# CONSTRUCTING UNITIZING: THE CRITICAL STRATEGIES AND MODELS THAT BUILD THIS ESSENTIAL MATHEMATICAL CONCEPT

by

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**Abstract**

The focus of this longitudinal case study was to investigate the progressive development of unitizing in a cohort of students receiving reform-oriented mathematics instruction. One-on-one videotaped mathematics interviews were conducted twice annually for 4 years from Grade 1 to Grade 4 with a varying number of participants from 61 to 45 respectively. Multiplication and quotative division questions were analyzed for correctness, the physical model used to solve the problem, and the computation strategy used to solve the problem. Multiplication and adding up for division models that required the development of more sophisticated unitizing included modelling only one group, modelling just the groups, modelling the groups with the composite numeral, and modelling the new whole. Multiplication and adding up for division strategies that indicated the development of a unitizing structure included rhythmic counting, starting with a doublet, skip counting, regrouping to form a new composite, and splitting the composite and then iterating the sub-composites to find the total. The varying levels of simultaneity demonstrated by students were also noted as they pertain to the development of a unitizing structure. A theoretical landscape of the development unitizing is proposed.

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# CHAPTER ONE

# INTRODUCTION

## Context of the Study

Proportional reasoning is a watershed concept in mathematics (Lesh, Post, & Behr, 1988), and comprehension of this idea is pivotal for future mathematical understanding. Many students who struggle in mathematics, particularly in the intermediate grades and beyond, lack true proportional reasoning. Lamon (1999) contended that more than half the adult population cannot reason proportionally across situations. Multiplicative reasoning, which begins to develop for some students as early as Grade 1, is foundational to the development of the concepts of ratio and proportion (Mulligan & Watson, 1998). The failure to develop multiplicative thinking skills hinders mathematical development and the later use of algebraic reasoning, including an understanding of functions and graphs.

One of the necessary mechanisms discussed in the research as central to the development of multiplicative thinking has been termed *unitizing* (Lamon, 1996; Mulligan & Watson, 1998). Unitizing identifies the ability to simultaneously consider a group of objects or a number as an entity in itself and as a countable group or unit in a greater whole (see Figure 1). For example, a student who understands three packs of gum with four sticks in each pack as being three 4s is able to consider the number 4 as representing the number of pieces of gum in a single pack and the number that will yield the total number of pieces of gum in all three packs when it is iterated three times.

It might be that unitizing is a cornerstone that must be addressed directly in elementary classrooms in order to foster multiplicative thinking and lead to the development of proportional reasoning in older students. There has been very little research on how the potentially crucial understanding of unitizing develops over time and across situations in students throughout their elementary schooling. Fortunately, a larger body of research in the area of number sense development has been able to serve as the foundation on which to build unitizing research.

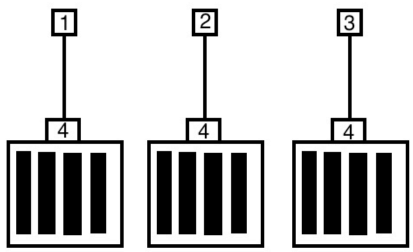



*Figure 1.* A graphical representation of unitizing in the context of three packs of gum with four pieces in each pack.

## Purpose of the Study

The purpose of this longitudinal case study was to investigate the development of unitizing in children from Grade 1 to Grade 4 in reform-oriented classrooms at a public school in Mississauga, Ontario. Unitizing is a critical mathematical concept that is needed to achieve multiplicative reasoning (Fosnot & Jacob, 2010; Mulligan & Watson, 1998). Multiplicative reasoning is necessary to fully develop higher order mathematical understanding. This research was designed to discover critical computation strategies that students need to develop in order to fully construct a unitizing structure as well as to uncover any strategies that might hinder students as they develop this important concept.

## Research Question

1. What are the critical models and cognitive strategies used by a cohort of 34 children in a reform-based mathematics program as they develop their understanding of and ability to flexibility use unitizing structures from Grade 1 to Grade 4?

## Significance of the Study

Multiplicative thinking, proportional reasoning skills, and algebraic logic are important mathematical reasoning processes that are stressed in mathematical curricula. Although multiplication problems can be solved through repeated addition, using multiplicative thinking is qualitatively different from and more sophisticated than using additive thinking (e.g., Nesher, 1988). Although unitizing is one of the key mechanisms in the development of multiplicative thinking (Lamon, 1996), it is surprisingly not mentioned in the current curriculum document for Ontario, *The Ontario Curriculum, Grades 1-8: Mathematics, 2005* (Ontario Ministry of Education [OME], 2005).

Aside from the importance of unitizing in the development of advanced mathematical ideas, unitizing might also be important for students who struggle in mathematics. Grobecker (1997) discovered that students who had a learning disability were less likely than their peers with no learning disability to successfully solve multiplicative tasks specifically designed to elicit unitizing. Recent research has continued to explore the idea that unitizing is an important missing piece for struggling students (McCandliss et al., 2010). This study offers insight into the development of unitizing, which can be used to better identify and create educational supports for students who struggle with this area of mathematics.

## Limitations of the Study

The participants in this study were students who attended a public school in urban Ontario and who belonged to a highly transient, middle- to lower socioeconomic status (SES) population. The children received their mathematics instruction through a reform-oriented approach. Other research questions not reported here addressed the success of the reform approach with this population. Because of the transient nature of the target population, the original sample of 61 children was down to 39 by the spring of Grade 4.

Because this study took place over 4 years, many of the questions on the test instrument were changed to challenge the students at the various stages of their mathematical development. No question was asked in an identical manner at all interviews, so it was not possible to use a single question on the test instrument to gauge a student's development on an identical question over the 4 years. However, items on the various forms of the instrument were designed to map onto one another for the purpose of gathering information about a student's mathematical development over time.

In addition, the test instrument was not always administered in its entirety. If students were struggling with the addition or subtraction questions, the interviewer might have decided not to ask the multiplication or division questions. The assumption was that the children who struggled with foundation concepts such as addition would not yet have constructed unitizing. Some struggling students were interviewed with an instrument from the previous grade level.

The interviews were conducted only twice annually, so it is possible that certain interim strategies and models that assisted in the construction of unitizing might not have been captured in the data if interviews were not conducted at the precise time when the strategies were being used. In addition the participants in this study were actively in classrooms at the time of each interview. The timing of the various units of mathematical inquiry relative to the interview times likely impacted the students' use of particular strategies during the interviews. The longitudinal nature of this study exacerbated the difficulty in finding a control group for research in schools. As such, there was no control group in this study.

# CHAPTER TWO

# LITERATURE REVIEW

## Introduction

The definition of multiplication in elementary texts and mathematics dictionaries usually involves summarizing it as a faster way to perform repeated addition (de Klerk, 2009), but it does not take into account some of the complexities of multiplicative contexts. Moreover, this description might not be broad enough to include multiplication with rational numbers. Even though multiplication and division often are introduced to students separately, the difference between them lies partly in the location of the unknown in the problem. For multiplication, the unknown is the total number or amount, whereas division contexts ask students to find either the number of groups or the number or amount within each group. To fully appreciate the complexities of multiplication and division, it is essential to examine the types of problems and contexts in which they arise.

## Problem Types

Initially, much of the research on multiplication focused on the diverse semantic structure of multiplication and division problems. Construction of many of the multiplication problems seen in elementary education has been of the type called *extensive* quantities multiplied by *intensive* quantities (Schwartz, 1988), or *equal groups* (Greer, 1992). These typical questions also have been included in the more general categorizes termed *isomorphism of measures* (Vergnaud, 1988) and *mapping rule* (Nesher, 1988). In this type of problem, an amount of something (e.g., four sticks of gum in each pack) is mapped out over something else (e.g., three packs of gum). These types of problems reflect their direct link to proportional reasoning because they can be solved like a *rule-of-three* problem (Greer, 1992). Rule-of-three problems involve two equal ratios in which three of the four quantities are known. The solution is found by using the proportional relationship to determine the fourth quantity. In the previous gum problem, the two equivalent ratios were 4:1 and $x$:3. To solve the problem, one has to determine the number ($x$) that has the same proportional relationship to 3 as 4 does to 1. Problems involving rate and price are usually considered special cases of this same type of problem. *Multiplicative comparison* problems (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Greer, 1992; Nesher, 1988) are distinctly different because they involve a comparison between two related quantities, where one is $n$ times larger or smaller than the other.

These two general problem types, equal groups and multiplicative comparison, represent asymmetrical multiplication problems, where the two values in the problem would not yield the same picture if reversed. In the example of the packs of gum, a child might conceive of the problem as 4 + 4 + 4 =12 or 3 multiplied by 4. The reverse, 3 + 3 + 3 + 3 = 12, although mathematically accurate, is not possible for this problem because the quantity three and the quantity four do not have reversible roles in the question. For each type of equal groups problem, there are three distinct varieties: multiplication, partitive division, and quotative division. With multiplication, the result is unknown; with partitive division, the amount in each group is unknown; and with quotative division, the number of groups is unknown.

Although the former are asymmetrical, multiplicative problems also can be symmetrical (e.g., Carpenter et al., 1999). In the case of rectangular area problems (i.e., length × width = area), and Cartesian product problems (e.g., shirts × pants = outfits), the two numbers in the problems are equally weighted. These symmetrical problems give rise to only two distinct varieties of questions, namely, multiplication and division, because there is no distinction as to partitive or quotative division. As researchers began to examine the types of problems that had a multiplicative structure, they realized that a fundamental difference between additive problems and multiplicative problems was that the referent for the solution was different from those in the problem.

**Changes in Referent**

Unlike addition and subtraction, where the two quantities being added or subtracted are of the same type, (e.g., three pieces of gum + four pieces of gum = seven pieces of gum), in multiplicative situations (i.e., problems that can be solved using multiplication or division), the numbers have different referents. In most cases, one of the referents is actually derived as a relationship between two things through division. For example, a problem such as, “There are four pieces of gum in each pack. If I have three packs of gum, how many pieces do I have?” gives rise to the multiplication problem 3 × 4. The “3” in the problem is an extensive quantity; in other words, it is a quantity that can be counted or measured, namely, three packs. At first glance, the “4” also might seem to be an extensive quantity, four pieces, but it is an intensive quantity that is generated through division. In reality, the four is 4/1, that is, “4” pieces in “1” pack.

Schwartz (1988) suggested that these two types of quantities, extensive and intensive, can be combined in multiplicative structures in a variety of ways: E × E` = E`` (meaning an extrinsic value multiplied by a different extrinsic value resulting in a third extrinsic value), I × I` = I`` (meaning an intrinsic value multiplied by a different intrinsic value resulting in a third intrinsic value) and E × I = E` (meaning an extrinsic value multiplied by an intrinsic value resulting in a second extrinsic value) and their related division problems (see Table 1). The most popular triad, according to Schwartz, is the latter, that is, E I E`. In all cases, he argued that because the multiplicative structure has a referent transforming effect, it must be distinguished from additive structures.

Table 1

*Multiplication and Division Questions With Extensive and Intensive Quantities*

| Multiplicative triad | Calculation | Sample word problem |
|---|---|---|
| (E E`E``) | E × E` | If you own 4 different shirts and 3 different pairs of pants, how many different outfits can you make? |
| (E E`E``) | E``÷ E` | If a rectangle with an area of 12 $cm^2$ has a length of 3 cm, what is its width? |
| (I I` I``) | I × I` | A hybrid car that gets 25 km/ Litre of fuel on the highway burned an average of 3.0L/hour on a road trip. What was the average speed traveled during the trip? |
| (I I` I``) | I``÷ I | A hybrid car averaged a speed of 75 km/hour on a road trip. If the car usually gets 25km/Litre of fuel, what was the rate of fuel consumption during the trip? |
| (I I` I``) | I``÷ I` | A hybrid car averaged a speed of 75 km/hour on a road trip. If the car usually burns 3.0L/hour, what was the fuel efficiency of the car during the trip? |
| (E I E`)<br>(rate) | E × I | Coffee costs $4.00 per pound. If I buy 3 pounds of coffee, how much will it cost? |
| (E I E`)<br>(partitive division) | E`÷ E | I paid $12.00 for 3 pounds of coffee. How much is each pound of coffee? |
| (E I E`)<br>(quotative division) | E`÷ I | I paid $12.00 for a bag of coffee. I know the coffee was priced at $4.00 for each pound. How many pounds were in the bag I bought? |

Although changes in the referent when solving multiplication and division questions is undeniable, many adults become quite capable and proficient in solving problems using a multiplicative structure without giving much thought to this change. Another fundamental difference between multiplicative and additive thinking requires a major shift in understanding, namely, a change in the unit used to navigate the number system.

## Changes in the Unit

In addition to a change in the referents, one of the themes that Hiebert and Behr (1988) felt rose from the initial research on multiplication was that during this critical period of mathematical development, children move beyond simply counting. In the primary grades, children initially count using the whole number 1 as their primary way of navigating and understanding the number system. As children move from addition and subtraction to multiplication and division, and from whole numbers to rational numbers, they have to reconstruct their understanding of the nature of the unit. As stated by Hiebert and Behr, “It is difficult to overestimate the significance of this basic shift” (p. 2).

Steffe (1988) provided case study evidence to demonstrate how children might move from the ability to only move up and down the number system with units of 1 to being able to navigate the system with what he termed *composites units*, which are units that are more than 1. Hiebert and Behr (1988) saw this shift as a change in what is understood as a number. When this shift in thinking occurs, the new unit itself can be counted. For example, the unit can be 3 and 1 simultaneously; as they stated, “A change in the nature of the unit is a change in the most basic entity of arithmetic” (p. 2). Fosnot and Dolk (2001b) believed that this shift in thinking required a new understanding of the structure of mathematics. They called this new structure or big idea *unitizing*.

## Developing Unitizing

Although there has been very little research on the development of unitizing, reference to the development of a composite unit (e.g., Anghileri, 1989; Mulligan &

Watson, 1998; Steffe, 1988); unitizing (e.g., Lamon, 1996); and equivalent sets (e.g., Kouba, 1989) as a part of the development of multiplication and division, or multiplicative thinking, offered a point from which I could begin. Table 2 summarizes the related research and the variety of terms used to describe the strategies and structures constructed and represented by children as they develop the operations of multiplication and division. The following section summarizes the work in this area, focusing on the strategies that children use to solve multiplicative word problems.

**Frameworks for the Linear Development of Multiplicative Thinking**

In 1989, Anghileri conducted a cross-sectional study of the strategies that children in Grades 1 to 7 used to solve a variety of multiplication problems. The analysis included a progression through what Anghileri conceived of as three broad strategy categories: modelling strategies, calculation strategies, and use of a multiplication fact. The calculation strategies were further broken down into a progression of three distinct counting strategies: unitary counting; rhythmic counting in groups; and number patterns, or skip counting.

At the same time, Kouba (1989) conducted another cross-sectional analysis of young children's (Grades 1-3) solution strategies for multiplication and division problems. Kouba's analysis acknowledged that children might use the support of objects in a variety of ways, including using them as tallies to support more sophisticated forms of calculation. For this reason, Kouba noted the use of objects, but did not include it within her progression of strategies. In order of increasing abstractness, Kouba reported the following strategies: direct representation, double counting, transitional counting (skip counting), additive or subtractive strategies, and recalled number facts.

Table 2

*Terminology Used in Prior Research on the Development of Unitizing*

| scription | Steffe (1988) | Anghileri (1989) | Mulligan & Mitchelmore (1997) | Fosnot & Dolk (2001b) | Carpenter et al. (1999) | Kou (198 |
|---|---|---|---|---|---|---|
| y | | Cardinal meaning | | The number they end on is the no. of objects in the set | | |
| cal inclusion | Explicitly nested number sequence | Cardinal meaning | | This means that amounts nest inside each other: 6 includes 5, plus 1 | | |
| nodel the | | Unitary counting | Direct counting | Represents groups and objects in the groups and counts by 1s; tries to make equal-sized groups through trial and error | Grouping strategy | Direct representa |
| 5, 6, 7, 8, 9, . | So many more than | Rhythmic counting in groups | Rhythmic Counting | | | |
| one two 12, three | Abstract composite units | Tallying the groups | Double count | | | Double co Occurred division |
| | Iterate a unit along a number line | Number pattern | Skip counting | Doubling, skip counting | Skip counting | Transition counting |
| = 12 2 | | Addition Facts; Unary Operations | Additive Calculation | Repeated addition; Repeated Subtraction | Doubling | Additive subtractiv |
| are 12 | | Multiplication facts; binary operations; retrieval of related number triple | Multiplicative calculation | Using doubling or halving, using partial products, using five times, using known facts | Derived or known facts | Recalled number fa |
| es unable to use es of on ously | Numerical composites | Information-processing difficulties | | | | |

A decade later, Mulligan and Watson (1998) generated a more sophisticated theory of the development of multiplicative thinking that acknowledged the importance of the development of the composite unit. Their longitudinal research outlined a linear development of multiplicative thinking in children in Grade 2 and Grade 3. They used the Piagetian-based structure of the observed learning outcome model that distinguishes five structural shifts in learning: sensorimotor, ikonic, concrete symbolic, formal-1 and formal-2. The sensorimotor mode is grounded in real objects. The ikonic mode uses mental images of real objects to support thinking. In the concrete symbolic mode, students use symbols to represent objects, and in the formal-1 and formal-2 modes, symbols are used in increasingly abstract ways.

Within each of these modes, five levels form the cycles of learning: prestructural, unistructural, multistructural, relational, and extended abstract. A prestructural response would use no elements from the problem and may appear tangential or unrelated to the problem. A unistructural solution would use only one element of the problem, a multistructural solution would use multiple elements from the problem, and a relational solution would relate all problem elements to create a cohesive proof. A solution categorized as extended abstract would contain some difference from a relational solution, indicating that it is just a step away from the next mode.

Mulligan and Watson (1998) concluded that the development of multiplicative thinking includes 12 levels that range from prestructural up to relational (concrete symbolic). Like Anghileri (1989), they included the use of objects as part of their lower ikonic level strategy; however, similar to Kouba (1989), they concluded that some solution strategies are multimodal, drawing on ikonic level modelling to support concrete symbolic thinking. This conclusion might have stemmed from their decision to group strategies and models together so that if a student were to use concrete materials to support a repeated addition strategy, this strategy would be coded as multimodal. This and other previous research focused primarily on the strategies that children use when solving multiplicative situational problems, but there is more to understanding the development of mathematical understanding than simply examining a progression of strategies.

**Identifying Structures or Investigating Structuring**

On their own, the strategies that involve various degrees of unitizing do not fully explain how children construct their understanding of the unitizing structures possible within the number system. Mathematical computation is only part of the picture. The other part of the picture, highlighted by Fosnot (2001a, 2001b) and Fosnot and Jacob (2010), is the mathematical development of children. Simply identifying strategies, models, and big ideas alone is not enough to understand the development of mathematical structures. Fosnot (2001b) harkened back to Piaget by suggesting that through the process of equilibration, that is, the complex process of restoring and creating a new balance after a new idea or problem challenged the understanding, the developing cognitive structures become more and more dense as connections and relationships are made among situations in which children use the structure to understand their world. It is not enough to say that children have or have not built the structure of unitizing by investigating unitizing based solely upon the strategies that children are able to wield. To understand the development of this mathematical structure, it is necessary also to understand the development of children because the interaction between the structures of mathematics and the mathematical development of children is of concern. To understand how children might develop mathematical understanding, it is essential to investigate learning in general and then choose a framework of learning through which to examine the development of unitizing.

## How Children Learn Mathematics

### The Pitfalls of Direct Instruction

Until recently, instruction in mathematics used to begin with an explanation by the teacher on how to execute a specific math skill. If manipulatives or hands-on activities were used, students were instructed on different ways to use them. Students learned to follow the rules and procedures transmitted by their teachers (Van de Walle & Folk, 2008). After computations were mastered, application of word problems would follow (Fuson, 2003). It was held and continues to be held by some researchers that working with the basic facts and algorithms will eventually lead to deeper understanding (Mighton, 2007), but research with adults by Ma (1999) and Simon (1993), among others, has demonstrated that deep conceptual understanding does not usually develop through this progression. International comparison studies such as the Third International Mathematics and Science Study (Office of Educational Research and Improvement [OERI], 1996) were pivotal in uncovering the shortcomings of the traditional focus on procedures at the expense of conceptual understanding for most students. This traditional approach focuses only on the structure of mathematics as understood by adults, not on the diverse ways in which children think about, understand, and develop mathematically (Fosnot & Dolk, 2001b).

In 1999, Ma uncovered a lack of conceptual understanding in many American adult students as compared to adult students in China, a country that focuses on the development of conceptual understanding. Battista (1999) referred to a focus on procedure-based mathematics without concept development as "mathematical miseducation" (p. 425). Because traditional, direct instruction of mathematics, which is based upon the assumption of students as empty vessels into which teachers can simply deposit knowledge, had such poor mathematical outcomes, theoreticians challenged not only the fundamental beliefs about what it means to teach and to learn but also the core beliefs about what knowledge is and how it is acquired.

**Theoretical Conceptions of Knowledge**

For centuries, educators have built an understanding of learning around the idea that knowledge is concrete; an object that can be transmitted or constructed; and once attained, that knowledge can be generalized and used across a variety of situations and contexts. In order to examine the ways in which children might come to understand unitizing, it became imperative to address the underpinning conception of knowledge to frame an understanding of what it means to build a unitizing structure.

In an article originally published in 1983 and republished in 2008, von Glasersfeld challenged the core belief that knowledge is something that exists outside of human beings. If what individuals think that they know about the world is an interpretation of their senses, then how do people know that their representation of some true reality, or knowledge, is accurate or valid? Instead, von Glasersfeld suggested an alternate conception of knowledge that did not raise this conundrum, but had far-reaching consequences in the field of education. He conceptualized knowledge as students' organization of the world in which they live. This change in the conception of knowledge would necessarily change the purpose of teaching from helping students to represent a situation in the same way adults have decided that it should be represented, to a role in facilitating students' ability to organize their worlds in their own increasingly complex ways.

Von Glasersfeld's (2008) conception of knowledge is foundational to the constructivist theory of learning mathematics and has implications for teaching. Where other cognitive theories hold that knowledge is a representation of an independent reality, von Glasersfeld (2005) distinguished the constructivist model as identifying knowledge as residing within individuals, who then have the capacity to change and develop new structures through experience.

