of the fractions was close to one, so a reasonable estimate must be two. Instead, over half of the 13-year-olds and a third of the 17-year-olds added either numerators or denominators and chose an estimate of 19 or 21. Since the procedures for adding fractions were not linked to a conceptual understanding of fractions, these students incorrectly chose a procedure related to whole number addition.

The report by the National Research Council (2001), *Adding It Up*, discussed these findings. Their review of the NAEP assessments over the previous twenty-five years showed that American students had some procedural proficiency but they had less developed conceptual understanding. Students may have been able to use algorithms for whole numbers, but they had difficulty with rational numbers and solving problems. Students also had difficulty making appropriate connections among concepts and situations and were often unable to justify their answers. The report concluded that, "during the past 25 years mathematics instruction in U.S. schools has not sufficiently developed mathematical proficiency" (p. 145).

In addition to standardized test results, the research literature is replete with examples of students' typical mistakes in specific mathematical topics (Even & Tirosh, 2002; Sfard, 2003; Siegler, 2003; Smith, diSessa & Roschelle, 1993). These misunderstandings can take a number of different forms. First, students may inappropriately apply an algorithm if they do not fully understand the concept. An example of one such "buggy" algorithm occurs with subtraction. When asked to solve the problem: 60 – 42 = ____, students who have been taught the traditional algorithm may give an answer of 22. They treat 0 – 2 as 2 – 0 because they know they cannot subtract a larger number from a smaller number. Second, students may use their prior knowledge inappropriately. For example, when asked to compare the relative sizes of two numbers such as 3.74 and 3.285, many students will say that 3.285 is larger because it has more digits to the right of the decimal point. They extend their knowledge of whole numbers, where the number with more digits is larger, to decimals. Others students believe that the number with the fewest digits to the right of the decimal is larger. Thus, they believe that 3.41 is larger than 3.962 because hundredths are bigger than thousandths. Finally, students may overgeneralize their prior knowledge. They may believe that when you multiply, your answer is always bigger and when you divide, your answer is always smaller than what you started with. When students encounter multiplication and division with fractions, these beliefs may be difficult to change (Sfard, 2003; Smith et al., 1993).

Call for Change

In light of these results, the National Council of Teachers of Mathematics (NCTM) provided both impetus and direction for the reform in mathematics education (Battista, 1999). Their documents promote a vision of mathematics teaching and learning based on the development of deep understanding of mathematical concepts and relationships rather than the rote memorization of rules or procedures (NCTM, 1989, 1991, 1995, 2000). Attention of mathematics educators is shifting from focusing on the mathematics content to focusing on how children can best learn mathematics with understanding (Hiebert, 2003). Thus, teaching mathematics from the reform perspective is no longer the traditional "show and tell"; instead, it fosters meaning making and understanding through engaging problem solving contexts and discussions.

Understanding in Mathematics

The concept of *understanding*, however, is complex. We use the term, both in education and in our everyday lives, to convey a range of meanings. In the mathematics class, some would relate understanding to successful performance on tests and homework, while others would say understanding is about knowing the *why* and not just the *how* (Brown et al., 1990). Teachers, parents, and administrators all want students to understand mathematics. The current NCTM reform documents emphasize the importance of understanding, stating, "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM 2000:20). But what exactly does it mean to understand mathematics? Is understanding an all or nothing phenomenon or a continuum? In this section I will explore how our understanding of this concept has been refined over the past century.

Throughout the various reform movements of the past century the meaning of understanding has often been predicated on ways mathematicians understood and taught mathematics (Carpenter & Lehrer, 1999). However, many of our current perspectives on understanding are grounded in the developing body of research on how students construct mathematical meaning. For the first half of the twentieth century mathematical understanding was essentially seen as an all or nothing phenomenon. An emphasis on skills and procedures led to an assumption that being able to produce the correct answer was indicative of understanding. While lack of understanding could be seen in rote memorization or meaningless manipulation, understanding was rarely defined (Byers, 1980).

Brownell (1935), an early advocate of concepts today called reform, provided some illumination on the concept of understanding. He argued that mathematics should make sense to students. Students needed to be able to explain their thinking when solving problems. For Brownell, understanding developed by recognizing the relationships within mathematics. Brownell argued that arithmetic should be "less a challenge to the pupil's memory and more a challenge to his (sic) intelligence" (p. 31).

A pivotal shift in the discussion of understanding came with Skemp's (1976) two-tiered model that differentiated between *instrumental* and *relational* understanding. According to Skemp, relational understanding is a deeper understanding since it refers to knowing "what to do and why," and thus is more likely to promote transfer. On the other hand, instrumental understanding is simply the possession and use of rules and procedures. While Skemp's work provides insight into the complex nature of understanding, explaining why simply knowing procedures cannot be sufficient for mathematical understanding, his model encourages a dichotomy between learning algorithms and mathematical understanding. Some see this distinction as superficial and misleading (Even & Tirosh, 2002). Skemp somewhat tempers the dichotomy between the two different kinds of mathematics, when he acknowledges that partial understanding can occur when some connections are formed but the new knowledge cannot yet be used appropriately in different situations (Byers, 1980).

Other theorists such as Byers and Herscovics (1977) recognize that there are differing levels and degrees of understanding. Byers and Herscovics extended Skemp's (1976) model by adding the categories of *intuitive* and *formal* understanding. For them, intuitive understanding refers to the ability to solve a problem instinctively without any instruction, and formal understanding refers to the ability to use logical reasoning and to make connections between mathematical symbols and ideas. In this model the four modes of understanding interact, so it is possible to have a mixture of these different types of understanding for a particular topic at a particular time. While being able to do mathematics gives evidence of understanding mathematics, Byers and Herscovics characterize understanding as a state of cognition.

Hiebert and Lefevre (1986) revisit the historic distinction between understanding and skill, yet bring a new dimension to the discussion. They define *conceptual understanding* as knowledge that is rich in relationships. When a new concept is learned, it is linked to part of the existing cognitive network, which allows for flexibility in using and accessing the information. In contrast, *procedural understanding* consists of a sequence of actions. The standard algorithms for addition or multiplication exemplify the step-by-step procedures of this type of knowledge. While conceptual knowledge is characterized by many different kinds of relationships, procedural knowledge is linear and requires minimal connections to link each step of the procedure. As a result, procedural knowledge can be learned by rote but conceptual knowledge cannot. Contrary to historic discussions, Hiebert and Lefevre believe that it is important to explore how these two types of knowledge are related rather than viewing them as dichotomous.

Both skills and understanding are essential in the development of mathematical competence. Hiebert and Lefevre maintain that a distinction between conceptual and procedural understanding helps us to recognize the process of learning mathematics. Fuson (2003) agrees and states that conceptual and procedural aspects are "continually intertwined and potentially facilitate each other" (p. 68). Both types of knowledge are required for mathematical expertise. In a similar vein, others view conceptual and procedural knowledge as two ends of a continuum, and recognize that these two aspects cannot always be treated as separate elements that develop independently (Rittle-Johnson & Alibali, 1999). Understanding mathematical concepts necessarily includes elements of conceptual and procedural knowledge.

Pirie and Kieren (1989, 1994) expanded the notion of levels of understanding by providing insight into how mathematical understanding grows and develops. For them, understanding, as a whole, is dynamic, levelled, and recursive. Understanding is dynamic in that it requires constant reorganization of one's knowledge structures (Pirie & Kieren, 1994). The eight embedded levels range outward from *primitive knowing* to *inventising* and are applicable for any topic or person. Each level grows out of and transcends the previous level. Yet growth is not linear. It involves *folding back* to prior levels to "re-member and to re-construct new understandings" (Pirie & Kieren, 1994, p. 84). Pirie and Kieren (1989) summarize this process:

> Mathematical understanding can be characterized as levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication…. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without. (p. 8)

Rather than attempting to identify different types of understanding, Haylock (1982) suggested that understanding something involves making cognitive connections that lead to varying degrees of understanding. "The more connections the learner can make between the new experience and previous experiences, the greater and consequently the more useful the understanding" (Haylock, 1982, p. 54). This model of connections representing understanding is not new. According to Hiebert and Carpenter (1992) this theme has run through some of the classic works of mathematics education and there is much agreement that "understanding involves recognizing relationships between pieces of information" (p. 67). They suggest that "mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections" (p. 67). In other words, "we understand something if we see how it is related or connected to other things we know" (Hiebert et al., 1997, p. 4). Each individual has his or her own understanding of particular mathematics concepts; some are rich and some are limited. Given this perspective, procedural knowledge would be viewed as a less connected understanding.

In the past half century, our awareness of the concept of understanding in mathematics has expanded beyond an all or nothing phenomenon. Mathematical understanding is no longer limited to procedural competencies. Conceptual understanding must provide a solid foundation that enables effective problem solving and the development of efficient procedures. Thus, mathematics is about doing and understanding. This expanded conception of mathematical understanding has moved beyond the theoretical discussion into sites of research study and policy development.

Mathematical Proficiency

Traditionally, proficiency in mathematics was limited to procedural fluency. Furthermore, mathematics teachers maintained that procedural skills should be taught first, before application could occur. Yet, the interplay between conceptual and procedural knowledge is complex and difficult to separate. Carpenter and his colleagues (1999; 1986; 1992) argue that teaching should be planned so that students first develop conceptual knowledge and then link this knowledge to procedures and symbols. Few researchers have directly examined the relationship between conceptual and procedural knowledge, yet many provide suggestive evidence about the relationship. In an overview of the literature, Siegler (2003) reported that several studies have shown that children who can successfully execute subtraction 'borrowing' procedures are more likely to understand the concept, while children who lack a conceptual understanding are more likely to develop "buggy" subtraction procedures. Hiebert and Wearne (1996), in a study of 70 children over their first three years of school, suggested that instruction that emphasizes conceptual understanding and procedural skills is more effective that instruction that only focuses on the skills. As a final example, Rittle-Johnson and Alibali (1999) directly examined the relations between conceptual and procedural knowledge of mathematical equivalence in Grade 4 and 5 children. Their findings point to a causal relationship between the two types of knowledge and suggest that conceptual knowledge may have a greater influence on procedural knowledge than vice versa. In light of these findings, we see support for the contention that in order to develop proficiency in mathematics, learners must be able to link their conceptual and procedural knowledge. Recent policy documents in the United States and Canada reflect this argument.

Reform documents from the National Council of Teachers of Mathematics recognize that proficiency in mathematics includes both conceptual and procedural understanding. The authors acknowledge that "students who memorize facts or procedures without understanding often are not sure when or how to use what they know, and such learning is often quite fragile" (NCTM, 2000, p. 20). Thus, it is important to link factual knowledge, procedural proficiency, and conceptual understanding to make mathematics useable in powerful ways (NCTM, 2000). As such, conceptual understanding is key to this new notion of proficiency in mathematics. It enables students to use their knowledge flexibly and to apply it to novel problems and settings. When students learn with understanding, they can become autonomous learners, one of the goals of education (Kamii, 2000; NCTM, 2000).

In a similar manner, the Ontario Ministry of Education (2005) promotes a balanced program of mathematics that includes learning concepts and procedures, acquiring skills, and learning and applying mathematical processes. Their curriculum is structured to help students both develop a solid conceptual understanding of mathematics and acquire the necessary operational skills. It is based on the belief that

> in order to learn mathematics and to apply their knowledge effectively, students must develop a solid understanding of mathematical concepts…. An investigative approach, with an emphasis on learning through problem solving and reasoning, best enables students to develop the conceptual foundation they need (Ontario Ministry of Education, 2005, p. 24).

The National Research Council's (2001) report, *Adding It Up: Helping Children Learn Mathematics*, introduced the term *mathematical proficiency* to encapsulate what it means to learn mathematics successfully. In doing so, the authors expanded the idea of mathematical understanding, illustrating its complexity by using the analogy of an intertwined cord. This concept of mathematical proficiency also sidesteps the controversy embedded in a focus on conceptual understanding at the expense of procedural fluency. According to the NRC, mathematical proficiency consists of five intertwining strands:

1. Conceptual understanding – comprehension of mathematical concepts, operations and relations
2. Procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
3. Strategic competence – ability to formulate, represent, and solve mathematical problems
4. Adaptive reasoning – capacity for logical thought, reflection, explanation, and justification
5. Productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (p. 116)

Even though these categories do not align perfectly with those identified in the literature, they reflect a sizeable portion of the content. Mathematical proficiency is a complex entity and each of the strands represents a different aspect of that whole. None of the strands can stand alone to produce mathematical proficiency; all five strands are entwined and interdependent. Yet the strands are not all of equal weight. Conceptual understanding provides the necessary foundation for the other strands. As the various competencies are interwoven and moored to a conceptual understanding, deep understanding develops. Learners develop the ability to make connections between pieces of knowledge in such a way that they can use them effectively in solving problems.

In summary, synthesizing the recent theoretical discussion, research findings, and policy documents, understanding in mathematics is more than getting the right answer; it is more than knowing the correct facts and procedures. Understanding is about constructing relationships. Students who have a deep conceptual understanding of mathematics understand the central or big ideas of the domain; they have a sense of proportion and size; they are able to see the relationship among ideas; they have an elaborated and detailed knowledge; and they have an ability to reason, analyze and solve problems (Kennedy, 1998). Moreover, understanding is not a static notion. Understanding changes, grows, and can be described from different points of view (Hiebert, et al. 1997). The notion of mathematical proficiency attempts to capture this complexity of mathematical understanding. Conceptual and procedural knowledge are both necessary in the development of mathematical proficiency.

How is Proficiency in Mathematics Achieved? Theories of Learning

While educators would agree that student understanding is a key goal of mathematics instruction, not everyone agrees on how this goal is best achieved. Even though proficiency in mathematics consists of the intertwined aspects of conceptual and procedural understanding, some stress conceptual understanding while others stress procedural fluency, at times to the detriment of each other. Different theories of learning, particularly historically, highlight either the conceptual or the procedural knowledge of mathematics. When mathematics education focused on memorizing procedures and factual knowledge, behaviourist learning theory seemed to provide insight into the overt behaviours of students. Student understanding was assessed by the correctness of their behaviour. However, as our notions of mathematical understanding changed, it became apparent that this learning theory did not include aspects of students' internal thinking that were necessary to assess conceptual understanding. The development of constructivist learning theory enabled the mathematics educator to gain clearer insights into students' thought processes and development of knowledge structures.

Reformers suggest that the traditional mathematics instruction, which grows out of a behaviourist perspective and focuses on rote memorization, contributes to a lack of understanding in students. Moreover, they contend that learning environments grounded in the constructivist perspective, which encourage students to create their own understanding through dialogue, logical arguments, and justification of mathematical ideas, promote a deeper conceptual understanding of mathematics.

Limits of Behaviourist Learning Theory

For the past century, mathematics education in Canada and the United States has been dominated by a traditional approach to instruction whereby teachers transmit information, skills and procedures to students who then have the opportunity to practice what was shown to them. The assumptions about teaching and learning in this paradigm are derived from the behaviourist learning theory. This theory focuses solely on observations of behaviour, excluding unobservable mental processes such as thoughts or feelings (De Corte, Greer & Verschaffel, 1996). Proponents, such as Thorndike and Skinner, used animals to examine the *association* between external events and the resulting behaviours (Sternberg, 1999). Extending this work to humans and to mathematics education in particular, the teaching of arithmetic became a "program of stimulus-response associations with appropriate rewards to act as reinforcement" (De Corte et al., 1996, p. 493). As a result of Thorndike's work, drill and practice became an accepted method of mathematics instruction.

From the behaviourist perspective, human learning is simply the acquisition of newly acquired behaviour. Knowledge exists independently, outside of people. This body of accepted information and skills, established by others, is transmitted to students as part of the education process (Scheurman, 1998). The associations between ideas or stimuli and responses form the basis for student knowledge (Greeno, Collins & Resnick, 1996). In the learning process associations are formed, strengthened, or adjusted using reinforcement or reward (Greeno et al., 1996). Students demonstrate their acquisition of knowledge by responding with the correct behaviour, or written response, in testing situations. Theorists believe this type of learning occurs because human beings are "wired" so that reinforced behaviour is more likely to occur (Phillips & Soltis, 1998).

In light of behaviourist learning theory, many teachers use a transmission mode of instruction and focus on student behaviour, or procedural learning, rather than understanding. The goal of instruction, according to these principles, is to convey facts and procedures efficiently to students (Even & Tirosh, 2002). The teacher, who is the source of knowledge, will often break down the information and skills into small increments for students to master before they present units that are more complex (Greeno et al., 1996). Practicing the new skills and procedures are an important part of the classroom, given that students must acquire knowledge before they can apply it in problem solving situations (De Corte et al., 1996). Since associations are strengthened through practice, it is important that mistakes are prevented and that students are not exposed to errors made by their peers (Even & Tirosh, 2002). Because learning depends on reinforcement of stimulus-response bonds, motivation is an important issue for behaviourists (Greeno et al., 1996). Behaviourist theory assumes student engagement in learning occurs because of extrinsic motivation, in the form of rewards, punishment, or incentives. Thus, reinforcement, either positive or negative, must be closely tied to a particular behaviour in order to be effective. In summary, in the behaviourist classroom students attempt to replicate the reality transmitted by their teacher and text book, by listening, practicing, and reciting what they have learned (Scheurman, 1998).

Constructivist Learning Theory

Constructivist theory rejects the behaviourist premise that learning is a stimulus-response phenomenon. In contrast, constructivists focus on the cognitive processes of the learner and how internal knowledge structures are built up. Early influences of constructivist thought on mathematics educators and researchers began to be felt in the 1960s and 1970s, while the full force of this revolution started in the early 1980s (Steffe & Kieren, 1994). The constructivist movement, as a whole, is complex and consists of a number of different factions (Geelan, 1997; Matthews, 2000; Phillips, 1995).

At its most basic, constructivism states that all knowledge is constructed. Contrary to the behaviourist position in which knowledge is independent of the knower and is transmitted unchanged from teacher to student, constructivists hold that "knowledge is actively constructed by the cognizing subject, not passively received from the environment" (Kilpatrick, 1987, p. 7). No matter what type of pedagogy is used, learners construct their own ways of knowing. The focus is on the nature or quality of those constructions (Cobb, 1994). Furthermore, as learners come to know, they use the process of organizing and adapting to respond to their experiential world (Kilpatrick, 1987). As people encounter new experiences or ideas that interact with their prior knowledge or beliefs, they create knowledge.

Within mathematics education research, the continuum of constructivist perspectives on the nature of knowledge construction ranges from individual cognition to sociocultural processes (Cobb, 2005). According to the cognitive or 'Piagetian' or personal (Geelan, 1997) or psychological (Phillips, 2000) constructivists, the development of knowledge is an individual process which takes place in the mind of the learner (Cobb, 2005). While the idea that knowledge is not the result of passive transmission dates back to Socrates (von Glasersfeld, 1991), cognitive constructivism finds its roots in Piaget's genetic epistemology (Steffe & Kieren, 1994).

According to Piaget (1977), change in cognition occurs through the process of *equilibration*. When learners encounter new experiences that conflict with their current knowledge or understandings, they enter the "nonlinear, dynamic 'dance' of progressive equilibria, adaptation and organization, growth and change" (Fosnot & Perry, 2005, p. 18) . Piaget posited that learners respond in one of three ways to this perturbation:

> They might ignore the contradictions and persevere with their initial scheme or idea; (2) they might waver, holding both theories simultaneously and dealing with the contradiction by making each theory hold for separate specific cases; or (3) they might construct a new, more encompassing notion that explains and resolves the prior contradiction. (pp. 19-20)

Growth and learning is characterized by the development of cognitive mental systems, or *structures*. Cognitive development and deep understanding are a result of the active reorganization of learner constructions through the process of *reflective abstraction* (Fosnot & Perry, 2005; Noddings, 1990). This growth is sporadic and uneven rather than smooth and predictable (Hiebert & Carpenter, 1992). Thus, learning is both complex and non-linear (Fosnot & Perry, 2005). In summary, the cognitive constructivist perspective typically stands for a gentle commitment to mentalism—a commitment to the belief that mental structures exist, that such structures shape the ways individuals see the world, and that people build those structures through interactions with the world around them. (Schoenfeld, 1992, p. 290)

One of the cognitive or psychological constructivists, Ernst von Glasersfeld (1990; 1991; 1995) has promoted a 'radical' version of constructivism. He maintains that knowledge is created in the mind of the individual, but also asserts that this knowledge is not an objective representation of reality. Von Glasersfeld defines knowledge as an *adaptive function* that helps us function in our experiential world rather than an objective truth of the real, observer-independent world (Hardy & Taylor, 1997). Since knowledge is in the mind,

> the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. What we can make of experience constitutes the only world we consciously live in…all kinds of experience are essentially subjective, and though I may find reasons to believe that my experience may not be unlike yours, I have no way of knowing that it is the same. (von Glasersfeld, 1995, p. 1)

While an objective reality does exist for von Glasersfeld, he believes that we are not able to know it (Hardy & Taylor, 1997). Rather than attempting to determine if a construction matches some ontological reality, von Glasersfeld suggests that cognitive structures are tested for viability against the world. The cognitive structure must fit with the world well enough in order to meet its goals. Nevertheless, according to Kilpatrick (1987) this version of constructivism is radical "because it rejects the metaphysical realism on which most empiricism rests. It requires its adherents forgo all efforts to know the world as it truly is" (p. 7).

Social constructivists disagree with the focus on the individual held by the cognitive constructivists (both Piagetian and the more radical shades) and claim that the development of knowledge is a social undertaking (Cobb, 2005). The process of knowing does not take place in a vacuum, since learners belong to social groups with whom they communicate and interact (Phillips & Soltis, 1998). Knowledge is constructed as individuals interact with the social milieu in which they are situated. Knowing, thus, takes place in a specific historical, social, cultural, and economic context (O'Loughlin, 1992). Constructivists at this end of the nature of knowledge continuum view learning as a process of enculturation into a community of practice (Cobb, 2005).

Social constructivism finds its roots in the work of Lev Vygotsky (1896-1934). While Vygotsky (1978) viewed knowledge as a human construction, he stressed the importance of social interaction.

> Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals. (p. 57)

Vygotsky also studied the role of dialogue, both as inner speech and as conversation with adults and peers, on the learning of concepts. He recognized the importance of language in the articulation of ideas and the development of cognition. He theorized that when learners interact with a more experienced member of a culture, their naturally developing functions or 'spontaneous concepts' are transformed into higher functions or 'scientific concepts' (De Corte et al., 1996). Vygotsky called this distance between the learner's independent or actual developmental level and the level of potential development under guidance as the zone of proximal development (ZPD). Thus both personal developmental level and social environment are important in the development of knowledge. Vygotsky described this development as follows:

> What lies in the Zone of Proximal Development at one stage is realized and moves to the level of actual development at a second. In other words, what the child is able to do in collaboration today he will be able to do independently tomorrow. (as cited in Greenes, 1995, p. 89)

As with cognitive constructivism, social constructivist views range in their degree of radicalism (Lawson, 2002). Some of this diversity can be related to relative emphases on the nature of reality (Geelan, 1997). Ernest (1992), for example, believes that although mathematical knowledge is an invention of humanity and thus a "social construction of our knowledge of reality" (p. 96), objective reality in mathematics exists as the knowledge in which a majority believes. In contrast, more radical social constructivists hold that knowledge is created through discourse and in societies rather than existing in the real world (Geelan, 1997). In order to address the lack of a traditional objectivity, these constructivists use the concept of intersubjectivity, or the shared meanings people construct in their interactions. By speaking about “conceptions that are *taken-as-shared*, which are built through *negotiation of meaning*, and which are accepted by the *community of practitioners”* (Sfard, 1998, p. 500), Sfard suggests that social constructivism responds to the charge of relativism levelled against the radical constructivists.

In response to the division between cognitive and social constructivists in education research, Cobb (2005) contends that each of these viewpoints provides the necessary background for the other. Depending on one’s unit of analysis, the focus is either on the cognizing individual or on the social and cultural context. Because “learning is both a process of self-organization and a process of enculturation that occurs while participating in cultural practices, frequently while interacting with others” (Cobb, 2005, p. 50), it is possible to coordinate the perspectives of social and cognitive constructivism. Fosnot and Perry (2005) further suggest that the focus should not be on which view takes precedence when analyzing learning, but rather the focus should be on the interplay between the two.

> We cannot understand an individual’s cognitive structure without observing it interacting in a context, within a culture. But neither can we understand culture as an isolated entity affecting the structure, since all knowledge within the culture is only to use Cobb’s terminology, “taken-as-shared”. (p. 28)

Regardless of the form of constructivist theory, it has had a strong influence on the way teachers are expected to teach in Canada and the United States.

Behaviourist and constructivist theories provide different models for understanding the nature of learning. Given our current insights into mathematical understanding, we have come to see the limits of the behaviourist model. With its focus on the behaviour of the learner, this theory suggests a transmission method of teaching which focuses on the procedural aspects of mathematics. Constructivism, on the other hand, acknowledges the importance of the cognitive processes of the learner and examines how the internal knowledge structures are developed. It provides a means for mathematics educators to gain insight into the conceptual understanding of learners. Consequently, this theory of learning significantly influences the thinking about current mathematics teaching practices.

Implications for Teaching

Many researchers contend that, "much of the failure in school mathematics is due to a tradition of teaching that is inappropriate to the way most students learn" (National Research Council, 1989, p. 44). Reformers argue that behaviourist methods of direct instruction, which emphasize procedural learning, directly contribute to a lack of understanding in mathematics. As a result, they look to constructivist learning theory for guidance in teaching for understanding.

Some critics argue that since constructivism is a theory of learning, it has a limited role in making meaningful decisions for teaching (Davis & Sumara, 2002; Lesh, Doerr, Carmona & Hjalmarson, 2003). Others acknowledge that one cannot abstract a set of instructional techniques from the theory of constructivism, yet believe that it is possible to derive general principles for educational practice (Cobb, Yackel & Wood, 1993; Fosnot & Perry, 2005; Pirie & Kieren, 1992). While teachers cannot transmit ideas to passive learners (Van de Walle, Folk, Karp, & Bay-Williams, 2011), teaching can be seen as 'perturbation' that initiates the process of equilibration in students. The following values, derived from constructivist learning theory by Fosnot and Perry (2005), can be used to inform and reform educational practices:

- Learning is not the result of development; learning *is* development. It requires invention and self-organization on the part of the learner…such as raising questions and generating hypotheses and models.
- Disequilibrium facilitates learning. "Errors" need to be perceived as a result of learners' conceptions, and therefore not minimized or avoided.
- Reflective abstraction is the driving force of learning…. This may be facilitated through journal writing, representation in multisymbolic form and/or discussion of connections across experiences or strategies.
- Dialogue within a community engenders further thinking.
- As learners struggle to make meaning, progressive structural shifts in perspective are constructed – in a sense, "big ideas". (p. 33-34, adapted)

Rather than speaking of 'constructivist teaching' we can use the concept of 'creating a constructivist environment' (Pirie & Kieren, 1992). This acknowledges that students actively create their own knowledge, while recognizing the changed role of the teacher. Pirie and Kieren also suggest that constructivist learning theory provides several foundational values for teachers who intend to create such an environment. First, teachers need to be aware that progress towards mathematics learning goals may not be achieved by all students and may proceed in unexpected directions for others. In light of this, teachers need to be continually re-creating the environment as they evaluate the learning taking place. Second, teachers' actions should reflect the belief that there are different paths to the similar mathematical understanding. No one path is best or suits all learners. Third, teachers need to recognize that different people hold different mathematical understandings. Each student has a unique way of understanding. Finally, teachers must be aware that achieving understanding is not 'once and for all' since there are different levels of understanding. Learners may revisit or 'fold back' to previous understandings when they encounter new concepts.

Reform-Based Mathematics Instruction

The current reform movement in mathematics education, as initiated by the National Council of Teachers of Mathematics, has been influenced by the work of Piaget and Dewey. It emphasizes the use of problem solving to develop understanding. Reformers suggest that major changes need to be made to the traditional mathematics classroom in order to foster greater understanding. The NCTM Standards outline the necessary shifts in instructional practice to make the change from a traditional behaviourist approach to a one that creates a constructivist environment:

- toward classrooms as mathematical communities – away from classrooms as simply a collection of individuals;
- toward logic and mathematical evidence as verification – away from the teacher as the sole authority for right answers;
- toward mathematical reasoning – away from merely memorizing procedures;
- toward conjecturing, inventing and problem solving – away from an emphasis on mechanistic answer-finding;
- toward connecting mathematics, its ideas and its applications – away from treating mathematics as a body of isolated concepts and procedures. (NCTM, 1991, p. 3)

Classrooms of teachers who make these shifts, and promote teaching and learning with a focus on understanding, share some foundational characteristics (Hiebert et al., 1997). Hiebert et al. elaborate on these features and provide glimpses into reform classrooms that promote understanding. At the core of reform-based instruction is the mathematical task. Genuine problems that connect to where students are at, in both interests and mathematical understanding, engage students in thinking about important mathematics. In this process, teachers no longer transmit knowledge to students but act as guides, providing appropriately sequenced tasks that move students towards mathematical goals. Teachers use questions to prompt students to look for connections with prior knowledge and to challenge them to make sense of contradictions in thinking. Students are encouraged to develop their own methods for solving problems and are further challenged to share and defend their mathematical ideas within the classroom. In this context, mistakes become sites for further learning. Communication within the classroom fosters reflection on methods and mathematical ideas, since correctness is determined by the logical mathematical argument. As students work on problems, they are encouraged to make use of mathematical tools to help them think about the mathematics and to record their work. In the reform classroom, all students have the right to participate and are expected to share their thinking with the class.

Evidence of Understanding

Reformers claim that students will develop a deeper understanding of mathematics as a result of this new teaching environment. Yet, because of the variety in program design and instructional goals, it has been difficult to compare the relative effectiveness of reform programs in contrast to traditional instruction (Hiebert, 2003). As standards-based curricula (based on the reform philosophy) were developed, empirical studies explored initial trends in student achievement (e.g., ARC Center, 2002; Carroll, 1997; Fuson, Carroll & Drueck, 2000; Riordan & Noyce, 2001; Schoen, Fey, Hirsch & Coxford, 1999). These researchers found that students in the standards-based programs performed at least as well as students in traditional programs on basic skills, refuting the concern of those in the 'back to basics' movement. Nevertheless, implementation of a new program may vary from teacher to teacher, and does not guarantee a shift to a constructivist-oriented learning environment.

In early studies, researchers used teaching experiments, case studies, or student interviews to describe and assess the efficacy of reform or constructivist-based instruction (Lawson, 2002). Individual studies have shown positive results in specific domains (grade level or topic) but research has not been conducted in all areas to give the needed convergence of findings (Hiebert, 2003). Nevertheless, substantial research has been done in reform-based programs of arithmetic at the primary level (e.g., Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Cobb et al., 1991; Kamii, 2000; Kamii, 1989; Kamii, 1994; Wood & Sellers, 1996; Wood & Sellers, 1997). Carpenter et al. (1989) used an experimental treatment to determine if teachers' understanding of children's thinking about mathematics, as facilitated by participation in a month long workshop, had an effect on student achievement. Experimental teachers were encouraged to implement principles of instruction that reflected a constructivist learning theory, while control teachers continued using their traditional methods. Pre and post tests were administered to students in all classes using both standardized tests and experimenter-constructed scales to assess problem solving abilities. After observing 40 grade 1 teachers in 24 different schools over the course of a year, Carpenter et al. found that students in the experimental classes achieved higher results in number fact knowledge and problem solving than did students in the control classes.

Similarly, Cobb and his colleagues (Cobb et al., 1991) compared the mathematics achievement of grade 2 students in ten constructivist-oriented classes with eight traditional classes over the course of one year. Two different arithmetic tests were administered in order to assess the students in the three different schools. The first, ISTEP, a standardized multiple choice test, consisted of two subtests; one for computation, and the other for concepts and applications. The second instrument was an arithmetic test developed for the project, which consisted of two scales, Instrumental and Relational, based on Skemp's (1976) distinctions. The relational items assessed student's conceptual understanding while the instrumental items assessed the ability to solve computational algorithms. Cobb and his colleagues found that both groups of students performed equally well on the computation items, but the students in the constructivist-oriented classrooms showed significantly higher levels of conceptual understanding and had distinctly different solution methods. This program of study was extended to include grade 3 students after two years of project instruction (Wood & Sellers, 1996) and a longitudinal analysis (Wood & Sellers, 1997) was conducted to deepen the knowledge of students' mathematical understanding and achievement at various points in the project. In these projects, Wood and Sellers found that students in the constructivist-oriented classrooms scored significantly higher on all the achievement tests than did the control students in the traditional textbook-based classrooms.

While Cobb and his colleagues operate from a social constructivist theory, others, such as Kamii (1989, 1994, 2000) employ a cognitive or Piagetian constructivist framework. As a result, even though Kamii acknowledges the need for social interaction, she stresses the development of a child's logico-mathematical knowledge or cognitive reasoning since mathematical understanding occurs in the mind. Therefore, the classroom practices of both cognitive and social constructivists are similar, despite the different emphasis on the role of social norms. Kamii found that students in Grades 1, 2 and 3, 'reinvented' algorithms for addition, subtraction, and multiplication in the course of problem solving. These students were not taught traditional algorithms, but were able to invent their own methods based on their understanding of numbers and place value. In order to assess mathematical understanding in a constructivist-oriented environment Kamii conducted research in single classrooms at Grades 1, 2, and 3. She matched these classrooms with control classes that were traditional textbook-based environments and found that students in the constructivist-oriented classes had a stronger conceptual understanding of problem solving and were able to reason more logically about their solution methods.

The call for reform or constructivist based mathematics instruction and the subsequent evaluation of programs is not just a North American phenomenon. For example, in England, Boaler (1998) compared the mathematical understandings of students in a reform-oriented school with those in a traditional one over the course of three years. She found that students in the reform-oriented school performed as well as students in the traditional school on content based, closed questions. Furthermore, students in the reform-oriented school showed a greater conceptual knowledge and were more able to apply their knowledge to real-life or novel situations than were students in the traditional school. In Germany, Staub and Stern (2002) explored the relationship between teachers' pedagogical content beliefs and student learning. Students in 27 classrooms were assessed on word problems and arithmetic tasks at the end of grades 2 and 3. Teachers of these 496 students completed a questionnaire to determine the degree to which their pedagogical content beliefs fit the cognitive constructivist or direct transmission model of learning. Staub and Stern found that students made larger gains on the word problems with teachers who were constructivist oriented. Moreover, students did equally well in computational fluency in both types of classrooms.

The results of these various studies give credence to the assumption that teaching environments based on constructivist learning theory and the reform principles, as delineated by the NCTM and other reformers, lead to improved student learning and understanding in mathematics. In conjunction with the growing evidence of improved learning, we also have an emerging picture of the development of mathematical knowledge in children in these classrooms.

Development of Mathematical Knowledge in Reform-Based Environments

Carpenter, Fennema, Franke, Levi and Empson (1999) contend that children do not think about mathematics in the same way that adults do. They suggest that when children are encouraged to construct their own understanding they begin with acting out the situation that requires addition or subtraction. By directly modeling the situation, children are able to use different strategies to solve mathematics problems. Using their research with children, Carpenter et al. have identified different types of problems and the different strategies children use to solve them. Thus, Carpenter et al. have been able to map out how basic number concepts and skills develop in grade school children. They claim that children are able to construct the different strategies by themselves if they are in an environment that values children's thinking and creativity. Children have a wealth of intuitive or informal knowledge of mathematics that serves as the basis for their developing understanding of the basic operations of addition, subtraction, multiplication and division. While children begin by directly modeling the actions and relations in the problems, eventually they develop more efficient and abstract counting strategies that in time become number facts. Carpenter et al. maintain that number relations should develop out of children's understanding of the concepts, not out of rote memorization. Children need the opportunity to develop a strong foundation through modeling and using counting strategies in order to develop a conceptually rich sense of number and place value.

Fosnot and Dolk (2001a; 2001b; 2002) build upon Carpenter et al.'s (1999) model, but they do not have the same perspective on children's mathematical learning trajectories. They believe that when viewed through the lens of constructivism, mathematical learning and development in a reform environment is messy-- not linear. There is no ultimate developmental progression or fixed recipe for teaching and learning mathematics, which will lead to the best understanding in children (Fosnot & Dolk, 2002; Van Den Heuvel-Panhuizen, 2003). Children learn at different rates and think in different ways. As they learn new material, children revisit prior concepts, reorganizing and restructuring the big ideas, coming to higher levels of understanding. This theory has been referred to as folding back (Pirie & Kieren, 1992) or *progressive mathematization* (Fosnot & Dolk, 2002; Van Den Heuvel-Panhuizen 2003). Yet, Sfard (1998) maintains that there are some invariant commonalities in the development of children's mathematical understanding.

> When students' learning is watched with an unprejudiced eye, one notices time and again that certain ways of thinking about mathematical ideas, far from being mere 'mistaken conceptions', are natural, inevitable stages in concept development. To put it differently, there are learning invariants that can hardly be changed by the way concepts are taught and, more often than not, display a striking similarity to early notions suggested by mathematicians themselves through history. (p. 499)

Fosnot and Dolk (2001a; 2001b; 2002) use the metaphor of *landscape* and *horizon* to capture this reality. For them, mathematics learning is developmental and goal oriented, always looking towards the next horizon. It takes place in a flexible and open context that has room for individual divergence and exploration. Fosnot and Dolk root this framework in three main concepts: *strategies*, *big ideas*, and *models*. Learning may proceed differently for children, yet there are certain landmarks they need to encounter and construct before they can successfully develop new concepts. Children employ a variety of strategies as they work within the context of their mathematical landscape. Models provide children with the necessary tools to help them organize and interpret their world mathematically. As children progress in their development of strategies, they begin to understand and construct big ideas. According to Schifter and Fosnot (1993), big ideas are "the central, organizing ideas of mathematics – principles that define mathematical order" (p. 35). However, big ideas are not limited to the structure of mathematics; they also reflect key shifts in students' thinking and reasoning about mathematics. When they encounter these big ideas, students are forced to struggle with new ways of interpreting relationships. Thus, these ideas are critical in helping students progress in their mathematical understanding.

In the case of division, for example, children will use a variety of strategies. Fosnot and Dolk (2001a) present the problem of finding out how many video games a friend sold, when she made $328 by selling them for $8 apiece at a yard sale. Many children began by using repeated addition; adding eights by doubling until they reach 328. Some even used tally marks to count groups of eight. Other children used repeated subtraction, beginning with 328 and subtracting 8 until they reach zero. Still others recognized that they could be more efficient if they used multiplication. Several students realized they knew that six times eight was forty-eight. They added groups of 48 until they reached 288; then added a group of 40 to make 328. While this strategy was more efficient, the numbers were still difficult to deal with. A key strategy recognizes the importance of 10 in our number system. Students who understand place value will find multiplying by ten to be a friendlier strategy, since they recognize the resulting pattern. In this example, some children knew ten times eight was eighty. They added 80 together 4 times to get 320, and knew that 40 games plus one more must have been sold to earn $328. Finally, the next strategy shows the beginnings of the long division algorithm. One student knew that four times eight was thirty-two. He then knew that ten times this is 320, so 40 games were sold, plus one more for a total of 41 games. In these strategies, students' thinking shifts when they are able to recognize the importance of multiplying or dividing by ten. They are constructing a big idea when they can connect place value to multiplying and dividing by ten. The learning paths individual students take will be different as they explore and come to understand the various strategies for themselves. Together, the strategies, models, and big ideas provide landmarks for the teacher to help her students explore the landscape of learning and to plan for the next mathematical horizon. Supporting students along this continuum is a demanding process for teachers.

Teacher Knowledge in the Reform Learning Environment

As teachers shift to a more constructivist-based practice, they begin to realize that they cannot simply 'transmit' mathematical ideas to their students. They need to provide problem-solving experiences where students can explore and wrestle with concepts to develop their own understanding. However, "such a practice cannot rely on predetermined scripts but depends on one's capacity to respond spontaneously to students' questions and discoveries" (Schifter & Fosnot, 1993, p. 197). This requires a rich and well-connected understanding of mathematics (Ma, 1999; Manouchehri, 1997). Teachers need to have sufficient content knowledge, pedagogical content knowledge, as well as a special knowledge of mathematics for teaching. As Schifter and Fosnot (1993) assert, this "new paradigm for mathematics instruction can be enacted only when teachers themselves grasp the big ideas, internalize the models, and then put them into play" (p. 197).

Content Knowledge

Teachers need an understanding of mathematics that is rich in the type of connections that Hiebert and Carpenter (1992) have described. Liping Ma (1999) elaborates on these connections in her model of *profound understanding of fundamental mathematics* (PUFM). Based on her interviews with 23 American and 72 Chinese elementary mathematics teachers, Ma depicts a content knowledge that has breadth, depth and thoroughness. "Breadth", according to Ma, refers to connections made between ideas of similar or less conceptual power, while "depth" refers to links made with ideas that are more profound and powerful. In addition, the topics need to be interwoven to provide a thoroughness that enables teachers to see mathematics as a conceptual whole. Teachers who have a profound understanding of fundamental mathematics will not only stress the connectedness of the mathematics, they will also encourage flexibility through the use of multiple perspectives and will recognize the "simple but powerful basic concepts and principles of mathematics" (Ma, 1999, p. 122) (similar to Fosnot and Dolk's (2001, 2002) "big ideas"). This understanding is dynamic and driven by context. Ma suggests that teachers who have a profound understanding of fundamental mathematics are like proficient taxi drivers who have a thorough and flexible knowledge of the city that enables them to drive efficiently to specific locations, using alternate routes as necessary.

Pedagogical Content Knowledge

In addition to the mathematical content knowledge, teachers also need to know the pedagogical strategies that will enable their students to develop understanding of specific concepts. Shulman's (1986, 1987) framework of teacher knowledge identifies pedagogical content knowledge as a category of special interest that blends content and pedagogy. Pedagogical content knowledge includes

> the ways of representing and formulating the subject that make it comprehensible to others…an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring

> with them to the learning of those most frequently taught topics and lessons. (Shulman, 1986, p. 9)

This special type of knowledge bundles mathematical content knowledge with knowledge of pedagogy, learners, and learning (Ball & Bass, 2000). Pedagogical content knowledge assumes a strong and explicit understanding of mathematical concepts. This explicit knowledge helps teachers decide what is important to teach, what sequencing is most beneficial, and what problems will help to facilitate student understanding of specific concepts (1998).

Liping Ma's (1999) work also contributes to our understanding of pedagogical content knowledge in mathematics. She describes the "knowledge packages" held by the 72 Chinese elementary teachers she interviewed. These knowledge packages display the organization, connections, and development of related ideas in a specific area of arithmetic. They also intertwine conceptual and procedural topics. The knowledge package provides a sequence for developing ideas. The package for division by fractions begins with the meaning of addition, proceeds to the meaning of multiplication of whole numbers, then to the meaning of multiplication of fractions, and ends with the meaning of division with fractions. Within the packages key ideas "weigh more" than other pieces. For example, special attention is given to the first time a concept or skill is introduced. Another key part of the knowledge package is the "concept knot". This idea ties the other important pieces together. In the division by fractions example, the Chinese teachers identified the meaning of multiplication of fractions as the idea that linked the concepts pivotal for the meaning of division by fractions: meaning of multiplication, models of division by whole numbers, concept of a fraction, concept of a whole, and the meaning of multiplication with whole numbers (Ma, 1999, pp. 115). This understanding of knowledge packages "represents a particularly generative form of and structure for pedagogical content knowledge" (Ball et al., 2001, p. 449).

While pedagogical content knowledge can help teachers anticipate effective teaching and learning strategies for specific concepts, Ball and Bass (2000) argue that it may fall short in the classroom. No one can anticipate all possible ways students may think, no matter how extensive their pedagogical content knowledge repertoire. Furthermore, it is possible for teachers to "know" the pedagogical concepts of reform mathematics and still not appreciate the need for guided discussion to foster mathematical understanding and development (Grouws & Schultz, 1996). Teachers need to be aware that their conceptions of mathematics and their preconceptions of students may hinder their appropriate use of pedagogical content knowledge.

Mathematical Knowledge for Teaching

Pedagogical content knowledge provides a model for discussing the intersection between content and pedagogy in mathematics teaching. However, it does not elaborate on the specific mathematics knowledge that is necessary for teaching. Unfortunately, there has been no consensus on the requisite mathematical knowledge for teaching (Fennema & Franke, 1992). Historically, researchers empirically examined the number of mathematics courses teachers had taken as a proxy for content knowledge. The more mathematics courses a teacher had taken the more (or better) content knowledge they were assumed to have. The best known of these studies, that of Edward Begle (1979), suggested these assumptions were inaccurate. His meta-analysis of the studies carried out between 1960 and 1976 concluded that advanced mathematical understanding, that is, additional mathematics courses past calculus, had little impact on teacher effectiveness. Nevertheless Begle also noted that the studies assumed that the teachers had a thorough understanding of the subject matter, even though researchers did not discuss the necessary depth of understanding teachers needed. This focus on teachers and their mathematical qualifications does not adequately address the issue of the nature of teachers' mathematical knowledge (Ball et al., 2001).

In the 1980s attention in mathematics education shifted to focus on teachers' knowledge of mathematics in and for teaching rather than on qualifications. Scholars began to observe and videotape teachers in context to analyze their practice and to infer their mathematical knowledge for teaching based on this evidence (Hill, Sleep, Lewis, & Ball, 2007). Researchers such as Borko et al. (1992) and Leinhardt and Smith (1985), used rich description to highlight teacher knowledge in areas such as fractions, multiplication, rate, and place value. A second line of research methodology used mathematical tasks and interviews, where participants may have been asked to think aloud as they solved the problem. Here the focus was not on the performance of individual teachers but rather on understanding the nature and extent of teachers' mathematical knowledge in specific content areas (Hill et al., 2007). The variety of tasks included tasks that mirrored those given to students (Graeber, Tirosh, & Glover, 1989), tasks that emphasized possible misconceptions (Tirosh & Graeber, 1989), and tasks that highlighted specialized mathematics for teaching (Ball, 1990a; Simon, 1993)

More recently, Ball and her colleagues (Ball & Bass, 2000, 2003; Ball, Hill & Bass, 2005; Hill, Rowan & Ball, 2005;) argue that there is a professional knowledge of mathematics for teaching that is different than the "common" math knowledge of a well educated adult. This specialized body of knowledge is also different from that needed in mathematically demanding occupations, such as carpentry, accounting, engineering, or physics (Ball et al., 2005; Hill et al., 2005). Furthermore, Ball and Bass (2000) argue that the historical gap between content and pedagogy, as articulated by Dewey, still challenges teachers today, since many teachers do not have a useable content knowledge of mathematics. While some teachers may understand the content, they are often not able to help all students learn by hearing students flexibly, using multiple representations of ideas, or choosing good mathematical problems. Thus, it becomes important to differentiate between "knowing how to do math and knowing it in ways that enable use in practice" (p. 94). In other words, *how* teachers hold their mathematical knowledge is as important as how much knowledge they hold (Hill & Ball, 2004).

Using an analysis of teaching practice, Ball and Bass (2000, 2003) were able to identify elements of mathematical content knowledge that were not readily apparent from an examination of a school curriculum. They discovered that teaching was more than knowing content and having the appropriate pedagogical content knowledge; teaching, itself, was a type of mathematical work. Choosing tasks to assess student understanding, interpreting and evaluating students' non-standard mathematical ideas, and making and evaluating explanations are a few of the activities that require teachers to deeply understand and use mathematics as they solve these problems of teaching. In addition, mathematical language features prominently in teaching. Teachers need to have a specialized fluency in this language, knowing what is an appropriate mathematical explanation, knowing when informal vocabulary is permissible and when technical vocabulary is appropriate, and knowing how to use symbols fittingly (Ball et al., 2005).

Several essential elements of knowing mathematics for teaching become apparent within the context of teaching as mathematical work (Ball & Bass 2000, 2003). First, teachers need to be able to decompress or unpack their own mathematical knowledge from its compressed symbolic form (Ball & Bass, 2000). This means they may need to "think from the learner's perspective and to consider what it takes to understand a mathematical idea for someone seeing it for the first time" (Ball et al., 2005, p. 21). Second, mathematical knowledge for teaching must be connected across the different domains of mathematics and lead to the building of links and coherence for students (Ball & Bass, 2003). For example, teachers may help students see that regrouping in subtraction is similar to renaming fractions when finding equivalent forms. Teachers must also see the connectedness of mathematical knowledge as they anticipate how mathematical ideas change and grow over time. For example, teachers often tell their primary students that they cannot subtract a larger number from a smaller number. Yet this statement, which may be true when dealing with whole numbers, becomes false when students learn about integers. Finally, mathematical practices, such as representing ideas, noticing patterns, and developing and using definitions, are important components of mathematical knowledge (Ball & Bass, 2000, 2003). Teachers need to pay attention to these practices to facilitate mathematical learning in students.

In summary, mathematical knowledge for teaching is the mathematical knowledge teachers use as they work in the classroom. It is distinct from the common content knowledge held by many adults and consists of the actual mathematical content or common knowledge of mathematics, and the specialized content knowledge (Hill & Ball, 2004; Hill et al., 2005; Ball et al., 2005). Teachers need to have a solid foundation of mathematical content in order to develop the requisite pedagogical content knowledge and mathematical knowledge for teaching. All three of these areas require a deep understanding that enables teachers to make connections among content, domains, and representations that will help them teach for understanding.

Limitations of Pre-service Teachers' Mathematical Knowledge

Unfortunately, numerous studies have shown that elementary teachers often lack an in-depth conceptual understanding of the mathematics content they are required to teach, even though they may have strong procedural knowledge (e.g., Fennema & Franke, 1992; Heaton, 1992; Izsak, Orrill, Cohen, & Brown, 2010; Ma, 1999; Mewborn, 2003; Post, Harel, Behr & Lesh, 1988). Not surprisingly, this lack of rich understanding of mathematics is also present in pre-service teachers (e.g., Ball et al., 2001; Brown et al., 1990; Newon, 2008) and contributes to their inability to incorporate teaching practices that differ from those they experienced as learners (Frykholm, 1999). The next sections will summarize the research on pre-service teacher knowledge related to fraction concepts, beginning with whole number multiplication, followed by division and concluding with rational numbers. Similar studies can be found in the other strands of mathematics (e.g., Ball et al., 2001; Baturo & Nason, 1996; Cooney, 1999; Mayberry, 1983; Reinke, 1997; Schmidt, 2002; Van Dooren, Verschaffel & Onghena, 2002).

Multiplication

Pre-service teachers, for the most part, have a solid procedural understanding of multiplication with whole numbers (Ball, Lubienski, & Mewborn, 2001). They are able to perform the procedures correctly and have little difficulty with multi-digit multiplication. However, some may have a limited understanding of the underlying concepts. Ball (1988, 1991, 2001) interviewed 19 pre-service teachers on the issue of multiplication. Using a three-digit multiplication question to elicit their understanding of place value in context, Ball asked them what they would do if their students did not 'move the numbers' (partial products) for each line in the multiplication process. Some of the pre-service teachers had an incomplete understanding of place value. They talked about making sure the digits lined up correctly, moving the numbers over, adding a zero, or using zero as a "placeholder". Since the traditional algorithm uses 'short cuts' that hide the conceptual meaning of the procedure, some of the pre-service teachers had difficulty articulating the use of place value and the distributive property in the partial products.

Sowder et al. (1998) outline the challenges for prospective teachers when teaching and learning multiplicative concepts. Learners need to reconceptualize the unit when they transition from additive to multiplicative reasoning. This means that they must begin to think in terms of composite units; using two levels of units they must first understand that the number three, for example, can be thought of as three single units or one group of three. Then, using multiplicative thinking, learners must simultaneously reason with three levels of units (Kamii, 1994; Izsak, 2008). This hierarchical or nested thinking can be seen in the example of 4 x 3; the operation can be thought of as a single group of 12, composed of four units where each of the four units is a composite unit composed of three separate units. Learners must begin to reason with *intensive* quantities, such as 3 candies per bag, in addition to the *extensive* or counting quantities, such as 12 candies and 4 bags, with which they are familiar, (Sowder et al., 1998). For these reasons, multiplicative thinking is slow to develop (Lamon, 1995).

In a teaching experiment with a class of prospective elementary teachers Simon and Blume (1994) focused on the development of multiplicative relationships. They found that the pre-service teachers struggled to connect area and multiplication conceptually because they focused on the memorized formula. These prospective teachers fell back on additive reasoning because they had difficulty with the fundamental multiplicative process of coordinating two dimensions.

Division

In a similar manner, pre-service teachers may have a strong procedural knowledge of division, but have a limited conceptual understanding of the process. Simon's (1993) study of 33 prospective elementary teachers revealed this lack of connectedness when he found that none of the participants were able to explain correctly why the traditional division algorithm worked. Likewise, they had difficulty interpreting or understanding the remainder and identifying its units.

Pre-service teachers' understanding of division tends to be limited to the partitive (how many are in each group?) or sharing model rather than the quotative (how many groups?) or measurement model (Ball, 1990b; Graeber et al., 1989; Simon & Schifter, 1993; Tirosh & Graeber, 1991). In the same study, Simon asked the participants to write three different story problems that would be solved by dividing 51 by 4 and for which the answer would be, respectively, 12 3/4, 13 and 12 (p. 239). Seventy-four percent of the resulting word problems created by the pre-service teachers reflected a partitive conception of division, while 17% reflected a quotative interpretation. This reliance on the partitive model of division may produce conceptual difficulties when encountering fractional divisors (Ball et al., 2001).

In another study involving 129 pre-service teachers, Graeber et al. (1989) found that pre-service teachers were influenced by the primitive models of division and multiplication, in the same manner that middle school children were. These teachers performed better on partitive division problems where the divisor was a whole number and less than the dividend. Some of the prospective teachers also held the implicit belief that division makes smaller. Many of the errors made by the pre-service teachers were similar to those made by schoolchildren. For example, just as some students overgeneralize the commutative property and believe that $15 \div 3$ is the same as $3 \div 15$, so did one of the interviewed pre-service students.

Tirosh and Graeber (1990) continued to explore pre-service teachers' thinking about division in another study. Three pencil and paper test instruments were administered to 58 pre-service teachers, and 21 teachers who agreed that the quotient must be less than the dividend, but were able to correctly solve a problem that violated this belief, participated in interviews. Tirosh and Graeber found that many of the prospective elementary teachers inappropriately applied their knowledge of whole numbers division to division with decimals. They suggest the traditional long division algorithm may reinforce the constraints of the primitive partitive model for those pre-service teachers who have a limited conceptual understanding of division. The algorithm only allows for whole number divisors. When dividing by a decimal, one "moves the decimal over" to generate whole numbers, so the final answer becomes less than the dividend. If the prospective teacher does not fully understand the equivalence-preserving effect of moving the decimal for both the divisor and dividend, they may reinforce the incorrect belief that one can only divide by a whole number and that division makes smaller.

Rational Numbers

Pre-service teacher knowledge. Pre-service teachers encounter many challenges when working with rational numbers that include fractions, decimals, and percents. Their fraction knowledge is often composed of memorized procedures or algorithms. In a study of 252 pre-service teachers from five different sites, Ball (1990a) examined knowledge of division of fractions. She found that only 31% of the prospective teachers were able to choose an appropriate representation for the problem $1\frac{3}{4} \div 1/2$. Furthermore, when intensive interviews were conducted with 35 of the participants, only four (11%) were able to develop an appropriate representation, while 54% were unable to create any representation. Many of the teacher candidates saw this question as primarily about fractions rather than division. They were not able to distinguish between dividing in half and dividing by one half. As stated above, these teacher candidates were limited by their use of the partitive model of division. In a separate study, Ball (1990b) conducted interviews with 19 prospective teachers about their understanding of division by fractions. Using the same task, $1\frac{3}{4} \div 1/2$, participants were asked to both solve and develop a representation for the problem. Seventeen of the prospective teachers were able to solve the problem correctly; however, only five were able to generate an appropriate representation and eight were unable to develop a representation. Their understanding of this concept tended to be rule bound and compartmentalized.

In another study, Graeber and Tirosh (1988) explored pre-service elementary teachers' performance on decimals in multiplication and division. A pencil and paper test instrument, similar to one used with middle school students, was administered to 129 pre-service teachers. Results show that the presence of a decimal was not necessarily a problem, but the role the decimal played in the word problem was critical. For example, 99% of the prospective teachers could give the proper expression for a problem solved by multiplying 2.25 by 15, but when the problem was solved by an expression with a decimal operator, such as 15 x 1.25 or 15 x .75 only 72% and 59 %, respectively, of the participants were able to give the correct expression. The results in both multiplication and division suggest that the pre-service teachers were influenced by the primitive models, which hold that multiplication makes bigger and division makes smaller. Thirty-three prospective teachers, including the ten highest and ten lowest scorers, were then interviewed and asked to explain their thinking on several similar word problems. Graeber and Tirosh found that some participants believed the numbers in the word problem influenced the operation used. "For example, the problem 'How far will a motorcycle travel on x liters of gas, if it travels 15.5 miles on 1 liter?' is solved by multiplication if x = 3 but by division if x = .75" (p. 266). The choice between multiplication and division depends on whether the students believe the answer should be larger or smaller. They would multiply to get larger numbers or divide for smaller results. The rote understanding of using the division algorithm to divide by a decimal contributes to the misunderstanding. After moving the decimal to get a whole number, as stated in the previous sections, some of the pre-service teachers believed the decimal must be moved back the appropriate number of digits in the quotient. Once again, their reliance on the primitive model leads them to believe that the answer must be smaller than the dividend.

Both Tirosh, Fischbein, Graeber, and Wilson (1998) and Stoddart, Connell, Stofflett, and Peck (1993) conducted broader studies of pre-service teachers' understanding of rational numbers. In a study of 147 prospective elementary teachers, Tirosh et al. found that most of the participants could successfully use algorithms to add, subtract, and multiply with decimals and fractions. However, they were less successful at dividing with fractions and decimals. Their understanding of the rational number concepts was procedural and bound to rules and algorithms. These prospective teachers were not able to explain or justify the various steps of the algorithms, nor were they able to generate appropriate representations of rational numbers or of operations with rational numbers. These results are similar to those of Stoddart et al. (1993).

Using a sample of 83 pre-service students, they found that less than 10% of the participants were able to give an accurate explanation of their solutions to the four conceptual questions that were posed. However, they also found that the teacher candidate had difficulty solving some of the more challenging procedural questions. For example, only 37% were able to make the following equivalent fraction statement true: ____/4 = 2/5.

Behr, Khoury, Harel, Post, & Lesh (1997) used the problem of finding 3/4 of a pile of 8 bundles of 4 counting sticks to explore pre-service teachers' understanding of the operator sub construct of fractions. They interviewed 30 prospective elementary teachers, posing three tasks based on this problem. The second and third tasks included the scenario that each bundle of sticks represented the boards used by one carpenter on the job in one day. The teachers were asked to use manipulatives to represent their solutions for: a) 3/4 of the pile of sticks; b) the number of boards used if only 3/4 of the carpenters came to work one day; and c) the number of boards used if all 8 carpenters came to work but only worked 3/4 of the day. While the researchers sought to link the strategies used by the participants to their in-depth analysis of the operator function, what is of note for this study is that the pre-service teachers found it difficult to distribute the fraction as operator across the conceptual unit. Only 12 of them used a strategy that found 3/4 of each bundle. In other words, more pre-service teachers focused on the number of units when dealing with units of units (finding 3/4 of 8 - bundles of 4) rather than operating on the size of the unit in a unit of units (finding 3/4 of each bundle of 4) even when the context seemed to elicit this type of strategy.

Foundational fraction concepts. Many of the pre-service teachers in these previous studies may not have internalized the foundational big ideas of fractions. Fractions are one of the more complex and challenging concepts for students to learn (Behr et al., 1993; Charles & Nason, 2000; Mack 1990). Numerous studies have shown that children have trouble initially understanding fractional concepts, and furthermore, this difficulty continues throughout their schooling and on into adult life. The multifaceted nature of fractions contributes to this complexity (Kieren, 1992; Lamon, 1999). Our traditional mathematics education has focused on the part-whole nature of fractions, even though different interpretations have been identified. A mature understanding of the rational number system, however, requires an integration of the part-whole, quotient, ratio, operator, and measure sub-constructs (Kieren, 1988; Pitkethly & Hunting, 1996). Because fractions are students' first experience with a set of numbers not based on some type of counting algorithm (Behr & Post, 1992) learners tend to use additive thinking as they engage in the concepts. Nevertheless, fractions have a multiplicative structure that reflects the relationship between the numerator and denominator (Empson, Junk, Dominguez, & Turner, 2005; Thompson & Saldanha, 2003). When learners attempt to create meaning out of fraction symbols and operations they often struggle with the relationships and the changing nature of the unit or the whole (Behr, Harel, Post, & Lesh, 1992; Fosnot & Dolk 2002; Fosnot 2007; Izsak, 2008). In addition, representing and interpreting fraction operations can be challenging for them since fractions occur in both continuous and discrete contexts (Behr & Post, 1992).

Since fractions are relations where the whole matters it is precisely the relationship between the parts and the whole, rather than the absolute magnitude of each part, that is important (Carraher, 1996; Clark, Berenson & Cavey, 2003; Clarke & Roche, 2009; Fosnot & Dolk, 2002; Post, Behr, & Lesh, 1986). When individuals do not understand this relationship they think in terms of *how many* instead of the more appropriate, *how much* (Mack, 1995). Individuals who think about fractions as relations are able to see them as meaningful objects rather than procedures to be carried out (Stephens, 2006).

In order to work meaningfully with fractions it is essential to see them as objects or quantities, recognizing the importance of the ratio and proportion interpretation (Empson, Levi & Carpenter, 2011; Steffe, 2002; Steffe & Olive 2010; Streefland, 1991; Tzur, 1999). In doing so learners shift from additive to multiplicative thinking (Streefland, 1993). They shift from focusing on the partitioned whole as a set to abstracting the invariant multiplicative relationship with respect to the whole (Tzur, 1999).

Empson et al., (2011) have extended this notion of fractions as relations to work within their relational thinking framework. Key elements to this understanding include the notions that "unit fractions are created by division or partitioning and that unit fractions are multiplicatively related to the whole" (p.414). When individuals have a relational understanding of fractions they are able to decompose and recompose fractional quantities when working with expressions or operations. Their thinking becomes anticipatory and goal oriented as they make choices about how to engage with the fractions. For example, when adding 1/2 + 3/4 these individuals may think of 3/4 as 1/2 + 1/4. They reason that 1/2 + 1/2 is equal to 1, then plus another 1/4 is 1¼ This is contrasted with individuals who use algorithms and simply think about what to do next.

Mathematical Knowledge for Teaching

Given a weak conceptual understanding of mathematical content and their lack of experience in the classroom, it can be assumed that many pre-service teachers have a limited special knowledge of mathematics for teaching. Most of these prospective teachers experienced the traditional mathematics classroom with direct instruction and an emphasis on procedures and rote memorization in their own education. As a result they may not have many positive models in using mathematics effectively for teaching.

