Immanants and Their Applications in Quantum Optics

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Abstract

The regular representation of S_n appears quite naturally in the combinatorial problem of the redistribution of quantum particles though an n-channel interferometer. By using tools from representation theory, it has been shown that the coincidence rate can expressed in terms of linear combinations of permuted immanants of the scattering matrix that describes the interferometer. This thesis introduces the delay matrix, whose entries are functions of the relative time delays between particles. The delay matrix is used with Gamas' theorem to determine exactly which immanants appear in the coincidence rate for a given set of time delays, which improves our understanding of the Hong-Ou-Mandel effect for many-particle systems. Both bosonic and fermionic systems are considered in this thesis.

Acknowledgements

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What one cannot fly to, one must limp to...
...The Scripture says that limping is no sin.

Friedrich Rückert's *Die beiden Gulden (The Two Coins)* [1]

1. Translated from the German by Gregory Richter as it appears in his 2011 translation of Sigmund Freud's *Beyond the Pleasure Principle*.

Contents

CHAPTER 1

Introduction

In this thesis, I investigate the combinatorial problem of the redistribution of quantum particles though an n-channel interferometer. An n-channel interferometer is an optical device where particles enter though n input channels, are scattered by an array of beam-splitters, and are then detected in n output channels. This system is modelled by a $n \times n$ unitary matrix U, called the scattering matrix. In this thesis two cases are studied; the particles are either all bosons, or they are all fermions; in each case the times of arrival of the particles, denoted by $\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots, \tau_n\}$, are arbitrary. The following figure is an example of an interferometer.

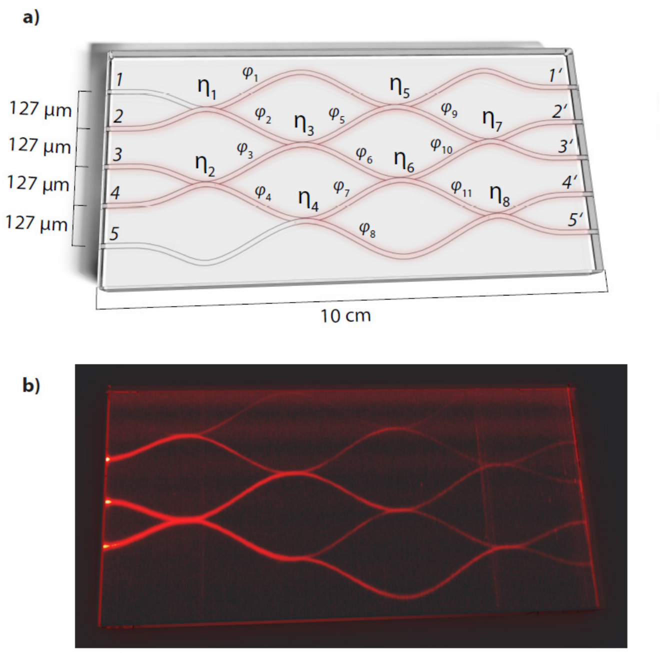


FIGURE 1.1. **a)** A schematic diagram of the interferometer used in [**22**]. The η_i's are beamsplitters and the φ_i's are phase shifters. **b)** A photograph of light scattering and propagating through the interferometer.

The original motivation for my work is the ground-breaking experiment of Hong, Ou, and Mandel [**15**], which demonstrates two-particle interference in a bosonic system. In this experiment, two photons (bosons) are injected into separate input channels of a

2−channel interferometer. The choice of interferometer in the experiment is a 50 : 50 beamsplitter. Hong, Ou, and Mandel found that when the photons enter at the same time in different channels (the particles are indistinguishable), then particles always exit from the same output channel; when the difference between the times of arrival of the two particles is large (the particles are completely distinguishable), then the photons exit from the same channel 50% of the time. Furthermore, when the time delay between the particles is continuously varied between the two extremes, the coincidence rate varies between 0 and $\frac{1}{2}$.

Instead of considering a 50 : 50 beamsplitter, the HOM effect can be generalized to an arbitrary 2-channel interferometer U. When the experiment is repeated with these new conditions it's found that when the particles enter at the same time, then the coincidence rate is given by the modulus square of the permanent of U; when the times of arrival are sufficiently far, then the coincidence rate has a 50% contribution from the modulus square of the permanent and a 50% contribution from the modulus square of the determinant. When the relative time delay between the particles is varied, the contribution of the determinant continuously varies from 0% to 50%. In this thesis I introduce the delay matrix $r(\boldsymbol{\tau})$, which is a matrix whose ij^{th} entry is a function of the relative time delay between the particles that entered from the i^{th} and j^{th} channel. Using the delay matrix, the coincidence rate for bosons can be expressed as a sum of permanents and determinants

$$\frac{1}{2}\operatorname{Perm}(r(\boldsymbol{\tau}))\left|\operatorname{Perm}(U)\right|^2 + \frac{1}{2}\operatorname{Det}(r(\boldsymbol{\tau}))\left|\operatorname{Det}(U)\right|^2. \tag{0.1}$$

In [**18**], the HOM experiment was generalized to n particles, and fermionic systems were also considered. However, it was assumed that each of the particles enter the interferometer at the same time. In the fermion case, it was found that the coincidence rate was given by the determinant of the scattering matrix, opposed to the permanent which appears in the boson case. If only two fermions with arbitrary times of arrival are considered, then the coincidence rate is

$$\frac{1}{2}\operatorname{Det}(r(\boldsymbol{\tau}))\left|\operatorname{Perm}(U)\right|^2 + \frac{1}{2}\operatorname{Perm}(r(\boldsymbol{\tau}))\left|\operatorname{Det}(U)\right|^2. \tag{0.2}$$

A 3-boson interference experiment with arbitrary arrival times is considered in [**22**]. It was shown that the coincidence rate can be expressed as a sum of linear combinations of permuted immanants of U. An immanant is a generalization of the permanent and determinant; for an $n \times n$ matrix A, there is an immanant defined for every partition of $\lambda \vdash n$. The λ-immanant of A is defined to be

$$\operatorname{Imm}^{\lambda}(A) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{0.3}$$

where χ_λ is the character of the λ-representation of S_n. Chapter 2 of this thesis covers the background material on partitions, Young tableaux, and representation theory needed to study immanants. In Chapter 3, I discuss immanants and their important properties.

In Chapter 4, I derive an expression for the coincidence rate for n particles. The respective expressions for bosons and fermions are

$$(0.4)\qquad \begin{aligned} P_b(\boldsymbol{\tau};n) &= \sum_{\lambda\vdash n}\sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda;i}\,\mathfrak{R}^\lambda\,\mathfrak{v}_{\lambda;i}, \\ P_f(\boldsymbol{\tau};n) &= \sum_{\lambda\vdash n}\sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda^*;i}\,\mathfrak{R}^\lambda\,\mathfrak{v}_{\lambda^*;i}. \end{aligned}$$

Every entry of the matrix $\mathfrak{R}^\lambda$ is a linear combination of permuted λ-immanants of the delay matrix; and each $\mathfrak{v}_{\lambda;i}$ is a vector where each entry is a linear combination of permuted λ-immanants of the scattering matrix U, describing the linear action of the interferometer on input particles. The notation $\lambda \vdash n$ means that we sum over all partitions of n, λ^* is the conjugate partition of λ, and s_λ is the dimension of the λ-representation of S_n. The linear combinations of permuted immanants can also be written in terms of elements of the invariant matrix [**3**] for the λ-representation of S_n - in physics language, the entries are called Wigner $\mathcal{D}$-functions.

The delay matrix $r(\boldsymbol{\tau})$ is a Gram matrix, so Gamas' theorem [**13**] can be used to determine which of the $\mathfrak{R}^\lambda$ terms are automatically zero based on the set of arrival times $\boldsymbol{\tau}$. A set of arrival times is associated to a partition $\mu = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_l\}$, where there are μ_i particles have the same time of arrival. I call this is the delay partition and denote it by μ_τ. Using Gamas' theorem, I will show that every entry of $\mathfrak{R}^\lambda$ is equal to 0 whenever λ is less than μ_τ in the dominance ordering of partitions. As a result, the coincidence rate can be expressed as

$$(0.5)\qquad \begin{aligned} P_b(\boldsymbol{\tau};n) &= \sum_{\lambda\trianglerighteq\mu_\tau}\sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda;i}\,\mathfrak{R}^\lambda\,\mathfrak{v}_{\lambda;i}, \\ P_f(\boldsymbol{\tau};n) &= \sum_{\lambda\trianglerighteq\mu_\tau}\sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda^*;i}\,\mathfrak{R}^\lambda\,\mathfrak{v}_{\lambda^*;i}. \end{aligned}$$

This is an important result because it allows for significant simplifications in computing coincidence rates.

CHAPTER 2

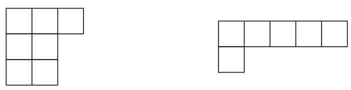
Background

1. Young Diagrams

A Young diagram is a combinatorial object that will be used extensively in this thesis. This section will be devoted to reviewing the key properties needed to discuss immanants. The references for this section are [**1**], [**2**], [**4**], and [**5**].

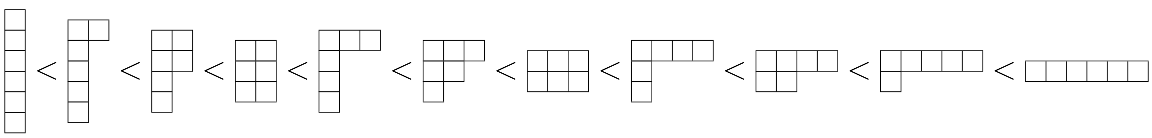
Recall that a **partition** of a positive integer n is a k-tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $\sum_{i=1}^{k} \lambda_i = n$; we call k the number of parts of a partition and n the size. The notation $\lambda \vdash n$ means that λ is a partition of n. A **Young diagram** is a visual way to write a partition using boxes. Given a partition λ, we construct the associated Young diagram by placing a row of λ_1 boxes at the top, then we add a left-justified row of λ_2 boxes is added directly below. This process is repeated for each λ_i. The partition λ will be referred to as the **shape** of the Young diagram.

Example 2.1. The following two Young diagrams are of shape $(3, 2, 2)$ and $(5, 1)$, respectively.



There are two orderings of partitions that we will refer to: they are **lexicographic order** and **dominance order**. We will use the symbol $\leq$ for lexicographic and $\trianglelefteq$ for dominance ordering to avoid confusion. The lexicographic order is a total ordering of all partitions given by $\lambda > \mu$ when the first non-zero $\lambda_i - \mu_i$ term is positive. If we are comparing two partitions that have a different number of parts, then we add zeroes at the end so the subtraction makes sense. This ordering can be used to compare any two partitions, but in this thesis we will only be concerned with comparing partitions of the same size.

Example 2.2. The partitions of 6 ordered lexicographically.

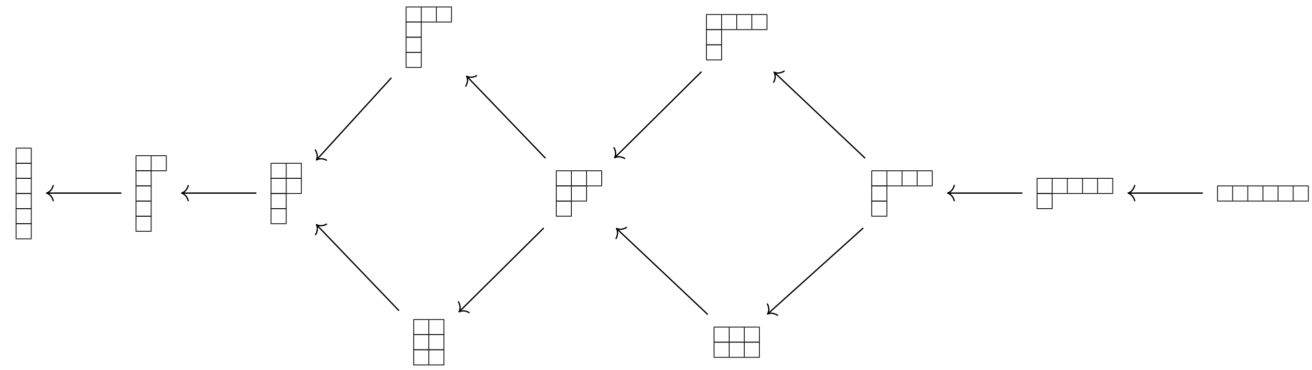


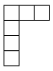
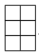
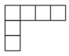
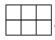
Dominance ordering is a partial ordering denoted by $\lambda \trianglerighteq \mu$ when

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad (1.1)$$

for all $i \geq 1$. The following example shows the dominance ordering of partitions of 6.

Example 2.3.

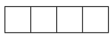


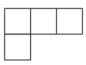
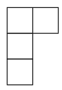
We see that this is only a partial ordering because there are pairs of partitions that cannot be compared:  and , as well as  and . For $n \leq 5$ lexicographic order and dominance ordering coincide.

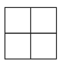
Given a partition λ, we can define its **conjugate partition**, denoted λ^*, by exchanging the rows and columns of λ. This is equivalent to reflecting the diagram λ about the main diagonal.

Example 2.4.

(1.2)

 is conjugate to 

 is conjugate to 

 is conjugate to itself

A **Young tableau** (plural tableaux) on a Young diagram of size n is a numbering of the boxes using entries from the set $\{1, 2, \ldots, n\}$. There are two specific types of tableaux that we will define here: **standard** and **semi-standard**. A standard tableau has its entries strictly increasing across rows and strictly increasing down the columns, as a result each integer from 1 to n appears exactly once in the tableau. We relax the conditions for the semi-standard tableaux: entries weakly increase along rows, but still must increase strictly down columns.

Example 2.5.

(1.3)

1	2	4
3	6	
5		

2	2	4
3	3	
5		

6	1	4
4	4	
1		

The tableau on the left is standard. The tableau in the centre is semi-standard, but fails to be standard because the entries along the first row are not strictly increasing. The tableau on the right is neither standard nor semi-standard.

For a diagram of shape λ, there are always going to be a finite number of tableaux when we consider entries in the set $\{1, 2, \ldots, n\}$. The number of standard and semi-standard tableaux are quite important and they will be referred to often in this thesis. Let $\mathcal{SYT}_\lambda$ and $\mathcal{SSYT}_\lambda$ be the sets of the standard Young tableaux and semi-standard Young tableaux of shape λ, respectively. We use the **hook-length formula** to calculate the size of these sets for a given $\lambda \vdash n$.

The number of standard tableaux of a partition $\lambda \vdash n$ is given by

$$|\mathcal{SYT}_\lambda| = \frac{n!}{\prod_{(i,j)} h_\lambda(i,j)}. \tag{1.4}$$

The sum is over every box in the diagram and $h_\lambda(i,j)$ refers to the hook length of the box in row i and column j. The hook-length of a box is the number of boxes in the same row and to the right of it, and in the same column and below it (while also counting the original box once).

Example 2.6. The following is a Young tableau where each entry is the hook-length of that box.

$$\begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \tag{1.5}$$

Applying the hook-length formula to a diagram of this shape yields $\frac{7!}{6*4*3*2} = 35$. Thus the number of standard Young tableaux of the partition $(4, 2, 1)$ is 35.

We can also compute the number of semi-standard Young tableaux by using Stanley's hook-content formula

$$|\mathcal{SSYT}_\lambda| = \frac{\prod_{i,j}(n + j - i)}{\prod_{i,j} h_\lambda(i,j)}. \tag{1.6}$$

Example 2.7. The $(i,j)^{th}$ entry of the following tableau is $(n + j - i)$

$$\begin{array}{|c|c|c|} \hline 9 & 10 & 11 \\ \hline 8 & 9 & 10 \\ \hline 7 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} . \tag{1.7}$$

The numerator is the product of these entries

$$\prod_{i,j}(9+j-i) = 239\,500\,800. \tag{1.8}$$

To find the denominator we compute the hook-lengths of each box

$$\begin{array}{|c|c|c|}\hline 6 & 4 & 2 \\ \hline 5 & 3 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}\;. \tag{1.9}$$

The product of these entries is 2160. It follows that the number of semi-standard Young tableaux of shape $(3,3,2,1)$ is $110\,880$.

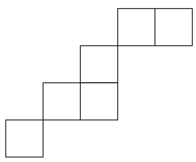
Computing the size of the sets $\mathcal{SYT}_\lambda$ and $\mathcal{SSYT}_\lambda$ by hand can be time-consuming for large λ. For partitions that are of shape $\lambda = (k,1,1,\ldots,1) \vdash n$ (hook-shaped partitions), there are closed-form expressions that can be used [**23**]

$$\begin{aligned} |\mathcal{SYT}_\lambda| &= \binom{n-1}{k-1} \\ |\mathcal{SSYT}_\lambda| &= \binom{n-k}{n-k,\,k-1,\,k-1}. \end{aligned} \tag{1.10}$$

Let λ and μ be Young diagrams with $\lambda_i \geq \mu_i$ for all i. The skew diagram of shape λ/μ is the set of squares that belong to λ but not to μ.

Example 2.8. The following is the skew diagram of shape $(5,3,3,1)/(3,2,1)$ is

(1.11)



A skew shape is disconnected if it can be partitioned into two shew shapes A and B such that none of the boxes of A share share an edge with any box in B. Otherwise the skew shape is connected. The example above fails to be connected in two places. A border strip is a skew diagram that contains no 2×2 square.