**Theoretical Conceptions of Learning**

Even if one adheres to a constructivist approach, the question remains as to how new knowledge is learned. Sfard (1998) summarized learning using the acquisition metaphor and the participation metaphor. According to Sfard, the acquisition metaphor of learning focuses on the development of concepts. Because learning theories have developed along with everyday language, Sfard contended that it is very challenging to talk or write about learning without referring to learning as obtaining possession of some knowledge object. Interpreting learning as something that is possessed by individuals can be problematic because of "the learning paradox" (Sfard, 1998, p. 7). If the only way to build knowledge is to do so in relation to what is already known, then no one can ever truly learn anything new. This paradox also leads one to wonder how learners who construct their own understandings can build concepts that are essentially the same as those built independently by others.

The pitfalls of this view of learning become the most noticeable when they are contrasted with a different paradigm. Sfard (1998) illuminated these issues with the acquisition metaphor for learning by comparing it to the participation metaphor for learning. In the participation metaphor, knowledge is not a commodity that is gained; instead, it is a part of the practice, activity, or discourse of a group. Knowledge grows through shared activities of a community in constant flux: The participants are not locked into being or acting one way because of what they possess; rather, they are able to be fluid, acting one way today and another way tomorrow.

Sfard (1998) suggested that the strongest research can draw from both metaphors, thus combining the best from both worlds. Current research on learning mathematics has roots in both  metaphors. The shift to reform instruction, as initially laid out in the National Council of Teachers of Mathematics' (2000) groundbreaking *Principles and Standards for School Mathematics*, had at the core von Glasserfeld's (2008) conception of mathematical knowledge as something that is constructed and Sfard's (1998) participation metaphor for learning mathematics. Classes were meant to be places where children worked together to build a mathematical community and the teacher served more as facilitator than sole expert. Mathematical norms and ideas were to be discussed, debated, and decided upon as a group. Mathematical knowledge was viewed as constantly changing to accommodate challenges to current understanding (Fosnot & Dolk, 2001a). Von Glasersfeld (2005) called for researchers to use longitudinal interviews with children solving mathematical operations to guide classroom instruction so that classroom teachers would know the direction that each child had to go in order to build an increasingly complex and useful mathematical organization.

## The Constructivist Model

Piaget (1977) maintained that knowledge does not serve a representative function, but an adaptive function. For Piaget, adaptive meant more than just increasing an organism's success: It also meant that knowledge could change the organism in some way, perhaps even physically, such that the organism would have an advantage through the process of natural selection. Piaget theorized that just as with evolution, the process of equilibration brings about changes in cognition. When individuals encounter ideas that disturb their understanding through contradiction, they enter into disequilibrium, a state of puzzlement and investigating in order to find order and understanding (Fosnot & Dolk, 2001a).

Through a process that Piaget (1977) termed *reflective abstraction*, individuals create new structures that are possible solutions to this disturbance. Reflective abstraction, according to Piaget, has two parts: assimilation and accommodation. Assimilation involves recognizing that the features of current situations are similar to previously encountered situations. This recognition allows people to call upon schemes or strategies that they have used or have seen being used before and to try to use them in current situations. The term *scheme*, sometimes called *scheme of action*, has been used synonymously with the term *strategy* by Fosnot and Dolk (2001b). Both terms refer to patterns of behaviours. In other words, through assimilation, organisms can determine in what ways strategies used before can help to serve in new situations.

Accommodation is the opposite of assimilation. With accommodation, people can recognize the differences between previous and current situations. Coming to terms with how old strategies will continue to work in new situations or changing the strategies to fit new situations is the work of reflective abstraction. According to Piaget (1977), several options might be developed. This work might require small changes in strategies, or it might require larger cognitive shifts that involve the altering not only of the strategies but also of the structures, or big ideas, upon which the strategies are built. Of these new strategies and/or structures, the ones that help to regain equilibrium are adaptive and selected (Piaget, 1977).

In mathematics classrooms guided by the constructivist learning theory, teachers acknowledge that children come to the classrooms and new situations with backgrounds of experience that form the children's current mathematical understanding (Carpenter et al., 1999). Any new understanding is formulated from this earlier understanding. When a teacher presents a problem that creates disequilibrium, as students struggle and grapple with possible solutions, there is opportunity for growth of mathematical structures. In a constructivist-informed classroom, various possibilities and conjectures are explored with the teacher taking the role as a catalyst, offering contradictions and supporting correspondences (Fosnot & Jacob, 2010). The teacher asks questions that will help the children either to discover flaws in their logic (further contradiction) or see the connections between this new idea and previous experiences (Fosnot & Perry, 2005; Schifter, 2005).

## Classrooms That Support Children's Thinking

In a traditional classroom where fact memorization and traditional algorithms are the normal progression, it is unlikely to find children who experiment with strategies and solve problems in a variety of ways. Traditional instruction, through its "teacher show and students follow" approach, generally results in most children attempting to solve problems in the same manner as was shown by the teacher (van de Walle, 2008). To study the development of a deep understanding of multiplicative thinking, it is essential to study mathematical development in the types of classrooms where children are supported to develop their own thinking and understanding, such as in constructivist-based, which build understanding together through the process of contradiction and correspondence of strategies. It is in this type of classroom that children are supported to develop rich, dense mathematical structures across situations and where they are assisted to develop increasingly more sophisticated mathematical structures. A framework of learning that captures both of these dimensions is essential to understand the dynamic nature of the development of mathematical thought.

## The Landscape of Learning

Fosnot and Dolk (2001b) introduced a framework for the development of mathematical ideas that they termed *landscape of learning*. Previous research had focused on the development of strategies for students to use to solve multiplication and division problems. Fosnot and Dolk took these progressions and wove them into a more comprehensive framework of learning, including more than just the strategies outlined in earlier continuums. Fosnot, who has authored and edited books on the application of constructivist theory to education, included in this framework not only the strategies that children use on the path to developing multiplicative thinking but also the big structural ideas of the number system that children need to construct in order to develop true multiplicative thinking. The constructivist underpinning of this landscape does not end at the inclusion of big ideas. It also includes the mathematical models, or representations, that Fosnot and Dolk felt would support children's development of strategies and big ideas. This initial framework was expanded by Fosnot in 2007(a).

**Big ideas (structures)**

Within the constructivist paradigm, big ideas (Fosnot & Dolk, 2001a), or structures (Piaget, 1977), are the human-generated cognitive systems that are characterized by three properties: wholeness, transformation, and self-regulation. The wholeness of a structure includes all the parts that it contains and also encompasses the idea that the whole that is created is greater than the sum of the individual parts (Fosnot & Perry, 2005). These parts are interrelated and have no meaning without each other and the structure as a whole. Transformation describes the relationship between and among the parts, as well as how one part changes to become another part. The self-regulation aspect of structures refers to the fact that organisms seek structures that are organized and closed (Fosnot & Perry, 2005).

In mathematics, as students deepen their understanding, certain possibilities or pathways to equilibrium are so adaptive that they open up whole new cognitive ideas not previously accessible. Fosnot and Dolk (2001b) wrote about these big ideas as enabling leaps in mathematical understanding. Once these big ideas have been constructed, students can develop a deeper understanding of the strategies that they have already been using; in addition, they might be able to access new strategies that are now within their grasp. For example, in the case of unitizing, children might have been using repeated addition to solve multiplication or division word problems, but once they have fully constructed the idea that they can change the unit within the number system, they can experiment with doubling strategies (e.g., $6 + 6 + 6 + 6 = 12 + 12$); halving and doubling strategies (e.g., $6 \times 4 = 12 \times 2$); and strategies involving the use of the distributive property (e.g., $6 \times 4 = [3 \times 2] \times 4 = 3 \times [2 \times 4]$). In classrooms where such ideas are supported, children also will employ models to represent their big ideas and facilitate their strategies.

**Models (representations)**

Models can be representations through images, symbols, or language that are used to communicate ideas and actions (Fosnot & Dolk, 2001a), but the power of models enables people to do more than simply represent their thinking. Because a model allows people to take a representation of a situation away from the immediacy of the concrete, they are able to draw on that model when they encounter a similar circumstance. Through assimilation, people can see the ways in which this previously used model is similar to the current situation, and through accommodation, they also can see the ways in which the two situations are different. This dichotomy between the attraction of categorization and generalization and the simultaneous appreciation of the individual differences in situations creates the conditions for Piaget's reflective abstraction (Fosnot & Perry, 2005).

As people wrestle to include all aspects of a broadening variety of situations, models become more than just a representation of their thinking; they also become a mechanism or tool for thinking and learning (Fosnot & Dolk, 2001a). For example, the array is a powerful model that can be used to represent multiplication and division problems (see Figure 2). This model also can aid in discovering deeper mathematical understanding, such as the link between partitive and quotative division; the commutative property of multiplication (e.g., $6 \times 4 = 4 \times 6$); or the distributive property for multiplication and division. Models can represent big ideas and strategies as well as support the construction of new big ideas and strategies.

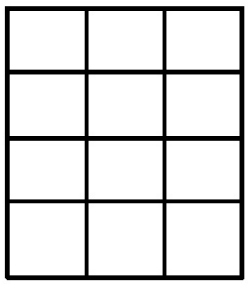


*Figure 2.* A $3 \times 4$ array.

**Strategies (schema)**

As children develop their mathematical understanding, the strategies, or schema, that they use to mathematize the world around them develop as their understanding of the number system deepens. Initially, they are able to consider the number system only in units of 1. The only way that they are able to solve a problem, even a multiplication or a division problem, is by applying the strategy of counting by 1s. However, as they experience more situations, they begin to develop a greater flexibility with the number system in conjunction with a deeper understanding of the structure built into the number system. This flexibility allows them to experiment with new strategies to see whether they fit their current understanding.

**The interplay among strategies, models, and big ideas**

Although it might be tidy and convenient to organize mathematical development in terms of these three aforementioned aspects, Fosnot and Dolk (2001b) indicated that the three parts must work together to form understanding. Sometimes, big ideas are constructed first and then followed by the strategies that they support. Other times, the strategies come first, and through representation and reflective abstraction, then the big ideas are constructed. Likewise, there are times when a model might be used to represent the way a child is thinking about something; at other times, it might be used as a tool for thinking, an aid in the solution process. When a model is used as an aid in the solution process, it can support the construction of new strategies and big ideas previously out of cognitive reach (Fosnot & Dolk, 2001a). All three aspects together build a deep understanding of mathematics.

**Unitizing is a big idea (structure)**

Unitizing fits the definition of a big idea because it involves a deepening of the understanding of the structure of numbers. The ability to simultaneously see a unit of 3 as both 3 units of 1 and one composite unit that can itself be counted requires a shift in the understanding of the part-whole relationship of numbers (Fosnot & Dolk, 2001a). Students struggle with the idea that six objects can be both 6 units of one and one unit of 6. Often through a period of perplexity, analysis, and disequilibrium, a new perspective emerges. This new perspective, the ability to simultaneously hold the idea of the units in a group and the group that itself can be counted as a unit, supports a whole new range of ideas and strategies using multiplicative thinking.

The landscape, therefore, offers a more nuanced, less rigid framework for understanding the development of unitizing over time than earlier continuums. Rather than unitizing being something that one attains, and has or does not have, it is an understanding of the underlying structure of the number system, a big idea, that is slowly constructed through a range of situations.

It seems doubtful that mathematical understanding is as linear as Mulligan and Watson (1998) contended. In a strong reform mathematics classroom, children will have access to a wide variety of strategies, whether or not they have sorted out all of the underpinning big ideas that support the strategy. To support this new strategy, they will draw upon other strategies or representations with which they have had success in the past, such as concrete modelling. It seems plausible that students will use different strategies at different times rather than learning them and using them in a particular succession.

Therefore, I have drawn upon two frameworks to interpret the development of unitizing, namely, the landscape of learning theory (Fosnot, 2007b; see Appendix A) and Harel and Confrey's (1994) book, *The Development of Multiplicative Reasoning in the Learning of Mathematics*, particularly the chapters by Lamon (1994), and Behr, Harel, Post, and Lesch (1994). These two works provided a lens through which to examine the data and to answer the question: How does multiplicative thinking and, more narrowly, unitizing, develop over time?

**Early strategies for multiplicative situations**

Initially, children cannot unitize multiplicative problems. Take the following problem for example: "Robin has three packages of gum with four pieces of gum in each package. How many pieces of gum does Robin have all together?” Children who are just beginning to form their understanding might count out three objects and four objects but be unsure how to relate the two groups to each other and count, “One, two, three, four, five, six, seven. Seven sticks of gum.” Anghileri (1989) discussed this type of error as children develop mathematically. She argued that this type of error is to the result of children’s inability to process the given information in a meaningful way. She suggested that this inability is due to the fact that in multiplication, the two numbers do not have the same roles (referent), so children need to sort out not only how to solve the problem but also what each of the given numbers means.

Use of a unit structure begins when children are able to model both numbers given in a problem with their appropriate roles (Mulligan & Watson, 1998). As children begin to construct the unitizing structure, they will be able to directly model the previously mentioned gum package situation using concrete objects. A typical solution strategy includes counting out three groups of four objects and then counting all of the objects again from 1 in order to determine how many pieces of gum Robin has all together (Carpenter et al., 1999; Kouba, 1989).

Children continue to construct the unitizing structure through quotative division problems such as the following example: “We went to the pet store and bought 21 fish. We have to put the fish into bowls, but we can only put three fish in each bowl. How many bowls do we need?” In the beginning, children generally solve this problem by making groups of 3 and building up until there are 21 in total (Fosnot & Dolk, 2001a). In order to determine the total, children will count all the objects from 1 and then count the groups to figure out how many bowls are needed. As children construct and reorganize these foundation strategies, they might begin to move away from unitary counting of the first group by using a counting-on strategy (e.g. Anghilieri, 1989; Carpenter et al., 1999). Instead of counting all 21 objects from 1, they will start at 3, after the first bowl, and then count on to 21. Lamon (1994) considered this strategy one of the early signs of the development of a composite unit.

**Foundation big ideas**

Until this point, the counting-by-1s strategy might be the only way that children have interacted with objects and numbers. In order to move beyond counting by 1s, children need to have constructed the mathematical structure of cardinality (Anghlieri, 1989). When children first begin counting, they often do not understand that the last number that they reach when counting a set is the number of objects in the set (Gelman & Gallistel, 1978; Fosnot & Dolk, 2001a). If there are five objects in a set, and if children can successfully count them, ending at 5, they might or might not have developed cardinality. If they demonstrate an understanding that they need all five objects to make 5, and not just the last one that they tagged and called “five,” then they have constructed cardinality, at least with small numbers.

Another important structure (big idea) of numbers is hierarchical inclusion, which refers to the relationship of the numbers in a number sequence (see Figure 3). Each number in the sequence is related to the one before as each number grows by exactly 1 each time. Steffe (1988) referred to both of these concepts together when he discussed the explicitly nested number sequence. Cardinality and hierarchical inclusion are constructed as children work through multiplication problems with manipulatives such as fingers and blocks. When they count all the groups together, the mathematical structure of cardinality means that each number includes all of the others up to that point. When solving multiplication problems, children often will count objects or fingers in groups as they directly model a problem. As they model and count, they are reinforcing the cardinal and hierarchical structure of numbers because they can visually see the number of items grow every time they add another object or group of objects.

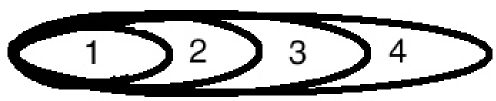



*Figure 3.* Graphical representation of hierarchical inclusion.

**Landmark strategies, unitizing structure on the horizon**

Steffe (1988) suggested that through the construction of cardinality and hierarchical inclusion, children begin to realize that there is great power in being able to create a composite unit from a segment of the number sequence. This segment can be treated as a new unit. Instead of moving up the number line by 1s each time, now it is possible to move up using jumps of this new unit. It is here that the beginnings of skip counting are found. Grounded initially in concrete objects, children now will begin to count objects or fingers using rhythmic counting (Anghileri, 1989). For example, when solving the earlier gum example, a child who was confident in the cardinal structure of number would have counted, “one, two, three, FOUR”; pause; “five, six, seven, EIGHT”; pause; “nine, ten eleven, TWELVE.” The emphasis on and the pauses after the 4, 8, and 12 would have indicated the new composite that the child was using when counting. Anghileri (1989) called this pause “tallying the groups” (p. 375), and she suggested that after stating the last number in each sequence, children must pause while they transfer the meaning of the number word they have just spoken from the counting meaning to the cardinal meaning to create a subtotal that they can then extend. Other researchers have referred to this strategy as many-to-one counting (e.g., Blöte, Lieffering, & Ouwehand, 2006).

Through repeated exposure to these rhythmic counting patterns, the interim words become internalized (e.g., whispered, mouthed, or paused), leaving only the sequence of emphasized words (Anghileri, 1989). Each time that a leap is taken along the counting sequence, the new number represents a subtotal of all the objects counted (or imagined) so far. In order to solve multiplication or division problems with rhythmic counting or skip-counting sequences, children must be able to keep two tallies going simultaneously. Anghileri (1989) suggested that the development of fluency with a number pattern (skip-counting sequence) lightens the information-processing load, enabling children to better keep track using a form of double counting. When double counting, children keep track of the number of iterations either mentally or by using fingers or tallies.

Skip-counting and double-counting strategies (note: double counting is skip counting while keeping track) might be critical to the development of unitizing (Mulligan & Watson, 1998) because these strategies set up the foundation for the iteration of a unit. The selection of the unit to skip count by is usually directly related to the structure of the problem. By looking again at the problem with the three packs of gum, each with four sticks of gum inside, it becomes clear that this problem could be solved with a skip counting strategy such as, “4, 8, 12, … 12 pieces of gum in all.” During this counting procedure, a double count might be used to keep track of how many times they counted. This second counting sequence represents the number of packs of gum that Robin has.

Skip-counting strategies also can be used to solve division problems. Quotative division problems lend themselves more naturally to these strategies because children are able to skip count using the number of objects in each group as the unit to skip count by. In order to determine the number of fish bowls that someone needs to buy to have enough bowls for 21 fish when only three can fit in a bowl, a skip-counting strategy might look like this: "Three,…one (puts up one finger), six,…two (puts up a second finger), nine …, three (puts up third fingers), 12, … four (puts up fourth finger), 15, … five (puts up fifth finger), 18, … six (puts up sixth finger), 21. You need seven bowls." Sometimes, if children know only the first few numbers in a skip-counting sequence, they might skip count initially and then finish their solution using unitary or rhythmic counting.

Partitive division problems also can be solved using skip-counting strategies, but because the size of the group is unknown, it is challenging for young children to know what to use as the skip unit. They usually use a trial-and-error strategy, guessing a possible number to skip by and checking to see if it works (Carpenter et al., 1999). They then adjust the number selection and try again.

Children also use computation strategies, such as repeated addition and repeated subtraction (division context) to solve multiplication and division problems. (Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1997; Mulligan & Watson, 1998). One of the difficulties for children using a repeated addition strategy, especially in a division context, is to know when to stop adding. In her study on understanding multiplication, Anghileri (1989) suggested that this computation strategy allows children to really only use one number in the problem at a time. One factor is used as the initial input, and this unit is added over and over again before the second number is considered. This second factor is used only to determine when to end the addition sequence.

Not all students execute repeated addition or subtraction sequences in the same way. Examples from Fosnot and Dolk (2001b) showed a variety of ways that problems like these can be solved. They published findings describing how children solved the division question 328 ÷ 8. Some children who were just beginning to make use of these strategies added (or subtracted) all of the numbers one at a time until they found the solution. Other students made groups using known facts. It is in this grouping of terms that one of the children cited by Fosnot and Dolk began to speak about the composite unit as something that also is countable.

> We started to add up eights, but after we wrote down six of them, we realized that we knew six times eight. It's forty-eight. So we wrote that" "Then two forty-eights. So that is … twelve eights … and we added forty-eight and forty-eight. That's ninety-six. (p. 66)

The children who were using this grouping strategy demonstrated that they were counting the units of 8. It was in this counting of units that the solution began to reflect a multiplicative structure.

Other researchers have suggested that the doubling or grouping of units can act as a bridge between additive and multiplicative thinking. Kouba (1989) suggested a connection because she grouped these solution strategies with the recalled number facts when analyzing her data. Lamon (1994) also included regrouping strategies in her theoretical chart exploring the increasing sophistication of the unitizing structure. She conceptualized these strategies as a part-part-whole relational understanding of unitizing when solving multiplication or division problems.

**Landmark strategies, multiplicative structure on the horizon.** Children who have constructed the unitizing structure will talk about the iterated unit as something that can be counted. Mulligan and Watson (1998) reported that these children in their study represented their thinking using words like “nine eights are seventy-two” (p. 77). With this deeper understanding, children can develop flexibility with the units, allowing them to reorganize their previous strategies. Two new strategies then emerge, namely, using known facts and using derived facts. Derived facts using distributive property require an understanding of unitizing and the multiplicative structure of number.

An understanding of unitizing and the multiplicative structure of number is also the case for using the algorithms for multiplication and division with understanding. The ability to model the binary structure of multiplication is essential to fully understanding the multiplicative structure. Clark and Kamii (1996) suggested that although additive thinking only requires one level of inclusion relationships, multiplication requires the composition of two levels of inclusion relationships. On one level, the many-to-one correspondence (e.g., four pieces per pack) creates one level of inclusion, and the other lies in the number of groups (e.g., three packs of gum). The multiplicative structure is powerful, enabling children to construct a deeper understanding of part-whole relationships and to use multiplication and division strategies to solve problems (Fosnot & Dolk, 2001a).

Although researchers have been able to see the possible route and nature of the development of unitizing in children, all of the results have come from cross-sectional studies, short studies, or a few studies of individual cases. To understand unitizing and its development fully, researchers must look at them over an extended period of time to gain a deep, rich, and connected understanding. Furthermore, researchers must do so in classes that support and develop the growth of unitizing rather than mask and undermine it with procedures. What does the development of unitizing look like in reform-oriented classrooms over a 4-year period from Grade 1 to the end of Grade 4? Is it similar to Fosnot (2007b) and Lamon's (1994) framework, or is it different?