Borko et al. (1992, 1993) present an example of what can happen when a prospective teacher has only a rote understanding of mathematics. In this case study, the pre-service teacher, Ms Daniels, is unable to provide an appropriate representation for division of fractions when responding to a student's question about the 'invert and multiply' algorithm. Instead, Ms. Daniels incorrectly presents a representation that depicts multiplication instead of division of fractions. When she recognizes her error, she does not tell the students that it models multiplication; rather, she simply encourages them to stick with the rule for now. Students are not encouraged to develop a deeper understanding of the concept because their teacher is unsure how to connect her procedural knowledge with a concrete representation. Ms Daniels does not have the necessary knowledge of mathematics for teaching.

As was shown earlier, many children who only have a procedural knowledge of mathematics concepts often develop 'buggy' algorithms. Similarly, pre-service teachers may be unaware of their incorrect application of partially understood algorithms. In a study that examined pre-service students' understanding of middle school mathematics, 52% of the 558 participants made one or more errors on the five fraction questions (Stienstra, 2005). Thirty-six of these prospective teachers incorrectly used a procedure or algorithm. For example, some incorrectly added fractions by adding across numerators and denominators; $3/4 + 2/3 = 5/7$. Others reversed the numerator and denominator when finding the decimal equivalent of a fraction; $3/4 = 4 \div 3 = 1.3$. If prospective teachers only have a rote knowledge of mathematical concepts they do not have the necessary mathematical knowledge for teaching to enable their future students to develop a deep conceptual understanding of mathematics.

Development of Pre-service Teachers' Mathematical Understanding

The preceding studies have demonstrated that many pre-service teachers tend to lack a conceptual understanding of the mathematics concepts they will eventually have to teach in the area of number sense. Some prospective teachers, upon recognizing their mathematical weaknesses, choose to engage in some form of remedial instruction. How will this learning differ from their previous experiences? These pre-service teachers have been through the mathematics education system and have experienced methods of instruction that were not beneficial for a conceptual understanding of certain mathematical ideas. Yet, they expect to learn the mathematical concepts at a deeper level this time around. Is this a realistic expectation? It may be that reform-oriented methods hold greater promise for adult students relearning mathematics.

Based on research into learning in the reform-based mathematics classroom, our conception of the development of mathematical learning and understanding has changed (Carpenter et al., 1999; Fosnot & Dolk, 2001a; Fosnot & Dolk, 2001b; Fosnot & Dolk, 2002). Students learn mathematics differently in a constructivist-oriented learning environment than in a traditional classroom. As a result of being encouraged to make sense of mathematical situations, students in reform-based classrooms tend to develop different strategies and mental models than students in traditional classes.

The concept of double-digit addition provides a concrete example of some of the differences that occur in mathematical development as a result of the learning environment. In the traditional mathematics classroom, double-digit addition is introduced as a set of rules where students first add the digits in the ones column and then add the digits in the tens column. For example, to solve 24 + 13, students would first add the 4 and 3 to get 7; then they would add the 2 and 1 to get 3; producing a final answer of 37. Students often have difficulty with place value since each column is thought of as single digits, rather than the actual value. When students are introduced to problems that require regrouping typical errors often occur. For example, when adding 24 + 17, students may respond with 311, since 4 plus 7 is 11; and 2 plus 1 is 3. They followed a correct procedure, but could not make sense of the necessary regrouping to account for the place value. When students do the regrouping correctly, many cannot explain what the "1" that was carried means (Kamii, 1989).

In contrast, a constructivist-oriented learning environment encourages students to invent their own procedures. Carpenter et al. (1999) illustrate how students may use different strategies, such as incrementing, combining tens and ones, or compensating, to solve multi-digit addition problems. To add 24 + 17, these students would work from left to right. Their thinking might be as follows: 20 and 10 is 30. Then I know 3 and 7 is 10. So 30 and 10 is 40. I have one more left over, so the answer is 41. In these instances the students have a knowledge of place value and use it as they combine the digits in the tens place. Fosnot and Dolk (2001b) reveal that students may use a number line to help them think about addition. For example, when adding 28 +15, students use several different strategies. In one of these strategies, the student adds 10 to 28 to get 38; then adds 2 to get 40; and finally adds 3 to get 43. Using the open number line helps the student envision the steps using convenient numbers.

In the reform-based environment students encounter different strategies and models as they work toward constructing an understanding of the big ideas in mathematics (Fosnot & Dolk, 2001a, 2001b, 2002). They have not memorized a set of rules or procedures. They develop greater autonomy as they come to a deeper conceptual understanding of the mathematical concepts. As pre-service teachers participate in a different environment to learn mathematics, it may be that they develop new mathematical models of knowledge similar to children in reform environments. They, like children, may develop a variety of strategies and models to replace their memorized rules and procedures. They will learn to use their own knowledge to generate procedures to solve new problems.

Little has been written on the nature and development of pre-service teachers' understanding of mathematical content knowledge within reform environments. A few researchers have found that reform-based mathematics methods courses can lead to improvement in mathematics content knowledge for elementary pre-service teachers (Newton, 2008; Quinn, 1997), but the pre and post test data do not indicate how this conceptual development takes place. Similarly, a growing number of other studies (Green, Piel, & Flowers, 2008; Osana & Royea, 2011) identify reform based interventions that improve specific aspects of mathematics content or pedagogy. Tirosh (2000), for example, has provided an extensive description of pre-service elementary teachers' knowledge of division of fractions and their knowledge of children's thinking processes before and after instruction. However, while many recent studies have explored the development of students' fraction knowledge from a reform-based perspective

(Empson et al., 2005; Fosnot & Dolk, 2002; Lamon, 2007; Steffe & Olive, 2010), we do not know if pre-service teachers' fraction knowledge develops similarly as they learn or relearn the fraction concepts they will eventually teach. It is this gap in the literature that the present study addresses.

Most of the focus in pre-service teacher development has been on their changing beliefs and attitudes rather than mathematical development. Ball (1990c) found that experiences in a reform-based methods course could help prospective teachers develop a new awareness of what it means to 'understand' something in mathematics. The following prospective teacher indicates that understanding does not mean merely knowing how to do something:

> mathematics to me was knowing how to get the right answer – not so much HOW to get it as merely GETTING it!! I felt that the best way to know math was to have a photogenic memory…. My greatest nuisance was story problems. I grew up hating them. Now my ideas are completely different. Mathematics is not memorization, it's more like methodical reasoning. Now one of my favorite kinds of problems is story problems because I can take it apart and figure out what it is asking for. Before I would take on the whole problem at once and *hope* I got it right. I was never able to look at it and say, "Yes! For sure this is right because…" Now I know WHY it's correct or if it's not. (p. 15)

Similarly, Wilcox, Schram, Lappan, and Lanier (1991) relate how the creation of a learning community in a reform-based environment contributed to a developing pedagogical content knowledge. As the pre-service students learned the mathematical content, they were able to reflect on their own learning and link it to children's learning. Wilcox et al. concluded that this type of learning environment contributed to significant changes in the teacher candidates' beliefs about what it means to know mathematics and how mathematics is learned.

The changes in beliefs and attitudes are important elements for pre-service teachers' developing conceptual understanding of mathematics for teaching, but they are only one part of the puzzle of effective professional development in mathematics instruction. In addition, we also need documentation of how pre-service students learn mathematics within a reform environment. Given that some adults have similar misconceptions and errors as children, it may be that these pre-service students will experience progressive steps similar to those shown in models of children's development in the reform classroom, as the prospective teachers deepen their understanding of mathematics. Some may need to begin with concrete, direct modeling experiences when they attempt to remediate faulty understandings. Others may begin with less basic landmark strategies or models as they move toward developing the big ideas. However, it may be that the pre-service students progress through the various stages more quickly or even skip some of the levels children pass through given their maturity and previous mathematical experiences. More information that documents how pre-service students' mathematical knowledge develops within reform environments will offer a foundation for designing effective professional development for improving teachers' content knowledge for teaching.

Summary

In summary, the NCTM and other current reformers promote a vision of teaching and learning that encourages a deep understanding of mathematics rather than the memorization of rules and procedures. The impetus for this change comes, in part, from the documented inability of students to make sense of mathematics or show sufficient proficiency. Students need to know more than the facts and procedures; they need to have a conceptual understanding that is rich in relationships. They need to develop strategies and reasoning that allows them to successfully solve and explain mathematical problems. In other words, mathematical reform calls for a mathematical proficiency rooted in deep conceptual understanding.

Since this type of understanding does not necessarily develop in the traditional transmission-based mathematics classroom, reformers suggest that a constructivist-oriented learning environment is preferable. A growing body of research suggests that reform-based instruction can lead to improved student learning and understanding in mathematics. In these learning environments, students are encouraged to construct their own understanding through the generation of models and hypotheses, dialogue, logical arguments, and the justification of mathematical ideas. Teachers act as guides rather than transmitters of knowledge. They provide genuine and engaging mathematical problem contexts and promote classroom environments that value inquiry, creativity, and communication. Here, invented solution methods are encouraged and all students are expected to share their thinking within their classroom community.

Children's mathematical knowledge develops differently within this reform-based setting than in the traditional classroom. When exploration and individual divergence are encouraged, children begin to employ a variety of strategies that relate to their prior experiences and their current level of understanding. They use models as tools to organize and make sense of their mathematical problem solving. As their thinking and reasoning shift, children begin to construct and understand big ideas. Their development of mathematical understanding continues as they struggle with new ways of interpreting relationships and move forward to the next mathematical horizon.

The process of supporting students along this developmental continuum is challenging for teachers. They need to have an understanding of mathematics that is well connected and profound. In addition to this solid foundation in the content, teachers need a firm knowledge of the specific pedagogical practices pertinent to mathematical learning. But even this is not enough. Teachers need a special knowledge of mathematics for teaching that enables them to unpack their own compressed mathematical knowledge, make connections across the domain, and represent ideas in ways that are meaningful to students.

Unfortunately, many elementary mathematics teachers and pre-service teachers lack this necessary in-depth conceptual understanding of mathematics. Researchers have documented limitations of pre-service teachers in the areas of multiplication, place value, division, fractions, and rational numbers. Pre-service teachers may have a solid procedural understanding of whole numbers, but many have a limited conceptual understanding of the process. For some this leads to the development of 'buggy' algorithms that are similar to those of children. In addition, mistakes and misconceptions with rational numbers often develop from an overgeneralization of whole number knowledge.

Pre-service teachers who attempt to remediate their mathematical knowledge by engaging in reform-based instruction expect to develop a deeper understanding of the concepts. Given that researchers have shown that mathematics learning occurs differently in a constructivist-oriented environment than in the traditional classroom, this is a realistic expectation. Yet little is known about how mathematical knowledge develops in these adults. Additional information about this development will provide a foundation for designing effective professional development for teachers who wish to improve their mathematics content knowledge for teaching.

Chapter Three: Methodology

This study explores prospective elementary teachers' emerging understanding of fractions over time, as they (re)learn foundational mathematics concepts in a reform-based learning environment. Participants were purposefully recruited from a remedial mathematics skills workshop in order to have the opportunity to observe growth from a number of different initial levels of understanding. The ensuing case study reflects both the critical nature of the fraction content knowledge and the ancillary specialized fraction knowledge for teaching, as well as the convenient nature of the sample (Creswell, 1998).

Research Setting

This research took place in a small Ontario university, where the Faculty of Education had an enrolment of approximately 750 students in its professional year education program. Students were either completing the last year of their four or five year concurrent undergraduate and Bachelor of Education degree or they had previously completed an undergraduate degree and were taking the one year education program in one of three streams, Primary/Junior (K-6), Junior/Intermediate (4-10) or Intermediate/Senior (7-12). All students in both the Primary/Junior and Junior/Intermediate programs were required to take a thirty-six hour mathematics methods course that spanned both the fall and winter terms.

In order to ensure that all its prospective primary/junior teachers have an adequate understanding of mathematics for teaching, the Faculty of Education required that these teacher candidates pass a mathematics content exam set at the Grade 6-7 level with a minimum of 75%. This exam consisted of 15 word problems, covering the five mathematical strands. It was written in September, on the second Friday of the fall term. While students were not permitted to use calculators, they were provided the use of manipulatives such as centicubes, ruler, fraction kit, and centimeter graph paper. Based on five years of practice, 20% of the primary/junior students fail this exam the first time they write it. They had an opportunity to retake it in the spring, at the end of the second term. Since demonstrating proficiency on this exam was a requirement for passing the mandatory primary/junior mathematics methods course, some students whose mathematics concepts were weak were motivated to enroll in mathematics tutoring to prepare for the exam.

The university offered a non-credit mathematics skills development course to help students prepare for the exam. This twenty-hour course emphasized teaching for understanding rather than the rote application of rules as it focused on basic mathematics concepts at the middle school level. The instructors of the skills workshop were experienced classroom teachers who sat in on the entire mathematics methods course in order to ensure consistency between the skills and methods courses. Typically, two sections of the skills development course were offered, each with a cap of twenty-five students. It is from this skills development course that the participants for this study were recruited.

Personal Location

My research questions were prompted by my experiences as a mathematics teacher, a mathematics tutor, and a mathematics teacher educator. As an instructor for various sections of the P/J mathematics methods course I wondered how students could make sense of the shift in pedagogical approaches to mathematics when they struggled with the content. As a mathematics tutor for the precursor to the current skills development course, I encountered many primary/junior education students who struggled with basic mathematics concepts. They faced challenges in the area of number sense. Many students were able to use memorized procedures or algorithms but often they did not understand why these procedures worked. In short, mathematics was frustrating and often mysterious for these students because it did not make sense. Yet, over the course of the tutorials, students were able to develop a deeper understanding of the mathematical concepts that had long eluded them. I became interested in the process of developing mathematical understanding. Were there specific concepts or progressions of ideas that contributed to the development of understanding? Why did students develop certain misconceptions? Did students progress through certain stages of understanding or misunderstanding? How did the context of reform-based teaching contribute to the development of understanding? In light of these experiences, my research focuses specifically on the development of mathematical understanding in these adults who have traversed the traditional mathematics education system and are still lacking a deep understanding of many foundational concepts.

Research Design

A qualitative case study design was used to explore and describe fourteen pre-service elementary teachers' developing understanding of basic fraction concepts. This design appropriately captures the intensive exploration of a system bounded by time and activity (Creswell, 2003) since it investigates the mathematical development that took place within the bounded setting of the reform-based mathematics skills development course for primary/junior education students during the 2006-07 school term. While some believe a *case* refers to the choice of what is to be studied (Stake, 1995) and others believe it is a process of inquiry or methodology (Yin, 2003), each instance requires the collection of rich data from multiple sources to provide an in-depth picture. In this study, the case is used instrumentally (Stake, 1995) to illustrate the process of developing a conceptual understanding of fractions, within the context of a constructivist-oriented environment. The design is an embedded case study model with multiple units of analysis (Yin, 2003). The 14 participants within the case become the first layer of analysis and are mini individual case studies (Patton, 2002), clusters of participants according to their mathematical abilities function as a second layer of analysis, while the group as a whole provides the larger picture necessary for the development of a continuum of conceptual understanding. Figure 1, adapted from Yin (2003), provides an overview of the case study design. In order to develop an in-depth understanding of the case it was necessary to collect multiple forms of data (Creswell, 2003; Stake, 1995; Yin, 2003). This study used pre and post tests, interviews, modified think aloud protocols, problem solving journals, and open ended questionnaires as sources of data. These multiple sources of evidence serve to provide construct validity for the study (Yin, 2003).

Data collection was designed to ascertain the developmental progression of mathematical understanding in the participants. Figure 2 provides a flow chart of the data collection process, which began in September 2006. Each piece of data gives a snapshot of the participants' understanding at that given moment in time. Thus, the various pieces of data were collected at significant times throughout the course of the school year. For example, the first paired problem-solving interview took place after the intensive first half of the math skills course, which addressed fraction concepts, while the second paired problem-solving interview took place after both the first practice teaching experience and the completion of the numeracy topics in the skills course.

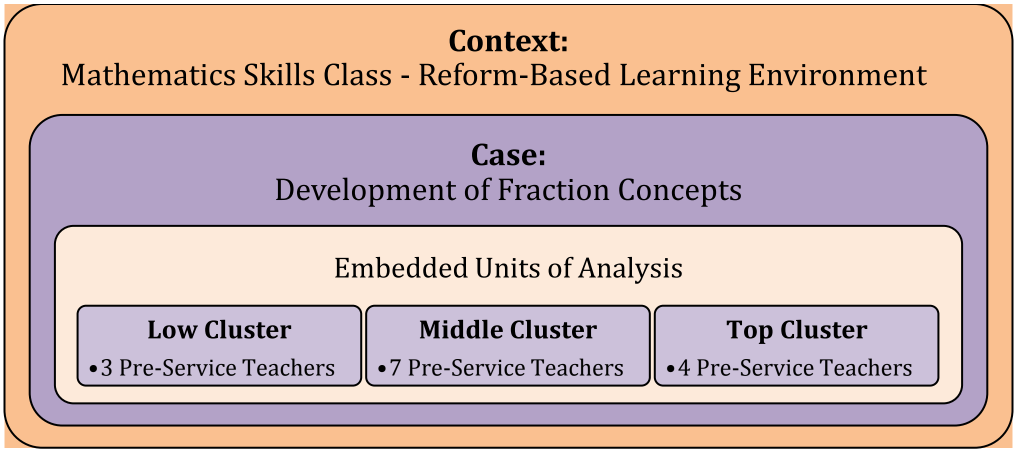


Figure 1. Case Study Design

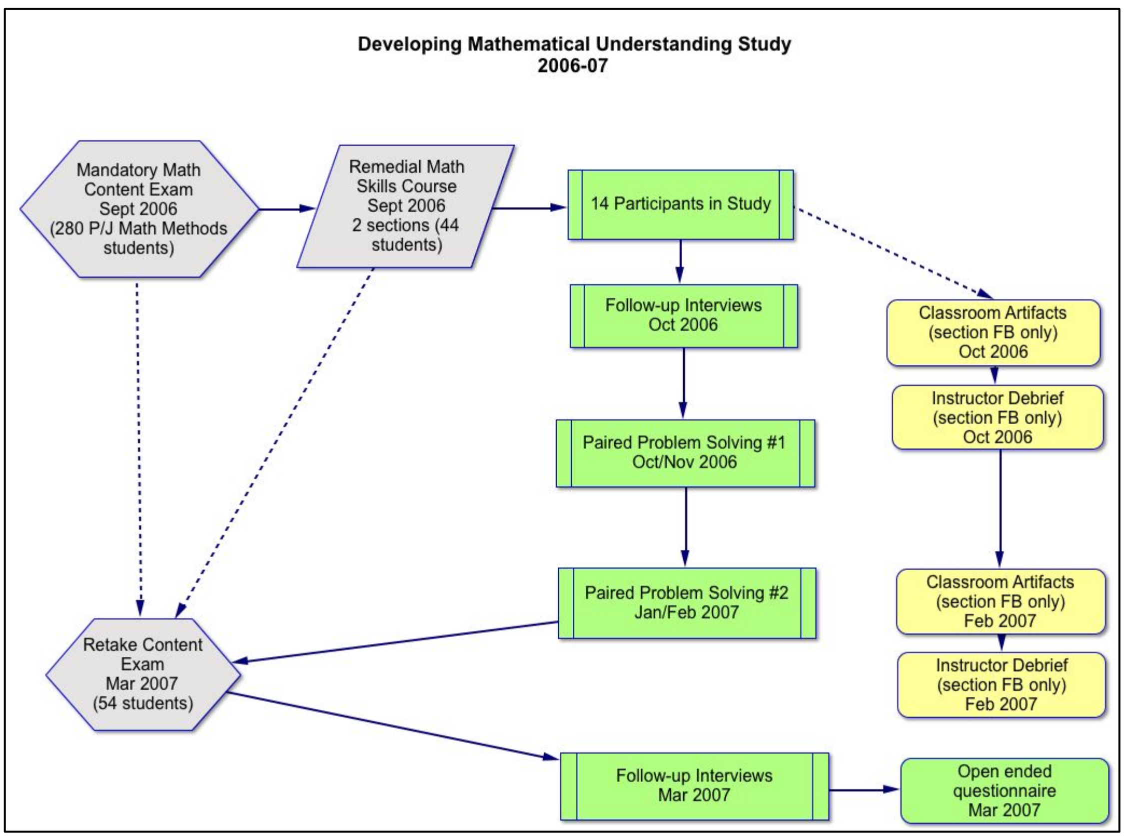


Figure 2. Data Collection Flow Chart

Participants

Participants were recruited from the 44 education students enrolled in the two sections of the Mathematics Skills Development Course for the 2006-07 school year. A minimum of ten participants was required in order to develop a landscape of learning that reflected the broader case rather than the individual participants. Fourteen students volunteered and all were accepted into the study. The participants ranged in age from 22 to 43 years old and all but one were female. Most of the participants were enrolled in the skills course because they had failed the initial mathematics content exam; however, three of them did not need to retake the content exam and were there for personal growth in understanding mathematics and children's understanding of mathematics. Two of these participants were from the Junior/Intermediate (4-10) program, which, at the time, did not have the same requirement for the mandatory mathematics content exam. Appendix A provides the complete demographics of the participants.

As an incentive, participants who completed the entire study received $60 each for their participation. One participant refused this money because she felt she had benefited so much from the interviews and problem solving sessions. All fourteen participants completed the study. However, one participant did not engage in the final follow-up interview due to timing constraints.

Course Overview

The 20-hour non-credit mathematics skills course provided opportunities for the pre-service teachers to increase their understanding of math concepts through hands-on activities, discussion, and problem solving. Basic mathematical concepts at grade levels four (4) through seven (7) were addressed; however, algorithms and memorization were not taught. The class generally met for 1.5 hours once a week; six times during the fall term, from the end of September to the beginning of November, and six times during the winter term, from the middle of January to the middle of February. A final 2-hour class provided participants an opportunity to take a practice review test and receive immediate feed back on the results. In the fall term, classes 2, 3, and 4 focused directly on fraction concepts. The pre-service teachers constructed a fraction kit and solved word problems that included addition, subtraction, division, and multiplication by using fractions as operators. Classes 5 & 6 addressed the concepts of decimals and percents. In the winter term, class 7 addressed the concepts of ratio and proportion. Class

12 was a mixed review that also included fraction concepts. The remaining classes focused on concepts of measurement, geometry, algebra, probability, and data management.

Participants worked with fractions concepts before both of the paired problem-solving interviews. They completed three classes (4.5 hours) on fraction concepts in the mathematics skills course before they engaged in the first paired problem-solving interview. They also completed six hours on issues of teaching and learning fraction concepts in their methods course before the second problem-solving interview. In addition, before this second interview participants engaged in at least 3 hours of work on decimals, percents, ratios, and proportions in their skills course.

Ethical Considerations

This study complied with all ethical procedures as outlined by the University. The project underwent an ethical review by the Research Ethics Board at Lakehead University prior to data collection. All participants signed letters of informed consent (see Appendix B). Pseudonyms are used in this report to maintain their anonymity. Participants were free to withdraw from the study at any time without any penalty. No potential harm was foreseen for the participants. The study was carried out in order to develop a continuum of conceptual understanding based on reform methods of mathematics instruction rather than to evaluate the performance of individual participants.

During the research project I taught two of the eight sections of the primary/junior mathematics methods course. This provided the potential for a conflict of interest between my research and my authority as an instructor. Since five of the participants in the study were also students in my methods course, to minimize possible conflict I did not analyze any data until the methods course had finished and these participants had passed the content exam.

A couple of months after data collection was completed participants also received a graduated use protocol for the video data (Derry et al., 2010). With full knowledge of their engagement on the mathematical research tasks, as well as the fact they passed the requisite content exam, participants could give additional consent for use of the four video clips of their interviews in contexts beyond the dissertation (see Appendix C). Participants could state which of the four specific contexts they agreed to and they could express whether or not they wanted their face to be blurred in these instances. Eight of the participants gave this additional consent.

Data Collection Procedures

Data collection took place from September 2006 to March 2007. Each of the various types of data focused on the concepts of division, rational numbers, and ratio and proportion. Big ideas or key concepts, such as that fractions are relations where the whole matters, or the meaning of the remainder in division, appeared in at least three different instruments or interviews to ensure that progression of thinking and understanding can be determined. Data were collected from the following sources: 1) pre and post tests; 2) follow-up interviews to explain thought processes on tests; 3) paired problem solving interviews; 4) classroom artifacts; 5) open ended questionnaire; and 6) debriefing interviews with the math skills course instructor.

A pilot project, conducted as part of another research project in 2005, provided an opportunity to field test instrument items and interviewing techniques. At that time, a sample content exam was administered as a pre-test to 23 pre-service students enrolled in a fall semester math skills development course. Follow-up interviews, videotaped with one camera that focused on the participants' hands, documenting their mathematics work, were conducted with 16 of these students.

Pre/Post Tests

The mandatory mathematics content exam was used as a pre/post test instrument to provide evidence that an increase in conceptual understanding took place while these pre-service elementary education teachers participated in the mathematics skills development course. The content exam was administered to all primary/junior education students on September 15, 2006. Those who failed had a second chance to write a parallel version of the exam on March 9, 2007. The test was a free response, pencil and paper, basic mathematics concepts assessment. This instrument consisted of 15 word questions, based on the five content strands, which participants had a maximum of two hours to complete. Of these 15 questions, ten dealt with numeracy concepts; seven questions on the pre-test and six on the post-test focused on fraction concepts. Calculators were not permitted, but manipulatives, such as centicubes, ruler, centimeter graph paper, and fraction kits were made available. The numeracy items from the exams are attached in Appendix D.

The exam was developed out of problems from the Grade 6 *Tops Problem Solving Deck* (Greenes, Immerzeel, Ockenga, Schulman & Spungin, 1980). The mathematics content exam instrument had been used with over 900 preservice elementary teachers to ascertain their proficiency with Grade 6-7 mathematical concepts. The instrument had been validated through its use in the previous three years (Lawson & Stienstra, 2007). All of the 120 (out of 900) students who failed the exam met with the developer of the exam or me to determine if the failure represented a genuine misunderstanding of the concepts or test anxiety. None of these students was able to demonstrate greater understanding of the mathematical concepts outside of the exam setting. In addition, we have found that the greater the failure, the greater the students' misunderstanding of the mathematics concepts.

Items on the content exams were scored on a seven level scale (0 to three points, including half points). No points were given for inappropriate solutions. Three points were awarded for correct solutions. One half a point was deducted for small calculation errors. For incorrect solutions that demonstrated some understanding of the relevant concepts the number of partial marks awarded depended on the degree of understanding shown. The three instructors for the p/j mathematics methods course, of which I was one, marked the exams. In order to ensure interrater agreement, the instructors marked a number of exams together and the basis for partial marks for each item was established. Each instructor then marked the same 10 exams and scores were compared. Once the instructors were confident of their interrater reliability, they each marked the exams from their particular sections of the course. If any questions about scoring occurred during this process the item was brought to the other two instructors and a joint decision was made on the scoring.

Content Exam Interviews

Participants took part in an hour-long interview after the September 2006 mathematics content exam to discuss how they interpreted and thought about the mathematics questions that focused on the number sense strand. In addition, participants were asked questions to ascertain their understanding of the big ideas about multiplication, division, and fractions. See Appendix E for the specific questions. This interview was video recorded using two cameras; one focused on the participant, and the other focused on the participant's hands and the mathematics she or he was demonstrating.

A second interview was held after the retake of the content exam in March 2007. Once again participants were asked to explain their thinking on the number sense portion of the exam.

In addition participants were asked about their perceptions of their mathematical learning during the mathematical skills development course. This interview also lasted approximately one hour and was video recorded with the two camera angles.

The interviews provided an opportunity to gain insight into participants' thinking processes on the mathematics content exam, ascertaining how fully they understood the concepts. Nevertheless, there are several limitations with this procedure. First, participants were not always able to articulate their mathematical thinking. Some may have been intimidated by the fact that they failed the content exam in September 2006. Second, since the first interview took place several weeks after the exam it is possible that the participants were not be able to recall accurately their thinking on the test. This is not as large an issue for the second interview since most of the participants were interviewed the day after the retake exam in March 2007. Finally, participant responses to the questions on the big ideas may reflect new thinking rather than baseline understanding since the first interviews may have taken place after the intensive first week of the mathematics skills development course. Nonetheless, these interviews both expand and validate the information gathered on the test instrument.

Paired Problem-Solving Interviews

In these interviews, participants were asked to work in pairs to solve novel word problems from the numeracy strand. The word problems presented in these interviews reflected both problems initially geared to children and those posed to pre-service teachers in the literature. In addition to the Greenes et al. (1980) resource mentioned above, the following resources served as the basis for many of the word problems because they provided examples of children's responses and used a reform-based approach: e.g., Burns, 2001; Clark & Kamii, 1996; Fosnot & Dolk, 2001, 2002; Wickett & Burns, 2003. Both problem-solving interview consisted of 10 questions; on the first interview six problems focused on fraction concepts while on the second interview there were seven. See Appendix F for the test instrument problems posed in each of the interviews. Not all participants completed the set of problems in the hour time frame. In some instances items from the first problem-solving session were carried over and completed in the second session. Appendix G shows the completion of the problem-solving fraction items by the participants.

A modified think aloud protocol (van Someren, Barnard & Sandberg, 1994; Leapard, 2009) was used to capture the pre-service teachers' thinking and reasoning about the problem. Once again two cameras were used to record the interview. One camera focused on the participants, while the other focused on their hands to capture their writing or solution procedures. I posed the problems verbally to the participants and also gave them a copy of the problem typed in large font. I then asked them to verbalize their thinking about the problem, to describe their solution process, and to use pencil and paper or manipulatives, such as a fraction kit, in ways that were helpful to them. At that point I let go and simply listened to the participants. At the end of the problem, or at critical points in the solving process, I would ask the participants to explain aspects of thinking more fully or to explain how they knew something was true. If participants were stuck, I might pose a question to prompt their thinking, but I did not intervene or guide their solution process. Two sets of these interviews were held, the first in October and November 2006 and the second in January and February 2007.

Participant responses to the word problems were not numerically scored. Instead they were coded for elements such as correctness, types of strategies and models used, and explanations.

These problem-solving interviews were paired as opposed to individual for a number of reasons. First, paired interviews may have helped to reduce the anxiety level of the students since they were able to help each other in solving the problems. Second, the participants were able to discuss their thinking together. This can produce richer and deeper mathematics thinking since participants have to be able to defend their reasoning to each other (Fosnot & Dolk, 2002). Finally, since the purpose of this study is to develop a continuum of understanding, rather than to evaluate the individual students, it is not essential that all sources of data be from the individual participants.

The problem-solving interview format does have several limitations. According to Creswell (2003), interviews can only provide 'indirect' information filtered through the interviewees; the information gathering no longer takes place in the natural field setting; the presence of the interviewer may bias the responses; and not all people are equally able to articulate their thinking. Nevertheless, since the participants could not be observed directly in the remedial math class, these interviews provide an alternative means of gaining access to their thinking processes as they work on developing mathematical understanding through a problem solving methodology (Creswell, 2003).

Open Ended Questionnaire

At the end of the mathematics skills development course participants were given a short open-ended questionnaire in which they were asked about their developing understanding of mathematics. The questions can be found in Appendix H.

This questionnaire provided the opportunity for another perspective on the shifts in mathematical understanding. It served to alert the researcher to possible overlooked changes of understanding or to confirm the documented changes. It is possible that participants over reported or under reported their mathematical learning. The instrument allows for the introduction of affect and participants' perceptions of their experiences, which may or may not be related to the actual changes in conceptual understanding.

Classroom Artifacts

As part of the mathematics skills development course students were asked to keep a problem-solving log. Four times over the duration of the course students used the learning logs to solve a word problem using a two-column method as part of the course review. In the first column they solved the problem and in the second they explained their thinking about the problem. Typically, the problem posed covered a concept dealt with in the previous class, focusing on division, fractions, ratio and proportion, and decimals and percents. These artifacts were to be collected at the end of the course from those students who chose to participate in the research study. Unfortunately, only the artifacts from one of the two sections were collected; the others were taken home by the students.

These seven learning logs provide an additional window into the various degrees of conceptual understanding of the participants. They are an unobtrusive source of information that reflects the thoughts and words of the participants (Creswell, 2003). However, some students may not have been able to articulate their thinking. They may have solved the mathematical problem, but not known why they used the procedures they did. In addition, participants may not be accurate in their explanations. Nevertheless, according to constructivist learning theory, reflection is a necessary part of learning (Fosnot, 2005). These learning logs provided an opportunity for participants to make further connections across their experiences.

Instructor Interviews/Debriefing

I met with one of the instructors of the mathematics skills development course several times to discuss his perceptions on the developing mathematical understanding he observed in his students. These discussions focused on the types of strategies, models, and representations the students used, as well as which misconceptions and big ideas students grappled with and what shifts occurred in students' thinking. These unstructured interviews, as well as additional reflections by the instructor, were audio taped. They provided further insight into the classroom and the development of mathematical understanding given that I could not be an observer because of possible conflict of interest considerations.

Data Analysis and Theoretical Framework

In order to analyze the participants' development of mathematical knowledge I use both a constructivist view of learning and a particular landscape of learning that highlights strategies, big ideas, and models (Fosnot & Dolk, 2002). As discussed in the literature review, mathematics teaching and learning in a reform-based environment takes a different shape from learning that occurs in a traditional mathematics classroom. Students are expected to make sense of mathematics problems and construct their own understanding, using their developing sense of number and relationships rather than memorized algorithms and procedures.

Within a constructivist framework learning is characterized by the development of cognitive mental schemes that reflect organized patterns of behavior (Piaget, 1977). As learners encounter new experiences their initial schemes are differentiated and coordinated (Fosnot & Dolk, 2002). The development of new strategies takes place as learners reorganize their mental constructions through a variety of assimilatory schemes. In this process they may experience perturbation when their mental scheme does not provide anticipated results. The process of reflective abstraction facilitates this *progressive schematization* (Treffers, 1987). However, the growth that takes place is sporadic and uneven rather than smooth and predictable (Hiebert & Carpenter, 1992). In Pirie and Kieren's (1989, 1994) model, understanding *folds back* to earlier levels when new concepts are encountered. Understanding is thus characterized as the process of making connections (Hiebert & Carpenter, 1992).

In order to connect aspects of teaching and learning Simon (1995) proposed a framework called a hypothetical learning trajectory. This model enables teachers to choose mathematical tasks with an awareness of the learner's conceptions and schemes, in the context of specific mathematical ideas. Teachers continually adapt and adjust their trajectories so that transformational learning takes place. Fosnot & Dolk (2002) used this model to develop their landscape of learning for children. They situate mathematics as a human activity, using Freudenthal's (1968) notion of *mathematizing*, rather than as memorized content. Big ideas as structures, strategies as schemes, and models as tools for thought provide key landmarks for teachers and students as they journey across the landscape of learning.

Fosnot & Dolk's *Landscape of Learning for Fractions, Decimals, and Percents* (2002, 2007), as found in Appendix I, served as the basis for exploring the development of the participants in this study. I used the strategies, models, big ideas, and roadblocks encountered by the pre-service elementary teachers to identify the different paths these participants took on their journey towards a deeper understanding of fraction concepts. I expected that as the adult participants engaged in the reform-based skills course, they would move from an unquestioned reliance on poorly understood algorithms to a broadened repertoire of strategies and models based on personal understanding. Similar to children, I expected that as these adults encountered and developed new strategies they would construct some of the big ideas underlying the organization of fractions. However, since these adult participants had already encountered the mathematics concepts in the traditional educational system, I also anticipated that they would not follow the same learning path as children encountering the ideas for the first time.

Video Data

Since all interviews used two camera angles, one focused on the participant(s) and the other zoomed in to record their hands and written work, I first organized and prepared the data for analysis. Using *Final Cut Pro*, I edited the videos by adding a title with the given problem, then embedding the second camera handwork into the main frame, and finally embedding any necessary digital pictures of the written work. An example of the edited video can be seen in Figure 3. I then segmented the videos into clips corresponding to each of the problems posed on the four test instruments. These segments became the primary data comprising of 556 video clips and 24 scans. Primary documents are referenced by the abbreviation *pd* followed by the particular document number. For example, pd121 refers to primary document file 121, which is Lynsey's work on item 4, the bag of peanuts problem. The list of primary documents for the fraction items can be found in Appendix J. In the few instances where there was no discussion about a given test item, the written results were scanned and the digital picture was used as data.

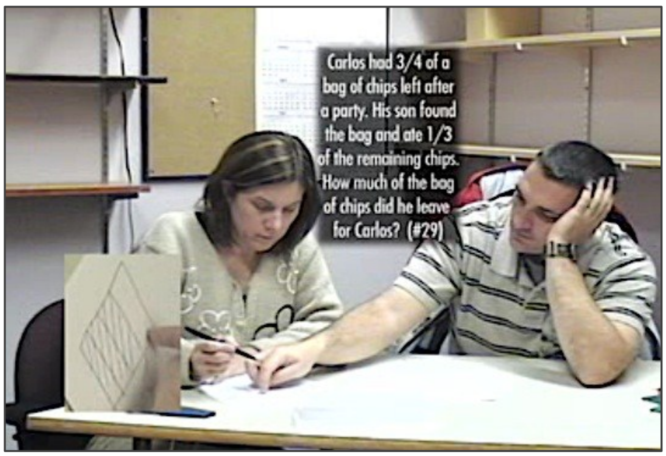


Figure 3. Screen shot of edited video

I used the qualitative software program *Atlas.ti* to code, organize, and analyze the data. Using the program as a database I could systematically select the video clips based on my needs for analysis, such as, for a participant, a cluster of participants, a particular problem, or all the problems for a particular interview session. This enabled me to compare and contrast the data with relative ease.

Data Analysis

Data analysis proceeded through several non-linear, interacting phases, and followed an iterative process of viewing and reviewing the data, similar to the approach described by Powell, Francisco, and Maher (2003). Initially, I viewed the video data to gain an overall picture of the content and context without imposing a specific analytic lens. In the second phase I coded the video data using a scheme based on Fosnot & Dolk's (2002) categories of strategies, models, and big ideas as well as Post et al.'s (1988) categories of explanation, sense making, and roadblocks (see Appendix K for the list of codes). In phase three I developed a descriptive profile for each of the participants based on their verbal and written responses to the individual fraction problems. The intent was to describe each participant's approach to the problems and to identify critical elements in their solution methods and thinking that would provide insight into their understanding of the fraction concepts. In this process several themes emerged that helped to frame the data. Phase four involved grouping the participants into clusters according to their mathematical abilities. Within the clusters I analyzed video data, participant profiles, and codes to develop a thick, rich description of the critical elements of the developing fraction understanding. In the final phase of analysis I closely analyzed the clusters of data, paying specific attention to the critical elements that reflected various stages of development in fraction understanding. Interpreting the data I developed a landscape of learning that reflected the participants' shifts in understanding fractions over time.

The initial intent of this study was to explore the participants' development of mathematical understanding in the number sense areas of division, rational numbers, and ratio and proportion; however, the first phase of analysis revealed that the scope was too broad to provide a meaningful analysis. Thus, the focus of the study was limited to the development of fraction concepts, even though data were collected for the other areas of number sense.

Limitations

While the findings of this study will contribute to the literature there are a few limitations. First, the results will not necessarily be generalizable to all pre-service teachers in the same situation. The majority of the participants are women from the dominant culture. While one participant was First Nation, and two others were new Canadians who had immigrated when they were in high school, these differences were not the focus of this study. It may be that different cultures have different learning trajectories.

Second, the landscape of learning is not necessarily prescriptive of a particular developmental trajectory; rather, it reflects the journey of the 14 participants in the case study. Even though the initial mathematics skills of the participants could be clustered to represent the weak, medium, and strong skills, there may be gaps in both the starting points and in the learning trajectories.

Third, the protocols for the modified think aloud during the problem-solving interviews may not have fully captured the mathematical reasoning of the participants. Some participants needed little guidance to verbalize their thinking processes while others needed continual prompting. Furthermore, additional information might have been garnered if participants were asked to give a second solution method for various problems.

Finally, the landscape is only applicable in contexts where the pre-service teachers are undertaking reform-based instruction.

Validity

Validity refers to the extent to which the findings are accurate from the standpoint of the participant, the researcher, and the reader of the account (Creswell, 2003). In order to increase the accuracy of the findings I used multiple sources of evidence so that I could triangulate the different data sources (Stake, 1995; Yin, 2003). I maintained a chain of evidence to increase the reliability of the evidence (Yin, 2003). Finally, I also ensured that my coding methods are outlined with enough detail to enable close inspection by others.

Chapter Four: Participant Profiles

This chapter provides an overview of participant profiles and perceptions at both the beginning and end of the study. The opening profiles are based on their scores on the pre/post-test, demographic information, and comments during the first interview. The final reflections are based on their written responses to the open-ended questionnaire.

The initial mathematical skills of the 14 participants covered a broad range. An examination of the results of the numeracy portion of the baseline content exam revealed three clusters of scores, indicating a likely grouping of mathematical skills for these participants. The top four scores ranged between 80-100%. This cluster of participants is categorized as having stronger mathematical skills. The middle seven scores ranged between 50-70%. This cluster of participants is categorized as having medium mathematical skills within the context of these findings. The lowest three scores ranged between 10-30%. This cluster of participants is categorized as having weaker mathematical skills. Table 1 provides an overview of these results. Each of the ten numeracy items on the pre-test was marked out of three points, with partial marks given for various degrees of understanding. The results of the retake content exam show how participants in each of the clusters progressed over the course of the study. The three cluster groupings will be used as a means to present the findings of the study in a consolidated manner. A brief description of the participants in each cluster will be presented in the following sections. The specific demographics for each cluster can be found in Appendix A.

Low Cluster Participants

The three participants in the low cluster were all females in the primary/junior education program. Two were in their mid twenties while the other was in her mid thirties. Two of the participants had taken either a mathematics course for elementary teachers or a statistics course at the university level. The oldest of these three participants had not taken any mathematics courses since Grade 12.

A common characteristic of this cluster of participants was their lack of conceptual understanding and their initial reliance on memorized procedures. In addition, each of the three participants had their unique qualities and situations. Valery struggled with both number facts and number sense. She often resorted to counting up on her fingers to complete a calculation.

This lack of automaticity meant she took extra time to solve problems and often had small calculation errors. Isabelle was a francophone and for the first time was taking her education in English. Even though she was fluent, at times the language provided another layer of challenge to her mathematics problem solving skills. Isabelle often lamented that she was only doing what she had been taught. Grace was eager to please and tried to quickly identify the procedure necessary for doing the problem correctly the next time. In order to consolidate her learning, Grace usually gave an unprompted, oral summary of her understanding at the end of each problem. Grace was extremely thankful for the one-on-one discussions of the research. She refused the honorarium for participating in the research study because she felt that she had learned so much as a result and did not feel justified in receiving money for the "tutoring" she had received.

Table 1

Percentage Scores for Numeracy Items on Content Exams

Cluster	Name	Baseline	Retake
Low	Valery	10	83
	Grace	22	83
	Isabelle	30	78
Middle	Erica	53	92
	Irene	57	88
	Lynsey	57	100
	Gabriela	60	98
	Tanya	62	97
	Brenda	67	93
	Olivia	67	83
Top	Bryann	83	91
	Diane	87	100[a]
	Mark	97	100[a]
	Megan	100	100[a]
	Average	61%	92%

Note. All names are pseudonyms.
[a] Did not need to write retake exam, but completed 8 of 10 items during the last interview.

The participants in the low cluster were only able to solve 10% of the fraction and ratio questions on the baseline exam. In many instances they were not able to recreate their thinking on the exam and had limited insights into concepts they encountered. This meant that during the initial follow-up interviews much of the time was spent on a guided questioning designed to develop the strategies and big ideas necessary to solve these types of questions. The participants willingly engaged in this process of exploring and communicating their mathematical thinking. By the end of the term, after completing the mathematics skills course, these participants solved 68% of the fraction and ratio problems correctly.

Middle Cluster Participants

The seven participants in the middle cluster were all females in the primary junior education program. Two of them were in the concurrent education program whereas the other five were in the post-degree program. While four of the participants took a mathematics course in university, either statistics or mathematics for elementary teachers, two of them had not taken mathematics since Grade 12. The remaining participant, the oldest, had not taken a mathematics course since Grade 11. Five of these participants were in their twenties and two were in their thirties.

Even though the focus of the interviews was on mathematical problem solving and the explanation of ideas, the participants revealed glimpses of their personal interactions and attitudes towards mathematics that are worth noting. An overview of these responses reveals that most of these participants were secure in their knowledge of basic addition and multiplication facts. However Brenda was challenged by her poor knowledge of the multiplication tables. She stated, “I just can’t remember them, like, I have to actually count them out. I have a lot of problems with nine, three and some six” (pd1[1]).

Five of the participants in this middle grouping overtly expressed frustration with mathematics. They believed they were not math people. Erica stated, “I feel that math is not my strong suit whatsoever. Like, my sister got the science and math, I got the arts and projects” (pd49). They especially found word problems and exams to be challenging. Irene felt that word problems “twisted her mind” so that she became “totally lost” (pd7). Likewise Gabriela found mathematics exams so frustrating and intimidating that even though she retook her Grade 12 math course to improve her grade of 54% she ended up with a lower grade of 52%. “I get very frustrated at exams, especially” (pd66). Lynsey found this content exam to be especially

[1] Refers to the primary document file (see Appendix J)

challenging. “I just froze. I was never, in my four years I have never been so nervous for a test, like I was physically nervous” (pd127). Two of them went on to identify the area of fractions as their nemesis. Lynsey acknowledged that, “fractions have always been my worst enemy” (pd123). Similarly Olivia identified her areas of weakness, “If there’s something I hate more than ratios, it would be fractions” (pd170).

Most participants in this cluster willingly engaged in the challenging process of communicating their ideas even though their previous experience with mathematics had focused on memorization. Erica’s comment is typical. “Look at all the mistakes I’ve made here, [pointing to the baseline content exam], but how much I, like, I’ve learned so much since writing this exam” (pd49). Similarly Tanya, who initially just wanted to be told what methods and formulas to use so she could memorize them, made attempts to explain her thinking. Her initial apologies for making mistakes or not being able to remember or explain her thinking reveal not only her insecurities about doing mathematics the correct way; they also demonstrate her willingness to risk being uncomfortable to learn mathematics at a deeper level.

The participants in this middle cluster of results solved just over half of the fraction questions on the baseline exam (57%) correctly, but by the end of the term they solved almost 83% of these questions correctly.

Top Cluster Participants

The top cluster was markedly different from the other two clusters. First, it included the only male in the study. Second, two of the women were in the primary/junior education program, while the other two participants were in the junior/intermediate program. Finally, unlike most of the group, three of the participants were in their late thirties and early forties, while the other participant was in her late twenties. Only the male participant had taken any mathematics courses at the university level. The remaining participants had not taken mathematics since high school.

Most importantly, only one of the participants in this cluster was taking the mathematics skills course because she had failed the content exam. Even though Bryann had strong enough numeracy skills to place her in this top cluster, her mathematics skills in areas such as measurement and geometry were much weaker. The other three participants took the course because they wanted to develop a deeper understanding of the mathematics they would have to teach. Diane also wanted to develop a greater confidence in her abilities. She would often lament that too much time had passed since she engaged in mathematical thinking.

> It's hard for me because I'm older, so I'm supposed to know how to do it and I don't, so it's, I've forgotten. So I know I can get it but that also for me creates, [indicates a barrier in front of her face] uh, I can't relax over it. (pd36)

Mark, on the other hand, was confident in his mathematical abilities, especially his proficient use of algorithms, and had a strong capacity for mental mathematics. He was switching careers from nursing to education and was taking the skills course in order to have a better understanding of children's thinking about mathematics. The remaining participant, Megan, signed up for the skills course even before she took the content exam. Megan recognized her weaknesses and knew they needed to be addressed before she could teach others about mathematics. Surprisingly, Megan had a perfect score on the exam and yet struggled with some of the key big ideas of fractions. Her ability to reason through situations and relate them to her prior experiences of measurement in carpentry and baking stood her in good stead.

Because Mark and Diane were in the Junior/Intermediate education program they were not required to write the content exam in September as were all students in the Primary/Junior education program. They wrote the exam several weeks later when they volunteered to participate in the study. While this time difference did not seem to affect the baseline results for Diane, it did somewhat compromise the results for Mark since his section of the skills course had an intensive week of classes before he wrote the baseline exam. When asked about the effect of these classes he replied,

> I would have been able to do all the questions, but if you would have asked me to explain in detail how I came, 'cuz I was taught traditionally, like you know when I, uh, like for some of the questions I wrote, there is probability ones I would, um, I would invert, like depending on how many dice I'm throwing, I would invert and multiply and I would just do it in my head and then write it out. And I wouldn't necessarily be able to, well, why does that work, um, I don't know, because that was the way I was taught. (pd142)

It is apparent from his use of models and diagrams on the baseline exam that his strategies do not necessarily reflect his traditional mathematics education. Nevertheless, he was able to make strong connections with his procedures and demonstrated a depth of understanding on this baseline exam that exceeded the post test understanding of many of the participants.

The participants in this top cluster solved 93% of all the fraction and ratio questions correctly on the baseline content exam. The greatest change for these participants over the course of the study was their shift from an unquestioned reliance on algorithms to a deeper understanding of the mathematics concepts as shown by their use of a variety of strategies and models. On the post-test the top cluster participants solved 95% of the fraction and ratio questions correctly.

Participant Reflections on Learning

After completing the mathematics skills course and the final retake content exam, participants reflected on their mathematical learning during the remedial skills course using the open-ended questionnaire instrument. They were unanimous in their appreciation for the mathematics skills class. Even the participants in the top cluster who did not need the remedial help spoke of the benefit of the course. “I needed to know basic math skills that I never learned in school. I learned how to do things and why; not just formulas that I didn’t understand and forgot” (Megan). Two of the participants in the middle cluster were glad they had failed the content exam and had to take the skills course.

> I think, actually, that it was better that I didn’t pass the first time. Because if I had just squeezed by with a 75% I wouldn’t have learned everything that I learned; I feel more comfortable [now]. I would have still hated math; I know I still would have hated it. And [now] I would feel more comfortable. (Olivia)

While the perceptions of the participants do not necessarily correspond with the development of their conceptual understanding, it is nonetheless important to hear their opinions of the growth process. The top three major shifts identified by the participants focused on the process of doing math, affect, and content. The most prevalent shift was the recognition that there are many different strategies for solving a problem. Twelve of the 14 participants commented on the importance of knowing that mathematics can be understood through the use of different strategies rather than simply memorizing algorithms.

> The main thing that I am taking away from this is that there is no one way of solving these problems. In fact, there are many ways to solve them and you don’t even need the formula in most cases! They can all be figured out without rules. (Lynsey)

The second significant shift was an increase in confidence and attitude. Nine participants echoed Diane's sentiments.

> My confidence and my recognizing that it's OK <u>not</u> to know the rules. I can get the answers by drawing, using manipulatives or taking my time – before that, I was so frozen and afraid of math because I know I'm not as strong as others. (Diane)

Embedded in this response is a recognition of the importance of visual models and manipulatives. Many of the pre-service teachers marveled at this realization; however, they framed it in terms of being a "visual learner". They did not appreciate that a key element of developing mathematical understanding for all learners might be connecting concrete contexts with abstract symbols.

The third important shift, identified by six participants, was an increased understanding of fraction concepts. Brenda's comment highlights the lack of initial understanding that many participants had. The most significant change for her was "my understanding of fractions. If someone said 2/3 to me before, I would have no idea how to picture that in my head. Now all fractions are clear to me" (Brenda). Many participants linked that fractional understanding to manipulatives just as Gabriela did. "I now know how to use manipulatives to further understand fractions."

Participants were also asked to identify areas they felt they did not understand well enough. Here four pre-service teachers were confident in their current level of understanding, one from the top cluster and three from the middle cluster. The areas most frequently identified by the remaining participants were those of ratios, decimals, or percentages. Five of the participants, across the clusters, identified one or more of these concepts. Participants from the low cluster identified additional concepts that challenged them. Valery still struggled with number lines. "I cannot figure out a number line. When I look at it, it makes no sense to me. I would rather deal with decimals as fractions." Grace, on the other hand, still felt "uneasy" with fractions; but believed that "with more practice I believe I will get better" (Grace). The remaining concepts for concern, area and spatial sense, were outside the general strand of number sense.

The pre-service teachers were unanimous in their appreciation for the mathematics skills course. Participants from all three clusters agreed that their understanding of the mathematical concepts deepened in ways that would not have occurred if they had only taken the required mathematics methods course. How did that learning and deepening transpire in the area of fractions?

Chapter Five: Results and Analysis – Fractions as Relations

Organization of the Results

The development of the pre-service teachers' mathematical understanding over the course of the study will be presented in terms of three significant themes that emerged from the data: a) understanding fractions as meaningful objects, b) the use of models, and c) roadblocks to understanding. First, in order to engage with fractions as meaningful objects rather than procedures the pre-service teachers worked to develop a deep understanding of the three big ideas that, *fractions are relations*, *fractions can be thought of as operators*, and *fractions can be thought of as ratios.* These big ideas proved to be pivotal for organizing and structuring the participants' meaningful understanding of fractions. The results of pre-service teachers' engagement with these big ideas will be presented in Chapters Five, Six, and Seven.

As the pre-service teachers engaged with the fractions problems they began to use models both to represent the context and to think about the meaning of fractions. However, in some instances models became procedural and did not enable the participants to make sense of the mathematical relationships. The results of this theme will be presented in Chapter Eight.

Finally, many pre-service teachers encountered roadblocks, which included conceptual gaps and a focus on superficial procedural thinking, as they engaged with the fraction problems. These roadblocks will be explored in Chapter Eight. The results for each theme will be presented according to the three different clusters of participants, beginning with the low cluster and ending with the top cluster. In order to show the progression of understanding over time, within each cluster the results of the four test instruments and interviews will be presented sequentially.

Fractions as Relations: Introduction

In spite of what many people remember from elementary school, fractions are not simply two separate numbers such that the numerator and denominator reflect the number of equal sized parts indicated out of a total number of parts. The whole matters. Fractions are relations; and it is precisely the relationship between the parts and the whole, rather than the absolute magnitude of each part, that is important (Post et al., 1986). Students who think about fractions as relations are able to see them as meaningful objects rather than procedures to be carried out (Stephens, 2006). This concept of relations is foundational to the development of a deep understanding of equivalent fractions, all four fraction operations, and comparing and ordering fractions (Fosnot & Dolk, 2002).

When the operations of multiplication and division are used on fractions two distinct wholes must be considered. In these instances the situations involve *relations on relations* (Fosnot & Dolk, 2002). The fractional amount of one whole becomes the new whole for the second fraction. For example, when sharing half a chocolate bar with three people, the half of a chocolate bar becomes the new whole and each person receives 1/3 of it. This piece is 1/3 of 1/2, or 1/6 of the original whole chocolate bar. In this context, the fractions are relations on relations.

Table 2

Test Items Reflecting Fractions as Relations

Instrument	Item #	Development of Relational Ideas
Content Exam & Follow-up 1	5	Ratio of equal parts to a whole
	15	Defining fractions
	6	Common whole needed to combine fractions; relative size of fractions; use of benchmarks
	4	Division by fraction; the whole matters; relations on relations;
	8	Relate fraction to part; the whole matters; part-part-whole ratio
Problem-Solving 1	25	Common whole is needed to combine; relative size of fractions
	28a	Common whole is needed to compare; relative size of fractions; use of benchmarks
	28b	Common whole is needed to compare; relative size of fractions; use of benchmarks
	29	Multiplication by fraction; the whole matters; relations on relations
Problem-Solving 2	33	Relate fraction to part; the whole matters; part-part-whole ratio
	35	Only denominator matters when comparing common numerators
	36	Use of benchmarks when comparing
	37	Inverse relation between size of fraction and gap piece
	41	Ordering fractions; equivalent fractions; fractions as measures
	42	Ordering fractions; equivalent fractions; fractions as measures
	43	Division by fraction; the whole matters; relations on relations
Retake Exam	49	Common whole needed to compare fractions; relative size of fractions; use of benchmarks

Each of the four test instruments had items that assessed the participants' understanding of the relational nature of fractions. Table 2 gives an overview of the pertinent items along with the big ideas each questions develops. The fourteen participants began with various degrees of understanding of the relational nature of fractions and all progressed towards deepening that understanding. But the degree to which they progressed depended more on their willingness to set aside their procedural knowledge than their starting point. It is not surprising that the low cluster of participants began the study with a weak understanding of this concept; however, even participants in the top cluster had gaps in their understanding.

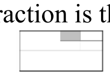
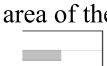
Low Cluster Results

Baseline Content Exam

The three participants in the low cluster began the study with an extremely limited understanding of fractions as relations. Their responses to the test items on the baseline assessment, which focused on the nature of fractions, combining and comparing fractions, division of fractions as relations on relations, and recognizing a fraction part in order to find the whole, as seen in Table 3, revealed a whole number understanding of fractions that did not recognize that the whole matters in fraction relationships. Their prior learning emphasized memorization of procedures rather than understanding, so these participants struggled when they encountered non-routine fraction problems.

Table 3

Content Exam – Fraction as Relations Questions

Item #	Code	Question
5	Fraction picture	What fraction is the shaded area of the whole (i.e. the largest rectangle)?  *(or*  *)*
15	Interview	What are fractions?
6	Sum fractions	Of the following fractions: 3/4, 5/12, 2/3, 3/2, 2/5 5/8 which two fractions (and only two) will give a sum that is less than 1?
4	Bag of peanuts	Rita had 6 kilograms of peanuts. She wanted to make bags of peanuts for her friends. She put 2/3 kg of peanuts in each bag. How many bags did she make, if she used all the peanuts? (4kg; 10kg)
8	Track team	There are 20 women on the university track team. One sixth of the team is men. How many students are on the university track team? (15; 25)

Defining fractions. None of the three participants in the low cluster recognized that fractions are equal parts of the whole. When they attempted the fraction picture question (#5) two of the participants focused on the number of pieces rather than the relationship, while the third did not attempt the question because she did not know how make sense of the picture. In all instances these participants used whole number thinking rather than a relational understanding of the parts and the whole. Isabelle suggested that this might be a result of her prior experiences with fractions. “This really threw me off because I saw the ones that are circled and you go three out of four, three colours out of four…, But this one, I don't think I've seen it before that way" (pd111).

While they were able to define fractions (#15) as (equal) parts of the whole during the first content exam interview, the participants in the low cluster were not able to extend this definition to include relations. Grace still thought about fractions as shading parts of the whole, revealing her focus on the magnitude of the parts rather than the relationship.

> You need to find out parts of a certain something, so if I drew a pizza, and I wanted to know the fraction of that then I could cut it into separate parts and that would give me the different fractions of the whole, different parts of the whole and then I would know a fraction if one of them would be shaded. (pd91)

Isabelle focused more on the rules and procedures she had learned in the past than the relations. Finally, Valery still needed visual models to understand the need for equal parts and thus continued to use her whole number thinking.

> The parts have to be equal … which I just learned…. People have tried to explain it to me but I just couldn't get it until I actually saw it. That's why the fraction kit helped a lot, because then I saw it and then I understood what it meant. (pd213)

Combining and comparing fractions. Participants in the low cluster also demonstrated a limited understanding of the importance of the whole when combining or comparing fractions. All three used memorized procedures to solve the sum of fractions problem (#6) and did not seem to think about the meaning of the fractions. Both Valery and Isabelle used common denominators in a trial and error approach to solve the problem, but neither thought about the relative sizes of the fractions. Isabelle was able to find the correct answer after several attempts, but Valery gave up before finding a solution. Valery's attempt at using 3/2, a fraction already larger than the expected answer, indicates a rote implementation of procedures, rather than a contextualized understanding of fractions and the need for a common whole (Figure 4).

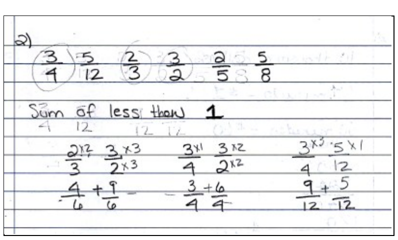


Figure 4. Valery (pd204)

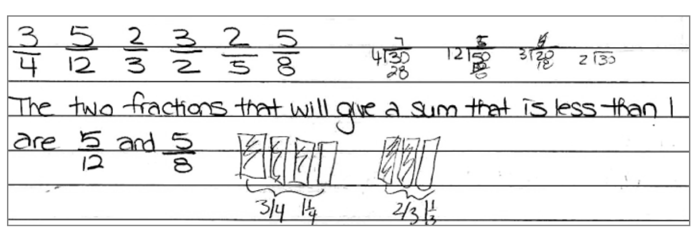


Figure 5. Grace (pd83)

Similarly, Grace's rote use of procedures reveals her limited understanding of fractions as relations. Grace attempted to convert each fraction to decimals, but quit when she could not make sense of the answers (Figure 5). "I thought that dividing the bottom number, the bigger number into the top number, would give me the answer of what it would be, less than one" (pd83). Grace then tried to draw pictures, but still did not know how to compare or combine the fractions. So she just wrote down two fractions without knowing why. Grace used whole number thinking that hindered her understanding of how to combine fractions.

Relations on relations. The bag of peanuts question (#4) presented a quotative division problem that required participants to differentiate between the two different wholes. All three of the participants in the low cluster used a type of whole number thinking and thus could not solve the problem correctly. Grace and Isabelle both used procedural algorithms in a way that demonstrated their limited understanding of the relationship between the fraction and the total given in the problem. Grace "wasn't sure what to do at all" (pd81) with this problem. After finding the decimal equivalent for the fraction Grace did not know how to relate the answer to her pictorial representation of the fraction and thus labelled each of the two shaded pieces as 0.66. She then incorrectly multiplied the total number of kilograms by the decimal fraction committing several calculation errors. Similarly, Isabelle essentially found 2/3 of the total number of kilograms when she incorrectly set up a ratio and cross multiplied (Figure 6). Neither Isabelle nor Grace identified the correct whole for the fraction.

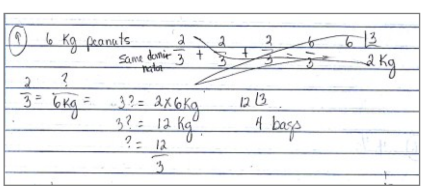


Figure 6. Isabelle (pd 564)

Valery, on the other hand, was "scared away by the kilograms" (pd202) and did not attempt the problem. In the follow-up interview, however, she decided to give it another try. Her method of adding up measures of 2/3 until reaching the total amount revealed that Valery was able to differentiate between the two wholes because of her choice of strategies. Unfortunately her understanding of fractions was not robust enough to successfully complete the problem. Her part whole understanding of fractions, rooted in whole number thinking, led to difficulties with understanding improper fractions.

Relating fraction part to unknown whole. None of the participants in the low cluster were able to correctly solve the track team problem (#8). While they recognized the male to female ratio, they did not realize that it was a part-part-whole ratio, men to women to team. As such they incorrectly assumed the whole for the fraction was the only number stated in the problem. None of these participants recognized the importance of determining what fraction of the team was women. Instead each of them attempted to multiply the number of women by the fraction 1/6 and when their results did not make sense they attempted to draw a picture to model the situation. Their limited understanding of the importance of the whole hindered their ability to correctly model the situation. Both Grace and Isabelle correctly showed the ratio of 1/6 men but they used the number of women as the total for the team (Figure 7). They did not attempt to determine if the numbers in their answer were proportional to the fraction given. Valery also attempted to model the problem with a picture, but she did not know how to relate the fraction to the ratio of men and women. All three drawings demonstrated the participants' reliance on whole number thinking, rather than an understanding of fractions as relations.