Let λ and μ be partitions of n. The border-strip tableaux of shape λ and type μ, denoted $BST(\lambda,\mu)$, is defined as follows. We begin with a Young diagram of shape λ and fill it with the integers $1,2,\ldots,k$ where the integer i occurs μ_i times, such that each label i forms a connected shape with no 2×2 square. For example, let's consider $BST(\ydiagram{4,2},\ydiagram{4,2})$. We need to fill the diagram $\ydiagram{4,2}$ with the integers $1,1,1,1,2,2$ such that each number forms a valid border strip within $\ydiagram{4,2}$ and no 2×2 squares of the same integer are formed. There is only one such border strip tableau, pictured below, which we denote by Z.

(1.12)

1	1	1	1
2	2		

The following tableaux are invalid: the former because it contains a 2×2 square that contains all ones, and the latter because the skew shape of twos isn't connected.

(1.13)

1	1	2	2
1	1		

1	1	1	2
1	2		

The height of a border strip is one less then its number of rows. Given some $T \in BST(\lambda, \mu)$, $ht(T)$ is the sum of the heights of all of its border strips. For example, the border strip tableaux Z contains two border strips, one has boxes filled with ones and the other filled with twos. Both border strips contain just one row, so they both have height 0. Adding these we get that $ht(Z) = 0$.

2. The Representation Theory of Finite Groups

I will start with a brief review of the representation theory of finite groups over $\mathbb{C}$, and then discuss the representations of S_n. References for this section are [**1**], [**6**], and [**7**].

A **representation** of a finite group G on a finite-dimensional vector space V is a homomorphism $\rho : G \to GL(V)$ from the group G to the set of automorphisms of V. Unless otherwise stated, V will be over the field $\mathbb{C}$. We call V the **representation space**. The **dimension** of a representation is the dimension of the associated representation space. Recall that $GL(V)$, sometimes written as $Aut(V)$, is the set of all bijective linear transformations $V \to V$. When a basis for V is given, then the elements of $GL(V)$ are given by matrices. To see this, suppose that V is $n-$dimensional with basis $\{e_1, e_2, \dots e_n\}$ and $T \in GL(V)$ is a linear transformation. We have that T acts on each basis vector by $T(e_i) = \sum_{j=1}^{n} a_{ij} e_j$ and the matrix corresponding to the transformation T is the matrix $(a_{ij})_{1 \leq i,j \leq n}$. When we have a basis, we have a map $\rho : G \to GL(n; \mathbb{C})$. We call such a representation a **matrix representation** of G. Before discussing subrepresentations, we need to define two terms.

Definition 2.9. Let ρ be a finite-dimensional representation of the group G acting on the vector space V. A subspace W of V is called **G-invariant** if $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$.

Definition 2.10. Let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be representations of G and $\phi : V \to W$ a map between the representation spaces. We say that ϕ is **G-linear** if the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \Big\downarrow{\scriptstyle \rho_1(g)} & & \Big\downarrow{\scriptstyle \rho_2(g)} \\ V & \xrightarrow{\phi} & W \end{array}$$

The diagram commuting is the same as saying that

$$\phi(\rho_1(g)(v) = \rho_2(g)(\phi(v)) \quad \text{for all } \ g \in G \ \text{ and } \ v \in V. \tag{2.1}$$

Suppose that $\rho : G \to GL(V)$ is a representation. If W is a G-invariant subspace of V, then $\rho|_W : G \to GL(W)$ is called a **subrepresentation** of ρ. The map ρ_W is the restriction of ρ to the subspace W, that is $\rho|_W(g)(w) = \rho(g)(w)$ for all $g \in G$ and $w \in W$. Given any representation $\rho : G \to GL(V)$, the **trivial representation** ρ_{triv} will always be a subrepresentation. The trivial representation acts on a vector space by sending every vector to itself $\rho_{triv}(g) : v \to v$ for all $g \in G$. We have the following proposition regarding subrepresentations.

PROPOSITION 2.11. *Let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be representations of G and let $\phi : V \to W$ be a G-linear map. We have that $ker(\phi)$ and $Im(\phi)$ are G-invariant subspaces of V and W, respectively.*

PROOF. To show that $ker(\phi)$ is a G-invariant subspace of V, we need to show that $\rho_1(g)(v) \in ker(\phi)$ for all $v \in ker(\phi)$. Since ϕ is G-linear we have that $\rho_2(g)(\phi(v)) = \phi(\rho_1(g)(v))$. If we assume that $v \in ker(\phi)$, then we get $\rho_2(g)(0) = \phi(\rho_1(g)(v))$. Linear transformations must map 0 to 0, which gives us $0 = \phi(\rho_1(g)(v))$. Thus $\rho_1(g)(v) \in ker(\phi)$, as desired.

To show that $Im(\phi)$ is a G-invariant subspace of W, we need to show that $\rho_2(g)(w) \in Im(\phi)$ for all $w \in Im(\phi)$. Let $w \in Im(\phi)$ and consider $\rho_2(g)(w)$. Since $w \in Im(\phi)$, we get that $w = \phi(v)$ for some $v \in V$. By the G-linearity of ϕ we have $\rho_2(g)(\phi(v)) = \phi(\rho_1(g)(v))$. Clearly the RHS of the preceding is in $Im(\phi)$, which implies $\rho_2(g)(w) \in Im(\phi)$, as desired. ☐

When we have two representations, we want to know under what conditions they are ”the same”. This is given in the following definition.

DEFINITION 2.12. Let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be representations of G. Then ρ_1 and ρ_2 are **equivalent** when there exists a vector space isomorphism $f : V \to W$ that is G-linear: $f(\rho_1(g)(v)) = \rho_2(g)(f(v))$.

Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be representations of G. We can construct larger representations from these two called the **direct sum** $\rho_1 \oplus \rho_2$ and the **tensor product** $\rho_1 \otimes \rho_2$. These representations are defined as follows. Recall that $V_1 \oplus V_2 = \{(v_1, v_2) | v_1 \in V_1 \text{ and } v_2 \in V_2\}$. The representation given by the direct sum $\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$ is via the map

$$\rho_1 \oplus \rho_2(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2)) \tag{2.2}$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. If we have matrix representations, then we can say the following about the direct sum. Given representations $\rho_1 : G \to GL(n;\mathbb{C})$ and $\rho_2 : G \to GL(m;\mathbb{C})$, their direct sum $\rho_1 \oplus \rho_2 : G \to GL(n+m;\mathbb{C})$, is given by the following map, where $\mathbf{0}$ denotes a matrix of zeroes of the appropriate size,

$$g \to \begin{bmatrix} \rho_1(g) & \mathbf{0} \\ \mathbf{0} & \rho_2(g) \end{bmatrix}. \tag{2.3}$$

Clearly, $\rho_1 \oplus \rho_2$ is $n+m$-dimensional.The tensor product of two representations $\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ is given by the map

$$\rho_1 \otimes \rho_2(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2). \tag{2.4}$$

If $\rho_1(g)$ and $\rho_2(g)$ are matrix representations that are respectively n and $m-$dimensional, then the matrix representation of the tensor product is

$$\rho_1 \otimes \rho_2(g) = \begin{bmatrix} \rho_1(g)_{1,1}\rho_2(g) & \dots & \rho_1(g)_{1,n}\rho_2(g) \\ \vdots & & \vdots \\ \rho_1(g)_{n,1}\rho_2(g) & \dots & \rho_1(g)_{n,n}\rho_2(g) \end{bmatrix}. \tag{2.5}$$

Here $\rho_1(g)_{i,j}$ denotes the ij^{th} entry of the matrix $\rho_1(g)$. Each block of this matrix is $m \times m$, which makes $\rho_1 \otimes \rho_2$ an $nm-$dimensional representation.

We will now discuss the reducibility and decomposability of representations. A representation is **irreducible** when it has no non-trivial subrepresentations, and reducible when it has a non-trivial subrepresentation. For brevity, we will refer to irreducible representations as **irreps** from this point forward. A representation is **decomposable** if it can be written as a direct product of two subrepresentations, and indecomposable if no such decomposition exists. Decomposability is a stronger condition, meaning that if a representation is decomposable, then it must be reducible, but the reverse implication need not hold. We illustrate this with the following example.

EXAMPLE 2.13. Let $\rho : \mathbb{R} \to GL(\mathbb{R}^2)$ be a representation of $\mathbb{R}$ given by the map

$$\rho(g) = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}. \tag{2.6}$$

This is a representation since $\rho(g+h) = \rho(g)+\rho(h)$. We see that $V_1 = \left\{ \begin{bmatrix} v \\ 0 \end{bmatrix} \middle| \, v \in \mathbb{R} \right\}$ is a G-invariant subspace since $\rho(g)(v_1) = v_1$ for all $v_1 \in V_1$. It follows that $\rho|_{V_1}$ is a subrepresentation, hence ρ is reducible; however, there is no complementary subspace V_2 such that

$V_1 \oplus V_2 = \mathbb{R}^2$ where V_2 is a G-invariant subspace, meaning this ρ is indecomposable. To show this, note that ρ is two-dimensional and $\rho|_{V_1}$ is one-dimensional. If there did exists a G-invariant subspace V_2 with $\rho = \rho|_{V_1} \oplus \rho|_{V_2}$, then V_2 would need to be one-dimensional, also. Moreover, we would require that a basis vector from each of V_1 and V_2 would span $\mathbb{R}^2$, which implies that $V_2 = \left\{ \begin{bmatrix} 0 \\ v \end{bmatrix} \middle| \; v \in \mathbb{R} \right\}$. Notice that for any $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V_2$, we get

$$\rho(g)\left(\begin{bmatrix} 0 \\ v \end{bmatrix}\right) = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} gv \\ v \end{bmatrix} \notin V_2, \tag{2.7}$$

which is a contradiction. Thus the representation ρ is reducible, yet indecomposable.

Notice that in the example $\mathbb{R}$ is an infinite group. We needed to reach for such an example because for finite representations of finite groups over a field of characteristic zero, reducibility and decomposability are equivalent. There are many counterexamples where we consider representations over finite fields; this area of study is called modular representation theory, and it is not within the scope of this thesis. Reducibility and decomposability being equivalent is a result of Maschke's theorem, which is proven as follows.

Theorem 2.14. *(Maschke's Theorem) Let $\rho : G \to GL(V)$ be a representation of a finite group G and let $\rho|_W : G \to GL(W)$ be a subrepresentation. Further assume that the characteristic of the underlying field $\mathbb{F}$ of V does not divide $|G|$. Then there exists a complimentary representation $\rho|_U : G \to GL(U)$ such that $\rho = \rho|_W \oplus \rho|_U$.*

Proof. Let V and W be as above. There exists a complimentary vector space U such that $V = W \oplus U$. Let $\pi_0 : V \to W$ be the projection map given by the direct sum above. This map need not be G-linear, so consider the map $\pi : V \to W$ given by

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(\pi_0(\rho(g)^{-1}(v))). \tag{2.8}$$

We claim that this map is G-linear and has the property $\pi(w) = w$ for all $w \in W$. First we prove G-linearity. We start with

$$\pi(\rho(h)(v)) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(\pi_0(\rho(g^{-1}h)(v))). \tag{2.9}$$

Instead of taking the sum over g, we can take the sum over $k = g^{-1}h$. Defining k in this way means that $g = hk^{-1}$. Putting these together we get

$$\pi(\rho(h)(v)) = \frac{1}{|G|}\sum_{k\in G}\rho(hk^{-1})(\pi_0(\rho(k)(v))). \tag{2.10}$$

Next express $\rho(hk^{-1})$ as $\rho(h)\rho(k^{-1})$, which allows us to take $\rho(h)$ outside of the sum. We are left with $\pi(\rho(h)(v) = \rho(h)(\pi(v))$, as desired. Next we need to show that $\pi(w) = w$ for all $w \in W$. Let $w \in W$, we have that

$$\pi(w) = \frac{1}{|G|}\sum_{g\in G}\rho(g)(\pi_0(\rho(g)^{-1}(w))). \tag{2.11}$$

By assumption W is a G-invariant subspace, so $\rho(g)^{-1}(w)) \in W$ for all $w \in W$. Recall that π_0 is a projection mapping, so $\pi_0(\rho(g)^{-1}(w)) = \rho(g)^{-1}(w)$. The g and g^{-1} terms make a 1, so we are left with $\pi(w) = \frac{1}{|G|}\sum_{g\in G} w$, which simplifies to $\pi(w) = w$, as desired. We have that $Im(\pi) = W$ and $ker(\pi) = U$. From proposition 2.11, we have that U is a $G-$invariant subspace and the result follows.

In the above proof it is assumed that V is a vector space over a field $\mathbb{F}$ of characteristic 0 (recall that $char(\mathbb{F})$ is the smallest $n \in \mathbb{N}$ such that $\underbrace{1+1+\cdots+1}_{n \text{ times}} = 0$, where 1 is the multiplicative identity of $\mathbb{F}$). If we were to take the characteristic of $\mathbb{F}$ to be prime, then we have a problem at the end of the proof where we have the sum $\pi(w) = \frac{1}{|G|}\sum_{g\in G} w$, in which there are $|G|$ summands of w. Over a field of characteristic 0, this is simply equal to w, but we need to be more careful in the general case. If the characteristic of $\mathbb{F}$ divides $|G|$, then $\pi(w) = 0$ for some non-zero $w \in W$, meaning our result would no longer hold. ☐

The second key theorem that we will discuss is Schur's Lemma, which tells us that we are very limited when considering homomorphisms between irreps. Proving it requires one lemma.

LEMMA 2.15. [**7**] *Every linear transformation on a finite-dimensional, complex, non-zero vector space has an eigenvalue.*

THEOREM 2.16. *(Schur's Lemma) Let V and W be vector spaces over $\mathbb{C}$, let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be irreducible representations, and let $\phi : V \to W$ be a $G-$linear map. The following statements hold.*

- *Either ϕ is an bijection or $\phi = 0$*
- *If $\rho_V \cong \rho_W$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity transformation.*

PROOF. We begin with the first statement. Assume that ϕ is not the zero map, and we will show that ϕ is a bijection. From Proposition 1 we know that $ker(\phi)$ and $Im(\phi)$ are $G-$invariant subspaces of V and W, respectively. Since ϕ is not the zero map, we have that $ker(\phi) \neq V$ and $Im(\phi) \neq \{0\}$. Since ρ_V and ρ_W are irreducible, it follows that $ker(\phi) = \{0\}$ and $Im(\phi) = W$, hence ϕ is a bijection.

For the second part, start with ϕ a bijection. From the lemma we know that ϕ must have an eigenvalue, so $\phi v = \lambda v$ for some non-zero $v \in V$ and thus the linear map $\phi - \lambda I : V \to V$ has a non-zero kernel. By assumption ϕ is G-linear and $\phi - \lambda I$ also has this property. It follows from the first part of Schur's Lemma that $ker(\phi - \lambda I) = V$, which means $\phi = \lambda I$. □

We note that the first part of this theorem doesn't use the fact that V and W are complex vector spaces, so this result holds for arbitrary vector spaces. A corollary of Maschke's theorem and Schur's lemma is the following theorem on complete reducibility.

THEOREM 2.17. *Let $\rho_V : G \to GL(V)$ be representation. There exists a decomposition of V:*

$$\rho_V = a_1\rho_1 \oplus \cdots \oplus a_k\rho_k \tag{2.12}$$

where the ρ_i are irreps and the coefficients denote how many times they occur. This decomposition is unique.

The existence of such a decomposition comes from Maschke's theorem and the uniqueness comes from Schur's lemma.

3. Character Theory

Next, character theory will be discussed, which is an invaluable tool for working with finite group representations. Given a representation $\rho : G \to GL(V)$, its **character** χ_ρ, is a function from G to $\mathbb{C}$ given by $\chi_\rho(g) = Tr(\rho(g))$. Here Tr denotes the trace of a linear transformation. The trace of a linear transformation is also not dependent on the basis of the representation space; in fact, the trace can be defined even when there is no basis. The trace in such a case is simply the sum of the eigenvalues of the linear transformation. Note that the character of a representation is a class function, meaning that it is constant on conjugacy classes. Next, a few statements on characters will be provided, including the character orthogonality theorem.

PROPOSITION 2.18. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G. The characters of these representations satisfy the following properties. Here $*$ denotes ordinary multiplication.*

- $\chi_{\rho_V \oplus \rho_W} = \chi_{\rho_V} + \chi_{\rho_W}$
- $\chi_{\rho_V \otimes \rho_W} = \chi_{\rho_V} * \chi_{\rho_W}$

PROOF. Let $g \in G$ be an arbitrary group element and consider $\chi_{\rho_V}(g)$ and $\chi_{\rho_V}(g)$. Suppose that $\{\lambda_i\}_{i\in\mathbb{I}}$ are the eigenvalues of $\rho_V(g)$ and $\{\mu_j\}_{j\in\mathbb{J}}$ are the eigenvalues of $\rho_W(g)$. When we take the direct sum of the matrices we see that $\rho_V(g) \oplus \rho_W(g)$ has eigenvalues

$\{\lambda_i + \mu_j\}_{i\in\mathbb{I}, j\in\mathbb{J}}$. Similarly, the tensor product $\rho_V(g) \otimes \rho_W(g)$ has eigenvalues $\{\lambda_i * \mu_j\}_{i\in\mathbb{I}, j\in\mathbb{J}}$. The result follows since the trace of an element is the sum of its eigenvalues. □

We can display all the information about the characters of the irreps of a group in a character table. The following is a character table for S_4. In the table, the top row is refers to the cycle decomposition of an permutation. For example, $(2, 1^2)$ refers to the class of all two-cycles.