# CHAPTER THREE

# METHOD

## Purpose

The purpose of this longitudinal case study was to investigate the development of unitizing in children from Grade 1 to Grade 4 in reform-oriented classrooms. The theoretical landscape of learning developed from the psychological theories of Piaget (as cited in Fosnot & Dolk, 2001b; Fosnot & Perry, 2005) and Lamon's (1994) theoretical development of the unitizing process were used as tools to assess the development of unitizing in this group of children. The data analysis and interpretation helped to extend these theories for the development of unitizing for this group.

## Design of the Study

Creswell (2007) defined a case study as a methodology requiring the gathering of multiple sources of data over time from a “bounded system” (p. 73). Merriam (1988), who wrote specifically about case studies in the field of education, defined a case study as “an intensive, holistic, description and analysis of a single instance, phenomenon, or social unit” (p. 21). Conducting a longitudinal case study allowed me to explore the complexity of the development of unitizing by students over time as well uncover themes as they developed unitizing (Creswell, 2007). To complete this study, I used previously collected data from the Elementary Mathematics Research Project, headed by Alex Lawson. I analyzed the video data of students solving multiplication and division problems to explore the development of unitizing over the span of 4 years.

## Research Sample

In 2005, a school in Mississauga, Ontario, was selected as the research site for the research ethics board (REB)-approved longitudinal study by Lawson. The school was purposefully selected because of the willingness of the principal and teachers to learn about and implement reform-oriented mathematics instruction. An additional consideration was the school's diverse ethnic and lower SES student population.

In the fall of 2006, Lawson conducted private and separate interviews with 30 Grade 1 students from two classrooms. In the fall of 2007, another 31 Grade 1 students were interviewed because it became apparent that a second cohort of students would be required to have sufficient numbers to complete the 5-year-long study. By the final interview session, the students were in the spring of Grade 5. Lawson conducted the final interviews with the second cohort in the spring of 2012. For the purposes of this study, only interview data from the Grade 1 fall interview for both cohorts (2006 and 2007) until the Grade 4 spring interview for both cohorts (2010 and 2011) were used.

## Procedure

After receiving ethics approval from Lakehead University's REB and the Peel District School Board, Lawson selected a school based upon the eagerness of the school's principal to be involved in the study and support her teachers, the keenness of the individual teachers involved, and the cultural and SES demographics of the school. Teachers were offered the opportunity for professional development in the area of reform mathematics instruction. This instruction was provided by Lawson, the primary researcher of the longitudinal study. Teacher participation in the project was optional. Participation in the professional development and implementation of reform-based mathematics varied among the individual teachers. Because all teachers in Ontario have the choice whether or not to take professional development and the autonomy to implement new ideas into their classrooms in their own ways, the situation in the research school was reflective of any other school in the province.

Data collection began in the fall of 2006. Parents or guardians were sent a letter of information and a request asking their permission for their children to participate in the study. Similar permission was requested of the additional cohort of 31 students in the fall of 2007. Each September, a similar letter was sent home requesting continued permission for each child's participation in the project.

Children were interviewed twice each school year, that is, once in the fall and again in the spring, by either Lawson or a member of the research team. Each interview lasted between 20 and 80 minutes, with the interviewer remaining sensitive throughout to participant success, engagement, and affect. The interviewer checked in with the participants about any desire to end the interview. The interviews were videotaped using two distinct angles: a medium shot and a bird's eye shot. The written artifacts from each interview were collected and catalogued. Video footage was then edited using Final Cut Pro, and segmented clips were uploaded into ATLAS.ti. The full data set comprised in excess of 9,500 video clips.

**Data Collection and Development of the Codes**

To answer the research question (i.e., What are the critical strategies and models used by a cohort of 34 children in a reform-based mathematics program as they develop unitizing from Grade 1 to Grade 4?), I searched and sorted through the full interview instrument for questions that could have revealed information about the development of unitizing (see Appendix B). From this list of possible questions, five were selected for analysis. No single question was asked in exactly the same way over the 5 years of interviews, but pairs of similar questions with similar attributes were grouped. The selection included one set of paired multiplication questions, one additional multiplication question, and one set of paired quotative division questions.

Initially, each item was coded on a 4-level scale of correctness. For students who independently solved an item correctly or picked up on a mistake as they explained their thinking, the item was coded as correct. When students made a small calculation error as they solved the problem, but their solution strategy would work to solve the problem, the item was coded as incorrect but close. When students received a level of assistance from the interviewer that directed them in some way toward the solution or aided them in solving the problem, the item was coded as correct with help. When students were unable to solve the problem, opted to pass on a problem, or made large calculation or strategy errors, the item was coded as incorrect.

For each item or pair, I used the literature to create a list of expected strategies children might use to solve the calculation. I compiled these items or pairs and the expected strategies as well as the grades in which each question was asked (see Appendix C). In addition to strategies, I looked for the models that the children used to represent and organize their thinking. Models are representations of the strategies that children are using to solve a problem (Fosnot & Dolk, 2001a). These models can be a representation of their thinking or a tool to help them solve the problem. The models I expected to be used by the children in this study included concrete modelling, modelling with an arithmetic rack, an open number line, a closed array, an open array, a T-chart or a ratio table, a money model, and modelling with symbols.

Initially, I created a priori codes using Fosnot's (2007b) landscape of learning for multiplication and division as well as her 2007(a) landscape of learning for addition and subtraction. The list of codes for strategies, models, and correctness can be found in Appendix D. After completing the initial coding, I analyzed across the question types in order to examine the development of unitizing across the variety of question types.

For the purposes of this analysis, the responses coded as correct and incorrect but close were combined because the errors in the category of incorrect but close were extremely minor (e.g., miscounting by 1) and did not constitute a mathematical misunderstanding. The percentages reported are the percentage correct for the sample only where indicated. All frequencies listed are the number of correct responses of a particular type, and the percentages for these frequencies are compared to the total number of correct responses for that item at the time of the given interview. The co-occurrence values reported include all coded solutions, regardless of the level of correctness.

## Verification

Three distinct verification strategies were used: triangulation; member checking; and rich, thick descriptions. Triangulation was achieved in two ways. First, the development of unitizing was analyzed for each question or pair. Second, the questions or pairs selected for analysis were of two different types. The development of unitizing for each type of question was analyzed and compared.

During the course of the interviews, the interviewer often asked the students to explain their thinking or strategies. After this was done, it allowed me to check during the coding process whether what I saw in the video was reiterated by the students when they explained their thinking. When the analysis began, Lawson assisted with the initial coding to ensure that each strategy and model was well defined. If at any time I was uncertain of the strategy or model code, I asked her to assist until we came to a consensus about the implemented strategy or model.

During the reporting process, I used rich, thick descriptions of the students' solutions, along with scans, where useful, when giving examples of various strategies and models. The use of video made this possible. All descriptions were cited with a primary document number linked to the video clip to allow others to follow the qualitative trail of my analysis.

**Ethical Considerations**

The REB approved the research project under which this study fell. Permission for the video data to be used by other members of the research team had already been obtained from the parents of the children in the fall of each year of the longitudinal study. Because members of a vulnerable population were involved in this research, extra care was taken to ensure that the children were treated ethically in all respects. Letters of consent were sent home and signed for each year of participation in the study. The children and their families were able to withdraw from the project at any time without any negative repercussions. The children's raw video data and edited data are stored on secure hardware in locked cabinets at Lakehead University in Thunder Bay, Ontario. When citing examples by specific children, only their first names were used. During the interview process, extra care was taken to ensure that the children felt comfortable and were not overly stressed. To address the issue of benefits to the participants, Lawson provided ongoing professional development and support to the participating classroom teachers and the principal at the research site. The inevitable classroom interruptions that arose during the interviews were minimized.

# CHAPTER FOUR

# FINDINGS

## Organization of the Findings

### Overall Organization

The following section is categorized by the type of question asked, that is, multiplication or division. Each section is further organized by the question posed. The items are then sequenced by grade in which the question was asked, beginning with early primary and ending with early junior. Student responses were coded as correct, incorrect, incorrect but close, or correct with help.

### Organization of Strategies and Models

Organization of the strategies and the models in this study is different from that of most other research (e.g., Anghileri, 1989; Muligan & Mitchelmore, 1997) because I analyzed the strategies and models separately. This separation during the analysis phase also was done by Kouba (1989), albeit to a lesser degree. Although some of the models were used exclusively with particular strategies, many of them were used with a variety of strategies. Likewise, although a couple of strategies were used exclusively with a particular model, many of them were used with a variety of models. Through the use of the qualitative data analysis software ATLAS.ti, I coded the solutions in very fine detail and examined in depth the development of unitizing in this population of students. In Chapter 4, I report the interplay among the models and strategies that were revealed through this process.

Typically, researchers have focused more on the diverse strategies that students use to solve problems, that is, by identifying models as simply concrete or symbolic. As discussed in Chapter 3, researchers have noted that some students use more ikonic, or concrete, models to support more sophisticated strategies (Anghileri, 1989; Mulligan & Watson, 1998). Graphical representations and explanations of the models and strategies for multiplication can be found in Appendix E. Graphical representations and explanations of the models and strategies for multiplication can be found in Appendix F.

## Examination of Multiplication Problem Findings

### Primary Multiplication Problems

**Overview of results for 3 × 4.** At the first interviews in the fall of 2006 and 2007 with the two cohorts of Grade 1 students, 53 (87%) of the 61 students were asked to solve a word problem that required the calculation 3 × 4 similar to the following: "Robin has 3 packages of gum. There are 4 pieces of gum in each package. How many pieces of gum does Robin have all together?" Thirty-three of the 53 students (62%) were able to solve the problem correctly (see Table 3). By the second interview in the spring of 2006 and 2007 with the two cohorts of Grade 1 students, the number of students with correct solutions had risen to 43 (81%).

Table 3

*Percentage of Correct Responses Compared to Total No. of Students Interviewed and No. of Students Posed 3 × 4 or 6 × 4*

| Interview | Total no. of students interviewed | Total no. of students posed 3 × 4 or 6 × 4 question | No. of students who answered correctly | % correct of those who were asked |
|---|---|---|---|---|
| 1st interview: Grade 1 fall (2006 & 2007) | 61 | 53 | 33 | 62% |
| 2nd interview: Grade 1 spring (2007& 2008) | 61 | 53 | 43 | 81% |
| 3rd interview: Grade 2 fall (2007 & 2008) | 56 | 56 | 36 | 64% |
| 4th interview: Grade 2 spring (2008 & 2009) | 54 | 23 | 20 | 87% |

**Models used in correct solutions for 3 × 4.** The comparison of models used in correct solutions to the problems involving the calculations 3 × 4 and 6 × 4 are presented in Figure 4. Twenty-five (76%) of the 33 Grade 1 students who responded correctly in the fall of 2006 and 2007 used a model in which they physically represented each object in each group. To represent the problem using this model, students had to be able to compose groups of 4 while also counting or subitizing that they had constructed three groups. Lamon (1994) suggested that this composition might indicate the very beginning of unitizing. Some students needed to count individual cubes and groups, but others were able to use subitizing to see the quantity of cubes in each group and/or the number of groups. However, data about this distinction were not coded.

At the spring (2007 & 2008) interview, although 19 (44%) of the 43 Grade 1 students continued to solve the problem correctly by representing each object in each group, there was an increase in the number of students who modelled only one group to solve the problem. The incidence of the use of this model rose from two of 33 students (7%) in the fall (2006 & 2007) to eight of 43 students (19) in the spring (2007 & 2008) interview. When using this model, children must keep some type of mental double (simultaneous) count. If they are modelling only one group and then repeatedly counting that group, they have to mentally track both the subtotal and the number of groups they have counted so far. The number of Grade 1 students using mental math with no physical model to solve the problem also increased from two (6%) of 33 at the time of the first fall interview (2006 & 2007) to nine (21%) of 43 at the time of the spring interview (2007 & 2008). Automatic responses also increased from one student (3% of 33) to three students (7% of 43) across the same time span.

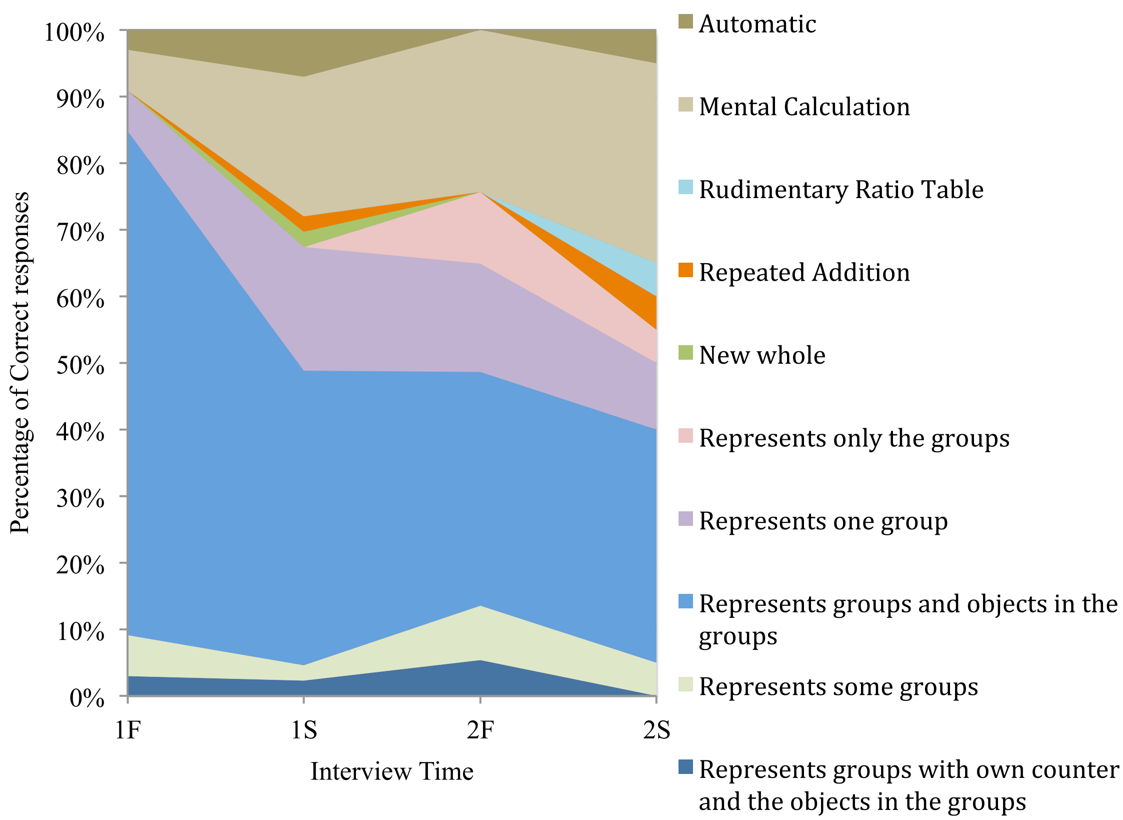



*Figure 4.* Percentage of correct responses for the various models used to represent the multiplication problem 3 × 4 or 6 × 4.

**Strategies used in correct solutions for 3 × 4.** A comparison of strategies used in correct solutions to the problems involving the calculations 3 × 4 and 6 × 4 is presented in Figure 5. The most frequent strategy used correctly by 19 of 33 Grade 1 students (58%) for solving this problem in the fall (2006 & 2007) was counting-by-1s. By the spring (2007 & 2008), the prevalence of this strategy had fallen, with only eight (20%) of 43 students using a counting-by-1s strategy. At each of the two interviews, a small number of students (one [3%] of 33 who answered correctly in the fall of 2006 & 2007 and two [5%] of 43 who answered correctly in the spring of 2007 & 2008 ) also used a unitary counting strategy, but they began by counting on after the first set. According to Lamon (1994), the ability to count on is reflective of a very basic development of unitizing. Related to subitizing, the ability to consider the first group of 4 as both four pieces of gum and as one pack demonstrated the very beginning step toward unitizing.

At the time of the second interview with 61 students in Grade 1 in the spring of 2007 and 2008, there was an increase in the number of students who knew the first doublet (i.e., 4 + 4 = 8) and then used rhythmic counting to count on the last group. In order to use this strategy, students had to be able to consider two levels of units simultaneously, that is, the two packs of gum and the eight pieces inside those packs. Students either could not consider a third pack in the same way or they did not yet know 8 + 4 without counting. The number of Grade 1 students who used this strategy rose from four (12%) of 33 in the fall to 12 (28%) of 43 in the spring. There also was an increase in the number of students using strategies involving skip counting by the composite (i.e., 4, 8,…) from zero to seven (16%) of 43 students.

**Overview of results for 6 × 4.** At the time of the third interview in the fall of 2007 and 2008 with 56 Grade 2 students, the item was altered to a word problem that required a calculation of 6 × 4. At this time, with this increased challenge came a decrease in the percentage of correct responses to 64% of the 56 students. By the fourth interview in the spring (2008 & 2009) with 54 Grade 2 students, the percentage of correct responses had risen to 87%, or 20 of only 23 students who answered the question.

**Models used in correct solutions for 6 × 4.** Although this new question was more challenging, there was no increase in use of the model in which students physically represented each object in each group. Instead, there was a slight decrease in its use, with only 36% (13) of the 36 correct responses from Grade 2 students reflecting this model. Six students (16%) continued to demonstrate the ability to model only one group in order to correctly solve the problem.

In the fall (2007 & 2008) of Grade 2, a new model emerged in which four of 36 students (11%) used counters, fingers, or tallies solely to represent only the groups themselves, not the objects within the groups. By the spring interview, the prevalence of this model had dropped to one (5%) of 20 students. To model the problem in this way, students had to be able to simultaneously see a concrete object such as a finger not only as a group but also as a representation of a group of objects. In the case of the current problem, the finger was both one package of gum and four pieces of gum.

In addition, at the spring (2008 & 2009) interview, use of a ratio table in rudimentary form by one student (5%) was evident. This model was strongly related to the previous model, except that the tracking of the groups and the subtotals were done through writing the new subtotal for each group (see Appendix F). Each new number written not only represented the total number of pieces of gum so far but also served to track the number of packages.

Despite the shift to a more challenging computation in the third interview with Grade 2 students (fall 2007 & 2008), the number of students who correctly used mental calculations remained relatively constant from the spring of Grade 1 to the fall of Grade 2, with nine (21% of 43) Grade 1 students and 9 (25% of 36) Grade 2 students using this strategy at both interview times, respectively. At the time of the fourth interview with the Grade 2 students in the spring (2008 & 2009), there was an increase in the number of students mentally calculating the solution to 6 (26%) of 23 students.

**Strategies used in correct solutions for 6 × 4.** The downward trend in the use of the counting-by-1s strategy seen at the second interview for 3 × 4 continued at the third and fourth interviews, despite the increased challenge of the problem. Only 6 (17%) of the 36 students who answered correctly used that strategy in the fall of Grade 2, and by the interview in the spring of Grade 2, no students used that strategy. Strategies involving rhythmic counting continued to be correctly used with some frequency in the Grade 2 fall interview (2007 & 2008), varying only by whether students started counting from 1 (six [17%] of 36 students); after the first set (two [5%] of 36 students); or after the first doublet (three [8%] of 36 students).

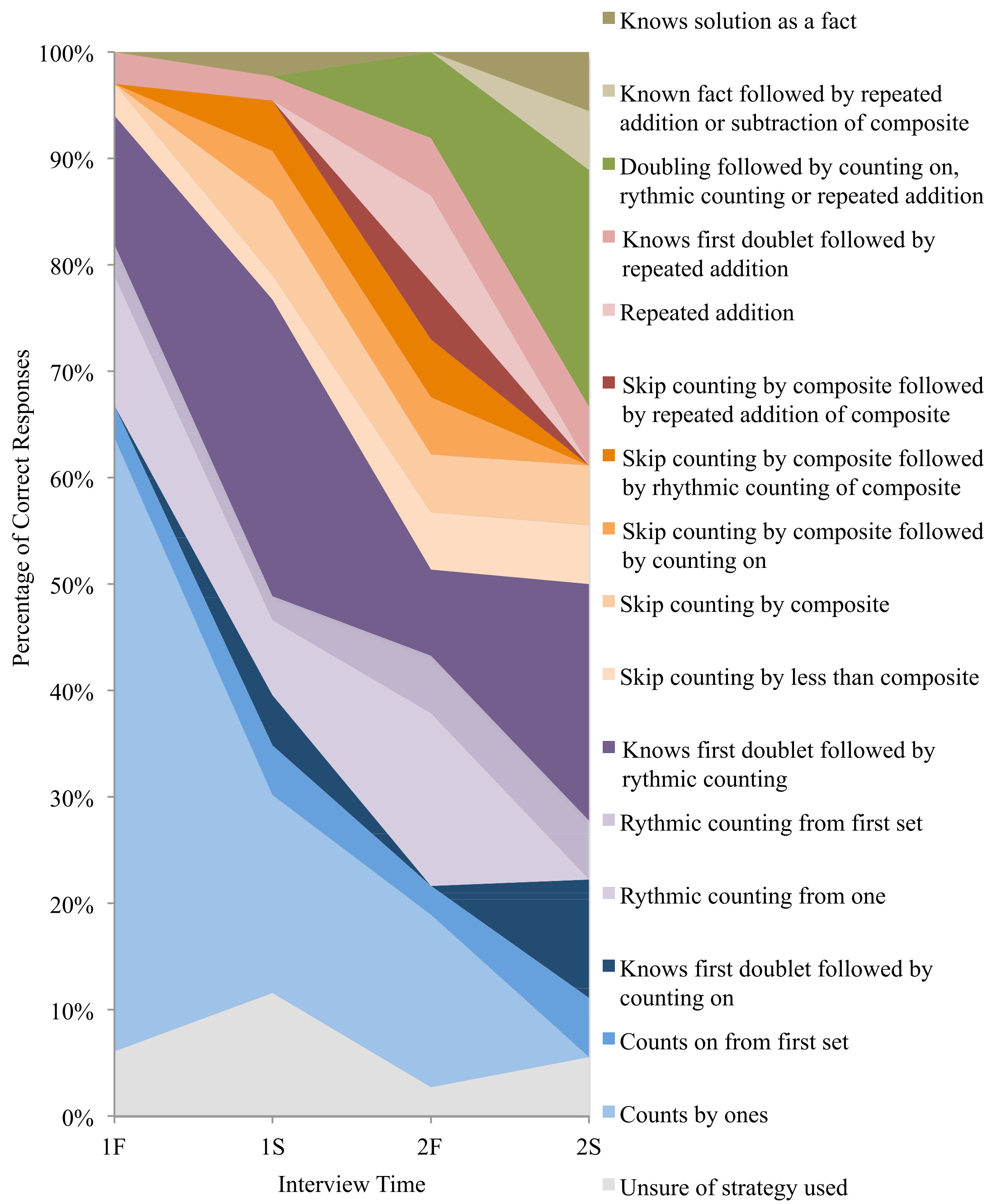



*Figure 5.* Percentage of correct responses for each calculation strategy used by students to solve 3 × 4 and 6 × 4 problems.