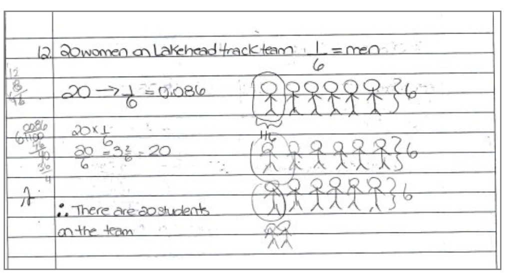


Figure 7. Grace (pd84)

Overall on the baseline content exam and follow-up interview, the participants in the low cluster demonstrated a limited understanding of fractions as relations. Each had their own particular compensation strategies. Grace relied on converting fractions to decimals; Isabelle used cross multiplication to solve for the unknown; and Valery used memorized procedures such as common denominators or when overwhelmed did not attempt the problem. Nevertheless, they all shared a reliance on whole number thinking that viewed fractions as two independent numbers with little relational meaning. They focused on the absolute magnitude of the numbers rather than the relationship. As such all three of the participants had difficulty choosing the correct whole for fractions.

First Problem-Solving Interview

In the first problem solving interview participants in the low cluster worked on two problems that highlighted the theme of fractions as relations. These problems focused on comparing and combining fractions as well as the need for a common whole. None of the participants had enough time to address the relations on relations question (see Table 4). While all three participants still wanted to use familiar algorithms and procedures, when challenged, two of them willingly engaged in the process of modeling and explaining their thinking. They demonstrated slow gains in their understanding of fractions as relations. The third participant engaged in the process with more reluctance, wanting the certainty of efficient procedures more than in-depth understanding. For these problem-solving sessions Grace and Isabelle each worked with a partner from the middle cluster while Valery worked on her own.

Table 4

Problem-Solving Interview #1 – Fractions as Relations Questions

Item #	Code	Question
25	1/2 +1/3 = 2/5	Is 1/2 + 1/3 = 2/5? Explain. Can you give me at least two different explanations?
28	Order fractions	Put the following fractions in order from smallest to largest: 5/6, 6/7, 4/11, 4/12, 7/5, 6/8. Explain your reasoning. a) 4/11 or 4/12; b) 5/6 or 6/7
29	Bag of chips	Carlos had 3/4 of a bag of chips left after a party. His son found the bag and ate 1/3 of the remaining chips. How much of the bag of chips did he leave for Carlos?

Combining fractions. The two participants in the low cluster who completed the 1/2 + 1/3 = 2/5 question (#25) during the first problem solving session each experienced difficulty in using a model correctly to demonstrate that the statement was false due to their limited understanding of the nature of the whole. Isabelle began by using common denominators to demonstrate that the correct answer should be 5/6 rather than 2/5. For her second strategy Isabelle wanted to use the fraction kit pieces but she had difficulty modeling the equation (Figure 8). Once this was completed with her partner's help, Isabelle struggled to demonstrate that 5/6 was not equal to 2/5. Because the fifths had a different orientation to the whole than the other pieces Isabelle mistakenly thought that 2/5 was larger than 1/2 + 1/3. "So you need one half plus one third plus one sixth to make two fifths" (pd283). Isabelle knew she needed to do more than just count the parts in order to compare the fractions, but she still did not comprehend what aspect of the part whole relation was important. She assumed equal lengths rather than equal areas were needed. Isabelle concluded this problem by stating, "All I know is that they have to have the same denominator" (pd283).

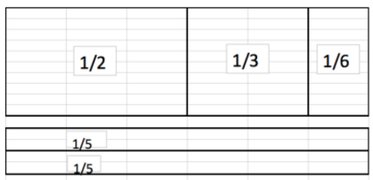


Figure 8. Isabelle (pd283)

Valery, on the other hand, recognized that she needed to compare the areas of the pieces in order to compare fractions. She correctly used the fraction kit pieces to demonstrate that the statement was false, comparing the two different areas. Unfortunately this understanding was contextually bound. For her second explanation Valery drew a picture and incorrectly concluded that the statement must be true (Figure 9). In her drawing Valery used a set model where each individual part was the same size, rather than the wholes or units. Valery combined the parts as if she were using whole numbers, making the whole the total number of pieces. This focus on counting the discrete pieces clouded her understanding of the relational nature of fractions. She was not able to solve the seeming contradiction without further guidance and exploration.

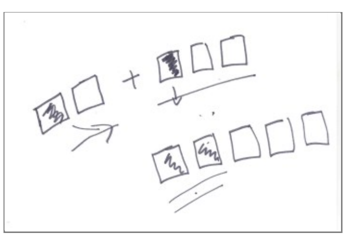


Figure 9. Valery (pd305)

Comparing fractions. Only one participant from the low cluster completed the questions on comparing fractions (#28). Grace and her partner spent half of the hour session working on this problem so they did not complete any of the other questions that addressed fractions as relations during this time. Even though Grace would have preferred to use her familiar procedure of converting the fractions to decimals, she willingly engaged in the process of thinking about the fractions to make sense of them first. She demonstrated a beginning understanding of fractions as relations but fragility was evident as she often deferred to her partner's thinking even if it was incorrect. When comparing fractions with a common numerator (4/11 and 4/12), Grace recognized that she only needed to compare the denominators. She understood the need for a common whole and reasoned that the fraction with the smaller denominator had larger pieces in relation to the whole. But when her partner suggested that 4/12 was larger Grace agreed. Her whole number thinking took precedence over her newly forming understanding of fractions as relations. "The twelfths [4/12] are bigger [than 4/11] because there are more pieces left" (pd261). While Grace expressed frustration with the process she did not give up. "It's so frustrating. I'm used to just having the answer!" (pd261). Eventually Grace reaffirmed her original thinking about the inverse relationship between the number of pieces in the denominator and the size of the pieces. However, she generalized this knowledge to all fractions rather than just those with a common numerator. Thus when comparing fractions that were both one piece away from the whole (5/6 and 6/7) Grace incorrectly chose the fraction with the smaller denominator as larger. She could not see the relationship between the one piece missing from the whole and the size of the fraction. Grace still focused on the magnitude of parts rather than the relationship between all of the parts and the whole. In spite of the limitations in her understanding, or maybe because of those limitations, Grace showed a positive determination to make sense of fractions without simply resorting to efficient procedures.

The three participants in the low cluster demonstrated a limited development of their understanding of fractions as relations during the first problem-solving interview. Each revealed contradictions, or aspects of dissonance, in their thinking about fractions. Grace recognized the importance of size of the denominator in relation to the whole, but continued to use whole number thinking when relating all of the parts to the whole. Valery recognized the relationship between parts and whole when using an area model, but reverted to whole number thinking when using her more familiar set models. Isabelle recognized that a common whole was needed to compare fractions, but lacked the ability to differentiate between different wholes. As they worked within this cognitive dissonance Grace and Valery embraced the opportunity to build new understandings, but Isabelle preferred the comfort of known procedures.

Second Problem-Solving Interview

Participants in the low cluster were challenged by the concepts found in the questions addressing the relational nature of fractions in the second problem-solving interview (see Table 5). They continued to struggle with identifying the fraction whole and using relational rather than whole number thinking when comparing fractions. Only one of the participants engaged in the problem exploring relations on relations in multiplying fractions and none of them completed the problems on ordering fractions as measures. As these participants engaged more fully in the process of making sense of fractions they began to let go of their entrenched beliefs that fractions were simply procedures to be carried out.

Table 5

Problem Solving Interview #2 – Fractions as Relations Questions

Item #	Code	Question
33	Rows to knit	Roberta has knit 260 rows of the afghan she is making for a friend. She still has 1/5 of the afghan to finish. How many rows does she still have to knit?
35	3/7 or 3/8	Without using division (i.e., converting to a decimal) determine which of the two fractions is larger. Explain your reasoning. a. 3/7 or 3/8
36	7/15 or 11/20	b. 7/15 or 11/20
37	14/15 or 17/18	c. 14/15 or 17/18 (use 5/6 or 6/7 if not used in interview #1)
41	Halfway between	What fraction is exactly halfway between 1/2 and 1/3?
42	2nd between	Find a second fraction between 1/2 and 1/3.
43	Ribbon	Avery has 5½ metres of ribbon. She would like to cut up small ribbons for her friends. How many ribbons can she make if each small ribbon is 3/8 of a metre? How much ribbon will be left over?

Relating fraction part to unknown whole. The rows to knit question (#33), a part-part-whole problem based on the track team problem (#8), proved to be problematic for the participants in the low cluster. Even though they all used a model that clearly identified the 1/5 part to finish, these participants did not understand that the remaining parts represented the complement fraction, 4/5. Their whole number thinking led to a generalization of the need to divide by the denominator, which incorrectly related the number of rows completed with the whole rather than the fraction part.

Thus they did not understand how both the numerator and denominator related to the numbers given in the problem. Because of their entrenched part whole understanding these participants had difficulty recognizing the inherent contradiction between their solution and the statements in the problem. Grace thought an extra section needed to be added to her picture to accommodate the difference (Figure 10). Isabelle struggled to follow her partner's reasoning when she eventually divided by four instead of five.

> I'm confused. It's not what I would have done, because it's 1/5. I remember doing just pictures [fills in circles in the air]. But, the remainder, she still has 1/5 of the afghan to finish, so you're looking at the four there [points to diagram] so that's why you're dividing by four. But see on the exam you would have got me. I would have divided by five. (pd368)

Even though all three of these participants were guided to recognize the importance of the numerator, and thereby differentiate between the different wholes, it is unclear how much they actually internalized. As Valery noted, "we've had problems like that in class" (pd391). Both the nature of the different parts in relation to the whole and the size of the numbers used may have contributed to the participants' fragile shift from whole number to relational thinking.

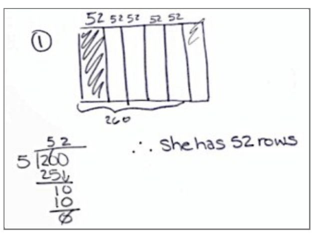


Figure 10. Grace (pd349)

Combining fractions. Grace continued to demonstrate the fragility of her understanding of fractions as relations when she addressed the 1/2 +1/3 = 2/5 question (#25). She began by incorrectly thinking only about the parts rather than their relationship to the whole and concluded the statement was true, because two of the total five parts were shaded. However, when she used the fraction kit pieces to model the statement she began to see a contradiction in her reasoning. She realized that the shaded parts were not all the same size.

> I just don't, like see two-fifths of it this way…. It's not, because this [2/5] is less, and you know what, it makes sense, because, look at our picture, this [pointing to shaded 1/2] isn't as big as that [pointing to 1/3]. (pd345)

Grace's personal modeling of the fractions was not sufficient. She needed the concrete representation of the fraction kit pieces with their explicit common whole structure in order to shift her thinking from the parts to the fraction relations.

Comparing fractions. The participants in the low cluster once again demonstrated the limits of their understanding of fractions as relations when comparing three different pairs of fractions. The more complex the fractions, the more likely the participants were to revert back to their whole number thinking. When comparing fractions with the same numerator (#35) two of the participants in the low cluster used a common whole to determine that the fractions with the smaller denominator had larger pieces because the whole was divided into fewer sections. Both Grace and Isabelle used an area model to help them think correctly about the relative sizes. The third participant, Valery, used a set model for each of the fractions and incorrectly focused on the parts rather than the need for a common whole. But she was not completely satisfied and chose a second method to check her thinking. This time Valery drew on her knowledge of the fraction factory pieces and reasoned that the whole must be the same for each fraction. She then used an area model to draw each of the fractions and correctly determined the larger fraction. "When I draw it, I draw it as the same [the whole for each fraction], my spaces for 3/8, these little sections are smaller than they would be for 3/7" (pd392). Valery needed the relational model to help her articulate the need for the common whole since her set model seemed to reinforce her whole number thinking.

Whole number thinking surfaced again when the three participants compared 7/15 and 11/20 (#36). Each of them began by simply comparing the size of the parts rather than the relation of the parts to the whole. They incorrectly assumed that because the fraction with the smaller denominator had the larger pieces, it must be the larger fraction. "I think it is 7/15…. Well 20, like if you divide everything into 20 that's a lot of pieces. They'd be smaller pieces than fifteenths" (pd371). Both Valery and Isabelle needed to draw a model in order to think about the fractions. Valery made sure both wholes were the same size but Isabelle did not. Nevertheless they each had difficulty accepting that a third fraction could be used as a benchmark to compare sizes. Valery acknowledged the reasoning but would have preferred to use a ruler to draw an accurate model. Isabelle began to shut down when she initially did not understand her partner's use of the benchmark. "I can't think today. I really, I can't. I just can't think. I can't even do my fractions anymore. I would know normally, I guess, but I am totally confused" (pd371). Once she recognized that it was not the size of the pieces but their relationship to the whole Isabelle accepted the reasoning. Grace also had difficulty making sense of her partner's use of 1/2 as a benchmark, but in the process of trying to put words to the concept she began to make sense of it.

> I don't know. I think I just see it…. [long pause] I don't know. I think I just see it in my head that that one [11/20] would be bigger than the other one [7/15]; because of the 11, maybe because it would be one more than the half, that's why…. Because I would just think that 7 + 7 = 14 and 10 + 10 = 20, but that's 11. So I would see that that's bigger. (pd352)

The final comparison of two fractions that were both one part away from being whole (#37) proved to be the most challenging for the participants in the low cluster. They all assumed the fractions must be equal since each was missing one part. "They are the same, 'cuz there's only one apart" (pd353). Even though they could recognize 1/15 was larger than 1/18 the participants found it difficult to think relationally about the entire fraction. Each responded uniquely to the challenge. Grace drew pictures of pies to help think through the logic of the inverse relationship. Isabelle remained focused on the size of the pieces and thus chose 14/15 as the larger fraction. Valery used her original rectangular models to think about the missing piece but found it difficult to separate counting the piece from determining its size. All three participants used whole number thinking to begin thinking about the problem but they were not all equally successful in shifting to a more relational understanding.

Relations on relations. The final question for the second problem-solving interview, bag of chips (#29), used a multiplication context to focus on fractions as relations on relations. Valery was the only participant in the low cluster who completed this question. She demonstrated an emerging understanding of the importance of the whole that surpassed many of those in the other clusters who relied on memorized procedures. After struggling with the meaning of the question Valery recognized the changing whole and clearly used her discrete model to act out the problem (Figure 11). Because the numbers were small enough Valery did not struggle with her poor numeracy skills and focused on the concepts, solidly demonstrating an understanding of what it means to find a fraction of a fraction (1/3 of 3/4). This understanding is juxtaposed with her predilection for using a whole number thinking that is embedded in her use of the set model.

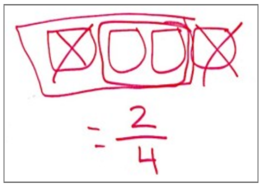


Figure 11. Valery (pd388)

Participants in the low cluster showed limited gains in their understanding of fractions as relations during the second problem-solving interview. They seemed to vacillate between whole number thinking and relational thinking as they compared fractions. In certain instances they had difficulty identifying the correct whole for the fraction. Yet these participants demonstrated progress in how they approached the fraction questions. For the most part they willingly let go of their memorized rules and procedures and tried to give meaning to concepts that had previously lacked sense. They used models to concretize their understanding. Unfortunately in some instances the use of models became a memorized procedure as participants in the low cluster generalized a concept such as dividing by the denominator.

Post-Test Retake Exam

The final question that highlighted fractions as relations focused on the need for a common whole in order to compare fractions (see Table 6). Participants once again worked individually on the problem that was part of the retake content exam. Only one of the participants in the low cluster solved the order fractions question (#49) correctly. Grace used her conceptual understanding to compare the fractions to the benchmarks 1/2 and 1. "Well the first one was 3/8, so I knew that half of 8 would be four, so 3/8 is obviously the smallest" (pd459). She felt confident in her developing abilities to meaningfully think about fractions. Grace still needed visual representations of the fractions to confirm her thinking, but she no longer needed to automatically convert each fraction to a decimal as she did prior to the skills course.

Table 6

Post Test Retake Exam – Fractions as Relations Question

Item #	Code	Question
49	Order fractions	Put the following fractions in order from smallest to largest: 5/4, 4/5, 4/3, 3/8, 7/12.

The remaining two participants had not internalized an understanding of fractions as relations. Although Isabelle used drawings instead of common denominators, as she would have done earlier, she still had difficulty ordering the fractions (Figure 12). Her use of the set model indicates a fragile understanding of the importance of the whole when comparing fractions. It may be that Isabelle used the size of the denominator when incorrectly ordering 7/12 and 3/8.

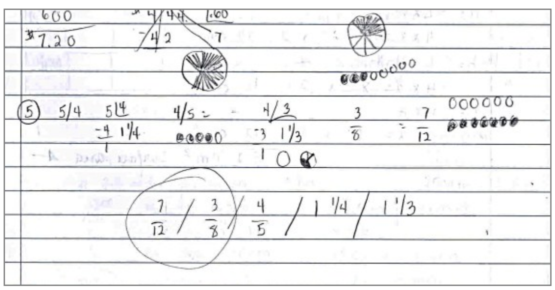


Figure 12. Isabelle (pd573)

Within the stressful context of the retake exam Valery could not think relationally about fractions. Instead she returned to her memorized procedures for finding a common denominator, but was not successful.

> I skipped it at first and I came back to it and I thought for some reason I was going back to the way that I learned it before. And I thought I need to get all of these in a common denominator, and I couldn't do it, like I couldn't figure out, so I tried, but I still couldn't do it. I still don't know how I would figure that one out… Was I wrong to try to find a common [denominator]? (pd549)

While Valery did recognize the need for a common whole in order to compare fractions, outside of the need for a common denominator she could not make sense of the fractions. In the guided follow-up she painstakingly used the fraction kit pieces to concretely compare the fractions. But even then Valery still wanted confirmation that the common denominator procedure could have worked. Given her weak level of understanding of the nature of fractions, Valery had more confidence in rote procedures than her personal insights on this high stakes assessment.

Concluding Overview

All three of the participants had a consistently weak understanding of the relational nature of fractions. They struggled with strongly entrenched whole number thinking and reliance on procedural thinking. While they learned the importance of equal parts they still had challenges correctly identifying the fraction whole. Throughout the study these participants experienced the dissonance between viewing fractions as procedures or meaningful relations. In the struggle to integrate this deeper understanding Grace and Valery displayed a willingness to articulate and reflect on the process while Isabelle tended to be less engaged. Nevertheless, each of the participants made subtle gains in their approach to fractions. Grace moved from an avoidance of fractions and converting to decimals to an emerging understanding of the importance of 1/2 as a benchmark. While Isabelle preferred to use her familiar rules, in certain contexts she recognized the value of using a model to visualize the relationship. Similarly, when working within simple contexts Valery demonstrated an ability to model or act out relational concepts. But ultimately these participants remained weak in their understanding of fractions as relations.

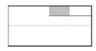
Middle Cluster Results

Baseline Content Exam

The test items on the baseline instrument focused on the nature of fractions, combining and comparing fractions, division of fractions as relations on relations, and recognizing a fraction part in order to find the whole. An overview of these questions can be found in Table 7. The seven participants in the middle cluster began the study with a limited understanding of fractions as relations. Most of them viewed fractions as procedures to be carried out rather than meaningful objects. Even though many of these participants recognized the importance of parts and the whole, they often reverted to whole number reasoning that focused on the magnitude rather than the relationship.

Table 7

Content Exam – Fraction as Relations Questions

Item #	Code	Question
5	Fraction picture	What fraction is the shaded area of the whole (i.e. the largest rectangle)?  (or )
15	Interview	What are fractions?
6	Sum fractions	Of the following fractions: 3/4, 5/12, 2/3, 3/2, 2/5 5/8 which two fractions (and only two) will give a sum that is less than 1?
4	Bag of peanuts	Rita had 6 kilograms of peanuts. She wanted to make bags of peanuts for her friends. She put 2/3 kg of peanuts in each bag. How many bags did she make, if she used all the peanuts? (4kg; 10kg)
8	Track team	There are 20 women on the university track team. One sixth of the team is men. How many students are on the university track team? (15; 25)

Defining fractions. Six of the seven participants in the middle cluster correctly recognized that all parts of a fraction need to be equal. When they solved the fraction picture questions (#5) these participants demonstrated a solid understanding of fractions as relations of parts and wholes. Five of these six participants subdivided the diagram to construct pieces of equal sizes, but one of them used multiplication. Erica worked with fractions as procedures rather than meaningful entities. She knew the multiplication algorithm would give the correct answer but did not question why it worked. “That was just the way I was taught, to multiply fractions to get that” (pd53). However, when challenged Erica realized that cutting each of the four rectangles in half and counting the parts would give the same answer as multiplying 1/4 times 1/2. In this way she demonstrated an understanding that the parts must all be an equal size.

The remaining participant in the middle cluster, Tanya, used whole number reasoning to incorrectly count all boxes, without recognizing the need for equal parts. Tanya acknowledged her thinking with embarrassment. “Oh, I’m like, there’s one part shaded and how many spots is there: [laughs] and it’s not equal, so I was totally off. So I know I have to break it down into equal parts” (pd187). Tanya’s reliance on memorized formulas and definitions did not enable her to make sense of the problem she initially thought was too easy.

When asked to define fractions (#15) the majority of the participants in the middle cluster reaffirmed that fractions were parts of a whole, with all parts of equal size. Even though they clearly understood the part whole nature of fractions, none of these participants extended their definition to include any other aspect of fractions. Two participants, however, simply focused on the fraction symbol and defined fractions as “a number over a number, the numerator over the denominator” (pd181). Both recognized that the bottom number told how many parts were needed to make a whole, but they did not seem to attach a meaning to the symbol as a whole. It is unclear if any of the seven participants in this cluster understood the relational nature of this ratio of parts to a whole or if they simply thought of fractions in terms of counting the parts using whole number reasoning.

Combining and comparing fractions. Participants in the middle cluster revealed a rule-bound understanding of the need for a common whole when combining fractions. Even though four of the seven participants correctly solved the sum of fractions question (#6), only one participant demonstrated an ability to see the parts in relation to the whole. Brenda drew pictures of each fraction (Figure 13), which clearly reflected a same sized whole, “so that if I needed to cut the pieces out, they would all have to be the same size in order to fit in, cause the whole is the same” (pd6). She then visually determined which two fractions could be combined to give a sum that was less than one and confirmed her assessment by using common denominators to add the fractions.

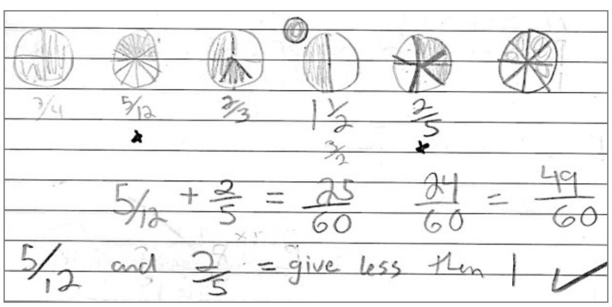


Figure 13. Brenda (pd6)

All of the remaining six participants in the middle cluster demonstrated a weaker understanding that focused on whole number thinking embedded in memorized procedures. Five of these participants used the traditional common denominator algorithm. While the need for a common whole and a common unit is embedded in the algorithm, these participants did not necessarily understand the connection. Gabriela’s response exemplifies this thinking. “You always have to have a common denominator if you want to add or subtract. I don’t know why but you do…. There’s no reasoning behind my thinking” (pd68). Many randomly combined fractions without thinking about the relationship between the parts and the whole. Thus, like Irene (Figure 14), they used 3/2 even though the sum needed to be less than one.

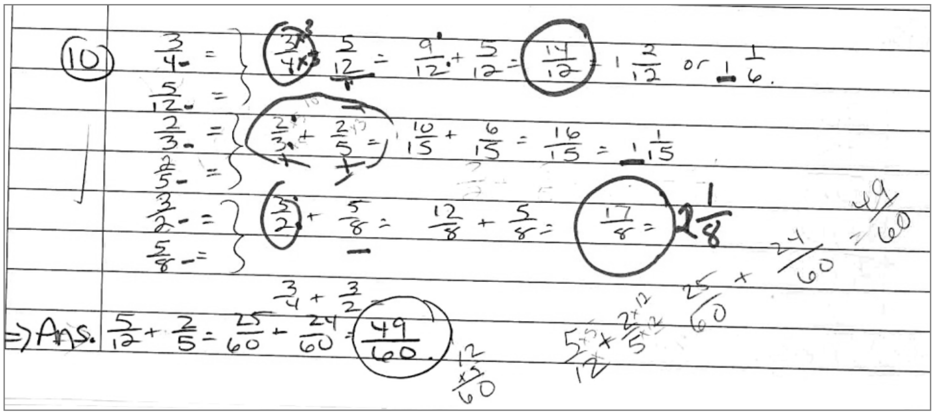


Figure 14. Irene (pd98)

The sixth participant, Erica, used whole number thinking to calculate how many parts each fraction was away from being whole. Since she did not understand that sum meant to add she looked for the fractions that were closest to one. However, even when she correctly understood the problem in the follow-up interview Erica continued to think of fractions as procedures and randomly combined them using common denominators. When challenged to think relationally about the parts and wholes by using the benchmark of 1/2, Erica still needed to verify the answer using common denominators. Her rigid reliance on procedures typifies the thinking of participants in this cluster.

Relations on relations. Almost all of the participants in the middle cluster had difficulty identifying the appropriate fraction whole in the quotative division by a fraction problem, bag of peanuts (#6). Only one participant in the cluster solved this problem correctly. After several false starts Olivia recognized that the fraction referred to one kilogram rather than the total number of kilograms. "I was trying to figure out how to put the fraction with the unit of kilograms" (pd170). She then used a measurement strategy to count out the groups of 2/3 kg (Figure 15). Nevertheless her thinking still revealed some limitations in her understanding of fractions as relations. While she could combine like fractions to make a whole, Olivia could not make sense of adding fractions in general and added across numerators and denominators, revealing the fragility of her understanding of the need for a common whole when combining fractions as well as her use of whole number reasoning.

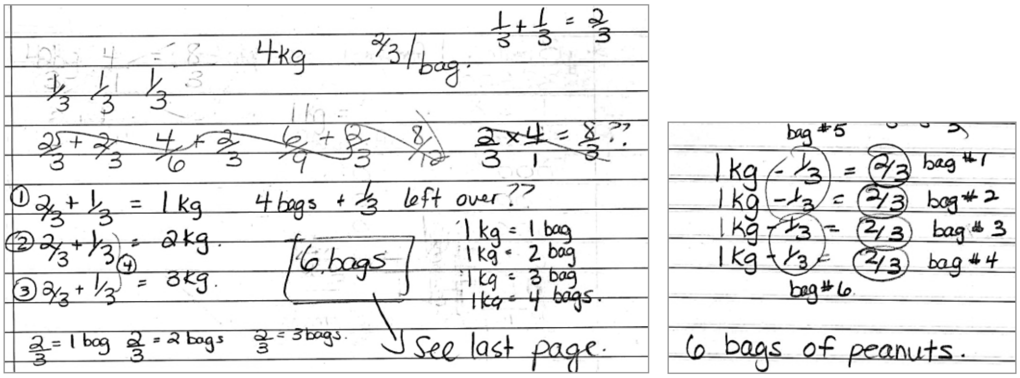


Figure 15. Olivia (pd170)

None of the six other participants in the middle cluster recognized that there were two different wholes in the question. They assumed the fraction referred to the total number of kilograms. Gabriela's response was typical of many of these participants. "I didn't even think of 2/3 of 1 kg. I don't know what I was thinking, but I forgot that it was 2/3 of 1 kg not 2/3 of [total] kg" (pd66). Because these participants treated fractions as procedures they did not consider the importance of the whole. Many just followed prescribed rules. Erica stated she multiplied, "because I was taught in elementary school to do fractions [this way] so I took the 2/3 and I wanted to times it by 6…. That's the way I would approach fraction problems…. It just made sense for me to multiply them" (pd52) (see Figure 16). In addition, others over generalized their whole number thinking to work with fractions. Irene manipulated the numbers rather than reasoning about the meaning of fractions. "I'm trying to find more of something and I think multiplying would show me more, whereas if I divide it wouldn't" (pd96).

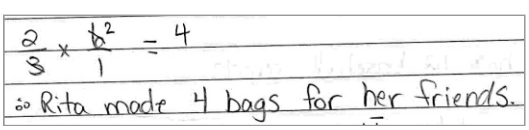


Figure 16. Erica (pd52)

Relating fraction part to unknown whole. The participants in the middle cluster struggled with relating the fraction part to the unknown whole in the track team question (#8). The majority of the seven participants either treated the fractions as procedures or incorrectly identified the part whole relationship. Only one of them correctly understood the need to find the complement fraction in this part-part-whole ratio problem. Brenda properly identified that 5/6 of the team was women and then used a picture to evenly distribute the number of women over the five parts of the fraction (Figure 17).

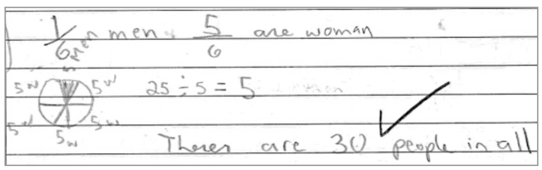


Figure 17. Brenda (pd8*)*

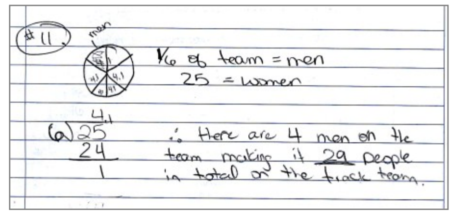


Figure 18. Lynsey (pd125)

Two of the participants recognized the importance of the part whole ratio but incorrectly identified the relationship. They over generalized the need to divide by the denominator and did not comprehend the necessity of finding the fraction complement in this instance. So they divided the total number of women by the denominator (1/6) not recognizing the number represented the parts rather than the whole (Figure 18). In the follow-up interview Lynsey described this confusion with distinguishing the magnitude of the parts and whole. “Oh but in my other questions that we had done, I had the whole…. But I wouldn’t do that here, because this isn’t the whole, this is part of it” (pd125).

The remaining four participants in this cluster also had difficulty identifying the appropriate whole for the given fraction. Because they could not make sense of the problem they treated the fraction as a procedure without thinking about the relationships between the parts and the whole. Erica’s response is typical of the thinking of these participants.

> My easy, let’s-get-out-of-here solution was to multiply the fractions [20 x 1/6 = 10/3]; and even though I would get, like, a mixed fraction I just decided to times the 10 by 3; and I figured that that [30] was close enough to 1/6 of the team. (pd56)

Similarly, Gabriela manipulated numbers, converting the number of women to the fraction 1/4 using 100 as the whole, and then combining the fractions using common denominators (Figure 19). These participants did not realize that since the whole was unknown and one of the two parts was given as a fraction, they needed to determine the fractional amount of the other part. Their use of partially understood rules and procedures prevented them from seeing the proportional relationship between the number of men and women.

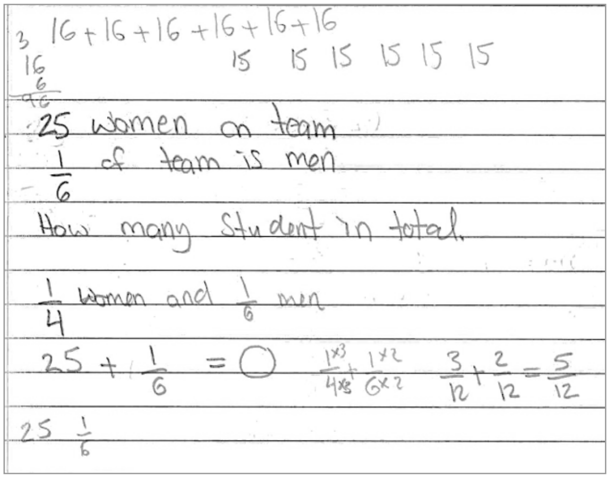


Figure 19. Gabriela (pd70)

Overall on the baseline content exam and follow-up interview the seven participants in the middle cluster demonstrated a moderate understanding of the part whole nature of fractions. However for most of these participants both whole number reasoning and a procedural view of fractions limited this understanding. They did not try to make sense of fractions by using familiar contexts or reasoning about the relations. Their relational understanding of fractions was limited to the magnitude of the parts and whole and did not fully recognize the importance of the whole in different contexts.

First Problem-Solving Interview

The questions pertaining to the theme of fractions as relations in the first problem-solving interview dealt with the concepts of combining fractions, comparing fractions, and relations on relations in multiplying fractions (Table 8). Participants in the middle cluster continued to prefer working with fraction rules and procedures during this problem-solving session; however, they willingly attempted different ways of thinking when challenged for a second method. The limits of their understanding of the relation between parts and whole were revealed in several of their responses. Participants worked with a partner of their choice for the two problem-solving interviews. Brenda worked with Olivia and Erica worked with Lynsey. Both Gabriela and Tanya worked with partners from the low cluster while Irene worked on her own.

Table 8

Problem-Solving Interview #1 – Fractions as Relations Questions

Item #	Code	Question
25	1/2 +1/3 = 2/5	Is 1/2 + 1/3 = 2/5? Explain. Can you give me at least two different explanations?
28	Order fractions	Put the following fractions in order from smallest to largest: 5/6, 6/7, 4/11, 4/12, 7/5, 6/8. Explain your reasoning. a) 4/11 or 4/12; b) 5/6 or 6/7
29	Bag of chips	Carlos had 3/4 of a bag of chips left after a party. His son found the bag and ate 1/3 of the remaining chips. How much of the bag of chips did he leave for Carlos?

Combining fractions. All six of the participants in the middle cluster who completed the 1/2 + 1/3 = 2/5 question (#25) in the first problem-solving interview recognized the statement was false. The remaining participant, Gabriela, addressed this question in the second problem-solving interview. Responses were divided between those who appealed to rules and procedures and those who compared the relative sizes of fractions. Three participants used the common denominator algorithm and simply stated that 3/6 + 2/6 = 5/6 so the statement was false. They could describe the procedure but could not extend their reasoning beyond the process itself. For example, when prompted for another explanation Tanya drew a picture using a circular whole to illustrate 1/2 and a larger rectangular whole to illustrate 1/3. She then became stuck and could not proceed. "You just know that you can't add them together because they have to be the same. Each piece has to be the same and it's not. You'd have to go back to do it this way [indicates algorithm]" (pd283). Neither Tanya nor the other two participants could think relationally about the fractions and the need for a common whole outside their experience with the common denominator procedure.

The other three participants in the middle cluster visually compared the size of both sides of the equation using the fraction kit model with the embedded common whole. These participants worked with fractions as objects rather than procedures. Yet the degree to which they let go of their procedural or whole number reasoning varied. Irene used her knowledge of common denominators and equivalent fractions in the visual model to show that 10/12 was greater than 4/10. Olivia's second explanation used a circular model that demonstrated her understanding of the need for a common whole. However, neither of these participants could extend their thinking to explore the relations embedded in the equation. Irene simply stated that the equation did not follow the rule, while Olivia called the question a trick. "This is trying to trick people that don't know the rule, because you think people are going to go 1 + 1 is 2 and 2 + 3 is 5. That doesn't work" (pd223).

Lynsey, on the other hand, moved beyond the concrete modeling when she began to draw a circular model for her second explanation. Even though she was tentative, Lynsey willingly explored a different type of reasoning.

> I don't know if this is right, but I'm going to try it here…. [draws circular area models for each fraction.] Two-fifths isn't that big and 1/2 is more than 2/5. So yeah…, yeah…, 2/5 isn't big at all and 1/2 is big, it's quite big; and so is 1/3, not as big as 1/2, but it's still pretty big. So there is no possible way that those two [1/2 + 1/3] could add up to make that number [2/5], because it's not the same ratios or the same size. (pd247)

Lynsey began to develop a relational understanding of fractional size as she drew her models. She moved beyond a rote procedural and whole number thinking to a deepening understanding of fractions as objects where the relation between part and whole matters.

Comparing fractions. Only four of the seven participants in the middle cluster completed the comparing fractions question (#28) in the first problem-solving interview. These participants were not used to reasoning about fractions as objects and thinking about the comparative relations between parts and whole. They began by thinking about fractions as procedures and most made limited shifts in their reasoning. When comparing fractions with a common numerator (#28a) all four participants wanted to either convert the fractions to decimals or find common denominators. When challenged to find a second method they all choose to draw a concrete representation of the fractions. Two of these participants correctly reasoned that the fraction with the larger denominator had smaller pieces and was the larger fraction. In using a sharing-a-pie analogy they demonstrated an understanding of the need for a common whole. The other two participants had difficulty coming to this reasoning. Erica used discrete sets to model the fractions and struggled comparing sizes because she did not address the need for a common whole.

> For me to do this, I'd have to make them into decimals, for me to figure it out…. See this [referring to her fraction models] is just I don't know, like I get it, but it's kind of confusing me…. I'd also have to see them in their actual size, not in our drawings. (pd250)

Similarly, Gabriela used whole number reasoning and incorrectly concluded that 4/12 > 4/11 because it had more pieces "left over" after shading the appropriate pieces in her circular area model. "It's 4/12, no? Because if you have 11, 12 [pieces] and four of them are coloured; if you do 11 and colour four then 1, 2, 3, 4, 5, 6 are left. So the 4 over 12 is bigger" (pd261). She worked with fractions as two separate numbers and had difficulty thinking about fractions as relations rather than procedures. Even after working with her partner to conclude that 1/11 was greater than 1/12 Gabriela still needed to check this reasoning by converting the fractions to decimals.

These participants found it more challenging to compare fractions that were both one piece away from being whole (#28b). Because of their weakly developed relational understanding three of these participants began by incorrectly generalizing their knowledge of the role of the denominator in determining relative fraction size. Both Erica and Lysney agreed, "I would say 5/6 is bigger [than 6/7] because you get bigger pieces" (pd250). Similarly Gabriela began by comparing the denominators, but she changed to whole number thinking by assuming the fractions were equal because they both had one piece missing from being whole. "I think they are the same now, there is only one difference between them" (pd261).

Only Irene correctly identified the larger fraction, but her thinking still focused on the size of the denominator. Using two identical wholes she carefully drew a rectangular model of each fraction and visually compared their sizes (Figure 20). While Irene had solid partitioning skills and understood the need for a common whole she could not generalize about part whole relations when comparing fractions. She still needed to confirm her reasoning by converting the fractions to common denominators.

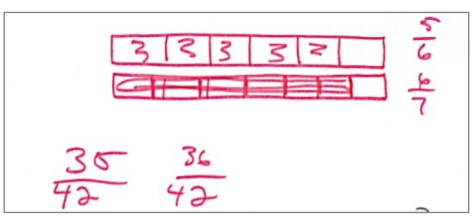


Figure 20. Irene (pd273)

Relations on relations. Only three participants in the middle cluster completed the bag of chips problem (#29) during the first problem-solving interview and none of them solved it correctly. Two participants correctly recognized the whole changed but they did not know how to interpret the answer. Irene used a procedure because she knew that *of* meant to multiply; she found 1/3 of 3/4, but did not subtract this amount from the original amount. Even though Irene demonstrated flexibility by using percentages as a second method her confusion became evident as she gave her final answer, "A quarter of the bag or 33% or something" (pd274). While Irene's procedural use of fractions was correct she did not know how to interpret the response in this two step question. Lynsey, on the other hand, used whole number reasoning dealing only with the number of pieces rather than the fractional amount (Figure 21). Even though she correctly determined the number of parts, Lynsey did not know how to state it as a fraction; she did not know to what whole the parts related.

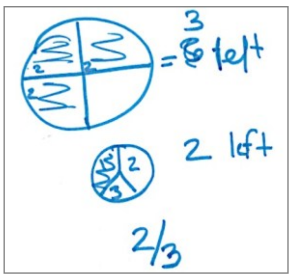


Figure 21 Lynsey (pd251)

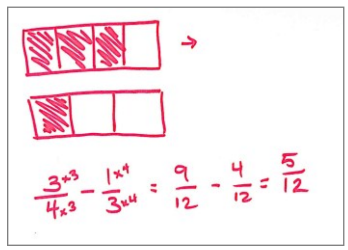


Figure 22. Erica (pd251)

In contrast the remaining participant did not understand that the multiplication context involved two different wholes. Even though Erica recognized the need to subtract the two different fractional parts her drawing revealed an assumption that they both had the same whole (Figure 22). Rather than thinking about the part whole relations Erica also focused on fractions as procedures. She converted both fractions to common denominators and subtracted. Even though she incorrectly interpreted the problem, not recognizing the relations on relations, Erica demonstrated some progress as she checked her calculation by using manipulatives to model her solution.

The seven participants in the middle cluster demonstrated varied degrees of progress in their understanding of fractions as relations during the first problem-solving interview. They continued with a part whole view of fractions that was moderated by either whole number reasoning or a notion of fractions as procedures. Even though they were more comfortable with familiar fraction rules most participants in this middle cluster willingly engaged in thinking meaningfully about fractions when challenged. They moved from a strict use of procedures to including the use of visual or concrete representations. However the use of these representations did not necessarily indicate a deeper understanding of fractions as relations. In a couple of instances the use of a discrete model or the use of whole numbers impeded the development of relational concepts. Many continued to have difficulty recognizing the importance of the whole, especially in contexts with more than one whole. Nevertheless, one participant used the visual model to begin thinking about fractions as objects.

Second Problem-Solving Interview

Participants in the middle cluster demonstrated their developing understanding of fractions as relations on questions that focused on the ideas of the importance of the whole, comparing fractions and relations on relations in multiplying and dividing fractions (Table 9). While they continued to be challenged to think beyond their memorized procedures and rules most of these participants began to reason about the nature of the relation between part and whole. However, in some instances the fragility of this understanding became apparent.

Combining fractions. The one participant in the middle cluster who did not complete the 1/2 +1/3 = 2/5 question (#25) in the first problem-solving interview demonstrated the fragility of her understanding of fractions as relations. Gabriela and her partner used a picture to illustrate both fractions and incorrectly concluded that the statement was correct because in total two out of five parts were shaded. Gabriela focused on counting the parts, rather than recognizing the size of the parts in relation to the whole. However, after using the fraction kit pieces to model the statement, Gabriela concluded the statement was false because a half is bigger than 2/5. She concluded by using common denominators to verify the statement was false. Gabriela seemed to need the concrete modeling and accurate visualization of the fractions in order to integrate a relational understanding of fractional size.

Table 9

Problem Solving Interview #2 – Fractions as Relations Questions

Item #	Code	Question
33	Rows to knit	Roberta has knit 260 rows of the afghan she is making for a friend. She still has 1/5 of the afghan to finish. How many rows does she still have to knit?
35	3/7 or 3/8	Without using division (i.e., converting to a decimal) determine which of the two fractions is larger. Explain your reasoning. a. 3/7 or 3/8
36	7/15 or 11/20	b. 7/15 or 11/20
37	14/15 or 17/18	c. 14/15 or 17/18 (use 5/6 or 6/7 if not used in interview #1)
41	Halfway between	What fraction is exactly halfway between 1/2 and 1/3?
42	2^{nd} between	Find a second fraction between 1/2 and 1/3.
43	Ribbon	Avery has 5½ metres of ribbon. She would like to cut up small ribbons for her friends. How many ribbons can she make if each small ribbon is 3/8 of a metre? How much ribbon will be left over?

Relating fraction part to unknown whole. The rows to knit question (#33) required that participants find the value of a fractional part given the other part. Participants in the middle cluster struggled with identifying the whole in this context and only one of the seven solved this problem correctly. Using a diagram Lynsey recognized that the rows completed must equal the complement fraction rather than the whole as her partner stated. So after shading in four sections she stated they needed to divide the number of rows completed by the numerator rather than the denominator. "This all [indicating fours sections on drawing] equals to 260 so we have to divide by four" (pd336). In doing so Lynsey demonstrated an understanding of how the numerator and denominator related to the parts of the whole as well as to the absolute quantities given in the problem.

The other participants in this cluster did not fully understand these relations. As they began to concretize the part whole relation through modeling many over generalized the need to divide by the denominator to find the size of each part. Five of the remaining six participants incorrectly identified the number of rows completed with the whole rather than with a part of the whole and thus the complement fraction, 4/5. Consequently, like Tanya, they divided by the denominator rather than the numerator (Figure 23). When asked to explain their thinking these participants found it challenging to make sense of the contradiction between their results and the information given in the problem. Some, like Brenda, were able to articulate a shift in their understanding of the fraction relations.

260 should only have been put into four of these squares; so we have to start over… because she still has 1/5 left [to complete]. So we needed to leave one [indicates one of five boxes in drawing] open. That's what she still has to do. But she did 260 in these, this 4/5 [indicates 4 or the 5 boxes]. (pd311)

But others still struggled to understand the relationship of numerator and denominator in this context. Olivia, for example thought it was a "sneaky" question and found it difficult to think when the cameras were on her. "I did it in class the other day and it didn't trick me" (pd311). She still thought of fractions as procedures rather than objects that made sense.

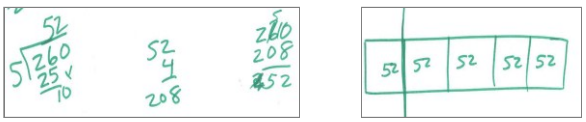


Figure 23. Tanya (p368)

The final participant in this cluster, Irene, could not make sense of the fraction relations because of the large numbers used in the problem. Switching to percents she readily understood that the completed rows represented 80% of the total. Nonetheless, Irene still multiplied this amount by 20% to find the remaining fifth. She focused on the procedural rule that percents mean to multiply rather than recognizing the need to determine a portion of the known part rather than the whole. In this context Irene, along with most of the other participants in this cluster, struggled to understand the relation between the two fraction parts and the unknown whole.

Comparing fractions. Participants of the middle cluster tentatively used their developing understanding of fractions as relations when comparing and ordering three different pairs of fractions. As the fractions increased in complexity many of them struggled to make sense of the relationship between the numerator and denominator because the relative size of the denominator no longer dictated the ordering principle. Even though many of these participants exhibited a fragile and partially formed understanding of the nature of fractions most willingly struggled to extend their limited procedural understanding.

All participants in the middle cluster recognized that the smaller denominator would give the larger fraction when comparing fractions with the same numerator (#35). Six of these participants focused on the size of the unit fractions in the context of an area model. In Olivia's words, "If I had a pie and I put it into eight pieces, and I got three of them, the pieces would be smaller than if I had cut the pie into seven pieces, they'd be bigger" (pd313). Only Irene had difficulty articulating her understanding of the fractions. She needed to draw a picture in order to visually see the difference between the two fractions. While she recognized that larger denominators had smaller pieces in the context of a common whole, Irene seemed to use the procedure of drawing to replace her preferred use of common denominators to compare fractions.

> I would say, just by looking at it [the drawing] 3/7 is more than 3/8… I don't think it is necessarily because it [the denominator] is smaller, because we're looking for the same amount of space, but is smaller in terms of size, than what I'd have to fit them into. (pd359)

In light of this as well as her responses to comparing more challenging fractions, it seems that Irene had difficulty internalizing the relational nature of fractions. She preferred the security of known procedures to her own reasoning.

When asked to compare 7/15 and 11/20 (#36) several of the participants in the middle cluster lost sight of fractions as relations because of the more challenging numbers. Some felt unprepared to deal with these fractions. "We haven't worked with twentieths in class" (pd339). Nevertheless, three of the seven participants in the middle cluster immediately compared the unfamiliar fractions correctly to the benchmark of 1/2 revealing their ability to think relationally about the fractions. Of the remaining four participants, two were able to eventually think about the fractions as relations. In the process of drawing and shading the parts of one of the fractions Brenda recognized that it was not necessary to finish the pictures since she could use the relationship of 1/2 to reason about both of the fractions. Similarly, as Lynsey talked through her thinking she rejected her assumption that both fractions were equal and focused the comparison of the fractions to the benchmark of 1/2. The final two participants in the middle cluster did not embrace the use of relationships to compare the fractions. Olivia simply agreed with her partner's reasoning but did not display conviction with the response. Erica, on the other hand, used her procedural knowledge. She needed to complete the division of both fractions in order to definitively know which was larger. "Well this [11/20] is obviously bigger because it is 55% and this [7/15] is … a failing mark" (pd339). Erica preferred to use procedures than trust her developing understanding of fractions as relations.

The final comparison of 14/15 and 17/18 (#37) challenged the participants in the middle cluster. None of the participants solved it correctly in their initial response. In this instance, because each fraction is one part away from being whole, the size of the individual part is inversely related to the size of the fraction. Three of the participants in this cluster reverted to whole number thinking and stated that the fractions were equal because both were one piece away from being whole. Even though they recognized the size of the parts for each of the fractions were different, these participants could not recognize the dissonance in their thinking. Gabriela typified the reasoning of these participants. "But then the 18 pieces they are smaller than the 15, they are each the whole. I'd say they are the same [size]" (pd353). Three of the remaining participants in this cluster focused on the size of the denominator. They too did not think about the relationship of the part to the whole. They simply assumed the larger parts would yield the larger fraction. It is possible they over generalized the case for common numerators to the case where both fractions are an equal number of parts away from being whole. Only Brenda shifted her thinking to focus on the size of the gap piece and corrected her initial thinking without any external prompting, reflecting the deeper development of her relational understanding. The remaining participant, Irene, used her part/whole knowledge to draw the fractions (Figure 24). Unfortunately her drawing was not accurate enough to provide a correct answer. Irene focused on the procedure of partitioning the wholes rather than thinking about how the relationship between part and whole in this instance related to the size of each fraction.

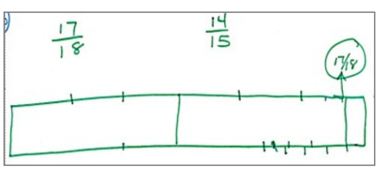


Figure 24. Irene (pd361)

Ordering fractions as measures. Only one participant in the middle cluster completed the set of questions that used a measurement understanding of fractions. In this context participants were asked to order fractions by finding a fraction exactly halfway in between two other given fractions. By drawing a strip model of the fractions (Figure 25), Irene used her knowledge of partitioning and equivalent fractions to determine that 3/8 was between 1/2 and 1/3 (#41). Because of the imprecision of her sketch Irene did not know if her answer was correct. "3/8 is more than 1/3 and less than 1/2… I don't know if it will be exactly in between, well, maybe, if my diagram was correct, but I don't know" (pd365). In this novel context Irene relied on her procedural knowledge of fractions and did not know how to extend her thinking to the relational aspects beyond the need for a common whole. She needed to physically compare the fractions using the fraction model kit to determine the correct answer.

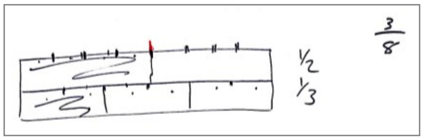


Figure 25. Irene (pd365)

Relations on relations. Two participants of the middle cluster completed the bag of chips problem (#29), which focused on understanding multiplication of fractions as relations on relations, during the second problem solving session. Contrary to the participants of this cluster who worked on the problem during the first problem-solving interview, these participants correctly solved the problem. However, both of them struggled to make sense of the two different wholes. While they each recognized the need for 3/4 to become the new whole, they faced different challenges interpreting their models. Olivia recognized that since the 3/4 was in three pieces, 1/3 of this amount would be one of those three quarters. But she rejected her answer, not being able to make sense of the result. "I always do that in my class; I always get the right answer, but I can't say why I have the right answer" (pd307). Brenda used twelfths to model the problem, possibly thinking of the algorithmic process of multiplying denominators, and surprisingly confirmed her partner's response. "Well, I was surprised because when she went like this to take out a third, I thought, no, because it's a fourth of a whole, so how can it be a third? But yeah, it's a third [of what's left]" (pd307). Brenda struggled to reconcile that the same fraction piece could be both a fourth and a third, depending on which whole was under consideration.

Over the course of the second problem-solving interview the seven participants in the middle cluster continued to demonstrate a willingness to let go of their memorized procedures and to reason about fractions instead. They readily reasoned about the relationship between part and whole when working with simple and familiar fractions. But when faced with larger fractions or novel contexts the fragility of their understanding became apparent. Some reverted to whole number thinking while others used their knowledge of the denominator in a procedural fashion, simply dividing by the denominator with little regard for the context. However, several of these participants used concrete representations of the fractions to think first about the context and then to generalize a deeper understanding of the relationships and thus to begin to see fractions as objects.

Post-Test Retake Exam

One item on the final test instrument focused on the relational nature of fractions (Table 10). It highlighted the need for a common whole in order to compare fractions (#49). Since this was the retake of the content exam participants worked individually on the problems. Their solutions revealed some solid connections to various aspects of fractions as relations but many of them still lacked a fully developed understanding of the essential concepts. Nevertheless, all but one of the seven participants in the middle cluster solved this problem correctly. Of these, one participant had a solid understanding of fractions as relations and simply reasoned about the fractions using her knowledge of benchmarks, such as 1/2, to order the fractions. The need for a common whole was implicit in her reasoning. She thought of the fractions as objects rather than merely focusing on the pieces. Tanya demonstrated a shift from her previous reliance on memorized formulas. "In the class we used a lot of manipulatives and I was able to picture things rather than before I strictly knew formulas and how to calculate and figure things out, which didn't really make sense to me" (pd538).

Table 10

Post Test Retake Exam – Fractions as Relations Question

Item #	Code	Question
49	Order fractions	Put the following fractions in order from smallest to largest: 5/4, 4/5, 4/3, 3/8, 7/12.

The other participants in this cluster who solved the problem correctly displayed some limitations in their developing understanding of fractions as relations. Two of the participants needed to use decimals to confirm their ordering of the fractions. Erica correctly reasoned about the relative sizes of the fractions, however her explanations emphasized the size of the denominators and the number of pieces and not the relations. She did not trust her reasoning abilities and needed the division procedure to confirm her thinking. The fractions were procedures and not meaningful objects for her. "I find with fractions, if I can see them in a decimal or a percent it's better for me" (pd436). Gabriela had a deeper understanding of the relational nature of some of the fractions and reasoned about the smaller fractions by comparing them to the benchmark of 1/2. However she needed to convert the improper fractions to decimals in order to compare them because of her weak comprehension. Improper fractions held little meaning for her and so she was not even able to use the fraction kit pieces to model the improper fractions without guidance. Like Erica, Gabriela had a procedural understanding of some fractions.

Another couple of participants used drawings to visualize the size of the fractions. They needed the concrete representation to think about the relationship between the quantity and size of the pieces. While both participants used a common whole to represent the fractions neither of them could integrate the use of benchmarks in their thinking without prompting. Olivia focused on the size of the pieces (Figure 26), whereas Brenda used doubling strategies to reason about size. Nevertheless Brenda stated, "I really need to visualize them still" (pd399). In working with the improper fractions, Brenda correctly used two wholes, but Olivia did not know how to represent relationship involving more than a whole and thus overgeneralized the use of the common whole.

Figure 26. Olivia (pd527)

The last participant who solved the problem correctly found it intimidating to deal with five different fractions. Lynsey recognized that as the size of the denominator increased the size of the pieces decreased. Yet she could not determine if 3/8 was larger than 7/12 without using concrete fraction kit pieces to model each fraction.

> I get fractions a lot more than I did, but I still had trouble deciding if 3/8 and 7/12, like I knew 12ths were a lot smaller, but I didn't know if it would be bigger or smaller. I was confused with that. (pd481)

In the stressful exam context she also reverted back to whole number thinking as she initially thought 5/4 and 4/3 were equal because they were both one more than a whole, even though she knew 1/3 was larger than 1/4. Nevertheless Lynsey correctly ordered the fractions using the manipulatives. She also readily recognized the efficiency of using the benchmark of 1/2 when prompted in the follow-up interview.

The remaining participant in the middle cluster did not solve this problem correctly. Irene could not visualize the fractions as relations because she persisted in seeing fractions as procedures. Initially Irene attempted to use a common whole to compare the fractions but she did not know how to represent the improper fraction on her measurement model (Figure 27). So she turned to the known procedure of finding a common denominator. However, Irene found it difficult to find a common denominator that included fifths.

> I had a hard time. I'm used to finding a common denominator but you can't. This [indicates 4/5], this thing threw me off all the time, because I could take 12, 8, 3, and 4 out of 24, but I can't do 5. So I just gave up after, because I thought this is not going to work. (pd470)

Because the fraction 4/5 did not fit within her procedural framework, Irene could not make sense of it as a meaningful object. Instead she placed the fraction as the smallest, even though she recognized that one of the other fractions was less than half. In doing so Irene demonstrated the fragility of her relational understanding of fractions.

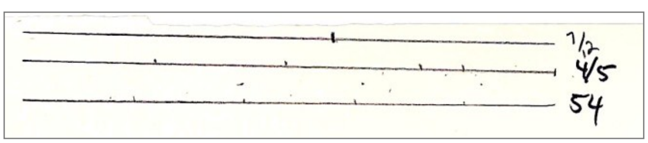


Figure 27. Irene (pd470)

Concluding Overview

The participants in the middle cluster demonstrated a moderate understanding of fractions as relations over the course of the study. All seven of these participants began the study with a strong reliance on memorized procedures. Many of them simply followed the rules and did not attempt to make sense of the fractions. During the study these participants willingly engaged in dialogue and used visual representations to deepen their understanding of fractions. Most of them displayed growth in understanding the relational concepts, however their understanding was fragile and often context dependent. Brenda, Lynsey, and Olivia moved from a procedural approach, tempered with some reasoning about parts and wholes, to a visual representation of fractions that enabled them to think about the inherent relationships. They were on the cusp of internalizing the notion of fractions as meaningful objects that are not limited to quantity of parts. Tanya moved from an unquestioned procedural approach to a solid ability to reason the size of fractions and use benchmarks. Erica, and Gabriela shifted from a solely rule-based approach to using visual representations to help them think about the relationship between fraction parts and wholes. They still felt more comfortable using their knowledge of decimals to confirm their shifts in thinking. These six participants held their relational understanding tentatively and tended to revert briefly to whole number thinking when they encountered novel problems or fractions with larger numbers. The final participant in this cluster continued in her understanding of fractions as procedures. While Irene shifted from simply using a formulaic approach her use of visual models sometimes became procedural. Her strong partitioning skills and proportional reasoning often masked her ability to see fractions as meaningful relations.

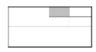
Top Cluster Results

Baseline Content Exam

The questions on the content exam served to establish participants' baseline understanding of fractions as relations. The questions, as seen in Table 11, focused on the nature of fractions, combining and comparing fractions, division of fractions as relations on relations, and recognizing a fraction part in order to find the whole. On the baseline content exam and follow-up interview participants in the top cluster demonstrated a partially formed understanding of fractions as relations that was either rule bound or contextually determined. While they were not able to clearly articulate this understanding, the responses of the participants to the four test items and interview question listed in Table 11 helps us delineate the boundaries of their relational knowledge of fractions.

Table 11

Content Exam – Fraction as Relations Questions

Item #	Code	Question
5	Fraction picture	What fraction is the shaded area of the whole (i.e. the largest rectangle)?  *(or*  *)*
15	Interview	What are fractions?
6	Sum fractions	Of the following fractions: 3/4, 5/12, 2/3, 3/2, 2/5 5/8 which two fractions (and only two) will give a sum that is less than 1?
4	Bag of peanuts	Rita had 6 kilograms of peanuts. She wanted to make bags of peanuts for her friends. She put 2/3 kg of peanuts in each bag. How many bags did she make, if she used all the peanuts? (4kg; 10kg)
8	Track team	There are 20 women on the university track team. One sixth of the team is men. How many students are on the university track team? (15; 25)

Defining fractions. At a basic level all four of these participants recognized that fractions are equal parts of a whole. They recognized that for the fraction picture (#5) the entire rectangle needed to be subdivided into parts equal to the shaded part in order to correctly determine the fraction. This understanding was confirmed with their responses in defining fractions (#15) as (equal) parts of a whole. Only two participants went beyond this definition. Megan could only think about fractions in terms of her context for using them, most especially baking. She noted that fractions were related to division.

> I guess fractions are if you're, you can use them if you are dividing something, sort of. So you can think in terms of dividing something into threes or thirds, if you have three people you can share something. (pd165)

Bryann, on the other hand, linked fractions to rules. She described fractions as her

> nightmare… I see a fraction and I break right down…. Math and fractions were so scary for me, 'cause all we would ever do was add, you had to have your common denominator and had to remember these rules and when to add and when to have a common denominator and it just, as soon as I see one I just like [pushes arms out and looks the other way]. (pd31)

Combining and comparing fractions. All participants in the top cluster were able to solve the sum of fractions question (#6) correctly, however, their use of a relational understanding of fractions differed. Two of the participants used the traditional procedural approach of common denominators to combine fractions to find a sum that was less than one. Mark used his numerical facility to find a common denominator of 120 for all six fractions.

Bryann, on the other hand, combined two fractions at a time in a trial and error approach. Neither of them, however, thought about the relative sizes of the fractions. They focused on the procedure. Thus Mark converted 3/2 to 180/120 and Bryann combined 5/8 + 3/2 even though 3/2 was clearly larger than the sum of 1 specified in the problem. While they recognized the need for a common whole in the algorithmic procedure, neither of these participants demonstrated a meaningful understanding of this concept within the given context.

In contrast, the other two top cluster participants used a conceptual understanding of fractions as relations to solve this problem. Megan needed to work within a familiar context in order to make sense of the problem.

> That was really hard. I went back to baking again because I sat there and tried to think of what my mom had told me about the denominator stuff and it just, I couldn't, I couldn't wrap my head around it, so I went back to the measuring cups. (pd156)

By using her measuring cup model Megan compared each fraction to a common whole; she recognized the relative nature of fraction sizes. However, she was not able to articulate the importance of this concept. So when asked why she used the same sized whole for each fraction picture, Megan replied, "I just think I'm a control freak. I don't know. It wasn't important; really, it just ended up that way" (pd156).

Diane demonstrated the strongest relational reasoning about fractions as she compared each fraction in the problem to the benchmark of 1/2 in order to find the smallest ones. She highlighted the importance of the whole, comparing the size of the numerator to that of the denominator.

> It's just the whole idea again of the whole. And so that's three-quarters, that's a half plus a half; a 1/2 and half of 1/2. And then this one [5/12] is less than a half, because it's all about the bottom number. If 12 were on the top then you would have a whole and you only have five of something divided into 12 so that's less than 1/2…. So I pulled out the lowest fractions and tried those first. (pd28)

Even though she was able to reason through the problem, Diane did not fully trust her understanding. She used the common denominator algorithm to confirm her thinking.