Character \ Class	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$	(4)
$\chi_{\yng(4)}$	1	1	1	1	1
$\chi_{\yng(3,1)}$	3	1	-1	0	-1
$\chi_{\yng(2,2)}$	2	0	2	-1	0
$\chi_{\yng(2,1,1)}$	3	-1	-1	0	1
$\chi_{\yng(1,1,1,1)}$	1	-1	1	1	-1

FIGURE 2.1. Character Table for S_4

From this table we can see that $\chi_{\yng(3,1)}[(2,2)] = -1$ and $\chi_{\yng(2,1,1)}[(3,1)] = 0$.

THEOREM 2.19. *(Character Orthogonality Relation) Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G. The following equation holds, where $\cong$ denotes equivalent representations.*

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\rho_V}(g)}\chi_{\rho_W}(g) = \begin{cases} 1 \text{ if } \rho_V \cong \rho_W, \\ 0 \text{ else} \end{cases} \tag{3.1}$$

COROLLARY 2.20. *Let $\rho : G \to GL(V)$ be a representation of G. The following equation holds*

$$\sum_{g\in G}\chi_\rho(g) = \begin{cases} |G| & \text{if } \rho \text{ is the trivial representation} \\ 0 & \text{if } \rho \text{ is non-trivial} \end{cases} \tag{3.2}$$

THEOREM 2.21. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Since characters are a class function, Theorem 2.21 implies that the set of characters of the irreps of G forms a basis for the space of all class functions of G.

The characters of S_n up to $n = 10$ can be found in the appendix of [**2**]. One method to compute characters is to use the Murnaghan–Nakayama rule, which states that

$$\chi_\lambda(\mu) = \sum_{T\in BST(\lambda,\mu)} (-1)^{ht(T)}. \tag{3.3}$$

EXAMPLE 2.22. Let's compute the characters $\chi_{\tiny\yng(3,1)}[(1^4)]$ and $\chi_{\tiny\yng(3,1)}[(2,1^2)]$ and from the table on the previous page. There are three Young diagrams in the set $BST(\tiny\yng(3,1),\tiny\yng(1,1,1,1))$

$$\young(123,4) \quad \young(124,3) \quad \young(134,2) \;. \tag{3.4}$$

In all cases, the border strips that make up the diagrams are just one box and thus have height zero. By the Murnaghan–Nakayama rule we have that

$$\begin{aligned} \chi_{\tiny\yng(3,1)}[(1^4)] &= \sum_{T \in BST(\tiny\yng(3,1),\tiny\yng(1,1,1,1))} (-1)^{ht(T)} \\ &= (-1)^0 + (-1)^0 + (-1)^0 \\ &= 3. \end{aligned} \tag{3.5}$$

There are also three Young diagrams in $BST(\tiny\yng(3,1),\tiny\yng(2,1,1))$

$$\young(112,3) \quad \young(113,2) \quad \young(123,1) \;. \tag{3.6}$$

The first two diagrams only consist of border strips of height 1, but in the third diagram the border strip of 1s has height 2. It follows that

$$\begin{aligned} \chi_{\tiny\yng(3,1)}[(2,1^2)] &= \sum_{T \in BST(\tiny\yng(3,1),\tiny\yng(2,1,1))} (-1)^{ht(T)} \\ &= (-1)^0 + (-1)^0 + (-1)^1 \\ &= 1. \end{aligned} \tag{3.7}$$

4. The Representation Theory of the Symmetric Group

The simplest non-trivial example of a representation of a finite group over $\mathbb{C}$ is the symmetric group S_n. Recall that in S_n conjugacy classes are completely determined by cycle shapes. For example, the set of all two-cycles form a conjugacy class, as do the cycles of the form $(a,b)(c,d,e)$. Cycle shapes in S_n are just partitions of n, so a more convenient notation is to use Young tableaux of n boxes. For example here are the five conjugacy classes of S_4: $\tiny\yng(4), \tiny\yng(3,1), \tiny\yng(2,2), \tiny\yng(2,1,1), \tiny\yng(1,1,1,1)$. Starting from the left these correspond to the class of four-cycles, three-cycles, double two-cycles, two-cycles, and the identity. From proposition 2.21 we know that each of these tableaux correspond to an irrep of S_4. We use the notation ρ_λ to be the irrep corresponding to the partition λ.

The following is an example of a matrix representation of $\rho_{\tiny\yng(2,1)} : S_3 \to GL(2,\mathbb{C})$, it is irreducible and two-dimensional.

$$\rho_{\yng(2,1)}[e] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho_{\yng(2,1)}[(12)] = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \qquad \rho_{\yng(2,1)}[(13)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.1}$$

$$\rho_{\yng(2,1)}[(23)] = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \rho_{\yng(2,1)}[(123)] = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad \rho_{\yng(2,1)}[(132)] = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

This representation corresponds to the second row in the S_3 character table.

Character \ Class	e	(a,b)	(a,b,c)
$\chi_{\yng(3)}$	1	1	1
$\chi_{\yng(2,1)}$	2	0	-1
$\chi_{\yng(1,1,1)}$	1	-1	1

Using the character table we can easily give an example of the orthogonality relations for S_3. Recall that characters must satisfy

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_V}(g)} \chi_{\rho_W}(g) = \begin{cases} 1 \text{ if } \rho_V \cong \rho_W, \\ 0 \text{ else.} \end{cases} \tag{4.2}$$

We have the following in the case of S_3

$$\begin{aligned} \frac{1}{6} \sum_{g \in S_3} \overline{\rho_{\yng(2,1)}(g)} \rho_{\yng(2,1)}(g) &= \frac{1}{6} [(2)(2) + (0)(0) + (0)(0) + (0)(0) + (-1)(-1) + (-1)(-1)] \\ &= 1 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \frac{1}{6} \sum_{g \in S_3} \overline{\rho_{\yng(2,1)}(g)} \rho_{\yng(1,1,1)}(g) &= \frac{1}{6} [(2)(1) + (0)(1) + (0)(1) + (0)(1) + (-1)(1) + (-1)(1)] \\ &= 0 \end{aligned} \tag{4.4}$$

as desired.

An irrep ρ of a group G may be reducible with respect to subgroups of G. The formula for expressing ρ as a direct sum of irreps of subgroups of G are referred to as the **branching rules** of G. We use the notation $G \downarrow H$ for the branching rules from G to H. In the case of S_n we have an elegant set of branching rules. Given a partition $\lambda \vdash n$ and $\mu \vdash (n-1)$, $\lambda \downarrow \mu$ is the set of all μ that are a valid Young diagram obtained by deleting one box from λ. For example, given $\lambda = \ydiagram{2,1}$, we have: $\lambda \downarrow \mu = \{\ydiagram{2}, \ydiagram{1,1}\}$. The following figure displays the successive branching rules for a specific S_5 representation down to S_2.

(4.5)
$$\ydiagram{3,1,1} \downarrow \begin{cases} \ydiagram{3,1} \downarrow \begin{cases} \ydiagram{3} \downarrow \ydiagram{2} \\ \ydiagram{2,1} \downarrow \begin{cases} \ydiagram{2} \\ \ydiagram{1,1} \end{cases} \end{cases} \\ \ydiagram{2,1,1} \downarrow \begin{cases} \ydiagram{2,1} \downarrow \begin{cases} \ydiagram{2} \\ \ydiagram{1,1} \end{cases} \\ \ydiagram{1,1,1} \downarrow \ydiagram{1,1} \end{cases} \end{cases}$$

In the example, $\rho_{\ydiagram{2,1}}$ is irreducible as an S_3 representation, but when restricted to the subgroup S_2 it is a direct sum of the S_2 irreps $\rho_{\ydiagram{2}}$ and $\rho_{\ydiagram{1,1}}$. The following notation is used to denote this

(4.6)
$$\rho_{\ydiagram{2,1}}|_{S_2} \cong \rho_{\ydiagram{2}} \oplus \rho_{\ydiagram{1,1}}.$$

The final topic for this chapter is the **regular representation of S_n**. Each basis vector of $\mathbb{C}^{n!}$ can be indexed by an element of S_n: $\{v_g \,|\, g \in S_n\}$. Let S_n act on an arbitrary vector in $\mathbb{C}^{n!}$ by

(4.7)
$$h \cdot \sum_{g \in S_n} a_g v_g = \sum_{g \in S_n} a_g v_{hg}.$$

This action defines the left representation $\rho_{reg} : S_n \to GL(n!; \mathbb{C})$. The right regular representation comes from the action

(4.8)
$$h \cdot \sum_{g \in S_n} a_g v_g = \sum_{g \in S_n} a_g v_{gh^{-1}}.$$

As an example, let's construct the (right) regular representation of S_3. Consider the basis vectors

(4.9)
$$v_e = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_{(12)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_{(13)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_{(23)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad v_{(123)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad v_{(132)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We know that ρ_{reg} must map the identity element e to the identity matrix, so let's begin by checking where it will map the permutation (12) to. We have that

$$\begin{aligned} e \cdot (12)^{-1} &= (12) \\ (12) \cdot (12)^{-1} &= e \\ (13) \cdot (12)^{-1} &= (123) \\ (23) \cdot (12)^{-1} &= (132) \\ (123) \cdot (12)^{-1} &= (13) \\ (132) \cdot (12)^{-1} &= (23) \end{aligned} \tag{4.10}$$

Thus $\rho_{reg}((12))$ maps the basis vectors to the following

$$\begin{aligned} v_e &\to v_{(12)} \\ v_{(12)} &\to v_e \\ v_{(13)} &\to v_{(123)} \\ v_{(23)} &\to v_{(132)} \\ v_{(123)} &\to v_{(13)} \\ v_{(132)} &\to v_{(23)}. \end{aligned} \tag{4.11}$$

Thus ρ_{reg} maps the permutation (12) to the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4.12}$$

This process can be easily repeated for each element in the group, the matrices are as follows

$$\rho_{reg}(e) = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix} \quad \rho_{reg}((12)) = \begin{bmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix} \tag{4.13}$$

$$\rho_{reg}((13)) = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0 \end{bmatrix} \quad \rho_{reg}((23)) = \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{bmatrix}$$

$$\rho_{reg}((123)) = \begin{bmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{bmatrix} \quad \rho_{reg}((132)) = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0 \end{bmatrix}.$$

By construction each of these matrices is a permutation matrix, which is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere. The regular representation of S_n is reducible; it can be written as a direct sum of irreps by the formula

$$\rho_{reg} = \bigoplus_{\lambda \vdash n} \bigoplus^{s_\lambda} \rho_\lambda. \tag{4.14}$$

Each irrep ρ_λ occurs as many times as its dimension. The transformation TMT^{-1}, where

$$T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \end{bmatrix} \tag{4.15}$$

brings all of the above matrices M to their block-diagonal form. See chapter 3, section 9 of [**10**] on how to construct the matrix T. The block-diagonal matrices are

(4.16)

$$\rho_{reg}(e) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho_{reg}((12)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\rho_{reg}((13)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \rho_{reg}((23)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\rho_{reg}((123)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \quad \rho_{reg}((132)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

We can see that for S_3 the regular representation decomposes as

$$\rho_{reg} = \rho_{\yng(3)} \oplus \rho_{\yng(1,1,1)} \oplus \rho_{\yng(2,1)} \oplus \rho_{\yng(2,1)}, \tag{4.17}$$

as anticipated. The change of basis T that was chosen makes the irreps in the block-diagonalization unitary.

CHAPTER 3

Immanants

1. Definition and Basic Properties

In this section we will recall the definitions of permanent and determinant of a matrix, and then define the generalization of these functions: the immanant. We will then discuss some properties of immanants. The two most important properties that we will cover are Gamas' theorem, which tells us when the immanant of a Gram matrix is zero, based on the linear independence relations of its basis functions; and secondly we will discuss how, from a computational complexity perspective, some immanants are easy to evaluate while others are difficult. The main references for this section are [**7**], [**8**], [**9**], [**23**], and [**24**].

Let $A = (a_{ij})$ be an $n \times n$ matrix. Recall that the Laplace expansion of the determinant of A is given by

$$\text{Det}(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{1.1}$$

where

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases} \tag{1.2}$$

The determinant is one of the most famous functions in linear algebra. To name a few key properties: the determinant being zero or non-zero tells us about the linear independence of the system of equations given by the entries of A; the determinant is the product of the eigenvalues of A; and the determinant is the signed volume of the n-dimensional parallelepiped spanned by the rows or columns of A. In combinatorics, there is a similar-looking function called the permanent that comes up quite frequently. The permanent is given by

$$\text{Perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{1.3}$$

Its best-known application is as follows. Let G be a bipartite graph and let A be the adjacency matrix of G, we have that $\text{Perm}(A)$ is equal to number of perfect matchings of G.

For an $n \times n$ matrix A, there is an immanant defined for every partition of n. The λ-immanant of A is defined to be

$$\text{Imm}^{\lambda}(A) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{1.4}$$

The characters of the trivial representation are always 1 and the characters of the alternating representation are $sgn(\sigma)$, so we see that the permanent and determinant are special cases of immanants. The simplest non-trivial example is the $\yng(2,1)$ immanant of a 3×3 matrix. Referring to the S_3 character table in our previous section, we see that $\chi_{\yng(2,1)}(e) = 2$, $\chi_{\yng(2,1)}((1,2)) = \chi_{\yng(2,1)}((1,3)) = \chi_{\yng(2,1)}((2,3)) = 0$, and $\chi_{\yng(2,1)}((1,2,3)) = \chi_{\yng(2,1)}((1,3,2) = -1$. Thus the $\yng(2,1)$ immanant is

$$\text{Imm}^{\yng(2,1)}(A) = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}. \tag{1.5}$$

The determinant has the property that when we permute any two rows or columns of A, then we pick up a negative sign. This statement easily generalizes to all permutations of rows or columns; the proof is as follows.

Proof. Let A^γ be the matrix A where the columns are permuted by γ, that is

$$A^\gamma = \begin{bmatrix} a_{1,\gamma(1)} & a_{1,\gamma(2)} & \cdots & \cdots & a_{1,\gamma(n)} \\ a_{2,\gamma(1)} & a_{2,\gamma(2)} & \cdots & \cdots & a_{2,\gamma(n)} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n,\gamma(1)} & a_{n,\gamma(2)} & \cdots & \cdots & a_{n,\gamma(n)} \end{bmatrix}. \tag{1.6}$$

The determinant of A^γ is

$$\text{Det}(A^\gamma) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma\gamma(i)}. \tag{1.7}$$

We choose to sum over $\sigma' = \sigma\gamma^{-1}$ to get the following

$$\begin{aligned} \text{Det}(A^\gamma) &= \sum_{\sigma' \in S_n} \text{sgn}(\sigma\gamma^{-1}) \prod_{i=1}^{n} a_{i\sigma(i)} \\ &= \text{sgn}(\gamma) \sum_{\sigma' \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \\ &= \text{sgn}(\gamma)\, \text{Det}(A). \end{aligned} \tag{1.8}$$

Thus the determinant picks up a negative sign if, and only if, γ is an odd permutation. An identical argument holds if we choose to permute the rows.

□

The same argument shows that the permanent is invariant under permutations of its rows or columns. The natural question to ask is whether all immanants have these properties. Unfortunately, the answer is no. If we use the same proof, then we run into problems when we want to evaluate $\chi_\lambda(\sigma\gamma^{-1})$; it's not the case that we can write the character as $\chi_\lambda(\sigma)\chi_\lambda(\gamma^{-1})$. For higher-dimensional immanants, rather than a multiple of A, $\mathrm{Imm}^\lambda(A^\gamma)$ will be a linear combination of immanants of other permuted matrices. For example

$$\mathrm{Imm}^{\boxplus}(A^{(132)}) = -\mathrm{Imm}^{\boxplus}(A^{(123)}) - \mathrm{Imm}^{\boxplus}(A). \tag{1.9}$$

A theorem that tells us what the linear combination is for a given λ and γ is currently under investigation.

The next statement that we present is true for all immanants. Let $A' = (a'_{ij})$ denote the matrix A where a row or column has been multiplied by a constant c. We have that

$$\mathrm{Imm}^\lambda(A') = c\,\mathrm{Imm}^\lambda(A). \tag{1.10}$$

The proof is very simple. We know that for all $\sigma \in S_n$ the product $\prod_{i=1}^n a'_{i\sigma(i)}$ contains exactly one element of each row and each column, so we can factor out the constant c from each term in the immanant.

Let $A(i|j)$ denote the $n \times n$ matrix obtained from A where the i^{th} row and j^{th} column are replaced with zeroes, expect for a_{ij} which is equal to one. For example,

$$A(3|2) = \begin{bmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 1 & 0 \end{bmatrix}. \tag{1.11}$$

We can write an immanant of A as a linear combination of immanants of these $A(i|j)$ matrices. For some fixed $1 \leq i \leq n$ we have

$$\mathrm{Imm}^\lambda(A) = \sum_{j=1}^{n} a_{ij}\,\mathrm{Imm}^\lambda(A(i|j)). \tag{1.12}$$

This equation is needed to prove the last statement of this section: the determinant is invariant under adding a multiple of a row to another row.

Proof. Let A be an $n \times n$ matrix; let B be the matrix A where row j is replaced by row i; and let C be the matrix A where row i is multiplied by the constant c and added to row j. We will show that $\text{Det}(C) = \text{Det}(A)$. We begin by writing the determinant of C using equation 1.12 and choosing to expand along row j,

$$\begin{aligned}
\text{(1.13)} \qquad \text{Det}(C) &= \sum_{k=1}^{n} c_{jk}\,\text{Det}(C(j|k)) \\
&= \sum_{k=1}^{n} (ca_{ik} + a_{jk})\,\text{Det}(C(j|k)) \\
&= c\sum_{k=1}^{n} a_{ik}\,\text{Det}(C(j|k)) + \sum_{k=1}^{n} a_{jk}\,\text{Det}(C(j|k))
\end{aligned}$$

Since $a_{ik} = b_{jk}$ for all $1 \leq k \leq n$ and the matrices A,B, and C only differ along row j, we can write

$$\begin{aligned}
\text{(1.14)} \qquad \text{Det}(C) &= c\sum_{k=1}^{n} b_{jk}\,\text{Det}(B(j|k)) + \sum_{k=1}^{n} a_{jk}\,\text{Det}(A(j|k)) \\
&= c\,\text{Det}(B) + \text{Det}(A) \\
&= \text{Det}(A) \quad (\text{Det}(B) \text{ is zero since it has repeated rows})
\end{aligned}$$

□

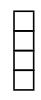
The determinant is the only immanant where a matrix having two repeated rows implies that the immanant is zero, so the last line of this proof will only be true for the determinant.