Rhythmic counting is a form of counting by 1s that is structured around the composite. There is a pause at the point where the student has completed each group. To use correctly rhythmic counting, students had to be able to consider two levels of units. In this problem, they had to be able to consider each group of four pieces of gum as a new pack, and they had to keep track of how many packs they had counted. The strategy did not quite require the simultaneity that some of the other strategies required because the students could count the individual units (i.e., the pieces of gum) and then, during the pause, calculate the second level of units (i.e., the packs). By the Grade 2 spring interview (2008 & 2009), the most commonly used strategy that involved rhythmic counting was the one that began with a doublet (four [20%] of 20).

New strategies also emerged at the time of the third interview (fall of 2007 & 2008). Three (8%) of the 36 Grade 2 students used a regrouping or reunitizing strategy. This number increased to four (20%) of 20 students at the following spring interview. To use a regrouping strategy correctly, students had to be able to take two packs of gum, each pack comprising four pieces, to create groups of 8. The students had to keep track of the original six groups of 4 in order to know that each of the new groups of 8 represented two packs of gum. Because each new group constituted two packs, adding three of them was the same as adding six of the original packs of gum. According to Behr et al. (1994), correct execution of this strategy requires the conceptualization of units in units in units (i.e., three groups, two gum packs per group, four pieces of gum per pack; see Figure 6).

**Issues with the posing of 6 × 4.** Unfortunately, 6 × 4 was not posed uniformly to all 43 students in the Grade 2 spring interview who were posed this problem. Twenty-three students were asked to calculate 6 × 4 but twenty were asked 4 × 6. Although the difference between these questions appeared slight, four groups of 6 could easily have been solved with doubling, so there might have seen a larger trend toward regrouping to doubles than if the question had remained six groups of 4. Although the two forms did not appear to alter model or strategy choice, I decided to use only the data from the question when asked as 6 × 4 to ensure a reliable comparison over time.

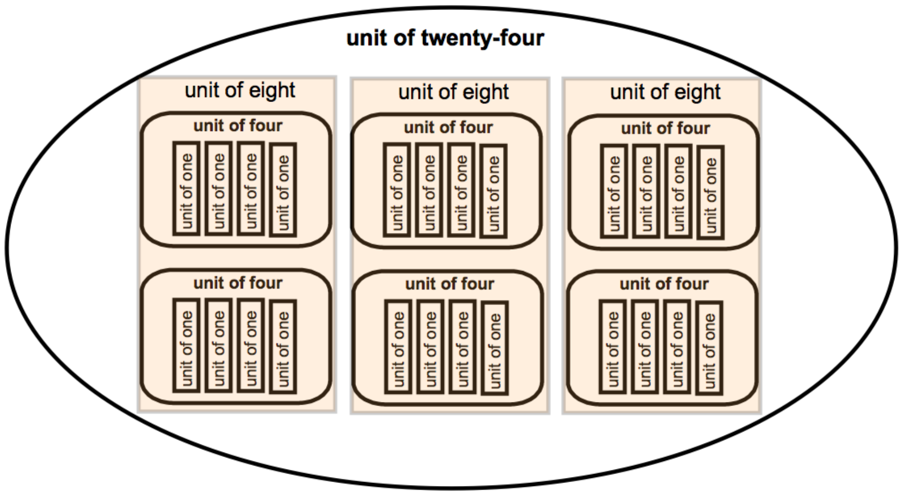



*Figure 6.* Depiction of the way 24 would be unitized when solving a multiplication problem with six groups of 4.

**Overview of results for 4 × 10.** A word problem that required the calculation 4 × 10 was included at the time of the Grade 1 fall interview for 30 students (see Table 4) similar to the following: "Corrine has 4 packages of baseball cards. There are 10 baseball cards in each package. How many baseball cards does Corrine have all together?" At that time, 22 (73% ) of the 30 students asked were able to answer the question correctly. By the spring of Grade 1, 91% (43) of the 47 students posed gave a correct response to the problem. This question also was asked at Interviews 3, 4, 5, and 6: fall of Grade 2, spring of Grade 2, fall of Grade 3, and spring of Grade 3. The percentages of correct responses during these interview times were 87% (45 of 52 students), 92% (48 of 52 students), 93% (37 of 40 students), and 97% (37 of 38 students), respectively.

**Models used in correct solutions for 4 × 10.** A comparison of models used in correct solutions to the problems involving the calculations 4 × 10 are presented in Figure 7. As with 3 × 4, 15 (68%) of the 22 students with correct solutions at the Grade 1 fall interview primarily modelled out all the objects in all the groups to find the solution. Four (18%) of 22 students automatically knew the solution, and two (9%) of 22 students used mental calculation at the first interview.

By the second interview of Grade 1 students in the spring, the percentage of correct solutions modelling all the objects and all the groups had dropped to 17 (40% of 43 students), and the number of automatic student responses had risen to 15 (35%) of the 43 students. At the Grade 1 spring interview, four (9%) of 43 students modelled only the groups; this model was not seen when students were asked 3 × 4 at the same interview time.

Table 4

*Percentage of Correct Responses Compared to Total No. of Students Interviewed and No. of Students Posed 4 × 10*

| Interview | Total no. of students interviewed | Total no. of students asked 4 × 10 | No. of Students who answered correctly | % correct of those who were asked |
|---|---|---|---|---|
| 1st interview: Grade 1 fall | 61 | 30 | 22 | 73% |
| 2nd interview: Grade 1 spring | 61 | 47 | 43 | 91% |
| 3rd interview: Grade 2 fall | 56 | 52 | 45 | 87% |
| 4th interview: Grade 2 spring | 54 | 52 | 48 | 92% |
| 5th interview: Grade 3 fall | 45 | 40 | 37 | 93% |
| 6th interview: Grade 3 spring | 45 | 38 | 37 | 97% |

At each of the three subsequent interviews (Grade 2 fall, Grade 2 spring, and Grade 3 fall), the number of students who modelled all the objects in all the groups continued to decline, with 10 (22%) of 45 fall Grade 2 students, eight (17%) of 48 srping Grade 2 students, and two (5%) of 37 fall Grade 3 students, respectively. There was a slight increase in the number of students using this model at the Grade 3 spring interview from two to three students (8% of 37). Conversely, the number of students with correct automatic responses generally rose with 19 (42%) of 45 fall Grade 2 students, 27 (56%) of 48 spring Grade 2 students, 21 (57%) of 37 fall Grade 3 students, and 31 (84%) of 37 spring Grade 3 students observed knowing the solution automatically across the same four interview periods. Similarly, the number of students who used mental calculations also rose across Interviews 3, 4, and 5, with three (7%) of 45 fall Grade 2 students, five (10%) of 48 spring Grade 2 students, and eight (22%) of 37 fall Grade 3 students observed using mental calculations respectively. The number of mental calculations used at Interview 6 decreased to two (5%) of 37 Grade 3 students. This decrease was likely the result of the high number of students who now knew the solution automatically.

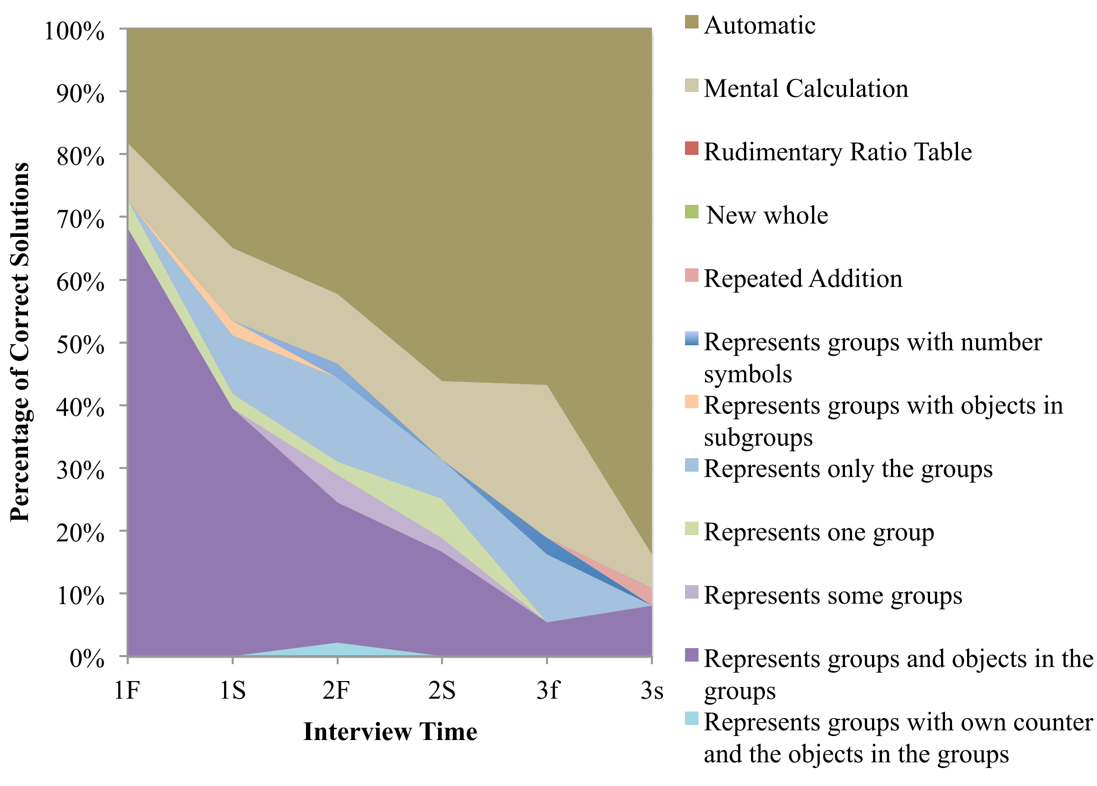



*Figure 7.* Percentages of correct solutions that use the various models as students solve the problem $4 \times 10$.

**Strategies used in correct solutions for 4 × 10.** A comparison of strategies used in correct solutions to the problems involving the calculations 4 × 10 is presented in Figure 8. Nine (41%) of the 22 fall Grade 1 students used a counting-by-1s strategy to calculate the total. An additional two (9%) of these 22 students counted objects by 1s, but they counted on after the first set of 10. Two more of the 22 students (9%) demonstrated a rhythmic counting strategy starting from one. Four (18%) of the 22 students skip counted up the number sequence with jumps of 10. Skip counting is similar to rhythmic counting in terms unitizing. However, the children's knowledge of the skip counting sequence removed some of the cognitive load required and allowed for a greater level of simultaneity between the calculation of the subtotals of individual units (i.e., candies) and the units of one-units (i.e., bags). In the fall of Grade 1, two (9%) of these 22 students already knew the solution as a math fact.

By the Grade 1 spring interview, the number of students correctly using the counting-by-1s strategy had fallen to five (12%) of 43 students. There was no evidence of counting on from the first set or of rhythmic counting from one. Instead, 16 (37%) of the 43 spring Grade 1 students used the strategy of skip counting by 10s. This result was congruent with Anghileri's (1989) conclusion that as students become more familiar with the skip counting sequence (number pattern), they will replace rhythmic counting with a skip counting sequence. Eight (19%) of the 43 students reported knowing 4 × 10 as a math fact.

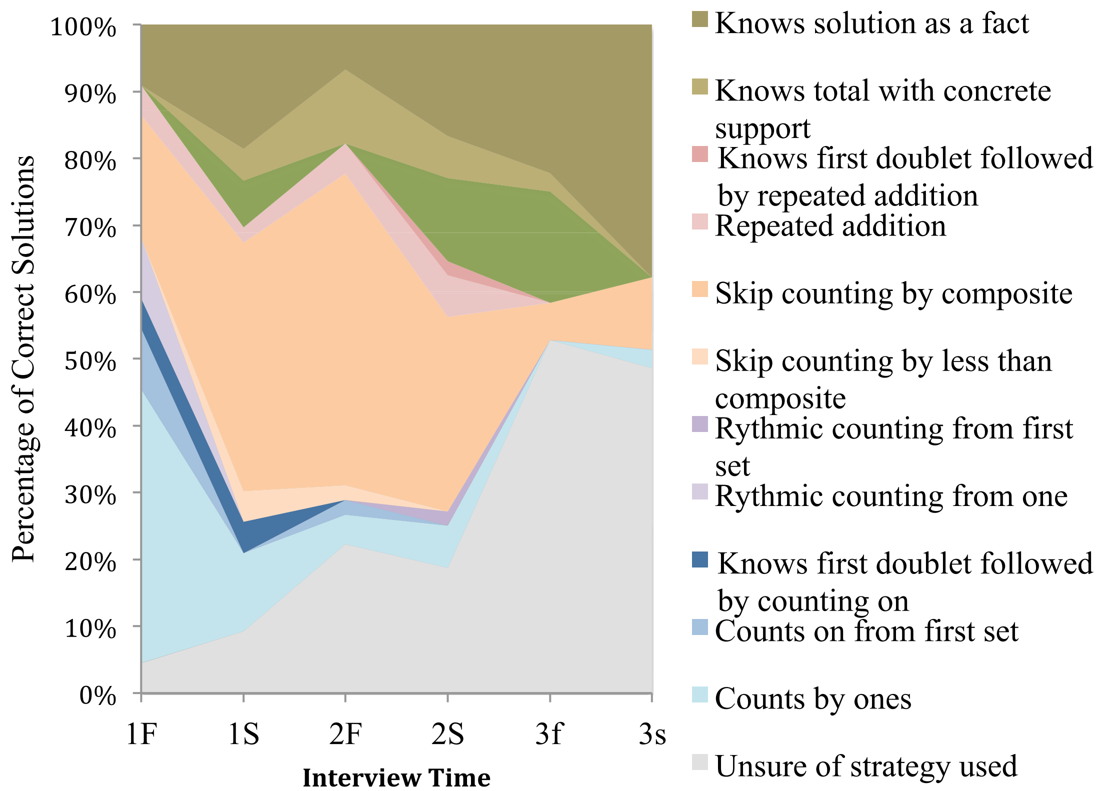



*Figure 8.* Percentages of correct solutions where students used various strategies to solve the 4 × 10 problem.

The use of these same two strategies (i.e., skip counting and knowing the solution as a fact) continued to be notable in the four subsequent interviews. Skip counting by 10 was utilized by 20 (47%) of the 45 fall Grade 2 students, 14 (29%) of the 48 spring Grade 2 students, two (5%) of the 37 fall Grade 3 students, and four (11%) of the 37 spring Grade 3 students, respectively. Simply knowing the solution as a math fact was the reasoning given by an increasing number of students; three (7%) of 45 fall Grade 2 students, eight (17%) of 48 spring Grade 2 students, eight (22%) of 37 fall Grade 2 students, and 14 (38%) of 37 spring Grade 3 students gave this justification across the same interview period. The apparent decrease in the use of a skip counting strategy might have been due to the increase in automatic responses and the choice of the interviewer not to ask students who responded automatically about their strategies. Responses were determined to be automatic if the student's response to the question was so rapid that I felt that no calculation took place. In traditional mathematics classrooms, this speedy recall was called memorization and it was achieved through repeated drill of isolated facts with classroom tools such as flash cards. In the classrooms where the participants in this study learned, math facts were automatized. This means that through practice of relating facts to each other and calculating these facts and related facts over and over the process of calculating the fact becomes automatic and so fast that the recall of the fact is almost immediate. In the fifth and sixth interviews with the fall and spring Grade 3 students, the number of student responses coded as method unsure was around 50%. Most of the occurrences of the code of unsure for this calculation across these two interviews co-occurred with automatic student responses (84% of 37); therefore it is likely that where these codes co-occurred, students knew the solution as a fact.

## Interplay Among the Strategies and Models in Primary Multiplication

To better understand the relationships between the models and strategies, co-occurrence tables were generated in ATLAS.ti that allowed me to examine when particular strategies and models occurred together (see Table 5). The strategies of counting by 1s and counting on from the first set occurred primarily when students used a model in which they represented all the objects in all the groups. Students flexibly used the model representing all the objects in all the groups, as demonstrated by the fact that each main strategy to solve these problems was used in conjunction with this model. Rhythmic counting primarily occurred with the model of representing just one group, and rhythmic counting was the most common strategy used with that model. When students represented the problem by representing only the groups, the most common strategy used in the primary grades was skip counting by the composite. This strategy was the only one used alongside the rudimentary ratio table model.

Table 5

*Total No. of Occurrences of Various Strategies Used for All Solutions to Problems 3 × 4, 6 × 4, and 4 × 10*

| Model strategy | Represents all objects in all groups | Represents one group | Represents only groups | Rudimentary ratio table | Other | Total |
|---|---|---|---|---|---|---|
| Counts by 1s | 72 | 0 | 0 | 0 | 7 | 79 |
| Counts on from first set | 10 | 0 | 0 | 0 | 0 | 10 |
| Rhythmic counting | 10 | 21 | 2 | 0 | 20 | 52 |
| Skip counting by composite | 21 | 7 | 16 | 3 | 34 | 78 |
| Skip counting < composite | 8 | 1 | 0 | 0 | 1 | 10 |
| Doubling/Regroup-ing | 6 | 0 | 0 | 0 | 18 | 24 |
| Other | 40 | 1 | 7 | 0 | | |
| Total | 167 | 30 | 25 | 3 | | |

**Primary/Junior Multiplication Problems**

**Overview of results for 6 × 7.** At the Grade 3 and 4 interviews (see Table 6), students were posed a word problem that required the calculation 6 × 7 similar to the following: “Josh has six fish bowls. There are seven fish in each bowl. How many fish does Josh have?” In the fall and spring Grade 3 interviews, 36 students, respectively, were asked to use mental strategies to solve this problem. The children found this calculation very challenging to complete mentally, as reflected by the 50% (18) and 58% (21) success rates on this problem. By the Grade 4 interviews, students were allowed to use modelling on paper or manipulatives to support their thinking. The success rate rose to 33 (85%) of 39 fall Grade 4 students and 35 (95%) of 37 spring Grade 4 students.

Following is a report on the models used by students; it is important to keep in mind that students were not given free choice as to the model to use in the Grade 3 interviews.

Table 6

*Percentage of Correct Responses Compared to Total No. of Students Interviewed and No. of Students Posed 6 × 7*

| Interview | Total no. of students interviewed | Total no. of students asked 6 × 7 | No. of students who answered correctly | % correct of students who were asked |
|---|---|---|---|---|
| 1st interview: Grade 3 fall (2006 & 2007) | 45 | 36 | 18 | 50 |
| 2nd interview: Grade 3 spring (2006 & 2007) | 45 | 36 | 21 | 58 |
| 3rd interview: Grade 4 fall (2006 & 2007) | 41 | 39 | 33 | 85 |
| 4th interview: Grade 4 spring (2006 & 2007) | 39 | 37 | 35 | 95 |

**Models used in correct solutions for 6 × 7.** A comparison of models used in correct solutions for the problems involving the calculations 6 × 7 is presented in Figure 9. As expected, regardless of the mental calculation constraint, the number of students who knew the solution automatically grew across the interviews from one (6%) of 18 students in the fall of Grade 3 to seven (20%) of 35 students by the spring of Grade 4. Sixteen (76%) of the 21 correct student strategies in the spring Grade 3 used completely mental calculations, and three (14%) of these 21 students modelled only the groups, using their fingers to keep track of their sevens.

The Grade 4 interviews were likely more indicative of the students' true models because they were once again free to select their own models. This freedom saw mental and automatic strategies drop to a total of 10 (30%) of 33 fall Grade 4 students, and the frequency of students representing all the objects in all the groups rise sharply to 9 (27%)

of 33 students in the fall Grade 4 interview. Although students' use of automatic and mental models began to rise again in the spring Grade 4 interview to a total of 18 (51%) of 35 students, the modelling of all the objects in all the groups continued to be employed, as noted in eight (23%) of the 35 correct responses. Students continued to model only the groups with three (9%) of 33 and two (6%) of 35 student solutions reflecting this strategy in the Grade 4 fall and Grade 4 spring interviews, respectively.

At the Grade 4 fall interview, students began to use a series of similar models that involved the writing of the number symbol 7 iterated six times; two (6%) of 33 students at this interview used the strategy of representing groups with number symbols, three (9%) students used the model of repeated addition, and two (6%) students used the new whole model. By the spring Grade 4 interview, these related models were used by four (11%) of 35 students, no (0%) students, and one (3%) student, respectively.

To represent the problem with any of the above three models, students had to be able to see the number symbol 7 as representative of the number of fish inside each bowl as well as a single bowl in order to track the number of bowls being combined. Later, I discuss whether the models drew out different strategies when examining the interplay between strategies and models for primary and junior multiplication. Also seen was the emergence of the use of a rudimentary ratio table to model the problem, with three (9%) Grade 3 students using this representation in the fall interview and one (3%) Grade 3 student continuing to represent the problem in this way at the spring interview.