Relations on relations. In the bag of peanuts question (#4), a quotative division by a fraction problem, all of the top participants correctly differentiated between the two different wholes. They identified that the fraction referred to the one kilogram rather than the total number of kilograms. Three of these participants used a measurement approach to determine the number of bags need, illustrating a solid understanding of the meaning of fractions in this context. Only one of these three was not able to arrive at the correct solution. Bryann used numbers rather than pictures to illustrate the problem (Figure 28). Because she focused on the fractions as thirds, rather than using the context, she did not know what to do with the remainders and combined them into wholes instead of groups of two 1/3s. Megan did not have this problem because her measuring cup context helped her count units of 2/3s. Bryann got caught up in her procedural definition of fractions rather using the context to provide meaning. Nonetheless she, along with the others participants who correctly solved the problem, demonstrated a beginning awareness that the division of fractions involves relations on relations.

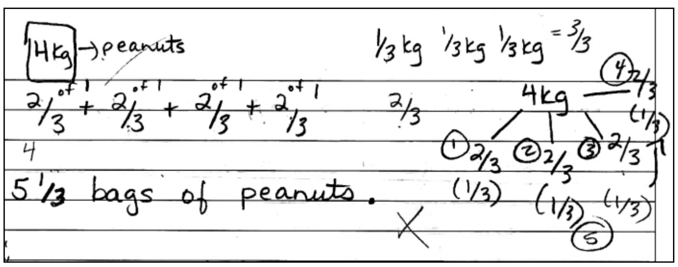


Figure 28. Bryann (pd20)

The final participant in this cluster, Diane, attempted to use an algorithm to divide the total number of kilograms by the fraction of kilograms, but did not succeed because she forgot the procedure. "I learned, years ago when I learned math, I learned by formula or the rules, and so I tend to get stuck on, well I can't remember the rule" (pd36). She did not know how to make sense of fractions as relations in a division context. In the follow-up interview Diane used a measurement model to demonstrate her growing understanding of these contexts. However, similar to Bryann, she had difficulty with relating the remainders to the changing unit.

Relating fraction part to unknown whole. On the track team question (#8), a part-part-whole ratio, all of the participants in the top cluster correctly related the number of women given to the complement of the fraction stated in the question (5/6). They recognized that the fractions representing the two parts of the whole must combine to give a total of 1. In this way they understood that the fractions represented a portion of the whole rather than an absolute value. Three of these participants used a fair shares model (Fosnot & Dolk, 2002) to distribute the number of women evenly over the five parts of the fraction, recognizing the correct portion of the whole. The fourth participant, Bryann, had more difficulty identifying the fraction whole, since she began with a more procedural approach. Bryann used a procedural approach to determine that the number of men was 2.5 by simply multiplying the numbers given in the problem (1/6 x 15). When this did not make sense she drew a model putting 2.5 in each of the six parts and eventually determined the correct answer (Figure 29). Her understanding of these concepts was fragile at best.

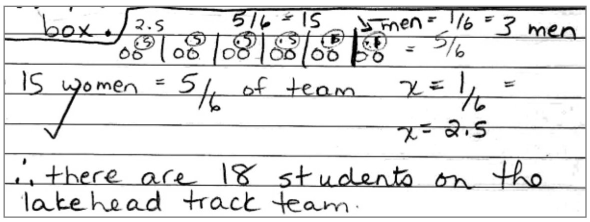


Figure 29. Bryann (pd24)

Overall on the baseline content exam and follow-up interview the participants in the top cluster demonstrated a partially formed understanding of the relational nature of fractions. In some instances the rote use of procedures hindered the development of sense making. In other instances the context and prior experiences helped participants make sense of challenging content.

First Problem-Solving Interview

In the first problem-solving interview the questions that highlighted the theme of fractions as relations expanded on the ideas of combining fractions, comparing fractions, and relations on relations in multiplying fractions (Table 12). This set of questions took place a month after the content exam and after participants had experienced several sessions of the mathematics skills workshop. Participants in the top cluster demonstrated growth in their relational understanding of fractions but this growth for some was limited by their reliance on procedural thinking. Others were challenged when the problems took them in areas outside their previous experiences. In the two paired problem-solving interviews Mark worked with Diane and Megan worked with Bryann.

Table 12

Problem-Solving Interview #1 – Fractions as Relations Questions

Item #	Code	Question
25	1/2 +1/3 = 2/5	Is 1/2 + 1/3 = 2/5? Explain. Can you give me at least two different explanations?
28	Order fractions	Put the following fractions in order from smallest to largest: 5/6, 6/7, 4/11, 4/12, 7/5, 6/8. Explain your reasoning. a) 4/11 or 4/12; b) 5/6 or 6/7
29	Bag of chips	Carlos had 3/4 of a bag of chips left after a party. His son found the bag and ate 1/3 of the remaining chips. How much of the bag of chips did he leave for Carlos?

Combining fractions. All participants in the top cluster knew that 1/2 + 1/3 = 2/5 (#25) was a false statement. Three of them immediately appealed to known procedures and rules as proof. As Mark stated, "What they've done, they've added the numerators and the denominators and they've come to answers, which is wrong…They've used the wrong rule" (pd234). These participants believed that using common denominators to combine the fractions and showing that the result was 5/6 not 2/5 was sufficient. This use of the algorithm alone does not convey the degree to which these participants understood fractions as relations. Bryann, for example, was limited by her quotative understanding of fractions. She found it difficult to compare fractions given her visualization scheme. When asked if 2/5 had any meaning for her Bryann replied, "Yes, 2/5 of something, I would draw it, and I would break it into five and I would cover up two" (pd293). This use of whole number reasoning fragmented her understanding about fractions and limited her ability to think relationally about them.

Megan, the only participant in this cluster who avoided the use of common denominators, visualized fractions in relation to her measuring cup and thus 2/5 was clearly less than half. She then reasoned that the result of the addition statement could not be less than what you started out with. Megan's ability to place fractions in a meaningful context enabled her to compare fractions using a common whole. When Mark and Diane were challenged to go beyond the algorithm, they also used the comparative sizes of the fractions to reason about the statement. They, however, first needed to work with concrete manipulatives to model the equation before they made the switch to comparing 2/5 to 1/2. In this process their thinking moved from rules and procedures, to modeling the procedure, to meaningfully thinking about each fraction in relation to the others and a common whole.

Comparing fractions. When comparing fractions with a common numerator (#28) once again two different approaches emerged. Diane and Mark reasoned that since the numerators were the same, the fraction with the larger denominator was the smaller fraction because the whole was cut into slightly smaller pieces. Similarly Megan realized that 1/12 was smaller than 1/11 and when she recognized that both fractions had the same numerator she quickly determined that 4/11 was the larger fraction. All three of these participants thought about the size of each part in relation to the whole. Bryann, on the other hand, had difficulty using this type of reasoning. While she immediately chose 4/11 as the larger fraction Bryann disengaged when her partner explained her relational approach. As Bryann stated, "oh, I would just totally multiply the bottoms together and find out; I can't see it like this, it frustrates me… yeah, I have a really hard time" (pd296). Bryann trusted her memorized procedure more than trying to reason about the relative size of the fraction pieces. Ironically, when she made a multiplication error in finding common denominators and both fractions seemed to be the same size, she didn't know how to proceed until she found the error and certainty was restored. "That's the only way I would, yeah, be able to do it; because then you see, you have a common denominator... [I] cannot visualize" (pd296).

Comparing fractions that were both one piece away from the whole (#28) was more challenging and not all of the participants in this group were able to solve it correctly. Diane and Mark were able to set aside their use of common denominators to think relationally about the fractions. Diane immediately recognized the inverse relationship between the size of the gap piece and the size of the fraction.

> Because the whole has been cut into slightly smaller pieces for this one [6/7] and the whole has been cut into slightly bigger pieces for this one [5/6] so having one less, this one piece that's missing is slightly bigger than the one piece that's missing here [6/7], so it leaves more of the whole. (pd237)

Mark, however, needed to explore the reasoning visually using physical manipulatives to ensure that it made sense. Using a simpler example of 2/3 and 3/4, he saw the inverse relationship between the size of the piece each fraction was missing and the size of the fraction itself. The physical movement of the pieces seemed to help solidify and generalize his reasoning.

Megan and Bryann were not able to solve this problem correctly. Megan's reasoning focused on the size of the denominator rather than the whole fraction. Thus she determined that the fraction with the larger pieces (i.e. smaller denominator) had to be the larger fraction. When simpler fractions illustrating the same concept were used Megan could readily draw on her measuring cup knowledge and correctly determine the larger fraction. Even though she was looking for patterns and trying to understand the relationship Megan missed the inverse connection that was present and could not generalize to the larger numbers. Unlike Mark, Megan used her model to examine the size of the pieces within each fraction, rather than the size of the piece that was missing from the whole. By focusing on the denominator, Megan lost sight of the whole and the complement fraction. In contrast, Bryann only focused on the number of pieces that were missing. Since each fraction was one piece away from being whole she assumed they were the same size. "Because six sixths is one whole and then seven sevenths is one whole and they are the same portion of each whole [that is missing]" [p296]. Bryann attempted to think relationally but focused on the absolute quantity rather than the relative size of the portion. She would have preferred to go straight to the common denominators so she could unequivocally compare sizes.

Relations on relations. Only Diane and Mark completed the bag of chip problem (#29) during the first problem-solving interview. Mark immediately recognized that the whole changed from the entire bag of chips to the amount left in the bag so he visually found 1/3 of the three portions. He understood that for this multiplication problem a relationship existed on the fractions relations. In contrast, Diane did not recognize the changing whole. She used a procedure that allowed her to work within the one whole of the entire bag of chips (Figure 30). She was surprised to find out that her area model of multiplication of fractions gave the same answer as Mark's quick method.

> But I couldn't see it unless I did that [draw the area model]. Isn't that wild? I have to do that. But I can't just immediately say that that's like 1/4, because I think it's going to be something trickier…. It's so easy. But I have to [indicates drawing model] find it. Wow! (pd238).

While her use of the area model for fraction multiplication represented a deeper understanding than the traditional algorithm for fraction multiplication, Diane still did not fully grasp the fraction relationships in the problem. In some sense the use of this model limited Diane's understanding of the changing whole.

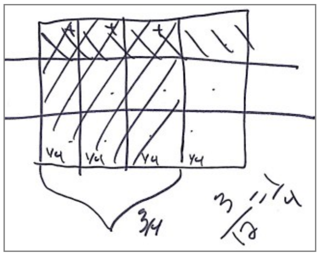


Figure 30. Diane (pd238)

Over the course of the first problem-solving interview participants in the top cluster developed their understanding of fractions as relations to varying degrees. Of the three participants who began the study with a more rule bound view of the nature of fractions only Bryann demonstrated a reluctance to let go of the familiar procedures. She had difficulty visualizing fraction relations and was hampered by a part-whole understanding that at times prioritized the absolute quantities rather than the relative size of fractions. In contrast, Diane and Mark willingly looked beyond their known fraction procedures. Diane demonstrated a strong sense of the relational nature of fractions when comparing different types of fractions. But limitations were revealed when she encountered more complex situations demanding a choice of the correct whole. Here her reliance on procedures was still prevalent. Mark was able to use concrete models of fractions to shift his efficient numerical facility with fractions to a deeper relational understanding. He readily distinguished between the two different fraction wholes in a multiplicative context. The remaining participant in the top cluster, Megan, pushed the limits of her contextually driven measuring cup model. She willingly looked for patterns and successfully used her model to compare fractions, but the limitations of the model became apparent in more complex situations.

Second Problem-Solving Interview

The questions in second problem solving interview that highlighted the theme of fractions as relations expanded on the ideas of the importance of the whole, comparing fractions, and relations on relations in multiplying and dividing fractions (Table 13). Participants in the top cluster continued to be challenged to think beyond their memorized procedures. Many employed models to help make these shifts in their thinking. Most of them demonstrated a deepening understanding of fractions as relations in a variety of contexts. Nevertheless, in a number of instances reliance on procedures revealed the fragility of parts of this understanding.

Table 13

Problem Solving Interview #2 – Fractions as Relations Questions

Item #	Code	Question
33	Rows to knit	Roberta has knit 260 rows of the afghan she is making for a friend. She still has 1/5 of the afghan to finish. How many rows does she still have to knit?
35	3/7 or 3/8	Without using division (i.e., converting to a decimal) determine which of the two fractions is larger. Explain your reasoning. a. 3/7 or 3/8
36	7/15 or 11/20	b. 7/15 or 11/20
37	14/15 or 17/18	c. 14/15 or 17/18 (use 5/6 or 6/7 if not used in interview #1)
41	Halfway between	What fraction is exactly halfway between 1/2 and 1/3?
42	2nd between	Find a second fraction between 1/2 and 1/3.
43	Ribbon	Avery has 5½ metres of ribbon. She would like to cut up small ribbons for her friends. How many ribbons can she make if each small ribbon is 3/8 of a metre? How much ribbon will be left over?

Relating fraction part to unknown whole. On the rows to knit question (#33), a part-part-whole problem modeled after the track team problem (#8), all participants in the top cluster correctly related the number of rows completed with the complement fraction rather than the whole. The use of larger numbers did not deter their understanding. Since the fraction represented one of the parts rather than the whole, both groups divided the total number of rows by the numerator in order to find the portion of rows in each fifth of the afghan. They understood how the numerator and denominator related to the parts of the whole as well as to the absolute quantities given in the problem. Nevertheless, two of the participants demonstrated some fragility in their understanding. Bryann did not give any constructive input while her partner was solving the problem. Even though she nodded her head slightly at certain points to indicate agreement, Bryann sat back as an observer rather than participant, until she answered a concrete addition question at the conclusion of the problem. It may be that she was not confident in verbalizing her understanding of this problem. Diane also demonstrated fragility understanding these fractions relationships. While she readily determined the complement fraction Diane was uncertain whether she needed to divide the number of rows by the numerator or the denominator. While Diane focused on the procedure and the equation, Mark concretely modeled the problem clearly demonstrating the relationships. The videotaping as well as her partner's quick grasp of the problems flustered Diane. When asked why she would have divided by the denominator, Diane replied, I felt pressured; I couldn't think. I knew that I was supposed to know it and I just couldn't get there. At home if I were alone doing my homework then I would know that 4/5 = 260 and then I would probably draw the thing like Mark did, and then I would see it. But I just, I can't do it fast…. Like I'm very laborious [in doing all the steps]. (pd319)

Nonetheless it was apparent that both partners contributed to the correct thinking in solving this problem. Mark acknowledged that he needed Diane's initial insight about the complement fraction in order to understand the problem. "So when she said 4/5, that's why I reread the question; that's the only reason why I picked up on that. So I may have done it wrong" (pd319). Mark intuitively recognized the different aspects of fraction relationships while Diane needed to follow the logical steps of a procedure in order to make sense of the relationship.

Comparing fractions. When comparing three different pairs of fractions the participants in the top cluster mainly used their understanding of fractions as relations to determine which was larger. But as the fractions became more complex some preferred to use the familiar common denominator procedure to verify their thinking, not fully trusting that this understanding could be sufficient. When comparing fractions with the same numerator (#35) all participants in the top cluster recognized that the smaller denominator would give the larger fraction. Three of them focused on the size of the unit fraction recognizing that, in Megan's words, "each individual piece is smaller, so the three pieces together would still be smaller than those three pieces [pointing to 3/8 < 3/7]" (pd380). Bryann, however, did not focus on the denominator but the portion of the whole that the 3 represented (or covered). Even though she understood the proportional relationship of the fraction, recognizing that 3/7 was more than 3/8, Bryann mainly thought about fractions as two discrete numbers. While Bryann and Megan needed to use a drawing to confirm their thinking because they got caught up in the language of smaller and larger, Mark and Diane were confident in their reasoning about the relative sizes of the denominators.

All participants in the top cluster wanted to use common denominators to compare 7/15 and 11/20 (#36). When challenged to think about the meaning of the fractions first, Diane immediately thought about relating the fractions to her knowledge of 1/2.

D: I usually go to the halfway, like half of 15 is 71/2 and half of 20 is 10, and so 7/15 would be a little bigger, is that right? Because 7.5 would be half and 10 would be half…

M: This is really close, 'cuz 7 is almost half, but 11 is over half, so 11/20 is bigger, because it is over the halfway mark. (pd322)

Megan and Bryann needed to draw pictures in order to think about the factions. Initially Megan incorrectly assumed that only the denominator was important and so the small denominator would be the larger fraction. But as she drew the fractions, Megan recognized that 11/20 was just over half and thus also thought to compare 7/15 to 1/2. After articulating this relationship, all participants felt confident in their reasoning and did not need to confirm their answer using a common denominator.

The final comparison of 14/15 and 17/18 (#37) challenged the participants in the top cluster, even though they had completed a similar problem in the first problem-solving interview. While all participants eventually reasoned correctly about the size of the fractions three of them began with an incorrect understanding. Bryann focused on the number of missing pieces and, since both fractions missed one piece from being whole, thought they must be the same size. In contrast Megan and Diane began by focusing on the size of the denominator and assumed that the smaller denominator gave the larger fraction. Each then took a different route to change their reasoning. Megan used a pair of simpler fractions that enabled her to use her measuring cup model. She recognized that since they both had one piece missing the fraction with the smaller sized pieces, or the larger denominator, would be the larger fraction. As Megan generalized this pattern to the larger fractions, Bryann could also affirm this reasoning. “I like how you said that, 'cuz you’re missing a smaller piece so that fraction [indicates 17/18] would be bigger” (pd382). Diane, on the other hand, was able to reason about the inverse relationship between the size of the gap piece and the remaining fraction without using a model. Mark needed to work through this reasoning by drawing a picture that illustrated the different sized missing pieces in relation to the remainder. The tentative nature of his understanding of this inverse relationship was underscored by his need to use cross multiplication to compare numerators with common denominators.

Ordering fractions as measures. The next set of problems extended the idea of comparing fractions with the measurement understanding of fractions since participants needed to find a fraction between two given fractions. To begin this section all participants in the top cluster knew that 1/4 was not between 1/2 and 1/3 (#40). While Mark and Diane had several strategies for comparing the fractions that reflected an understanding of the relation to a common whole and to decimals, Megan and Bryann just knew that 1/4 was smaller than both of the other fractions. When challenged to use a number line Bryann had difficulty placing 1/3 in relation to 1/4 and 1/2. Her procedural view of fractions worked well for computations, but did not provide a foundation for understanding the relational nature of fractions in a measurement context. Megan, on the other hand, readily extended her relational understanding of fractions to a measurement context because of contextual experience with measuring cups.

All participants in the top cluster experienced some challenges when asked to find a fraction exactly halfway between 1/3 and 1/2 (#41). They had varying degrees of success in transferring their understanding about fractions as relations to this measurement context as they implemented known strategies and procedures in this unknown context. Two participants began using common denominators or equivalent fractions but neither of them knew how to complete the process. After converting the fractions to 2/6 and 3/6, as she would do for addition, Diane deferred to her partner's thinking rather than exploring what 21/2 /6 might mean. Megan followed a similar process but extended the equivalent fractions to twelfths. Even though she knew that 5/12 was between the fractions she did not know how to determine if it was exactly halfway between the two. Mark took a much more procedural approach to the problem using a logic that worked for whole numbers. He simply found the difference between the two fractions, divided this in half, and then added this amount to the smaller fraction. Even though Mark did not use his understanding of fractions as relations to simplify the process, he demonstrated a solid understanding of the procedure as he explained his thinking using a number line. But because of this focus on procedure both Mark and Diane struggled to find half of 1/6.

Bryann, on the other hand, could not work meaningfully with fractions in this measurement context. She wanted to use decimals but the repeating decimal hampered this process. She tried to use the fraction kit but did not know how to use it to model the problem. Even when her partner had an answer and they were looking for a way to verify it with the fraction kit pieces, Bryann struggled to make sense of the problem.

> I don't even know how to see what's halfway between that though, like there is no way for me to use them to see what's halfway between that one [1/2] and that one [1/3], yeah, I can't see it. (pd385)

Bryann could make sense of fractions when she converted them to decimals or when she used a model that followed her quotative thinking (e.g., three out of five) but she did not understand the relational nature of fractions in a measurement or number line context. Thus when asked to find a second fraction between 1/2 and 1/3 (#42) Bryann did not think there would be any and indicated through gestures that this went over her head. Megan immediately found an equivalent fraction for her previous answer, but she also did not think there would be any other possible fractions between the two.

Mark and Diane still struggled with the measurement understanding of fractions when looking for a second fraction. As Mark stated, "This is hard; this is making me use my mind" (pd328). But both showed progress in their reasoning as they no longer used a mechanical process to find the answer; instead they eventually used their knowledge of equivalent fractions and the number line model to find a second fraction.

Relations on relations. The final question for this second problem-solving interview focused on understanding relations on relations – fractions in multiplication or division contexts. Each pair completed a different question. Megan and Bryann worked with the multiplication context, bag of chips (#29), that Mark and Diane completed in the first problem-solving interview. Megan immediately recognized that the two fractions each referred to different wholes and used her measuring cup model to illustrate the problem. But she did not know how to make sense of her answer of 2/3 of 3/4. Bryann also had difficulty interpreting the changing whole. She attempted to model the problem with fraction kit pieces, using the 3/4 as the whole, but did not understand that each fraction referred to a different whole rather than a common whole. Thus she changed the problem to subtraction and found that 1/4 plus 1/6 of a bag were left. The partners then focused on using common denominators, stating incorrectly that 5/12 was left, which left Megan confused.

> But aren't you sort of saying that 3/4 is the whole though? Well you're not starting with the full bag, you're starting with 3/4 of the bag so 3/4 sort of is the whole of that piece. The number line would probably work but I can't use number lines, they confuse me. But I just think that's the sort of thing [the workshop instructor] uses. (pd374)

In the guided follow-up Megan and Bryann explored the meaning of 1/3 when 3/4 was the whole. While Bryann was able to shift her thinking, Megan still wanted to use one as the whole. Megan struggled as she thought through the shifting wholes. "Each quarter is a third. That is really confusing" (pd374). Even though Megan initially recognized the meaning of the change in the whole, once the common denominator process was introduced she lost sight of this idea.

Since Diane and Mark had completed the bag of chips problem in the previous interview they worked on the ribbon problem (#43). They correctly solved this division problem by drawing a linear model, dividing it into eighths and measuring out groups of 3/8s. While Mark and Diane understood conceptually what it meant to divide by a fraction they still encountered a couple of challenges. Diane did not know how to link the process to symbols and algorithms. She went from converting to improper fractions, to multiplying, to cross-multiplying and finally to wondering if she needed a common denominator. Mark explained that since they were dividing fractions they needed to cross-multiply. Neither made any connections between their memorized procedures and the process they actually used. As Diane stated,

> I know that's the better way of doing it [points to model] and if I were teaching it I think it's way better to do it like that because you show the kids, you can internalize it and figure it out; but I just saw the fractions and I like working with the fractions and I just messed something up over here [points to calculations]. (pd329)

After completing the calculations Mark and Diane also had difficulty reconciling the two different responses they got for the remainder. In their visual model they had two of the eighths left over, so they knew the remainder was 1/4 of a metre. But using their algorithm they had a remainder of 2/3 and did not know how to interpret it. They struggled to understand that the unit determined the remainder. So in this case since they were counting groups of 3/8, Mark eventually recognized that, "You would be able to make 2/8s of the 3/8s, that would be 2/3" (pd329).

Over the course of the second problem-solving interview the four participants in the top cluster showed ongoing development of their understanding of fractions as relations in ways similar to those revealed in the first problem-solving session. Megan continued to use her measuring cup model to make sense of challenging problems and demonstrated a willingness to look for patterns to generalize her understanding about fractions relations. Even though she often began with an incorrect assumption, Megan was able to reason through the situation using a model and come to the correct answer without external prompting. However when she tried to use memorized procedures Megan encountered confusion that could not be resolved. Diane continued to demonstrate a strong relational sense when comparing fractions. She thoughtfully reasoned about the inverse relationship between the gap piece and the size of the fraction without a physical model. Nevertheless Diane encountered difficulty making links from her conceptual understanding to the memorized algorithms for fraction operations. Mark continued to use his numerical facility to do mental calculations to verify his shift from rote procedures. He willingly used visual models to help him think about the relationships. In unfamiliar contexts Mark preferred to use known procedures than be challenged to think about fraction relations; but he readily shifted his approach to include a deeper understanding of the relationship. Bryann continued to be limited by her reliance on memorized procedures. Even though she thought about fractions proportionally and recognized when the whole changed, Bryann found it difficult to visualize fraction relations and thus give a contextualized meaning to the fractions. Her quotative understanding of fractions seemed more conducive to converting fractions to decimals. Bryann seemed quick to agree with her partner's deeper understanding about fraction relations but was slow to integrate them into her own active understanding.

Post-Test Retake Exam

The final test instrument had one question that highlighted fractions as relations (Table 14). It focused on the need for a common whole in order to compare fractions. For this instrument the participants once again worked individually on the problems. Three of the participants in the top cluster used their conceptual understanding of fractions to correctly complete the order fractions question (#49). They used the benchmarks of 1/2 and 1 to compare fractions and reasoned about the relative sizes of unit fractions as necessary. All three showed growth in their understanding of fractions as relations over the course of the math workshop. At the beginning of the study both Diane and Mark would have focused on finding a common denominator for all fractions without thinking about the actual meaning of the fractions and their relation to the whole. They shifted from thinking about fractions as procedures to fractions as relations. As Diane stated,

> I would have thought that I should instantly find the common denominator and go from there; I wouldn't have understood that it's, how to relate it to the whole. I wouldn't have done that before taking the math workshop; I wouldn't have, even in my JI [math methods] class I wouldn't have done that. It's the math workshop. (pd419)

Megan shifted from needing to draw out fractions using her measuring cup model to having an internalized sense of fraction size based on a more explicit understanding of parts in relation to the whole. She no longer saw fractions as simply actions; they had meaning in and of themselves.

Table 14

Post Test Retake Exam – Fractions as Relations Question

Item #	Code	Question
49	Order fractions	Put the following fractions in order from smallest to largest: 5/4, 4/5, 4/3, 3/8, 7/12.

The final participant in the top cluster, Bryann, experienced difficulty with this problem. She tried to use some of what she had learned in the math workshop but her understanding about fractions as relations was fragmented at best.

> That one hurt…. The 4/3 and the 5/4, like I know there was more than a whole there and I did not know what to do with it. And there were so many different denominators that it just wasn't making sense to work with the manipulatives for me, to make the whole piece because there were so many different wholes. So I said let's try to get all decimals here. I went by my rule that I know to put the bottom into the top. (pd410).

Even though she correctly calculated all the decimal equivalents, Bryann still made an error in ordering the two improper fractions. She did not know how to reason about the relative sizes of fractions since they still were two separate numbers that had no real meaning unless they were manipulated into one number as a decimal. Bryann revealed more of her limitations of relational knowledge when challenged to find another way to compare fractions. After drawing a rectangular area model of 3/8 and 5/12 Bryann explained her thinking.

> These pieces [pointing to 3/8] are bigger than these ones [7/12], so this fraction [circles 3/8] would be bigger than these smaller pieces [7/12]; because these pieces over here [points to each eighth] are bigger than these ones [points to twelfths]. (pd410)

Bryann mistakenly related the size of each part to the size of the fraction and when confronted with this contradiction she pondered for several moments before sheepishly stating that more of the whole has been covered with the 7/12. Bryann lacked a coherent model for making sense of fractions as individual objects. She did not believe that it was possible to compare fractions without drawing a picture or converting to decimals and so using benchmarks, as the other participants in this cluster did, did not resonate with her conceptual understanding. In spite of all of these limitations, Bryann did make some progress during the study. She now looked for alternative ways to understand fractions before resorting to her memorized procedures. Nonetheless Bryann still needed more opportunities to explore the nature of fractions as relations.

Concluding Overview

As anticipated, the majority of the top cluster of participants demonstrated a fairly strong understanding of fractions as relations over the course of the study. Diane and Mark moved from a strong procedural approach to an internalized understanding of the importance of the whole in part-whole relations. Similarly, Megan moved from a reliance on a visual representation of a particular model to an internalized understanding of part-whole relations. All three of these participants engaged with fractions as meaningful objects. Surprisingly, the remaining participant demonstrated a much weaker understanding of the relational nature of fractions. Bryann understood the proportional nature of fractions, especially because of her use of decimals, but at the same time she continued to maintain a limited part-whole understanding. While she attempted to use a variety of strategies to make sense of fractions Bryann struggled to visualize them and preferred to continue with familiar algorithms. Thus, the degree to which participants in the top cluster solidified their understanding of the relational nature of fractions seemed to depend on their willingness to let go of procedural thinking and engage with fractions as meaningful objects.

Pre-service teachers in all three clusters demonstrated gains in their understanding of fractions as relations. They willingly engaged in oral reasoning and in the use of visual models. Nevertheless, in many instances this growth was fragile as pre-service teachers folded back to their part-whole understanding of fractions. Did this differ from their understanding of fractions as ratios?

Chapter Six: Results and Analysis – Fractions as Ratios

Ratios are an inherent aspect of fractions. As relations, fractions highlight certain relationships such as those between the number of the parts in the whole and the size of those parts, those between the number of parts and the unit whole, and those between equivalent fractions. While the notion of ratio encompasses more than fractions, all fractions are ratios, since ratios are defined as the comparison of two quantities in a multiplicative relationship. Fractions typically represent a part-to-whole relationship between similar types of quantities; however, ratios of part-to-part comparisons of quantities with the same whole may also be represented in fractional form. Thus context plays an important role for understanding fractions as ratios.

Students who have a solid understanding of fractions as ratios understand the notion of relative amounts in contrast to absolute amounts. (Charalambous & Pitta-Pantazi, 2007; Lamon, 1993, 2007). They use multiplicative rather than additive thinking. These students understand that as one quantity of the ratio changes the other quantity must also change so that the relationship between them remains constant. Nevertheless students who think about fractions as procedures rather than relations may still correctly solve problems with ratios because they know an appropriate algorithm. These students simply see ratios as the separate quantities rather than the relationship between the quantities and therefore they do not recognize the ratio as a distinct meaningful unit. In order for students to develop a solid understanding of fractions as ratios they must also develop their proportional reasoning skills (Lamon, 2005; 2007). They can accomplish this, in part, by being willing to partition and iterate composite units (Smith, 2002).

Each of the four test instruments had items that assessed the concept of fractions as ratios. Table 15 presents an overview of the items on the various instruments. Participants began the study with a mixture of additive and multiplicative thinking as they worked with the ratios. Many of them could not ascertain the scalar constant that operated on the ratio, nor did they interpret the ratio as a distinct meaningful unit.

Table 15

Test Items Reflecting Fractions as Ratios

Instrument	Item #	Code
Baseline Content Exam & Follow-up Interview	9	ratio professors
Problem-Solving 1	20-21	food ratios
	27	stamp collection
Problem-Solving 2	38	ratio cookies
Post Test Retake Exam & Follow-up Interview	46	Muffins
	53	ratio boxes

Low Cluster Results

Baseline Content Exam

The three participants in the low cluster began the study with a limited understanding of fractions as ratios. Their responses to the baseline question, which focused on a part-part-whole ratio context (Table 16), revealed a reliance on partially remembered procedures. They did not recognize the ratio as a meaningful object but instead worked with the individual quantities being compared. As a result they did not know how to represent or interpret the given relationships.

Table 16

Content Exam – Fractions as Ratios Questions

Item #	Code	Question
9	ratio professors	There is a ratio of three male professors to two female professors on faculty. If there are 20 professors in all how many are women and how many are men? (three – one, 20; 25)

None of the participants in the low cluster correctly solved the ratio professors (#9) question. They could not make sense of the nature of the ratio and ensuing proportion. Valery did not attempt the problem because she "did not like it" (pd207). The other two participants both used partially remembered procedures that did not adequately capture the relationships. Grace did not understand the nature of the ratio as a comparison that yielded a constant relationship between the two parts and incorrectly assumed the total referred to the absolute value of the parts. Thus she created part whole fractions and attempted to use algebraic procedures to solve the problem (Figure 31). Isabelle expressed the constant relationship between the two parts as a fraction, but she was simply following a rule and did not know how this relationship functioned within the proportion (Figure 32).

Somehow I only remember special rules and that's what stuck to my mind. And that's where I went, okay, what do I do now, because I'm going, it doesn't make sense. (pd112)

None of these participants understood ratios as meaningful entities separate from the items being compared.

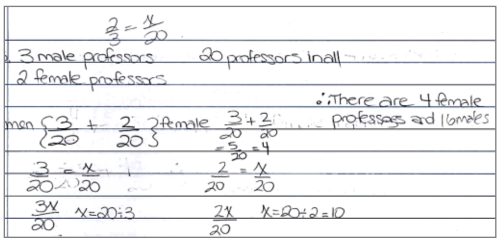


Figure 31. Grace (pd85)

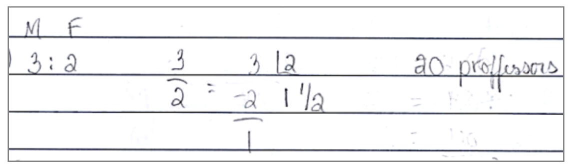


Figure 32. Isabelle (pd112)

First Problem-Solving Interview

The first problem-solving interview contained two questions that highlighted the theme of fractions as ratios (Table 17). These problems focused on the multiplicative relationships in a part-whole setting as well as the unknown complement part. Participants in the low cluster demonstrated slight gains in their work with ratios. They no longer struggled with partially remembered procedures but instead focused on making sense of the context. Nevertheless these participants continued to focus on the comparative values rather than the ratio relationships. For both problem-solving sessions Isabelle and Grace each worked with a partner from the middle cluster while Valery worked on her own.

Table 17

Problem Solving Interview #1 – Fractions as Ratios Questions

Item #	Code	Questions
20-21	food ratios	The Vancouver Aquarium has three beluga whales: small, medium and large. The medium whale eats 2 times as much as the smallest; the largest eats 3 times as much as the smallest. a) If the total amount of food was 60 kg, … b) If the total amount of food was 90 kg, … how many kilograms did each of the whales receive?
27	stamp collection	One out of every eight stamps in Arnie's stamp collection is Canadian. He has 720 stamps in his collection. How many foreign stamps does he have?

The two participants in the low cluster who completed the food ratios (#20-21) problem, both solved it correctly. They recognized the multiplicative relationship between the parts but they did not know how the parts related to the total. Both Isabelle and Valery used a guess and check approach. Isabelle became frustrated when she knew there must be an easy way to approach the problem. “I feel like those little kids on TV wondering, what do we do next?” (pd278). She started working with manipulatives and groups of ten when her partner discovered that this guess for the smallest gave the correct total. Similarly Valery began guessing with 10, “just because I like 10s” (pd300). Both Isabelle and Valery continued with their trial and error strategy for the second part of the question. While they maintained the given proportions these participants only worked with the absolute values of the quantities and did not understand the ratio as a meaningful entity. They could not extend their reasoning to relationships with the ratio.

Only Isabelle completed the stamp collection (#27) question during the first problem-solving session. She used a model to represent the given part-whole ratio and readily determined the complement fraction for the unknown part using this model. Isabelle demonstrated the fragility of her understanding of ratios when she suggested they divide the total by 7, the number of parts for foreign stamps, rather than the denominator. Within the context of ratios Isabelle had difficulty recognizing the relationship between parts and whole. Nevertheless, with the help of her partner Isabelle correctly solved the problem.

Two participants in the low cluster demonstrated some limited growth in their understanding the nature of ratios during the first problem-solving interview. Isabelle set aside her procedures and attempted to reason about the relationships. Valery willingly engaged with the problems, trusting her ability to reason and make sense of the contexts. Both participants worked with the absolute values of the quantities maintaining the given ratios; but they did not understand the ratio as a meaningful unit and could not extend their knowledge of the ratio parts to the whole.

Second Problem-Solving Interview

Participants in the low cluster were challenged by the several aspects of fractions as ratios in the second problem-solving interview. Some found the large numbers of the part whole ratio from the previous session difficult while others found the non whole number multiplier in a part-

part ratio (Table 18) problematic. Participants who clung to procedural thinking showed little development in their understanding of the nature of ratios. But those who used concrete representations or identified relationships demonstrated some gains in their understanding of ratios.

Table 18

Problem Solving Interview #2 – Fractions as Ratios Questions

Item #	Code	Questions
38	ratio cookies	Maria and her son Owen were decorating cookies for the bake sale. For every 8 cookies that Owen decorated, Maria decorated 30. If Maria completed a total of 75 cookies by herself, how many cookies did Owen complete?

Two participants completed the stamp collection (#27) question from the previous session in this second problem-solving session. Each of these participants encountered a different challenge. Grace used the cross multiplication procedure, 1/8 = x/720, without thinking about the meaning of the ratio and simply assumed *x* would give the answer. She did not recognize the need for the complement part until challenged by her partner.

> I thought, we know that 1/8 is already Canadian and we have 720 stamps altogether. So I thought the *x* would just give me the foreign ones. But it would make sense that if there is 90, obviously that is not, that's not the ones that are foreign because there are so many and that's only 1/8 that's Canadian.

Grace focused on the procedure and did not work with the ratio as a meaningful unit. Valery, however, did see the importance of the composite ratio unit. She used concrete modeling of this unit as the basis of her process to build up to the required quantity, by adding groups of four iterations (Figure 33). Nevertheless, Valery's poor number skills reflected additive rather than multiplicative thinking about the ratio.

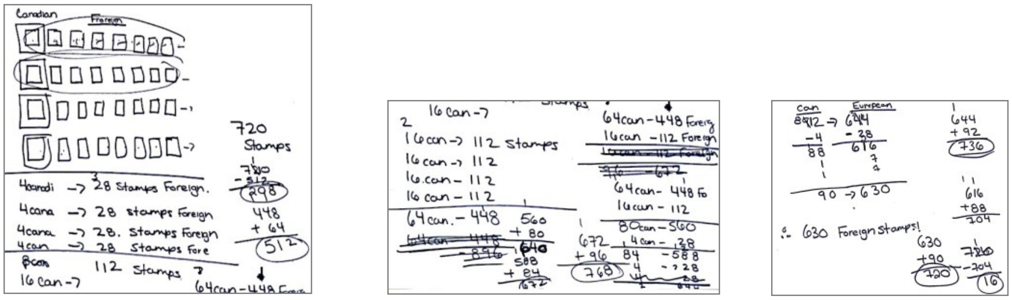


Figure 33. Valery (pd387)

The ratio cookies (#38) problem proved to be challenging for the participants in the low cluster who continued to hold on to procedural thinking. Grace and Isabelle struggled to make sense of the ratio as a meaningful entity and could not solve the problem correctly without help. Both assumed past procedures would continue to work, regardless of changing contexts. Grace used a cross multiplication procedure based on the part-whole relationship even though this problem was a part-part ratio (Figure 34). "I'm upset because it has been working for me all along…. I guess it doesn't work all the time" (pd354). She struggled with the multiplicative reasoning her partner used when finding half of the ratio and instead relied on additive thinking. "So you just added whatever it would take to get 75, to 60?" (pd354). Similarly, Isabelle could only understand the whole number multipliers of the ratio. She used a ratio table in a procedural fashion, doubling the ratio, but quit when she could not find a whole number multiple that yielded the necessary total. "The ratio doesn't work here" (pd373). She expected that procedures she used in the past would continue to work in all circumstances. "Why is it so complicated, though? See I never seen it that way" (pd373). Isabelle used additive thinking and could only build up the ratio in whole number iterations.

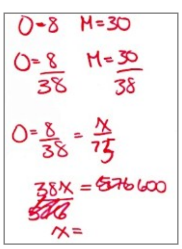


Figure 34. Grace (pd354)

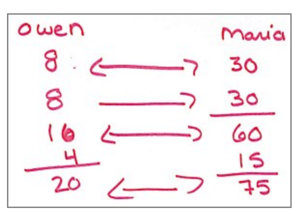


Figure 35. Valery (pd395)

The final participant, Valery, used the ratio table in a meaningful way that reflected an understanding of the ratio (Figure 35). While Valery used additive thinking as she built up the ratio, she kept the composite unit intact and appropriately partitioned this unit.

Participants in the low cluster demonstrated limited gains in their understanding of fractions as ratios during the second problem-solving session. Those who maintained their procedural approach experienced difficulty working with the ratio as a meaningful unit. Grace and Isabelle continued to focus on the absolute values of the quantities without a full understanding of the ratio relationships. Yet each demonstrated a willingness to explore a concrete representation of the ratio after their methods failed. Valery, on the other hand, began by working concretely with the ratio on each of these questions. She demonstrated gains in her ability to iterate and partition the composite ratio unit.

Post-Test Retake Exam

The two questions on the final test instrument that addressed the theme of fractions as ratios focused on part-part and part-part-whole ratios (Table 19). Participants in the low cluster demonstrated a deepening understanding of ratios as they worked with the composite ratio unit rather than memorized procedures. The fragility of their understanding, however, became evident in the use of additive thinking by some of the participants as well as the need for concrete visual representation of the ratio by others. For this test instrument participants once again worked individually to solve the problems.

Table 19

Post Test Retake Exam – Fractions as Ratios Questions

Item #	Code	Questions
46	Muffins	You are baking muffins. The recipe for 12 muffins calls for 11/2 cups of flour. You want to make 18 muffins. How much flour will you need?
53	ratio boxes	Sue and her younger sister Amy work together boxing cookies for sale at the local bakery. Sue can make up 4 boxes of cookies for every 3 that her younger sister does. At the end of their shift they had 49 boxes in total. How many boxes did Sue make?

Only one of the participants in the low cluster solved the muffin (#46) problem correctly. Isabelle recognized that since the recipe increased by half the amount of flour also increased by half (Figure 36). Even though Isabelle reasoned about the ratio relationship she still needed to use procedures to combine the fractions. The other two participants struggled to make sense of the mixed fraction in the ratio. While both Grace and Valery could readily double the recipe and the amount of flour they struggled to visually represent the ratio relationships. Grace identified the scalar multiplier but she did not know how to use it with a fraction. Instead she switched to decimals and stated that .75 cups of flour was needed. Because she did not know how to combine the decimal and the fraction, Grace simply stated this was the total amount required. “When I was trying to think of it in my head I was confused because I didn’t know how to really show it” (pd456).

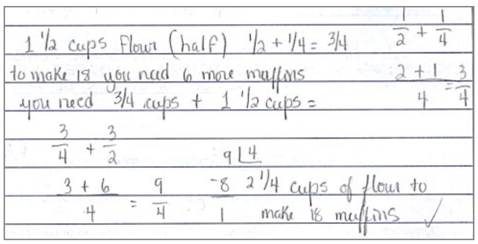


Figure 36. Isabelle (pd569)

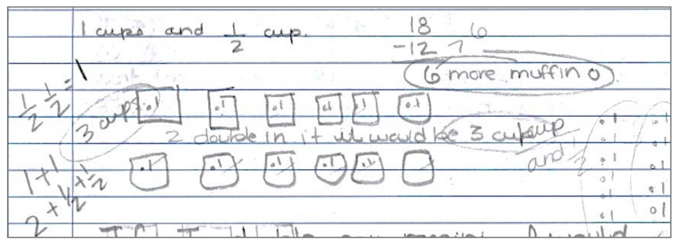


Figure 37. Valery (pd546)

In contrast to the first two participants Valery could not identify the multiplicative relationship in the ratio. She thought in additive terms, knowing only that six more muffins were needed. Furthermore since Valery struggled to visualize the composite ratio unit she could not build up the ratio using this unit. She attempted to find an alternative method to represent the ratio, but did not succeed (Figure 37).

> I think what threw me off was the fact that it was 1 and 1/2 cups…. Any problem like this I always have to draw it out to solve it and that one just threw me off because I couldn’t draw it. So I had to think of it. I think if I would have been able to draw it I would have been better able to see it better. (pd546)

Valery’s understanding of the ratio was limited by her difficulty with the nature of fractions as relations. She only understood fractions that were less than one and struggled with the whole shifting from the mixed fraction to the unit cup. Even though Valery could find half of 1 and also find half of 1/2, she did not understand that combining them would give half of the total quantity.

All participants in the low cluster correctly solved the ratio boxes (#53) problem. Two of these participants continued to demonstrate additive thinking as they built up successive iterations of the composite unit. Neither Valery nor Isabelle extended the ratio to include the whole as well as the two given parts and thus did not recognize the scalar relationship. Even though Isabelle included the multiplier in her ratio table this was added to simplify her calculations rather than to reflect the structural relationships (Figure 38).

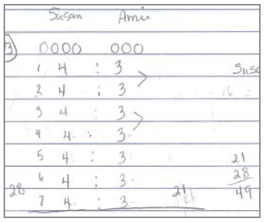


Figure 38. Isabelle (pd577)

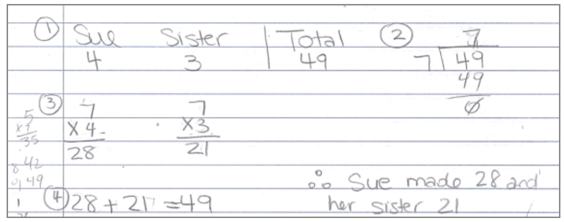


Figure 39. Grace (pd463)

The final participant, Grace, did recognize the importance of the whole for this ratio. Rather than using a procedure Grace reasoned about the relationship between the total of the ratio and that of the boxes to determine the multiplicative constant (Figure 39). In this manner Grace did not simply focus on the absolute quantities but worked with the ratio as a meaningful entity.

Concluding Overview

Participants in the low cluster demonstrated a moderate growth in their understanding of fractions as ratios over the course of the study. These participants began with a strong reliance on procedures and a limited understanding of the nature of ratio relationships. They shifted to using reasoning to think about the ratio as a composite unit. While their understanding remained limited and fragile each participant made gains in their approach to ratios. Grace seemed to make the greatest gain. Even though she held on to her procedural approach throughout most of the study, by the end Grace engaged meaningfully with whole number ratios, recognizing the inherent multiplicative relationship. Yet at the same time she struggled with both fractional multipliers and quantities. Isabelle shifted from a procedural approach that focused only on the quantities being compared to a meaningful use of the composited unit in the context of a ratio table. In certain contexts she could determine the simple fractional portion of this composite unit, but Isabelle did not extend her understanding to multiples of the unit. In those settings she used additive thinking to build up the ratio parts to the total. Valery moved from not engaging with ratios to a solid use of the ratio table to visualize and make sense of the composite ratio unit. She thoughtfully used strategies of iteration and partitioning when working with ratios.

Nevertheless, her additive thinking and poor number skills limited Valery's deepening understanding of fractions as ratios.

Middle Cluster Results

Baseline Content Exam

Many of the participants in the middle cluster began the study with a basic understanding of the composite ratio unit. Their responses to the to the baseline question, which focused on a part-part-whole ratio (Table 20), revealed a reliance on concrete representations to make sense of the context. While the majority of the seven participants demonstrated additive thinking or relied on partially remembered procedures one participant used multiplicative thinking.

Table 20

Content Exam – Fractions as Ratios Questions

Item #	Code	Question
9	ratio professors	There is a ratio of three male professors to two female professors on faculty. If there are 20 professors in all how many are women and how many are men? (three – one, 20; 25)

Five of the seven participants in the middle cluster correctly solved the ratio professors (#9) problem. All of these participants recognized the importance of the composite ratio unit, but only one extended her knowledge to the multiplicative constant. After drawing two iterations of the ratio unit Brenda found the multiplicative constant using the total of the ratio (Figure 40). She went beyond the quantitative values and thought of the ratio as a distinct meaningful unit. The remainder of these five participants used additive thinking as they concretely displayed successive iterations of the ratio unit (Figure 41). Erica, Gabriela, Olivia, and Tanya all focused on the quantities being compared rather than using the ratio to find relationships.

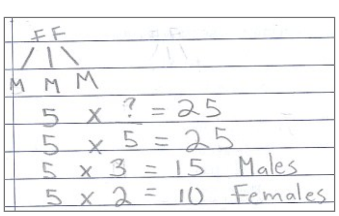


Figure 40. Brenda (pd9)

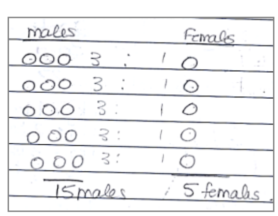


Figure 41. Olivia (pd175)

Two of the participants in the low cluster did not work with the composite ratio unit and did not solve the problem correctly. Lynsey used additive thinking and increased each part of the ratio maintaining the difference of one until she reached the correct total (Figure 42). In doing so she did not maintain the constant relationship between the two parts. In the follow-up interview Lynsey began to recognize the importance of the composite unit.

> In my mind I was thinking just add one, so that's 4 males and 3 females. But every time I do that the ratio is like, it's not very big ratio anymore. It's not making sense…. When I'm adding just one I'm not adding the ratio, 'cuz the ratio is 3 to 2 and I was just adding one, which is not what they're asking at all. (pd126)

Irene, on the other hand, attempted to use a partially remembered procedure and simply dealt with the quantities (Figure 43). While she correctly expressed the ratio as a fraction Irene did not realize her answer did not make sense because the part was larger than the total. In the follow-up interview Irene also recognized the importance of representation. "It's the visual stuff that will help me…. I think I'm one that has to see it and do it" (pd101).

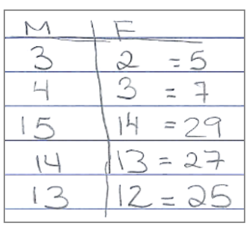


Figure 42. Lynsey (pd126)

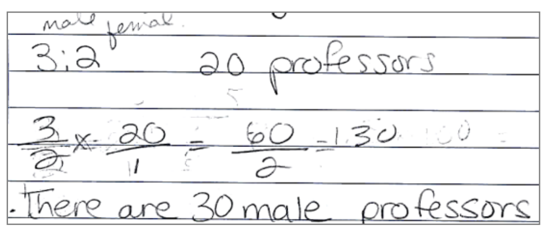


Figure 43. Irene (pd101)

First Problem-Solving Interview

The first problem-solving interview contained several questions that addressed the theme of ratios in part-whole contexts (Table 21). The first explored multiplicative relationships and the second explored the unknown complement part. Participants in the middle cluster continued to use the composite ratio unit, but most could not recognize the ratios as separate entities distinct from the quantities being compared. During both problem solving sessions Brenda worked with Olivia, Erica worked with Lynsey, Irene worked one her own, and Gabriela and Tanya each worked with a partner from the low cluster.

Table 21

Problem Solving Interview #1 – Fractions as Ratios Questions

Item #	Code	Questions
20-21	food ratios	The Vancouver Aquarium has three beluga whales: small, medium and large. The medium whale eats 2 times as much as the smallest; the largest eats 3 times as much as the smallest. a) If the total amount of food was 60 kg, … b) If the total amount of food was 90 kg, … how many kilograms did each of the whales receive?
27	stamp collection	One out of every eight stamps in Arnie's stamp collection is Canadian. He has 720 stamps in his collection. How many foreign stamps does he have?

All participants in the middle cluster who completed the food ratios (#20-21) problem in this session solved it correctly. They all recognized the multiplicative relationships between the three different parts but only one participant also understood the relationship with the whole. Irene used the ratio as an entity distinct from the quantities. She determined that 6 units of food were needed in total and the multiplicative constant in the first instance would be 10 (Figure 44). In contrast the other participants focused on the quantities using a guess and check approach. Brenda, Olivia, and Tanya only used the ratio to determine if their choices fit the parameters. None of them recognized the potential of finding the total of the ratio. They focused on quantities being compared. So, even though Brenda and Olivia attempted to use a strategy, finding the average amount of food and then "chunking" amounts from the small to the large, they did not look within the ratio itself. Similarly, Tanya used an algebraic representation for the parts of the ratio but only she only used it to confirm her guess (Figure 45).

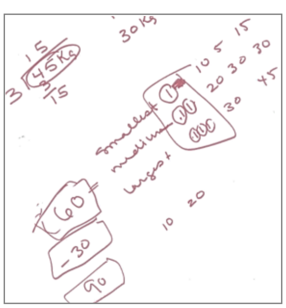


Figure 44. Irene (pd265)

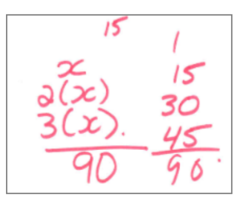


Figure 45. Tanya (pd279)

Even though the stamp collection (#27) problem used larger numbers for the ratio, the six participants in the middle cluster who completed the question during this session did not find it problematic. All of these participants used quotative division to determine how many groups of the composite ratio unit were in the total (Figure 46). They used multiplicative reasoning to build up the ratio and then act out the ratio statement, by subtracting the 90 groups of one from the total. Most of these participants did not identify the complement part and only worked with the values given in the problem. They did not fully recognize the distinctness of the ratio apart from the values being compared. Only Tanya and Lynsey identified the complement part and used the multiplicative constant of 90 to directly find the missing quantity. In this way they used the ratio as a distinct entity to help them think about the relationships between the quantities.

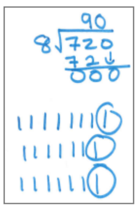


Figure 46. Lynsey (pd249)

Most participants in the middle cluster demonstrated some development in their understanding of ratios during the first problem-solving interview. Those who began with an additive approach continued to build up the ratio using the composite unit, but they shifted to a multiplicative, *groups-of* approach. All participants now used the composite ratio unit rather than procedures, but the degree to which they recognized the ratio as a distinct entity varied. Many had difficulty going beyond the given quantities and extrapolating additional relationships inherent in the ratio, so they focused on the quantities being compared. Even the participant who had initially demonstrated a strong multiplicative understanding could not extend the ratio relationships in these new situations.

Second Problem-Solving Interview

The second problem-solving instrument contained a part-part ratio problem that used a non whole number multiplier (Table 22). Participants in the middle cluster continued to struggle with ratios as meaningful entities during this session. Those participants who built up to the total using multiples of the parts found it challenging to believe the ratio could be maintained when the parts were reduced. These participants continued to focus on the absolute quantities rather than ratio relationships.

Table 22

Problem Solving Interview #2 – Fractions as Ratios Questions

Item #	Code	Questions
38	ratio cookies	Maria and her son Owen were decorating cookies for the bake sale. For every 8 cookies that Owen decorated, Maria decorated 30. If Maria completed a total of 75 cookies by herself, how many cookies did Owen complete?

Three participants completed problems from the previous session (Table 21). Like the other participants Erica and Lynsey started with a guess and check method on the food ratios (#20-21) problem that focused on the absolute values rather than the ratio. But they extended their thinking to look for patterns among their table of values.

> L: The small and the medium equal what the large eats…. The total kilograms of food, it's half of, the large whale eats half of it and then once you know that number you know that the medium eats half of what the large eats, and this eats, the small eats half of what the medium eats. So it's just kind of halving it as you down. It just took us a long time figure that out.
>
> E: Yeah, I know. Once we hit 120, then I was catching on but I wasn't too sure how to explain it. (pd333)

Lynsey and Erica used the absolute values to identify a relationship, yet they still did not know how this relationship related to the ratio. They were on the verge of seeing the ratio as a meaningful and distinct entity.

Even though Gabriela used a method that differed from the one used by other participants on the stamp collection (#27) problem, like many of them she did not identify the complement part. Gabriela and her partner used the cross multiplication procedure $1/8 = x/720$ to find a portion of the total quantity. Gabriela clearly understood how to use the procedure and interpret the results but she could not give it deeper meaning until her partner provided a visual model which they used for partitive fair shares. Gabriela focused on the absolute quantities rather than the relationships.

Four of the participants in the middle cluster solved the ratio cookies (#38) problem correctly without additional help. They recognized that the composite unit needed to be doubled and then combined with half of the unit. These participants understood that the composite unit could be partitioned as well as replicated. All of these participants used a version of the ratio table to help them think about the relationships between the ratio and the absolute values. Brenda and Olivia needed to have visual representations of the values and thus focused more on the absolute value than the ratio (Figure 47). They used additive reasoning as they built up to the total quantity. Gabriela and Irene focused more on the multiplicative relationships recording the doubling and halving on their table and then combining the values.

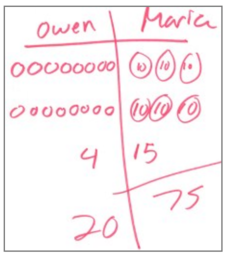


Figure 47. Brenda/Olivia (pd316)

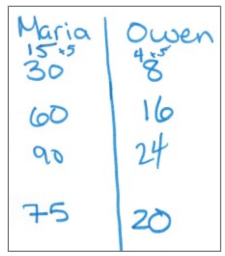


Figure 48. Erica/Lynsey (pd341)

The remaining three participants also used ratio tables but became stuck when they could not use a whole number multiplier to obtain the correct total. The fragility of their understanding of the nature of ratios became apparent as they worked with the composite unit as an absolute value that could not be partitioned. Remnants of additive thinking emerged when Erica suggested that the difference of 22 between the values might be the important pattern. When prompted to think about a portion of the unit Erica readily halved the values, but she then used a whole number multiplier to build up to the total (Figure 48). Tanya revealed elements of her procedural thinking when she expressed frustration that her previous methods did not work. "Yeah, we do them all the time, but because the numbers are smaller and they just normally work for us; I don't know why we struggled with that" (pd373).

Participants in the middle cluster deepened some aspects of their developing understanding of fractions as ratios during the second problem-solving session. While these participants continued to value the composite ratio unit they varied in the degree to which they engaged with the ratio as a meaningful distinct entity. Some used the ratio as a tool to build up to the required quantity, while others were on the verge of identifying direct relationships. Most of the participants used multiplicative reasoning when iterating groups of the composite unit. However, some of the participants limited this reasoning to whole number multiples.

Post-Test Retake Exam

The final test instrument contained two items that addressed the concept of fractions as ratios (Table 23). These questions focused on both associated sets and part-part-whole ratios. Participants once again worked individually for this test instrument.

Table 23

Post Test Retake Exam – Fractions as Ratios Questions

Item #	Code	Questions
46	Muffins	You are baking muffins. The recipe for 12 muffins calls for 11/2 cups of flour. You want to make 18 muffins. How much flour will you need?
53	ratio boxes	Sue and her younger sister Amy work together boxing cookies for sale at the local bakery. Sue can make up 4 boxes of cookies for every 3 that her younger sister does. At the end of their shift they had 49 boxes in total. How many boxes did Sue make?

All but one of the participants in the middle cluster correctly solved the muffins (#46) problem. Participants used three different strategies that reflected differing levels of understanding of the nature of ratios. Three participants used a strategy that reconceptualized the ratio as a distinct unit. They chunked the flour into convenient units and then used a partitive fair shares method to distribute the muffins. Once these participants determined the new unit of flour per muffins they could reason up to the required quantity using their diagram. Erica and Irene chunked the flour into 1/2 cup portions that gave the complex unit of 4 muffins per 1/2 c flour (Figure 49). Gabriela chose to use 1/4 cup portions so her unit became 2 muffins per 1/4 c flour (Figure 50). In using this approach these participants did not work with the scalar multiple between the ratios but focused on the functional relationship, building up to the total.

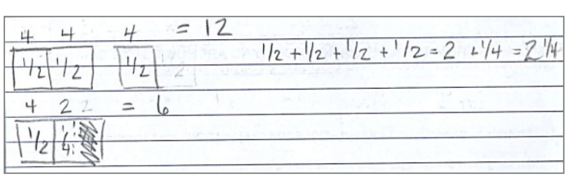


Figure 49. Erica (pd440)

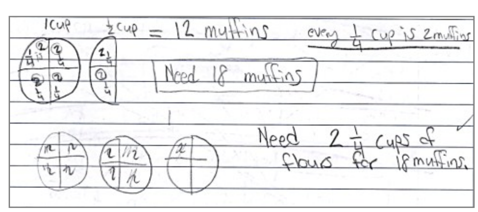
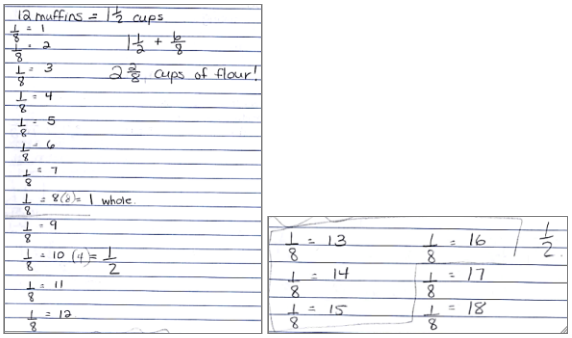


Figure 50. Gabriela (pd445)

One participant used the second strategy that found the scalar factor between the two ratios. Lynsey used relative thinking to recognize the multiplicative relationship, reasoning that since the muffins increased by half the flour also needed to increase by half. She used her knowledge of fractions to visually combine the amounts to determine the total. Lynsey used the context to help her identify the ratio as a distinct unit. Two participants used the third strategy that was more basic and labour intensive. Brenda and Olivia used trial and error to determine the amount of flour for each muffin, focusing on the absolute quantities rather than the relationship. Olivia first tried 1/4 cup, then 1/12 cup before settling on 1/8 cup as the amount of flour per muffin. Once she determined the new unit Olivia simply added on the required number of muffins and flour. Her visual representation reflects the distinctness of the quantities rather than relationships even though she grouped the amounts to reflect both 1 and 1/2 (Figure 51). Brenda randomly chose 1/16 to begin, but when it did not work she chose a more systematic approach. Brenda needed a visual representation of the cups in order to determine how to divide the quantities evenly into 12 sections (Figure 52). She struggled to make sense of the fractional value.

> Because it wasn't whole numbers there was a half that made it difficult. I wasn't practicing questions like that. Like, I didn't know what those extra muffins, what I would need and I don't remember doing that [in the skills course]. But I do, we did learn other things that helped me solve the problem. So dividing the 1 cup, that was the first step. That was easy, kind of. Then having to divide the 1/2 cup; that was really hard. So it was the 1/2 cup. I just imagined, I had to, like, actually make lines on this one cup and then so I had to do the exact same thing for this 1/2 cup, so it would just have to double, so the 12 had to double to 24. (pd396)

Once she determined that each muffin required 1/12 plus 1/24 cups of flour, she combined the amounts using her prior knowledge. Brenda then knew that 18 of these 1/8 cups of flour were required and she reduced the fraction to mixed terms. Even though she determined a unit amount for the flour, Brenda did not fully recognize the relationship inherent in the ratio. Even though the problem provided a familiar context of baking neither Brenda nor Olivia recognized the scalar relationship involved in increasing the recipe.



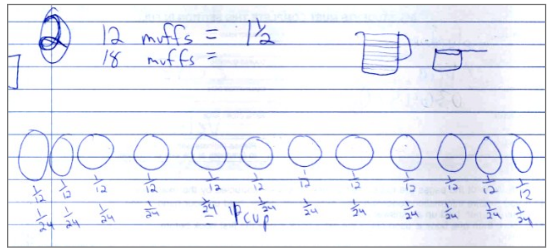


Figure 51. Olivia (pd524) *Figure 52.* Brenda (pd396)

The final participant attempted to use the unitizing strategy but was not successful. Tanya vacillated between two different interpretations of the ratio because she struggled with interpreting the fractional value. Ultimately she did not maintain the ratio relationships. While Tanya initially determined that 4 muffins needed 1/2 cup flour, dividing both the flour and the muffins into equal sections, she changed her mind divided the flour into two unequal sections that reflected the whole number and fraction (Figure 53).