2. Gamas' Theorem

Gamas' theorem tells us whether the λ-immanant of a Gram matrix will be zero by looking at the shape of the partition λ. The theorem is stated as follows.

Theorem (Gamas [**13**]): Let $G_{ij} = \langle f_i |\, f_j \rangle$ be a Gram matrix. We have that $\text{Imm}^{\lambda}(G) \neq 0$ if and only if there exists a Young tableau of shape λ whose columns can index sets of linearly independent functions f_i.

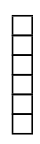
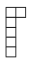
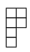
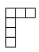
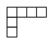
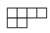
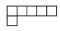
To understand this, let's look at a few examples. In the following charts, a 0 indicates that the immanant, for the corresponding set of basis functions, is 0. Gamas' theorem is a bi-conditional statement, so the blank boxes mean the corresponding immanant is non-zero.

					
0					$f_1\ f_1\ f_2\ f_3$
0	0				$f_1\ f_1\ f_2\ f_2$
0	0	0			$f_1\ f_1\ f_1\ f_2$
0	0	0	0		$f_1\ f_1\ f_1\ f_1$

Let's take a closer look at the second row of the preceding table. The immanants corresponding to  and  are zero because there is no way to fill the Young diagrams with the functions f_i such that each column contains all different functions. For the remaining immanants we have

(2.1) $$\begin{array}{|c|c|}\hline f_1 & f_1 \\ \hline f_2 & f_2 \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|c|}\hline f_1 & f_1 & f_2 \\ \hline f_2 \\ \cline{1-1} \end{array}\,, \qquad \begin{array}{|c|c|c|c|}\hline f_1 & f_1 & f_2 & f_2 \\ \hline \end{array}\,,$$

thus these immanants are non-zero. The table for 6×6 matrices is as follows.

											
0											$f_1\ f_1\ f_2\ f_3\ f_4\ f_5$
0	0										$f_1\ f_1\ f_2\ f_2\ f_3\ f_4$
0	0	0		0							$f_1\ f_1\ f_2\ f_2\ f_3\ f_3$
0	0	0	0								$f_1\ f_1\ f_1\ f_3\ f_4\ f_5$
0	0	0	0	0							$f_1\ f_1\ f_1\ f_3\ f_3\ f_4$
0	0	0	0	0	0		0				$f_1\ f_1\ f_1\ f_3\ f_3\ f_3$
0	0	0	0	0	0	0					$f_1\ f_1\ f_1\ f_1\ f_3\ f_4$
0	0	0	0	0	0	0	0				$f_1\ f_1\ f_1\ f_1\ f_3\ f_3$
0	0	0	0	0	0	0	0	0			$f_1\ f_1\ f_1\ f_1\ f_1\ f_3$
0	0	0	0	0	0	0	0	0	0		$f_1\ f_1\ f_1\ f_1\ f_1\ f_1$

Notice the relation [**14**]:

(2.2) $$\mathrm{Imm}^{\lambda}(G) = 0 \implies \mathrm{Imm}^{\mu}(G) = 0 \quad \forall\, \mu \trianglelefteq \lambda,$$

where $\trianglelefteq$ is the dominance ordering of partitions. Note that 6 is the smallest n for which the dominance ordering differs from the lexicographic ordering.

3. Computational Complexity

Perhaps the most interesting aspect of immanants is their computational complexity. It's well-known that the determinant is "easy" to compute exactly, taking only $\mathcal{O}(n^3)$ operations using Gaussian elimination, whereas the permanent is "hard" to compute exactly, taking $\mathcal{O}(2^{n-1}n^2)$ operations using Ryser's algorithm. Generally speaking, algorithms that run in polynomial time, that is $\mathcal{O}(n^k)$ for some constant k, are considered to be fast. We begin this section with a review of $\mathcal{O}$-notation and the algorithms for evaluating the permanent and determinant. We will then discuss Bürgisser's algorithm for arbitrary immanants, which provides an exact evaluation of an immanant using complex number arithmetic. A good reference for the computer science background is [**21**].

Algorithms are classified by their running time, which isn't measured in seconds, but rather the number of primitive operations needed to execute the algorithm. The running time depends on the size of the input; in this thesis we are only interested in computing the immanants of $n \times n$ matrices so the input size will be n in all cases. Knowing the exact runtime of an algorithm isn't important, it's much more useful to know how this run-time scales with n, a property usually denoted using the $\mathcal{O}-$notation. Given some function $g(n)$, the set $\mathcal{O}\left(g(n)\right)$ is the set of all functions $f(n)$ such that f is bounded above by g up to a constant factor as $n \to \infty$. In symbols this is

$$\mathcal{O}\left(g(n)\right) = \{f(n) \,|\, \exists\ c, n_0 > 0 \text{ such that } 0 \leq f(n) \leq cg(n) \quad \forall n \geq n_0\}. \tag{3.1}$$

Example 3.1. Suppose that an algorithm has an exact runtime of $3n^2 + n + 25$ operations. This algorithm runs in $\mathcal{O}(n^2)$ time.

It's very computationally expensive to compute any immanant by brute-force methods. It takes $n-1$ operations to multiply the product $\prod_{i=1}^{n} a_{i,\sigma(i)}$ and this has to be done for all $n!$ terms in the sum, so the time complexity of the naïve algorithm is $\mathcal{O}(nn!)$.

When computing the determinant there is a clever trick that can be used to greatly simplify the calculation. Recall that we can add a multiple of a row to another row without changing the determinant; this allows us to preform row reduction on a matrix and bring to row echelon form (all entries below the main diagonal are zero) by choosing constants that make zeroes appear in the matrix. We will give a simplified overview of this process and keep track of the number of operations used.

The basic idea is as follows. We begin with an arbitrary $n \times n$ matrix. The first step is to make the $(n-1)$ entries below $a_{1,1}$ all zero. To do this we choose a constant c such that $c \times r_1 + r_2$ has a zero in the first column and replace the second row with this linear combination. We repeat for each of the $(n-1)$ rows below the first. Multiplying r_1 by a constant takes n operations because there are n entries in the row; this is done $(n-1)$ times for a total of $n(n-1)$ multiplications. Similarly preforming $c \times r_1 + r_i$ takes n additions and this is repeated $(n-1)$ times for a total of $n(n-1)$ additions. Next we make all the entries below $a_{2,2}$ zero. We again choose a constant c such that $c \times r_3 + r_3$ has a zero in the second entry and replace row 3. Since we know that there is a zero in the first entry it only takes $(n-1)$ multiplications to preform $c \times r_2$; this is repeated $(n-2)$ times for a total of $(n-1)(n-2)$ multiplications. The additions needed for this step are $(n-1)(n-2)$ as well. This process is repeated for each column and the result is a matrix of the form

$$\begin{bmatrix} * & * & * & \dots & \dots & * \\ 0 & * & * & \dots & \dots & * \\ 0 & 0 & * & & & * \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \tag{3.2}$$

Using the well-known identities for $\sum i^2$ and $\sum i$, we find that the total number of multiplications needed is

$$\sum_{i=1}^{n-1}(i+1)i = \frac{n^3-n}{3}, \tag{3.3}$$

which is also the number of additions needed. The determinant of the row echelon matrix is simply the product of the diagonal entries which takes $(n-1)$ multiplications. The sum of all these operations is $\frac{2n^3+n-1}{3}$, thus the determinant can be computed in $\mathcal{O}(n^3)$ time.

Next, we'll discuss the complexity of the permanent. Ryser [**11**] showed that the permanent of an $n \times n$ matrix $A = (a_{ij})$ can be expressed as

$$\mathrm{Perm}(A) = (-1)^n \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|} \prod_{i=1}^{n} \sum_{j \in S} a_{ij}, \tag{3.4}$$

where $[n] = \{1, 2, 3, \dots, n\}$ and we sum over all $2^n - 1$ non-empty subsets S of $[n]$. We will count the number of operations needed to compute the permanent by Ryser's formula and show that the time complexity is $\mathcal{O}(2^{n-1}n^2)$. We will begin by counting the number of additions needed. Consider the innermost sum $\sum_{j \in S} a_{ij}$. Since there are $|S|$ numbers in the sum, it takes $|S| - 1$ operations to add them together. We compute this sum for every $i \in [n]$, so we are up to $n\,(|S| - 1)$ additions. For each non-empty subset $S \subseteq [n]$ we have $n\,(|S| - 1)$ additions, this gives us

$$\sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} n\,(|S| - 1) \tag{3.5}$$

total additions. Recall that there are $2^n - 1$ non-empty subsets of $[n]$ and there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ subsets of size k. Using this we have

$$\sum_{\substack{S\subseteq[n]\\S\neq\emptyset}} n\left(|S|-1\right) = \sum_{k=1}^{n} n(k-1)\binom{n}{k} \tag{3.6}$$
$$= n\sum_{k=1}^{n} k\binom{n}{k} - n\sum_{k=1}^{n}\binom{n}{k}.$$

Notice that

$$k\binom{n}{k} = \frac{kn!}{k!(n-k)!} \tag{3.7}$$
$$= \frac{n(n-1)!}{(k-1)!\left((n-1)-(k-1)\right)!}$$
$$= n\binom{n-1}{k-1}.$$

We use this identity to simplify our sum. We have

$$\sum_{\substack{S\subseteq[n]\\S\neq\emptyset}} n\left(|S|-1\right) = n^2\sum_{k=1}^{n}\binom{n-1}{k-1} - n\sum_{k=1}^{n}\binom{n}{k} \tag{3.8}$$
$$= n^2\left(2^{n-1}-1\right) - n\left(2^n-1\right)$$
$$= n^2 2^{n-1} - n2^n - n^2 + n.$$

Next, we count the the number of multiplications required for this algorithm. For each non-empty set $S \subseteq [n]$ it takes $n-1$ multiplications to evaluate the product $\prod_{i=1}^{n}\sum_{j\in S} a_{ij}$. There are $2^n - 1$ subsets, thus the total number of multiplications is $n2^n - n - 2^n + 1$.

We also need to compute $(-1)^{|S|}$ for each non-empty set S, so this is an additional $2^n - 1$ operations needed for the algorithm. When we add the number of these operations together, we get a total of $n^2 2^{n-1} - n^2$. The highest-order term is $n^2 2^{n-1}$, thus the time complexity is $\mathcal{O}(n^2 2^{n-1})$.

Bürgisser's algorithm [**23**], [**24**] uses a result from Kostant [**25**], which will be discussed in the next section, to evaluate the λ-immanant of a matrix; the algorithm does so with non-scalar complexity $\mathcal{O}(n^2 s_\lambda d_\lambda)$, where n is the size of the matrix, s_λ is the number of standard Young tableaux of shape λ and d_λ is the number of semi-standard Young tableaux of shape λ. Recall the dominance ordering of partitions, where the partition consisting of a single column (the determinant) is the smallest and the partition consisting of a single row (the permanent) is the largest. The general trend with Bürgisser's algorithm is that the immanants that are "near" the determinant in dominance ordering are easier to compute and the computational cost grows when moving towards the permanent. It was shown in [**29**] that all partitions that have all but a constant number of boxes in the first column can be evaluated in a polynomial number of operations.

Example 3.2. Consider the partition $\lambda = \{k, k, k, 1, 1, \ldots, 1\}$ of n, where k is some constant integer. All but $3(k-1)$ of the boxes are in the first column, so for any choice of k the λ-immanant can evaluated in polynomial time.

Another result from [**29**] is as follows.

Theorem 3.3. *For any constant integer $k \geq 2$, there exists a partition λ of width k such that the λ-immanant cannot be evaluated in polynomial time.*

This theorem is notable because it states that there are immanants of width 2, which are close to the determinant in dominance ordering, that are hard to compute.

CHAPTER 4

Applications in Quantum Optics

1. Quantum Mechanical Definitions and Notation

This section will begin with a review of some basic concepts from quantum mechanics that are needed to derive the coincidence rate in an $n-$particle interferometry experiment. The main references for this material are [**12**], [**16**], and [**17**].

The two types of objects that we are interested in are quantum states and the operators that act on them. States are denoted by angled brackets and are elements of some complex Hilbert space $|v\rangle \in \mathbb{H}$. Every ket $|v\rangle$ has a corresponding bra $\langle v|$ given by the conjugate transpose $|v\rangle^{\dagger} = \langle v|$, which lives in the dual space $\mathbb{H}^{*}$; as a result $\langle v|w\rangle \in \mathbb{C}$. The system we will be studying in this chapter is an n-channel interferometer: an optical device where particles enter though n input channels, are scattered by an array of beamsplitters and phase shifters, and are then detected in n output channels. The simplest state is the vacuum state

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \cdots \otimes |0\rangle_n\,, \tag{1.1}$$

which tells us that no particles have entered from any of the input channels. In general $|m\rangle_k$ means that there are m particles entering in the k^{th} channel. We use creation and destruction operators ($a_k^{\dagger}$ and a_k, respectively) to alter the number of particles in the k^{th} channel. The two equations that govern this change are

$$\begin{aligned} a_k\,|n\rangle_k &= \sqrt{n}\,|n-1\rangle_k \\ a_k^{\dagger}\,|n\rangle_k &= \sqrt{n+1}\,|n+1\rangle_k\,. \end{aligned} \tag{1.2}$$

The coefficients come from the fact that $\langle n|\,a^{\dagger}a\,|n\rangle = n$. Of course, we can't have a negative number of particles, so we declare that $a_k\,|0\rangle_k = 0$. For example, we have

$$a_2^{\dagger}\,|0\rangle = |0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 \otimes \cdots \otimes |0\rangle_n\,. \tag{1.3}$$

A pulse localized in time is described by a superposition of particles of different frequencies, so that $a_k^{\dagger}(\omega)|0\rangle$ creates a particle of frequency ω in channel k, where we have added the additional index ω to the creation and destruction operator to specify this frequency. A particle "created" at time τ_l in channel j is conveniently described by an operator $A_j^\dagger(\tau_l)$ acting on the vacuum state:

$$A_j^\dagger(\tau_l)\left|0\right\rangle := \int d\omega_l \phi(\omega_l) e^{-i\omega_l \tau_l} a_j^\dagger(\omega_l)\left|0\right\rangle . \tag{1.4}$$

The reason for the integral is because a pulse localized in time is Fourier-synthesized from excitations at different frequencies. The amplitude of each frequency is given by the spectral function $\phi(\omega)$. We assume this function is the same for all modes j. For simplicity we will assume $\phi(\omega_0)$ to be a Gaussian distribution with standard deviation 1 centred at μ_0:

$$\phi(\omega_l) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(\omega_l-\mu_0)^2}{2}}. \tag{1.5}$$

This function is normalized so that $\int d\omega \left|\phi(\omega_0)\right|^2 = 1$. For fixed τ, the states $\{A_j^\dagger(\tau)\left|0\right\rangle, j = 1, \ldots, n\}$ form a basis for the n-fold product of the defining representation of $U(n)$:

$$A_1^\dagger(\tau)\left|0\right\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_2^\dagger(\tau)\left|0\right\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad A_n^\dagger(\tau)\left|0\right\rangle \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{1.6}$$

Recall that the defining representation is simply when a matrix acts on a vector space by matrix multiplication: $\rho(U) : v \to Uv$. For later convenience, we recall that a vector $v \in \mathbb{C}^n$ has weight $w = (w_1, w_2, \ldots, w_n)$ when

$$\mathrm{diag}(t_1, t_2, \ldots, t_n)v = t_1^{w_1} t_2^{w_2} \ldots t_n^{w_n} v. \tag{1.7}$$

Clearly $A_j^\dagger(\tau)\left|0\right\rangle$ has weight $(0, 0, \ldots, 1_j \ldots, 0)$, with 0 everywhere except for a 1 in the j^{th} entry.

The time delay between the pulses determines the overlap of the wave packets, and thus the degree to which particles are distinguishable. There are three broad possibilities for this measure of distinguishability: the particles can be either indistinguishable, partially distinguishable, or distinguishable. When the particles are created at the same time, their wave packets completely overlap and it's impossible to tell them apart; when the time delay between two particles is small, then there is a partial overlap where we can't differentiate between the wave functions; when the time delay is sufficiently large, then the wave functions overlap by only a negligible amount. These three cases are illustrated in the following figure from [**12**].

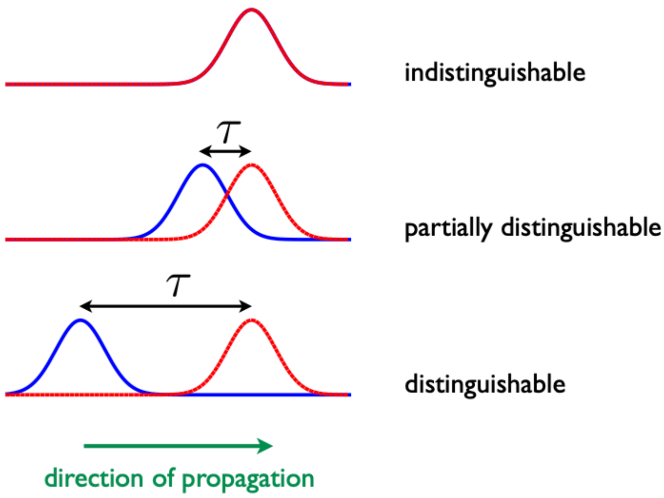


FIGURE 4.1. A pictorial representation of the levels of distinguishability between particles.