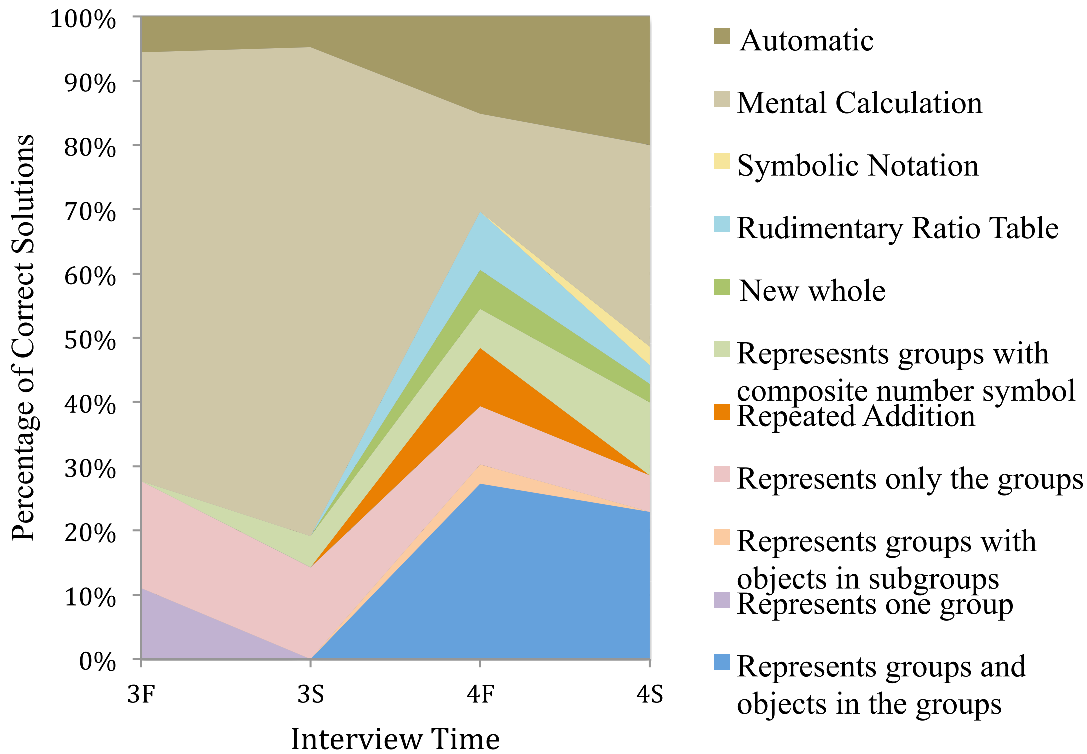



*Figure 9.* Percentages of correct responses used for each model by students answering question 6 x 7.

**Strategies used in correct solutions for 6 × 7.** A comparison of strategies used in correct solutions to the problems involving the calculations 6 × 7 is presented in Figure 10. In contrast to what was seen in early primary, in the fall of Grade 3, students did not use a strategy that involved counting by 1s starting from 1, after the first set, or after the first doublet to solve 6 × 7. This was likely to the result of the mental calculation constraint built in to this item.

Three (17%) of the 18 fall Grade 3 students used skip counting, and another three (17%) used repeated addition. Four (22%) of the 18 students used a strategy coupling the six groups of seven into three groups of 14. From that point, the four students used a variety of ways to determine the total (counting on, rhythmic counting or repeated addition), but for the purposes of analysis, these strategies were combined into a single group. The exception to this recombination (one student [6%]) was a strategy that involved doubling to form a new composite but then splitting the new composite along place value lines in order to add to the total.

Found in the analysis of the Grade 3 spring interview data were results showing that occurrences of skip counting had dropped to zero and that use of repeated addition had dropped to two (10%) of 21 of the Grade 3 students. The prevalence of the strategy of doubling to form a new composite and then calculating the total using iterations of this new composite rose to seven (33%) of 21 student responses. The students' solutions, making use of doubling to form a new composite, followed by a splitting strategy, also increased, with two (10%) of 21 students using this strategy.

Even after students were permitted to use a variety of models at the Grade 4 fall interview, few of the 33 students used unitary counting strategies (i.e., two [6%] of 33). Three (9%) of the 33 students successfully used skip counting strategies. The doubling strategies that had emerged in the Grade 3 interviews continued to be used by students in the Grade 4 fall interview, with nine (27%) of the 33 students using the strategy in which the new composite of 14 was added consecutively three times. The other strategy involving doubles, in which students then split the three 14s in order to combine them, also was used by two (6%) of the 33 Grade 4 students.

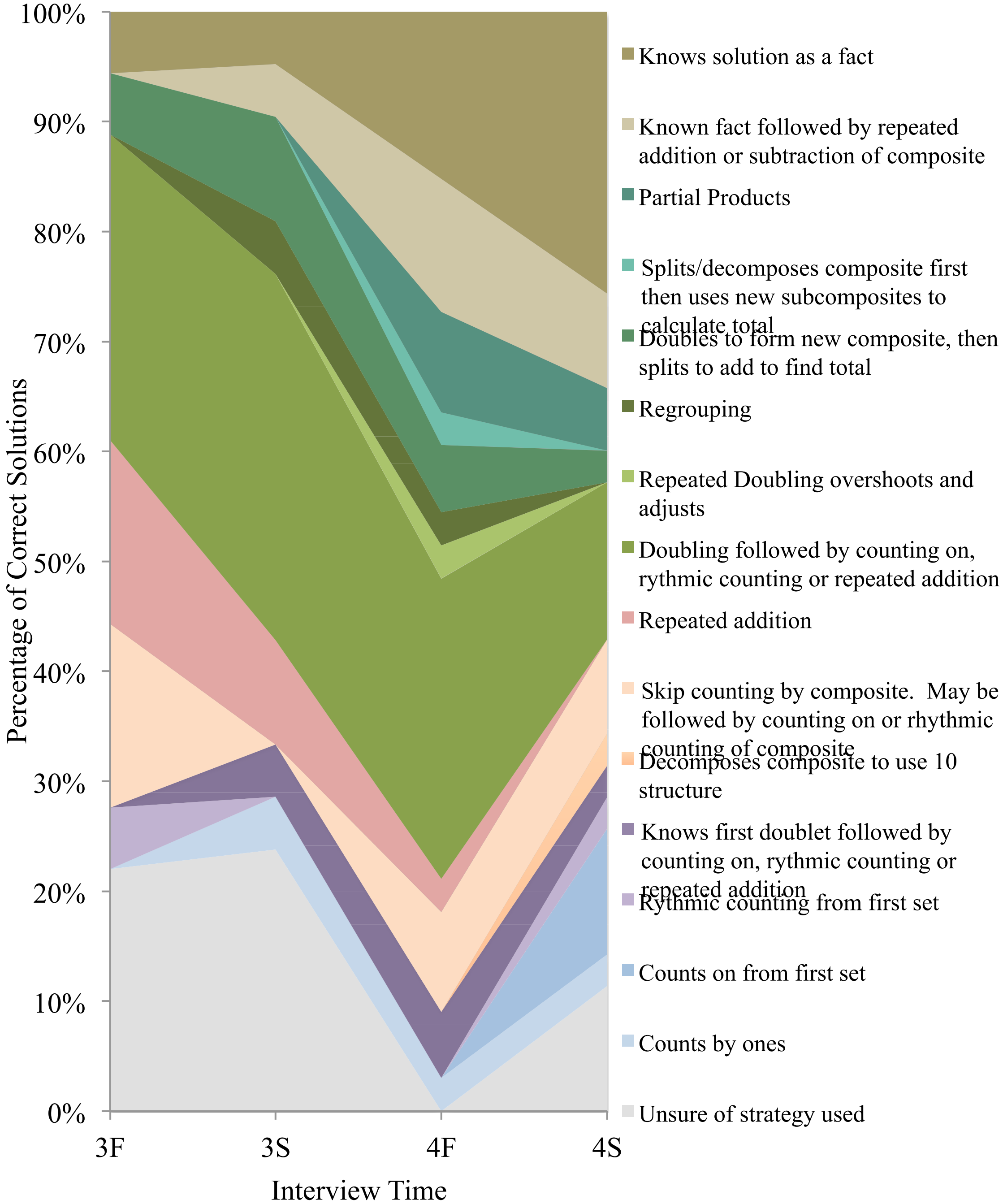



*Figure 10.* Percentage of correct responses used for each strategy by students when solving question 6 ×7.

At the Grade 3 spring interview, two (6%) of the 21 students used the commutative property of multiplication prior to using one of the doubling strategies to calculate the solution. In this way, they could work with the 6s and the 12s instead of working with the 7s and the 14s. The use of the commutative property of multiplication was evident in the remaining interviews at which this question was asked, with four (10%) of the 33 fall Grade 4 students using this property and only one (3%) of 35 students using it at the Grade 4 spring interview.

In each interview, some students reported knowing the solution as a fact. In the Grade 3 fall and spring interviews, this strategy was limited to one student, 6% of 18 and 5% of 21, respectively. By the Grade 4 fall interview, this number increased to 5 (15% of 33) students and then rose again to nine (26%) of 35 students at the Grade 4 spring interview. With this slightly more complex question, two strategies emerged that had not been seen in the primary multiplication problems. Students reported using a known fact to derive the new fact with either addition or subtraction (e.g., $6 \times 7 = [5 \times 7] + [1 \times 7]$). To build to a solution in this way, students had to understand that six groups of 7 comprised five groups of 7 and one group of 7 (i.e., the distributive property).

From a unitizing perspective, this strategy is very similar to a regrouping strategy. Students use a known math fact that is as close to the one being asked as they know, and then they build either up or down from there. This advanced regrouping strategy was utilized by one (5%) of 21 spring Grade 3 students, 4 (12%) of 33 fall Grade 4 students, and three (9%) of spring Grade 4 students, respectively. In addition, with a similar strategy, students built to the solution with two known facts. Three students (9% of 33) and two students (6% of 35) used a student-generated partial products strategy in the Grade 4 fall and spring interviews, respectively.

**Overview of results for 6 × 24.** At the four interviews, students were posed a word problem that required the calculation 6 × 24 similar to the following: “Sandy has 6 boxes of candy. There are 24 pieces in each box. How much candy does she have all together?" The challenging problem 6 × 24 provided an opportunity to see children work through a more complex multiplication problem that not only was unlikely to be automatic but also had large enough numbers that counting by 1s was not practical. Because of these two constraints, the weaker math students usually attempted calculation strategies instead of counting strategies, and the stronger math students continued to develop their solution strategies for the problem at all four interviews when this question was asked. Consequently, a wider variety of strategies was seen (see Table 7).

Table 7

*Percentage of Correct Responses Compared to Total No. of Students Interviewed and No. of Students Posed 6 × 24*

| Interview | Total no. of students interviewed | Total no. of students asked 6 x 24 | No. of students who answered correctly | % correct of students who were asked |
|---|---|---|---|---|
| 1st interview: Grade 3 fall | 45 | 41 | 21 | 51% |
| 2nd interview: Grade 3 spring | 45 | 39 | 27 | 69% |
| 3rd interview: Grade 4 fall | 41 | 38 | 22 | 58% |
| 4th interview: Grade 4 spring | 39 | 39 | 32 | 82% |

This question was first asked in the fall of Grade 3. At this time, 21 (51%) of 41 students who were asked the question solved the problem correctly. This number rose to 27 (69%) of 39 students who were asked the same question in the spring of Grade 3. It fell to 22 (58%) of 38 students in the fall of Grade 4. The number of students who were able to correctly solve this problem rose drastically to 32 (82%) of 39 students by the spring of Grade 4 (see Table 7).

**Models used in correct solutions for 6 × 24.** A comparison of models used in correct solutions to the problems involving the calculations 6 × 24 is presented in Figure 11. The most prevalent models that students used involved those representing the number symbol 24 iterated six times in one of three ways: in six groups defined by a shape such as a square; iterated six times as part of a repeated-addition sequence; and as a new whole, where they simply modelled with the number written out six times without any other symbols or grouping diagrams. The representation of the problem as repeated addition was used correctly by two (10%) of 21 fall Grade 3 students. Two different students (7%) of 27 spring Grade 3 students used this model, and three (14%) of 22 fall Grade 4 students correctly used this model for solutions. This model was not used to represent this problem in the Grade 4 spring interview. The model, representing the groups with a number symbol, was used by four (19%) of the 21 fall Grade 3 students, two (7%) of the 27 spring Grade 3 students, five (23%) of the 22 fall Grade 4 students, and four (13%) of the spring Grade 4 students. Over the same four interviews, the model of the new whole was used by four fall Grade 3 students (19% of 21), six spring Grade 3 students (26% of 27), six fall Grade 4 students (27% of 22), and three of spring Grade 4 students (9% of 32). The differences among these three models were very slight.

In contrast to the results for the primary multiplication problems, there was very little use of the model to represent all the objects in all the groups; however, students usually did not have enough manipulatives available in the interview to represent the problem with this model. This was done expressly to prevent counting by 1s; nonetheless, students who required a concrete full representation were able to draw out all the objects and groups on paper.

As with the earlier 6 × 7 question, students using a rudimentary ratio table to represent their thinking and help them to track the groups were recorded. This model was used by 3 (14%) of the 21 fall Grade 3 students, two (7%) of the 27 spring Grade 3 students, and one (3%) of the 22 fall Grade 4 students, respectively. No students used a ratio table at the Grade 4 fall interview.

With this problem came the emergence of jottings, a much less structured way to keep track of a predominantly mental calculation. With this model, students definitely wrote something down on paper, but just enough to support their mental work.

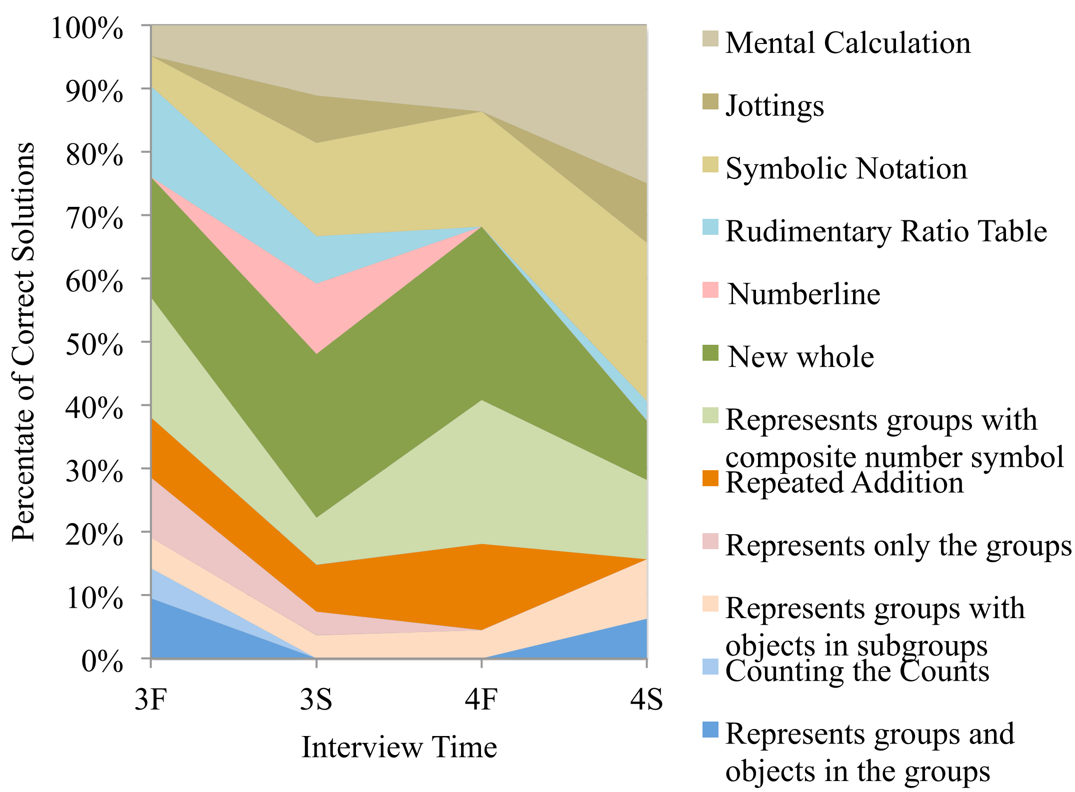



*Figure 11.* Percentage of correct responses for each model used to represent 6 × 24.

**Strategies used in correct solutions for 6 × 24.** A comparison of strategies used in correct solutions to the problems involving the calculations 6 × 24 is presented in Figure 12. In the fall of Grade 3, the 21 students used two familiar strategies, namely, (a) repeated addition, where the student added 24 six times, finding a subtotal after each new addition, and b) doubling to create a new composite of 48 and then reiterating the new composite three times in some way to find the total of 144. Three (17%) of the fall 21 Grade 3 students correctly used repeated addition. This strategy also was used by three (11%) of the 27 spring Grade 3 students and one (3%) of the 32 spring Grade 4 students. This strategy was not used in the fall of Grade 4.

Students used the doubling strategy to create a new composite with much success. Of the 21 fall Grade 3 students with correct solutions, two (11%) of them used this strategy. The doubling strategy to create a new composite was used by seven (26) of the 27 spring Grade 3 students, four (18%) of the 22 fall Grade 4 students, and nine (28%) of the 12 spring Grade 4 students, respectively.

The complexity of the problem 6 × 24 resulting from the size of the number 24 also revealed the presence of two related and more intricate strategies. In these strategies, students first split the 24 along place value lines into six 20s and six 4s. In one version, the students calculated the subtotal of the 20s and then began to recombine the six 4s one at a time to the subtotal from the 20s. This version was the most prevalent initially, used by seven (39%) of the 21 fall Grade 3 students correctly, but it was quickly dropped in usage and was used only by two students each at the Grade 3 spring (7% of 27), Grade 4 fall (9 of 22), and Grade 4 spring interviews (6% of 32), respectively.

In the related strategy, students calculated both the subtotal for the six 20s and the subtotal for the six 4s before recombining them to find the total (i.e., essentially using partial products). This strategy was executed correctly by only two (11%) of the 21 fall Grade 3 students, but it quickly increased in usage with nine (33%) of 27 spring Grade 3 students, eight (36%) of 22 fall Grade 4 students, and eight (25%) of 32 spring Grade 4 students, respectively. In terms of unitizing, the difference among these last three strategies was substantial. The difference could have been key to unlocking part of the way from unitizing to multiplicative thinking.

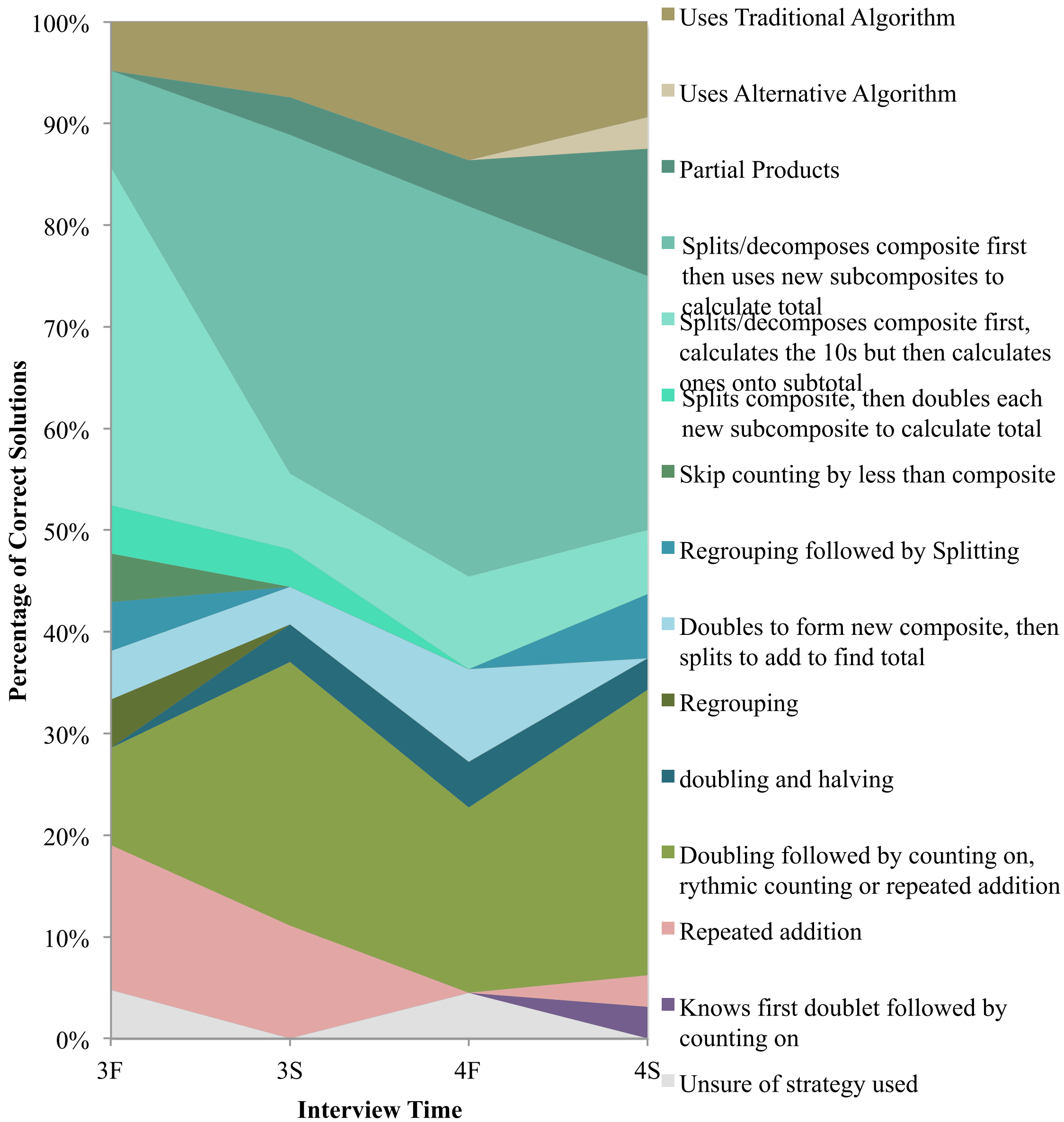



*Figure 12*. Percentage of correct responses for each strategy used to solve 6 x 24.

In 1994, Lamon published a table describing the first strategy of regrouping by combining groups as a middle level of unitizing that is used by students who are solving a multiplication or a division problem. In the same table, she described the last strategy as a more sophisticated understanding of unitizing that she theorized would be used in solving ratio problems. In the middle was the middle strategy, not quite one and not quite the other (see Figure 13). Likewise, in the same middle ground lies the small number of students who split the composite but then doubled the subgroups to find the totals.