> I thought 1 cup of flour for 6 muffins and just 1/2 cup for 6 but that doesn't make sense, 'cuz it's not, they're not equal. So I was totally thrown off. So I switched it. I did have it but I switched it. (pd535)

Tanya identified patterns within the ratio but because she did not fully understand the numerical relationship she could not decode them to make sense of the problem.

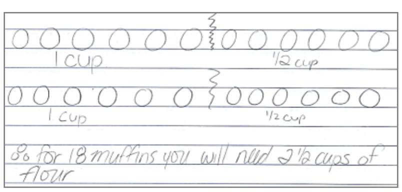


Figure 53. Tanya (pd535)

All of the participants in the middle cluster solved the ratio boxes (#53) problem correctly. While each of them maintained the ratio between the unit not all of the could determine the structural relationships inherent in the ratio. Participants used one of two different types of strategies that revealed the varying levels of thinking about the nature of ratios. The first strategy focused on the scalar relationship. Four participants divided the sum of the parts of the ratio into the given total to find the multiplicative constant of the relationship. They then multiplied this constant times the parts to determine the final quantities (Figure 54). While Brenda, Erica, Lynsey and Olivia each used multiplicative reasoning they did not necessarily recognize the ratio as a distinct entity. Some still thought of the parts of the two distinct numbers rather than a relationship. Brenda linked the multiplier directly to her concrete model, stating that it told how many successive iterations of the composite ratio unit were needed until the given total was reached. However, others could only describe the procedure, not its meaning.

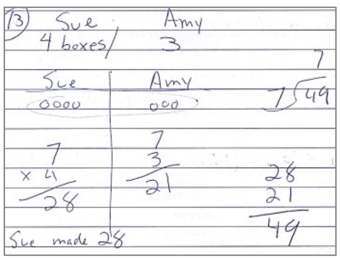


Figure 54. Brenda (403)

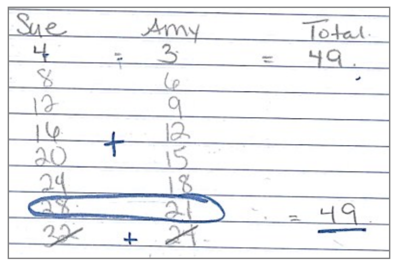


Figure 55. Irene (pd474)

Three participants used the second strategy that focused on building up to the total. Irene, Gabriela, and Tanya each used the ratio table to record successive multiples of the composite ratio unit until they reached the required quantity (Figure 55). They worked with the absolute value of the parts using a type of additive thinking as they built up to the total. They did not recognize any other structural relationships inherent in the ratio in part because they thought about each part of the ratio separately rather than as a distinct entity.

Concluding Overview

Most of the seven participants in the middle cluster demonstrated a deepening of their conceptual understanding of fractions as ratios over the course of the study. Two of the participants began without a strong sense of the ratio unit, using partially remembered procedures or additive strategies that maintained the difference. The remainder of the participants began with an additive understanding of ratios, strongly rooted in the context. They worked with the ratio as a composite unit of two distinct parts rather than a relationship and needed the physical representation to build up to the missing part. As participants encountered new challenges they often returned to this basic understanding, sometimes also using trial and error. Participants shifted to multiplicative reasoning and developed varying degrees of understanding of ratios as distinct entities as they determined the scalar multiple or developed a functional relationship through unitizing. All of the participants, except for Gabriela, found the multiplicative relationship between two ratios for at least one item in the study. Each of these participants demonstrated an ability to use the complement part or the sum of the ratio parts to determine the scalar constant. Yet some of the participants could not fully articulate why the procedure worked. Similarly all of the participants, except for Lynsey, worked with a functional relationship on one of the items. Erica, Irene and Gabriela demonstrated a solid understanding of this ratio relationship, using an appropriate and efficient unit. Brenda and Olivia, however, struggled with this key concept as they worked with a less efficient single unit. Finally, Tanya vacillated between a correct and incorrect interpretation of the ratio when attempting to unitize, indicating the fragility of her understanding.

Top Cluster Results

Baseline Content Exam

One question on the content exam served to establish participants' baseline understanding of fractions as ratios. This question, as seen in Table 24, is a part-part-whole ratio where the quantity of the parts must be determined from the ratio and the given whole or total. All four participants in the top cluster solved this problem correctly. However, they had varying degrees of understanding of the nature of ratios. Some of the participants used additive reasoning while others used multiplicative reasoning to work with the ratio.

Table 24

Content Exam – Fractions as Ratios Questions

Item #	Code	Question
9	ratio professors	There is a ratio of three male professors to two female professors on faculty. If there are 20 professors in all how many are women and how many are men? (three – one, 20; 25)

On the ratio professors (#9) question all four participants in the top cluster demonstrated an understanding of the ratio as a composite unit. However their understanding of the multiplicative relationships varied. Participants used two distinct approaches to solve the problem correctly. Two of the participants used additive thinking as they built up to the total quantity. Bryann drew out successive iterations of the 3 to 1 ratio adding up until she reached the appropriate total. Similarly, Megan used a model of skip counting by the larger quantity and adding up until she reached the total (Figure 56). Neither of these participants thought of the ratio as a meaningful unit nor did they recognize the larger relationship inherent in the problem. The remaining two participants did understand this relationship and thus used multiplicative thinking to solve the problem. Diane and Mark each recognized that in order to maintain the ratio both quantities needed to be multiplied by the same amount in order to reach the given total. While Mark used a more procedural and algebraic approach, solving the equation $3x + 2x = 25$, he still understood the nature of the multiplicative relationship. Diane intuitively understood the nature of the relationship. She "just knew" that the ratio must be 15 to 10, and had determined that the multiplier for this problem must be five (Figure 57). Her explanation identified the covariance-invariance nature of proportionality.

> Because it's like a fraction; as long as you multiply both number times the same amount, it's going to grow with the same relation. So it could even be the same as 30 to 20; or 12 to 8 (pd41)

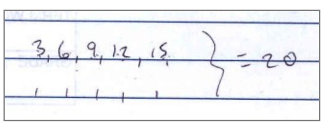


Figure 56. Megan (pd159)

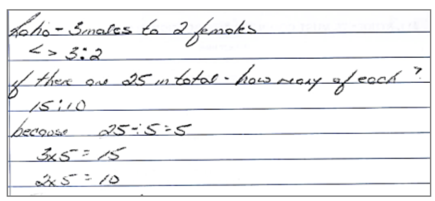


Figure 57. Diane (pd41)

First Problem-Solving Interview

The first problem-solving interview contained several questions that focused on the theme of ratios (see Table 25). First, a series of questions based on the same stem used the ratio language to explore a multiplicative relationship in a part-whole setting. Then a part-part-whole ratio explored the unknown complement part. Participants in the top cluster demonstrated a good understanding of the multiplicative relationships involved in the ratios. However, they could not always recognize the ratios as separate entities, distinct from the quantities being compared. During both problem-solving interviews Mark worked with Diane and Megan worked with Bryann.

All participants in the top cluster correctly solved the food ratios (#20-21) problem. They recognized the multiplicative relationship between the three different parts but only two of the participants also understood the relationship with the whole. Diane and Mark began by thinking about the total quantity of food to be apportioned in the ratio given. They knew that some number plus that number doubled plus that number tripled had to give the total amount of food. While Diane and Mark simply used mental calculations to add three numbers their reasoning was deeper than an instinctual trial and error guess. Mark articulated a solid understanding of the ratio as distinct from the quantities of food and determined the scalar operator of the ratio.

> Another way, I'd say that it has to be six portions; one portion for the smallest; two portions for the medium; three portions for the largest; that's six portions. So I take my 60 and divide that by 6 portions to get one portion, which would be 10 and just follow through from there. (pd229)

Table 25

Problem Solving Interview #1 – Fractions as Ratios Questions

Item #	Code	Questions
20-21	food ratios	The Vancouver Aquarium has three beluga whales: small, medium and large. The medium whale eats 2 times as much as the smallest; the largest eats 3 times as much as the smallest. a) If the total amount of food was 60 kg, … b) If the total amount of food was 90 kg, … how many kilograms did each of the whales receive?
27	stamp collection	One out of every eight stamps in Arnie's stamp collection is Canadian. He has 720 stamps in his collection. How many foreign stamps does he have?

Megan and Bryann also recognized the relationship between the three amounts of food, but they did not understand the ratio as a separate and meaningful entity. They only dealt the concrete quantities given in the problem. Bryann immediately knew the answer for the first total because she started guessing with 10 and it worked. However, she could only give a limited explanation of her thinking. "It broke up evenly into three, and I knew that one was double the first one" (pd288). For the second total Megan and Bryann continued with the guess and check strategy based solely on the relationships between the parts of the ratio. Even though Megan explored the difference of 30 between the two totals and used the same pattern to correctly apportion the additional food she still did not recognize the relationship between the parts and the whole.

> Well, we have 30 extra, you could just give each 10; you could give 5, then 10, then…. Ya it does work. You could just add 5 to the small one; 10 to the next one; 15 to the next one on top of the 60. Then it does work out evenly. (pd289)

The stamp collection problem (#27) used larger numbers for the ratio and this caused problems for some of the participants in the top cluster. Nevertheless all these participants solved the problem correctly even though they had different degrees of understanding the nature of ratios. Three of the participants used a partitive fair shares model to interpret the ratio. They readily determined that the unknown part must be 7/8 of the whole. After determining the size of each partition they calculated the necessary number of parts. Mark and Diane had a strong understanding of the ratio distinct from the number of stamps, using multiplicative thinking. Megan's understanding, however, was more fragile. Her model reflected the context of the problem and became her process for thinking. When her partner used a different method Megan no longer trusted her own thinking.

> So it did work both ways. But it didn't look like it was going to work my way 'cuz I got a different answer and that bogged me down; 'cuz you got 80 and I got 90. (pd295).

The final participant in this group used a quotative model to reflect the ratio, drawing nine groups of 8. Bryann needed to physically act out the ratio to think about the relationships, writing 8 – 1 = 7 in each of the boxes. For her the ratio did not have meaning outside of the context of comparing the given quantities. Bryann struggled to make sense of the multiplication given the large size of the total. As a result she dropped the zeros to make it easier to compute. But then Bryann could no longer interpret her diagram and the ratio without help from her partner.

> That was working with the 8 and 7, I was going to add a zero to the end of it after I had done it, 'cuz I like to drop that zero, you know that. (pd295)

Participants in the top cluster demonstrated minor gains in their understanding of fractions as ratios during the first problem-solving interview. The two participants who began the study with multiplicative thinking continued to develop a stronger view of ratio as meaningful entities. Mark moved from his procedural and algebraic thinking about ratio relationships to making sense of the scalar constant and identifying the relationship between the ratio parts and whole. Diane began to articulate her intuitive understanding of ratios using her understanding of fractions as relations. The two participants who began with additive thinking experienced more challenges in deepening their understanding. Megan used the context to help her understand the specific ratio relationships. She willingly explored patterns but could not always determine the multiplicative relationship. Bryann remained embedded in additive thinking. While she could identify certain relationships in the ratio she could not consistently extrapolate all of the relationships embedded in the ratio.

Second Problem-Solving Interview

One question in the second problem-solving interview highlighted the theme of fractions as ratios using a non whole number multiplier in a part-to-part ratio (Table 26). Participants in the top cluster continued to be challenged to think of ratios as meaningful entities. These participants displayed a willingness to go beyond their initial procedural focus and look at the relationships rather than the individual numbers.

Table 26

Problem Solving Interview #2 – Fractions as Ratios Questions

Item #	Code	Questions
38	ratio cookies	Maria and her son Owen were decorating cookies for the bake sale. For every 8 cookies that Owen decorated, Maria decorated 30. If Maria completed a total of 75 cookies by herself, how many cookies did Owen complete?

All participants in the top cluster correctly solved the ratio cookies question (#38), however, each of them employed some degree of procedural thinking as they engaged with the problem. Diane and Mark focused on the absolute amounts of the quantities rather than the relative amounts as they used the cross multiplication algorithm to solve the problem. They simply explained their calculations and algebraic procedures but did not make connections to the relationships. Nevertheless, when challenged to find a second method, Mark used a concrete model to think about the context (Figure 58). In doing so he recognized that the composite ratio unit needed to be doubled and then combined with half of the unit. Through the iterative process Mark's focus changed from the individual quantities to the relationship. Megan and Bryann, on the other hand, began by using the ratio table to think about the relationships in the proportion. Megan recognized the importance of multiplicative reasoning as she began by doubling and halving the composite unit. However, she confused the part-part relationship with the part-whole. Her placement of the total value at the bottom and middle of the chart (Figure 59), rather than as a part of the ratio, indicates limitations in her ability to see the entire part-part-whole relationship as a meaningful entity. While Bryann clarified the problem as a part-part relationship, she still used some additive thinking when she suggested they simply add all values in the table so that Owen completed 28. Bryann focused on the addition procedure rather than on understanding the relationship.

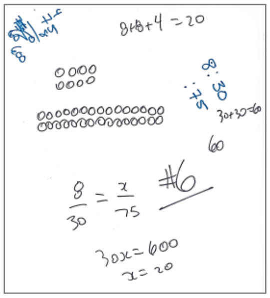


Figure 58. Mark (pd324)

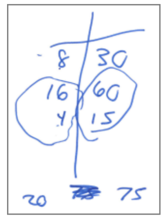


Figure 59. Megan (pd383)

Post-Test Retake Exam

The final test instrument contained two items that focused on fractions as ratios (Table 27). These questions addressed both part-part and part-part-whole ratios. Once again for this instrument participants worked individually. Most participants in the top cluster recognized the ratio as a meaningful entity distinct from the comparative quantities. However, the use of additive reasoning by some of the participants limited the deepening of their understanding of fractions as ratios, even though all of these participants solved both problems correctly.

Table 27

Post Test Retake Exam – Fractions as Ratios Questions

Item #	Code	Questions
46	Muffins	You are baking muffins. The recipe for 12 muffins calls for 11/2 cups of flour. You want to make 18 muffins. How much flour will you need?
53	ratio boxes	Sue and her younger sister Amy work together boxing cookies for sale at the local bakery. Sue can make up 4 boxes of cookies for every 3 that her younger sister does. At the end of their shift they had 49 boxes in total. How many boxes did Sue make?

On the muffins problem (#46) the participants in the top cluster used varying degrees of proportional reasoning to solve the problem. Each of them identified structural relationships inherent in the problem. Three of the participants determined the scalar relationship between the two ratios, recognizing that since the recipe increased by half, the amount of flour would also increase by half. Mark and Megan confidently used mental math to solve the problem. Their understanding of fractions readily enabled them to divide and to combine the fractional quantities, while the baking context enabled them to give meaning to the ratio apart from the quantities. Diane, on the other hand, used the ratio notation to facilitate her thinking about the relationship between the numbers (Figure 60). She was not fully confident in her reasoning and attempted to use fraction notation to confirm her answer. Eventually Diane used her rational number sense and the context, rather than procedures, to make sense of the ratio. “It feels right. If I were baking the muffins that’s probably what I would do” (pd418).

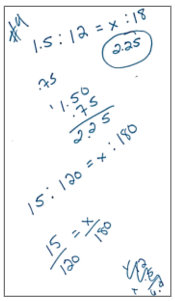


Figure 60. Diane (pd418)

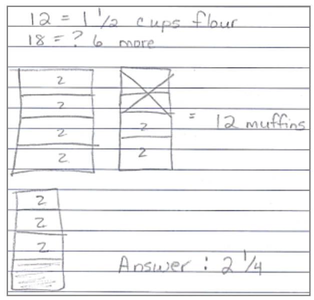
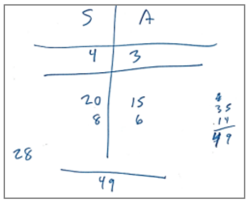


Figure 61. Bryann (pd407)

The remaining participant determined a functional relationship within the ratio. Bryann used a fair shares model to determine the amount of flour needed for every two muffins (Figure 61). Bryann reconceptualised the ratio as a distinct unit as she correctly identified this constant relationship. Nevertheless Bryann still used aspects of additive reasoning as she built up the required amount of flour, adding 6 more muffins. In this way Bryann demonstrated the fragility of her understanding of the relational nature of ratios.

All of the participants in the top cluster who completed the ratio boxes question (#53) used a building up strategy to solve it. Ultimately these participants maintained the constancy of the composite ratio unit, but their reasoning focused more on the quantities than the relationships. Megan built up to the total quantity using multiples of 5 and 2 times the composite unit (Figure 62). Even though she continued to use the visual placement of the total as a cue for its meaning, Megan did not recognize the relationship between the total of the parts of the ratio and the total quantity. Thus she could not determine the scalar constant. Mark attempted to mentally determine the correct multiple, but when he could not immediately find it he switched to additive thinking and listed the consecutive multiples of the ratio until he found the pair that gave the requisite total. He also focused on the quantities rather than the relationships and so did not recognize the importance of the sum of the parts of the ratio. Neither recognized the ratio as a distinct entity in this context.



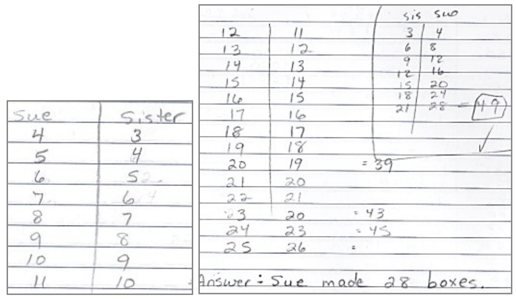


Figure 62. Megan (pd511)   *Figure 63*. Bryann (pd414)

Bryann revealed the fragility of her understanding of fractions as ratios when she began by using additive thinking. Rather than maintaining the constancy of the ratio, Bryann maintained the difference between the two parts (Figure 63). When her results did not add up to the correct total Bryann shifted her approach and looked at the successive multiples of the ratio.

> I started going up by one, and it wasn't making sense to me. I was, like, this is silly, 'cuz the ratio, going up by one it's obvious. And then when I got to this point [25 and 26 on table], I was like, I can't get 49 total. So then I said I'm going to double, I'm going to go up by the 3's and by the 4's, not double it. I was going to increase it on each side by 3 and 4, which I did and I got a total of 49 and it made sense. … It didn't make sense that it was a ratio of 3 to 4 and it's not making sense that it's going to be a ratio of 11 to 12 either. (pd 414)

Bryann needed to encounter the error in order to shift from thinking about the ratio as two separate numbers to understanding the relationship of the composite unit. Nevertheless she still used additive thinking as she built up to the required quantity and did not recognize the other relationships implicit in the ratio.

Concluding Overview

Participants in the top cluster demonstrated minimal gains in their understanding of fractions as ratios over the course of the study. Two of the participants began with a strong multiplicative understanding of fractions as ratios coupled with a procedural approach. Mark and Diane used their known procedures to identify the scalar relationships in the proportions.

They understood the ratio to be a distinct entity. Over time they slightly deepened their understanding of the nature of ratios as they focused on reasoning and making sense of the contexts. While they were able to determine the complement part of a ratio, they did not extend this reasoning to identifying the whole when working with a part-part ratio. The other two participants in this cluster began with an additive understanding of the ratios. While they recognized the importance of maintaining the ratio unit Megan and Bryann did not see the ratio as a unit separate from the quantities they compared and thus they used trial and error to build up to required amounts. During the study Megan developed her multiplicative reasoning. Within the baking context Megan used ratios as meaningful entities, however, outside that context Megan relied on the ratio table, which limited her ability to engage meaningfully with all aspects of the ratio. Bryann, on the other hand, remained embedded in her additive thinking. She continued to think of the absolute values rather than the unit. So even though Bryann began to use multiples she did so in a procedural manner that lacked an understanding of the embedded relationships.

Pre-service teachers in all three clusters demonstrated gains in their understanding of fractions as ratios, with deeper growth exhibited in the middle and low clusters. All participants worked with ratios as composite units; many developed multiplicative thinking. However, the fragility of this understanding became evident when participants encountered non-integer ratios. Did these elements of multiplicative thinking differ from their understanding of fractions as operator?

Chapter Seven: Results and Analysis – Fractions as Operators

When fractions are conceptualized as operators they act as transformers, enlarging or reducing the original whole (Behr et al, 1993; Lamon, 2005). The fraction serves as a rule that provides instructions for operating on the unit (Lamon, 2001). This rule can be carried out in several different ways. For example, if we are using the *2/3 of* operator then we could use a single operation to multiply the whole by 2/3, we could first multiply the whole by 2 and then divide the result by 3 or we could first divide the whole by 3 and then multiply the result by 2. In each of these instances the fraction 2/3 operates on the quantity to reduce, or find a portion of, the whole. Students who think about fractions as procedures rather than relations may correctly solve problems with operators because they know the multiplication algorithm for fractions. However, they may not fully understand the nature of the fraction and its relationship to the unit whole. This relationship is ultimately a comparison between the resulting quantity and the original quantity and, in our example of the *2/3 of* operator, can be thought of as a 2-for-3 exchange (Behr et al., 1992; Lamon, 2005).

The concept of operator can be represented by a partitive fair shares model that distributes the original quantity into the number of parts indicated by the denominator and chooses the number of parts indicated by the numerator. This model reflects Behr et al.'s (1993) duplicator/partition-reducer interpretation. Quotative modeling that partitions the unit whole into groups equal in size to the denominator and uses the numerator to reduce or expand each group is also possible, though less likely. Fractions as operators can also be represented more efficiently using a double number line that represents both the fraction and the whole, using the partitive interpretation (Fosnot, 2007). The multiplicative nature of the operator function becomes apparent when working with these concepts of "*groups-of*" (Kieren, 1976).

Each of the four test instruments contained items that addressed the concept of fractions as operators. Understanding of this concept was also explored in contexts using fractions as decimals and percentages. Table 28 provides an overview of the items on the various instruments. Participants began with a strongly mechanistic approach to the fractions as operator problems. Many knew they had to multiply simply because that was the rule they had learned; *of means multiply*. Others forgot the rule and used buggy algorithms that indicated their lack of understanding. Over the course of the study participants' growing understanding enabled them to use appropriate models to make sense of the problems. Participants in the top cluster developed a deeper understanding of the concept of operator than did the participants in the low cluster.

Table 28

Test Items Reflecting Fractions as Operators

Instrument	Item #	Code	
Baseline Content Exam &	7	Tribes	Fractions
Follow-up Interview	10	Class increase	Percent
Problem-Solving 1	26	Math course	Fraction, percent, decimal
Problem-Solving 2	39	Trail mix	Fraction, percent, decimal
Post Test Retake Exam &	47	Olympics training	Fractions
Follow-up	50	Pages book	Fraction
	55	Decrease class	Percent

Low Cluster Results

Baseline Content Exam

The three participants in the low cluster began the study with a limited understanding of the operator function. Their responses to the baseline questions, which focused on both fractions and percents as operators, as seen in Table 29, revealed a strong reliance on memorized procedures. While some of these participants recognized that percent operators required finding a portion of the whole, none of them worked with the fractions as meaningful objects to determine *groups-of* or partitioning. Fractions as operators simply meant multiplication for these participants.

Table 29

Content Exam – Fractions as Operators Questions

Item #	Code	Questions
7	Tribes	Two tenths of the P/J class of 240 went to 'Tribes'. Two thirds of the 240 went to a workshop on 'Portfolio Writing'. All the rest stayed home for extra sleep. How many stayed home? (3/10, 1/3; 270)
10	Class increase	We will be increasing the present class of 220 students in P/J by 30% next year. How many students will there be in next year's PJ class? (180, 25%; 160, 40%)

Fraction operators. None of the participants in the low cluster solved the Tribes (#7) problem correctly. Each of them used some form of multiplication, but none of them were able to make sense of the operator procedure as finding a portion of the whole. Even though Isabelle correctly set up a cross multiplication proportion, making only a small calculation error, she could not link the procedure to the fraction (Figure 64). For her this was a *formula* that provided the answer, as long as she put the question mark in the correct place. She did not know how to explain it otherwise.

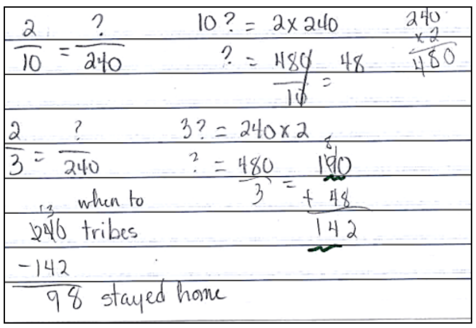


Figure 64. Isabelle (pd562)

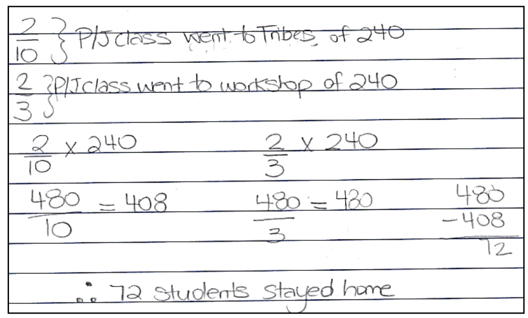


Figure 65. Grace (pd557)

Both Grace and Valery multiplied the total by the fraction yet neither of them could give meaning to the procedure. Grace multiplied across the numerators but was not able to correctly divide by the denominators (Figure 65). Even though both fractions were less than one the results of her procedures were both larger than the original whole. Grace did not realize the incongruity of her results, nor understand the nature of the operator function. She simply thought her final answer made some sense for the question. Likewise, Valery set up the multiplication procedures, but could not reconstruct her thinking. “It looks like I’m multiplying again, but I don’t know why I thought I always have to multiply. I’m not sure” (pd205). Even though her procedures were partially correct, Valery did not know what the product concretely represented and so did not complete the problem.

Percent operators. The participants in the low cluster had similar challenges with the class increase problem (#10). Here the percent operator was another multiplication procedure that held little meaning, even though two of them recognized that operator function indicated a portion of the whole. Isabelle correctly solved the problem using her cross multiplication “formula”. She struggled, however, to explain what those procedures represented. Even though she read the explanations in mathematics textbooks, she simply remembered how to find the answer.

> This would be a formula for me… I don't know how to explain it otherwise…. If I look at the book, and then it tells me, and I go ohh, okay it makes sense. But to memorize it I just look at the numbers. (pd113)

Valery, on the other hand, made connections to her prior knowledge of finding the sale price when shopping to give meaning to percent operators. She knew she had to find a portion of the whole. Yet she struggled with making sense of the procedure. She did not know how to multiply by a decimal and so did not complete the problem (Figure 66).

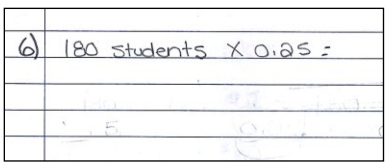


Figure 66. Valery (pd208)

The final participant in the low cluster did not fully understand the nature of operators. Even though Grace knew how to convert a percent to a decimal and multiply, she did not think about the context of meaning of the problem. She simply thought that using certain procedures would produce the answer.

> Honestly, I really don't know. I think I was just thinking that that was the right thing because I was given those two numbers and I thought in order to find out how many, um, students would be in the next year I would, I just thought I would need to multiply those two [numbers]. (pd86)

Grace did not realize that her answer of 66 needed to be added to the total number of students to give more than the original amount.

Overall, the participants in the low cluster demonstrated limited understanding of the operator nature of fractions on the baseline content exam and follow-up interview. All three of the participants took a procedural approach that focused on manipulating the numbers rather than understanding the context. Isabelle successfully used the cross multiplication procedure as the template for working with the operator function, yet she did not understand the exact nature of the relationship between the given operator and the ratio of the resulting amount to the original amount. Grace relied on partially remembered or buggy procedures; she did not question the inconsistencies in her answers. Valery simply remembered that she needed to multiply for operator problems; she did not know how to complete or make sense of the calculations. None of these participants recognized how their calculations might be linked to *groups-of* or partitions of the whole.

First Problem-Solving Interview

The operator question on the first problem-solving interview combined the use of fractions, decimals, and percents as operators (Table 30). The participants in the low cluster demonstrated a shift in their understanding of the operator function with fractions as they began to use a model to concretize the calculation. However, many still retained a procedural understand of decimals and percents as operators. For these sessions Isabelle and Grace each worked with a partner from the middle cluster while Valery worked on her own.

Table 30

Problem Solving Interview #1 – Fractions as Operators Questions

Item #	Code	Question
26	Math course	Jesse was planning the amount of time he would devote to each strand in his 120 hour math course. He planned to spend 0.4 of the time on measurement and data management, 15% of the time on algebra, 3/8 of the time on numeracy, and the remainder of the time on geometry. How many hours were spent on each of the four strands?

Participants in the low cluster found the combination of fractions, decimals and percents in the math course problem (#26) challenging. Two of them assumed all three operators had to be converted into the same format. These participants also had a strong procedural approach to the problem. Grace attempted to convert all the operators to decimals, but used the incorrect procedure for the fraction, equating 3/8 with 2.6 (Figure 67). The decimals seemed to have little connection to their meaning as operators for her since she did not realize the decimal operators needed to combine to make a total of 1; they were simply a necessary part of the procedure.

> The first thing we have to get them all in the same thing, right? And I just thought of putting, I always put them in decimals before anything else; I don't know why…. Because, maybe I think in order to do anything they have to be in the same. (pd260)

Nevertheless, when prompted to explain her assumption, Grace made a connection to the math skills course and realized she could use an area fair shares model to visually determine the portion indicated by the fraction operator. In spite of this connection for fractions, Grace still worked with the decimal and percent operators as procedures that held little inherent meaning. She did not know whether to divide or multiply and relied on her partner to provide confirmation of the correct approach.

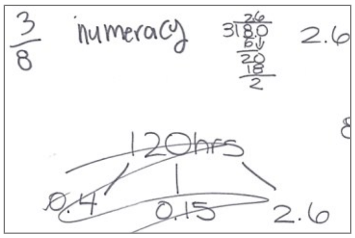


Figure 67. Grace (pd260)

Isabelle also assumed all operators needed to be in the same format, however, she was more flexible on the choice. After deciding to use percents, Isabelle and her partner successfully multiplied by the decimal equivalents to find the portions of the whole indicated by the decimal and percent operators. However, they encountered difficulty making sense of their procedural approach with the fraction operator. They confused the percent 37.5 with the decimal equivalent for 3/8 and ended up with a product that was too large. As a result Isabelle attempted to make a connection with the math skills course, drawing a pie to represent 3/8. “So you remember how to do that to make it 100? We did it in class and it all made sense” (pd284). Her partner then used the area fair shares model, but they could not link the partitions to portions of the whole, confusing them instead with percents (Figure 68). Isabelle did not understand that the operator function determined *groups-of* the whole. She was more comfortable with known procedures.

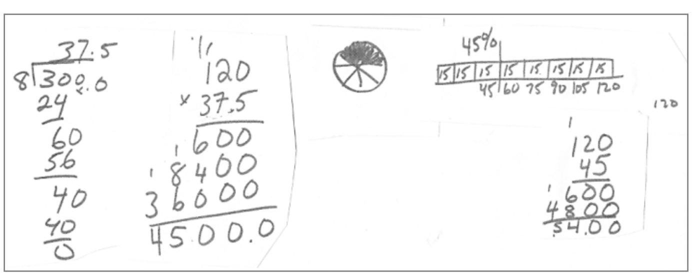


Figure 68. Isabelle (pd284)

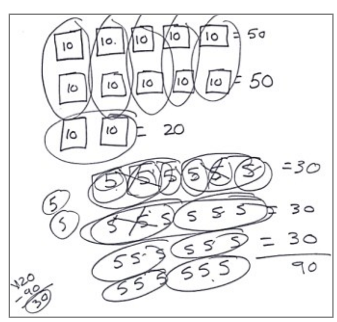


Figure 69. Valery (pd306)

The remaining participant chose to make sense of the operators by drawing concrete representations to act out the operation rather than using partially remembered procedures. Valery used a trial and error approach to represent the total number of hours, first as groups of 10 and then as groups of 5. She then used a fair shares model to divide the total number of hours into eight equal groups (Figure 69). Valery needed to work concretely with all the hours rather than using a rectangle to represent the total. She thought of each of the operators as fractions to facilitate the partitioning into *groups-of.* Because of her limited knowledge of percentages Valery chose to reduce the percent fraction to lower terms. Once again she represented the total number of hours, this time counting by groups of 3, and then partitioning them into 20 equal groups. She then combined three of these groups to find 3/20 or 15% of the total. Valery's lack of number sense meant that in each instance she had to start at the beginning, concretely representing the total number of hours. She could not, for example, use the model partitioned into 20 groups for working with 10 groups. Nevertheless, Valery was beginning to develop her understanding of operators as partitioning to find a portion of the whole.

The participants in the low cluster demonstrated varying degrees of development in their understanding of the operator function that depended on their willingness to think beyond their previously memorized procedures. Two of the participants who began with a procedural approach shifted to a visual model to help them make sense of the context when they encountered an error in their process. Grace readily appreciated the opportunity to give meaning to her procedures for fraction operators, but Isabelle incorrectly interpreted the model from the framework of her multiplication procedure. Both Grace and Isabelle continued to work with decimal and percent operators as procedures. Valery, on the other hand, embraced the concrete visual fair shares model as a way to make sense of the operator function. Due to her poor number sense, Valery needed to act out the entire process by using a trial and error based chunking of the total amount into requisite groupings, rather than using symbols to represent the whole or other aspects of the process. Within these constraints, however, Valery understood how the operator functioned to reduce the whole.

Post-Test Retake Exam

Since none of the participants in the low cluster completed the operator question on the second problem-solving interview the next time they encountered the operator function was on the final re-take exam. This instrument contained several items that focused on fraction and percent operators in a variety of multi-step contexts (Table 31). Once again participants worked individually on the problems. Two participants in the low cluster demonstrated a solid understanding of the transformative process of fraction operators. They used fair share models to illustrate the partitioning and reducing. Percent operators, however, remained procedures for two other participants.

Table 31

Post Test Retake Exam – Fractions as Operators Questions

Item #	Code	Question
47	Olympics training	Maria is training for the Olympics. She has a workout that is 2/10th running, 1/3 treadmill and the rest is weight training. If she works out 180 hours a month how much time does she spend weight training each month?
50	Pages book	I have read 3/8ths of my 560 page book. If I read at a rate of 10 pages a day, how many days will it take me to finish reading the remainder of the book?
55	Decrease class	We will be decreasing the present class of 180 students in P/J by 15% next year. How many students will there be in next year's PJ class? (25%)

Fraction operators. All participants in the low cluster correctly solved the two questions that addressed fractions as operators, the Olympics training question (#47) and the pages book question (#50). Two of these participants demonstrated a solid confidence in their understanding of fraction operators. Both Grace and Valery used a fair shares model to represent the partitioning of the different fractions and resulting portion. The calculations now held meaning for Grace since she related them directly to the pictures (charts) she drew (Figure 70).

> I really like [this kind of question]. I practiced it a lot and I knew the chart really helps me. I knew even division is the same thing, but I think, just the fact that I was able to visualize it really helped me. (pd457)

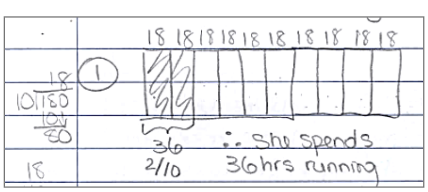


Figure 70. Grace (pd457)

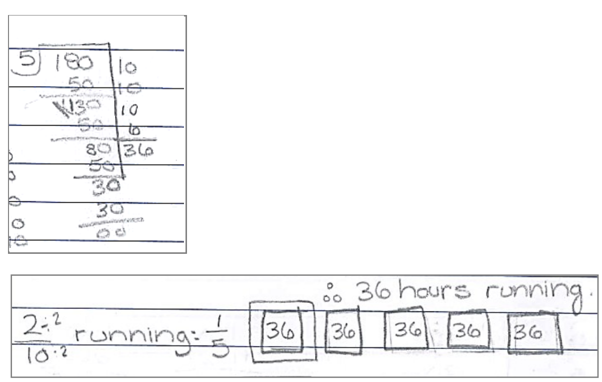


Figure 71. Valery (547)

While Grace used an area model, Valery used a discrete set model (Figure 71). Valery may still think of fractions as two separate numbers, rather than relations, but the modeling process helped her think about the whole as *groups-of*. Because of her poor calculation skills and number sense Valery preferred to reduce the fractions to have fewer parts for partitioning. Valery also needed the visual cues to help her make sense of the operator function. She could not simply interpret the numbers she needed to circle the portion indicated by the numerator.

> Then I circle what 1/5 is; I just circled the one square and then what 1/3 is; because I have to see that. Like I still have to circle them and know. I have to visually see it. (pd547)

The remaining participant in the low cluster used a combination of procedures and models to work with fraction operators. Isabelle continued to use the cross multiplication procedure as her formula for finding the direct portion of the whole (Figure 72). Yet when required to find the portion of the whole for a complement fraction Isabelle chose to use an area fair shares model (Figure 73). It may be that this extra step prompted her to use a visual model to make sense of the context. Since Isabelle did not participate in the final interview it is unclear if she understood how the calculations in her cross multiplication procedure related to the fair shares model she used. While Isabelle did use a similar ratio beside the fraction for her fair shares model, it may be that she used it either to confirm or guide her thinking. Finally, it is also unclear if she understood that the fraction operator represented the relationship between the final result and the original amount.

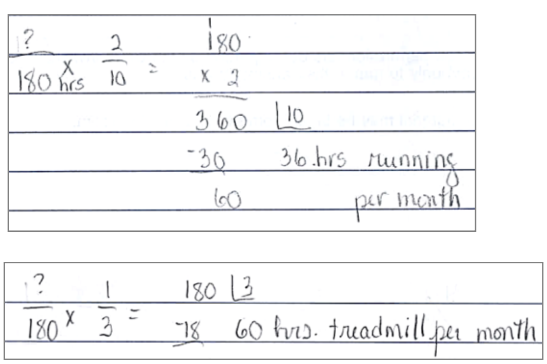


Figure 72. Isabelle (pd570-1)

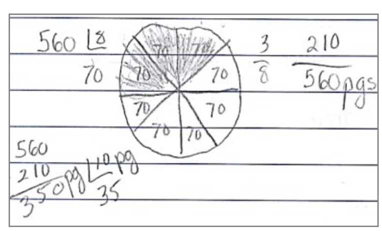


Figure 73. Isabelle (pd574)

Percent operators. Only one participant in the low cluster solved the percent decrease problem (#55) correctly; yet the presence or absence of errors did not necessarily reflect the individual participant's depth of understanding of percent operators. Two of the participants maintained their procedural approach to percent operators. Isabelle used cross multiplication to solve the problem correctly. Her unquestioned formulaic approach to percentages did not appear to change or deepen over the course of the study. Although Grace misread the question as increase rather than decrease, she correctly multiplied by the decimal equivalent to find the portion of the whole and added this amount to the total. Nevertheless Grace could not explain how the multiplication procedure she used related to the process. She simply followed memorized rules and just knew that what she did was right. This unquestioned use of rules led to her inefficient procedure of lining up decimals for multiplication (Figure 74). Both Grace and Isabelle understood that percent operators determined a portion of the whole, but they lacked a deeper understanding of the nature of this transformative function.

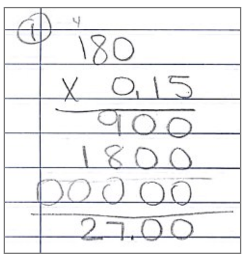


Figure 74. Grace (465)

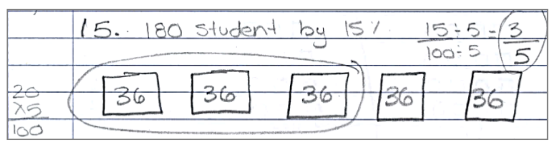


Figure 75. Valery (pd555)

The final participant in the low cluster used a fair shares model to determine the portion of the whole. Understanding percentages as fractions out of 100 Valery chose to reduce the fraction to make her partitive division model simpler. While she made several calculation errors that she was not able to find and correct due to time constraints, Valery did understand the basic nature of percent operators as *groups-of (*Figure 75). However, her poor number sense, as indicated by the error equating 15% with 3/5, limited the depth of her understanding of the relational nature of operators.

Concluding Overview

Most of the participants in the low cluster demonstrated solid growth in their understanding of the operator function. This growth depended, in part, on their willingness to move beyond their memorized procedures. In addition, these participants were more likely to engage meaningfully with fraction operators than with decimal or percent operators. Grace and Valery moved from using rote procedures to using visual models that clearly represented the partitioning and the equal *groups-of* necessary to determine the portion of the whole. Even though Grace used the models to deepen her understanding of the calculations for fraction operators, her understanding of percent and decimal operators remained procedural. Valery moved from using trial and error as she directly modeled the problems to using calculations to determine the fair share model. While Valery used this model for the fraction equivalent of all types of operators she still needed to reduce the fraction to simplest terms in order manage the calculations. This limited her understanding of the relational nature of operators. The final participant in this cluster preferred the security of known procedures. Isabelle continued to use rote cross multiplication, especially for percent operators. However, in certain contexts for fraction operators Isabelle recognized the value of the fair shares model. In these instances she demonstrated an understanding of the role partitioning and groups-of play in finding the operator transformation of the whole.

Middle Cluster Results

Baseline Content Exam

Most of the participants in the middle cluster demonstrated a procedural understanding of fraction operators at the beginning of the study. Their responses to the two questions on the baseline content exam that focused on fractions and percents as operators (Table 32) revealed that many of them relied on memorized procedures that held little meaning. This was especially true for percent operators. However, a couple of the participants used visual models to display a deeper understanding of fraction operators.

Table 32

Content Exam – Fractions as Operators Questions

Item #	Code	Questions
7	Tribes	Two tenths of the P/J class of 240 went to ‘Tribes’. Two thirds of the 240 went to a workshop on ‘Portfolio Writing’. All the rest stayed home for extra sleep. How many stayed home? (3/10, 1/3; 270)
10	Class increase	We will be increasing the present class of 220 students in P/J by 30% next year. How many students will there be in next year’s PJ class? (180, 25%; 160, 40%)

Fraction operators. Two of the participants in the middle cluster demonstrated a solid understanding of fraction operators on the Tribes question (#7). Both Brenda and Lynsey used a fair shares model to partition the whole and then determine the necessary portion of the whole. In this context they knew that the operator function determined *groups-of*. Even though Brenda made a small calculation error in the final answer she demonstrated a clear understanding of how her set model related to the fraction operator (Figure 76). Lynsey's use of an area circle model was a newly acquired approach (Figure 77). With this visual strategy she developed a clearer understanding of operators.

> With stuff like this I really didn't get [in school]. When I was studying with a friend of mine for this content exam, we were going over some stuff she showed me, like that it's easier to actually see it. 'Cuz I was never one to really. It's easier to see it to draw pictures. But I never thought like that. I was stuck on formulas. (pd124)

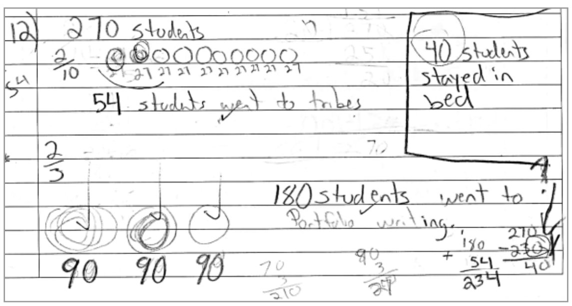


Figure 76. Brenda (pd7)

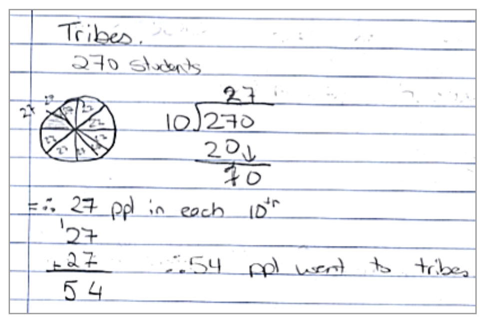


Figure 77. Lynsey (pd124)

The remaining five participants in the middle cluster took a procedural approach to the notion of operators. Yet the degree to which they could make sense of their rote process varied. Erica and Tanya both used the standard algorithm for multiplying fractions, simplifying before multiplying (Figure 78). While they knew that the fractions were used to determine a portion of the whole Erica and Tanya simply explained their calculation procedures and did not make any connections to the meaning of the operator function within the given context. Irene was less efficient in her use of the algorithm since she did not simplify before multiplying. She, however, stated some of her uncertainty in the process. While she could manipulate the numbers to determine a portion of the whole that made sense for the context, Irene could not give meaning to her calculations.

> I probably just put an answer down. I used a formula that I thought worked best and crossed my fingers…. I didn't cross multiply or anything. I could have, but I didn't. I

don't know why I didn't. I just multiplied straight across and divided this into that [points to denominator then numerator]. (pd99)

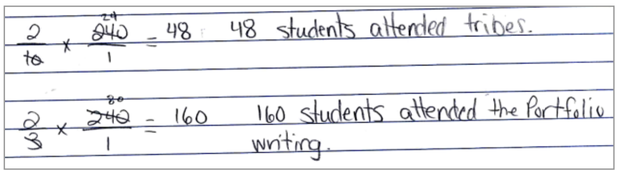


Figure 78. Erica (pd55)

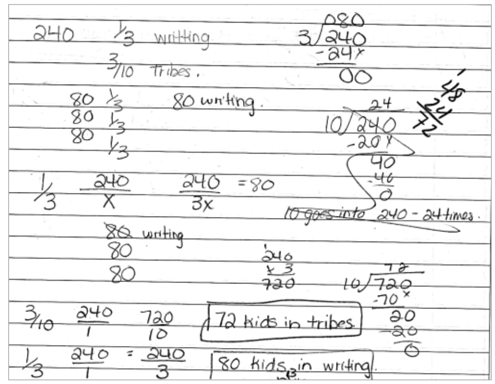


Figure 79. Olivia (pd173)

Olivia had a partial understanding of the operator function but she could not connect it to her calculations. With the unit fraction, 1/3, she correctly determined there would be three groups of 80 people. However, when faced with a common fraction Olivia could not make sense of the division calculation as a partitioning of the whole (Figure 79). "I probably just thought I was doing something wrong and I just started all over again" (pd173). Instead Olivia turned to the multiplication algorithm to determine the answer, confirming that it also gave the correct answer for the unit fraction. Olivia's understanding of the need for *groups-of* in order to transform the whole through the operator function was limited to unit fractions.

Gabriela did not fully remember the algorithm and instead merely divided by the denominator, assuming this provided the necessary portion of the whole (Figure 80). However, she could not relate the calculation to the operator notion of *groups-of* because the fractions had little meaning for her. In the follow-up interview Gabriela confirmed this limited understanding of fractions as operators. While she easily drew pictures of each of the fractions and shaded the correct number of parts, Gabriela could not relate the total number of people to the area models of the fractions. She could not connect her division calculations to her model without external prompting. Her part whole understanding of fractions as procedures restricted her understanding of the operator function.

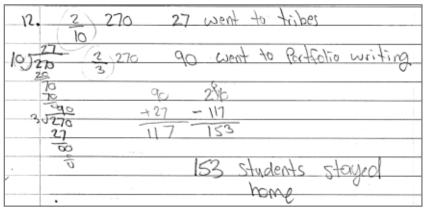


Figure 80. Gabriela (pd69)

Percent Operators. All of the participants in the middle cluster correctly solved the pj class increase problem (#10), yet none of them fully understood their use of percent operators. While these participants understood that the percentage indicated finding a portion of the whole their understanding seemed to be limited to the steps of the procedures they used. Only one participant could explain why the process worked. Because her version of the problem used 25% Olivia thought of the percent as a fraction and thus divided by the denominator (Figure 81). She knew that she had four equal groups that each represent 25% and so added one more group to transform the total. It is unclear how Olivia would understand a percent operator that did not convert into a simple unit fraction.

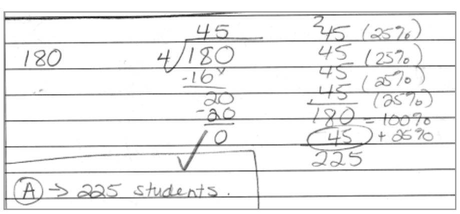


Figure 81. Olivia (pd171)

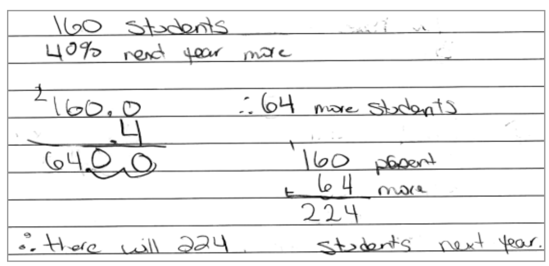


Figure 82. Lynsey (pd127)

The remaining six participants all used a multiplication procedure, converting the percent to a decimal, to find the portion of the whole. None of them were able to clearly explain the procedure. Like Tanya, many multiplied because, “that’s just what I was taught” (pd192). They learned the rules for working with decimals, but they did not know why the procedure worked. As Gabriela stated, “I just move the decimal over because it’s percentage…. I was taught that way, just move it over” (pd72). Some, like Brenda, recognized the percentage was a fraction out of 100, but still did not understand the procedure. “I just knew from my previous math course, because 40%, 40 out of 100, I don’t know, I just knew” (pd10). Others tried to make some sense out of the procedure. Lynsey connected the multiplication to the process she used to find the sales tax when working in a store. Even though she did not know why it worked, she knew the process would give the correct answer. The unnecessary lining up of the decimals accentuated her rote understanding of the multiplication procedure (Figure 82). Irene's procedural understanding ignored the partitioning or *groups-of* aspect of the operator function. She simply focused on multiplication making bigger.

> Because I want to increase something, I want to find something, I want to know that it has to be bigger than 220 because it's increasing, so I know that this word [increase] would symbolize *of* for me or product, which means to multiply. (pd102)

Even though Irene knew she had to add to get the final answer she did not seem to understand that multiplication by the decimal percent gave a portion of the whole, an amount that was smaller. She only looked at the final answer. As was true for many other participants in this cluster, Irene's focus on the procedure for percents limited her understanding of the operator function.

The participants in the middle cluster demonstrated a mainly procedural understanding of fractions as operators on the baseline content exam. Most of the participants focused on the rules for multiplying fractions and decimals to find the fraction or percent portion of the whole. Some, like Gabriela, relied on partially remembered or buggy procedures. Most did not recognize how their calculations might be linked to *groups-of* or partitions of the whole. Furthermore, they did not understand the relationship between the operator and the ratio of the resulting amount to the original amount. Nevertheless, when working with fraction operators Brenda and Lynsey revealed a solid understanding that linked a partitive fair shares model to the notion of *groups-of*, even though their understanding of percent operators remained procedural. Similarly, Olivia demonstrated a strong understanding of unit fractions as operators, clearly partitioning the whole into *groups-of*, even though she could not extend this understanding to common fractions.

First Problem-Solving Interview

One question on the first problem-solving interview addressed the theme of fractions as operators, focusing on the relationship between fractions, decimals, and percents (Table 33). For the most part, the participants in the middle cluster retained their procedural understanding of operators. Nonetheless some modest gains occurred as more participants explored a visual model to represent fraction operators. In addition one participant attempted to think meaningfully about percent operators using the definition of percentage and the distributive property. For the problem-solving sessions Brenda worked with Olivia, Erica worked with Lynsey, and Irene worked alone, while Gabriela and Tanya each worked with a participant in the low cluster.

Table 33

Problem Solving Interview #1 – Fractions as Operators Questions

Item #	Code	Question
26	Math course	Jesse was planning the amount of time he would devote to each strand in his 120 hour math course. He planned to spend 0.4 of the time on measurement and data management, 15% of the time on algebra, 3/8 of the time on numeracy, and the remainder of the time on geometry. How many hours were spent on each of the four strands?

Participants in the middle cluster found the combination of different types of operators in one the math course problem (#26) challenging. Most of them thought all three operators needed to be converted to the same format. These participants worked with the operators as numbers to be manipulated rather than meaningful objects. The focus became the mechanical procedure for finding the necessary portion of the whole rather than using what they knew about the nature of fractions, decimals, or percents as operators. Nevertheless, while three of the participants remained constant in their understanding of fractions as operators four of the participants demonstrated various degrees of growth in their understanding of the operator function.

Of the participants who demonstrated little growth in their understanding of operators, Erica and Irene both remained strongly entrenched in their procedural understanding. To find the portion of the whole they converted each of the operators to a decimal and multiplied by the whole. While both Erica and Irene could clearly explain the procedures they could not explain why they used the calculations or why the calculations worked. Irene attempted to explain, but could not get beyond the rule.

> Because I'm working with the one base number that's going to be the same for all of it. And I'm trying to find out 15% of 120, tells me that it is 18 hours…. That's, I have no idea, other than knowing that's just what I'm supposed to do. Because I'm trying to find a *product of* and in doing so you know you need to multiply. (pd271)

Lynsey also remained at her prior level of understanding of operators. She continued to use a fair shares model when working with fractions, clearly understanding the need to partition the whole and find the appropriate number of portions. Yet Lynsey maintained her procedural approach to decimal and percent operators. Once again her buggy multiplication algorithm indicates her focus on the memorized procedure (Figure 83).

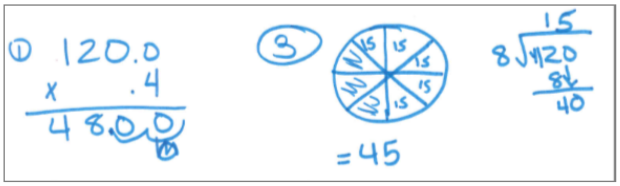


Figure 83. Lynsey (pd248)

The remaining four participants in the middle cluster demonstrated varying degrees of growth and success in their understanding of operators. Tanya continued with a procedural approach, multiplying by decimal equivalents, but when she encountered an error due to decimal placement she looked for an alternative method (Figure 84). She attempted to use a fair shares model with the fraction operator. Unfortunately, Tanya interpreted the model through her procedural lens of percentages and thought each partition was 15% rather than 15 hours. She understood the need for *groups-of,* but did not know what they represented. Thus she continued with her procedural approach of multiplication. It was only when she faced a contradiction in her final results that she was able to recognize that the model represented the actual hours.

Gabriela also made a shift in her procedural approach to fraction operators after encountering an error converting the fraction to a decimal. Once she and her partner determined they did not have to convert all operators to the same format, Gabriela confidently used the fair shares model to accurately represent the portions of the whole (Figure 85). She was less confident with the percent and decimal and was unsure about multiplying. Nevertheless Gabriela articulated an understanding of multiplication linked to her experience with sales tax.

> I don't know; I just multiplied because it's 15%; and I know when you buy something here, for every dollar it's 15%. So I just figured you'd multiply whatever this is by 15% to get the… I'm thinking sales tax; for every dollar it's 15 cents. So at the dollar store it's really $1.15. (pd260)

Gabriela understood concept of *groups of* in this decimal multiplication context, even if she did not know what the partitions were.

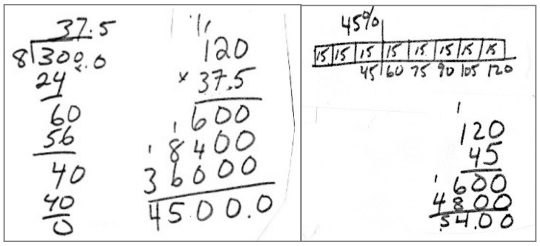


Figure 84. Tanya (pd284)

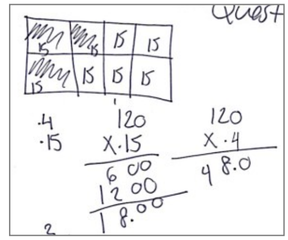


Figure 85. Gabriela (pd260)

Olivia and Brenda both demonstrated a willingness to let go of procedures and think about the meaning of the operators. But they struggled with the decimal operator spending almost 10 minutes trying different strategies. Olivia thought about the decimal as a percent and knew that 40% of 100 was 40, but she could not figure out the remaining 20. She attempted to use a number line and incorrectly thought that 40% of 20 must be 2, because 120 was two more tens than 100 (Figure 86). She did not understand the 2 represented two groups of 4. Brenda thought of the decimal as a fraction and tried to reason about the context using 12 groups of 10 (Figure 87). But she could only see the 12 groups, rather than the 10 in each group.

> One, two, three, four; four out of every ten, but then we have 12, so then I just don't understand this part, would it be a quarter of the next one?... If we imagined there were 8 more here, then there would be another 4 coloured, but there's not, there's only 2. (pd224)

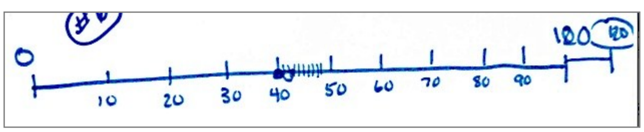


Figure 86. Olivia (pd224)

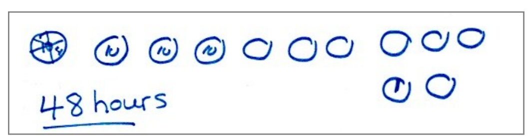


Figure 87. Brenda (pd224)

Ultimately the pair needed prompting to think about 40% of (100 + 10 + 10) in order to solve the problem correctly. For the remaining two operators Brenda took charge, while Olivia simply watched. With the fraction operator Brenda used the fair shares model to find the portion of the whole. Here she had no difficulties relating the fraction to her model. For the final operator Brenda avoided the challenges of the percent and instead reduced the equivalent fraction to lowest terms (Figure 88). In this way she could continue to use the fair shares model to find the portion of the whole.

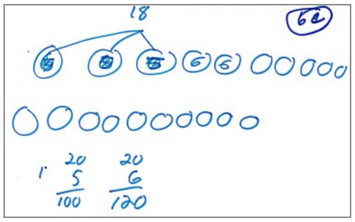


Figure 88. Brenda (pd224)

During the first problem-solving interview many participants in the middle cluster retained their procedural understanding of the operator function. This was especially true for their work with decimals and percents. Three of the participants demonstrated no growth in their understanding of operators in this session. Erica and Irene continued with their rote multiplication by decimals for all operators, while Lynsey still limited her understanding of partitions and *groups-of* to fraction operators, reverting to memorized procedures for decimal and percent operators. Another three participants attempted to use new ways to think about one of the operators but their understanding was only partially formed and they could not make sense of their models. Tanya attempted to use the fair shares model for the fraction operator but misinterpreted the portions as percents. Brenda attempted to use the decimal fraction as a ratio to find the portion of the whole but confused quotative and partitive division. Olivia attempted to use proportional reasoning based on the definition of percents, using a number line model, but focused on the absolute magnitude of the number rather than the total represented by the *groups-of*. The remaining participant in the middle cluster deepened her understanding of operators through the use of a model. Gabriela correctly used a fair shares model for the fraction operator and used her prior experience with money to give meaning to the multiplication procedure with percent operators.

Second Problem-Solving Interview

One question in the second problem-solving interview highlighted the theme of fractions as operators by using fractions, decimals and percents as operators in the context of larger numbers (Table 34). Participants in the top cluster who completed this question still maintained a procedural understanding of most operators, yet they showed small gains in their thinking.

Table 34

Problem Solving Interview #2 – Fractions as Operators Questions

Item #	Code	Question
39	Trail mix	Olivia is making a special trail mix for her next hiking trip. The recipe makes 640 grams of the mixture. It consists of 1/8 raisins, 0.3 sunflower seeds, 25% peanuts, with the remainder being a dried fruit mixture. How many grams must she use of each of the ingredients?

Only three participants in the middle cluster completed the trail mix (#39) question. Erica and Lynsey maintained a procedural approach to the percent and decimal operators. The rules and multiplication procedures seemed to provide a secure framework for them. Lynsey, for example, continued to line up the decimals and add extra zeroes in the multiplication procedure (Figure 89). "I know I add a lot of unnecessary zeroes; they keep me on track" (pd342). Even though it would have been more efficient to think of 25% as a fraction Erica and Lynsey did not have this flexibility when working with percent operators. They did, however, develop their understanding of fraction operators. Working with a unit fraction, Erica no longer needed to convert the fraction to a decimal and Lynsey did not need to draw a model, rather they internalized the notion of partitioning and divided the whole by the denominator. In this way both participants used the fair shares model to determine the portion of the whole.

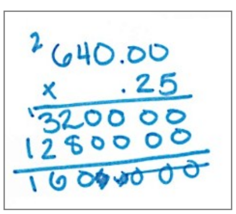


Figure 89. Lynsey (pd342)

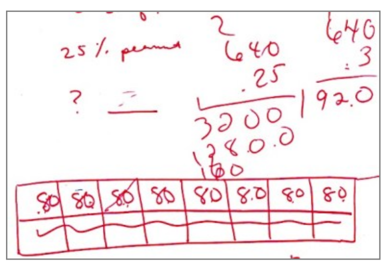


Figure 90. Irene (pd363)

Irene also maintained a mainly procedural understanding of operators. Even though she would have preferred to multiply with decimal equivalents for each of the operators Irene chose to use a visual model to represent the operator function because of the interview setting (Figure 90). "So I would take 640, I would multiply by 1/8, which is…[pause] or else I'll just draw it out; you might make me do it anyway" (pd363). Irene readily used the fair shares model to find the portion indicated by the fraction operator. She understood that the number of partitions related to the fraction operator. Nevertheless, Irene then applied this same model in a procedural manner to the decimal and percent operator.

> Point three is sunflower seeds or 30%, hmm, I'm not sure if I'd be doing it right but I'd say 1, 2, 3 [counts partitions of eighths] that's 240; 25% would be 160 plus 40 would be 200 [counts 2 1/2 portions of eighths]. (pd363)

Irene simply counted the portions of the given model because she thought it was important to use the same visual model for each of the operators in the problem. She chose to follow a procedure rather than think about the partitions and *groups-of*. Irene found it challenging to give meaning to the operator function and preferred to return to the security of the multiplication procedure to find the correct answer. She assumed the visual model could not be guaranteed to give the correct answer. "Showing it from a diagram is different than trying to figure it out by percent" (pd363).

The participants in the middle cluster made small gains in their understanding of the operator function on the second problem-solving interview, but continued to have a mainly procedural understanding. Irene attempted to use the fair shares model to represent the operator function. Unfortunately after using the model successfully with a fraction operator Irene proceeded to use the same model in a rote manner with different operators. Erica and Lynsey internalized the fair shares model for a unit fraction operator, but maintained their procedural understanding of decimal and percent operators.

Post-Test Retake Exam

The final test instrument contained several items that highlighted both fraction and percent operators in multi-step contexts (Table 35). Once again participants worked individually on the problems. All participants in the middle cluster demonstrated an understanding of the partitioning and groups-of in the fair shares model of fraction operators. However, depending on the nature of the fraction some of the participants continued to use a procedural approach. Percent operators remained a memorized procedure with little meaning for most of these participants.