The two types of particles that we will be considering are bosons and fermions. Their creation and destruction operators are denoted by $b^\dagger$ and b for bosons and $f^\dagger$ and f for fermions, respectively. The difference between these two operators is in how they commute. For bosons we have

$$\begin{aligned}
[b_i^\dagger(\omega_k), b_j^\dagger(\omega_l)] &= b_i^\dagger(\omega_k)b_j^\dagger(\omega_l) - b_j^\dagger(\omega_l)b_i^\dagger(\omega_k) = 0 \\
[b_i(\omega_k), b_j(\omega_l)] &= b_i(\omega_k)b_j(\omega_l) - b_j(\omega_l)b_i(\omega_l) = 0 \\
[b_i(\omega_k), b_j^\dagger(\omega_l)] &= b_i(\omega_k)b_j^\dagger(\omega_l) - b_j^\dagger(\omega_l)b_i(\omega_k) = \delta_{ij}\delta(\omega_k - \omega_l).
\end{aligned} \tag{1.8}$$

In particular it follows that $b_i^\dagger(\omega_k)b_j^\dagger(\omega_l) = b_j^\dagger(\omega_l)b_i^\dagger(\omega_k)$ and $b_i(\omega_k)b_j(\omega_l) = b_j(\omega_l)b_i(\omega_k)$. The fermionic relations use the anti-commutator $\{\cdot,\cdot\}$ rather than the usual commutator $[\cdot,\cdot]$. We have

$$\begin{aligned}
\{f_i^\dagger(\omega_k), f_j^\dagger(\omega_l)\} &= f_i^\dagger(\omega_k)f_j^\dagger(\omega_l) + f_j^\dagger(\omega_l)f_i^\dagger(\omega_k) = 0 \\
\{f_i(\omega_k), f_j(\omega_l)\} &= f_i(\omega_k)f_j(\omega_l) + f_j(\omega_l)f_i(\omega_k) = 0 \\
\{f_i(\omega_k), f_j^\dagger(\omega_l)\} &= f_i(\omega_k)f_j^\dagger(\omega_l) + f_j^\dagger(\omega_l)f_i(\omega_k) = \delta_{ij}\delta(\omega_k - \omega_l)
\end{aligned} \tag{1.9}$$

Thus $f_i^\dagger(\omega_k)f_j^\dagger(\omega_l) = -f_j^\dagger(\omega_l)f_i^\dagger(\omega_k)$ and $f_i(\omega_k)f_j(\omega_l) = -f_j(\omega_l)f_i(\omega_k)$.

An n-channel interferometer is modelled by a $n \times n$ unitary matrix $U = (U_{ij})_{1 \le i,j, \le n}$. The entries of the matrix depend on the transmission and reflection coefficients of the beamsplitters inside the interferometer. Following the convention used in quantum optics, the operator U acts on the input creation operators by the equation

$$Ua_j^{\dagger}(\omega_l) = \sum_{k=1}^{n} U_{jk} a_k^{\dagger}(\omega_l). \tag{1.10}$$

A **coincidence count** occurs when exactly one particle is detected in each of the n output channels; the **coincidence rate** is the relative probability of this event occurring; it is my goal to find an expression for this rate. Detecting multiple particles in a single output channel was considered in [**17**]. In this thesis, I will only be considering one detection in each of the n channels, to focus on the consequences of the permutation symmetries of the extinction terms in the expression of the coincidence rate. To make distinction between the boson and fermion cases, we will write $P_b(\boldsymbol{\tau}; n)$ and $P_f(\boldsymbol{\tau}; n)$ for the n-particle coincidence rates for bosons and fermions, respectively. Both $P_b(\boldsymbol{\tau}; n)$ and $P_f(\boldsymbol{\tau}; n)$ are functions of n variables: these variables are the times of arrival of the various particles at the input of the interferometer

$$\boldsymbol{\tau} = (\tau_1,\, \tau_2,\, \tau_3,\, \dots,\, \tau_n). \tag{1.11}$$

The **delay matrix** $r(\boldsymbol{\tau})$ is used to keep track of the relative time delays. The ij^{th} entry of the matrix is

$$r_{ij} := e^{\frac{-(\tau_i - \tau_j)^2}{2}}. \tag{1.12}$$

When the arrival times τ_i and τ_j are equal, then $r_{ij} = 1$. When the difference between τ_i and τ_j is sufficiently large, such that the particles are completely distinguishable, then $r_{ij} \approx 0$. When the particles are partially indistinguishable, then $0 \le r_{ij} \le 1$. We see that $r(\boldsymbol{\tau})$ is a Gram matrix by considering the basis functions

$$f_k(\omega) = e^{-i\omega\tau_k}, \quad k = 1, 2, \dots, n \tag{1.13}$$

with the inner product

$$\langle f_i | \, f_j \rangle = \int d\omega |\phi(\omega)|^2 f_i^*(\omega) f_j(\omega). \tag{1.14}$$

We get that $r(\boldsymbol{\tau})_{ij} = \langle f_i | \, f_j \rangle$.

Example 4.1. Consider $n = 3$. When all arrival times are distinct, we have

$$r(\boldsymbol{\tau}) = \begin{pmatrix} 1 & e^{-\frac{1}{2}(\tau_1-\tau_2)^2} & e^{-\frac{1}{2}(\tau_1-\tau_3)^2} \\ e^{-\frac{1}{2}(\tau_1-\tau_2)^2} & 1 & e^{-\frac{1}{2}(\tau_2-\tau_3)^2} \\ e^{-\frac{1}{2}(\tau_1-\tau_3)^2} & e^{-\frac{1}{2}(\tau_2-\tau_3)^2} & 1 \end{pmatrix}. \tag{1.15}$$

When two arrival times are equal, meaning that $\boldsymbol{\tau} = (\tau_1,\, \tau_1,\, \tau_2)$, we have

$$r(\boldsymbol{\tau}) = \begin{pmatrix} 1 & 1 & e^{-\frac{1}{2}(\tau_1-\tau_2)^2} \\ 1 & 1 & e^{-\frac{1}{2}(\tau_1-\tau_2)^2} \\ e^{-\frac{1}{2}(\tau_1-\tau_2)^2} & e^{-\frac{1}{2}(\tau_1-\tau_2)^2} & 1 \end{pmatrix}. \tag{1.16}$$

When all arrival times are the same $\boldsymbol{\tau} = (\tau_1,\, \tau_1,\, \tau_1)$, then every entry of the delay matrix is 1.

This section ends with a discussion of the input and output states. When one particle enters from each input channel, the input state is

$$\begin{aligned} |\psi_{in}\rangle &= A_1^\dagger(\tau_1)\,|0\rangle_1 \otimes A_2^\dagger(\tau_2)\,|0\rangle_2 \otimes \cdots \otimes A_n^\dagger(\tau_n)\,|0\rangle_n \\ &= A_1^\dagger(\tau_1) A_2^\dagger(\tau_2) \ldots A_n^\dagger(\tau_n)\,|0\rangle \end{aligned} \tag{1.17}$$

This input state thus lives in the n-fold product of the defining representation of $U(n)$, and is easily seen to have weight $(1, 1, \ldots, 1)$. The output state $|\psi_{out}\rangle$ is mathematically related to the input state by a linear transformation called the scattering matrix U,

$$\begin{aligned} |\psi_{out}\rangle &= UA_1^\dagger(\tau_1)\,|0\rangle_1 \otimes UA_2^\dagger(\tau_2)\,|0\rangle_2 \otimes \cdots \otimes UA_n^\dagger(\tau_n)\,|0\rangle_n \\ &= U\,|\psi_{in}\rangle\,. \end{aligned} \tag{1.18}$$

To compute the output state we use equation 1.10 with every term in the input state:

$$\begin{aligned} UA_1^\dagger(\tau_1) &= U_{11}A_1^\dagger(\tau_1) + U_{12}A_2^\dagger(\tau_1) + \cdots + U_{1n}A_n^\dagger(\tau_1) \\ UA_2^\dagger(\tau_2) &= U_{21}A_1^\dagger(\tau_2) + U_{22}A_2^\dagger(\tau_2) + \cdots + U_{2n}A_n^\dagger(\tau_2) \\ &\vdots \\ UA_n^\dagger(\tau_n) &= U_{n1}A_1^\dagger(\tau_n) + U_{n2}A_2^\dagger(\tau_n) + \cdots + U_{nn}A_n^\dagger(\tau_n). \end{aligned} \tag{1.19}$$

The output state will be the product of these n sums. To model an experiment where, over time, an experimentalist counts the number of times exactly one particle is detected at each output channel (a coincidence count), we project from the full output state $|\psi_{out}\rangle$ on to the state $|\psi_{out}\rangle_C$ which contains only terms with exactly one creation operator $A_j^\dagger$ for each index j; this is justified on physical grounds because detectors can be modelled as projection operators. Details of the modelling of detectors is beyond the scope of this thesis, for a discussion of this topic see [**17**] and [**19**]. The state $|\psi_{out}\rangle_C$ is therefore of the form

$$|\psi_{out}\rangle_C = \sum_{\sigma\in S_n} U_{1,\sigma(1)}U_{2,\sigma(2)}\dots U_{n,\sigma(n)}A^\dagger_{\sigma(1)}(\tau_1)A^\dagger_{\sigma(2)}(\tau_2)\dots A^\dagger_{\sigma(n)}(\tau_n)\,|0\rangle\,. \tag{1.20}$$

Note that the state $|\psi_{out}\rangle_C$ is also a state of weight $(1,1,1,\dots,1)$ in the same n-fold product space as the input state. The corresponding bra is the conjugate transpose

$$_C\,\langle\psi_{out}| = \sum_{\sigma\in S_n} U^*_{1,\sigma(1)}U^*_{2,\sigma(2)}\dots U^*_{n,\sigma(n)}\,\langle 0|\,A_{\sigma(n)}(\tau_n)A_{\sigma(n-1)}(\tau_{n-1})\dots A_{\sigma(1)}(\tau_1), \tag{1.21}$$

where

$$A_j(\tau_l) = \int d\bar{\omega}_l\phi^*(\bar{\omega}_l)e^{i\bar{\omega}_l\tau_l}a_j(\bar{\omega}_l). \tag{1.22}$$

2. The Hong-Ou-Mandel Effect for Two Particles

The Hong-Ou-Mandel effect [**15**] is a phenomena in quantum mechanics where in a 50 : 50 beamsplitter, the coincidence rate is directly related to the distinguishability of the two input photons. A 50 : 50 beam splitter is a device where an input photon has probability $\frac{1}{2}$ of remaining in the same channel, and probability $\frac{1}{2}$ of scattering to a neighbouring channel. If the photons enter the interferometer at the same time (completely indistinguishable) and from different input channels, then the coincidence rate is 0, i.e. the photons must exit from the same channel. If there is a large enough time delay so that the photons are completely distinguishable, then the coincidence rate becomes $\frac{1}{2}$. Recall that photons are bosons. If we generalize the Hong-Ou-Mandel effect to fermions, then we get a very interesting result: the coincidence rate changes [**18**], [**20**]. Having equal arrival times now corresponds to a coincidence rate of 1, and a sufficiently large time delay leads to a coincidence rate of $\frac{1}{2}$. In this section these results will be derived. We begin with the boson case. Our input state is

$$|\psi_{in}\rangle = A^\dagger_1(\tau_1)A^\dagger_2(\tau_2)\,|0\rangle \tag{2.1}$$

and the scattering matrix for a 50 : 50 beamsplitter is

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}. \tag{2.2}$$

It follows that the output state will be

$$
\begin{aligned}
|\psi_{out}\rangle &= \left(\frac{1}{\sqrt{2}}A_1^\dagger(\tau_1) + \frac{1}{\sqrt{2}}A_2^\dagger(\tau_1)\right)\left(\frac{1}{\sqrt{2}}A_1^\dagger(\tau_2) - \frac{1}{\sqrt{2}}A_2^\dagger(\tau_2)\right)|0\rangle \\
&= \left(\frac{1}{2}A_1^\dagger(\tau_1)A_1^\dagger(\tau_2) - \frac{1}{2}A_1^\dagger(\tau_1)A_2^\dagger(\tau_2) + \frac{1}{2}A_2^\dagger(\tau_1)A_1^\dagger(\tau_2) - \frac{1}{2}A_2^\dagger(\tau_1)A_2^\dagger(\tau_2)\right)|0\rangle . \qquad (2.3)
\end{aligned}
$$

The output state corresponding to the detection of one particle in each channel will contain only the terms that contain each of $A_1^\dagger$ and $A_2^\dagger$ exactly once.

$$
|\psi_{out}\rangle_C = \left(\frac{1}{2}A_2^\dagger(\tau_1)A_1^\dagger(\tau_2) - \frac{1}{2}A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)\right)|0\rangle . \qquad (2.4)
$$

Thus the coincidence rate is

(2.5)

$$
\begin{aligned}
P_b(\boldsymbol{\tau};2) &= \langle 0|\left(-\frac{1}{2}A_2(\tau_2)A_1(\tau_1) + \frac{1}{2}A_1(\tau_2)A_2(\tau_1)\right)\left(\frac{1}{2}A_2^\dagger(\tau_1)A_1^\dagger(\tau_2) - \frac{1}{2}A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)\right)|0\rangle \\
&= -\frac{1}{4}\langle 0| A_2(\tau_2)A_1(\tau_1)A_2^\dagger(\tau_1)A_1^\dagger(\tau_2)|0\rangle \\
&\quad + \frac{1}{4}\langle 0| A_2(\tau_2)A_1(\tau_1)A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)|0\rangle \\
&\quad + \frac{1}{4}\langle 0| A_1(\tau_2)A_2(\tau_1)A_2^\dagger(\tau_1)A_1^\dagger(\tau_2)|0\rangle \\
&\quad - \frac{1}{4}\langle 0| A_1(\tau_2)A_2(\tau_1)A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)|0\rangle
\end{aligned}
$$

We need to evaluate each term in the sum. These particles are bosons, so we get that

$$
P_b(\boldsymbol{\tau};2) = -\frac{1}{4}r_{12}r_{21} + \frac{1}{4}r_{11}r_{22} + \frac{1}{4}r_{22}r_{11} - \frac{1}{4}r_{21}r_{12}. \qquad (2.6)
$$

If the bosons have the same arrival time, then both of r_{12} and r_{21} are equal to 1. There is total destructive interference and the coincidence rate is zero

$$
\begin{aligned}
P_b(\boldsymbol{\tau};2) &= -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} \\
&= 0.
\end{aligned} \qquad (2.7)
$$

If the time delay is large enough so that the bosons are completely distinguishable, then both of r_{12} and r_{21} are equal to 0, thus

$$P_b(\boldsymbol{\tau};2) = 0 + \frac{1}{4} + \frac{1}{4} - 0 = \frac{1}{2}. \tag{2.8}$$

The coincidence rate continuously changes from 0 to $\frac{1}{2}$ depending on the level of distinguishability, as shown in the following figure from [**15**]. The horizontal axis is labelled 'position of beam splitter' because that is what controls the relative delays.