The added complexity of the problem provided the opportunity to see the use of a student-generated partial products strategy, commencing in the Grade 3 spring interview with one (4%) of 27 students correctly using this strategy. This frequency stayed about the same, with one (5%) of 22 students in the fall of Grade 4, but it rose to four (13%) of 32 students by the spring of the Grade 4 interview. Students also began to use the traditional algorithm along the same time line, with two (7%) of 27 spring Grade 3 students, two (14%) of 22 fall Grade 4 students, and three (9%) of 32 spring Grade 4 students (9% of *n* = 32), respectively, using it.

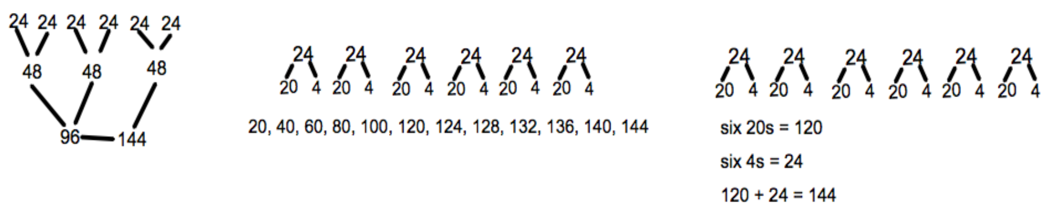



*Figure 13.* Representations regrouping, splitting with adding 1s to the subtotal, and splitting and using new subcomposites to calculate the total.

## Interplay Among Strategies and Models in Primary/Junior Multiplication

I examined five models and seven strategies across the two primary and junior multiplication problems that were analyzed to see whether particular models encouraged mental strategies; multiplication-level unitizing strategies like regrouping, followed by adding the groups; or a ratio understanding of unitizing, as indicated by the use of strategies where the students split the composite into smaller subcomposites and then used them to calculate the total, as seen with 6 × 24 (see Table 8).

Table 8

*Co-occurrences of Strategies and Models in Primary and Junior Multiplication for Correct And Incorrect Solutions*

| Model<br>strategy | Mental calculation | Repeated addition | New whole | Represents groups and comp. no. symbol | Other | Total |
|---|---|---|---|---|---|---|
| Doubles or regroups to create new composite | 23 | 5 | 12 | 8 | 17 | 65 |
| Doubles to create new composite then splits | 5 | 0 | 1 | 3 | 4 | 13 |
| Creates subcomposites by decomposing/splitting | 5 | 3 | 9 | 8 | 16 | 41 |
| Creates subcomposites by decomposing/splitting BUT recombines by adding 1s sequentially on to subtotal | 1 | 1 | 5 | 3 | 7 | 17 |
| Repeated addition | 5 | 1 | 0 | 0 | 35 | 41 |
| Other | 41 | 3 | 0 | 7 | | |
| Total | 80 | 13 | 27 | 29 | | |

As seen in Table 8, students who used a mental calculation to solve the problem did so primarily with doubling or repeated addition strategies. As mentioned previously in Figure 13, this tendency supported Lamon’s (1994) suggestion that splitting strategies are more cognitively demanding than regrouping or repeated addition strategies. Most students who used splitting strategies did so by either modelling with a new whole or representing the groups with a number symbol, arguably very similar models that appeared to be utilized similarly by students across the time span studied. These two models did not co-occur in the questions analyzed with the strategy of repeated addition. A shift in the model alone, however, was not enough to ensure a transition to the splitting solution strategies that required the most complex level of unitizing as the new whole and representing the groups with the composite number symbol were also used by a large number of students who used a doubling strategy.

## Summary of Multiplication Findings

### Unitizing Through Models and Strategies in Multiplication Contexts

The students used various models through the primary and junior grades to solve multiplication problems. In previous research, these models have generally been categorized as either concrete modelling or symbolic. In this study, four different models or representations that students used to solve multiplication problems that had some degree of concrete modelling emerged. These models included concretely modelling the whole situation by representing all the objects in all the groups, representing only one group, representing only the groups, and representing groups with objects in subgroups. Representing the groups and the objects in the groups was the most widely used model by the students, particularly in the primary grades.

Students appeared to use this model in a way that Fosnot and Dolk (2001b) would have described as a model of their thinking because they represented the action of the problem, and they also used unitary counting to determine the solution. However, this model also was widely used with more complex strategies, demonstrating that students used this model more generally to support their construction of mathematical relations and their increasingly sophisticated structures. In this way, this model supported the development of unitizing structures, as evidenced by its use in regrouping and splitting strategies.

## Examination of Division Problem Findings

### Primary Division Problems

**Overview of results for 15 ÷ 3.** In the fall and spring Grade 1 interviews, students were posed a quotative division word problem that required the calculation 15 ÷ 3 similar to the following: “Tad had 15 guppies. He put 3 guppies in each jar. How many jars did Tad put guppies in?” This type of division problem showed whether the students were able to iterate a given composite unit, in this case a 3-unit, to determine how many groups of 3 were in the total. This task proved quite difficult for many of the Grade 1 students. In the fall interview, 43 Grade 1 students were asked to solve this problem, but only 21 (49%) were able to correctly find the solution (see Table 9). At the spring interview, of the 53 Grade 1 students asked, 41 (77%) of them solved the problem correctly.

Table 9

*Percentage of Correct Responses Compared to Total No. of Students Interviewed and No. of Students Posed 15 ÷ 3 or 21 ÷ 3*

| Interview | Total no. of students interviewed | Total no. of students asked 15 ÷ 3 or 21 ÷ 3 | No. of correct | % correct of those asked |
|---|---|---|---|---|
| 1st interview: Grade 1 fall | 61 | 43 | 21 | 49% |
| 2nd interview: Grade 1 spring | 61 | 53 | 41 | 77% |
| 3rd interview: Grade 2 fall | 56 | 45 | 30 | 67% |
| 4th interview: Grade 2 spring | 54 | 43 | 34 | 79% |
| 5th interview: Grade 3 fall | 45 | 40 | 34 | 85% |
| 6th interview: Grade 3 spring | 45 | 39 | 34 | 87% |

**Models used in correct solutions for 15 ÷ 3.** A comparison of models used in correct solutions to the problems involving the calculations 15 ÷ 3 and 21 ÷ 3 is presented in Figure 14. Students used models from three broader categories of models to represent their thinking: adding up, repeated subtraction, and partitioning to represent the problem. At the first interview, 12 (57%) of the 21 Grade 1 students used an adding-up model, where they used the composite 3 to build up to the dividend of 15. These students either used manipulatives to model out each of the individual units inside each group of 3 or they used drawings, symbols, or a combination of the two to add up from zero, all the way up to 15. Three (14%) of the 21 Grade 1 students modelled the division by repeated subtraction. These students started with 15 and removed the composite three units repeatedly until they reached zero. Five (24%) more of the 21 Grade 1 students built the dividend of 15 with manipulatives or drawings and then partitioned off groups of 3.

By the second interview in the spring, 18 (44%) of 41 Grade 1 students were using an adding-up model; two (5%) of the 41 students were using a subtraction model to represent their thinking; and 18 (44%) of the 41 students were using a partitioning model, up from only five students in the fall.

The students used three main adding-up models at the Grade 1 fall interview. Four (19%) of the 21 Grade 1 students began their solution by drawing out a trial number of empty groups. Although the students demonstrated a large variation in their execution of this strategy (e.g., number of groups in the first trial, representation of the composite, and simultaneous tracking of the number of groups), the students first drew or otherwise represented the fish bowls in the problem. The students then proceeded to fill the groups with the composite of three objects using manipulatives, drawings, symbols, or imaginary objects. Students also were different in their processes, that is, by whether or not they tracked the subtotals and/or the number of groups as they added more groups and objects. No single model stood out because only one or two students used any one variation. By the second interview in the spring of Grade 1, the number of students using this cluster of strategies had fallen to three (7%) of 41 students.

At the Grade 1 fall interview, five (24%) of 21 students began by drawing out a trial number of groups, but this time, they drew in the objects inside the groups as they added each new group, using manipulatives, drawings, or the composite numeral. They began with a trial number of these composite-filled groups and then they proceeded to count up to see how many individual objects they had all together. There was a lot of variety in the ways that individual students used the model (e.g., number of groups in the first trial, how the composite is represented, and simultaneous tracking of the number of groups). None of these specific models was used by more than one or two students. By the spring interview in Grade 1, five (12%) of the 41 students were observed using this model.

At the Grade 1 fall interview, the final adding-up model was used by three 14%) of 21 students to solve this division problem correctly. This model also involved representing all the groups and all the objects in the groups. However, this time, students did not guess a trial number of groups to start; instead, they began at 1 and tracked the subtotals while they added each new composite-filled group. There was far less variety with this strategy, but students were different on one important feature of simultaneity, which I discuss later. By the spring interview, the number of Grade 1 students who used this strategy had risen to seven (17%) of 41. This increase could have accounted for some of the decreases in the other adding-up models.

At the second interview of Grade 1 students in the spring, a new adding-up model emerged where three (7%) of 41 students modelled out only the groups while they used a counting strategy (i.e., rhythmic counting, skip counting) to count the composites in the groups. To use a strategy where they modelled only the groups, the students needed to have a more complete understanding of the structure of number. The students had to be able to unitize the first two levels required by the problem simultaneously because the students were able to represent both the fish (three 1-units) and the bowls (one 3-unit) with a single item. Each item (e.g., finger, cube, or tally) simultaneously represented both types of units. As the problem was solved, each representation was counted first as three units of 1 and then as one unit of 3.

The four aforementioned adding-up models were different in terms of the depth of unitizing required for each, cardinality, hierarchical inclusion, and the simultaneous consideration of up to three levels of units. In the first model described, where students modelled out empty groups, they were able to consider the groups or fish bowls as separate, countable entities. When they went back and put the three fish in each empty bowl, they did not need to simultaneously consider the bowls (one unit of 3) and the fish (three units of 1). It was only once they reached the dividend that they simultaneously held these two levels of units in their heads as they considered the bowls and the total number of fish. When they counted up each unit of 3, or each bowl, it was not necessary for them to see the various levels of unitizing in the situation in order to solve the problem in this way.

In the second model described, the students simultaneously considered the three units of 1 inside each unit of 3 as they built up to the dividend, but they did not have to also simultaneously consider the dividend as a third type of unit. To model out their thinking in this way, the students had to keep two types of units and their relationship in mind throughout the entire process.

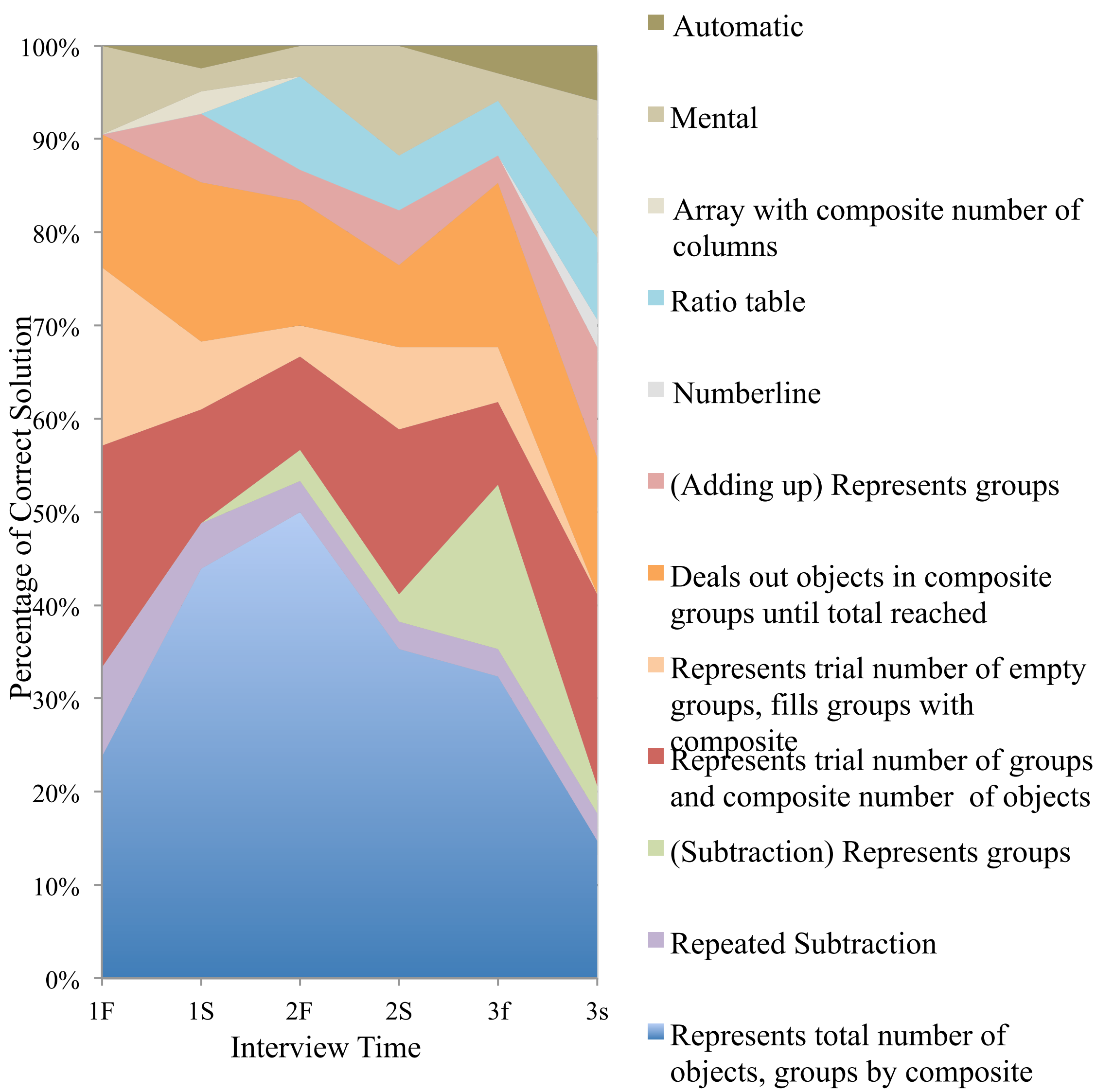



*Figure 14.* Percentage of correct responses for each model used to represent problems 15 ÷ 3 and 21 ÷ 3.

In the third model described, the students held all three levels of units in their minds throughout the entire process. They considered the fish (the units of 1) that went into each bowl (the units of 3) and the total number of fish that they had reached so far after the addition of each new bowl (i.e., one unit of 3, one unit of 6, one unit of 9, one unit of 12, and one unit of 15). When analyzing the students' solutions using the third model, a few students also demonstrated the ability to track the number of groups simultaneously as they built up to the dividend. This was essentially an oral ratio table. This simultaneity occurred only in two of the seven correct solutions that used the dealing-out model.

In the fourth model, which emerged in the spring interview, the students represented only the groups. This was similar to the third model with respect to the levels of unitizing because the students need to be able to consider all three levels of units present in division problems simultaneously in order to use this model to support their thinking.

Interestingly, some of the variations on the first and second models mentioned earlier included students who began with the first or second model, but their trial number of groups was less than the quotient. When some of these students then added more groups, they transitioned to the third model and began tracking the subtotals as they added more groups, perhaps realizing the importance of tracking this third level of unit as well as the other two. I will revisit these variations when examining 21 ÷ 3, but for 15 ÷ 3, this transition happened seven times, four of which were from the second model, when students already had considered two levels of units simultaneously as they built their solutions, and only three of which were from the first model, when students had represented only one level of unit at a time.

If these three models did provide insight into the ability of students to unitize one, two, or three levels simultaneously, it is important to note that in many cases, the student artifacts for these three different models looked identical. In many cases, without the video analysis, there would have been no way of knowing the differences among the three models unless the students overshot the number of groups required and then adjusted by crossing out groups. In all other instances where the initial trial number of groups was either less than or equal to the quotient, the student artifacts looked identical for the first three models already mentioned. This is important for teachers and researchers because it indicated that paper-and-pencil artifacts cannot differentiate among three very different models in terms of students' understanding of unitizing.

The same statement can be made about the simultaneity of counting the groups. If the presence of simultaneously counting the groups also provided insight into students' understanding of the unitizing structure, it is troubling from the perspectives of teachers and researchers' because it cannot be determined from looking at a paper artifact whether the counting occurred concurrently with the formation of new groups or after the fact. It is impossible to know for certain without hearing the students in the process of solving the problem.

Repeated subtraction was another category of models used by the students to solve the problem. At the time of the first interview with Grade 1 students in the fall, two (10%) of 21 students used a repeated subtraction model. One (5%) of the 21 students was able to do so with just the number symbols and his own ability to subtract, but the other required concrete support to help him to rhythmically count back by the composite. In the spring, the number of Grade 1 students correctly using the repeated subtraction model for division remained at two (5%) of 41.

Mulligan and Michelmore (1997) suggested that repeated subtraction is an intuitive model for division for some students. By intuitive model, they meant a model that came naturally to students due to the context or structure of the problem without instruction. The results of my study supported this finding. Very few of the Grade 1 students used repeated subtraction to solve the division problem. As in the Mulligan and Michelmore study, far more students used a model based upon adding up or partitioning rather than repeated subtraction. Like Mulligan and Michelmore, I separated partitioning models, for quotative division problems (they referred to these as direct counting methods).

I decided to group this model separately as a partitioning model for two reasons. The students who used this method initially did not necessarily go on to use repeated subtraction, as discussed in the results. In addition, a few students attempted a partitioning model with the more complex problem 64 ÷ 16. The analysis of the student solutions to that problem also influenced this decision.

The third category of models used by the students included all models, where the students first built the dividend using manipulatives, drawings, or tallies and then broke this total number up into groups of equal sizes. Although this model might have appeared to be a precursor to repeated subtraction, the way the groups were handled was very different because there was no tracking of subtotals. The total number of objects simply was divided into equal groups. For this reason, I reported this model as a separate category.

At the Grade 1 fall interview, five (24%) of the 21 students who correctly solved this problem represented their thinking with this model. Although errors were not analyzed, it was noted that this partitioning model was associated with eight incorrect responses, twice as many as any other single model. In the spring, the number of Grade 1 students who represented their thinking with this model increased to 18 (44%). At this time, the number of students using this strategy incorrectly fell to six (15%) of 41.

At the time of the Grade 1 fall interview, two (10%) of the 21 students used mental calculations to solve the problem. No students knew the solution in the Grade 1 fall interview. By the second interview in the spring, the number of Grade 1 students who used a mental calculation had fallen to one (2%) of 41, and one (2%) of the 41 students demonstrated knowing the solution automatically.

**Strategies used in correct solutions for 15 ÷ 3.** A comparison of the strategies used in correct solutions to the problems involving the calculations 15 ÷ 3 and 21 ÷ 3 is presented in Figure 15. Like the models, the strategies also were mainly used for adding up, removing composite groups, or partitioning the whole. The exception was the most frequently used strategy of counting by ones. At the Grade 1 fall interview, in total, 11 (53%) of the 21 students with correct solutions counted by 1s in order to reach the dividend. Of these, seven (33%) of the 21 students used counting by 1s with an adding-up model, and four (19%) used it with the partitioning model.

At the spring interview, 19 (48%) of the 41 Grade 1 students used a counting-by-1s strategy to count to the dividend. Of these, only seven (17%) of 41 students used an adding-up model, and the remaining 12 (29%) of 41 used a partitioning model. At this interview time, 5 (24%) of 41 students were found to be using rhythmic counting.

As described earlier with multiplication, rhythmic counting demonstrates not only that the student knows in order to solve the problem, it is important to group the one units into new groups of three 1-units, but also that they are able to consider both of those levels of units simultaneously. The pause that is apparent when rhythmic counting is required to help students to define the end of one unit of 3 and the beginning of the next.

At the Grade 1 spring interview, 20 (48%) of the 41 students were observed using a counting-by-1s strategy, but only three (7%) of the 41 students were using rhythmic counting. The decrease in the number of students using rhythmic counting was likely the result of the increase in the number of students using skip counting from zero students to seven (17%) of the 41 students. In separate studies, Anghileri (1989) and Steffe (1988) postulated that rhythmic counting is the precursor to a skip counting sequence and that once the sequence or number pattern is learned, the middle numbers drop out, leaving only the skip counting sequence.

At the Grade 1 fall interview, five (24%) of the 21 students correctly used a partitioning model to solve the problem. Four (19%) of the 21 students counted by 1s up to 15 using some representation for the fish and then removed or grouped three together to make the bowls. One (5%) of the 21 students built up to 15. He was able to take advantage of the 10 sticks (10 pop cubes stuck together to form a stick of 10) and build to the 15 without having to count individual cubes by ones.

This building or counting up to the dividend and then removing or grouping the units of three does not require students to consider the three levels of units simultaneously because the total number does not have to be held and considered at each step. The total can be discarded during the process of making the groups of 3. At the time of making the groups, one (5%) of the fall 21 Grade 1 students demonstrated the ability to make the groups and track the number of groups simultaneously. To do this, she took three cubes off and said, “One.” In order to take three cubes, but say the number 1, the student had to be able to simultaneously consider the three 1-units also as the one 3-unit.