Table 35

Post Test Retake Exam – Fractions as Operators Questions

Item #	Code	Question
47	Olympics training	Maria is training for the Olympics. She has a workout that is 2/10th running, 1/3 treadmill and the rest is weight training. If she works out 180 hours a month how much time does she spend weight training each month?
50	Pages book	I have read 3/8ths of my 560 page book. If I read at a rate of 10 pages a day, how many days will it take me to finish reading the remainder of the book?
55	Decrease class	We will be decreasing the present class of 180 students in P/J by 15% next year. How many students will there be in next year's PJ class? (25%)

Fraction operators. The participants in the middle cluster demonstrated varying degrees of understanding of fractions as operators as presented in the Olympic training question (#47) and the pages book question (#50). While all of these participants used a fair shares model with partitioning to find the appropriate portion of the whole for at least one of the fractions, several still maintained a procedural approach with certain fractions. Both Brenda and Erica used a procedural approach for the decimal fraction, 2/10, even though they used a visual model for the other fractions (Figure 91). Brenda's procedural focus is reflected in her first attempt to multiply, inserting all the zeroes. Both Brenda and Erica clearly described their process but they could not explain how their multiplication procedure related to finding the portion of the whole. Similarly, Irene multiplied by the decimal equivalent for both 2/10 and 1/3 even though she used a fair shares model with the fraction 3/8. Because she still focused on the procedure rather than understanding Irene did not realize that rounding the decimal would give a less accurate answer (Figure 92).

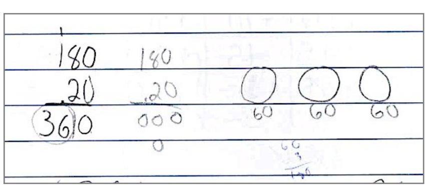


Figure 91. Brenda (pd397)

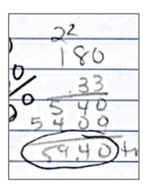


Figure 92. Irene (pd471)

The remaining four participants represented the operator function with a fair shares model for all of the fractions. Gabriela demonstrated her shift from procedural understanding as she used a visual model to confirm her initial multiplication by the decimal fraction. Even though Gabriela did not fully understand the decimals she multiplied because the fraction was equal to a percent and that was the method she learned.

> This is why I do this [points to drawing]. The whole decimal thing; I forget the decimal and I just do this [points to drawing], 'cuz this makes sense to me; 10 pieces make up the 180; this I understand. The .2 I still don't really understand. I just move them over. That's why I did this [points to drawing] to confirm. (pd446)

While most of these participants used an area model and filled in the portion of each partition, Olivia used a set model and only filled in the necessary parts to represent the fraction operator (Figure 93). She became more efficient, knowing that each portion was the same size and thus did not need to complete them all. Lynsey also began to internalize her understanding of the nature of fraction operators. Even though she drew visual models to determine the appropriate portion for each fraction, she no longer needed them (Figure 94).

> To be honest, I don't think I really need to [draw them out] any more. Like when I was doing a lot of the practice exam ones, I wouldn't always draw it out because I can just, like, I can visualize it now in my head. But for this I drew it out, just cause it says do all your work, 'cause if I got it wrong I might have gotten part marks. (pd479).

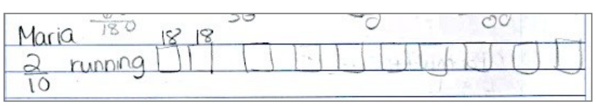


Figure 93. Olivia (pd525)

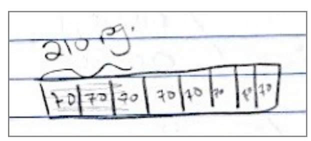


Figure 94. Lynsey (pd482)

Many of the participants in the middle cluster moved from a strongly procedural understanding of fraction operators to a deeper understanding of this notion and its links to partitioning and *groups-of*. Tanya exemplified this shift as she felt more confident with the visual method than her previous use of formulas.

> This way I actually understand it. Just actually understanding the concepts; drawing it out and having it visually there, rather than just doing a formula which really you know that it, you know you're doing it right, but you don't know why you're doing it. (pd536)

Percent operators. All of the participants in the middle cluster used a procedural approach with percent operators on the percent decrease (#55) problem. Only one of them could not solve the problem correctly. Initially, even though Olivia divided by the percent rather than multiplying she was confident with her answer (Figure 95). She used a procedure and did not think about the meaning of her answer. In the follow-up Olivia knew her answer did not make sense when she reasoned with a benchmark percentage.

> I was thinking if it was 15% of 100 I knew it would be 15 students. And that's when I, I should have realized that this was wrong, because 15% of 100 is 15 students and this is 180. So it would have to be higher than 12. But I thought this makes sense. I'd have to multiply. (pd533)

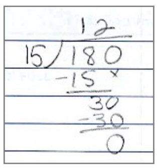


Figure 95. Olivia (pd533)

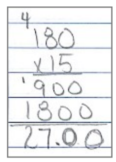


Figure 96. Erica (pd442)

While the other participants correctly multiplied by the decimal equivalent of the percent, they also simply followed the procedure and did not think about the meaning of the calculation. They just knew that it gave the correct answer, "because when it's a percent you multiply" (pd487). Most of these participants knew why the percent could be written as a decimal but beyond that they simply manipulated the numbers. Erica's response represented the procedural thinking of these participants (Figure 96).

> Those are the only two numbers you really have [in the problem] and I know that multiplication works. Yeah, I don't really have a solid answer or backing up my reasoning. Well there's probably another way you could do it, but that's my method. It's fast, efficient and I got it right. (pd442)

Concluding Overview

Participants in the middle cluster demonstrated a solid growth in their understanding of fractions as operators, but their understanding of percent and decimal operators remained static. Four of the participants began the study with a strong reliance on memorized procedures for all operators. They interpreted the operator function as a multiplication procedure that produced a portion of the whole; but they did not know why their calculations worked. Tanya and Gabriela moved to the use of a fair shares model to represent the distribution of the whole over partitions. This visual representation helped them give meaning to the calculations for finding the necessary portion of the whole. Erica and Irene made smaller gains in their understanding of fraction operators. While they moved to using the fair shares model with certain fractions, such as eighths, they preferred the security of working with decimals and memorized procedures. Nevertheless they both understood how partitioning and the development of *groups-of* determined the portion of the whole.

Three of the participants in the middle cluster began with a more developed understanding of fraction operators. Olivia could make sense of unit fractions, adding up to partition the whole, but she worked procedurally with common fractions, not understanding why her calculations worked. Olivia moved to understanding the partitive fair shares model for all fractions. Lynsey moved from using a visual model to think about the operator function to internalizing her understanding of this function to such a degree that she no longer depended on the drawing and her calculations meaningfully reflected the partitioning process. While Brenda began using a visual fair shares model her understanding of fraction operators seemed to plateau as she did not appear to internalize the process by making connections with her calculations.

All of the participants in the middle cluster retained their procedural understanding of percent operators. Even though Brenda and Olivia attempted to find another way to make sense of percents, their struggles simply accentuated their need for the security of procedures, rather than understanding.

Top Cluster Results

Baseline Content Exam

Two questions on the content exam served to establish participants' baseline understanding of fractions as operators. The questions, as seen in Table 36, used simple fractions and percentages that required participants to find a portion of the whole. This amount was then added to or subtracted from the whole depending on the context. All participants in the top cluster solved both of these problems correctly. However, the degree to which they relied on rote procedures or understanding varied. These participants in the top cluster readily used their reasoning skills to make sense of the numbers within the given contexts. But those who relied on procedures could not necessarily state why the procedure worked or what the steps represented.

Table 36

Content Exam – Fractions as Operators Questions

Item #	Code	Questions
7	Tribes	Two tenths of the P/J class of 240 went to 'Tribes'. Two thirds of the 240 went to a workshop on 'Portfolio Writing'. All the rest stayed home for extra sleep. How many stayed home? (3/10, 1/3; 270)
10	Class increase	We will be increasing the present class of 220 students in P/J by 30% next year. How many students will there be in next year's PJ class? (180, 25%; 160, 40%)

Fraction operators. On the Tribes question (#7) two of the participants in the top cluster demonstrated a solid understanding of fractions as operators while the other two participants struggled to make sense of the procedures they used. Both Mark and Megan used the unit fraction to partition the whole into the requisite number of groups and then multiplied according to the number of groups needed. While Mark used pictures, Megan explained her calculations.

> At first off I was breaking it into tenths saying there was ten groups in 240. So I was breaking it into tenths to find out how many kids were in three-tenths. So I divided the 240, of the whole class, by 10 to get the tenths. So there is 24 in each tenth. So 24 times 3 meant there are 72 kids who went to Tribes. (pd157)

Mark acknowledged that he could have multiplied by the decimal equivalents, but he chose to use the pictorial representation because of the fraction 2/3. He realized that multiplying by a repeating decimal was more difficult than dividing the whole into three equal parts.

> I could have just multiplied because 2/10 is the same as decimal 2. And, thirds are hard to work with, that's why I did it this way [points to diagram of three equal parts], cause it's like, a third is like decimal 33333, it's not a nice decimal to work with. The two-thirds; that way it was easier to show than trying to multiply it because you get a decimal that's a funny number. (pd140)

Diane and Bryann both focused on the procedural use of fractions as operators and encountered difficulty making sense of their initial responses. While they knew the fractions represented portions of the whole, they each found the portion in different ways. Diane began by using the traditional algorithm to multiply each fraction times the unit whole. When she made a mistake in her calculations, Diane used different procedures to try to make sense of her answer (Figure 97). First, she confused the whole with 100 percent and subtracted the portions from 100. When her resulting answer did not make sense, Diane used an algebraic approach to find the fraction representing the unknown part. She then used her initial procedure to find the resulting portion of the whole. In both instances Diane used her knowledge of procedures more than concepts to solve the problem correctly. Ultimately she did not fully know what the operator procedure meant. “I’m not quite sure right now how I knew how to do that; at the time I guess I was just in the zone and I knew that I was supposed to divide by the bottom” (pd39).

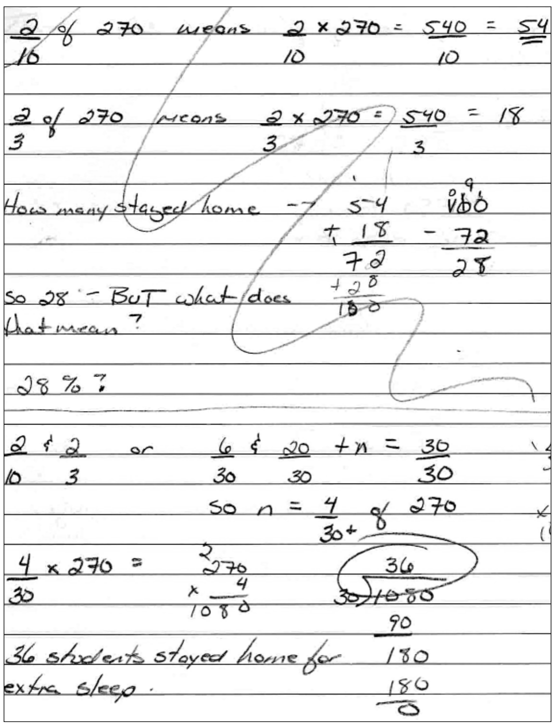


Figure 97. Diane (pd39)

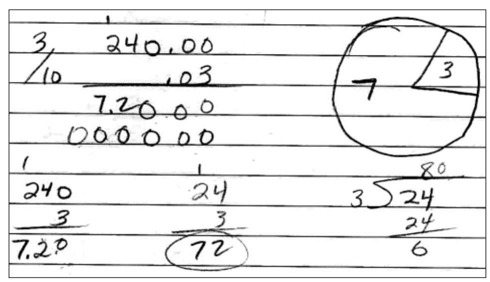


Figure 98. Bryann (pd23)

Similarly, Bryann focused more on procedures and manipulating numbers to make sense in the context of the problem. After incorrectly converting 3/10 to a decimal Bryann, simply tried different combinations of numbers to get an answer that was appropriate (see Figure 98). “I think the number was too small, not sure, not 100% sure, so I tried a couple of things, and that one [indicates 3 x 24 = 72] was a heck of a lot more logical than 7.2” (pd23). Bryann could complete the process but she did not know how fraction tenths worked as operators, even though she eventually explained how she divided the total amount into three equal thirds to find the 1/3 operator.

Percent operators. Participants worked with percents as operators on the pj class increase problem (#10). While all participants in the top cluster could explain their process for finding the answer only one of them went beyond the process to explain why it worked. Megan thought of the percent as a fraction and her understanding of fractions as operator was embedded in her knowledge of shopping and money.

> 25% of a dollar is a quarter, so, anyway, so I divided it [the total] by 4, because that would make it into 4 groups that were each 25%, so then I just had to add another 45, which was 25%, [to the total]; yeah, I thought of shopping with that one. (pd160)

The three other participants used a multiplication procedure to find the portion of the whole, but, in varying degrees, their understanding was limited to the process, using the rules they had been taught. Diane converted the percentage to a fraction out of 100 and then multiplied the total by the fraction, because "by usually means multiply" (pd42). She tried to make sense of this process but could not fully articulate how the *groups-of* related to 40% and incorrectly ignored the importance of 100, focusing instead on the numerator.

> Basically there are 160 and we want to divide that into, want to increase that by 40%, so that would mean dividing it by 4, 40 [is unsure]; and then adding one of those 40's to the 160. (pd42)

Diane explained the operator concept in terms of division even though her process clearly used multiplication.

Neither Mark nor Bryann attempted to articulate their understanding of the percents as operators. It was simply a deeply ingrained process. Mark tried to explain his understanding of the procedure for decimal placement, but could not completely follow through with it.

> Because it's a tenth; I'm, I'm multiplying the 160 by a, like a tenths place, so my, my question is going to be smaller than, like it's um, no; I just know…. Only because I was multiplying by tenths, I have to have a tenths in there, like in my answer, I know I have to have a tenth in there somehow, so um, the only way to do that is by moving the decimal over one decimal place. (pd144)

For Bryann converting a percentage to a decimal and multiplying was simply a memorized procedure that gave the correct answer for finding the portion of the whole. It held no meaning for her (Figure 99). Thus she used a buggy form of decimal multiplication that required both factors to have the same number of decimal places.

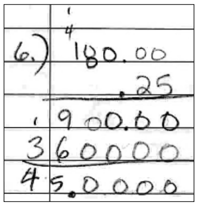


Figure 99. Bryann (pd26)

The participants in the top cluster demonstrated a range of understandings of the operator nature of fractions. Megan revealed a solid understanding that linked her prior experiences to a partitive understanding of the notion of *groups-of*. Mark similarly understood the partitive model in the context of fraction operators, but had a more procedural view of percentages. Both Diane and Bryann had a procedural understanding of operators that generally focused on the rule of multiplication for finding a portion of the whole.

First Problem-Solving Interview

The first problem-solving interview contained one question that addressed the theme of fractions as operators, highlighting the relationship between fractions, decimals, and percents (Table 37). Participants in the top cluster demonstrated growth in the development of their understanding of operator as they used models rather than algorithms to explain their thinking. However, the flexibility of shifting among percents and decimals and fractions varied between the participants. In both problem-solving interviews Diane worked with Mark and Bryann worked with Megan.

Table 37

Problem Solving Interview #1 – Fractions as Operators Questions

Item #	Code	Question
26	Math course	Jesse was planning the amount of time he would devote to each strand in his 120 hour math course. He planned to spend 0.4 of the time on measurement and data management, 15% of the time on algebra, 3/8 of the time on numeracy, and the remainder of the time on geometry. How many hours were spent on each of the four strands?

All participants in the top cluster correctly solved the math course problem (#26); however, their understanding of the operator function depended on how the fraction was expressed. Two of the participants in this cluster worked flexibly with fractions in all three forms. They understood that the both the decimal and percent could also be represented as fractions. In each instance Diane and Mark used an area fair shares model to represent the parts of the whole and to determine the value of each part. They recognized that the given operator functions required that they partition the whole and reduce it by taking only a certain number of the parts. Mark and Diane demonstrated the depth of their understanding when they used their model for decimal tenths to help them think about finding 15% of the whole.

> M: We can say that we are going to take 11/2 of these [points to the tenths partitions of 120].
>
> D: Yup. And so that's going to be 6 plus 12
>
> Int: Can you explain your thinking?
>
> M: 15 over 100 is like, uh, 1.5 over 10…. I'm dividing by 10. I divided the bottom by 10 and I divided the top by 10. And then when I used that information, I know I have one whole tenth, which is 12, and half of a tenth, which is 6. So I'm going to use the 12 and the 6, which is 18. (pd235)

The other two participants in the top cluster struggled to make sense of percents and decimals as operators. Both Megan and Bryann were confident when using fractions as operators since they could use the area fair shares model to determine the correct portion of the whole. Yet they did not know how to model and therefore, make sense of the operator concept for decimals or percents. Bryann focused on her procedural knowledge and simply wanted to multiply by the decimal. However, she did not understand the meaning of the operation when challenged by her partner. Both recognized that .4 also meant 40%, but they didn't know how to use that knowledge.

> B: Just multiply 120 by .4.
>
> M: Okay. [writes down calculation] Why? Why are we doing that? Is that going to give us our percentage?
>
> B: I don't know. And then times 100…. Or no, wait…
>
> M: Something, yeah, but it's the something part that I forget…. Or is it 100 times .4 divided into.., let's see now; if I had a formula I could figure it out.
>
> B: I have one on my calculator; it's called a percent button. [laughter]. (pd294)

These participants willingly engaged in the struggle to reason through the problem instead of relying on rote procedures. Megan attempted to use proportional reasoning based on the definition of percents. Since she knew that 40% of 100 was 40, she tried to reason out 40% of the remaining 20, knowing that "5" was important. Bryann, on the other hand, simply used a benchmark, 50% of 120 was 60, to reason about the problem. Unfortunately neither of them could complete their line of reasoning using the percents (Figure 100). Instead they chose to think about the decimal as the fraction 4/10. Once they made this connection Megan and Bryann used the area fair shares model to determine the correct portion of the whole.

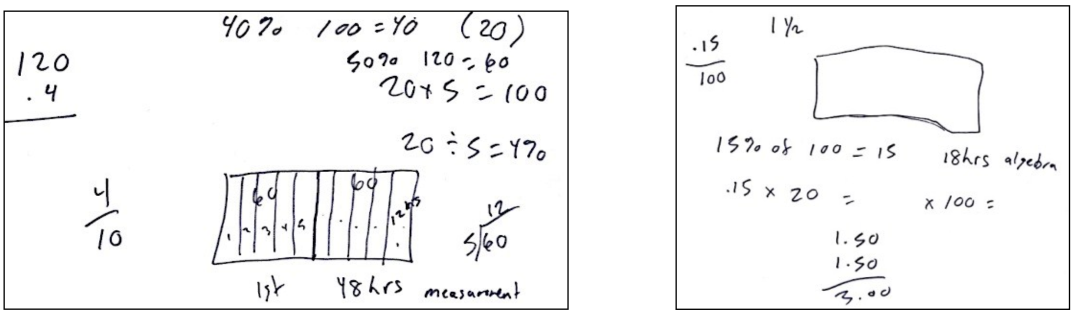


Figure 100. Megan/Bryann (pd294)

Bryann and Megan continued to struggle with the final percent. Megan did not know how to represent 15% as a fraction, while Bryann started to "freak out" over the question and missed the security of using a calculator. Because they did not know how to represent 100 on their fair shares model Bryann suggested the context of figuring out sales tax. Eventually Megan decomposed 120 to 100 + 10 + 10 and they simply found 15% of each of the components. While these participants understood the ultimate intent of percents as operators, they struggled to find a model that enabled them to meaningfully use percents to find a portion of the whole.

The participants in the top cluster demonstrated varying degrees of development in their understanding of fractions as operators over the course of the first problem-solving interview. The three participants who began the study with a procedural notion of operators willingly shifted their approach and attempted to reason through the meaning of the concept of operator. Mark and Diane effectively used the fair shares model to demonstrate the partitioning and reducing operator function for fractions, decimals, and percents. Bryann also used the fair shares model to concretize her understanding of fractions as operators. However, she had difficulty conceptualizing decimals and percents as operators and needed the prompts of her partner to think beyond the procedures. Although the remaining participant in this cluster began the study with a strong understanding of the partitive notion of *groups-of* for fractions, her understanding for percents was contextually bound. Megan eventually linked decimal tenths to her understanding of fraction operators, but she struggled with the exact meaning of percentages. She looked for patterns and proportions between numbers in an attempt to understand and represent the concept of operators.

Second Problem-Solving Interview

The question in the second problem-solving interview that highlighted the theme of fractions as operators explored the interchangeability of fractions, decimals, and percents as operators in the context of larger numbers (Table 38). Participants in the top cluster who completed this question demonstrated a shift in their understanding to an internalization of the concept of operators as they eliminated their use of a concrete model to think about partitioning and reducing the whole.

Table 38

Problem Solving Interview #2 – Fractions as Operators Questions

Item #	Code	Question
39	Trail mix	Olivia is making a special trail mix for her next hiking trip. The recipe makes 640 grams of the mixture. It consists of 1/8 raisins, 0.3 sunflower seeds, 25% peanuts, with the remainder being a dried fruit mixture. How many grams must she use of each of the ingredients?

Only two participants in the top cluster completed the trail mix (#39) question. Mark and Diane worked flexibly with the different forms of rational numbers, converting them all to fractions. They no longer needed a visual model to make sense of the operator function, but instead used mental calculations, clearly explaining their reasoning. In each instance they used the fair shares model to partition and reduce the whole. For example, Diane no longer worked with the numbers as procedures to follow, multiplying first and then dividing. When determining 3/10 of the amount she first divided by 10, finding the value of each partition and then multiplied by 3. Both of these participants demonstrated continued growth in their understanding of the operator function of fractions as they internalized the notion of partitioning.

Post-Test Retake Exam

The final test instrument contained several items that focused on fractions as operators (Table 39). These questions highlighted different multi-step contexts that contained both fraction and percent operators. Once again participants worked individually on this final test instrument and engaged in a follow-up interview to explain their thinking. Most of the participants in the top cluster no longer relied on a procedural understanding of operators, but used mental or concrete models that stressed partitioning the whole and determining the requisite number of parts. However, one of the participants still reverted to known procedures in the stressful context of the exam. Nevertheless even she demonstrated growth in her ability to conceptualize fraction operators.

Table 39

Post Test Retake Exam – Fractions as Operators Questions

Item #	Code	Question
47	Olympics training	Maria is training for the Olympics. She has a workout that is 2/10th running, 1/3 treadmill and the rest is weight training. If she works out 180 hours a month how much time does she spend weight training each month?
50	Pages book	I have read 3/8ths of my 560 page book. If I read at a rate of 10 pages a day, how many days will it take me to finish reading the remainder of the book?
55	Decrease class	We will be decreasing the present class of 180 students in P/J by 15% next year. How many students will there be in next year's PJ class? (25%)

Fraction operators. All four participants in the top cluster correctly solved the Olympics training question (#47) and the pages book question (#50), both of which addressed fractions as operators. Two of these participants demonstrated a strong understanding of the operator function. Both Diane and Mark used the fair shares model to represent the partitioning and portions inherent in this concept. Mark's observations on his use of the "box" model reflect the shift in understanding that took place for both of these participants, from numbers and procedures to meaningful models.

> A lot of time working with fractions I like making the boxes, so it just simplifies things. It really did. I was absolutely amazed by that whole process…. I'd never seen it that way, I'd never seen, I always just, it was always mental math for me; and if I would have been asked to show that at the beginning of the year, I wouldn't have been able to; I wouldn't know why; I just would have known that that was the answer; I wouldn't have

> been able to visually show you that or I wouldn't have know how to visually show you that. But once, you know, just from things that I did in the course and um, I mean it's so easy, I wouldn't have come up with that on my own…. To visually show something as far as I'm concerned, it doesn't get any easier than that; it just explains itself. (pd498)

Another participant's understanding of operators was more contextually determined. Megan chose to combine the fractions, using a common denominator, in order to determine the remaining amount. She reduced the resulting fraction of 14/30 to 7/15, but then was unsure of what this meant. "I don't know if that works out to every 15 minutes, but let's see" (pd508). Since Megan did not know how to interpret the larger denominator, "her brain went slamming shut on that version", she chose to use a second method. With the smaller denominators of tenths and thirds, Megan could meaningfully, and successfully, reason about the fractions as operators. She used fair shares thinking to partition the whole into thirds, by dividing the total by 3 and into tenths by mentally dividing by10 and then doubling. The more familiar and simpler fractions enabled Megan to make sense of the challenging context of hours.

> As soon as you get the hours you're thinking 60 minutes and does that mess up the hours? … I try to think of the hours as a whole unit and not think of the 60 minutes because that messes me up…. Whenever you deal with hours it just gets complicated. The 60 minutes thing, trying to think what's a fraction of that; how do you do 1/10 of 60? You can't, you know, and that's what starts to mess me up and then I just sort of put it aside and went, you can't think of 60, you have to think of it as a whole; which I probably wouldn't have done before the course, either, so, you know. I find any of the things with the hours just get bogged down, what's the second, what's the minute? And I know now to label everything to death. (pd515)

The final participant in the top cluster continued to hold on to elements of her procedural approach to operators, but also began to see contexts where a visual model would be helpful. Bryann preferred to convert fractions such as tenths to decimals and multiply to find the operator function. However, she did not fully understand how or why these procedures worked. "The only thing I know about that is that the bottom number has to go into the top number when I do a decimal. I don't know why" (pd408). When confronted with a repeating decimal Bryann chose instead to use the area fair shares model to partition the whole. Similarly when working with 3/8, Bryann chose to avoid the conversion to decimals. "As soon as I saw the eighths, I knew that there were going to be 8 parts that I could work with" (pd411). Bryann was beginning to make sense of fractions as operators by using visual models, recognizing that at times fractions might be easier to work with than decimals.

> I didn't even know the difference [between fractions and decimals] before that [the skills course]; it was just numbers to me. [Now] it was so nice that I could visualize that fraction and be able to do that and use different strategies to work it out. (pd408)

Percent operators. On the percent decrease (#55) problem three of the participants in the top cluster used a form of the partitioning and fair shares model to work with percents as operators. Because they all worked with the familiar percentage of 25% they simply used their knowledge of quarters to find one fourth of the total. Nevertheless they all articulated a solid understanding of the partitioning and portions in their explanations. Diane divided by four to calculate the portion. Mark used a visual area model to represent his thinking and record his mental calculation. Megan used a halving strategy to find the portion of the total. They all recognized the importance of the partitioned *groups-of*.

The final participant, Bryann, continued to work with a procedural understanding of percents as operators. While she understood that percentages are special fractions Bryann simply converted the percentage to a decimal and multiplied to find the portion of the whole (Figure 101). Even though she no longer used the buggy algorithm when multiplying by a decimal Bryann did not know why she multiplied or what the procedure represented. She just knew that it produced the correct answer.

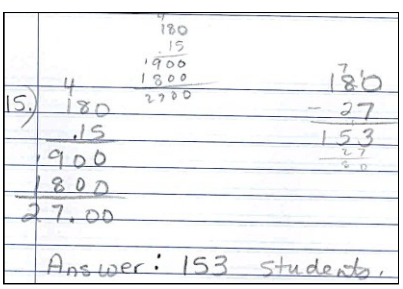


Figure 101. Bryann (pd416)

Concluding Overview

Most participants in the top cluster demonstrated solid growth in their understanding of the operator function over the course of the study. This growth in part, reflected the participant's willingness to engage with operators as meaningful operations rather than rote multiplication. While the majority of participants in this cluster demonstrated a fairly strong understanding of the operator function, one of the participants displayed only a moderate understanding of the concept. Mark and Diane moved from a procedural approach to the use of visual models to represent the distribution of the whole over the partitions. Furthermore, they internalized their understanding of the operator function to such a degree that they were no longer dependent on the visual model and their calculations meaningfully reflected the partitioning process. Both of them also developed their ability to think about fractions, decimals, and percents interchangeably. Megan developed her understanding of operators as *groups-of* and extended it to include decimal tenths. In order to make sense of percent operators that did not readily convert to fractions Megan used her reasoning skills to decompose the numbers to ones she could work with, intuitively using the distributive property. Bryann, however, retained her procedural understanding of percent operators and preferred the security of the multiplication procedure for most decimal operators. Nevertheless, she came to appreciate the added insight the visual representation of the fair shares model gave and used it to give meaning to fraction operators.

Pre-service teachers in all clusters demonstrated solid growth in their understanding of fractions as operators, but participants in the middle and low clusters maintained a procedural understanding of decimal and percent operators. The previous chapters have analyzed pre-service teachers' changing understanding of fractions in the context of the big ideas that, *fractions are relations*, *fractions can be thought of as operators*, and *fractions can be thought of as ratios*. From this perspective it was possible to see how the strategies the pre-service teachers used reflected their understanding of fractions as meaningful objects. Did an examination of the visual models pre-service teachers used provide similar information about their mathematical growth?

Chapter Eight: Results and Analysis – Models and Roadblocks

Use of Models

The previous chapters have shown how the theme of understanding fractions as meaningful objects emerged in the development of the big ideas that, fractions are relations, fractions can be thought of as operators, and fractions can be thought of as ratios. The second theme that emerged from the data was the pre-service teacher's use of models in their (re)learning trajectory. Within the landscape of learning models are used to see, organize, and interpret mathematical relationships (Fosnot 2002). While models can be concrete materials, visual sketches, diagrams, schemes, or even symbols, they are not restricted to these manifestations (van den Heuvel-Panhuizen, 2003). Models reflect the mathematical awareness of the user, providing a mental map to think about the specific concepts. They can also provide a way to support the learning process in a shift from informal to formal understanding. Learners generally begin by using models to act out or represent a particular situation. As they begin to think about the mathematical relations the model shifts to become a tool for thinking about more formal and generalized solutions. The model thus bridges the developing levels of understanding of the learner as it shifts from a *model of* to a *model for* (Gravemeijer, 1999). An examination of this development in modeling, from a) actions in the situation, to b) representation of strategies, and finally to c) representation of number relations, can provide insight into the mathematical growth of the pre-service teachers that goes beyond their use of strategies. The modelling may reflect the degree to which the pre-service teachers generalized their understanding of the big ideas in fractions through the process of reflective abstraction (Piaget, 1977).

Most of the pre-service teachers began the study with a reliance on equations, symbols, and memorized procedures to solve fraction problems. Because many of them simply followed rules, these models did not help the pre-service teachers make sense of the context or think about the mathematical relations. Instead their initial models limited the development of understanding. As participants progressed through the study many of them, to varying degrees, let go of their procedural algorithms and developed visual models that enabled them to think about the relations embedded in fractions in a concrete and accessible manner. Once they felt confident in their deeper understanding of the fraction concepts, several of the pre-service teachers lessoned their dependence on the visual models as a support for their mathematical reasoning. On the other hand, a number of the pre-service teachers used their new models in a procedural fashion that limited the deepening of their understanding. They used models for notating rather than as a mental reflection on the relations (Gravemeijer, 1999).

Types of Models Used

While the use of a visual model potentially helped pre-service teachers to think about the relationships, the type of models used depended on the nature of fraction problem they encountered. To facilitate an examination of the different models, the fraction problems from the test instruments can be grouped into three broad categories: a) comparing and ordering fractions; b) finding an unknown portion or whole; and, c) ratios. Within each grouping, examples of the different types of models used by the pre-service teachers will be described.

In the first group of problems, when comparing or ordering fractions, many pre-service teachers began by using models to show the size of the individual fractions. Because they did not have a relational understanding they focused on counting the parts of the whole, using circles and rectangles for either an area (Figure 102) or set (Figure 103) fraction model. As can be seen in these examples, some pre-service teachers encountered problems with modeling improper fractions or comparing equal sized wholes rather than parts. Only one participant used a fraction model that grew out of personal prior experiences; i.e. baking. Using a margarine container as a measuring cup (Figure 104) this pre-service teacher understood the importance of the whole and was beginning to think more relationally about fractions. In the first problem-solving session participants also began to use the premade fraction kit pieces they encountered in their skills course to directly represent the fraction. Because all the pieces of this model related to the same whole, participants could simply compare the size of the parts without necessarily thinking about the relationships. As the pre-service teachers progressed through the study they also began to use mental imagery when they compared fractions. Some used the image of pie pieces while others used the benchmark fraction of 1/2. Finally, most pre-service teachers did not readily use number lines because they did not fully understand the relationships, as can be seen in Figure 105. Number lines were only used when participants had to find a fraction between two given fractions. In summary, when comparing fractions, many pre-service teachers initially used models to focus on the magnitude of the parts because they did not understand the relational nature of fractions. Some used their models in a rote fashion for notation, while others began to reflect on the relationships embedded in their models.

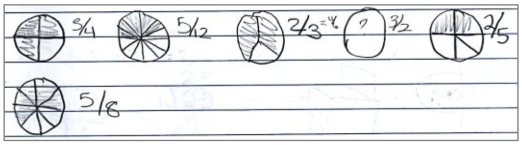


Figure 102. Area model (pd123)

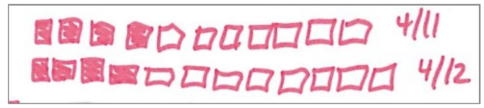


Figure 103. Set model (pd250)

Figure 104. Measuring cups model (pd156)

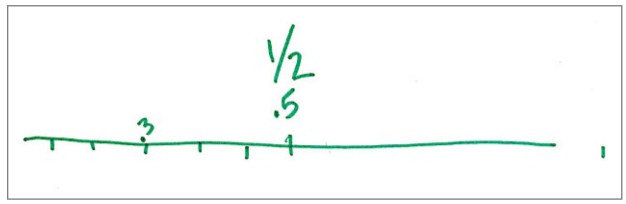


Figure 105. Number line model (pd385)

In the second group of problems, when solving for an unknown portion or whole, many participants used a partitive fair shares model, especially when the fraction acted as an operator. In these instances the pre-service teachers drew a picture, using circles, rectangles or sets, to represent the whole and used division, or some equivalent operation, to find the value of each part (Figure 106). One participant used number chunking as a precursor to the more efficient fair shares model (Figure 107). In these models pre-service teachers used a part-whole interpretation of fractions that once again focused on both the size and number of parts. Few of the participants extended their thinking to see the fraction as a meaningful object rather than the procedure.

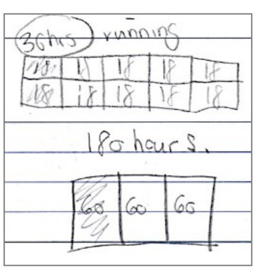


Figure 106. Partitive fair shares model (pd449)

Figure 107. Number chunking (pd306)

In instances where the whole was unknown or two different wholes were given some of the pre-service teachers attempted to directly model the context, either using fraction kit pieces or with a drawing (Figure 108). Some participants used a numerical diagram to act out the problem, measuring out the portions (Figure 109). Others used an area model to illustrate multiplication (Figure 110). In all three of these instances, many pre-service teachers struggled to make sense of the context. Simply using a model did not necessarily help because they did not know how to interpret the numbers. In both the direct modeling and area multiplication examples the participants did not know how to interpret the model in order to make sense of the context because they did not fully understand the relationship between the whole and the parts. In summary, when solving for an unknown portion or whole, pre-service teachers used models that continued to emphasize the part-whole relationships. Because many of them continued to focus on the absolute magnitude they encountered difficulty interpreting their models in contexts with either an unknown whole or two different wholes.

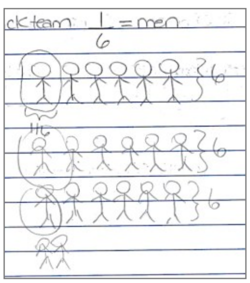


Figure 14. Direct modelling (pd84)

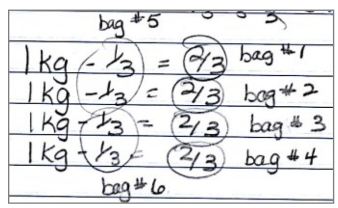


Figure 109. Quotative measuring model (pd170)

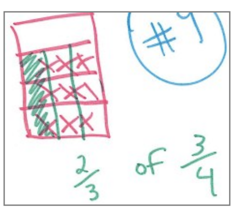


Figure 110. Area multiplication (pd307)

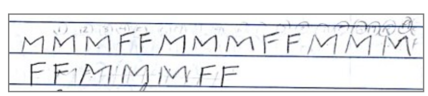


Figure 111. Direct modeling of ratio (pd191)

In the final group of problems, when solving ratios, many pre-service teachers used direct modeling of the composite unit, in a diagram or drawing, as their initial solution method (Figure 111). Participants worked with the ratios as separate quantities, focusing on the absolute amounts. As the size of ratios increased many participants shifted to a ratio table to record values or multiples of the ratio (Figure 112). Using this type of model, pre-service teachers needed to change their thinking from additive to multiplicative relationships. In contexts with a fractional ratio, some participants partitioned the composite unit, using a fair shares approach, to determine the size of each part, chunking the ratio into a new unit (Figure 113). This ability to reconceptualize the unit through partitioning proved a key element for helping pre-service teachers to recognize that fractions are meaningful relations. In summary, when working with fractions as ratios, many pre-service teachers continued to think of the absolute amounts of each of the parts rather than the constant relationship embedded in the ratio. Using the ratio table helped to develop their multiplicative thinking and partitioning the composite unit furthered their proportional reasoning skills.

Figure 112. Ratio Table (pd341)

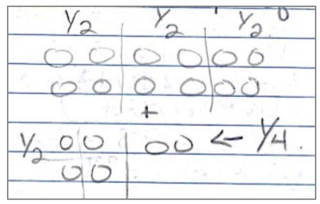


Figure 113. Chunking ratio to form new unit (pd467)

Changes in Use of Models

As the pre-service teachers progressed through the study their use of models shifted. Not only did they begin to use models more frequently and effectively, the gap in usage between the different clusters also diminished. To see this shift, descriptive statistics on the use of models by cluster will be presented for each of the four test instruments. These statistics will show: a) how many participants used a visual model for each of the instrument items; b) on what percentage of the instrument items participants in each cluster used visual models; and, c) what percentage of these visual models were used correctly by the participants in each cluster.

Baseline content exam. While participants used a visual model on 41% of the test items on the baseline content exam, this usage varied dramatically between the different clusters (Table 40). Pre-service teachers in the top cluster used visual models more frequently and more effectively than did participants in either of the other clusters. These pre-service teachers used models on just over half of the items and used them effectively in 87% of the instances they used the models. On the other hand, pre-service teachers in the low cluster used models for less than a quarter of the items and did not use them effectively on any of the items. Pre-service teachers in the middle cluster used visual models on 41% of the items and used them effectively in 75% of the instances they used the models. This extreme discrepancy in ability to use visual models correctly, from 87% to 0%, points to the importance of analyzing models as well as strategies. In this context, the participants in the low cluster revealed the limitations of their mental maps for thinking about fraction concepts.

Two of the individual items on the baseline content exam stand out in terms of the use of visual models. First, the greatest number of participants used a visual model on the track team question (#8), a part-part-whole ratio problem that required participants to relate the fraction part to an unknown whole. This challenging problem was novel for most participants. It was the only item for which all participants in both the top cluster and low cluster used a visual model. The second item, the class increase problem (#10), stands out because none of the participants in any of the clusters used a visual model to solve it. Since the problem involved finding a percentage of the total, participants relied on their known procedures.

Table 40

Baseline Content Exam – Use of Visual Models by Cluster

#	Item (Theme)	Top (n=4)	Middle (n=7)	Low (n=3)	Total (n=14)
5	fraction picture (relations)	4	5	0	9
6	sum of fractions (relations)	1	2	1	4
4	bag of peanuts (relations)	3	2	1	6
8	track team (relations)	4	3	3	10
7	tribes (operator)	1	2	0	3
10	class increase (operator)	0	0	0	0
9	ratio professors (ratio)	2	6	0	8
	% Visual Model	53.6	40.8	23.8	40.8
	% Visual Model Correct	86.6	75.0	0.0	70.1

First problem-solving interview. Use of visual models increased for all clusters on the first problem-solving instrument, with an overall use of models on 58% of the items. Table 41 shows that the top cluster still used visual models on the greatest number of items, 74%, and the low cluster used these models on the least number of items, 40%. Effective use of the models increased for both of these clusters with rates of 93% and 50% respectively. While the participants in the middle cluster also increased their use of visual models to over half of the items, their effective use decreased by 11% to 64%.

Even though most of the participants worked in pairs for the problem solving sessions, the data in the tables reflect the thinking of each individual participant. Given the nature of the paired problem solving sessions, not all of the participants completed each of the items on this instrument. Many of them completed the unfinished items during the second problem solving session. The following two tables reflect these variations with the sample size for each item dependent on the number of participants who completed it. Appendix G provides an overview of the completion of the problem-solving fraction items by participants.

The greatest number of participants used a visual model on item #25. This is not surprising because participants were required to give two different ways of explaining their thinking about the statement 1/2 + 1/3 = 2/5. Participants had to go beyond any initial use of procedural thinking. The least number of participants used visual models for the total food ratios (# 20-21). Many simply used a guess and check method. They understood the multiplicative relationships for each part, but did not know how to model the complete relationship.

Table 41

Problem Solving Interview #1 – Use of Visual Models by Cluster

#	Item (Theme)	Top (n=4)	Middle (n=7)	Low (n=3)	Total
25a	1/2 + 1/3 = 2/5 (relations)	3	2^a	1^c	6
b	1/2 + 1/3 = 2/5 (relations)	4	5^a	2^c	11
28a	order fractions (relations)	3	5^a	0^d	8
b	order fractions (relations)	2	6^a	1^d	9
29	bag of chips (relations)	2^c	2^b	--	4
26a	math course .4 (operator)	3	2	1	6
b	math course 15% (operator)	3	2	1	6
c	math course 3/8 (operator)	4	4	1	9
20	total food 60 (ratio)	1	1^a	0^c	2
21	total food 90 (ratio)	2	1^a	0^c	3
27	stamp collection (ratio)	4	3^a	1^d	8
	% Visual Model	73.8	53.2	40.0	58.1
	% Visual Model Correct	93.4	63.7	50.0	74.8

[a]n=6; [b]n=3; [c]n=2; [d]n=1; [--]n=0

Second problem-solving interview. The use of visual models continued to increase in the second problem solving session, especially in the middle and low clusters, with models being used overall for 75% of the items. However, the effective use of these models decreased by 11% to 64%. Table 42 shows that the use of models by cluster began to equalize. Both the top and low clusters used visual models on about 70% of the items while participants in the middle cluster used models on 79% of the items. The major difference between the clusters occurred in the effective use of the models. Participants in the top cluster continued to be most effective, with 88% of their models being correct, while the middle cluster was only effective in 58% of the items and the low cluster in 43% of the items.

In this session, like the previous, not all of the participants completed each of the instrument items. While some participants completed items from the first problem-solving session, as indicated on the table, the data reflect the use of visual models during the second problem-solving session, indicating the shift in usage over time, rather than strictly by test instrument. The dotted line on the table separates the items from the different instruments.

Table 42

Problem Solving Interview #2 – Use of Visual Model by Cluster

#	**Item** (Theme)	**Top (n=4)**	**Middle (n=7)**	**Low (n=3)**	**Total**
25a	1/2 + 1/3 = 2/5 (relations)	--	1[a]	1[a]	2
b	1/2 + 1/3 = 2/5 (relations)	--	1[a]	0[a]	1
29	bag of chips (relations)	2[b]	2[b]	1[a]	5
20	total food 60 (ratio)	--	0[b]	--	0
21	total food 90 (ratio)	--	2[b]	--	2
27	stamp collection (ratio)	--	0[a]	1[b]	1
33	rows to knit (relations)	3	7	3	13
35	3/7 or 3/8 (relations)	4	6	3	13
36	7/15 or 11/20 (relations)	2	6	2	10
37	14/15 or 17/18 (relations)	3	4	1	8
41	halfway between (relations)	3	1[a]	--	4
42	2nd between (relations)	2[b]	--	--	2
43	ribbon (relations)	2[b]	--	--	2
39	trail mix (operator)	1[b]	1[c]	--	2
38	ratio cookies (ratio)	2	7	2	11
	% Visual Model	70.6	79.2	70.0	74.5
	% Visual Model Correct	87.5	57.8	42.8	64.4

[a]n=1; [b]n=2; [c]n=3; [--]n=0

Post-test retake exam. The use of visual models decreased by 10% (from the last session) on the final retake content exam; nonetheless, each cluster still showed an increase from the initial baseline exam. Overall participants used models on 65% of the items. As seen in Table 43 the usage of models equalized amongst the clusters, with the top cluster using slightly less than the middle cluster and the low cluster using slightly more. In addition, the effective use of models increased dramatically to 93% overall. Not surprisingly, participants in the low cluster were the least effective, with 79% of their models used correctly. Nevertheless, this is a considerable gain from their baseline content exam, where they were unable to use visual models effectively. Two of the items show adjusted sample sizes. Several of the participants in the top cluster, who did not have to take the retake exam, were not able to complete these items on the instrument in the follow-up session. The decrease class question (#55) stands out as the item with the least number of participants using a visual model. As with its counterpart on the initial baseline content exam, this problem required participants to find a percentage of the total. Most participants still preferred to use their known procedures for this type of question.

Table 43

Retake Content Exam – Use of Visual Model by Cluster

#	Item (Theme)	Top (n=4)	Middle (n=7)	Low (n=3)	Total
47a	Olympics training 2/10 (relations)	2	4	2	8
b	Olympics training 1/3 (relations)	3	6	2	11
49	order fractions (relations)	3	4	2	9
50	pages book (operator)	1[a]	7	3	11
55	decrease class (operator)	1	0	1	2
46	Muffins (ratio)	2	7	2	11
53	ratio boxes (ratio)	3[b]	4	2	9
	% Visual Model	62.5	65.3	66.7	64.9
	% Visual Model Correct	100.0	96.9	78.6	93.3

[a]n=1; [b]n=3

Concluding overview. Over the course of the study pre-service teachers began to use visual models more frequently and more effectively. They shifted from a primary reliance on procedures to a general willingness to use visual models to act out or represent a particular situation. The use of visual models, however, did not necessarily ensure that participants understood the fraction concepts. Especially in the problem-solving interviews participants struggled to use the models effectively. Some still used their models in a procedural fashion, for notation rather than reflecting on relations. Others struggled to extend their thinking beyond a part-whole understanding of fractions. Many of the pre-service teachers in the top cluster, however, succeeded in using the models to help them think more deeply about fraction relations. They began to internalize the visual models. While there was an initial disparity in usage between the clusters, with the top cluster using the visual models more frequently and effectively, this evened out on the final retake exam; however, the low cluster remained the least effective in using the models.

Just as the pre-service teachers use of models evolved over time so, too, did their roadblocks to understanding. An examination of these roadblocks provides additional insight in the development of the pre-service teachers' understanding of fractions. Did an analysis of the persistent roadblocks reveal a similar fragility in understanding as did the visual models?

Roadblocks to Understanding

As the pre-service teachers engaged with fraction problems they encountered various challenges that inhibited the development of their deeper conceptual understanding of the concepts. This theme of roadblocks occurred in various forms throughout the study. In some instances the roadblocks prevented the pre-service teachers from making sense of the problem and solving it correctly, but in other instances, even though participants could correctly solve the problem, the roadblocks revealed the limits of their understanding. These different roadblocks can be grouped into the two broad categories – conceptual gaps and procedural thinking. Both types of roadblocks were evident in the responses to each of the test instruments; yet, as the pre-service teachers progressed through the study the profile of these roadblocks shifted. Using different strategies, visual models, and oral reasoning, the pre-service teachers deepened their understanding, filling gaps as they began to focus on making sense of the concepts rather than using memorized procedures.

Baseline Content Exam and Follow-up Interview

The main barriers to an integrated understanding of fraction concepts on the baseline content exam took the form of procedural thinking and conceptual gaps. Table 44 provides an overview of the different types of roadblocks exhibited by each cluster. The number in parentheses indicates the number of participants who encountered this roadblock, if it was more than one; while the bullets identify specific algorithms or procedures. Most of the roadblocks for this instrument consisted of procedural thinking as pre-service teachers attempted to use their knowledge of memorized procedures to solve the test items. In many instances this use of algorithms served to inhibit their further understanding because the participants did not know why the rules or procedures worked, even though they could find the correct answer. In other instances the pre-service teachers used *buggy* or improperly remembered algorithms that indicated a lack of understanding of the foundational concepts. Participants in all clusters, regardless of level of understanding, demonstrated this reliance on procedures rather than understanding.

Table 44

Content Exam – Roadblocks by Cluster

Roadblock	Top Cluster (n=4)	Middle Cluster (n=7)	Low Cluster (n=3)
Conceptual Gaps	-identifying fraction whole -modelling concepts	-identifying fraction whole (7) -multiplying makes bigger -modelling concepts (2)	-identifying fraction whole (3) -modelling concepts
Procedural Thinking	-procedural algorithms (2) • powers of ten • common denominator • percents -limited sense making -buggy procedures (2) • decimal multiplication • fraction to decimal • procedures forgotten	-procedural algorithms (6) • common denominator • percents • fraction to decimal • fraction multiplication • decimal multiplication -procedure vs sense making (4) -buggy procedures (5) • definition of fractions • decimal multiplication • cross multiplication • fraction as operator	-procedural algorithms (3) • common denominator • percents • cross multiplication -procedure vs sensemaking (3) -buggy procedures (3) • definition of fractions • decimal multiplication • fraction ratios • division algorithm • procedures forgotten

Procedural thinking. On the baseline content exam 11 out of the 14 participants used algorithms in a rote or procedural manner. When they were asked to explain their thinking these participants simply stated the rule and described their procedure. They could not link their calculations to a full understanding of the concepts. These participants remembered the rules and knew that the procedures consistently gave the correct answer. For example, Lynsey, a member of the middle cluster, explained her work with percentages and decimals as something she just knew. When pressed to explain her thinking on how she found a percentage increase Lynsey also related her procedures to finding a 15% sales tax.

L: I don't know why I know this, but if I'm looking to find; like to times it by a percentage I just times it by .4, 'cuz that's just like, and that will give me 64 students. So I know that 160, 40% of 160 is 64, I guess. And you just add those two together.

I: How do you know .4 is 40%?

L: Because it's, 40, I don't know how I know that. I just know that if I was to find the, I was thinking back, if I were trying to find the tax of something, I would take, like let's say it's $160 and times it by .15 and that's just how I would find, 'cuz I work in a store and if our system was down that's just how I did it. And so I just knew that that's what I would do here. I don't even know why I would do that, I guess. But I just know that that's how I would do it.

I: How did you determine where to put the decimal place?

L: 'Cuz I know that you just add up how many numbers are after the decimal in your first, in the beginning of your question. So let's say I had two more [decimal places], let's say it was 160.00 and then .40, then I would have, even though those zeroes are useless, I know to cut them out, then I would have them there. I would have made my little lines all the way over and I would have done that four times. But this time I only need two.

I: Do you know why that's the case? (pd127)

L: That's just how I was taught to move my decimals. That's just an easy way to remember. So there's two here [points to factors]; just move it two over [points to product]. So I always make two little pockets to move it over.

Lynsey, like many of the other participants, made use of memorized rules that held little meaning. Her "little pockets" or curved lines under the numbers to indicate movement of the decimal represented a knowledge of the rules rather than an understanding of place value concepts or the meaning of multiplication by a number less than one. Since the rules had little meaning for many of these participants their continued use effectively blocked further development of the concepts. Participants also used this type of procedural thinking when multiplying fractions, combining fractions using common denominators, converting fractions to decimals, and using shortcuts when working with powers of ten. Even though some of the participants did consider whether their final answer made sense within the context of the question, they still focused their attention on the procedure rather than the underlying concepts.

Some participants, especially from the middle or low cluster, focused on procedures to the exclusion of sense making in certain contexts. These eight participants could not see contradictions inherent in their responses to problems nor could they identify or eliminate incorrect options within the problem. For example, when asked to find two fractions whose sum was less than one, Tanya, a member of the middle cluster, could not eliminate fractions greater than one (Figure 114). Instead she, and those like her, used trial and error to combine all possible fractions using common denominators until she found the correct answer. These participants focused on the procedure without thinking about the meaning of the fractions. They did not attempt to make sense of the context but simply followed the rules. Similarly, Irene, also a member of the middle cluster, did not challenge the reasonableness of her answer when she determined that 120 people were on a team when 20 of them were female and 1/6 were men (Figure 115). She simply followed a procedure that she thought would produce the correct answer.

> I wanted to find *x*. So there's my *x*. And I said okay there's 20 of them. And I figured if I multiplied [by 1/6] I'm going to come up with an answer. I'm just going to use the numbers they give me and make up a formula. Surely I can't add, 'cuz it just didn't seem like I should add. And I can't divide 'cuz if I divide I would be multiplying which I'm doing the same thing anyways. So I thought, well I'll just multiply. Cross multiply, mind you. (pd100)

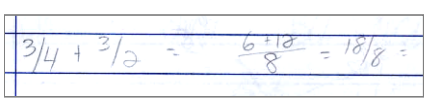


Figure 114. Tanya (pd188)

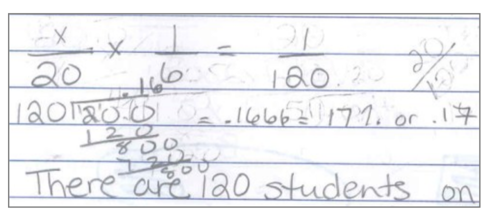


Figure 115. Irene (pd100)

Because the procedures did not have inherent meaning for the some of the participants, ten of them worked with improperly remembered processes. In the decimal multiplication process several of the participants inefficiently lined up the decimals, mimicking the rules for addition and subtraction (Figure 116). Even though they still obtained the correct answer, their process reflected their focus on the rules rather than meaning. Other participants used a partially remembered definition of fractions as the number of parts out of the whole. They simply counted the number of parts without ensuring the parts were equal. These participants focused on the process for counting the two separate numbers rather than the relational meaning of fractions. Still other participants encountered difficulties when multiplying by a fraction operator (Figure 117). These participants only divided by the denominator, neglecting to multiply by the numerator. One participant in the low cluster had difficulty with the traditional division algorithm, losing track of the place value and meaning of the process (Figure 118). Additional errors in procedures also occurred in cross multiplication procedures and when converting fractions to decimals. In all instances participants used the buggy procedure because they did not fully understand the underlying concepts and were simply trying to use a partially memorized procedure.

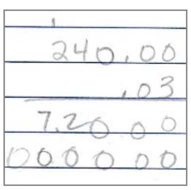


Figure 116. Bryann (pd23)

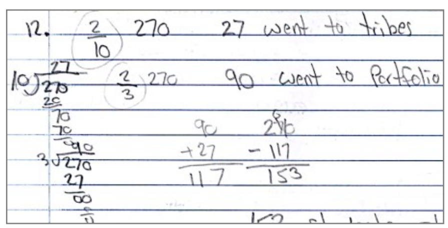


Figure 117. Gabriela (pd69)

Figure 118. Grace (pd557)

Conceptual gaps. All participants in the low and middle clusters as well as one participant in the top cluster encountered difficulty with identifying the fraction whole. Participants often assumed the whole number given in the problem must be the whole for the fraction. They especially had difficulty when the problem contained two different wholes. Gabriela's thinking on such a problem, which required participants to determine how many 2/3 kg bags would be needed for 10 kg, typified that of many participants. She assumed the fraction referred to the 10 kg (Figure 119). Even though her work contains several other errors it serves to illustrate this conceptual gap of identifying the whole.

> See I didn't even think of 2/3 of 1 kg. I just went along with, I don't know what I was thinking, but I forgot that it was 2/3 of 1 kg not 2/3 of 10 kg. I don't know what I did. …

I think I just misread the part that she put 2/3 kg into each bag. I just assumed that was out of the 10. I thought it was 2/3 of 10 kg. (pd66)

This error was more than a simple misreading of the problem. Most participants did not realize that there could be two different wholes in a given problem. Their procedural approach did not facilitate making sense of fractions as relations where the whole is important.

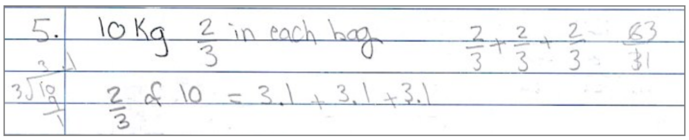


Figure 119. Gabriela (pd66)

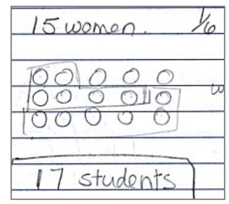


Figure 120. Olivia (pd174)

Participants also encountered difficulties identifying the whole when it was the unknown in the problem. Olivia's response typified that of many of these participants. Olivia made an assumption that the whole must be the number given in the problem, even though the number and the fraction clearly identified the parts, not the whole (Figure 120).

O: I have 15 women and I have 1/6. I think I tried to use the 15 beads and then 1/6 of them. So I circled 6, 'cuz men. And you want to know how many students are on the team. Okay, so I, what I did was, I was 1/6 here, and this is 1/6 here, then how many do I have left over.

I: Can you explain for me again why you circled the 6?

O: Because I was looking at the bottom of 1/6. I'm pretty sure. 'Cuz I grouped them as 1 and then in six, and then I grouped them in one group in 6 [circles each group]. (pd174)

Olivia used the 15 as the whole for the fraction as she circled her groups of six. In this way she could add one male for every group of six. This inability to correctly identify the fraction whole provided a roadblock to a deepening of participants' understanding of fractions as relations. Like Olivia, many of them incorrectly used part-whole thinking.

The other conceptual gaps that surfaced as barriers to a deeper understanding of the nature of fractions were less pervasive. One participant held to the naïve notion that multiplying makes bigger. Because she knew the context of the problem indicated an increase in the quantity Irene chose to multiply by a fraction even though her final answer gave a smaller amount.

> I don't even know why I did that. I think I wanted to find out, um. Again I just used the questions there and I thought okay, I need to multiply. Because I'm trying to find more of something and I think multiply will show me more whereas if I divide it wouldn't. I would still multiply, but it would be flipped over or something. I don't know. I just thought that multiplying there would be a bigger quantity or something. (pd96?)

This belief inhibited Irene from thinking about the meaning of fractions and instead she focused on the procedures.

Two of the participants struggled with finding models to make sense of fraction concepts. They did not know how to reflect the relationships inherent in the problems. Olivia spent a lot of time on the ratio question as she wrestled with representing the concepts.

> I just can't picture it. 'Cuz I don't like ratios. And so then once I stared at it and stared at it and stared at it forever, I was like, uh, oh my god, I don't know how to do this. So I was so upset about it. Like I was hoping I got the right answer. But I was like how come it takes me that long? It should have taken me two minutes to do that, if that. But anyways, I got it right. (pd175)

While Olivia eventually succeeded in visually representing the concepts it was only the high stakes context of the exam that pushed her to continue searching for the model. Valery, on the other hand, did not succeed. She tried to draw a picture to find the unknown whole but did not know how to relate it to her incorrect equation. "I was trying to do it by multiplying, but I did also draw a picture…. When I see this now I'm confused. I would look at it totally differently now" (pd206).

Two additional participants commented on their reluctance to use the manipulatives during the exam. Diane felt too uncomfortable. "I didn't want to use the manipulatives because that guy was sitting there and I felt embarrassed" (pd39). Irene, on the other hand did not know they could be used. "When we were told we had those manipulatives, I thought, why on earth would anyone want those? But you know they're actually very, very helpful, to draw out stuff" (pd101). The use of manipualtives or models was outside the realm of normal or procedural problem solving for these participants.

First Problem-Solving Interview

Participants continued to experience roadblocks due to their procedural thinking during the first problem-solving interview. However, many of the participants began to let go of the procedures and also used concrete ways to model and make sense of the fraction concepts. As a result the specific conceptual roadblocks became more apparent (Table 45).

Table 45

Problem-Solving Interview #1 – Roadblocks by Cluster

Roadblock	Top Cluster (n=4)	Middle Cluster (n=7)	Low Cluster (n=3)
Conceptual Gaps	-identifying fraction whole -large numbers for ratio	-identifying fraction whole -large numbers for ratio (2) -flexibility between fraction, decimal, percent (3)	-identifying fraction whole -flexibility between fraction, decimal, percent (3) -limited facility with number -limited use of friendly numbers (i.e 5, 10)
Procedural Thinking	-procedural algorithms • percents • decimals • powers of ten -model limits thinking -buggy procedure • compares denominator not relation	-procedural algorithms (4) • decimals • fraction multiplication • percents -buggy procedures (5) • fraction to decimal • procedures forgotten • compares denominator not relation	-procedural algorithms • decimals -model limits thinking -buggy procedures • cross multiplication • compares denominator not relation

Procedural thinking. Six of the 14 participants encountered roadblocks to deepening their understanding of the fraction concepts during the first problem-solving interview due to their reliance on procedural algorithms. In certain contexts these participants continued to focus on the procedure rather than making sense of the concepts. They used procedural thinking when working with decimals or percents, when multiplying fractions, and when using shortcuts in calculations with powers of 10. These participants could describe their procedures and the rules, but they did not know why they used the particular procedure or why it worked. For example, Irene could not explain why she multiplied with the decimal or percent. "I have no idea, other than knowing that's just what I'm supposed to do. Because I'm trying to find a *product of*. And in doing so you know you need to multiply" (pd271). Similarly Erica and Lynsey could not explain their work with percents. "I just know to find the percent, to find .4 of something is like finding the tax. I just know to multiply. I don't know why I do it, I just do" (pd248). Nor could

Bryann explain to her partner why 15 % was out of 100 but .4 was not. "I can't explain why but it wouldn't be over 10. Because 15 over 100 is 15%" (pd294). All six of these participants relied on their known procedures and struggled with understanding the underlying concepts.

Two of the participants used their model as a procedure rather than as a way to deepen their understanding of the concepts. Isabelle used the fraction kit pieces to show that 1/2 + 1/3 + 1/6 = 2/5. Because the lengths of the pieces were equivalent she assumed the equation was true. Isabelle focused on the procedure of lining up the pieces and did not think about the area of the pieces. Diane, on the other hand, used an area model to correctly multiply two fractions. She also focused on the procedure and could not make sense of her partner's conceptual model of the context. "But I couldn't see it unless I did that. I have to do that. I can't just [understand my partner's model]. I think it's going to be something trickier" (pd238). In each instance the way the participants procedurally used the model became a roadblock to a deeper understanding of the fraction concepts.

The types of buggy procedures used by the participants began to shift during the first problem-solving interview. As participants began to use new ways of thinking about fractions, some of them used their reasoning in a procedural way that led to errors. Over half of the eight participants who used buggy procedures only did so with the new concepts they were learning. As they compared fractions these participants generalized the importance of the denominator to the exclusion of the relationship between parts and whole. Lynsey's reasoning typifies that of these participants.

> If you think of quarters, like if you have a whole pie and there are four of us we each get a quarter. And then somebody comes in and now we have to bring that down to fifths, and that's smaller. So even if they have 6/7 it's still not as big as 5/6. (pd250)

The remainder of the buggy procedures used by the participants related to their procedural use of traditional algorithms. Neither Tanya nor Grace remembered how to convert a fraction to a decimal. Both began by incorrectly dividing the denominator by the numerator. When working with a cross multiplication equation Grace assumed the x would automatically give her the answer, even though she needed the complement part. Finally Brenda could not remember whether she needed to add or multiply when calculating common denominators for 1/3 and asked for her partner's help. "Three goes into six twice. Do you do two plus one or two times one? What do you do, on top?" (pd223). In each instance the participants struggled with the procedures because they could not fully make sense of the concepts.