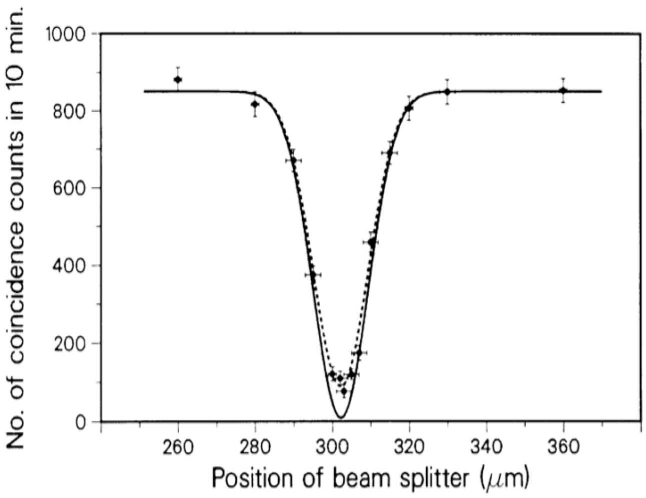


FIGURE 4.2. The Hong-Ou-Mandel dip

This result easily generalizes from a 50 : 50 beamsplitter to an arbitrary one. The input state will be the same as the previous case, but we will instead act on it by an arbitrary unitary matrix

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}. \tag{2.9}$$

The output state will be

$$\begin{aligned} |\psi_{out}\rangle &= \left(U_{11}A_1^\dagger(\tau_1) + U_{12}A_2^\dagger(\tau_1)\right)\left(U_{21}A_1^\dagger(\tau_2) + U_{22}A_2^\dagger(\tau_2)\right)|0\rangle \\ &= U_{11}U_{21}A_1^\dagger(\tau_1)A_1^\dagger(\tau_2)|0\rangle + U_{11}U_{22}A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)|0\rangle \\ &\quad + U_{12}U_{21}A_2^\dagger(\tau_1)A_1^\dagger(\tau_2)|0\rangle + U_{12}U_{22}A_2^\dagger(\tau_1)A_2^\dagger(\tau_2)|0\rangle. \end{aligned} \tag{2.10}$$

The output state corresponding to the coincidence rate is

$$(2.11) \qquad |\psi_{out}\rangle_C = \Big(U_{11}U_{22}A_1^\dagger(\tau_1)A_2^\dagger(\tau_2) + U_{12}U_{21}A_2^\dagger(\tau_1)A_1^\dagger(\tau_2)\Big)\,|0\rangle\,.$$

Thus the coincidence rate is

$$(2.12) \qquad \begin{aligned} P_b(\boldsymbol{\tau};2) &= \langle 0|\,\Big(U_{11}^*U_{22}^*A_2(\tau_2)A_1(\tau_1) + U_{12}^*U_{21}^*A_1(\tau_2)A_2(\tau_1)\Big) \\ &= \quad \times \Big(U_{12}U_{21}A_2^\dagger(\tau_1)A_1^\dagger(\tau_2) + U_{11}U_{22}A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)\Big)\,|0\rangle \\ &= U_{11}^*U_{22}^*U_{12}U_{21}\,\langle 0|\,A_2(\tau_2)A_1(\tau_1)A_2^\dagger(\tau_1)A_1^\dagger(\tau_2)\,|0\rangle \\ &\quad + U_{11}^*U_{22}^*U_{11}U_{22}\,\langle 0|\,A_2(\tau_2)A_1(\tau_1)A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)\,|0\rangle \\ &\quad + U_{12}^*U_{21}^*U_{12}U_{21}\,\langle 0|\,A_1(\tau_2)A_2(\tau_1)A_2^\dagger(\tau_1)A_1^\dagger(\tau_2)\,|0\rangle \\ &\quad + U_{12}^*U_{21}^*U_{11}U_{22}\,\langle 0|\,A_1(\tau_2)A_2(\tau_1)A_1^\dagger(\tau_1)A_2^\dagger(\tau_2)\,|0\rangle\,. \end{aligned}$$

Evaluating the bosonic sequences yields

$$(2.13) \qquad \begin{aligned} P_b(\boldsymbol{\tau};2) = U_{11}^*U_{22}^*U_{12}U_{21}r_{12}r_{21} + U_{11}^*U_{22}^*U_{11}U_{22}r_{11}r_{22} \\ + U_{12}^*U_{21}^*U_{12}U_{21}r_{22}r_{11} + U_{12}^*U_{21}^*U_{11}U_{22}r_{21}r_{12}. \end{aligned}$$

We can express the coincidence rate as

$$(2.14) \qquad P_b(\boldsymbol{\tau};2) = v^\dagger R_b(\boldsymbol{\tau})v$$

where $v = \begin{bmatrix} U_{11}U_{22} \\ U_{12}U_{21} \end{bmatrix}$ and $R(\boldsymbol{\tau}) = \begin{bmatrix} r_{11}r_{22} & r_{12}r_{21} \\ r_{21}r_{12} & r_{22}r_{11} \end{bmatrix}$. We know that $r_{ij} = r_{ji}$ and $r_{ii} = 1$, so the coincidence rate can be written as

$$(2.15) \qquad \begin{bmatrix} U_{11}U_{22} \\ U_{12}U_{21} \end{bmatrix}^\dagger \begin{bmatrix} 1 & r_{12}^2 \\ r_{12}^2 & 1 \end{bmatrix} \begin{bmatrix} U_{11}U_{22} \\ U_{12}U_{21} \end{bmatrix}.$$

The 2×2 delay matrix carries the regular representation of S_2 since it can be written as a sum of permutation matrices

$$(2.16) \qquad \begin{bmatrix} 1 & r_{12}^2 \\ r_{12}^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_{12}^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix

$$(2.17) \qquad T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

block diagonalizes the regular representation, so we can write the coincidence rate as

$$P_b(\boldsymbol{\tau};2) = v^\dagger T^{-1}\left(TR_b(\boldsymbol{\tau})T^{-1}\right)Tv = (Tv)^\dagger \begin{bmatrix} 1+r_{12}^2 & 0 \\ 0 & 1-r_{12}^2 \end{bmatrix}(Tv) \tag{2.18}$$

where

$$Tv = \begin{bmatrix} \frac{1}{\sqrt{2}}\left(U_{11}U_{22}+U_{12}U_{21}\right) \\ \frac{1}{\sqrt{2}}\left(U_{11}U_{22}-U_{12}U_{21}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\operatorname{Perm}(U) \\ \frac{1}{\sqrt{2}}\operatorname{Det}(U) \end{bmatrix}. \tag{2.19}$$

Recall that the 2×2 delay matrix is

$$r(\boldsymbol{\tau}) = \begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix} \tag{2.20}$$

and notice that $\operatorname{Perm}(r(\boldsymbol{\tau})) = 1 + r_{12}^2$ and $\operatorname{Det}(r(\boldsymbol{\tau})) = 1 - r_{12}^2$. We get that

$$P_b(\boldsymbol{\tau};2) = \frac{1}{2}\operatorname{Perm}(r(\boldsymbol{\tau}))\left|\operatorname{Perm}(U)\right|^2 + \frac{1}{2}\operatorname{Det}(r(\boldsymbol{\tau}))\left|\operatorname{Det}(U)\right|^2. \tag{2.21}$$

When the arrival times are equal, the coincidence rate is

$$P_b(\boldsymbol{\tau};2) = \left|\operatorname{Perm}(U)\right|^2 \tag{2.22}$$

and when the arrival times are sufficiently far we have

$$P_b(\boldsymbol{\tau};2) = \frac{1}{2}\left|\operatorname{Perm}(U)\right|^2 + \frac{1}{2}\left|\operatorname{Det}(U)\right|^2. \tag{2.23}$$

As arrival times change, the resulting wavepackets go from fully overlapping to completely distinguishable, and the coincidence rate continuously changes from the permanent to an equal contribution of permanent and determinant. This behaviour is summarized in the following graph from [**22**].

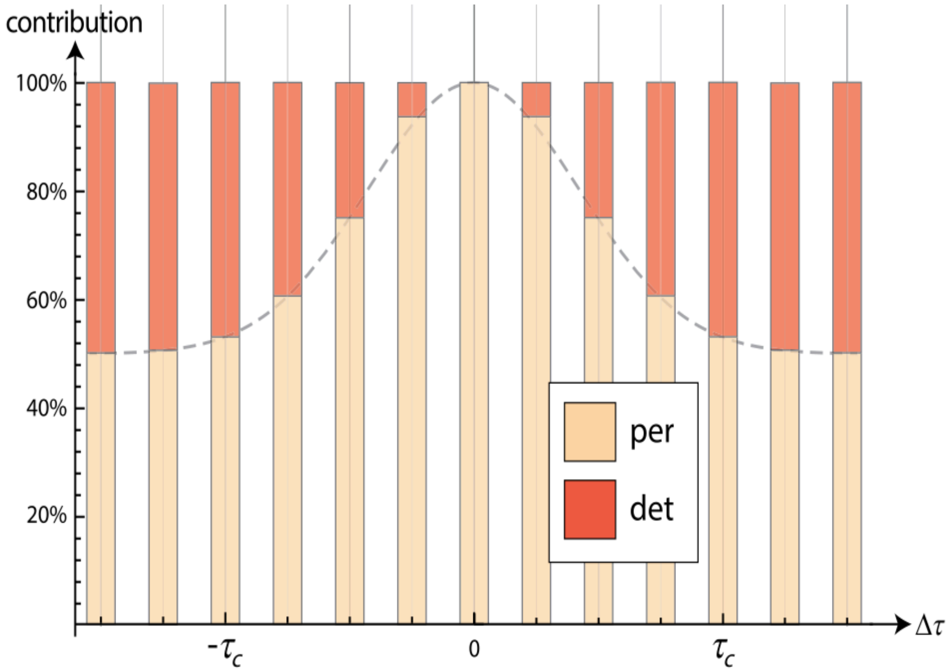


FIGURE 4.3. A graph showing how the contribution of the permanent and determinant to the coincidence rate for varying $\boldsymbol{\tau}$.

It's easy to derive the fermion coincidence rate from the boson case. Like before, the coincidence rate can be expressed as follows, except the $A_i^\dagger(\tau_j)$ terms are superpositions of fermionic operators:

$$
\begin{aligned}
P_f(\boldsymbol{\tau};2) = {} & U_{11}^* U_{22}^* U_{12} U_{21} \langle 0| A_2(\tau_2) A_1(\tau_1) A_2^\dagger(\tau_1) A_1^\dagger(\tau_2) |0\rangle \\
& + U_{11}^* U_{22}^* U_{11} U_{22} \langle 0| A_2(\tau_2) A_1(\tau_1) A_1^\dagger(\tau_1) A_2^\dagger(\tau_2) |0\rangle \\
& + U_{12}^* U_{21}^* U_{12} U_{21} \langle 0| A_1(\tau_2) A_2(\tau_1) A_2^\dagger(\tau_1) A_1^\dagger(\tau_2) |0\rangle \\
& + U_{12}^* U_{21}^* U_{11} U_{22} \langle 0| A_1(\tau_2) A_2(\tau_1) A_1^\dagger(\tau_1) A_2^\dagger(\tau_2) |0\rangle .
\end{aligned} \tag{2.24}
$$

Evaluating the first and fourth term in the above equation involves an odd number of interchanges of fermionic operators, so these terms will pick up a negative sign. We get that the fermionic version of equation 2.13 is

$$
\begin{aligned}
P_f(\boldsymbol{\tau};2) = {} & -U_{11}^* U_{22}^* U_{12} U_{21} r_{12} r_{21} + U_{11}^* U_{22}^* U_{11} U_{22} r_{11} r_{22} \\
& + U_{12}^* U_{21}^* U_{12} U_{21} r_{22} r_{11} - U_{12}^* U_{21}^* U_{11} U_{22} r_{21} r_{12}.
\end{aligned} \tag{2.25}
$$

These negative signs mean that

$$(2.26)\qquad P_f(\boldsymbol{\tau};2) = \begin{bmatrix} \frac{1}{\sqrt{2}}\,\mathrm{Perm}(U) \\ \frac{1}{\sqrt{2}}\,\mathrm{Det}(U) \end{bmatrix}^{\dagger} \begin{bmatrix} 1-r_{12}^2 & 0 \\ 0 & 1+r_{12}^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}\,\mathrm{Perm}(U) \\ \frac{1}{\sqrt{2}}\,\mathrm{Det}(U) \end{bmatrix}$$
$$= \frac{1}{2}\,\mathrm{Det}(r(\boldsymbol{\tau}))\,|\mathrm{Perm}(U)|^2 + \frac{1}{2}\,\mathrm{Perm}(r(\boldsymbol{\tau}))\,|\mathrm{Det}(U)|^2 .$$

When the particles have equal arrival times we have $\mathrm{Det}\,(r(\boldsymbol{\tau})) = 0$ and $\mathrm{Perm}\,(r(\boldsymbol{\tau})) = 2$, thus the coincidence rate is $|\mathrm{Det}(U)|^2$. When the scattering matrix is unitary, then this is equal to 1. When the particles have sufficiently separated arrival times, then $\mathrm{Det}\,(r(\boldsymbol{\tau})) = 1$ and $\mathrm{Perm}\,(r(\boldsymbol{\tau})) = 1$, which means the coincidence rate is $\frac{1}{2}\,|\mathrm{Perm}(U)|^2 + \frac{1}{2}\,|\mathrm{Det}(U)|^2$. For a $50:50$ beamsplitter, then this is equal to $\frac{1}{2}$.

3. Computing n-Particle Coincidence Rates

In this section, the coincidence rate for n partially-indistinguishable particles entering an interferometer with arbitrary arrival times are derived. To begin, we know that the output state is

$$(3.1)\qquad |\psi_{out}\rangle_C = \sum_{\sigma\in S_n} U_{1,\sigma(1)}U_{2,\sigma(2)}\dots U_{n,\sigma(n)} A^{\dagger}_{\sigma(1)}(\tau_1)A^{\dagger}_{\sigma(2)}(\tau_2)\dots A^{\dagger}_{\sigma(n)}(\tau_n)\,|0\rangle .$$

The probability of an outcome is obtained by taking the modulus square of the output state. We have

$$(3.2)\qquad {}_C\langle\psi_{out}|\,\psi_{out}\rangle_C$$
$$= \langle 0|\sum_{\sigma,\gamma\in S_n} U^*_{1,\gamma(1)}U^*_{2,\gamma(2)}\dots U^*_{n,\gamma(n)}U_{1,\sigma(1)}U_{2,\sigma(2)}\dots U_{n,\sigma(n)}\,\times$$
$$\left(A_{\gamma(n)}(\tau_n)\dots A_{\gamma(2)}(\tau_2)A_{\gamma(1)}(\tau_1)\right)\left(A^{\dagger}_{\sigma(1)}(\tau_1)A^{\dagger}_{\sigma(2)}(\tau_2)\dots A^{\dagger}_{\sigma(n)}(\tau_n)\right)|0\rangle$$
$$= \int d\bar{\omega}_1 d\omega_1 d\bar{\omega}_2 d\omega_2\dots d\bar{\omega}_n d\omega_n \phi^*(\bar{\omega}_1)\phi(\omega_1)\phi^*(\bar{\omega}_2)\phi(\omega_2)\dots\phi^*(\bar{\omega}_n)\phi(\omega_n)\times$$
$$e^{-i\tau_1(\omega_1-\bar{\omega}_1)}e^{-i\tau_2(\omega_2-\bar{\omega}_2)}\dots e^{-i\tau_n(\omega_n-\bar{\omega}_n)}\times$$
$$\sum_{\sigma,\gamma\in S_n} U^*_{1,\gamma(1)}U^*_{2,\gamma(2)}\dots U^*_{n,\gamma(n)}U_{1,\sigma(1)}U_{2,\sigma(2)}\dots U_{n,\sigma(n)}\times$$
$$\langle 0|\,a_{\gamma(n)}(\bar{\omega}_n)a_{\gamma(n-1)}(\bar{\omega}_{n-1})\dots a_{\gamma(1)}(\bar{\omega}_1)a^{\dagger}_{\sigma(1)}(\omega_1)\dots a^{\dagger}_{\sigma(n-1)}(\omega_{n-1})a^{\dagger}_{\sigma(n)}(\omega_n)\,|0\rangle .$$

The first step of this computation is evaluating the sequence of creation and destruction operators. We begin by assuming the particles to be bosons, the simpler of the two cases as there will be no negative signs to keep track of. The last line of equation 3.2 can be expressed as

(3.3)
$$\langle 0|\, b_n(\bar{\omega}_{\gamma^{-1}(n)})b_{n-1}(\bar{\omega}_{\gamma^{-1}(n-1)})\dots b_1(\bar{\omega}_{\gamma^{-1}(1)})b_1^\dagger(\omega_{\sigma^{-1}(1)})\dots b_{n-1}^\dagger(\omega_{\sigma^{-1}(n-1)})b_n^\dagger(\omega_{\sigma^{-1}(n)})\,|0\rangle\,.$$

Next we commute $b_n(\bar{\omega}_{\gamma^{-1}(n)})$ with its neighbours until it is directly to the left of $b_n^\dagger(\omega_{\sigma^{-1}(n)})\,|0\rangle$. We then use the commutation relations on these and are left with the delta function $\delta(\bar{\omega}_{\gamma^{-1}(n)} - \omega_{\sigma^{-1}(n)})$. This is repeated for all of the other indices to get

$$\delta(\bar{\omega}_{\gamma^{-1}(1)} - \omega_{\sigma^{-1}(1)})\delta(\bar{\omega}_{\gamma^{-1}(2)} - \omega_{\sigma^{-1}(2)})\dots\delta(\bar{\omega}_{\gamma^{-1}(n)} - \omega_{\sigma^{-1}(n)}). \tag{3.4}$$

When this expression is substituted into equation 3.2 the coincidence rate becomes

$$P_b(\boldsymbol{\tau};n) = \sum_{\sigma,\gamma\in S_n} U^*_{1,\gamma(1)}U^*_{2,\gamma(2)}\dots U^*_{n,\gamma(n)}U_{1,\sigma(1)}U_{2,\sigma(2)}\dots U_{n,\sigma(n)}\times$$
$$e^{\frac{-(\tau_{\gamma^{-1}(1)}-\tau_{\sigma^{-1}(1)})^2}{2}}e^{\frac{-(\tau_{\gamma^{-1}(2)}-\tau_{\sigma^{-1}(2)})^2}{2}}\dots e^{\frac{-(\tau_{\gamma^{-1}(n)}-\tau_{\sigma^{-1}(n)})^2}{2}}. \tag{3.5}$$

Using the r_{ij} notation we can express the coincidence rate as

$$P_b(\boldsymbol{\tau};n) = \sum_{\sigma,\gamma\in S_n} U^*_{1,\gamma(1)}U^*_{2,\gamma(2)}\dots U^*_{n,\gamma(n)}U_{1,\sigma(1)}U_{2,\sigma(2)}\dots U_{n,\sigma(n)}\times$$
$$r_{\gamma^{-1}(1),\sigma^{-1}(1)}r_{\gamma^{-1}(2),\sigma^{-1}(2)}\dots r_{\gamma^{-1}(n),\sigma^{-1}(n)}. \tag{3.6}$$

For fermions we pick up a negative sign every time we commute two operators so the last line of equation 3.2 can be expressed as

(3.7)
$$\mathrm{sgn}(\sigma\gamma)\,\langle 0|\, f_n(\bar{\omega}_{\gamma^{-1}(n)})f_{n-1}(\bar{\omega}_{\gamma^{-1}(n-1)})\dots f_1(\bar{\omega}_{\gamma^{-1}(1)})f_1^\dagger(\omega_{\sigma^{-1}(1)})\dots f_{n-1}^\dagger(\omega_{\sigma^{-1}(n-1)})f_n^\dagger(\omega_{\sigma^{-1}(n)})\,|0\rangle$$