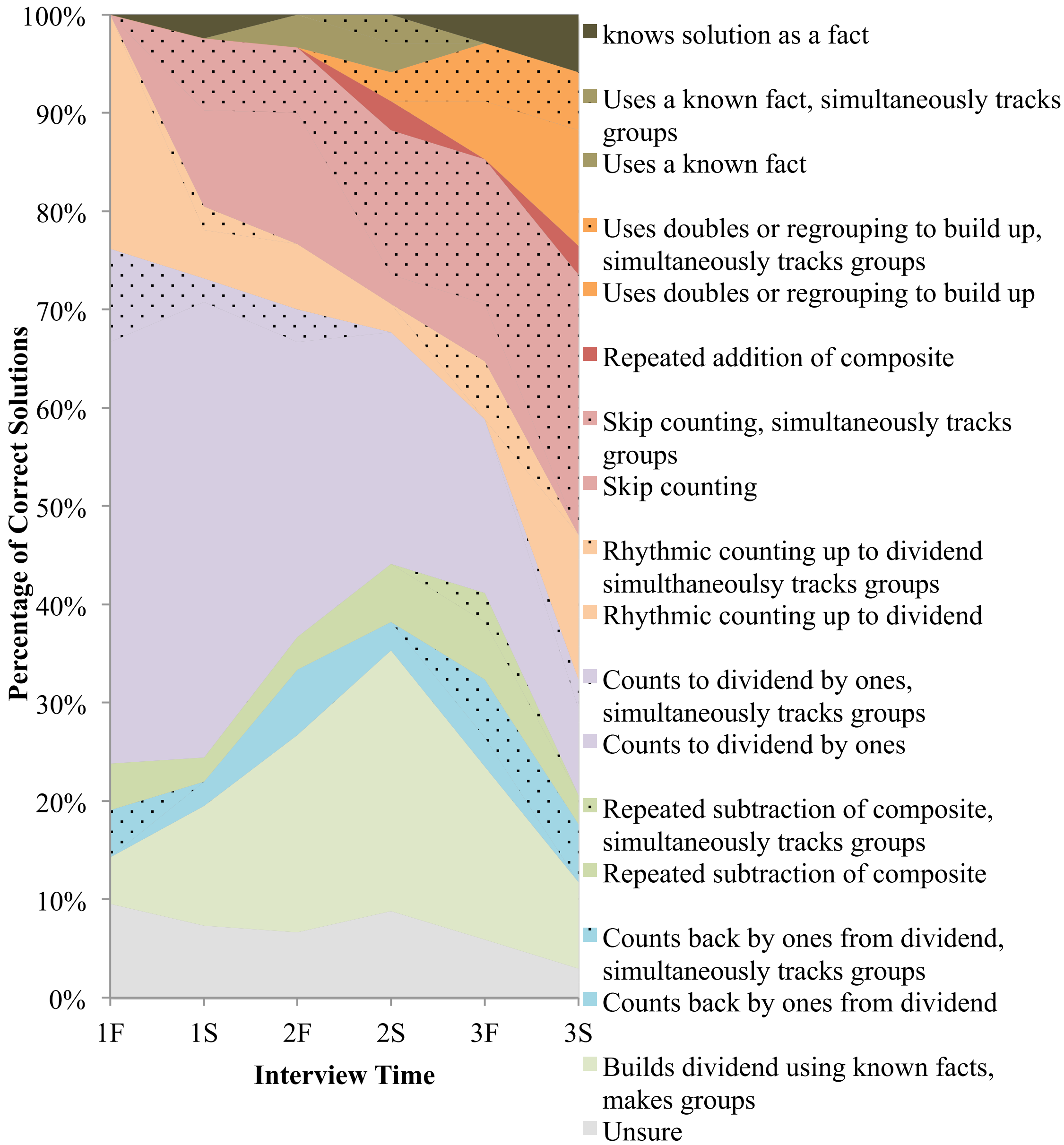



*Figure 15.* Percentage of correct responses for each strategy used to solve 15 ÷ 3.

It was unclear whether students were holding both of these units in their heads if they were not simultaneously tracking the number of groups. I think that some students likely were considering both levels of units as representing their thinking in this way, whereas others were using this model to help them to solve the problem. They were simply removing the three fish, not thinking of them simultaneously as a bowl.

Two (10%) of the 21 fall Grade 1 students who used a repeated subtraction did so more symbolically by writing the number 15 and then proceeding to subtract groups of three. One student used subtraction to find the solution after the removal of each group of three, and the other drew out the fish bowls and counted back to find the remaining number of fish each time. As with the partitioning strategies, the students who chose this strategy did not have to hold the 15 in their heads because by starting with 15, they knew when to stop grouping or subtracting groups when they got to zero or ran out of objects. As they grouped or subtracted, they did have to consider that that each group of three or subtraction of three was also countable as one 3-unit. They also kept track of the new subtotal of fish after each bowl was removed.

**Overview of results for 21 ÷ 3.** I determined that a slightly more challenging quotative division problem was needed commencing at the third interview (i.e., fall Grade 2 students), so the numbers in the word problem were changed such that the word problem now resulted in the calculation 21 ÷ 3. This new problem was asked at the Grade 2 fall, Grade 2 spring, Grade 3 fall, and Grade 3 spring interviews. As seen in Table 9, at the Grade 2 fall interview, 30 (67%) of the 45 students who were asked this new question were able to solve it correctly. By the fourth interview, this number had risen to 34 (80%) of 43 Grade 2 students. This level held through Interviews 5 and 6, with 34 (85%) of 40 fall Grade 3 students and 34 (87%) of 39 spring Grade 3 students, respectively.

**Models used in correct solutions for 21 ÷ 3.** At the Grade 2 fall interview, students were observed modelling the situation with one of the three categories of models: adding up, repeated subtraction, or partitioning (see Figure 12). At the Grade 2 fall interview, the total number of correct solutions that the adding-up models accounted for was 12 (40%) of 30 students. At the same interview, a subtraction model accounted for 2 (7%) of 30 students, and the partitioning model was used by 15 (50%) of the 30 Grade 2 students. The percentage of correct solutions where students used an adding-up model remained relatively consistent at the next two interviews, with 14 (41%) of 34 students at the Grade 2 spring interview and 12 (35%) of 34 students at the Grade 3 fall interview. At the Grade 2 fall interview, the number of correct solutions that used a repeated subtraction model was consistent with only 2 (7%) of 30 students.

By the fall of Grade 3, this number had risen to 7 (21%) of 34 students. The number of correct solutions represented using a partitioning model decreased slightly at the time of the Grade 2 spring interview to 12 (35%) of 34 students. This number fell again to 11 (32%) of the 34 fall Grade 3 students. There was a substantial increase in the use of adding-up models at the time of the Grade 3 spring interview (23 [68%] of 34 students) and a related decrease at this interview in the use of both the subtraction model (3 [9%] of 34 students) and the partitioning model (5 [15%] of 34 students).

At the Grade 2 fall interview, nine (30%) of the 30 students used one of the four adding-up models introduced earlier in the section discussing the 15 ÷ 3 models, but the distribution of the number of students using each type of model had shifted. A higher percentage of students used models that required more simultaneous consideration of different layers of units. Only one (3%) of the 30 fall Grade 2 students used a trial number of empty groups to begin the solution, three (10%) of the students represented the problem with a trial number of groups with the composite number of objects inside, four (13%)) of the students dealt out the objects in composite groups, and one (3%) student represented only the groups as the student built up toward the composite.

A new model, the ratio table, emerged at the time of this interview. Three (10%) of the 30 fall Grade 2 students used this new model at this time. To use a ratio table, students were only representing each new subtotal that they reached when were adding on a new 3-unit. They had to group three 1-units into one 3-unit to successfully represent their solution in this way. Simultaneous consideration of each group of three 1-units as part of a new layer of units was not required with this model because the students could go back and count their subtotals to determine how many groups of three were in 21. Two of the students who used this model did track the number of groups as they went. In order to do this, the students had to be able to consider all three levels of units simultaneously, namely, the 1-units, the 3-units, and the units of three 1-units. One 21 unit also is seven 3-units and 21 1-units. To use a ratio table, students also had to demonstrate the ability to hold the cardinal meaning for each number along the sequence to 21. The number symbol 9, for example, first had to be understood as nine fish or nine 1-units as well as three 3-units.

As indicated earlier, subtraction models were not observed very often throughout the interviews. At the Grade 2 spring interview, the emergence of a new model, in which the students represented only the groups for subtraction, was observed. They did so with either fingers; counters; tallies; the composite numeral (i.e., 3); or with number symbols to count the groups as they tracked. Although only one (3%) of the 34 spring Grade 4 students, six (18%) of 34 students used it at the Grade 3 fall interview. At the time of the Grade 3 spring interviews, only two (6%) of 34 students used a subtraction model, with one (3%) using repeated subtraction, and the other representing only the groups to track as he subtracted.

At the Grade 2 fall interview, 15 (50%) of the 30 students used a partitioning model by representing the total number of objects and then grouping by the composite. This model was always the most frequently observed, with 12 (35%) of 34 spring Grade 2 students and 11 (32%) of 34 fall Grade 3 students using it. This changed at the Grade 3 spring interview, when only five (15%) of 34 students used this model.

As seen with multiplication, the number of fall Grade 2 students who knew the solution automatically was zero, when this problem was introduced. Automatic responses were not observed on this problem until Interview 5 (one [3%] of 34 fall Grade 3 students). This percentage increased slightly by Interview 6 (two [6%] of 34 spring Grade 3 students). Mental calculations were inconsistent, with only one (3%) of the 30 fall Grade 2 students calculating mentally, four (12%) of the 34 spring Grade 2 students calculating mentally, one (3%) of the 34 fall Grade 3 students calculating mentally, and five (15%) of the 34 spring Grade 3 students calculating this way.

**Strategies used in correct solutions for 21 ÷ 3.** A comparison of strategies used in correct solutions to the problems involving the calculations 15 ÷ 3 and 21 ÷ 3 is presented in Figure 14. Counting by 1s was the most frequent strategy at the Grade 2 fall interview, with nine (30%) of the 30 students observed calculating their solutions using this strategy. At the Grade 2 spring interview, this number remained high, with eight (24%) of the 34 students using this strategy. However by the fall of Grade 3, only 6

(18%) of the 34 students were using this strategy, and by the final interview at which this question was asked, in the spring of Grade 3, only 3 (9%) of the 34 students were using this strategy.

As noted earlier, this strategy was used by students who used the subtraction model as well as by students who used the adding-up model. At the fall of Grade 2 interview, five of the nine students who were using this strategy did so to subtract. At the interview in the spring of Grade 2, only three of the eight students who were using this strategy did so to subtract. Half of the six students at the Grade 3 fall interview who were using this strategy did so to subtract. Two of the three students who counted by 1s at the Grade 3 spring interview were using this strategy with a subtraction model. Overall, counting by 1s did not appear to be used more frequently in conjunction with an adding-up or a subtraction model, which lends support to the hypothesis that neither the adding-up nor the subtraction model was used predominantly by either weaker or stronger students. In fact, students with high levels of mathematics achievement were found to be using both of these models.

Only a small number of students were seen to be using rhythmic counting beyond the Grade 1 interviews, with two (7%) of 30 students using this strategy at the Grade 2 fall interview, one (3%) of 34 students using this strategy at the Grade 2 spring interview, and two (6%) of 34 students using it at the Grade 3 fall interview. There was a surprising increase in the use of this strategy at the Grade 3 spring interview, with five (15%) of 34 students using it.

It is unclear what the cause of this increase was. The five students who used this strategy on this question at the Grade 3 spring interview varied by gender, teacher, and mathematical ability. At the Grade 3 fall interview, two of these students had used a counting-by-1s strategy with an adding-up model, one had used a repeated-subtraction strategy, the fourth had used rhythmic counting, and the fifth had not been interviewed.

The percentage of students who used a skip counting strategy held relatively constant over Interviews 3, 4, and 5, with six (20%) of 30 fall Grade 2 students, six (18%) of 34 spring Grade 2 students, and seven (21%) of 34 fall Grade 3 students, respectively. As with rhythmic counting, there was an increase in the use of this strategy at the Grade 3 spring interview, with nine (26%) of 34 students observed using it to solve the problem. Over the four interviews, one student used a repeated-addition strategy on two occasions, one at the Grade 2 spring interview (3% of 34) and one (3% of 34) at the Grade 3 spring interview.

Two new strategies were observed, one at the Grade 2 fall interview and one at the Grade 2 spring interview. The first emerged at the fall of Grade 2 interview, when one (3%) of 30 students used a known multiplication fact (i.e., four 3s are 12) and then repeatedly added the composite unit (3) to derive the solution from a known fact. At the spring of Grade 2 interview, two (6%) of 34 students were able to use this strategy. This strategy was not observed at the Grade 3 interviews because of the increase in automatic responses and students moving out of the project. The second emerged at the interview in the spring of Grade 2. At this interview, one (3%) of 34 students was observed using doubling or regrouping to add up to the dividend and determine a solution.

The use of this strategy continued to grow through the remaining fall and spring Grade 3 interviews at which this question was asked, with four (12%) of 34 fall Grade 3 students using this strategy and six (18%) of 34 spring Grade 3 students using this strategy.

Both of the aforementioned strategies were quite advanced in terms of unitizing; in both cases, students were considering not only the 1-units (fish) but also the units of three 1-units (bowls), the 21 1-units that represents the whole, and the seven 3-units but also that that seven 3-units were made up of some kind of grouping of the t3-units, like two bowls is two 3-units (bowls), which is six 1-units (fish; see Figure 16). The use of reunitizing through doubles or other groups was difficult because it required the students to hold at least four levels of unitizing in their heads simultaneously. Surprisingly, this strategy did not appear error prone because no students were observed using it incorrectly; however, a few students did attempt it.

The same partitioning strategies observed for this more difficult problem had been observed for the Grade 1 problem requiring the calculation 15 ÷ 13, as already reported. The strategy of building up to the dividend and then making groups was the most frequently used partitioning strategy on this problem throughout the remaining interviews, with six (20%) of the 30 fall Grade 2 students, eight (24%) of the 34 spring Grade 2 students, six (18%) of the 34 fall Grade 3 students, and three (9%) of the 34 spring Grade 3 students observed using this strategy.

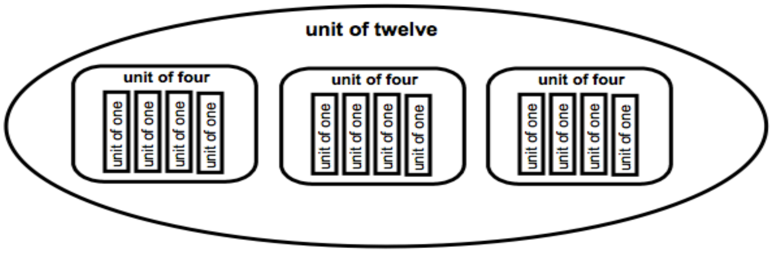



*Figure 16.* Depiction of the way 21 would be unitized when solving a quotative division problem with three in each group.

The frequencies of repeatedly subtracting the composite and rhythmically counting back by 1s fluctuated slightly but remained relatively low. In the Grade 2 fall interview, only one (3%) of the 30 students used repeated subtraction. This number rose to two students (6% of 34), respectively, at the Grade 2 spring and Grade 3 fall interviews, only to fall back to one (3%) of 34 students at the Grade 3 spring interview. Rhythmically counting back by 1s from the dividend was used with similar frequency. Two (7%) of the 30 fall Grade 2 students, but only one 3%) of the 34 spring Grade 2 students used this strategy. This number rose to three (12%) of the 34 fall Grade 3 students and then fell again to two (6%) of the 34 spring Grade 3 students.

**Junior Division Problem**

**Overview of results for 64 ÷ 16.** At Interviews 7 and 8 students were posed a quotative word problem that required the calculation 64 ÷ 16 similar to the following: “The teacher was packing pencils into boxes of 16. She has 64 pencils. How many boxes can she make?” In the fall interview, 34 Grade 4 students were asked to solve this problem, but only 68% (23) of the 34 students were able to find the solution correctly (see Table 10). At the spring interview, of the 35 Grade 4 students who were asked, 71% ( 25) of the 35 solved the problem correctly.

Table 10

*Percentage of Correct Responses Compared to Total No. of Students Interviewed and No. of Students Posed 64 ÷ 16*

| Interview | Total no. of students interviewed | Total no. of students asked 64 ÷ 16 | No. of students who answered correctly | % correct of students who were asked |
|---|---|---|---|---|
| 1st interview: Grade 4 fall | 45 | 34 | 23 | 68% |
| 2nd interview: Grade 4 spring | 45 | 35 | 25 | 71% |

**Models used in correct solutions for 64 ÷ 16.** The percentages of models used in correct solutions over the two sets interviews are shown in Figure 17. As seen with the multiplication problem 6 × 24, the students used a variety of different models to work with the larger numbers. The underpinning subtraction model was evident at the Grade 4 fall interview, but it was used by only 3 (13%) of the 23 students. Two (9%) of the 23 students were observed using repeated subtraction to represent the problem. The remaining student (4%) of the 23 Grade 4 fall students used a mental calculation.

At the Grade 4 spring interview, three (12%) of the 25 students were observed using a subtraction model to represent the problem. One (4%) of these 25 students was observed using repeated subtraction with the traditional algorithm to represent the problem. The remaining two (8%) students represented their thinking and underpinning subtraction model in a less structured way using only jottings.

At the Grade 4 fall interview, four (17%) of the 23 students represented this problem and solved it correctly using the partitioning model. These students represented the 64 by splitting it into six 10s and four 1s. They then decomposed some of the 10s to make groups of 16 with four left over. After making three groups of 16 in this way, they then added up the four remaining 4s to make the final group of 16 (see Figure 18). This model appeared quite cumbersome for the students. This model seemed to require the students to reunitize the dividend, 64, twice. They reorganized the 64 first as six 10s and four 1s, and then they had to reorganize it again in order to create the four units of 16. The unitizing itself also seemed more complicated with this model because the four 16s were not created in the same way. Three of them were made with one group of 10 and one group of 6, whereas the last unit of 16 was made with four groups of 4. The unitizing structure is not iterated in the same way at all three layers (see Figure 18). Instead, inside the 64 are four units of 16, but those units of 16 are not made up of 16 units of 1. Instead, three of them are constructed with one unit of 10 and one unit of 6, and one of them is constructed with four units of 4. This added level of complexity, coupled with the students' familiarity with larger numbers, could have accounted for the fact that this model was not used correctly in the Grade 4 spring interview.

The remaining 16 of the 23 students who solved the problem correctly at the Grade 4 fall interview used an underpinning adding-up model. At this time, the students were fairly spread over the various adding up strategies. Three (13%) of the 23 students used mental calculation, and another three (13%) used representing a trial number of groups filled with the composite number of objects. At this interview, the use of jottings (one student [4%]) as well as the use of symbolic notation (two students [9%]) to model their thinking using adding up was observed. Only one (4%) student of the 23 used a model where she tried multiplying the 16 by a trial number of groups in order to figure out the solution. Two (9%) of the students used the number line to add up to solve the problem.

By the Grade 4 spring interview, 22 (88%) of the 25 students who correctly solved the problem did so with an underpinning adding-up model. Some of the variety of model use had diminished. Now twice as many students used mental calculations to solve the problem with adding up (8 [32%] of 25). Two (8%) of the 25 students used jottings to represent the problem, and three (12%) used the number line. Two (8%) of the 25 students used a ratio table, two more (8%) represented their thinking with a trial number of composite-filled groups, and two more (8%) built up to the dividend with composite-filled groups. Again, as at the fall interview, one (4%) of the 25 students used a trial number to multiply 16 by in order to determine the solution.

**Strategies used in correct solutions for 64 ÷ 16.** The percentages of the strategies used by students who correctly solved the problem are shown in Figure 19. At the time of the Grade 4 fall interview, the three (13%) students of the 23 who answered correctly used an underpinning subtraction model for division did so using three variations of repeated subtraction of the composite. One student (4%) started with 64 and repeatedly subtracted 16 until reaching zero and then counted the number of subtractions made. The remaining two (9%) of 23 students tracked the number of groups as they solved the problem so that they did not have to go back and count them later. One of these students knew that two 16s were 32 and subtracted 32 from 64 to begin solving the problem.

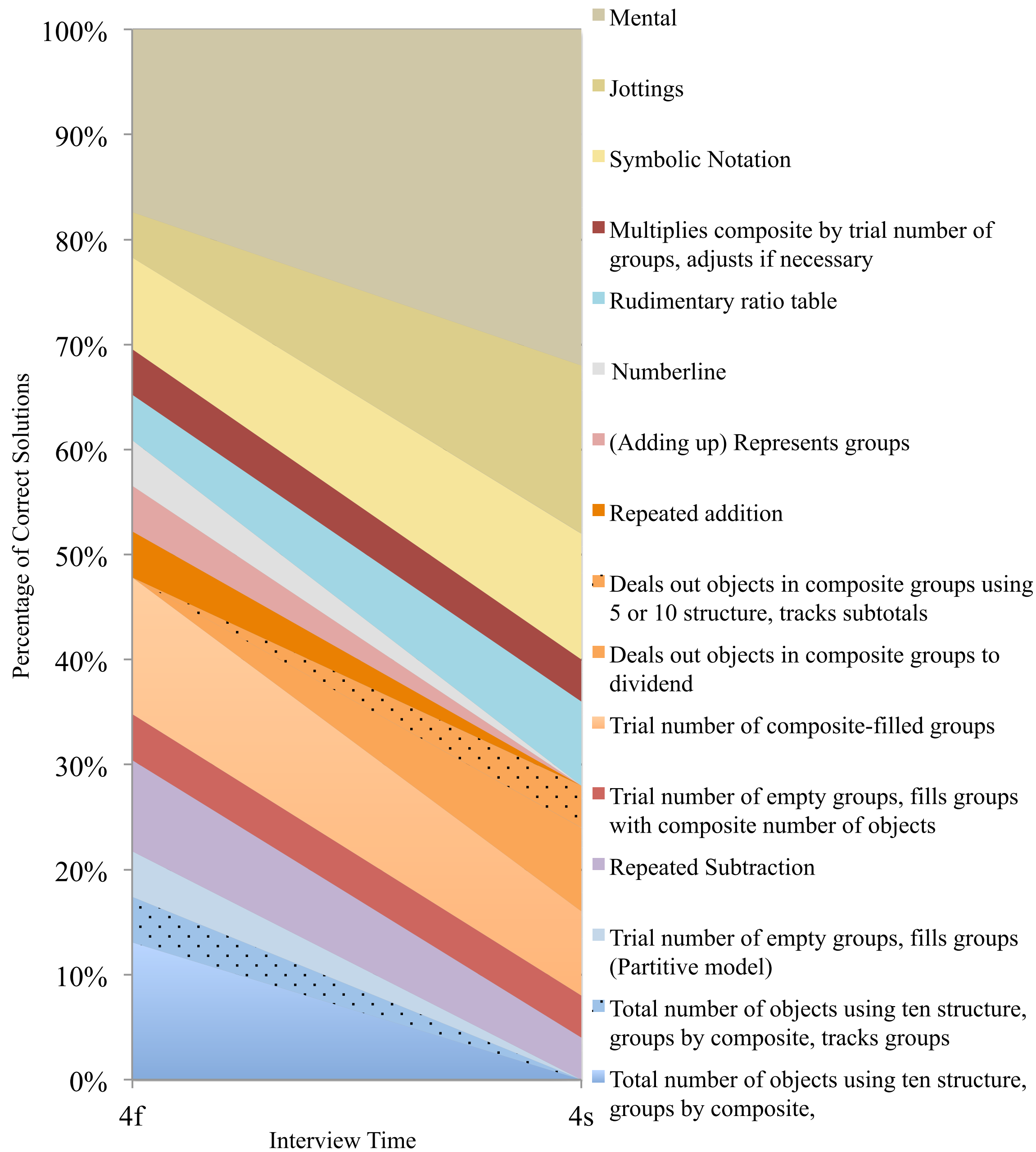
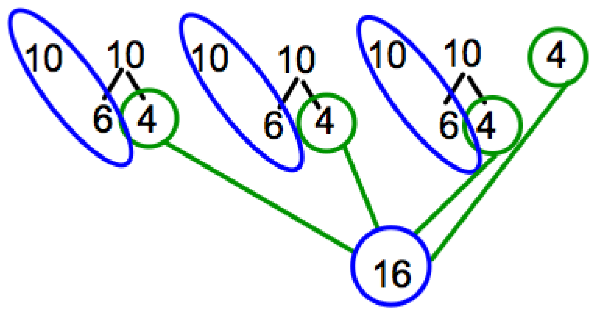



*Figure 17.* Percentage of correct responses for each model used to represent 64 ÷ 16.




$$64 = [(10) + (6)] + [(10) + (6)] + [(10) + (6)] + [(4) + (4) + (4) + (4)]$$

*Figure 18.* Partitioning model used to solve 64 ÷ 16 and unitization that correct use of this model would require.