Conceptual gaps. With the slight shift from procedures to making sense of fraction concepts during the first problem-solving session more conceptual gaps became apparent. Since several of the problems from this test instrument were completed during the second problem solving session by a number of the participants, the conceptual gaps for those participants are found in the next section. Four different types of conceptual gaps became apparent in this session. Three participants continued to struggle with identifying the fraction whole. When working with a fraction multiplication problem that contained two different wholes, both Erica and Diane experienced difficulty identifying which whole each fraction referred to. Diane could not understand her partner's work with the two different wholes because her model did not highlight the differences. Erica, on the other hand, simply assumed there was only one whole and thus she used common denominators for the calculations. Valery also failed to recognize the importance of the whole when combining fractions. She focused on the number of parts rather than common wholes. Using a set model to represent 1/2 + 1/3, Valery shifted the whole when she combined two parts and three parts, giving a new whole of five parts.

Three participants struggled with the shift to larger numbers. While Megan, Erica, and Irene all eventually solved the ratio problem with large numbers correctly, they lacked confidence working with the larger numbers because they could not draw it out as they did for simpler numbers. "But then when we had such a big number it bogged me down" (pd295). They became reliant on division procedures and struggled to give meaning to the calculations.

Six participants experienced challenges working in a context that combined fractions, decimals and percents as operators. "They're all different. This is point, this is decimal, this is fractions. So we'd all have to get them on the same thing first" (pd260). Like Gabriela and Grace, all these participants incorrectly believed each of the numbers needed to be converted to the same format, since they occurred in the same question. As a result the participants tended to work procedurally with the numbers. They did not necessarily realize that using 3/8 as a fraction operator could be more efficient or meaningful than converting it to a decimal or percent. These participants lacked the flexibility to shift meaningfully and appropriately between fractions, decimals and percents.

Finally, one participant struggled with a limited sense of number. Valery lacked a strong knowledge of basic number facts. She often resorted to using her fingers to count up and used doubling strategies instead of multiplication. "I have to count, because if I do the multiplication I end up doubting myself and I end up using my fingers anyway, 'cuz I always count up" (pd287). Thus working on problems with large numbers often took a lot of time and contained calculation errors. As she stated, "It's a process and I'm sure it's painful for some people to watch" (pd387). Valery also lacked consistency in using friendly numbers, such as five or 10, to simplify her calculations. For example, when adding up a ratio of 1 to 7 that totaled 720 Valery added chunks of 4, not recognizing that groups of 5 would have simplified her calculations. While other participants such as Bryann, Megan, and Brenda, expressed frustration with their poor knowledge of multiplication and division facts, they had sense of number relations that still eluded Valery.

Second Problem-Solving Interview

As participants continued to focus on deepening their understanding of fractions the types of roadblocks they encountered during the second problem-solving session also shifted. Participants used fewer procedural algorithms as they grappled with the underlying concepts. The struggles participants faced reflected the degree to which they engaged with the big ideas about fractions. Table 46 summarizes the roadblocks encountered by participants during the second problem-solving interview.

Table 46

Problem-Solving Interview #2 – Roadblocks by Cluster

Roadblock	Top Cluster (n=4)	Middle Cluster (n=7)	Low Cluster (n=3)
Conceptual Gaps	-identifying fraction whole (3) -modelling between fractions	-identifying fraction whole (5) -fractional ratios (3)	-identifying fraction whole (3) -fractional ratios -limited facility with number
Procedural Thinking	-buggy procedures • compares denominator not relation • division of fractions	-buggy procedures (5) • compares denominator not relation -model limits thinking -procedure vs sense making	-buggy procedures (3) • compares denominator not relation -model limits thinking (2)

Procedural thinking. Nine of the 14 participants struggled with a buggy procedure during the second problem-solving session. Many began to use partially understood fractions concepts in a procedural way. These participants focused on the size of the denominator when comparing fractions rather than examining the entire fraction relation, especially when comparing fractions such as 14/15 and 17/18. Some, like Diane and Brenda, recognized the limitations of their initial reasoning and continued to develop reasoning about the entire fraction. Others, like Tanya and Isabelle, found it difficult to move beyond this procedure without external prompting.

Only one other buggy procedure emerged during this session. As Diane struggled to understand and write an equation for dividing fractions, she wrote a multiplication statement and used cross multiplication to find the answer (Figure 121). Even though her buggy procedure gave the correct answer Diane did not understand the difference between multiplication and division as it was enacted in the context of the problem.

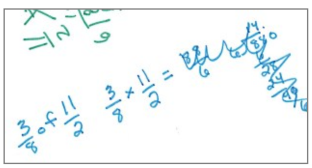


Figure 121. Diane (pd329)

Three participants used their model as a procedure and thus limited the development of their understanding of the concepts. Irene compared fractions using a double linear or bar model on which she drew or shaded the fractions. Even though she carefully maintained the common whole, Irene focused on the procedure of partitioning rather than thinking about fraction relations. The helpfulness of her model depended on the accuracy of her drawing. Both Valery and Isabelle used a set model to compare fractions. They did not maintain a common whole, but focused on the number of parts. Using her model Isabelle thought 7/15 was greater than 11/20. "Well 20, if you divide everything in 20 that's a lot of pieces. They'd be smaller pieces than 15. And this [7/15] is almost half" (pd371).

Finally, one participant found it difficult to reason and make sense of fractions. Even though her partner successfully compared two fractions using the benchmark of 1/2, Erica needed to convert the fractions to percentages, like she would for a test score, to confirm the answer. She needed the security of the familiar context to legitimize the reasoning.

Conceptual gaps. Eleven of the 14 participants continued to struggle with identifying the fraction whole during the second problem-solving session. Participants experienced different degrees of challenges that reflected their developing understanding of fractions as relations. When working with multiplication or division problems that contained two different wholes some of the participants recognized only one whole. Bryann assumed both fractions in a multiplication problem referred to the same whole so she incorrectly modelled the problem using fraction kit pieces (pd311). She did not realize that even though the whole of 3/4 was the unit one bag, the whole of 1/3 actually referred to 3/4 of a bag. Other participants recognized the two different wholes, but encountered difficulty interpreting their model or identifying the fraction remainder. For example, Brenda's visual model correctly shows 2/3 of 3/4 but she did not know how to interpret what she saw (Figure 122). She could only work with one whole at a time; she could not relate her results back to the original whole by extending the division of thirds to the final quarter. Similarly, Diane and Mark could not explain why their visual model gave a remainder of 1/4 and their calculation gave a remainder of 2/3 (pd329). They did not realize that the interpretation of the remainder depended on which whole was used.

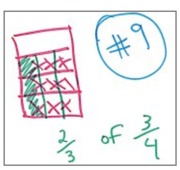


Figure 122. Brenda (pd307)

Problems with an unknown whole provided a second context for the challenge of identifying the whole. Here participants had to determine if the given quantity referred to the part or the whole. They also had to decide if the numerator or denominator reflected the correct number of parts of the stated quantity. Many participants simply assumed the given quantity referred to the whole so they consistently divided by the denominator rather than the numerator regardless of context. For example, Erica and Lynsey incorrectly assumed the 260 remaining referred to the whole even though 1/5 had already been completed (pd336). Thus they divided by 5 instead of 4.

Modelling fraction addition provided a third context for the challenge of identifying the whole. When participants focused on the number of fraction parts rather than the relationship they incorrectly changed the whole. Gabriela and Grace initially combined the total number of parts to say 1/2 + 1/3 = 2/5 (Figure 123). They did not maintain the common whole when combining parts. However, they eventually rejected this reasoning after experiencing dissonance when seeing the statement correctly modelled using the fraction kit pieces.

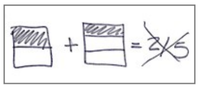


Figure 123. Gabriela/Grace (pd345)

Three additional roadblocks surfaced during the second problem-solving session; however, they were less pervasive. Four participants struggled with fractional ratios. These participants could only iterate ratios by whole number. They could not partition the complex unit. Isabelle and Tanya spent over five minutes trying to find the missing component of ratio with a scalar constant of 21/2 and during this time they could only double the ratio (Figure 124). They needed prompting to consider working with a portion of the ratio. When Lynsey and Erica become stuck on the same problem they considered additive rather than multiplicative thinking. "There's a 22 difference. Maybe it's just 75 minus 22" (pd341). They also needed prompting to consider partitioning the unit in half.

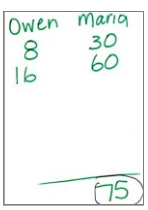


Figure 124. Isabelle/Tanya (pd373)

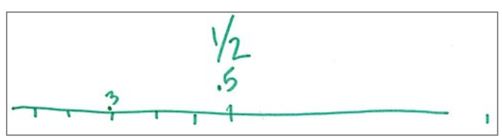


Figure 125. Bryann (pd341)

One participant struggled to use a model to find a fraction exactly halfway between two given fractions. Bryann struggled to place 1/3 and 1/2 on the same number line (Figure 125). She could not represent fractions as measures in relationship. Bryann also attempted to use fraction pieces to check her partner's response, but only became frustrated.

I don't even know what I'm trying to accomplish now. I don't even know how to see what's halfway between that though. Like it's just sitting in front of me. There's no way for me to use these [fraction pieces] to see what's halfway in between that one [1/2] and that one [1/3]. Yeah, can't see it. (pd385)

Finally, Valery continued to encounter roadblocks due to her limited facility with numbers. For example, when using fractions as an operator Valery needed to divide 260 by 4. Even though she used an alternative algorithm to divide, it still took her 3 minutes and two attempts to determine the correct answer (Figure 126). As a result of her focus on number calculations, at times Valery lost sight of the bigger fraction concepts.

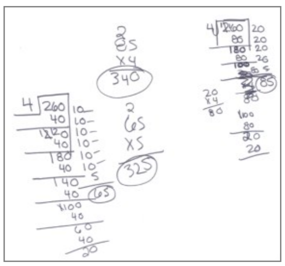


Figure 126. Valery (pd391)

Final Retake Exam and Follow-up Interview

On the final retake exam some of the participants returned to their procedural thinking with traditional algorithms and procedures. Under the stress of the high stakes exam these participants reverted to what was known and familiar. New conceptual gaps revealed fragility in participants' understanding of ratios with fractional parts. Table 47 provides an overview of the roadblocks that emerged during the final test instrument.

Procedural thinking. Five participants continued to use procedural thinking and algorithms with decimals. Some simply preferred the familiar and known rules while others looked for ways to avoid working with fractions. Bryann's reasoning typified that of these participants. She did not understand the decimals but she knew how to work with them.

I converted [fractions] into decimals. It's just easier for me to multiply that way. So I divided the 10 into the 2, 'cuz I just knew that that was how I could get a decimal for that. The only thing I know about that is that the bottom number has to go into the top number when I do a decimal. I don't know why. (pd408)

When multiplying by decimals many of these participants simply followed the rules for moving the decimal place over. They, like Gabriela, did not know why. "I just move it over as many times for the answer. I still don't know why. But I do know that I move it" (pd446).

Table 47

Final Retake Exam – Roadblocks by Cluster

Roadblock	**Top Cluster (n=4)**	**Middle Cluster (n=7)**	**Low Cluster (n=3)**
Conceptual Gaps	-multiplicative thinking	-ratio fractions (2) -improper fractions	-ratio fractions (2) -limited facility with number
Procedural Thinking	-procedural algorithm • decimals	-procedural algorithm (3) • decimals -buggy procedures • modelling improper fractions • percents -procedure vs sense making	-procedural algorithm • decimals -buggy procedure (2) • decimal multiplication • comparing fractions -procedure vs sense making

Several buggy procedures emerged on the final retake exam that reflected both traditional algorithms and new reasoning methods. This flawed procedural thinking reflected limitations in the understanding of the underlying concepts. Grace continued to use a buggy procedure when multiplying decimals (Figure 127). She lined up the decimal places following the rules for adding and subtracting decimals. Olivia used a common size shape to represent and compare fractions, even if they were greater than one (Figure 128). Even though Olivia found this format easier to understand than pie pieces, her focus was on the number of pieces rather than the relationships. Both Grace and Olivia used their procedures to correctly solve the questions, yet their process reveals fragile aspects of their understanding.

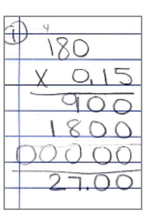


Figure 127. Grace (465)

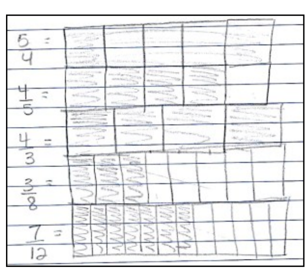


Figure 128. Olivia (pd527)

Two other participants used buggy procedures that reflected incorrectly memorized procedures and a weak understanding of the concepts. When finding a percentage of a quantity Olivia divided by the percent instead of multiplying (Figure 129). Even when she attempted to explain how to correctly solve the problem in the follow-up interview she still showed her fragility in understanding the meaning of 15%. "I should have multiplied by 1.5 times 180…. Because that's how you figure out taxes" (pd533). Isabelle incorrectly used a visual model to compare sizes of fractions, focusing on the number of pieces rather than the relationship (Figure 130). She counted the unshaded pieces and determined the larger fractions had fewer unshaded pieces. Both Isabelle and Olivia used incorrect procedures because they did not fully understand the concepts.

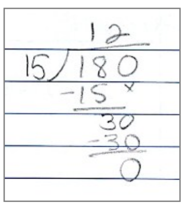


Figure 129. Olivia (pd533)

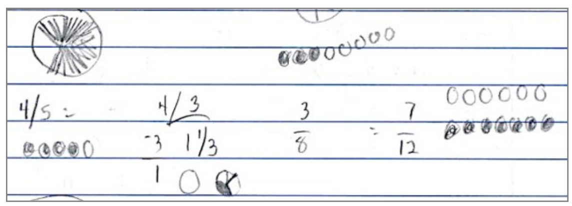


Figure 130. Isabelle (pd573)

Finally, two participants used procedural thinking as they chose to use procedures rather than making sense of fractions. Both Irene and Valery struggled to find a common denominator to compare five different fractions. They could only find a common denominator for some of the fractions. Ultimately neither of them could correctly order the fractions using the known procedure. Irene's final ordering shows that she did not make sense of the fraction relationships, since she stated 4/5 was smaller than 3/8 (Figure 131).

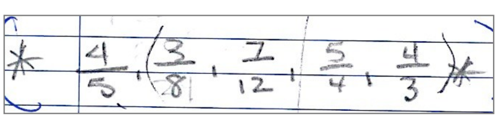


Figure 131. Irene (pd470)

Conceptual gaps. On the final test instrument and follow-up interview four different conceptual gaps came to light. Four participants revealed limitations in their understanding of fractions in the context of ratios. The shift from ratio parts as whole number to ratios challenged the participants understanding of both fractions and ratios. As Brenda stated, the fraction caused difficulty. “Because it wasn’t a whole number, there was a half that made it difficult. I wasn’t practicing questions like that…. Having to divide the 1/2 cup; that was hard” (pd 396). Valery echoed these sentiments. “What threw me off was the fact that it was 1 and 1/2 cups” (pd546). The other participants also struggled with partitioning the fraction. Tanya, for example, lost sight the ratio relationships as she partitioned the fraction into unequal parts (Figure 132).

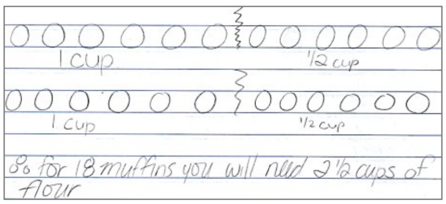


Figure 132. Tanya (pd535)

Another participant experienced difficulty with improper fractions. Gabriela knew they represented more than one but she did not know how much more than one. When asked to use a model to represent 5/4 she had “no idea” (pd448). Even though she had five quarter sized fraction pieces in front of her, Gabriela thought the pieces must be fifths. She lost sight of the relationship between the part and the whole when the fraction was greater than one.

One participant struggled with using the multiplicative thinking necessary to maintain ratio relationships. Bryann began her work on a ratio problem by using additive thinking and maintaining the difference between the ratio parts. Her reflection reveals the tentative nature of her understanding of the nature of ratios.

> The first thing here, I started going up by one, and it wasn’t making sense to me. I was like this is silly because, the ratio, going up by one it’s obvious. By the time I got to this point, I can’t get 49 in total. So then I said, I’m going to double. I’m going to go up by the 3s and the 4s. I’m going to increase it on each side by 3 and 4. Which I did, and got a total of 49. And it made sense. (pd414)

Finally, Valery continued to struggle with a limited facility with numbers. Working with small numbers she could concretely model and understand most fraction concepts. However, working with larger numbers inhibited the deepening of these concepts. During the follow-up interview Valery struggled to divide 180 by 20 while working with percentages (Figure 133). She found it difficult to keep track of her process and had to redo the calculation. Her lack of number sense prevents Valery from seeing certain patterns and ways to develop efficiency and understanding.

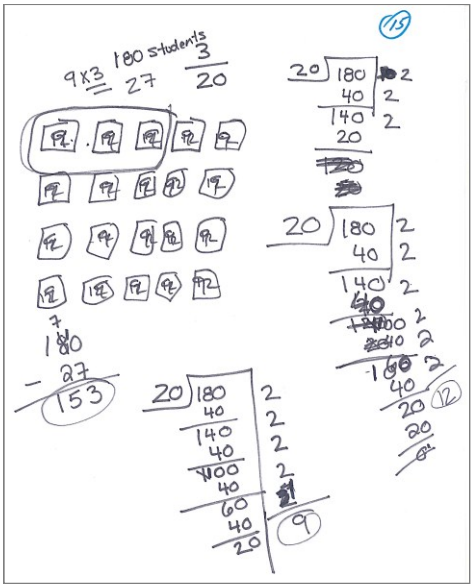


Figure 133. Valery (pd555)

Concluding overview. Participants in all clusters encountered roadblocks as they worked with fraction concepts. While these roadblocks have been grouped into two broad categories, conceptual gaps and procedural thinking, for many of the participants, the procedural thinking masked an underlying lack of conceptual understanding. As the pre-service teachers proceeded through the study many let go of their rote procedures; but, when they tried to find concrete ways to model and make sense of the fraction concepts their conceptual gaps became apparent. The most persistent conceptual gap was that of identifying the fraction whole. This roadblock appeared across all clusters. Similarly the most persistent form of procedural thinking related to the use of decimals and percents. Once again this roadblock appeared across all clusters. While all pre-service teachers grew in their understanding of fraction concepts over the course of the study, some continued to struggle and encounter roadblocks. The fragility of their understanding was revealed as the complexity of the fraction problems increased.

Chapter Nine: Discussion and Conclusions

This study was undertaken to identify the development of pre-service elementary teachers' understanding of fraction content knowledge over the course of their final academic year, while they were engaged in a remedial reform-based mathematics course. In this final chapter the results will be discussed by answering the opening research questions. This will be followed with concluding remarks and the implications for curriculum, professional development, and future research.

The Questions

One overarching question guided this dissertation and three additional sub questions contributed to the direction of the study. In the following sections each of these questions will be answered and discussed:

> How does conceptual understanding of mathematics content knowledge, specifically in the strand of number sense in the area of fractions, develop in pre-service elementary teachers enrolled in a reform-based mathematics skills development course?
>
> a) What changes occur in students' understanding of fractions?
> b) What specific concepts cause students great difficulty as they (re)learn?
> c) What are the typical stages of development of understanding and misunderstanding?

Changes in Understanding of Fractions

As anticipated, pre-service teacher understanding of fraction concepts increased over the course of the academic year. These students encountered new ways of engaging with fraction concepts in both the remedial mathematics skills course and their mathematics methods course. Initial results on the Grade 6/7 baseline content exam were low, which is not surprising, given that 11 of the 14 students scored less than the requisite 75% and were thus taking the mathematics skills development course. The overall average score for the fraction portion of the exam was 61%, with average scores for the different clusters ranging from 21% for those in the low cluster, 62% for those in the middle cluster, to 90% for those in the top cluster. Each of the clusters showed significant gains on the retake content exam post-test as the overall average rose to 93%. The participants in the low cluster demonstrated the greatest improvement, which is not surprising since they had the greatest room for growth. The average scores for the clusters ranged from 82% for the low, 92% for the middle, to 98% for the top. These findings of progress fit with the growing body of literature documenting that reform-oriented interventions do improve pre-service teachers' fraction knowledge (Green et al., 2008; Newton, 2008; Osana & Royea, 2011; Toluk-Ucar, 2009). As pre-service teachers are challenged to think about and engage with fractions as meaningful objects rather than just memorized procedures, their understanding of the big ideas deepens and they become more flexible in their use of strategies. While most participants made great gains in their capacity to correctly answer the fraction items on the post-test retake exam this nonetheless masked a continued lack of deep fractional understanding.

Roadblocks to Understanding

Two types of roadblocks, procedural thinking and conceptual gaps, prevented many pre-service teachers from developing a deeper understanding of fractions as meaningful objects. Pre-service teachers used elements of superficial procedural thinking in which they focused on algorithms and "doing next" (Empson et al., 2011) rather than anticipating how they might make sense of fractions as quantities. They used rules but did not know why or how the rules worked. This type of procedural thinking reinforced a fragmented and partial understanding of fractions and in some cases led to various errors in their use of the algorithms. Pre-service teachers also demonstrated specific gaps in their understanding of the foundational concepts. These conceptual gaps limited the degree to which they could develop a flexible understanding of fractions. The concepts that caused the pre-service teachers difficulty in both groups of roadblocks decreased in intensity but were not completely diminished over time.

Reliance on procedures. At the beginning of the study students in all three clusters relied heavily on procedural algorithms. For example, they found common denominators, converted fractions to decimals, or used the standard multiplication and division algorithms without thinking about fractions as meaningful objects. While most participants willingly let go of their rote procedures in the subsequent interviews, several resisted. Thus, on the post-test the procedural conversion of fractions to decimals still persisted. Results for two other elements of the procedural thinking roadblock followed a similar pattern. Participants, especially in the middle and low clusters, began by demonstrating limited sense making when working with fractions. They simply followed the steps of known procedures. For example, when asked to find two fractions from a list that combined to give a sum less than one, they did not consider that they could immediately exclude 3/2 because it was larger than one. Over time most participants willingly tried to make sense of fractions and reasoned about their meaning. Yet several participants struggled and chose to continue their reliance on procedures rather than to trust their ability to reason about fractions on the post-test. Similarly, participants in all clusters incorrectly made use of procedures on the pre-test; however, as students progressed through the year these errors transferred to the new concepts and procedures they were learning. For example, rather than incorrectly using the cross multiplication algorithm students now simply examined the denominators when comparing fractions, overgeneralizing the importance of the denominator rather than the relationship between the numerator and denominator. By the end of the study only three participants made errors in using procedures compared to the ten who had initially made these types of errors.

Each of the aforementioned roadblocks, the reliance on rote procedures, the lack of sense making, and the incorrect use of procedures, reflected the fragmentation and isolation of the participants' prior knowledge of procedures. Similar to Mack's (1990) findings with grade 6 students, this knowledge initially kept the pre-service teachers from drawing on their informal knowledge of fractions and making sense of the contexts. Some of the participants tended to trust their answers based on procedures more than those based on informal knowledge and as the study progressed needed to use procedures to confirm their reasoning. Yet, as the participants developed their informal reasoning and abandoned their rote algorithms, they developed a more connected understanding of fraction symbols and operations, in keeping with Mack's (1990; 1993) work with middle school students. The group of procedural roadblocks thus kept students from developing a full understanding of the big ideas related to fractions, but as the roadblocks were removed they began to deepen their understanding.

What is the whole? The most persistent and challenging conceptual gap for the pre-service teachers was that of identifying the fraction whole; that is, the appropriate referent unit for the fraction. When working in contexts that required finding the unknown whole from the parts, contexts that required the multiplication or division of fractions, or contexts that required the representation of the addition of fractions students from all three clusters struggled to make sense of the appropriate referent unit. This roadblock persisted through the first three test instruments. While it was not seen on the post-test this was because there were none of these type of problems on that test instrument. Research with children shows that reasoning with referent units is challenging and the complexity makes it difficult for them to connect meaning, symbols, and operations (Behr et al., 1992). Lamon (1994; 2005) extends the concept and stresses the importance of flexibility in unitizing in the development of an increasingly sophisticated understanding of mathematical ideas. Multiplicative rather than additive reasoning is required to coordinate these complex structures (Lamon, 1994). Even though the importance of the whole is pivotal for the development of understanding of rational numbers, only a few studies have explored the role of referent units in teacher understanding (e.g., Izsak, 2008; Izsak, Tillema, & Tunc-Pekkan, 2008; Lee, Brown, & Orrill, 2011) and even fewer in pre-service teacher understanding (Behr et al., 1997; McAllister & Beaver, 2012). The research with teachers shows that their lack of flexibility with referent units hinders their ability to understand drawn representations of fractions and to write appropriate word problems for fraction operations. These elements are important aspects of the specialized knowledge for teaching (Ball, Thames, & Phelps, 2008; Lee et al., 2011), which pre-service teachers need to have in order to teach effectively. This present study illustrates the challenges pre-service teachers have with understanding the changing nature of the referent whole and suggests that it may be linked to their fragility of understanding fractions as relations. This is in agreement with the research findings for children that those who interpret fractions as pairs of separate whole numbers have difficulty identifying and maintaining a fixed whole (Ball, 1993; Mack, 1995; Izsak et al., 2008; Streefland, 1991).

Additional conceptual challenges. The remaining conceptual challenges were less pervasive, but nonetheless, reflected pre-service teachers' fragility of understanding as the types of problems posed deepened in complexity. For example, as the context changed to include fractions, decimals, and percents in the same question students in the middle and low clusters lacked the flexibility to shift between the different representations and use strategies that fit best with the numbers. They assumed the same strategy must be used for all elements of the problem. Similarly, as the context for ratios changed to include larger numbers and non-integer components students from all clusters struggled to make sense of ratios as meaningful quantities. Finally, individual students struggled to reorganize their understanding about the place of additive thinking (Lamon 1993, 2005), the extent of the intuitive idea that multiplication makes bigger (Graeber & Tirosh, 1988), and the nature of improper fractions (Tzur, 1999). Each of these conceptual challenges reflected the fragility with which the students held the big ideas about fractions and parallels some of the challenges younger students may have when initially learning about fractions.

The importance of identifying roadblocks. The identification of roadblocks in this study is significant because it reframes the analysis of errors and misconceptions. Rather than analysing a single snapshot of pre-service teachers' work, or even pre and post intervention snapshots, this study explored their work and thinking about fractions over time. In this context it became possible to analyze the interaction between errors, strategies, and the development of big ideas. Previous studies have emphasized errors and limitations in pre-service teachers' performance with fractions (Ball, 1990; McAllister & Beaver, 2012; Newton, 2008; Tirosh et al., 1998). Algorithmically based errors, intuitively based mistakes, and misconceptions based on limited or inadequate knowledge have been all been identified. In addition, these studies have noted how pre-service teachers gave procedural rather than adequate conceptual explanations of their work with fractions. However, few connections have been made to the specific big ideas the pre-service teachers need to develop for a deeper understanding of the fraction concepts. While McAllister and Beaver (2012) did identify the lack of understanding of the concept of unit whole in their analysis of errors in the creation of word problems for fraction operations, most studies were not as specific and merely identified general topics, such as Ball (1990) who simply stated that the division by fraction errors indicated a narrow understanding of division. The roadblocks identified in this study indicate that the pre-service teachers struggled with understanding fractions as meaningful objects. At the most basic level they had not fully internalized an awareness of fractions as relations.

By exploring the roadblocks to deeper understanding this study acknowledges that pre-service teachers were not novices in fraction concepts at the beginning of the project – rather their understanding lacked cohesion and connection to foundational big ideas and fraction concepts (Fosnot & Dolk, 2002; Ma, 1999). This means that while the errors of pre-service teachers may at times mirror those of children learning fractions concepts for the first time, they reflect fragmented and entrenched, rather than novice, knowledge (Newton, 2008). While Newton identifies specific errors across the different fraction operations as evidence of knowledge fragmentation, this study identifies categories of roadblocks that either inhibit the development of a connected, coherent knowledge of fractions or indicate limits in conceptual understanding. The shift to a more finely grained analysis of the big ideas (Fosnot & Dolk, 2002) reflects a constructivist theory of learning that views knowledge as a complex system in which the development of understanding includes refining prior conceptions and making sense of contexts (Smith, DiSessa, & Roschelle, 1993). This identification of roadblocks and big ideas will assist teacher educators as they help pre-service teachers prepare to teach fraction concepts.

Development of Understanding: Adult (Re)Learning

As adults, the pre-service teachers progress in their development of understanding fractions in a different manner than children. The overall landscape of (re)learning, seen in Figure 134, differs significantly from that of children in a reform-based environment learning fractions for the first time. The intertwined concepts of fractions as relations, fractions as ratios, and fractions as operators, reflect the key big ideas the pre-service teachers struggle to develop. Given the prior learning experiences of the participants, these concepts overlap, so the (re)learning process does not reflect the progression seen in children's learning that begins with the foundational idea of fractions as relations. Instead the pre-service teachers need to unpack their initial understandings of fractions as procedures and work to develop an understanding of fractions as meaningful objects.

The journey of the pre-service teachers through the landscape proceeds from the bottom to the top, loosely following the direction of the dotted line and arrow. They move back and forth across the landscape using the two main strategies of oral sense making and visual modelling to develop various big ideas that support the development of their understanding of each of the three central concepts. For several of the participants, errors and roadblocks provide additional impetus for their shift from procedures to visuals and sense making. As the pre-service teachers develop new strategies and deepen their understanding of the big ideas many move from an additive or whole number approach to fractions to the use of multiplicative thinking and increased flexibility in their understanding of fractions.

Progression through the landscape, like that of children, is messy (Duckworth, 1987) and non-linear (Fosnot & Dolk, 2002). The specific elements or landmarks on the landscape are not necessarily sequential, nonetheless, just as for children, some landmarks are precursors to others. While children within Fosnot & Dolk's framework construct the big ideas about fractions through their developmental progression of strategies or *progressive schematization* (Treffers, 1987), the adult learners, often hampered by their previous knowledge of fractions, experience some challenges in the reconstruction and reorganization of each of the concepts. It is only as the pre-service teachers let go of their procedural strategies that they begin to move through the landscape. The degree to which they construct the big ideas about fractions is directly related to their willingness to engage in the process and to let go of the procedures. Even though their initial models and strategies are based on an incomplete notion of fractions, this weak modeling still enables some movement, albeit limited, through the landscape.

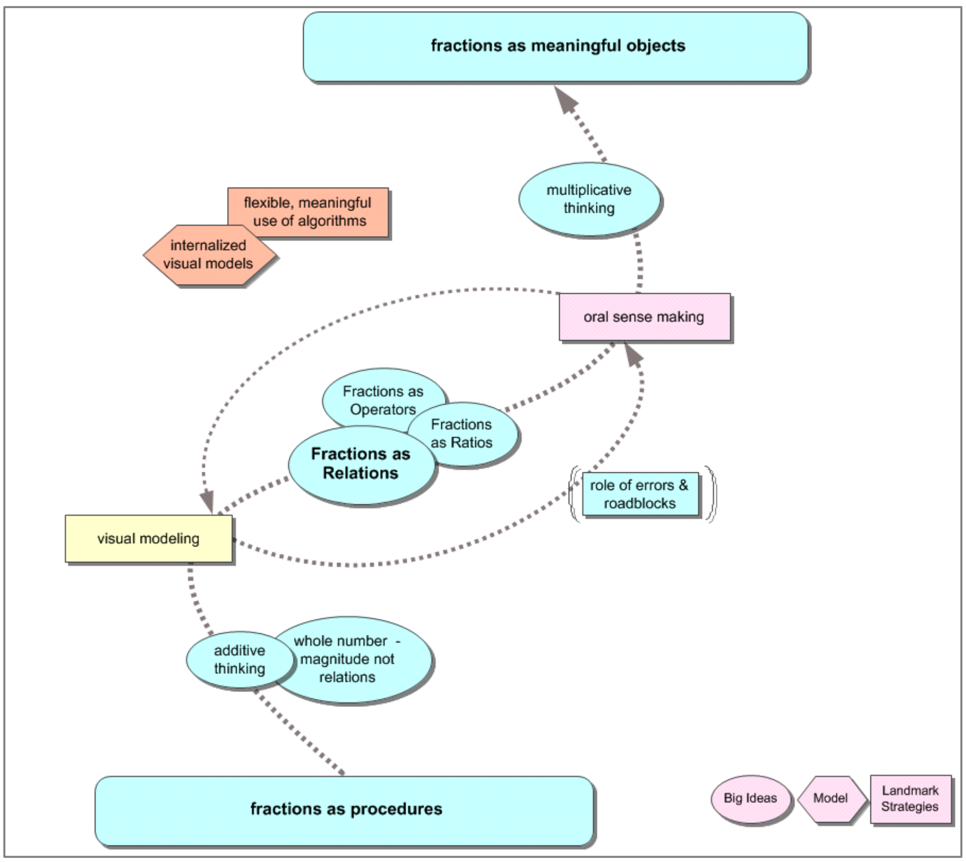


Figure 134. Pre-Service Teacher Landscape of (re)Learning Fractions

As the pre-service teachers encounter new big ideas or more challenging contexts and number structures, they often revert back to their original rudimentary part-whole models and additive thinking. Their strategies reveal the tentative and fragile nature of their understanding as they *fold back* (Pirie & Kieren, 1994) to previous levels in an effort to make sense of the new contexts and thereby deepen their big ideas. Yet, when the pre-service teachers experience fractions as meaningful objects, with a connected and deep relational understanding, they are able to use a variety of strategies in flexible and meaningful ways. It is this ability to recognize fractions as numbers or quantities that is key in developing an efficacious and flexible understanding (Empson et al., 2011; Lamon, 2001, 2005, 2007; Mack, 1995). The findings regarding the progression of the pre-service teachers' construction of this notion of fractions as meaningful objects in conjunction with the big ideas of fractions as relations, as ratios, and as operators will be discussed in the following sections.

Development of understanding: Fractions as relations. Understanding fractions as relations is foundational to making sense of these quantities and their operations. The landscape of learning for fractions as relations, as seen in Figure 135, highlights the key elements of the pre-service teachers' journey. As they gain understanding, they move up the landscape from working with fractions as procedures to recognizing relationships and seeing fractions as meaningful objects. To varying degrees, participants let go of their reliance on the rote procedures of finding common denominators, converting fractions to decimals, using the rule that when *of* is used with fractions it means multiply, and using cross multiplication. Yet many return to these procedures when they encounter new contexts and problems. Using strategies of visual modeling and oral sense making, pre-service teachers develop the big ideas and multiplicative thinking necessary for this shift to a relational understanding of fractions. Three different learning trajectories portray the various levels of understanding of the pre-service teacher clusters as they move through the landscape. Each of these trajectories will refer to big ideas, models, and strategies represented on the landscape; references to the big ideas will be italicized.

Low cluster trajectory. The first trajectory reflects the learning of the pre-service teachers in the low cluster. Beginning with a view that fractions are simply two whole numbers that reflect the number of parts in the whole, these pre-service teachers move to realize that fractions are *equal parts of a whole*. They prefer to use memorized procedures of common denominators, converting to decimals, and cross multiplication because they do not know how to model fractions and their operations correctly. Their initial modelling uses a set concept that focuses on the number of parts rather than relationships. Thus they can work with premade fraction models but cannot correctly draw their own model because they lose sight of the whole. As these pre-service teachers recognize that the *denominator is related to the size of the piece* they overgeneralize this relationship to a procedure for comparing fractions. One of them, nonetheless, begins to recognize the importance of *1/2 as a benchmark fraction*. None of the three pre-service teachers in this trajectory move beyond their additive thinking and their focus on the magnitude of the numbers, even though their reasoning does vacillate. Thus they continue to struggle with identifying the fraction whole.

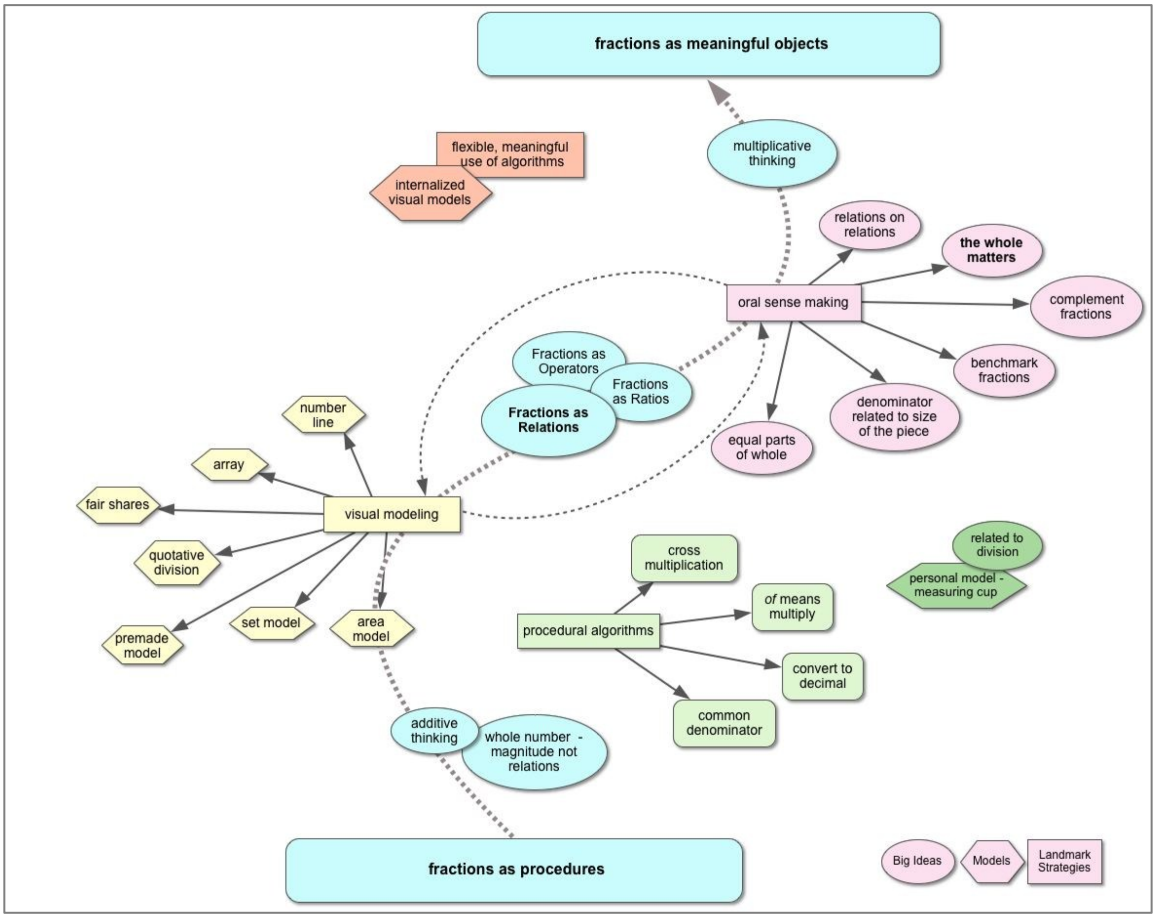


Figure 135. Fractions as Relations Landscape

Middle cluster trajectory. The second trajectory highlights the learning of the pre-service teachers in the middle cluster. These participants begin with a recognition of the need for fractions to have *equal parts of the whole* even though they do not fully understand the importance of the whole. They use procedural algorithms rather than making sense of the context, which reflects their whole number thinking. As these pre-service teachers move through the landscape most willingly let go of their memorized procedures and reason about fractions instead, often using visual models. They encounter difficulties attempting to model contexts with an unknown whole or two wholes, often misapplying the fair shares model. Their reasoning results in fragile and sporadic shifts to recognizing fractions as relations. They develop an emerging understanding that the *denominator is related to the size of the piece*, that *benchmark fractions* help visualize fractions as quantities, that *complement fractions* combine to make a whole, and that the *whole matters*. However, as these pre-service teachers encounter new contexts or larger fractions they often revert back to whole number thinking and focus on the magnitude rather than the relationship. Some even use their new strategies or ways of thinking in a procedural fashion as they look for rules to help them work with fractions. Of the seven pre-service teachers in this cluster, three are on the cusp of an understanding of fractions as meaningful objects, another three use reasoning and modeling to shift between whole number and relational thinking, and one continues with a procedural understanding of fractions that now includes models.

Top cluster trajectory. The final trajectory reflects the learning of the pre-service teachers in the top cluster. These participants begin with a stronger understanding of fractions that is still bound by procedures and influenced by context, prior experiences, as well as a personal measuring cup model of fractions. Yet they understand fractions as more than parts and wholes, articulating that *fractions are related to division*. As these pre-service teachers journey through the landscape they use concrete visual models to develop and confirm new ways of reasoning about fractions, such as the *denominator is related to the size of the piece* in varying contexts, and that *benchmark fractions* provide a means of comparing fractional quantities. When they encounter new or challenging contexts, they also demonstrate the fragility of their understanding by folding back to previous procedures or ways of thinking. These pre-service teachers readily use *complement fractions* in part-part-whole contexts because they are able to look beyond the magnitude of the parts. Thus they understand that the referent *whole matters* in all contexts. Most of these participants come to understand that multiplication and division of fraction are *relations on relations* that have two different wholes, even though some initially struggle to use models and to make sense of the array model when multiplying fractions. Some of these participants also encounter a final challenge as they work with fractions in a measurement or number line context. While three of the four pre-service teachers in this cluster engage with fractions as meaningful objects and internalize a visual understanding of the part-whole relations, one continues to rely on procedural thinking, converting fractions to decimals.

The importance of fractions as relations trajectories. Understanding fractions as relational quantities is essential for a mature understanding of fractions (Empson, et al., 2011). Yet, for pre-service teachers this concept develops only slowly over time. At the end of the study, only three of the fourteen pre-service teachers have a deep understanding of fractions as relations; not surprisingly, the same participants who initially demonstrated proficiency on the content exam – pre-test. While three of the remaining eleven participants are on the cusp of fully actualizing this understanding, the majority of the pre-service teachers do not have a consistently enacted and multiplicatively supported, relational understanding of fractions—even though they all progress over time. This slow development in pre-service teachers parallels the finding with children that a relational understanding of fractions develops gradually (Empson et al, 2011). It may also be that the pre-service teachers did not recognize that a relational understanding is more than another strategy: that it is a way of thinking about fractions (Stephens, 2006). Thus some may not see the need to switch from computational or procedural thinking.

This well developed understanding of fractions is also a critical component of the specialized mathematical content knowledge for teaching fractions (Ball et al., 2008). Pre-service teachers need to develop an understanding of fractions as meaningful objects in order to be effective in their teaching. Yet, similar to findings with children, the majority of the participants lack an anticipatory and goal oriented approach to engaging with fractions that comes with a relational understanding (Empson et al., 2011). Instead they simply focus on what to do next. As with children, these pre-service teachers think in terms of "how many" instead of the more appropriate, "how much" (Mack, 1995). The pre-service teachers who do have this relational understanding, like children, are able to decompose and recompose fractional quantities when working with expressions or operations, anticipating the end goal (Empson et al., 2011).

Development of understanding: Fractions as ratios. Exploring fractions as ratios adds another layer of depth to our understanding of the development of fraction knowledge in pre-service teachers. By focusing on ratio contexts with both part-part and part-whole structures we are able to investigate the interconnected development between proportional reasoning and fraction concepts. Multiplicative thinking is the foundation of a deep understanding of the intertwined relational concepts. The landscape of learning, as seen in Figure 136, highlights the big ideas, strategies, and models used by the pre-service teachers on their journey towards a deeper understanding of fractions as ratios. In this journey they move along the continuum from ratios as procedures to ratios as meaningful objects and shift from additive thinking to varying degrees of multiplicative thinking by using the two key strategies of visual modeling and oral sense making. Different than their work with fractions in general, many pre-service teachers model ratio concepts from the beginning, perhaps because they feel confident this will give the correct answer. Thus their initial procedural approach focuses on associating the two extensive quantities in a concrete representation in addition to some use of memorized procedures such as cross multiplication, division, and algebraic manipulation. Movement across the landscape entails a shift to a meaningful and flexible use of algorithms.

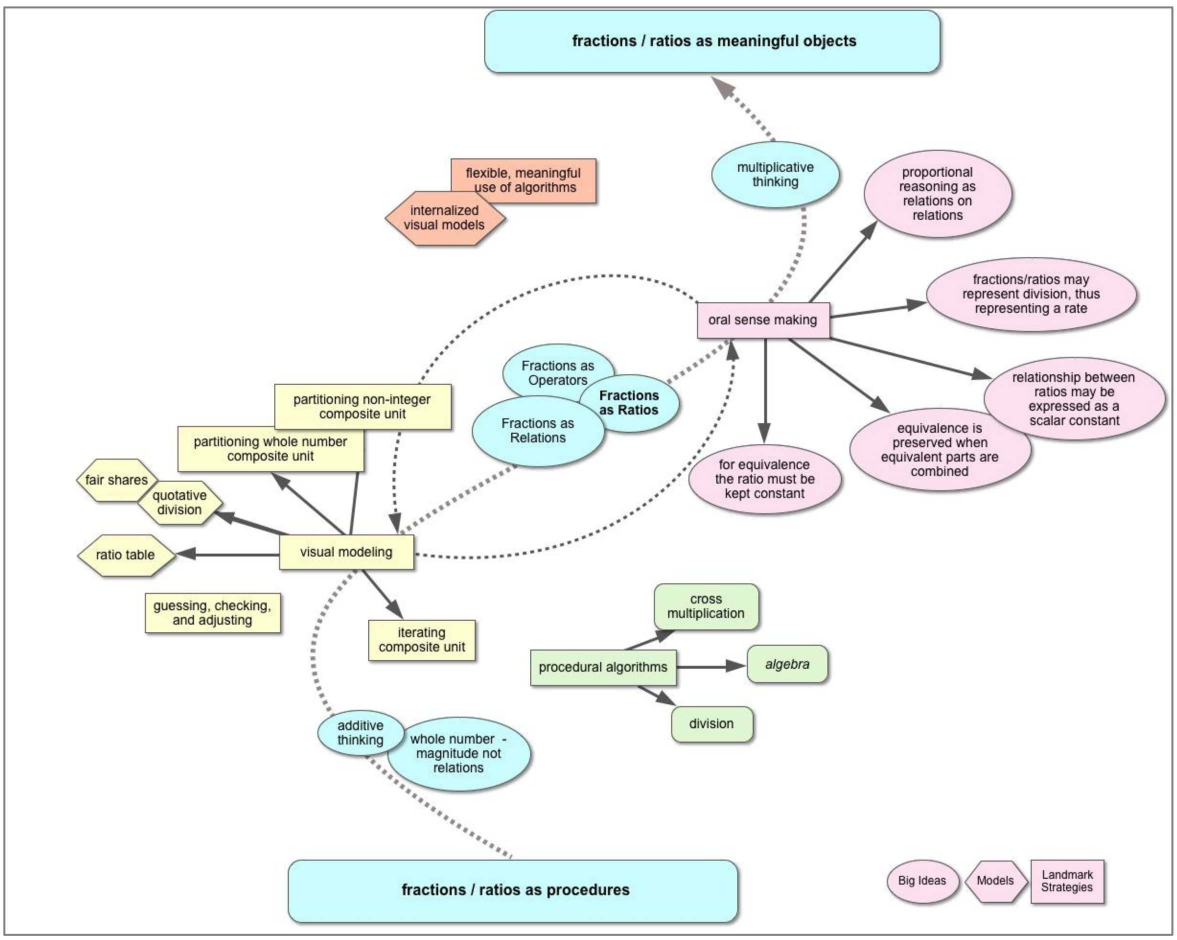


Figure 136. Fractions as Ratios Landscape

While the developmental landscape of learning for pre-service teachers parallels that of children in many ways, it also reflects the fragmented nature of the adults' prior knowledge. In the four learning trajectories that emerge from the data we see the extent of the connections made by the pre-service teachers to foundational big ideas and fraction/ratio concepts. Each successive trajectory encompasses the previous one and extends beyond it.

Low cluster trajectory. The first trajectory reflects the learning of two participants in the low cluster. Beginning with an incomplete understanding of the connection between fractions and ratios, these pre-service teachers move to a recognition that *for equivalence the ratio must be kept constant* and thus use composite units in their strategies. However, when encountering three-part ratios these participants do not know how to represent the complex unit and instead use a simple guess and check strategy. They display an emergent understanding that *equivalence is preserved when equal parts are combined* that is limited to whole number ratios. They use ratio tables to add single iterations of the composite unit or to double their results, maintaining a focus on the separate quantities of the ratio rather than the relationship. While they have different starting points and approaches, one relies heavily on memorized procedures and the other lacks basic number sense, both reach an impasse when attempting to partition ratios with fractional units.

Middle cluster trajectories. The second trajectory reflects the learning of five participants across the three clusters, with the majority from the middle cluster. These pre-service teachers begin by using composite units to physically model the ratio contexts or by using incorrectly memorized fraction/ratio concepts. The majority of these participants need scaffolding to make sense of initial contexts that require partitioning of the whole number composite unit. Their emerging understanding that *equivalence is preserved when equal parts are combined* does not include combining convenient multiples of the composite unit. It may be that they are unclear about the difference of combining equivalent parts and adding fractions. Nevertheless, when working with a part-whole unit ratio that mirrors their work with fractions, these same participants easily use either a fair shares or quotative division model. Even though they are now able to partition fractional units, these pre-service teachers still work with ratios in contextually dependent ways that focus on the individual unit quantities.

The third trajectory reflects the learning of four participants from the middle cluster. Their progress mirrors that of the previous group with a small extension. These pre-service teachers are beginning to use aspects of multiplicative reasoning to identify the relationship between ratios. While they do not combine convenient multiples of the composite unit, these participants use quotative division to find the *scalar constant between the ratios*. Yet at its core, this abbreviated build up process models replicative growth rather than true multiplicative growth (Kaput & West, 1994). These pre-service teachers still work with ratios as composite units and do not recognize the relationship within the ratio itself.

Top cluster trajectory. The final trajectory encompasses the learning of three participants from the top cluster. These pre-service teachers begin with a stronger understanding of ratios in which they use composite units, or reasoning about known algebraic procedures and intuitive understandings of equivalent fractions to find the scalar constant. While this initial use of the scalar constant does not indicate a full understanding of the multiplicative relationships, their later work reflects a deepening understanding as they identify the *scalar relationship between ratios* in non-integer contexts and use ratio tables to combine convenient multiples of the composite unit. These participants are on the cusp of developing an understanding of *relations on relations.* They no longer see ratios as merely unit quantities, but are beginning to recognize the relationship both within and between ratios (Carpenter et al., 1999). Yet, none of these participants interpret the *ratios as division and thus representing a rate*. It may be that the types of problems presented in the study did not facilitate the development of this idea.

The importance of fractions as ratios trajectories. The literature describes a three stage trajectory of strategies from qualitative, to building up using repeated addition of the ratio, to multiplicative (Behr et al.,1992; Kaput & West, 1994); however, the progression of the pre-service teachers' understanding of ratios did not strictly correspond to this path. Their learning paths reflect the complexity of the number structures used as well as strategy. For example, some pre-service teachers could only combine ratio units but could not partition them, others could partition ratio units, but only using integers, and still others could scale the ratio up and down with both integers and non-integers. This corresponds with a fuller picture of proportional reasoning development from more current studies with grade 4-5 children in a constructivist oriented setting (Carpenter, Gomez et al., 1999; Steinthorsdottir & Sriraman, 2009) based on Lamon's (1994, 1995) notions of unitizing and norming. Yet, most of the pre-service teachers did not use their multiplication knowledge when using the building up strategy, even though the children in these studies moved back and forth between additive and multiplicative strategies depending on context and facility with numbers. It may be that the pre-service teachers preferred to use strategies they could clearly explain (Steinthorsdottir & Sriraman, 2009). It may also be that these pre-service teachers had not fully constructed an understanding of the big idea that *equivalence is preserved when equivalent parts are combined* and did not know how to use the ratio table to facilitate this thinking. They may have been uncertain about the difference between combining fractions and combining equivalent parts.

The development of the pre-service teachers' understanding of ratios was constrained by their understanding of fractions as relations. Those who did not have a fully developed understanding of fractions as relations did not see proportions as relationships between two relationships. They did not recognize the complex numerical relationships both within and between the ratios. Rather, they simply recognized the equality between static ratios or unit quantities. This corresponds with the finding for children that fractional reasoning supports the development of proportional reasoning (Nabors, 2003). In a teaching experiment Nabors found that grade 7 students used their part-whole fraction knowledge when working with ratios in a way that paralleled children's use of whole number knowledge to build up their fraction knowledge. Similarly, as the pre-service teachers built up their knowledge of ratios they used their fragile or limited understanding of fractions as relations.

Only three of the fourteen pre-service teachers began to understand ratios as meaningful objects. These are the same participants who had constructed an understanding of fractions as relations. It is noteworthy that the majority of the pre-service teachers continued with a concept of ratios as composite units, an association of two extensive quantities, rather than the more robust relational understanding. Heinz' (2000) work with pre-service teachers reveals that while this composite unit type of reasoning is sufficient for many types of ratio problems, it is not sufficient to deal with problems of relative size, where the ratio measures a given attribute such as the "squareness" of a rectangle by comparing the lengths of two sides (Simon & Blume, 1994). Additional research with pre-service teachers shows that they lack both flexibility in the domain of proportional reasoning (Berk, Taber, Gorowara, & Poetzl, 2009) and flexibility in interpreting the developmental progression of middle school students' proportional reasoning strategies (Hines & McMahon, 2005). It seems likely that this lack of flexibility is connected with limitations in understanding ratios as meaningful objects and in recognizing proportional reasoning as relations on relations. All three of these aforementioned elements are necessary components of the specialized knowledge for teaching (Ball et al., 2008) which pre-service teachers need to develop. Finding multiple strategies to solving problems (Steinthorsdottir &

Sriraman, 2009), as well as exploring contexts in which ratios measure attributes, may both be necessary activities for pre-service teachers to shift their thinking about ratios.

Development of understanding: Fractions as operators. When learners understand fractions as operators they are able to find portions of a given unit whole. This process of determining fractional amounts provides a foundation for understanding multiplication of fractions where partitioning and the reconceptualization of composite units are critical (Behr, Harel, Post, & Lesh, 1993; Hackenberg & Tillema, 2009; Izsak, 2008; Mack, 2001). The landscape of learning, as seen in Figure 137, reflects the progression of pre-service teachers as they work with fractions, decimals, and percents. They begin with a procedural understanding of operators and multiplication that is constrained by memorized rules. As they journey up through the landscape using the strategies of visual modeling and oral sense making, the pre-service teachers begin to develop a deeper understanding of the multiplicative concepts embedded in the algorithms. Many begin to see fractions as meaningful operators, but only a few are able to extend this understanding to both decimals and percents. Three different learning trajectories emerge from the data. These diverse levels of understanding of the pre-service teacher clusters will be discussed using the big ideas, strategies, and models represented on the landscape.

Low cluster trajectory. The first trajectory reflects the learning of the three participants in the low cluster. These pre-service teachers begin with a procedural understanding that operators mean multiplication. They use partially remembered algorithms, converting fractions to decimals, or cross multiplying, to focus on manipulating numbers rather than understanding the fraction relations and context. Encountering the dissonance of their errors (Fosnot & Dolk, 2002; Fosnot & Perry, 2005; Piaget, 1977) provides impetus for these pre-service teachers to make sense of the context by shifting to a concrete visual representation of the fraction operators using a fair shares model. As they partition the given whole, the pre-service teachers focus on *counting the number of parts*. They readily distribute the original quantity into the number of parts indicated by the denominator, making *equal "groups of",* and choose the number of parts indicated by the numerator. The fair shares model provides a means for making sense of fraction operators, yet it becomes a procedure and only one of the participants in this low cluster extends the model to percents and decimals. These pre-service teachers do not see percents or decimals as a type of fraction.

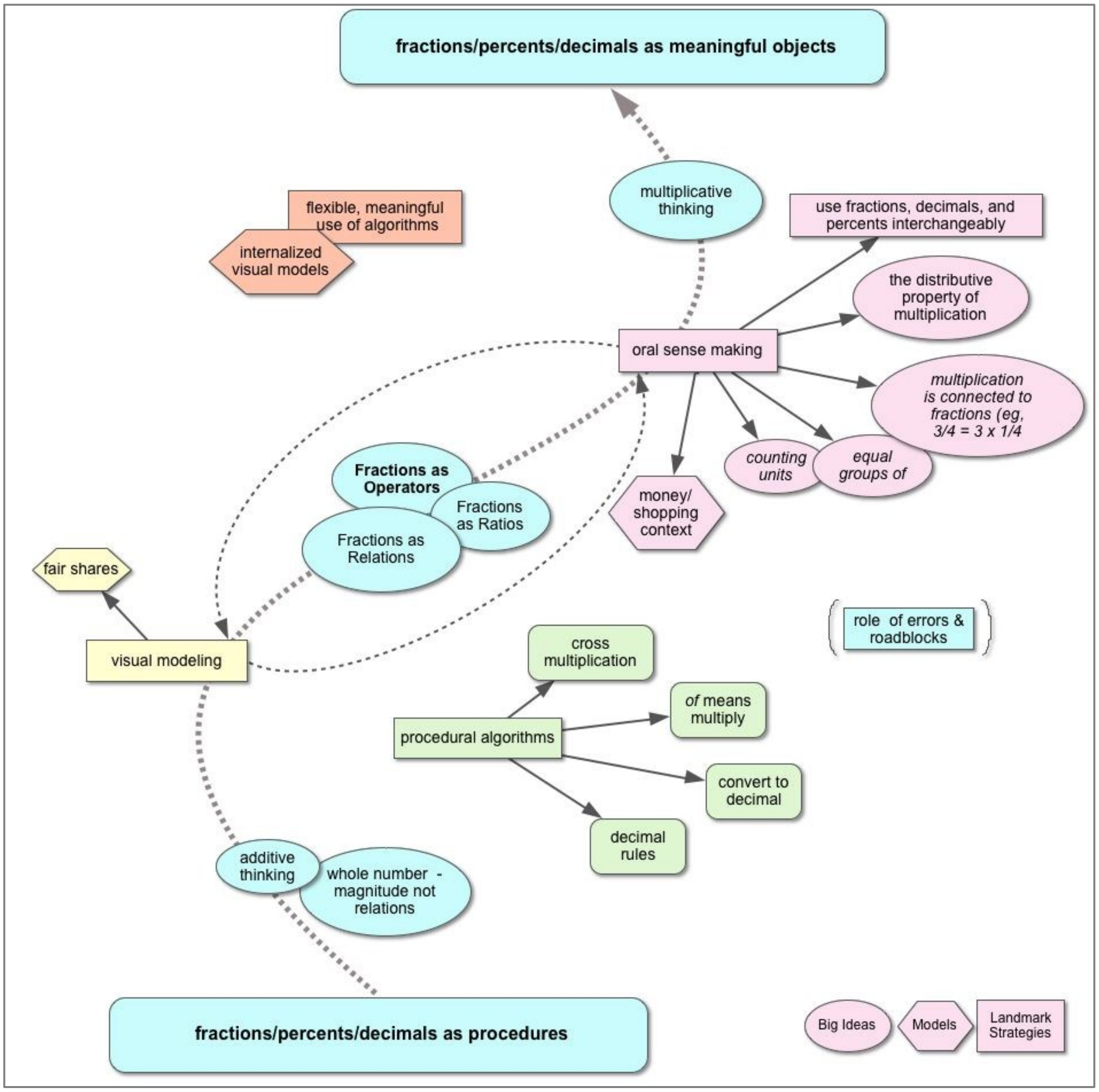


Figure 137. Fractions as Operators Landscape

Middle cluster trajectory. The second trajectory encompasses the learning of the seven participants in the middle cluster as well as one of the participants from the top cluster. While the majority of this group also begin with a limited procedural understanding of operators, three of the pre-service teachers immediately connect fraction operators to the fair shares model and determine *equal "groups of",* even though one of them is constrained by unit fractions. Movement through the landscape proceeds slowly. Several participants experience confusion as they attempt to reason about and model percent operators. For others, errors provide the impetus to shift from procedures to making sense of fraction operators through visual modeling. Still others use links to money and shopping contexts to provides a concrete model for reasoning about percent operators. All participants in this cluster demonstrate growth in their understanding of how to model fractions as operators. They use the fair shares model to determine the size of the equal parts and one participant even internalizes this model to make sense of her calculations. Yet, it is unclear how many of these participants move beyond a part-whole interpretation in which they simply count the parts, since some seem to use the model in an algorithmic manner. All continue to use a procedural approach to percents and decimals.

Top cluster trajectory. The third trajectory reflects the learning of three of the four participants in the top cluster. While they also begin with a mainly rule-based understanding of fraction and percent operators, one of these pre-service teachers links her calculations directly to the partitive fair shares model and goes beyond the part-whole interpretation to explicitly identify the groups as unit fractions and multiplies to find the total. In this context the fraction is a meaningful object and *multiplication is connected to fractions*. These pre-service teachers willingly shift to the use of models to explain their reasoning about operators. They progress to internalizing a visual model for the operator function and readily link the partitioning to their calculations. Additional growth is seen as they use the *distributive property of multiplication* to reason about percent operators. These same participants are able to work interchangeably with operators as fractions, decimals, and percents. For them, all forms of fractions are becoming meaningful objects.

The importance of fractions as operators trajectories. As can be seen in the learning trajectories, most of the pre-service teachers use their informal knowledge related to partitioning to make sense of fractions as operators. The foundational concepts embedded in equal sharing provide a touchstone for the pre-service teachers as they (re)learn and deepen these beginning concepts of fraction multiplication, just as it did for grade 5 children (Mack, 2001). With the partitive fair shares model the pre-service teachers focus on a part-whole perspective that emphasizes whole number counting units and the associated two levels of units rather than three nested levels of units connected with multiplicative thinking (Izsak, 2008; Hackenberg & Tillema, 2009). This modeling and concomitant interpretation of fraction operators, however, has the potential to limit their ability to reconceptualize and partition units. In Mack's (2001) study, children who focused on fractions in terms of the number of parts struggled with partitioning parts of the whole in multiplication contexts. Only children who considered the fractional amount each part comprised of the unit (e.g., 3/4 means 3 x 1/4) were able to use their informal knowledge to reconceptualise and partition units in a variety of ways. In this context the findings of this study parallels that of children. Only the three pre-service teachers in the top cluster with a relational understanding of fractions demonstrate a flexible understanding of fractional units.