The commutation relations are used an even number $2(n-1)$ times to move the operator $f_n(\bar{\omega}_{\gamma^{-1}(n)})$ from the far left until it is adjacent to $f_n^\dagger(\omega_{\sigma^{-1}(n)})$. Once they are adjacent the commutation relations are used once more to produce the delta function $\delta(\bar{\omega}_{\gamma^{-1}(n)} - \omega_{\sigma^{-1}(n)})$. This process is repeated for each index of the operators. We get that equation 3.7 becomes

$$\mathrm{sgn}(\sigma\gamma)\delta(\bar{\omega}_{\gamma^{-1}(1)} - \omega_{\sigma^{-1}(1)})\delta(\bar{\omega}_{\gamma^{-1}(2)} - \omega_{\sigma^{-1}(2)})\dots\delta(\bar{\omega}_{\gamma^{-1}(n)} - \omega_{\sigma^{-1}(n)}). \tag{3.8}$$

Thus the coincidence rate is

$$P_f(\boldsymbol{\tau}; n) = \sum_{\sigma,\gamma \in S_n} \operatorname{sgn}(\sigma\gamma) U^*_{1,\gamma(1)} U^*_{2,\gamma(2)} \dots U^*_{n,\gamma(n)} U_{1,\sigma(1)} U_{2,\sigma(2)} \dots U_{n,\sigma(n)} \times$$
$$r_{\gamma^{-1}(1),\sigma^{-1}(1)} r_{\gamma^{-1}(2),\sigma^{-1}(2)} \dots r_{\gamma^{-1}(n),\sigma^{-1}(n)}. \tag{3.9}$$

Rather than a large sum, we can write the boson and fermion coincidence rates as a matrix multiplied by a row and column vector

$$\begin{aligned} P_b(\boldsymbol{\tau}; n) &= v^\dagger R_b(\boldsymbol{\tau}) v \\ P_f(\boldsymbol{\tau}; n) &= v^\dagger R_f(\boldsymbol{\tau}) v. \end{aligned} \tag{3.10}$$

To do this, we choose some ordering for the elements γ_i of S_n and put them in the vector v. For brevity, we write

$$U_\gamma := U_{1\gamma(1)} U_{2\gamma(2)} \dots U_{n\gamma(n)}. \tag{3.11}$$

The vector v can then be written as

$$v = \begin{bmatrix} U_{\gamma_1} \\ U_{\gamma_2} \\ \vdots \\ U_{\gamma_{n!}} \end{bmatrix}. \tag{3.12}$$

The ij^{th} entry of the matrix $R_b(\boldsymbol{\tau})$, which we call the **bosonic rate matrix**, is

$$r_{\gamma_i^{-1}(1),\gamma_j^{-1}(1)} r_{\gamma_i^{-1}(2),\gamma_j^{-1}(2)} \dots r_{\gamma_i^{-1}(n),\gamma_j^{-1}(n)}. \tag{3.13}$$

When the matrix multiplication is carried out, it's clear that

$$P_b(\boldsymbol{\tau}; n) = v^\dagger R_b(\boldsymbol{\tau}) v. \tag{3.14}$$

The **fermionic rate matrix** is constructed similarly, the only difference is that the ij^{th} entry of the matrix $R_f(\boldsymbol{\tau})$ is

$$\operatorname{sgn}(\gamma_i\gamma_j) r_{\gamma_i^{-1}(1),\gamma_j^{-1}(1)} r_{\gamma_i^{-1}(2),\gamma_j^{-1}(2)} \dots r_{\gamma_i^{-1}(n),\gamma_j^{-1}(n)}. \tag{3.15}$$

This section concludes with an example for $n = 3$ particles.

EXAMPLE 4.2. Consider three particles. We choose the ordering

$$v = \begin{bmatrix} U_e \\ U_{(12)} \\ U_{(13)} \\ U_{(23)} \\ U_{(123)} \\ U_{(132)} \end{bmatrix}. \tag{3.16}$$

By using the fact that $r_{ii} = 1$ and $r_{ij} = r_{ji}$, we write $R_b(\boldsymbol{\tau})$ as

$$R_b(\boldsymbol{\tau}) = \begin{bmatrix} 1 & r_{12}^2 & r_{13}^2 & r_{23}^2 & r_{12}r_{23}r_{31} & r_{12}r_{23}r_{31} \\ r_{12}^2 & 1 & r_{12}r_{23}r_{31} & r_{12}r_{23}r_{31} & r_{23}^2 & r_{13}^2 \\ r_{13}^2 & r_{12}r_{23}r_{31} & 1 & r_{12}r_{23}r_{31} & r_{12}^2 & r_{23}^2 \\ r_{23}^2 & r_{12}r_{23}r_{31} & r_{12}r_{23}r_{31} & 1 & r_{13}^2 & r_{12}^2 \\ r_{12}r_{23}r_{31} & r_{23}^2 & r_{12}^2 & r_{13}^2 & 1 & r_{12}r_{23}r_{31} \\ r_{12}r_{23}r_{31} & r_{13}^2 & r_{23}^2 & r_{12}^2 & r_{12}r_{23}r_{31} & 1 \end{bmatrix} \tag{3.17}$$

For the fermionic case, we add a negative sign to the entries when $\gamma_i\gamma_j$ is odd; the result is as follows.

$$R_f(\boldsymbol{\tau}) = \begin{bmatrix} 1 & -r_{12}^2 & -r_{13}^2 & -r_{23}^2 & r_{12}r_{23}r_{31} & r_{12}r_{23}r_{31} \\ -r_{12}^2 & 1 & r_{12}r_{23}r_{31} & r_{12}r_{23}r_{31} & -r_{23}^2 & -r_{13}^2 \\ -r_{13}^2 & r_{12}r_{23}r_{31} & 1 & r_{12}r_{23}r_{31} & -r_{12}^2 & -r_{23}^2 \\ -r_{23}^2 & r_{12}r_{23}r_{31} & r_{12}r_{23}r_{31} & 1 & -r_{13}^2 & -r_{12}^2 \\ r_{12}r_{23}r_{31} & -r_{23}^2 & -r_{12}^2 & -r_{13}^2 & 1 & r_{12}r_{23}r_{31} \\ r_{12}r_{23}r_{31} & -r_{13}^2 & -r_{23}^2 & -r_{12}^2 & r_{12}r_{23}r_{31} & 1 \end{bmatrix}. \tag{3.18}$$

Each matrix can be written as a sum of permutation matrices, like in equation 4.13 of chapter 1. Hence, the rate matrix carries the regular representation of S_n. There exists a linear transformation T that block-diagonalizes the rate matrix. We choose a transformation that makes the irreps in the diagonalization unitary, so their rows and columns are orthonormal. We begin with the boson case

$$\begin{aligned} P_b(\boldsymbol{\tau}; n) &= v^\dagger R_b(\boldsymbol{\tau})\, v \\ &= v^\dagger T^{-1} \left(T R_b(\boldsymbol{\tau}) T^{-1}\right) T v \\ &= (Tv)^\dagger \left(T R_b(\boldsymbol{\tau}) T^{-1}\right) (Tv)\,. \end{aligned} \tag{3.19}$$

Gothic letters will be used to simplify notation: $\mathfrak{v} = Tv$ and $\mathfrak{R}_b = TR_b(\boldsymbol{\tau})T^{-1}$. We write $\mathfrak{R}^\lambda$ to denote the blocks of $\mathfrak{R}_b$ that carry the λ-representation of S_n. There are s_λ such blocks that appear in the diagonalization and each copy gets multiplied by s_λ entries of the vector $\mathfrak{v}$, these entries are denoted by $\mathfrak{v}_{\lambda;i}$. The index i ranges from 1 to s_λ to denote the different occurrences of the λ-irrep. The notation $\mathfrak{v}^j_{\lambda;i}$ refers to the j^{th} entry of the vector $\mathfrak{v}_{\lambda;i}$. For example, the coincidence rate for the 3 boson case is as follows.

$$
\begin{aligned}
(3.20)\qquad P_b(\boldsymbol{\tau};3) &= \mathfrak{v}^\dagger \mathfrak{R}_b \mathfrak{v} \\
&= \begin{bmatrix} \mathfrak{v}_{\ydiagram{3};1} \\ \mathfrak{v}_{\ydiagram{1,1,1};1} \\ \mathfrak{v}^1_{\ydiagram{2,1};1} \\ \mathfrak{v}^2_{\ydiagram{2,1};1} \\ \mathfrak{v}^1_{\ydiagram{2,1};2} \\ \mathfrak{v}^2_{\ydiagram{2,1};2} \end{bmatrix}^\dagger
\begin{bmatrix} \mathfrak{R}^{\ydiagram{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{R}^{\ydiagram{1,1,1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{R}^{\ydiagram{2,1}}_{1,1} & \mathfrak{R}^{\ydiagram{2,1}}_{1,2} & 0 & 0 \\ 0 & 0 & \mathfrak{R}^{\ydiagram{2,1}}_{2,1} & \mathfrak{R}^{\ydiagram{2,1}}_{2,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathfrak{R}^{\ydiagram{2,1}}_{1,1} & \mathfrak{R}^{\ydiagram{2,1}}_{1,2} \\ 0 & 0 & 0 & 0 & \mathfrak{R}^{\ydiagram{2,1}}_{2,1} & \mathfrak{R}^{\ydiagram{2,1}}_{2,2} \end{bmatrix}
\begin{bmatrix} \mathfrak{v}_{\ydiagram{3};1} \\ \mathfrak{v}_{\ydiagram{1,1,1};1} \\ \mathfrak{v}^1_{\ydiagram{2,1};1} \\ \mathfrak{v}^2_{\ydiagram{2,1};1} \\ \mathfrak{v}^1_{\ydiagram{2,1};2} \\ \mathfrak{v}^2_{\ydiagram{2,1};2} \end{bmatrix}.
\end{aligned}
$$

For any value of n, the coincidence rate can be expressed as

$$(3.21)\qquad P_b(\boldsymbol{\tau};n) = \sum_{\lambda\vdash n}\sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda;i}\,\mathfrak{R}^\lambda\,\mathfrak{v}_{\lambda;i}.$$

The fermion case has one key difference from the boson case. Recall that conjugate representations of S_n only differ by a sign in their odd permutations, and that the bosonic and fermionic rate matrices have this symmetry. As a result when the fermionic rate matrix is block-diagonalized into irreducible representations by T, every representation λ that appears in the block-diagonalization is replaced by its conjugate λ^*. Thus, coincidence rate for the n-fermion case is

$$(3.22)\qquad P_f(\boldsymbol{\tau};n) = \sum_{\lambda\vdash n}\sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda;i}\,\mathfrak{R}^{\lambda^*}\,\mathfrak{v}_{\lambda;i}.$$

4. Expressing Coincidence Rates as Functions of Immanants

In this section, it will be shown that the coincidence rates that were derived in the previous section can be written in terms of immanants. To show this, I will need a result of Kostant [**25**], which states that the λ-immanant of a $GL(n;\mathbb{C})$ matrix can be expressed as a sum of $\mathcal{D}$-functions, which are overlaps of basis states of the λ-representation of $GL(n;\mathbb{C})$ that have weight $(1,1,\ldots,1)$. The $\mathcal{D}$ stands for *Darstellung*, the German word for representation. Kostant's theorem will be used to prove two results. Firstly, it will be shown that if the λ-immanant of the delay matrix $r(\boldsymbol{\tau})$ is 0, then all entries of the matrix $\mathfrak{R}^\lambda$ will also be 0. Secondly, Kostant's result will be expanded on by showing that permuted immanants can also be expressed as a sum of $\mathcal{D}$-functions. This section will refer to the representation theory of the general linear group, for a more detailed discussion on the topic see [**28**]; for an interferometry-related discussion see [**27**].

As discussed in chapter 2, the partition $\lambda \vdash n$ labels an irreducible representation ρ_λ of S_n, which is of dimension $s_\lambda := |\mathcal{SYT}_\lambda|$; λ also labels a partition of the general linear group, which is of dimension $d_\lambda := |\mathcal{SSYT}_\lambda|$. Consider the representation D_λ of $GL(n;\mathbb{C})$

$$D_\lambda : GL(n;\mathbb{C}) \to GL(d_\lambda;\mathbb{C}) \\ A \to D_\lambda(A). \tag{4.1}$$

A $\mathcal{D}$-function is a function of the form

$$\mathcal{D}^\lambda_{ij}(A) = \langle \psi^\lambda_i | D_\lambda(A) | \psi^\lambda_j \rangle , \tag{4.2}$$

where $|\psi^\lambda_i\rangle$ is a basis vector of the representation space of the D_λ representation of $GL(n;\mathbb{C})$. Define $\mathcal{D}^\lambda(A)$ to be the matrix with $\mathcal{D}^\lambda_{ij}(A)$ as its ij^{th} entry. In this section we are only interested in the functions for which the states $|\psi^\lambda_i\rangle$ are basis vectors in the $(1,1,1,...,1)$ weight subspace of D_λ. Recall that a vector $v \in \mathbb{C}^n$ is of weight $w = (w_1, w_2, \ldots, w_n)$ when

$$\text{diag}(t_1, t_2, \ldots t_n)v = t_1^{w_1} t_2^{w_2} \ldots t_n^{w_n} v. \tag{4.3}$$

In section 4 of [**26**], it's shown that the n-fold product of the defining representation of $GL(n;\mathbb{C})$, when restricted to the $(1,1,\ldots,1)$ subspace is equivalent to the regular representation of S_n. In particular, when we take A to be the delay matrix $r(\boldsymbol{\tau})$, then $\mathcal{D}^\lambda(A)$ is simply $\mathfrak{R}^\lambda$, the λ-block of the diagonalized rate matrix. Now we state the aforementioned theorem from [**25**] that relates immanants and $\mathcal{D}$-functions.

Theorem 4.3. *(Kostant) Given $A \in GL(n;\mathbb{C})$, it follows that*

$$\text{Imm}^\lambda(A) = \sum_i \mathcal{D}^\lambda_{i,i}(A). \tag{4.4}$$

I use this result to prove the following two corollaries.

Corollary 4.4. *Let $r(\boldsymbol{\tau})$ be the $n \times n$ delay matrix and $\mathfrak{R}^\lambda$ be the block that carries the λ-representation of S_n in the block diagonalization of the rate matrix $R(\boldsymbol{\tau})$. If* $\text{Imm}^\lambda(r(\boldsymbol{\tau})) = 0$*, then every entry of the matrix $\mathfrak{R}^\lambda$ is also* 0*.*

Proof. Begin by assuming that $\text{Imm}^\lambda(r(\boldsymbol{\tau})) = 0$. Applying the previous theorem to the delay matrix, it follows that the trace of $\mathfrak{R}^\lambda$ is 0. From equation 4.2, we see that $\mathfrak{R}^\lambda$ can be written as an inner product $\mathfrak{R}^\lambda_{ij} = \langle \psi^\lambda_i, \psi^\lambda_j \rangle$, thus $\mathfrak{R}^\lambda$ is a Gram matrix and has non-negative eigenvalues. Since the trace of a matrix is the sum of its eigenvalues, the trace of $\mathfrak{R}^\lambda$ is 0, and $\mathfrak{R}^\lambda$ has all non-negative eigenvalues, it follows that all the eigenvalues of $\mathfrak{R}^\lambda$ are zero. Since all the eigenvalues of $\mathfrak{R}^\lambda$ are zero, it must be nilpotent; however, $\mathfrak{R}^\lambda$ is also symmetric and it's well-known that the only matrix that is both symmetric and nilpotent is the zero matrix. □

COROLLARY 4.5. *Permuted immanants are linear combinations of $\mathcal{D}$-functions (and vice versa).*

PROOF. Let P_σ be the permutation matrix corresponding to the permutation σ. Depending on whether P_σ acts on the left or the right of a matrix A, either the rows or columns of A get permuted. In this proof, P_σ will be acting on the left, which permutes the rows. To prove this corollary, the result of theorem 4.3 is applied to the row-permuted matrix $P_\sigma A$:

$$\begin{aligned}
\text{Imm}^\lambda(P_\sigma A) &= \sum_i \mathcal{D}^\lambda_{i,i}(P_\sigma A) \\
&= \sum_i \left\langle \psi_i^\lambda \right| D_\lambda(P_\sigma) D_\lambda(A) \left| \psi_i^\lambda \right\rangle \\
&= \sum_i \left(\sum_j a_{ij} \left\langle \psi_j^\lambda \right| D_\lambda(A)) \left| \psi_i^\lambda \right\rangle \right) \\
&= \sum_{ij} a_{ij} \left\langle \psi_j^\lambda \right| D_\lambda(A) \left| \psi_i^\lambda \right\rangle \\
&= \sum_{ij} a_{ij} \mathcal{D}^\lambda_{ji}(A).
\end{aligned} \tag{4.5}$$

Note that the coefficient can be expressed as $a_{ij} = \left\langle \psi_i^\lambda \right| D_\lambda(P_\sigma) \left| \psi_j^\lambda \right\rangle$. From the above computation, we see that an immanant of $P_\sigma A$ can be expressed as a linear combination of $\mathcal{D}$-functions, which in turn means that all the entries of $\mathcal{D}^\lambda(A)$ are linear combinations of permuted λ-immanants. In particular, for an arbitrary matrix A, this corollary shows that of the total $n!$ possible permuted immanants, there are exactly s_λ^2 linearly independent permuted immanants of shape λ, since there are that many $\mathcal{D}$-functions in $\mathcal{D}^\lambda(A)$. □

The same linear transformation T that block-diagonalizes the rate matrix also acts on the scattering matrix U. Here, $\mathcal{D}^\lambda(U)$ is the matrix whose i^{th} column is $\mathfrak{v}_{\lambda;i}$. This is illustrated in the following example. Consider the case of $n = 3$ bosons, the diagonalized rate matrix is equal to the following. To simplify notation, λ_σ^A is used to denote the λ immanant of the matrix A, whose rows are permuted by σ. There are $3! = 6$ permuted immanants, but only a linear combination of s_λ^2 of them are needed to write an entry of $\mathfrak{R}^\lambda$; I choose the permutations e, (12), (23), and (132).