The predominant use of the partitive fair shares model for fraction operators may have also been influenced by two other factors. First, the mathematics skills class focused primarily on this model as a concrete way to demonstrate understanding beyond the use of procedural calculations. The importance of partitioning parts of the whole and reconceptualising the unit as a means of deepening both the multiplicative and relational understanding of fractions was not directly addressed. Thus pre-service teachers were not challenged to use additional models. Second, the test instruments for this study focused on the use of fractions as operators on whole number quantities. Only one item, the bag of chips problem (#29), required participants to reconceptualise the unit in a multiplicative context; however, they did not need to partition the unit because the fractional amount was embedded in the composite unit (1/3 x 3/4). Consequently, there was no direct way to determine pre-service teachers' growth in understanding multiplicative fraction contexts.

In order to have the necessary flexibility to interpret and evaluate the different ways their future students may reason about fraction multiplication situations these pre-service teachers need to move beyond their focus on counting the equal sized parts of a whole. The literature provides several avenues for developing the necessary connections to multiplicative concepts that would be helpful for these pre-service teachers. First, the use of a double number line as an alternative model for thinking about operator contexts (Fosnot & Dolk, 2002; Fosnot, 2007) would enable the pre-service teachers to explicitly link fractions to parts of the given whole in a way that preserves and highlights the relationships. Second, providing contexts for pre-service teachers to focus on the fractional amount that results from partitioning a unit (Mack, 2001) would help them build on their informal knowledge to develop an understanding of multiplicative fraction relations. Finally, providing opportunities for pre-service teachers to reason explicitly about fraction units with a three level structure using tasks that involve parts of parts (Izsak, 2008) would enable them to interpret and assess the variety of drawings students might use to represent fraction multiplication.

Summary of the Major Findings

The major findings regarding the development of these pre-service elementary teachers' understanding of fraction concepts can be summarized as follows:

1. Although all of the pre-service teachers demonstrated growth in their capacity to correctly answer fraction word problem items, this masked a continued lack of deep fractional understanding for the majority of the participants. Only the three pre-service teachers who began the study with a strong understanding of fraction procedures and concepts developed an understanding of fractions as meaningful objects.
2. Understanding and identifying the fraction whole, that is, the appropriate referent unit for the fraction, proved to be the most challenging concept for the pre-service teachers to (re)learn. This difficulty persisted throughout the study and reflected the fragility of the pre-service teachers' understanding of fractions as relations.
3. The landscape of (re)learning fractions for adults is significantly different from the landscape for children in a reform-based learning environment. Pre-service teachers worked to make sense of their fragmented knowledge around the intertwined concepts of fractions as relations, ratios, and operators.
4. A deep understanding of the foundational concept of fractions as relations develops slowly over time. Only three of the pre-service teachers in the top cluster developed this understanding.
5. Understanding of fractions as ratios and operators is constrained by pre-service teachers' relational understanding of fractions. In these contexts participants needed to be able to identify the relationship between the part and the whole when reconceptualising and partitioning units.
6. Modeling is a necessary but not sufficient means for pre-service teachers to understand fractions as meaningful objects and to shift to multiplicative thinking. Gains were limited by the type of models used and the degree to which the modelling became a rote procedure.

Concluding Statements

Elementary teachers need to have a deep understanding of fraction concepts if they want their students to develop more than a mechanized understanding of the concepts. This idea that students learn more when their teachers possess a deep understanding of mathematics finds increasing support in the literature (Hill et al., 2005). Many teachers, however, do not develop this deep understanding when they initially learn about fractions. A small, but growing, number of studies identify reform-based interventions that improve pre-service teacher fraction knowledge (Green et al., 2008; Newton, 2008; Osana & Royea, 2011; Toluk-Ucar, 2009). This present study took a different perspective and explored how pre-service teachers' understanding of fraction concepts developed over the course of six months. The 14 pre-service teachers in this study were all enrolled in a remedial mathematics workshop that used a reform-based approach to develop an understanding of math concepts at the middle school level.

The results of the baseline content exam revealed a broad range of mathematical skills; for the purpose of analysis the participants were grouped into three levels according to their scores on the numeracy portion. Scores for the participants in the top cluster ranged between 80-100%; the middle cluster ranged between 50-70%; and the low cluster ranged between 10-30%. At the end of the study all participants showed improvement on the post-test retake exam with a combined average score of 93% on the fraction word problems. Unfortunately this ability to solve the word problems correctly masked a continued lack of deep fractional understanding. This tension has implications for teacher educators. The adult learners developed strategies and models to act out fraction contexts and make sense of them, and yet they were still grappling with constructing the inherent big ideas. The fragility of their understanding became apparent when they encountered more challenging contexts and number structures. Mathematics educators need to be aware that as pre-service teachers unpack their initial understanding of fractions as procedures and work to develop an understanding of fractions as meaningful objects it will take time to develop the foundational big ideas.

Fosnot & Dolk's (2002, 2007) landscape for fractions, decimals, and percents served as the starting point for exploring the development of the pre-service teachers' understanding of factions over the course of the academic year. However, as the pre-service teachers progressed through the study it became apparent that their learning landscape differed significantly from that of children in a reform-based environment learning fractions for the first time. The adult learners were often impaired by their previous knowledge and needed to let go of the safety of their preferred procedures in order to reconstruct and reorganize their fraction concepts. This reconstruction often occurred through the use of models. Nevertheless, progress depended on the type of models used and the degree to which the modelling became a rote procedure. Teacher educators need to be cognizant of these differences between child and adult learning trajectories and develop learning experiences that help the pre-service teachers engage with fractions as numbers and quantities rather than procedures to promote the development of an efficacious and flexible understanding (Empson et al, 2011; Lamon, 2007).

Learners with a fully realized understanding of fractions recognize that they are relational quantities (Empson et al., 2011). This foundational big idea challenged the pre-service teachers. They found it difficult to move from an additive or part-whole approach to fractions to a relational approach that emphasized multiplicative and anticipatory thinking. The extent of pre-service teachers' understanding of fractions as relations impacted their ability to make sense of referent units for fractions and to work in contexts of fractions as ratios and operators. However, when pre-service teachers experienced fractions as meaningful objects with a deep relational understanding they developed a greater flexibility in their approaches. Unfortunately, the concept of fractions as relations developed slowly over time and only three of the pre-service teachers demonstrated this deep understanding.

This case study focused on the developmental learning journey of 14 pre-service teachers in the area of fractions. The use of an embedded case study model with multiple units of analysis (Yin, 2003) enabled the development of a robust learning landscape. By identifying clusters of participants within the group as a whole, the learning trajectories reflect the progress of weaker, medium, and stronger skilled pre-service teachers and thus may be shared by other pre-service teachers beyond these participants. While the landscape for (re)learning fractions may have gaps, such as the lack of non- dominant culture participants, it may still serve as a theoretical basis for further studies. The use of new contexts and more participants will serve to confirm and extend the landscape.

Implications for Future Research

The findings and limitations of this study suggest a number of avenues for future research. First, while this study focused on pre-service teachers, the literature states that inservice teachers also struggle with fraction concepts (Ball, 1990; Ma, 1999; Post et al., 1988; Tirosh et al., 1998). How would the development of their understanding of fractions compare and contrast with that of the pre-service teachers, given the inservice teachers received professional development in fraction concepts? Would they experience similar roadblocks? How would the development of their conceptual understanding affect their teaching?

A second avenue of research could explore the longitudinal development of fraction understanding in pre-service teachers. This study focused on the development of understanding over the course of an academic year. Extending the time frame would provide the opportunity to follow pre-service teachers into their first year or two of teaching. How and when do these teachers develop a robust relational understanding of fractions? How (and when) does their fraction content knowledge deepen into the specialized mathematical knowledge for teaching fractions? What additional professional development is necessary to facilitate this shift?

Another area of research stems from some of the limitations of this study. The mathematics skills course was only 20 hours and did not provide targeted instruction on the development of multiplicative thinking. While students were encouraged to work with models they were not challenged to use models that went beyond the part-whole perspective. In addition, students had limited opportunities to explore the partitioning of parts and the reconceptualization of units. It would be interesting either to redesign this skills course or to find a comparable university mathematics course for elementary teachers and then research the development of pre-service teachers' understanding of fraction concepts in these settings. How do these changes in curriculum affect the development of student understanding?

A further area of research also stems from limitations of the study. Based on the findings, how could the test instruments be sharpened to have better alignment with the development of big ideas? What items need refinement to better discriminate shifts in understanding? How will these changes deepen our understanding of the landscape of learning?

A fifth area of research could explore the learning trajectories of non-dominant culture pre-service teachers. How is the (re)learning landscape for fractions similar or different for First Nations, Metis, or Inuit pre-service teachers in Canada?

Finally, the findings indicate that pre-service teachers' work with decimals and percents remained at a rote procedural level. Research could focus on the links between fractions, decimals, and percents. How does an understanding of these concepts develop? What strategies and models facilitate a deepening of these concepts?

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Appendices

Appendix A: Participant Demographics

	Name	Gender		Program		Age					Highest Level High School Math				University Math				Level	
		F	M	ConEd*	Post BEd*	20-24	25-29	30-34	35-39	40-44	Gr 10	Gr 11	Gr 12	OAC*	MET*	Statistics	Other	None	PJ	JI
Low	Valery	1			1		1					1				1			1	
	Grace	1		1		1							1		1				1	
	Isabelle	1			1				1				1					1	1	
Middle	Erica	1		1		1							1					1	1	
	Lynsey	1			1	1								1		1			1	
	Irene	1			1				1			1						1	1	
	Gabriela	1			1		1						1					1	1	
	Tanya	1			1			1					1		1				1	
	Brenda	1		1			1					1			1				1	
	Olivia	1			1		1					1				1			1	
Top	Bryann	1		1			1				1							1	1	
	Diane	1			1					1			1					1		1
	Mark		1		1				1					1		1	1			1
	Megan	1			1					1		1						1	1	
	Total	**13**	**1**	**4**	**10**	**3**	**5**	**1**	**3**	**2**	**1**	**5**	**6**	**2**	**3**	**4**	**1**	**7**	**12**	**2**

Note: All names are pseudonyms
*ConEd – Concurrent undergraduate and Bachelor of Education Program
Post B.Ed. – previously completed undergraduate degree and taking 1 year B.Ed. program
OAC – Ontario Academic Credit (equivalent of grade 13, phased out in 2003)
MET – Mathematics for Elementary Teachers: University Math content course
P/J – Primary/Junior: K-6
J/I – Junior/Intermediate: 4-10

Appendix B: Informed Consent: Math Skills Development

Dear Primary/Junior Education Student,

I am inviting you to participate in a research study concerning the mathematical understandings of preservice elementary teachers, entitled *Developing Understanding: Preservice Elementary Teachers' Changing Conceptions of Mathematics.*

As you know, I teach two sections of Ed 4050, the primary/junior mathematics methods course. I am also a doctoral student at Lakehead University in the PhD in Educational Studies program offered jointly by Brock University, Lakehead University, the University of Western Ontario, and the University of Windsor. I am conducting this study to understand the development of mathematical understanding in primary/junior education students. I will not be evaluating individual students, but hope to develop a general picture of how adult learners deepen their understanding of mathematics.

As a participant in this study, you will be asked to take part in a number of activities. First, you will engage in a 26 item, multiple-choice pre-test on mathematics knowledge for teaching. This assessment will take approximately 45 minutes and will be held during the third week in September. Second, you will engage in an interview to discuss how you interpreted and thought about the specific mathematics questions on the Content Exam. This interview will take place at a mutually agreed upon time in the next two weeks, and will be video recorded. Third, you will engage in a paired problem-solving interview, which will last approximately one hour. During this time, you and a partner will work together to solve a number of mathematics problems. This interview will take place at a mutually agreed upon time in the last two weeks in October, and will be video recorded. Fourth, in the second term, after your first student teaching practicum, you will participate in a second paired problem-solving interview. This will last approximately one hour and will take place at a mutually agreed upon time at the end of January or beginning of February. It will also be video recorded. Fifth, at the end of your mathematics skills course, you will be invited to complete a short questionnaire about your perceptions regarding your mathematical growth and development. This should take about half an hour to complete. Sixth, you will engage in a multiple-choice post-test on mathematics knowledge for teaching. This assessment will take approximately 45 minutes and will be held during the last week in February. Finally, after the retake of the Mathematics Content Exam, you may be invited to participate in a follow-up interview to discuss how you interpreted and thought about the specific mathematics questions and/or the questionnaire. This interview will take place at a mutually agreed upon time and will be held during the middle of March. It will last approximately one hour and will be video recorded. Through these various interviews, I anticipate that you will have the opportunity to develop your mathematical understanding as you engage in mathematical discussions.

In addition to these activities, I would like permission to talk to your course instructor about the mathematical strategies and thinking used during the course. These discussions would focus on the general development of mathematical understanding and would not focus on evaluating individuals. Finally, I would like your permission to have access to the following materials: your course problem-solving journal, the course tracking sheets, your mathematics content exam, and your retake of the mathematics content exam. Through these materials, I hope to develop a clearer picture of the different stages of mathematical understanding.

The total time commitment for this study is approximately six hours. Participants who complete the entire study will receive $60 each for their services.

In order to avoid any conflict of interest with students in my mathematics methods sections, I will not analyze their data until after they have successfully passed the mandatory content exam.

You may withdraw from this study at any time, without any penalty. I do not anticipate any negative consequences as a result of participation in this study. All data will remain confidential and be viewed only by the research team which consists of Alex Lawson, a trained research assistant, and me. The hard data will be stored in a locked cabinet at Lakehead University for seven years. Soft data will be securely stored on a hard drive and will be password protected. You will remain anonymous in the final report, with all identifying marks changed. A report of the research will be made available upon request. Should you have any further questions please contact me at 343-8722 or Dr. Alex Lawson at 343-8720.

Thank you for considering participation in the project.
Wendy Stienstra

Participant Consent Form
Mathematics Content Knowledge Exam

I, ________________________________, agree to participate in the research study:
(print name)

Developing Understanding: Preservice Elementary Teachers' Changing Conceptions of Mathematics, conducted by Wendy Stienstra of Lakehead University. I have read and understood the preceding project description. I am aware that there are several aspects to this project: I may complete a multiple-choice pre-test and post-test; I may volunteer to participate in a follow-up interview after the initial Mathematics Content Exam which will be video recorded; I may volunteer to participate in two paired problem-solving interviews which will be video recorded; I may complete a short questionnaire about my perspectives on my mathematics development; and I may volunteer to participate in a follow-up interview after the Mathematics Content Exam Retake, which will be video taped. I understand that I am free to withdraw from this study at any time, without penalty of any kind. I understand that all data will remain confidential and that my anonymity is guaranteed in the final report. I understand that there is no apparent risk to me and that the data will be securely stored at Lakehead University for seven years.

I give Wendy Stienstra permission to access the following materials for this study:

- ☐ course problem-solving journals
- ☐ course tracking sheets
- ☐ discussions with course instructor about mathematics strategies and thinking
- ☐ Mathematics Content Exam
- ☐ Mathematics Content Exam Retake

____
(Signature) (Date)

Appendix C: Additional Video Use Consent

Tel: (807)343-8964
Fax: (807) 346-7771
wendy.stienstra@lakeheadu.ca

June 2007

Dear Research Participants,

Thank you for your participation in my research study concerning the mathematical understandings of pre-service elementary teachers, entitled *Developing Understanding: Pre-service Elementary Teachers' Changing Conceptions of Mathematics*. Over the past year I have been privileged to see the great mathematical strides you have made in your understanding of number sense and numeration.

As a participant in this study you engaged in four video recorded interviews: the follow-up to the initial mathematics content exam, two paired problem-solving interviews, and the follow-up to the retake of the mathematics content exam. Through these interviews and the other activities of the project I hope to develop a clear picture of the different stages of mathematical understanding for adult learners engaged in upgrading their basic mathematical concepts.

The knowledge that can be gained from the written materials and video recordings of your work is invaluable for providing information that will be useful in improving mathematics instruction and learning for both pre-service and in-service elementary mathematics teachers. I would like to be able to show segments of the video recordings and written work at research conferences and in educational settings such as professional development or pre-service methods classes. I believe that seeing the video clips of the various ways of thinking about mathematical problems, rather than just reading about the methods used, is critical. The clips will be used to demonstrate the development of mathematical strategies, benchmarks, and big ideas in pre-service elementary teachers.

If you are willing to have video segments of you and your work shown at scientific conferences or in educational settings, please indicate this on the consent form. You may indicate what uses of the video recordings you are willing to consent to by initialling any number of the various options. I will only use the video recordings in ways that you agree to. If you do not want your data used for any other purposes than my written dissertation project and ensuing journal articles, it will be stored securely at Lakehead University and destroyed after seven years.

Please use the enclosed stamped and self-addressed envelope to return the video consent form. If you would like more information about the video data or my research project, please feel free to contact me at 343-8964 or Dr. Alex Lawson at 343-8720.

I have fully appreciated working with you on this project and want to congratulate you on your mathematical growth. I wish you the best in your future careers. Thank you again for your participation.

Sincerely,

Wendy Stienstra

Video Use Consent Form
Pre-service Elementary Teachers' Changing Conceptions of Mathematics

As a participant in the research study, *Developing Understanding: Pre-service Elementary Teachers' Changing Conceptions of Mathematics*, conducted by Wendy Stienstra, please indicate what additional uses of the video data you are willing to consent to by initialling any number of the options below. The video data will only be used in ways that you agree to. In any use of the video data your name will not be identified.

Segments of the video can be used for scientific/professional publications.

- ❖ *please initial*: ____Yes ____No

Segments of the video can be shown at research conferences.

- ❖ *please initial*: ____Yes ____No

Segments of the video can be used for educational professional development purposes for in-service teachers.

- ❖ *please initial*: ____Yes ____No

Segments of the video can be used in classrooms with pre-service teachers.

- ❖ *please initial*: ____Yes ____No

I wish to have my face 'blacked' out on the video-clip.

- ❖ *please initial*: ____Yes ***OR***,

I do not need to have my face 'blacked' out on the video-clip.

- ❖ *please initial*: ____Yes

Print Name: ____

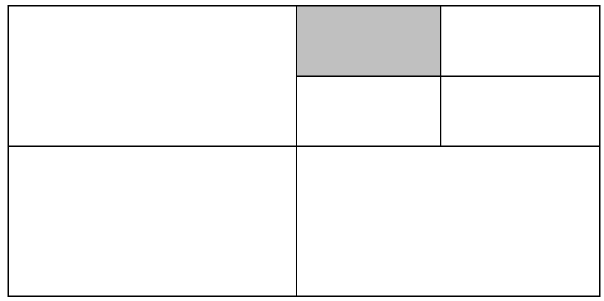
Signature: ___________________________________ Date________________

The extra copy of this consent form is for you to keep.

Appendix D: Pre/Post Tests

Mathematics Content Exam 2006 – Numeracy Items

1. What fraction is the shaded area of the whole (i.e. the largest rectangle)?



2. Three tenths of the P/J class of 240 went to 'Tribes'. One third of the 240 went to a workshop on 'Portfolio Writing'. All the rest stayed home for extra sleep. How many stayed home?

3. We will be increasing the present class of 180 students in P/J by 25% next year. How many students will there be in next year's PJ class?

4. Of the following fractions: 3/4, 5/12, 2/3, 3/2, 2/5 5/8 which two fractions (and only two) will give a sum that is less than 1?

5. Rita had 4 kilograms of peanuts. She wanted to make bags of peanuts for her friends. She put 2/3 kg of peanuts in each bag. How many bags did she make, if she used all the peanuts?

6. There are 15 women on the university track team. One sixth of the team is men. How many students are on the university track team?

7. There is a ratio of three male professors to one female professor on faculty. If there are 20 professors in all, how many are women and how many are men?

8. The average car is 6 metres long. How many whole cars can you park in a single lane bumper to bumper in 1.1 km?

9. Mr. Sander's Boy Scout troop made 900 pieces of fudge to sell. They plan to sell each box for $1.00. The total expense for the project was $25.00. They want to make a profit of $50.00. How many pieces should they put in each box? (Use all the pieces and put the same number in each box.)

10. How long would it take you to give away ten thousand dollars if you gave it away at a rate of one dollar a minute? Give your answer in days, hours, and minutes (e.g., 4 days, 5 hours and 3 minutes).

Mathematics Retake Exam – March 2007 – Numeracy Items

1. Maria is training for the Olympics. She has a workout that is 2/10th running, 1/3 treadmill and the rest is weight training. If she works out 180 hours a month how much time does she spend weight training each month?

2. I have read 3/8ths of my 560 page book. If I read at a rate of 10 pages a day, how many days will it take me to finish reading the remainder of the book?

3. We will be decreasing the present class of 180 students in P/J by 15% next year. How many students will there be in next year's PJ class?

4. Put the following fractions in order from smallest to largest: 5/4, 4/5, 4/3, 3/8, 7/12.

5. You are baking muffins. The recipe for 12 muffins calls for 11/2 cups of flour. You want to make 18 muffins. How much flour will you need?

6. Sue and her younger sister Amy work together boxing cookies for sale at the local bakery. Sue can make up 4 boxes of cookies for every 3 that her younger sister does. At the end of their shift they had 49 boxes in total. How many boxes did Sue make?

7. You have a roll of stamps that is 1.3 metres long. Each stamp is 2cm. long. How many stamps are in the roll?

8. I buy grapefruit at 60 cents each. How much do I have to sell each grapefruit for in order to make a total profit of $2.76 on a dozen grapefruit?

9. You have 23 kg of chocolates. Each individual chocolate weighs .2 kg. You want to make up as many bags as possible, each with exactly 3 kg of chocolate. How many chocolates should you put in each bag? How many chocolates are left?

10. Frieda runs at a rate of 2.5 metres per second. How long will it take her to run 5 km? Answer the question in minutes and seconds, for example, 20 minutes and 10 seconds.

Appendix E: Follow-up Interview Questions

1. What do you know about multiplication?

2. Explain how you would calculate 28 x 51. Can you think of any other ways to do this problem?

3. What is division? Can you think of a second way to explain what division is?

4. In long division carried out as in the example below, the sequence *divide, multiply, subtract, bring down* is repeated. Explain what information the *multiply* step and the *subtract* step provide and how they contribute to arriving at the answer.

```
       __59
  12 √ 715
        60
       115
       108
         7
```

5. What are fractions?

6. How would you represent the fraction 3/4? Can you think of any other ways to represent this fraction?

Appendix F: Paired Problem Solving Interview Questions

Paired Problem-Solving Interview #1 – October/November 2006

1. The Vancouver Aquarium has three beluga whales: small, medium and large. The medium whale eats 2 times as much as the smallest; the largest eats 3 times as much as the smallest.
   - If the smallest whale gets 9 kg of food, how many kilograms of food will the other two whales receive?
   - If the medium whale receives 24 kg of food, how many kilograms of food will the other two whales receive?
   - If the largest receives 45 kg of food, how many kilograms of food will the other two whales receive?
   - If the total amount of food was 60 kg, how many kilograms did each of the whales receive?

2. You purchased $18.00 worth of CD jewel cases. Each case is 5.2 mm thick and costs 24 cents. How tall is your stack of CD cases?

3. Here are two multiplication problems that have the same answer. 18 x 24; 24 x 18. Find at least two other multiplications which also have this answer. Explain your reasoning.

4. Find three different division problems with a quotient (answer) of 4 R3. Explain how you know they are correct.

5. Is 1/2 + 1/3 = 2/5? Explain. Can you give me at least two different explanations?

6. Jesse was planning the amount of time he would devote to each strand in his 120 hour math course. He planned to spend 0.4 of the time on measurement and data management, 15% of the time on algebra, 3/8 of the time on numeracy, and the remainder of the time on geometry. How many hours were spent on each of the four strands?

7. One out of every eight foreign stamps in Arnie's stamp collection is Canadian. He has 720 stamps in his collection. How many foreign stamps does he have?

8. Put the following fractions in order from smallest to largest: 5/6, 6/7, 4/11, 4/12, 7/5, 6/8. Explain your reasoning.

9. Carlos had 3/4 of a bag of chips left after a party. His son found the bag and ate 1/3 of the remaining chips. How much of the bag of chips did he leave for Carlos?

10. Maria has $330.00 to purchase cat food for the animal shelter. Bob's store sells 12 cans of cat food for $15.00. How many cans of cat food can she purchase?

    - Can you find a second method to solve this problem?

Paired Problem-Solving Interview #2 – January/February 2007

1. Our hockey team bought grapefruit at a cost of 55 cents each. We plan to sell them to earn money for additional equipment. We want to make a total profit of $3.00 on every dozen grapefruit sold. How much do we have to sell each grapefruit for?

2. Which of the following multiplication statements will give the same answer as 36 x 24? Explain your thinking. A) 35 x 25 B) 72 x 12 C) 9 x 96 D) 30 x 30

3. Roberta has knit 260 rows of the afghan she is making for a friend. She still has 1/5 of the afghan to finish.
   a. How many rows does she still have to knit?
   b. If she knits 14 rows per day, how many days, in total, will it take for her to knit the entire afghan?

4. Without using division determine which of the two fractions is larger. Explain your reasoning.
   a. 3/7 or 3/8
   b. 7/15 or 11/20
   c. 14/15 or 17/18 (use 5/6 or 6/7 if not used in interview #1)

5. Maria and her son Owen were decorating cookies for the bake sale. For every 8 cookies that Owen decorated, Maria decorated 30. If Maria completed a total of 75 cookies by herself, how many cookies did Owen complete?

6. Olivia making a special trail mix for her next hiking trip. The recipe makes 640 grams of the mixture. It consists of 1/8 raisins, 0.3 sunflower seeds, 25% peanuts, with the remainder being a dried fruit mixture. How many grams must she use of each of the ingredients?

7. Is 1/4 between 1/2 and 1/3? Explain your thinking in two different ways.

8. What fraction is exactly halfway between 1/2 and 1/3?
   a. Find a second fraction between 1/2 and 1/3.

9. Avery has 5 1/2 metres of ribbon. She would like to cut up small ribbons for her friends. How many ribbons can she make if each small ribbon is 3/8 of a metre? How much ribbon will be left over?

10. Rowan wants to download some music from the Internet. However, at home she only has a dialup connection, which downloads 6.8 kB every second.
    a. How long will it take her to download her favourite song, which is 5746 kB (5.6 MB)? (give your answer in minutes and seconds)
    b. How much time would she save if she had a high-speed internet connection with a download rate of 610 kB/sec?

Appendix G: Completion of Problem-Solving Fraction Items

Completion of Problem-Solving Interview Questions

Item		Brenda & Olivia	Diane & Mark	Erica & Lynsey	Gabriela & Grace	Megan & Bryann	Isabelle & Tanya	Irene	Valery
	Problem-solving #1								
20	total 60	✔	✔	✔ ☑	✔	✔	✔	✔	✔
21	total 90	✔	✔	☑		✔	✔	✔	✔
25	1/2 + 1/3 = 2/5	✔	✔	✔	☑	✔	✔	✔	✔
26	math course	✔	✔	✔	✔	✔	✔	✔	
27	stamp collection	✔	✔	✔	☑	✔		✔	☑
28	order fractions		✔	✔	✔	✔		✔	
29	bag of chips	☑	✔	✔		☑		✔	☑
	Problem-solving #2								
33	rows to knit	☑	☑	☑	☑	☑	☑	☑	☑
35	3/7 or 3/8	☑	☑	☑	☑	☑	☑	☑	☑
36	7/15 or 11/20	☑	☑	☑	☑	☑	☑	☑	☑
37	14/15 or 17/18 (5/6 or 6/7)	☑	☑	☑	☑	☑	☑	☑	☑
38	ratio cookies	☑	☑	☑	☑	☑	☑	☑	☑
39	trail mix		☑	☑				☑	
40	is 1/4 between		☑	☑		☑		☑	
41	halfway between		☑			☑		☑	
42	2nd halfway between		☑			☑			
43	ribbon		☑						

Note: ✔ - completed during the first problem-solving interview
☑ - completed during the second problem-solving interview

Appendix H: Open Ended Questionnaire

1. What do you now understand about primary/junior mathematical concepts from the number sense strand (for example, division, multiplication, fractions, decimals, percents, ratios, and proportion) that you did not realize before this class?

2. Are there any basic mathematical concepts in the area of numeracy that you feel you still do not understand well enough? Please explain your response.

3. What is the most significant change that happened in your mathematical understanding during this course?

4. When you compare your experiences in the mathematics skills development course with your experiences in the mathematics methods course (Ed 4050), how was your learning of mathematical concepts different in the two courses?

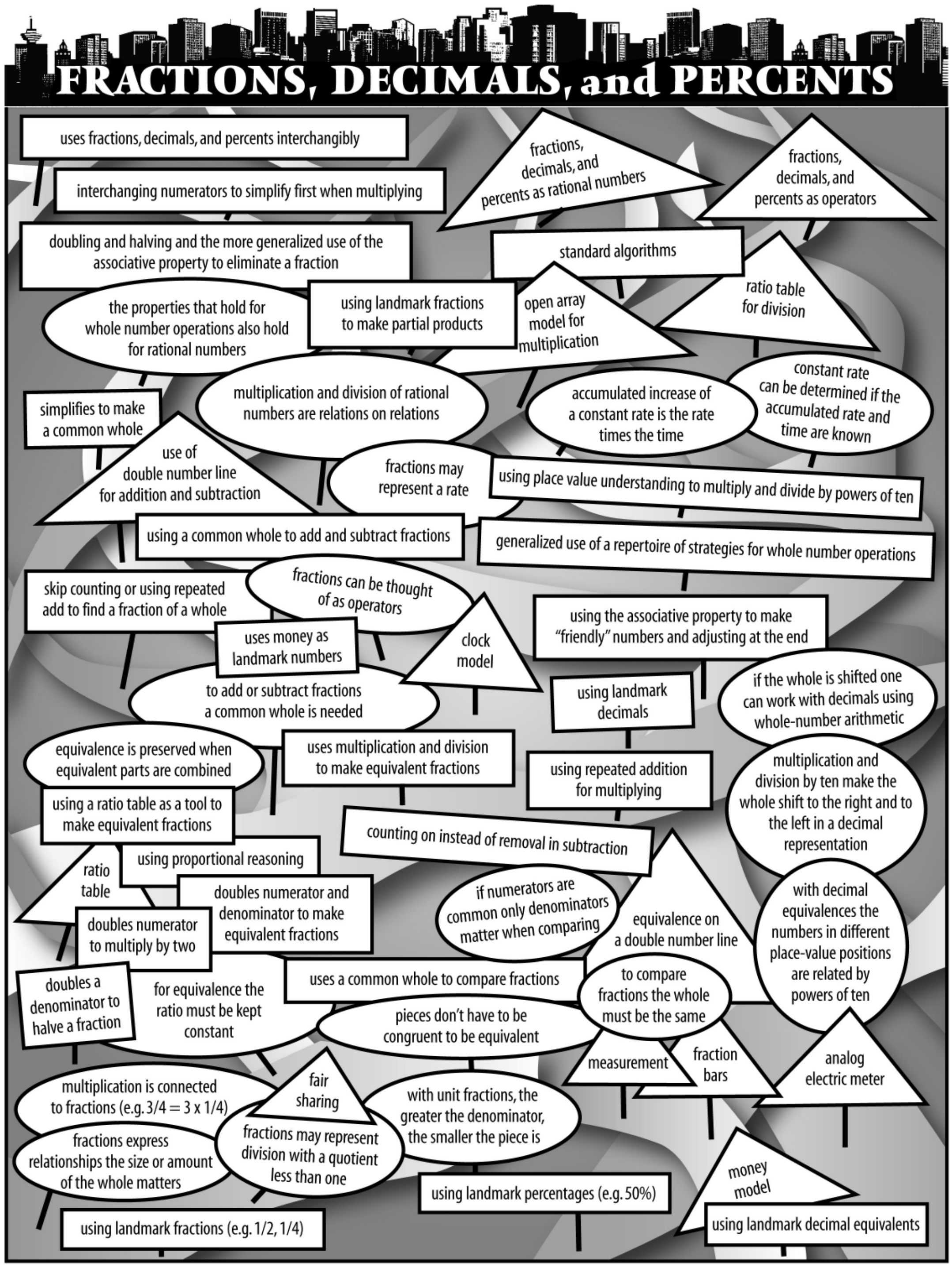
5. Do you think the mathematics methods course would have been sufficient for you to develop a deeper understanding of the mathematics concepts? Explain your response.

Appendix I: Fosnot and Dolk's (2002, 2007) Landscape of Learning Models

Fractions, Decimals, and Percents (Fosnot & Dolk, 2002, pp. 136-137)

Strategies	Big Ideas	Models
• uses landmark fractions • uses landmark percents • uses landmark decimals • uses common fractions • doubles numerator and denominator to make equivalent fractions • keeps ratio constant • uses ratio table as tool to make equivalent fractions • uses multiplication and division to make equivalent relations • doubles numerator to multiply by two • doubles denominator to halve a fraction • uses money as landmark numbers • uses a common whole to add and subtract • uses ratio as an operator to make equivalent fractions • gets rid of fraction when possible • multiplies numerators and denominators when multiplying fractions • swapping and reducing • uses a familiar landmark whole • invert and multiply when dividing • compensating with decimal addition and subtraction • uses fraction and decimals and percents interchangeably	• Fractions are division (e.g., 3 subs shared with 4 kids • Multiplication is connected to fractions • Fractions are part/whole relations • The whole matters: fractions are relations • For equivalence the ratio must be kept constant • To compare fractions, the whole must be the same • Pieces don't have to be congruent to be equivalent • The bigger the denominator the smaller the piece • If the denominators are common only the numerators matter when comparing • If the numerators are common only the denominator matters when comparing • Multiplication and division of rational numbers are relations on relations • Decimals are based on place value	• fair sharing partitive division • measurement equivalent division • percentage equivalents • decimal equivalents • ratio table • fractions as part/whole relations • double number line • clock model • arrays • generalizing the models • fractions, decimals, and percents as rational numbers • fractions, decimals, and percents as operators

Fosnot's (2007) Landscape of Learning Model



The landscape of learning: fractions, decimals, and percents on the horizon showing landmark strategies (rectangles), big ideas (ovals), and models (triangles).

Retrieved from: http://www.contextsforlearning.com/samples/46LandsapeofLearning.pdf

Appendix J: Primary Documents by Instrument Items

Item Number_ Question Code (# of pds)	Primary Documents: File Name	
Baseline Content Exam		
04_bag of peanuts (14):	P 4: ce1_Aev06_brenda_04.mpg P20: ce1_Aev06_bryann_04.mpg P36: ce1_Aji06_diane_04.mpg P52: ce1_Bal06_erica_04.mpg P66: ce1_Aws06_gabriela_04.mpg P81: ce1_Aws06_grace_04.mpg P96: ce1_Bwsr06_irene_04.mpg	P121: ce1_Aal06_lynsey_04.mpg P137: ce1_Bji06_mark_04.mpg P154: ce1_Aev06_megan_04.mpg P170: ce1_Bev06_olivia_04.mpg P186: ce1_Bwsr06_tanya_04.mpg P202: ce1_Bev06_valery_04.mpg P564: isa_ce_9b.jpg
05_fraction picture (14):	P 5: ce1_Aev06_brenda_05.mpg P21: ce1_Aev06_bryann_05.mpg P37: ce1_Aji06_diane_05.mpg P53: ce1_Bal06_erica_05.mpg P67: ce1_Aws06_gabriela_05.mpg P82: ce1_Aws06_grace_05.mpg P97: ce1_Bwsr06_irene_05.mpg	P111: ce1_Bwsr06_isabelle_05.mpg P122: ce1_Aal06_lynsey_05.mpg P138: ce1_Bji06_mark_05.mpg P155: ce1_Aev06_megan_05.mpg P171: ce1_Bev06_olivia_05.mpg P187: ce1_Bwsr06_tanya_05.mpg P203: ce1_Bev06_valery_05.mpg
06_sum of fractions (15):	P 6: ce1_Aev06_brenda_06.mpg P22: ce1_Aev06_bryann_06.mpg P38: ce1_Aji06_diane_06.mpg P54: ce1_Bal06_erica_06.mpg P68: ce1_Aws06_gabriela_06.mpg P83: ce1_Aws06_grace_06.mpg P98: ce1_Bwsr06_irene_06.mpg P123: ce1_Aal06_lynsey_06.mpg	P139: ce1_Bji06_mark_06.mpg P156: ce1_Aev06_megan_06.mpg P172: ce1_Bev06_olivia_06.mpg P188: ce1_Bwsr06_tanya_06.mpg P204: ce1_Bev06_valery_06.mpg P559: isa_ce_10b.jpg P560: isa_ce_10b2.jpg
07_tribes (14):	P 7: ce1_Aev06_brenda_07.mpg P23: ce1_Aev06_bryann_07.mpg P39: ce1_Aji06_diane_07.mpg P55: ce1_Bal06_erica_07.mpg P69: ce1_Aws06_gabriela_07.mpg P99: ce1_Bwsr06_irene_07.mpg P124: ce1_Aal06_lynsey_07.mpg	P140: ce1_Bji06_mark_07.mpg P157: ce1_Aev06_megan_07.mpg P173: ce1_Bev06_olivia_07.mpg P189: ce1_Bwsr06_tanya_07.mpg P205: ce1_Bev06_valery_07.mpg P557: grace_ce_13b.jpg P562: isa_ce_13b.jpg
08_track team (14):	P 8: ce1_Aev06_brenda_08.mpg P24: ce1_Aev06_bryann_08.mpg P40: ce1_Aji06_diane_08.mpg P56: ce1_Bal06_erica_08.mpg P70: ce1_Aws06_gabriela_08.mpg P84: ce1_Aws06_grace_08.mpg P100: ce1_Bwsr06_irene_08.mpg	P125: ce1_Aal06_lynsey_08.mpg P141: ce1_Bji06_mark_08.mpg P158: ce1_Aev06_megan_08.mpg P174: ce1_Bev06_olivia_08.mpg P190: ce1_Bwsr06_tanya_08.mpg P206: ce1_Bev06_valery_08.mpg P561: isa_ce_12b.jpg
09_ratio professors (15):	P 9: ce1_Aev06_brenda_09.mpg P25: ce1_Aev06_bryann_09.mpg P41: ce1_Aji06_diane_09.mpg P57: ce1_Bal06_erica_09.mpg P71: ce1_Aws06_gabriela_09.mpg P85: ce1_Aws06_grace_09.mpg P101: ce1_Bwsr06_irene_09.mpg P112: ce1_Bwsr06_isabelle_09.mpg	P126: ce1_Aal06_lynsey_09.mpg P142: ce1_Bji06_mark_09.mpg P143: ce1_Bji06_mark_09a.mpg P159: ce1_Aev06_megan_09.mpg P175: ce1_Bev06_olivia_09.mpg P191: ce1_Bwsr06_tanya_09.mpg P207: ce1_Bev06_valery_09.mpg

10_pj class increase (14):	P10: ce1_Aev06_brenda_10.mpg P26: ce1_Aev06_bryann_10.mpg P42: ce1_Aji06_diane_10.mpg P58: ce1_Bal06_erica_10.mpg P72: ce1_Aws06_gabriela_10.mpg P86: ce1_Aws06_grace_10.mpg P102: ce1_Bwsr06_irene_10.mpg	P113: ce1_Bwsr06_isabelle_10.mpg P127: ce1_Aal06_lynsey_10.mpg P144: ce1_Bji06_mark_10.mpg P160: ce1_Aev06_megan_10.mpg P176: ce1_Bev06_olivia_10.mpg P192: ce1_Bwsr06_tanya_10.mpg P208: ce1_Bev06_valery_10.mpg
15_fractions (13):	P15: ce1_Aev06_brenda_15.mpg P31: ce1_Aev06_bryann_15.mpg P47: ce1_Aji06_diane_15.mpg P63: ce1_Bal06_erica_15.mpg P77: ce1_Aws06_gabriela_15.mpg P91: ce1_Aws06_grace_15.mpg P107: ce1_Bwsr06_irene_15.mpg	P132: ce1_Aal06_lynsey_15.mpg P149: ce1_Bji06_mark_15.mpg P165: ce1_Aev06_megan_15.mpg P181: ce1_Bev06_olivia_15.mpg P197: ce1_Bwsr06_tanya_15.mpg P213: ce1_Bev06_valery_15.mpg
16_represent 3/4 (14):	P16: ce1_Aev06_brenda_16.mpg P32: ce1_Aev06_bryann_16.mpg P48: ce1_Aji06_diane_16.mpg P64: ce1_Bal06_erica_16.mpg P78: ce1_Aws06_gabriela_16.mpg P92: ce1_Aws06_grace_16.mpg P108: ce1_Bwsr06_irene_16.mpg	P133: ce1_Aal06_lynsey_16.mpg P150: ce1_Bji06_mark_16.mpg P166: ce1_Aev06_megan_16.mpg P182: ce1_Bev06_olivia_16.mpg P198: ce1_Bwsr06_tanya_16.mpg P214: ce1_Bev06_valery_16.mpg P568: isa_fuce_frac.jpg
Problem-Solving Interview #1		
20_total 60 (8):	P229:Ps1_AB06_dia_mar_20.mpg P243:Ps1_BA06_eri_lyn_20.mpg P256:Ps1_AA06_gab_gra_20.mpg P265:Ps1_B06_irene_20.mpg	P278:Ps1_BB06_isa_tan_20.mpg P288:Ps1_AA06_meg_bry_20.mpg P300:Ps1_B06_valery_20.mpg P332:Ps2_BA07_eri_lyn_20.mpg
21_total 90 (7):	P219:Ps1_AB06_bre_oli_21.mpg P230:Ps1_AB06_dia_mar_21.mpg P266:Ps1_B06_irene_21.mpg P279:Ps1_BB06_isa_tan_21.mpg	P289:Ps1_AA06_meg_bry_21.mpg P301:Ps1_B06_valery_21.mpg P333:Ps2_BA07_eri_lyn_21.mpg
25_1/2 + 1/3 = 2/5 (8):	P223:Ps1_AB06_bre_oli_25.mpg P234:Ps1_AB06_dia_mar_25.mpg P247:Ps1_BA06_eri_lyn_25.mpg P270:Ps1_B06_irene_25.mpg	P283:Ps1_BB06_isa_tan_25.mpg P293:Ps1_AA06_meg_bry_25.mpg P305:Ps1_B06_valery_25.mpg P345:Ps2_AA07_gab_gra_25.mpg
26_math course (8):	P224:Ps1_AB06_bre_oli_26.mpg P235:Ps1_AB06_dia_mar_26.mpg P248:Ps1_BA06_eri_lyn_26.mpg P260:Ps1_AA06_gab_gra_26.mpg	P271:Ps1_B06_irene_26.mpg P284:Ps1_BB06_isa_tan_26.mpg P294:Ps1_AA06_meg_bry_26.mpg P306:Ps1_B06_valery_26.mpg
27_stamp collection (7):	P225:Ps1_AB06_bre_oli_27.mpg P236:Ps1_AB06_dia_mar_27.mpg P249:Ps1_BA06_eri_lyn_27.mpg P272:Ps1_B06_irene_27.mpg	P295:Ps1_AA06_meg_bry_27.mpg P346:Ps2_AA07_gab_gra_27.mpg P387:Ps2_B07_valery_27.mpg
28_order fractions (5):	P237:Ps1_AB06_dia_mar_28.mpg P250:Ps1_BA06_eri_lyn_28.mpg P261:Ps1_AA06_gab_gra_28.mpg	P273:Ps1_B06_irene_28.mpg P296:Ps1_AA06_meg_bry_28.mpg
29_ bag of chips (6):	P238:Ps1_AB06_dia_mar_29.mpg P251:Ps1_BA06_eri_lyn_29.mpg P274:Ps1_B06_irene_29.mpg	P307:Ps2_AB07_bre_oli_29.mpg P374:Ps2_AA07_meg_bry_29.mpg P388:Ps2_B07_valery_29.mpg

Problem Solving Interview #2		
33_rows to knit (8):	P311:Ps2_AB07_bre_oli_33.mpg P319:Ps2_AB07_dia_mar_33.mpg P336:Ps2_BA07_eri_lyn_33.mpg P349:Ps2_AA07_gab_gra_33.mpg	P357:Ps2_B07_irene_33.mpg P368:Ps2_BB07_isa_tan_33.mpg P378:Ps2_AA07_meg_bry_33.mpg P391:Ps2_B07_valery_33.mpg
35_3/7 or 3/8 (8):	P313:Ps2_AB07_bre_oli_35.mpg P321:Ps2_AB07_dia_mar_35.mpg P338:Ps2_BA07_eri_lyn_35.mpg P351:Ps2_AA07_gab_gra_35.mpg	P359:Ps2_B07_irene_35.mpg P370:Ps2_BB07_isa_tan_35.mpg P380:Ps2_AA07_meg_bry_35.mpg P392:Ps2_B07_valery_35.mpg
36_7/15 or 11/20 (8):	P314:Ps2_AB07_bre_oli_36.mpg P322:Ps2_AB07_dia_mar_36.mpg P339:Ps2_BA07_eri_lyn_36.mpg P352:Ps2_AA07_gab_gra_36.mpg	P360:Ps2_B07_irene_36.mpg P371:Ps2_BB07_isa_tan_36.mpg P381:Ps2_AA07_meg_bry_36.mpg P393:Ps2_B07_valery_36.mpg
37_14/15 or 17/18 (8):	P315:Ps2_AB07_bre_oli_37.mpg P323:Ps2_AB07_dia_mar_37.mpg P340:Ps2_BA07_eri_lyn_37.mpg P353:Ps2_AA07_gab_gra_37.mpg	P361:Ps2_B07_irene_37.mpg P372:Ps2_BB07_isa_tan_37.mpg P382:Ps2_AA07_meg_bry_37.mpg P394:Ps2_B07_valery_37.mpg
38_ratio cookies (8):	P316:Ps2_AB07_bre_oli_38.mpg P324:Ps2_AB07_dia_mar_38.mpg P341:Ps2_BA07_eri_lyn_38.mpg P354:Ps2_AA07_gab_gra_38.mpg	P362:Ps2_B07_irene_38.mpg P373:Ps2_BB07_isa_tan_38.mpg P383:Ps2_AA07_meg_bry_38.mpg P395:Ps2_B07_valery_38.mpg
39_trail mix (3):	P325:Ps2_AB07_dia_mar_39.mpg P342:Ps2_BA07_eri_lyn_39.mpg	P363:Ps2_B07_irene_39.mpg
41_halfway between (3):	P327:Ps2_AB07_dia_mar_41.mpg P365:Ps2_B07_irene_41.mpg	P385:Ps2_AA07_meg_bry_41.mpg
42_2nd halfway between (2):	P328:Ps2_AB07_dia_mar_42.mpg	P386:Ps2_AA07_meg_bry_42.mpg
43_ribbon (1):	P329:Ps2_AB07_dia_mar_43.mpg	
Retake Content Exam		
46_muffins (14):	P396: c2_Aev07_brenda_46.mpg P407: ce2_Aev07_bryann_46.mpg P418: ce2_Aji07_diane_46.mpg P433: ce2_Bal07_erica_46.mpg P445: ce2_Aws07_gabriela_46.mpg P456: ce2_Aws07_grace_46.mpg P467: ce2_Bwsr07_irene_46.mpg	P478: ce2_Aal07_lynsey_46.mpg P491: ce2_Bji07_mark_46.mpg P507: ce2_Aev07_megan_46.mpg P524: ce2_Bev07_olivia_46.mpg P535: ce2_Bwsr07_tanya_46.mpg P546: ce2_Bev07_valery_46.mpg P569: isa_re_02.jpg
47_training (13):	P397: ce2_Aev07_brenda_47.mpg P408: ce2_Aev07_bryann_47.mpg P434: ce2_Bal07_erica_47.mpg P446: ce2_Aws07_gabriela_47.mpg P457: ce2_Aws07_grace_47.mpg P468: ce2_Bwsr07_irene_47.mpg P479: ce2_Aal07_lynsey_47.mpg	P508: ce2_Aev07_megan_47.mpg P525: ce2_Bev07_olivia_47.mpg P536: ce2_Bwsr07_tanya_47.mpg P547: ce2_Bev07_valery_47.mpg P570: isa_re_03.jpg P571: isa_re_03b.jpg
49_order fractions (14):	P399: ce2_Aev07_brenda_49.mpg P410: ce2_Aev07_bryann_49.mpg P419: ce2_Aji07_diane_49.mpg P436: ce2_Bal07_erica_49.mpg P448: ce2_Aws07_gabriela_49.mpg P459: ce2_Aws07_grace_49.mpg P470: ce2_Bwsr07_irene_49.mpg	P481: ce2_Aal07_lynsey_49.mpg P492: ce2_Bji07_mark_49.mpg P509: ce2_Aev07_megan_49.mpg P527: ce2_Bev07_olivia_49.mpg P538: ce2_Bwsr07_tanya_49.mpg P549: ce2_Bev07_valery_49.mpg P573: isa_re_05.jpg

50_pages book (11):	P400: ce2_Aev07_brenda_50.mpg P411: ce2_Aev07_bryann_50.mpg P437: ce2_Bal07_erica_50.mpg P449: ce2_Aws07_gabriela_50.mpg P460: ce2_Aws07_grace_50.mpg P471: ce2_Bwsr07_irene_50.mpg	P482: ce2_Aal07_lynsey_50.mpg P528: ce2_Bev07_olivia_50.mpg P539: ce2_Bwsr07_tanya_50.mpg P550: ce2_Bev07_valery_50.mpg P574: isa_re_07.jpg
53_ratio boxes (13):	P403: ce2_Aev07_brenda_53.mpg P414: ce2_Aev07_bryann_53.mpg P440: ce2_Bal07_erica_53.mpg P452: ce2_Aws07_gabriela_53.mpg P463: ce2_Aws07_grace_53.mpg P474: ce2_Bwsr07_irene_53.mpg P485: ce2_Aal07_lynsey_53.mpg	P494: ce2_Bji07_mark_53.mpg P511: ce2_Aev07_megan_53.mpg P531: ce2_Bev07_olivia_53.mpg P542: ce2_Bwsr07_tanya_53.mpg P553: ce2_Bev07_valery_53.mpg P577: isa_re_13.jpg
55_decrease pj class (14):	P405: ce2_Aev07_brenda_55.mpg P416: ce2_Aev07_bryann_55.mpg P422: ce2_Aji07_diane_55.mpg P442: ce2_Bal07_erica_55.mpg P454: ce2_Aws07_gabriela_55.mpg P465: ce2_Aws07_grace_55.mpg P476: ce2_Bwsr07_irene_55.mpg	P487: ce2_Aal07_lynsey_55.mpg P496: ce2_Bji07_mark_55.mpg P513: ce2_Aev07_megan_55.mpg P533: ce2_Bev07_olivia_55.mpg P544: ce2_Bwsr07_tanya_55.mpg P555: ce2_Bev07_valery_55.mpg P579: isa_re_15.jpg
57_time teaching (3):	P424: ce2_Aji07_diane_57.mpg P498: ce2_Bji07_mark_57.mpg	P515: ce2_Aev07_megan_57.mpg
65_interview re math growth (16):	P406: ce2_Aev07_brenda_65.mpg P417: ce2_Aev07_bryann_65.mpg P432: ce2_Aji07_diane_65.mpg P443: ce2_Bal07_erica_65.mpg P444: ce2_Bal07_erica_65a.mpg P455: ce2_Aws07_gabriela_65.mpg P466: ce2_Aws07_grace_65.mpg P477: ce2_Bwsr07_irene_65.mpg	P488: ce2_Aal07_lynsey_65.mpg P489: ce2_Aal07_lynsey_65a.mpg P490: ce2_Aal07_lynsey_65b.mpg P506: ce2_Bji07_mark_65.mpg P523: ce2_Aev07_megan_65.mpg P534: ce2_Bev07_olivia_65.mpg P545: ce2_Bwsr07_tanya_65.mpg P556: ce2_Bev07_valery_65.mpg

Appendix K: Coding List

Alg: div - conceptual
Alg: div - procedural
Alg: mult - conceptual
Alg: mult - procedural
Alg: multiple methods for multiplication
Ans: correct
Ans: correct, prompting provided
Ans: incomplete
Ans: incorrect
Ans: not attempted
Ans: stuck unable to proceed without help
Benefit of partner work
Bg: div - break into smaller pieces
Bg: div - extraction of groups from total
Bg: div - fair sharing
Bg: div - how many groups of
Bg: div - how many in each group
Bg: div - how many times # goes into other #
Bg: div - memorized facts
Bg: div - related to multiplication
Bg: div - separate into equal parts
Bg: fract - equal parts of whole
Bg: fract - larger denom smaller pieces
Bg: fract - measurement
Bg: fract - numbers with top and bottom
Bg: fract - parts of a whole
Bg: fract - related to division
Bg: fract - rules
Bg: mult - increasing number
Bg: mult - memorized facts
Bg: mult - number of groups
Bg: mult - related to division
Bg: mult - repeated addition
BigI: decimals
BigI: division
BigI: fractions
BigI: multiplication
BigI: percents
BigI: place value
BigI: ratio & proportion
dvmt: decimals
dvmt: division - time - remainder
dvmt: division by a fraction
dvmt: division remainder
dvmt: fraction concepts
dvmt: percentage concepts
dvmt: ratio & fraction concepts
dvmt: ratio concepts
Exp: clear and coherent
Exp: confusing/partial/may have errors
Exp: faulty or incorrect reasoning
Exp: partial/ sketchy/ on track
Exp: procedure clear and coherent
Exp: rules stated but no rationale
Exp: unable to reconstruct thinking
guided follow-up
guided follow-up: computation
impact of skills course
Mod: diagram
Mod: direct modeling
Mod: none
Mod: number line
Mod: picture
Mod: ratio table
Qte: book
Qte: formula
Qte: move decimal for %
Qte: process painful to watch
Qte: remind self to use 10s like kids
Rbk: 100ths not fully understood
Rbk: addititve vs multiplicative thinking
Rbk: assumes only one picture per problem
Rbk: buggy 3 part ratio calculation
Rbk: buggy cross multiplication for ratio
Rbk: buggy decimal multiplication
Rbk: buggy division algorithm
Rbk: buggy fraction to decimal algorithm
Rbk: buggy percent algorithm
Rbk: buggy ratio whole
Rbk: buggy ratios as fractions
Rbk: compares denominator not relation
Rbk: comprehension of word problem
Rbk: decimal calculation as procedure
Rbk: decimal division
Rbk: decimal multiplication
Rbk: div only by smaller number
Rbk: division makes less
Rbk: division of fractions procedures
Rbk: error - decimal thirds - rounding
Rbk: error - division remainder
Rbk: error - fraction to decimal
Rbk: error - invert divisor/dividend decimal
Rbk: error - km to m
Rbk: error - profit only
Rbk: error - remainder same as decimal value
Rbk: error - small calculation
Rbk: error - unequal pieces for fraction
Rbk: error - wrong divisor
Rbk: error converting units
Rbk: error km = 100 m
Rbk: fraction remainders
Rbk: fraction to decimal only as procedure
Rbk: generalized dividing by denominator regardless of whole
Rbk: how to model problem
Rbk: identifying fraction whole
Rbk: inefficient mult by 1000
Rbk: inefficient procedure
Rbk: interpret model of fraction multiplication
Rbk: language for div incorrect
Rbk: language for div inverted
Rbk: large numbers for ratio
Rbk: limited facility with number
Rbk: limited flexibility fractions, decimals, percents
Rbk: limited fraction as operator
Rbk: limited fractions as relations
Rbk: limited knowledge of decimal fractions
Rbk: limited knowledge of decimals
Rbk: limited knowledge of improper fractions
Rbk: limited knowledge of mixed fractions

Rbk: limited use of friendly numbers
Rbk: limits in multiplication knowldege
Rbk: meaning of sum is unknown
Rbk: memorized procedures forgotten
Rbk: misread question
Rbk: missing multiplicative thinking
Rbk: model limits thinking
Rbk: modelling improper fractions
Rbk: moving decimal place for div
Rbk: multiple steps
Rbk: multiplication of fractions as procedure
Rbk: multiplying gives more
Rbk: percent to fraction as procedure
Rbk: place value in division
Rbk: pressure of quick partner/interview
Rbk: procedural division by decimal
Rbk: procedural knowledge of percents
Rbk: procedure vs sense making
Rbk: relation of fractions on numberline
Rbk: rote use of fraction kit model
Rbk: same model for fraction, decimal, percent
Rbk: shortcuts for multiples of 10
Rbk: symbol for cents
Rbk: unknown value of decimal
Rbk: use of decimals to avoid fractions
Rbk: whole number thinking limits ratio
Rbk: working with tens
Rep: fract - area/ continuous
Rep: fract - clock
Rep: fract - decimal
Rep: fract - discrete
Rep: fract - equivalent
Rep: fract - measurement
Rep: fract - money
Rep: fract - percent
Sns: able to correct error
Sns: division by partitioning total
Sns: eliminates obvious incorrect possibilities
Sns: limited due to lack of knowledge of context
Sns: limited or none
Sns: looks for patterns
Sns: makes incorrect assumption
Sns: mental use of number line
Sns: personal benchmark to check
Sns: reason through errors
Sns: reasons through understanding of fractions
Sns: second solution method to check
Sns: strategy choice based on understanding of #s
Sns: thinks through context for numbers
Sns: used ruler to visualize problem
Sns: uses 2nd method when 1st doesn't work
Sns: uses inverse operation to check answer
Sns: uses prior knowledge of similar contexts
Str: used prior answer and adjusted for difference
Str: 1/2 > 2/5 with diagram
Str: 1/2 cup for 4 muffins
Str: 1/3 of 3/4 using picture model
Str: 15% = one tenth + half of one tenth
Str: 2 muffins = 1/4 cup using picture
Str: 4/5 completed = 260
Str: 60 kg div by 6 portions sm=1 med=2 lg=3
Str: 9 times table - fingers
Str: add 1/2 & 1/4 c for 6 muffins
Str: add 18 single portions flour
Str: add 3 groups of 1/4 c
Str: add 6 groups to given flour
Str: add decimal to get # choc in kg
Str: add ratios as fractions
Str: add small to med to get large
Str: add to find hours
Str: add up # choc per bag sub from total # choc
Str: add up fraction to get total kg
Str: add up m per sec to total number of m
Str: adding up
Str: addition
Str: algebra to find muliplier for ratio
Str: algebraic equation
Str: algorithm with understanding
Str: benchmark 1/2 & convert to decimals
Str: chart to show relative size of fraction pieces
Str: cm to m conversion as procedure
Str: cm to m conversion known fact
Str: cm to m mult by 100
Str: combine fractions solve for unknown group
Str: combined fractions mentally
Str: common denominator
Str: common denominator of sixths
Str: compare 1/2 + 1/3 to 2/5 using picture
Str: compare 1/2 + 1/3 to 2/5 with fraction kit
Str: compare 1/2 to 2/5
Str: compare size of piece missing when challenged
Str: compare to baking and nested cups
Str: compare to benchmark of 1/2
Str: compared relative size of denominator
Str: compares 4/10 to 10/12 with fraction kit
Str: compares 5/6 to 2/5 with fraction kit
Str: conceptual understanding of fractions
Str: construct equal pieces
Str: conversion unknown
Str: convert decimal to fraction
Str: convert fraction to decimal
Str: convert to 10ths, combine & subt from total
Str: converts decimal to fract to find # choc per kg
Str: coombines all parts - not relations
Str: counted # of sections
Str: counts out by 10s (denominator)
Str: cross multiplication
Str: developmental student generated
Str: diagrams & reasoning re fraction size
Str: distrib prop 15% of 120 = 100+10+10
Str: distribute women over 5/6
Str: div - both quotative & partitive
Str: div # women by fraction men
Str: div by 4 for 25 %
Str: div by 60 for hrs then 24 for days
Str: div by 60 for min
Str: div by denominator for each group
Str: div distance m by rate m per sec
Str: div for total per section, mult by # of sections
Str: div kg by fraction
Str: div kg choc by bag - remainder kg
Str: div large by 3 then mult by 2 for med
Str: div remainder by rate for seconds
Str: div rows by 4, so 65 still to finish

Str: div rows by 5, so 52 still to finish
Str: div to find #cases then mult by ht
Str: div to find how many groups of 8
Str: div to find mulitplicative factor
Str: div total # by # choc in bag
Str: div total cost by 12
Str: div total distance by m per min
Str: div total sec by 60 for min
Str: div women by fraction men
Str: divide by minutes in a day
Str: division
Str: division - partitive
Str: Division - quotative
Str: division and picture
Str: division as repeated subtraction
Str: double amount plus half
Str: double amounts
Str: double numberline with fractions
Str: doubling & adding ratio until total reached
Str: draw & compare fraction pictures
Str: draw and measure out portions
Str: draw fractions & visualize addition
Str: draw ratio parts until total is reached
Str: draw ratio women-men to total #
Str: draw total & circle groups
Str: equivalent fractions
Str: equivalent ratios combine for total
Str: find 1/2 of 1 and 1/2
Str: find amount of flour for 1 muffin
Str: find ave amt per whale & adjust
Str: find cost per doz
Str: find cost per doz and add to profit
Str: find fraction of remainder not whole gives 2/3
Str: find half of flour
Str: fluency - add up to subtract
Str: guess & check
Str: guess based on number of pieces
Str: half of med then mult by 3
Str: half then multiply by 5
Str: halving to get 25%
Str: identified wrong use of rule
Str: increase ratio parts increments of 1
Str: km to m conversion as procedure
Str: km to m mult by 1000
Str: km to m two part procedure whole and decimal
Str: lucky guess
Str: measures out amount for equal groups
Str: mental math
Str: Mental math - sees numbers in head
Str: mixed - understanding & algorithm
Str: model division of kg into groups
Str: mult - known facts and doubling
Str: mult by 12 - trial & error - to find cost for one
Str: mult by decimal for tenth but div for third
Str: mult by fraction for each group
Str: mult by min to find m distance
Str: mult decimal x pieces of choc
Str: mult min per day x # of days
Str: mult rate by 60 to find m per min
Str: mult rate to find m per hour
Str: mult rows by 20%, so 52 still to finish
Str: mult to find total # of seconds
Str: multiplication
Str: multiplication and addition
Str: multiplication instead of division
Str: multiplies fractions
Str: multiply by decimal for %
Str: multiply by fraction for %
Str: multiply fractions
Str: multiply kg by fraction
Str: multiply part by fraction
Str: multiply total by ratio
Str: picture represents 10ths - convert all to 10ths
Str: picture represents fraction groups
Str: prior knowledge, fraction kit, 1/4+1/4=1/2
Str: procedural algorithm
Str: procedure first choice
Str: profit & expense
Str: ratio cups flour to # muffins
Str: ratio small group to total
Str: reasoning re fractions
Str: reduces fraction then draws picture and divides
Str: remainder - did not ask
Str: remainder same unit as dividend
Str: same #number pieces missing so same size fractions
Str: size of missing piece different
Str: sub 1/3 of whole bag not remainder gives 5/12
Str: sub bag from total kg - remainder kg convert to # choc
Str: sub to find hours
Str: tentative compare to 1/2 but fractions unfamiliar
Str: use 12ths to take away 1/3
Str: use ratio parts to find operator & mult
Str: use ratios of known relationship to add up to total m
Str: use simpler problem
Str: used fraction kit to compare pieces
Str: uses ratio of 1man - 6 women
Str: visual 1/3 of 3 portions
Str: visualization of fractions
Str: write out all info given
Str:quick - times = multiply