(4.6)

$$\mathfrak{R}(\boldsymbol{\tau}) = \begin{bmatrix} \ydiagram{3}^r & 0 & 0 & 0 & 0 & 0 \\ 0 & \ydiagram{1,1,1}^r & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(\ydiagram{2,1}^r_e + \ydiagram{2,1}^r_{(12)}\right) & \sqrt{3}\left(-\frac{1}{6}\ydiagram{2,1}^r_e + \frac{1}{3}\ydiagram{2,1}^r_{(23)} + \frac{1}{6}\ydiagram{2,1}^r_{(12)} - \frac{1}{3}\ydiagram{2,1}^r_{(132)}\right) & 0 & 0 \\ 0 & 0 & \sqrt{3}\left(-\frac{1}{6}\ydiagram{2,1}^r_e + \frac{1}{3}\ydiagram{2,1}^r_{(23)} + \frac{1}{6}\ydiagram{2,1}^r_{(12)} - \frac{1}{3}\ydiagram{2,1}^r_{(132)}\right) & \frac{1}{2}\left(\ydiagram{2,1}^r_e - \ydiagram{2,1}^r_{(12)}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}.$$

The bottom 2×2 block is the same as the other 2×2 block. For the column vector $\mathfrak{v}$, we have

$$\mathfrak{v} = \begin{bmatrix} \frac{1}{\sqrt{6}}\ydiagram{3}^U \\ \frac{1}{\sqrt{6}}\ydiagram{1,1,1}^U \\ \frac{1}{2\sqrt{3}}\left(\ydiagram{2,1}^U_e + \ydiagram{2,1}^U_{(12)}\right) \\ -\frac{1}{6}\ydiagram{2,1}^U_e + \frac{1}{3}\ydiagram{2,1}^U_{(23)} + \frac{1}{6}\ydiagram{2,1}^U_{(12)} - \frac{1}{3}\ydiagram{2,1}^U_{(132)} \\ \frac{1}{6}\ydiagram{2,1}^U_e + \frac{1}{3}\ydiagram{2,1}^U_{(23)} + \frac{1}{6}\ydiagram{2,1}^U_{(12)} + \frac{1}{3}\ydiagram{2,1}^U_{(132)} \\ \frac{1}{2\sqrt{3}}\left(\ydiagram{2,1}^U_e - \ydiagram{2,1}^U_{(12)}\right) \end{bmatrix}. \tag{4.7}$$

5. Applying Gamas' Theorem to Coincidence Rates

In this final section, I show how Gamas' theorem can be used to determine which immanants will appear in the coincidence rate equation for a given set of arrival times $\boldsymbol{\tau}$. To do this, we need to make a correspondence between a set of arrival times $\boldsymbol{\tau}$ and a partition $\mu \vdash n$. This correspondence is as follows: for the partition $\mu = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_l\}$, there are μ_i particles have the same time of arrival. We call this the **delay partition** and denote it by μ_τ.

Example 4.6. The followings sets of arrival times correspond to the following delay partitions.

$$\begin{aligned} (\tau_1, \tau_1, \tau_1, \tau_1) &\to \mu_\tau = \ydiagram{4} \\ (\tau_1, \tau_1, \tau_1, \tau_2) &\to \mu_\tau = \ydiagram{3,1} \\ (\tau_1, \tau_1, \tau_2, \tau_2) &\to \mu_\tau = \ydiagram{2,2} \\ (\tau_1, \tau_1, \tau_2, \tau_3) &\to \mu_\tau = \ydiagram{2,1,1} \\ (\tau_1, \tau_2, \tau_3, \tau_4) &\to \mu_\tau = \ydiagram{1,1,1,1} \end{aligned} \tag{5.1}$$

Note that the ordering of the arrival times does not matter. Both $(\tau_1, \tau_1, \tau_1, \tau_2)$ and $(\tau_1, \tau_1, \tau_2, \tau_1)$ have three particles arriving simultaneously and one particle arriving at a different time, each of these sets of arrival times corresponds to the same delay partition. In this sense, a same delay partition is assigned to an equivalence class of arrival times, which differ only by a permutation. The indices are organized from lowest to highest to make the notation consistent.

Let τ be a set of arrival times that corresponds to delay partition μ_τ. From Gamas' theorem, we know that if $\lambda \lhd \mu_\tau$, then $\mathrm{Imm}^\lambda(r(\boldsymbol{\tau})) = 0$; furthermore, the entire matrix $\mathfrak{R}^\lambda$ will be zero. As a result, all terms containing these immanants are automatically zero. The coincidence rate equations can then be written as

$$
\begin{aligned}
P_b(\boldsymbol{\tau}; n) &= \sum_{\lambda \not\lhd \mu_\tau} \sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda;i}\, \mathfrak{R}^\lambda\, \mathfrak{v}_{\lambda;i}, \\
P_f(\boldsymbol{\tau}; n) &= \sum_{\lambda \not\lhd \mu_\tau} \sum_{i=1}^{s_\lambda} \mathfrak{v}^\dagger_{\lambda^*;i}\, \mathfrak{R}^\lambda\, \mathfrak{v}_{\lambda^*;i}.
\end{aligned} \tag{5.2}
$$

This thesis concludes with an in-depth example for the four-particle case. Here, the block-diagonalized rate matrix $\mathfrak{R}(\boldsymbol{\tau})$ is of the form

$$
\mathfrak{R}(\boldsymbol{\tau}) = \mathfrak{R}^{(4)} \oplus \mathfrak{R}^{(3,1)} \oplus \mathfrak{R}^{(3,1)} \oplus \mathfrak{R}^{(3,1)} \oplus \mathfrak{R}^{(2,2)} \oplus \mathfrak{R}^{(2,2)} \oplus \mathfrak{R}^{(2,1,1)} \oplus \mathfrak{R}^{(2,1,1)} \oplus \mathfrak{R}^{(2,1,1)} \oplus \mathfrak{R}^{(1,1,1,1)}, \tag{5.3}
$$

$$
\mathfrak{R}(\boldsymbol{\tau}) = \begin{pmatrix}
\mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X} & \mathbf{X} & \mathbf{X} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{X}
\end{pmatrix}.
$$

There are five possible cases for the arrival times.

- When $\boldsymbol{\tau} = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ the corresponding delay partition is $\ydiagram{1,1,1,1}$. There are no partitions that are dominated by $\ydiagram{1,1,1,1}$, so there are no $\mathfrak{R}^\lambda$ terms that can be eliminated from the rate calculation.
- When $\boldsymbol{\tau} = \{\tau_1, \tau_1, \tau_2, \tau_3\}$ the corresponding delay partition is $\ydiagram{2,1,1}$. The delay partition dominates $\ydiagram{1,1,1,1}$, so $\mathfrak{R}^{\ydiagram{1,1,1,1}} = 0$. The simplified coincidence rates are

(5.4)
$$\begin{aligned} P_b(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{4};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{4};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{3,1};i}\mathfrak{R}^{\ydiagram{3,1}}\mathfrak{v}_{\ydiagram{3,1};i} + \sum_{i=1}^{2}\mathfrak{v}^\dagger_{\ydiagram{2,2};i}\mathfrak{R}^{\ydiagram{2,2}}\mathfrak{v}_{\ydiagram{2,2};i} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{2,1,1};i}\mathfrak{R}^{\ydiagram{2,1,1}}\mathfrak{v}_{\ydiagram{2,1,1};i}, \\ P_f(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{1,1,1,1};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{1,1,1,1};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{2,1,1};i}\mathfrak{R}^{\ydiagram{3,1}}\mathfrak{v}_{\ydiagram{2,1,1};i} + \sum_{i=1}^{2}\mathfrak{v}^\dagger_{\ydiagram{2,2};1}\mathfrak{R}^{\ydiagram{2,2}}\mathfrak{v}_{\ydiagram{2,2};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{3,1};i}\mathfrak{R}^{\ydiagram{2,1,1}}\mathfrak{v}_{\ydiagram{3,1};i}. \end{aligned}$$

- When $\boldsymbol{\tau} = \{\tau_1, \tau_1, \tau_2, \tau_2\}$ the corresponding delay partition is $\ydiagram{2,2}$, thus the $\mathfrak{R}^{\ydiagram{1,1,1,1}}$ and $\mathfrak{R}^{\ydiagram{2,1,1}}$ terms are equal to zero since both $\ydiagram{1,1,1,1}$ and $\ydiagram{2,1,1}$ are dominated by the delay partition. The coincidence rates can be simplified further.

(5.5)
$$\begin{aligned} P_b(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{4};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{4};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{3,1};i}\mathfrak{R}^{\ydiagram{3,1}}\mathfrak{v}_{\ydiagram{3,1};i} + \sum_{i=1}^{2}\mathfrak{v}^\dagger_{\ydiagram{2,2};i}\mathfrak{R}^{\ydiagram{2,2}}\mathfrak{v}_{\ydiagram{2,2};i}, \\ P_f(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{1,1,1,1};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{1,1,1,1};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{2,1,1};i}\mathfrak{R}^{\ydiagram{3,1}}\mathfrak{v}_{\ydiagram{2,1,1};i} + \sum_{i=1}^{2}\mathfrak{v}^\dagger_{\ydiagram{2,2};i}\mathfrak{R}^{\ydiagram{2,2}}\mathfrak{v}_{\ydiagram{2,2};i}. \end{aligned}$$

- When $\boldsymbol{\tau} = \{\tau_1, \tau_1, \tau_1, \tau_2\}$, the corresponding delay partition is $\ydiagram{3,1}$. The delay partition dominates $\ydiagram{2,2}$, $\ydiagram{2,1,1}$, and $\ydiagram{1,1,1,1}$, so the coincidence rates are

(5.6)
$$\begin{aligned} P_b(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{4};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{4};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{3,1};i}\mathfrak{R}^{\ydiagram{3,1}}\mathfrak{v}_{\ydiagram{3,1};i}, \\ P_f(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{1,1,1,1};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{1,1,1,1};1} + \sum_{i=1}^{3}\mathfrak{v}^\dagger_{\ydiagram{2,1,1};i}\mathfrak{R}^{\ydiagram{3,1}}\mathfrak{v}_{\ydiagram{2,1,1};i}. \end{aligned}$$

- When all particles enter simultaneously $\boldsymbol{\tau} = \{\tau_1, \tau_1, \tau_1, \tau_1\}$, then the coincidence rates are

(5.7)
$$\begin{aligned} P_b(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{4};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{4};1} \\ &= \mathrm{Perm}(r(\boldsymbol{\tau}))\left|\mathrm{Perm}(U)\right|^2, \\ P_f(\boldsymbol{\tau};4) &= \mathfrak{v}^\dagger_{\ydiagram{1,1,1,1};1}\mathfrak{R}^{\ydiagram{4}}\mathfrak{v}_{\ydiagram{1,1,1,1};1} \\ &= \mathrm{Perm}(r(\boldsymbol{\tau}))\left|\mathrm{Det}(U)\right|^2. \end{aligned}$$

CHAPTER 5

Conclusion

In this thesis I have introduced the delay matrix $r(\boldsymbol{\tau})$ in order to derive an expression for the coincidence rate of n particles in an n-channel interferometer. The respective expressions for bosons and fermions are as follows.

$$P_b(\boldsymbol{\tau}; n) = \sum_{\lambda \vdash n} \sum_{i=1}^{s_\lambda} \mathfrak{v}_{\lambda;i}^\dagger \, \mathfrak{R}^\lambda \, \mathfrak{v}_{\lambda;i}, \tag{0.1}$$
$$P_f(\boldsymbol{\tau}; n) = \sum_{\lambda \vdash n} \sum_{i=1}^{s_\lambda} \mathfrak{v}_{\lambda^*;i}^\dagger \, \mathfrak{R}^\lambda \, \mathfrak{v}_{\lambda^*;i}.$$

Every entry of the matrix $\mathfrak{R}^\lambda$ is a linear combination of permuted λ-immanants of the delay matrix; and each $\mathfrak{v}_{\lambda;i}$ is a vector where each entry is a linear combination of permuted λ-immanants of the scattering matrix U. The second important result of this thesis is using Gamas' theorem to determine which immanants will automatically be zero in the coincidence rate equations for a given set of arrival times $\boldsymbol{\tau}$. This is achieved by introducing the delay partition μ_τ, which is a partition of n that corresponds to the set of arrival times. The simplified coincidence rates are as follows.

$$P_b(\boldsymbol{\tau}; n) = \sum_{\lambda \ntriangleleft \mu_\tau} \sum_{i=1}^{s_\lambda} \mathfrak{v}_{\lambda;i}^\dagger \, \mathfrak{R}^\lambda \, \mathfrak{v}_{\lambda;i}, \tag{0.2}$$
$$P_f(\boldsymbol{\tau}; n) = \sum_{\lambda \ntriangleleft \mu_\tau} \sum_{i=1}^{s_\lambda} \mathfrak{v}_{\lambda^*;i}^\dagger \, \mathfrak{R}^\lambda \, \mathfrak{v}_{\lambda^*;i}.$$

Recall that $\triangleleft$ is the dominance ordering of partitions, and that the sum is over all partitions that are not dominated by μ_τ.

In the future I aim to apply the work done in this thesis to anyons, a two-dimensional particle that generalizes bosons and fermions. Recall that a system with two indistinguishable particles that are in states Ψ_1 and Ψ_2, respectively, can be denoted by the state $|\Psi_1\Psi_2\rangle$. When the states of the two particles are exchanged in the clockwise direction, then the new state differs by a phase factor.

$$|\Psi_1\Psi_2\rangle \rightarrow e^{i\theta}\,|\Psi_2\Psi_1\rangle\,. \tag{0.3}$$

A counter-clockwise exchange results in an inverse of the previous phase factor

$$|\Psi_1\Psi_2\rangle \rightarrow e^{-i\theta}\,|\Psi_2\Psi_1\rangle\,. \tag{0.4}$$

For bosons $\theta = 0$ and for fermions $\theta = \pi$, which yields the following.

$$\begin{aligned} &\text{Bosons:} \quad |\Psi_1\Psi_2\rangle \rightarrow |\Psi_2\Psi_1\rangle \\ &\text{Fermions:} \, |\Psi_1\Psi_2\rangle \rightarrow -\,|\Psi_2\Psi_1\rangle \end{aligned} \tag{0.5}$$

Under two exchanges in the same direction, both fermionic and bosonic systems return to their original state. For anyons the θ term is arbitrary, so under two exchanges in the same direction the resultant state picks up a square of the phase factor

$$|\Psi_1\Psi_2\rangle \rightarrow e^{2i\theta}\,|\Psi_1\Psi_2\rangle\,. \tag{0.6}$$

The symmetric group describes the transformation from an initial configuration of particles to a final configuration.

$$|\Psi_1\Psi_2\ldots\Psi_n\rangle \rightarrow \left|\Psi_{\sigma(1)}\Psi_{\sigma(2)}\ldots\Psi_{\sigma(n)}\right\rangle. \tag{0.7}$$

In the above, a bosonic system picks up no phase factor and a fermionic system picks up a negative sign when σ is an odd permutation. Clearly, the symmetric group S_n is inadequate to describe anyons because it only permutes elements; to know the overall phase factor we need to know about every exchange between particles - not just the initial and final positions. To do this the more robust braid group $\mathcal{B}_n$ needs to be used. The presentation for this group is as follows.

$$\mathcal{B}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} |\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,\forall i \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ when } |i-j| \geq 2\rangle. \tag{0.8}$$

Elements of the braid group can be thought of as n strings oriented vertically. Strings can braid over or under a neighbouring string. The element σ_i denotes the braid where the strand in the i^{th} position crosses over the strand in the $i+1^{th}$ position. By convention, the first position is on the far left. The following figures show the relations for $\mathcal{B}_n$ on a braid diagram. The intuition behind these relations is that two braids are equivalent if one can be continuously deformed into the other.

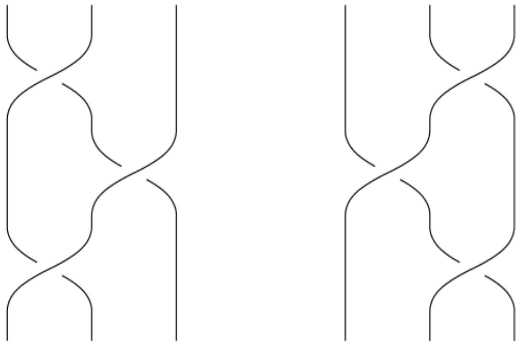


FIGURE 5.1. The relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

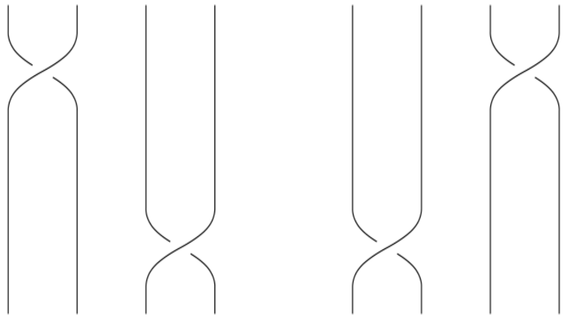


FIGURE 5.2. The relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i - j| \geq 2$

Interferometry with two anyons can been considered in [**30**] and [**31**]. I hope to explore the case when $n = 3$ and find an expression for the coincidence rate in terms of immanants.

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