At the spring interview, three (12%) of the 25 Grade 4 students used strategies based in the subtraction model. One (4%) student used the traditional subtraction algorithm. The remaining two (8%) students simultaneously tracked the number of groups, but one knew that there were two groups of 16 left once he reached 32.

At the Grade 4 fall interview, three of the four students (13% of 23 students) who used a partitioning model had strategies that were evident as I coded. One of the students (4% of 23 students) built up to the composite using fives and 10s. The remaining two (9% of 23 students) split the 64 into 60 and four and attempted to then make groups of 16. None of these strategies was used correctly at the spring interview; instead, the adding-up model was much more commonly used, as explored in the section on models. Unlike what was seen with the primary division problems, very few students used a counting-by-1s approach to solve this problem. In fact, none of the 23 fall Grade 4 students used this strategy, and only one (4%) of the 25 spring Grade 4 students used it.

The most frequently used cluster of strategies at the fall interview of 23 Grade 4 involved doubling (11 students [48%]). At the fall interview, only three (13%) students were able to track the groups simultaneously. By the spring interview of 25 Grade 4 students, the number of students using doubling had increased slightly to 12 (48%), but at this interview, seven (28%) of the students were able to track the groups simultaneously.

Some students split the composite in order to add up to 64. At the fall interview, two (9%) of the 23 Grade 4 students split the divisor (16) into a 10 and a 6 in order to add up to 64. At the spring interview, three (12%) of the students used this strategy. At this time, two (8%) more of the 25 students chose to use a similar strategy where they used skip counting by a value less than the composite to work their way up to 64. Two students (9% of 23 and 8% of 25) used a repeated addition strategy at the fall and spring interviews.

I also noted that one (4%) of the 25 Grade 4 spring students used a student-generated method. Although all the strategies observed were considered student generated, with the exception of traditional or alternative algorithms, this student's method was very similar to partial products, except for the organization.

**Interplay between strategies and models in primary/junior division.** I analyzed seven models and 10 strategies across the two primary/junior division problems and the one junior division problem that I selected for this research. This analysis was conducted to determine whether particular models were associated with strategies that indicated more complex unitizing structures. The results of this analysis can be seen in Table 11.

The number of models and strategies were far greater for division than for multiplication because there were three different underpinning models for division; within the underpinning adding-up model, four main models were used by students.

Therefore, examination of the interplay among the models and strategies was more complex than was seen for multiplication.

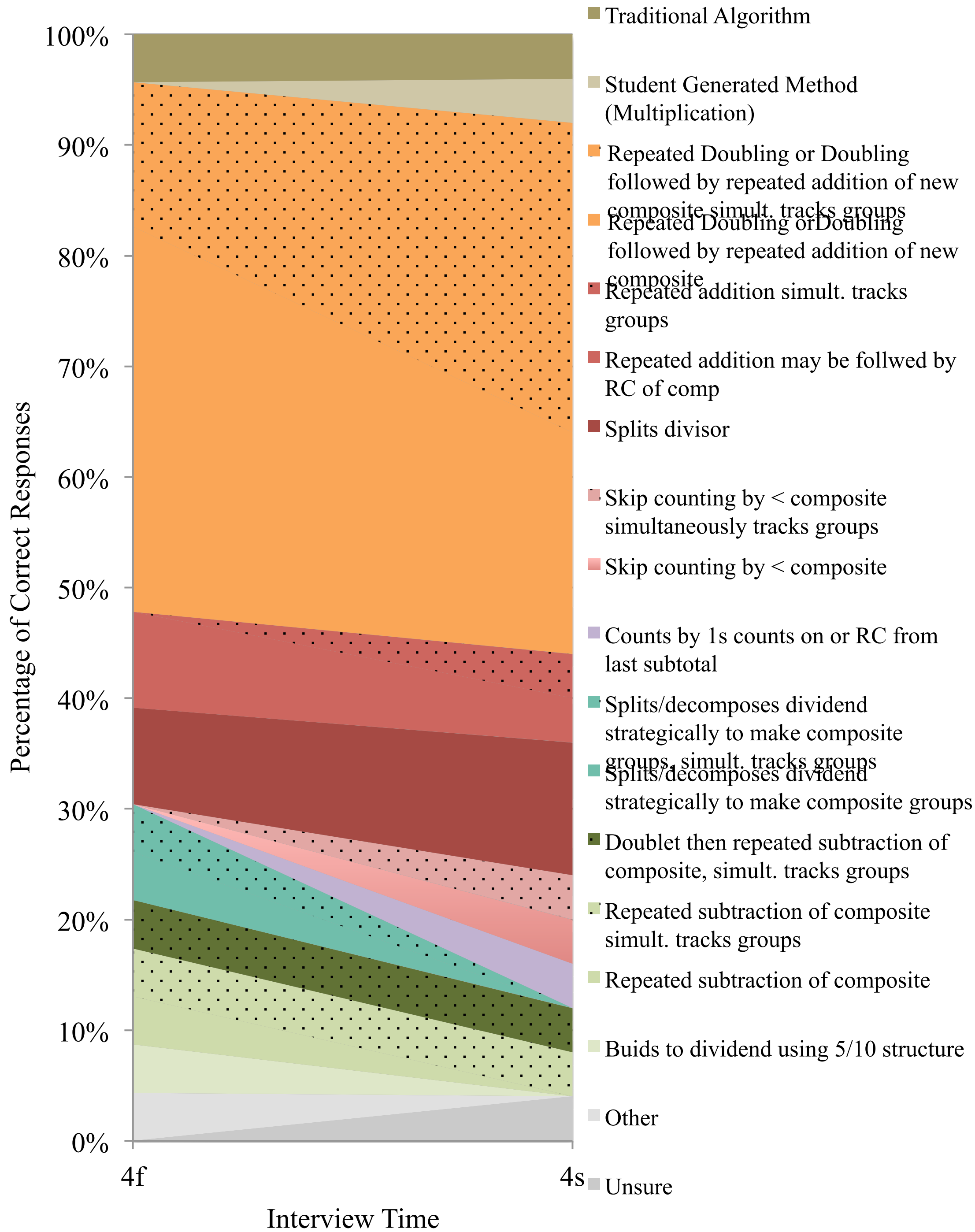



*Figure 19.* Percentage of correct responses for each strategy used to solve the problem 64 ÷ 16.

The transition from concrete modelling to the use of symbols is considered an important step on the path to a unitizing structure. Because there were far more models to consider in division, the subtleties of this nature were not explicitly analyzed. However, because a concrete model transitions to the use of symbolic notation, the strategies associated with it also shift in sophistication.

As with multiplication, students who used mental calculations primarily did so in conjunction with a strategy involving doubling; however, there was an attempt at mental calculations for most of the strategies examined. The most common strategy to occur with the trial number of empty groups model involved counting by 1s. Although there was some occurrence of more sophisticated strategies such as rhythmic counting and skip counting, indicating that some students continued to use a variation of this model with number symbols, the higher number of co-occurrences with counting by 1s lent support to the conclusion that this model is less sophisticated in terms of its required a unitizing structure.

The model of representing a trial number of composite-filled groups also co-occurred the most frequently with the strategies of counting by 1s and rhythmic counting. This model also co-occurred with skip counting, repeated addition, and doubling, indicating that many students used this model with number symbols. The high co-occurrence of this model with the simpler strategies as well as its occasional co-occurrence with several more complex strategies supported the finding that this model is slightly more advanced than the previous model. In terms of the adding-up models, the strategy of counting by 1s co-occurred almost exclusively with these two models, in which students began with a trial number of groups. As support for an intuitive sense that these two guess-and-check methods seem inefficient when compared to more sophisticated models, their high level of association with counting by 1s also indicates a reduced requirement for mathematical sophistication for their execution.

The model of dealing out objects in composite groups occurred only infrequently, with a counting-by-1s strategy indicating that students who were using this model to support their thinking no longer needed to count by 1, but have a slightly more complex unitizing structure supporting their mathematics. The co-occurrence of this model with rhythmic counting and skip counting further supported this conclusion.

The ratio table was the least used model that I examined in this analysis, with only 14 occurrences across all three problems. The use of this model is exclusive to the use of number symbols. The majority of the students who used this model utilized a skip counting strategy to create their ratio table, although a few also used repeated addition or a doubling strategy.

A repeated subtraction model was used by some students to represent and support their thinking with division problems. Two groups of strategies were used equally frequently to find the solution alongside this model, namely, counting back rhythmically from the dividend and using repeated subtraction.

The partitioning model of representing the total number of objects and then removing groups of the composite was the most frequently used primary division strategy with 105 occurrences. The majority of the students who used this model did so alongside one of two strategies, either counting by 1s to the dividend or the more complex strategy of building to the dividend. The two strategies are very similar, with the difference being when executing the latter, the students used knowledge of an underlying unit structure within the dividend, usually with units of 100-units to create a representation of the dividend without counting by 1 s.

Table 11

*Co-Occurrence of Strategies and Models for All Quotative Division Problems*

| Model strategy | Mental | Trial no. of empty groups | Trial no. of comp.-filled groups | Deals out in comp.-filled groups | Total no. of objects, groups by comp. | Repeat. Sub. | Ratio table | Other | Total |
|---|---|---|---|---|---|---|---|---|---|
| Counts to dividend by 1s | 0 | 9 | 15 | 5 | 45 | 0 | 0 | 1 | 76 |
| Rhythmic counting to dividend | 1 | 4 | 11 | 11 | 1 | 1 * | 0 | 1 | 30 |
| Skip counting | 1 | 0 | 4 | 12 | 2 | 0 | 8 | 9 | 36 |
| Skip counting < composite | 1 | 1 | 0 | 1 | 3 | 0 | 0 | 0 | 6 |
| Repeat. addition | 1 | 1 | 3 | 0 | 0 | 0 | 2 | 3 | 10 |
| Repeat. doubling | 9 | 1 | 3 | 3 | 0 | 0 | 3 | 8 | 27 |
| Builds to Dividend | 0 | 0 | 0 | 0 | 34 | 0 | 0 | 0 | 34 |
| Splits dividend to solve problem | 0 | 2 | 0 | 0 | 7 | 0 | 0 | 0 | 9 |
| Counts back from dividend | 2 | 0 | 0 | 0 | 0 | 13 | 0 | 1 | 16 |
| Repeat. sub. | 1 | 0 | 0 | 0 | 1 | 13 | 0 | 3 | 18 |
| Other | 11 | 7 | 14 | 13 | 12 | 2 | 1 | | |
| Total | 27 | 25 | 50 | 45 | 105 | 29 | 14 | | |

**Note.* This student counted back incorrectly but then counted up correctly

## Summary of Division Findings

**Unitizing in division contexts.** The students used various models and strategies through the primary and junior grades to solve quotative division problems. As noted, these models and strategies have generally been placed together, with the models themselves being put together into one of two categories: concrete or symbolic. In my analysis, I attempted to separate the two categories to identify how models and strategies together shed light on the development of a unitizing structure in children as they engage in mathematizing through quotative division contexts.

Through this in-depth analysis, I determined that the students used one of three underpinning models to solve division problems: an adding-up model, a repeated-subtraction model, or a partitioning model. The adding-up model was the most widely used across all grades and problems analyzed. I uncovered a difficulty for teachers and researchers who rely on paper artifacts to analyze students' understanding of unitizing through my analysis of the responses to the division problems because four of the models used for division might look exactly the same as finished products on paper (i.e., represents total number of objects groups by composite, represents trial number of empty groups then fills with composite, represents trial number of composite-filled groups, and deals out objects in composite groups until total reached). Although these four models might indicate very different levels of development in terms of the underlying unitizing structure, which is critical for increasing sophistication of strategies and models, they might be impossible to distinguish without a video or audio record of students' process.

Despite the fact that researchers have called repeated subtraction a primitive strategy for division (Mulligan & Michelmore, 1997), this model was used by only a small number of students to solve the division problems. This model was not associated with many of the more sophisticated strategies in terms of unitizing structures that were linked with the adding-up models.

Students in the early primary grades often used a partitioning model to divide. This simple model did not require a unitizing structure to execute unless it was accompanied by simultaneous tracking of the groups. Although this model was widely used in early primary, few students who attempted to adapt this model to use with numerals with the junior level problem did so successfully.

# CHAPTER FIVE

# CONCLUSIONS

## Development of Unitizing Across Primary Multiplication and Division

I undertook this study to examine the development of unitizing over 4 years in a cohort of children in a reform-based mathematics program. I analyzed their videotaped solutions to whole-number multiplication and quotative division questions to determine whether they used critical models and/or strategies as they developed their sense of numbers to include layers of unitizing structures. Figure 20 shows the development of unitizing constructed through this analysis.

### Models and Strategies Highlighting the Development of One Layer of Unitizing

To correctly set up and solve multiplication and division problems, the students had to have been at least on the cusp of developing a unitizing structure beyond that of units of 1 (termed *one-units* by Behr et al., 1994). They must have been able to form composite groups of units and also count those same groups by 1s in order to calculate correct solutions; however, they could have considered these two layers consecutively instead of simultaneously when they solved multiplication and division problems by using the model of representing the groups and the objects in the groups for multiplication or by using the model of representing the total number of objects and then grouping by the composite when solving division problems.

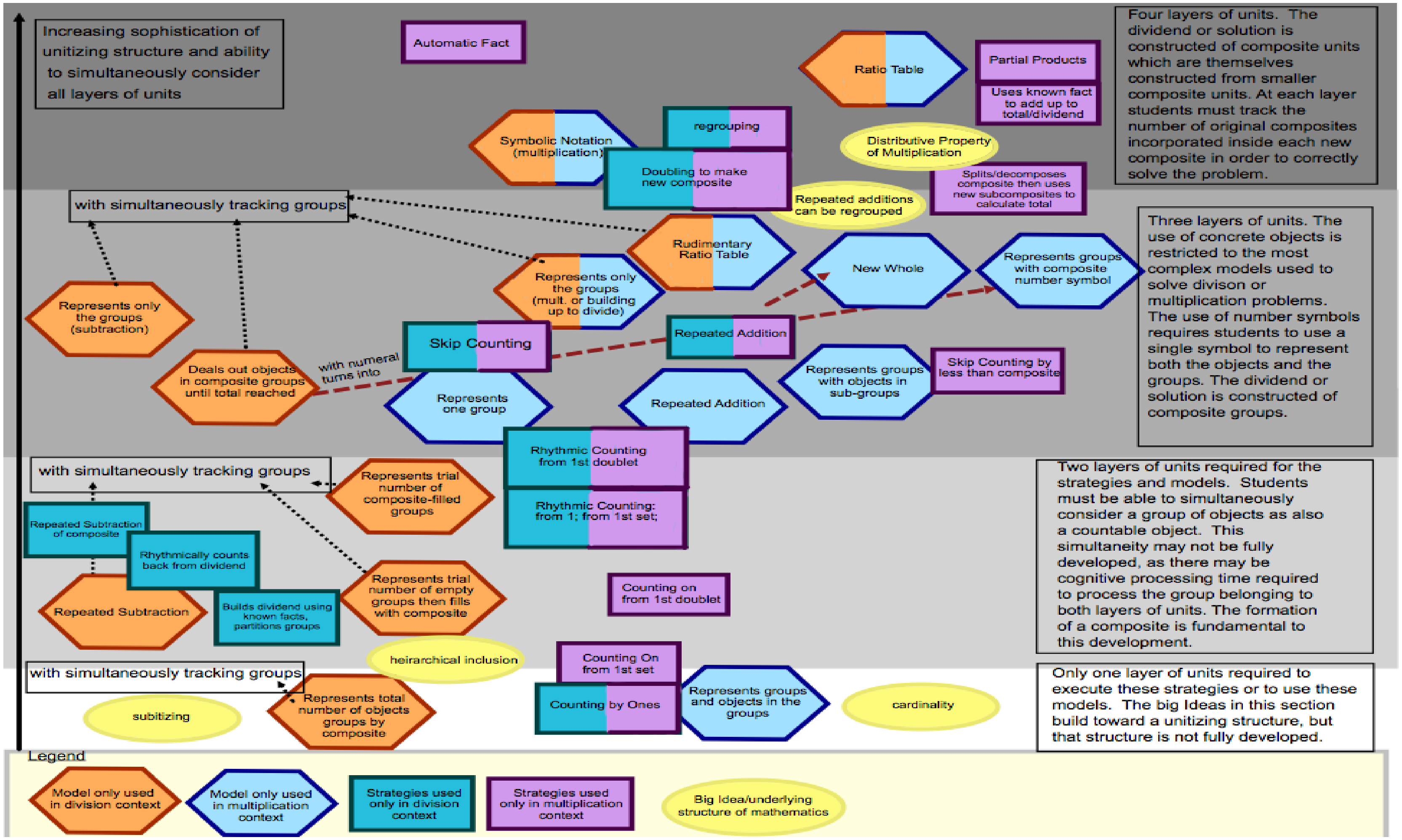



*Figure 20*. Strategies, models and big ideas used and constructed in the development of unitizing.

Models told only a portion of the story. When examining the strategies used and their associations with particular models, counting by 1s was one of the most frequently used strategies with both of these models in the early primary grades. When the students counted by 1s, they were not taking advantage of any of the inherent 10 structure of the number system or using any composite unit imposed by the questions. As a result, the students had to do each phase of the question consecutively instead of simultaneously.

By examining the strategies separately from the models, I began to see a shift toward simultaneity because some students were able to use the same model but use a strategy that required a small amount of simultaneity of two layers of units. In the case of multiplication, students who were able to count on from the first set demonstrated that they were able to consider the first set of 4 as both four units of 1 and one unit of 4 at the same time. This analysis supported the theoretical supposition put forward by Lamon (1994) that counting on is evidence of a primary level of unitizing. This preliminary ability to see a beginning group with two layers of units was even more apparent when students began counting on after the first doublet. To use counting on from the first doublet strategy, students had to be able to simultaneously consider 8 as eight units of 1 as well as two units of four units of 1. A similar progression was seen in their division strategies.

With division, the students began to build the total number of objects using known facts, such as the fact that a 10 stick contains 10 cubes and that in order to construct 21 cubes, they needed two 10 sticks and one more cube. This building required a different type of simultaneous construction of a number. In this construction, students had to be able to consider 21 to be 21 units of 1 and two units of 10 and one unit of 1 simultaneously. Although they did not have to simultaneously consider the 21 in terms of the composite unit imposed by the question with this model (i.e., 3), it was evident that the students could consider the whole 21 as well as the two 10-units within the 21.

**Models and Strategies Highlighting the Development of Two Layers of Unitizing**

As the students began to be able to consider two layers of units simultaneously, they were able to use slightly more complex models and/or strategies. For multiplication, the model of representing the groups and the objects in the groups persisted, but the selected numbers of the contextual problem also facilitated the use of fingers to solve problems. The use of fingers enabled the children to use a rhythmic counting strategy to solve the early multiplication problems, but because of the restriction of having only 10 fingers, a new model where the students took a bit of a shortcut and only represented one group that they repeatedly counted emerged. This new model was observed the most frequently with fingers. Close inspection of rhythmic counting revealed variations in the execution of this strategy with respect to the time lag between iterations; some students were able to proceed very quickly from one group to the next, but others had long pauses between groupings. In agreement with Anghileri (1989), I attributed this time lag to different levels of simultaneity as students tracked the subtotals as well as the composites as they counted.

New models and strategies that required the simultaneous consideration of two layers of units also arose for the division problems. When students were able to consider simultaneously or nearly simultaneously that the composite units also could be counted, they began to use three new division models. Use of a repeated subtraction model was seen with either rhythmically counting back from the dividend or with repeated subtraction of the composites. To model their solutions in this way, students had to already understand that within the dividends, there were groups of the composite units, even if they were unsure how many. This different type of layering of units in units begins with one unit that contains a large number of units of 1, forcing students to remove groups of units to determine how many of those groups can be removed.

Two similar models that the students used early on in division contexts were (a) representing a trial number of empty groups and then filling in the groups with the composite number of objects, and (b) representing a trial number of composite filled groups. Although the artifacts from these two models appeared identical, students who represented the trial number of empty groups and then filled those groups with the composites did not necessarily have to simultaneously consider two layers of units; instead, they first considered the groups of 3, in this case, the fish bowls and drew out an arbitrary number of them and then proceeded to fill those premade bowls with groups of three units that represent the fish in the bowls. Often, the children did this without tracking their total number of fish, so they either would go back and count the number of fish they had from 1 or would count on from the first set.

The distinction between these two models might seem insignificant, but in examples of both strategies, some students, whose initial trial number of groups was lower than the required number, were able to transition from using a trial number of groups to tracking subtotals as they added new composite-filled groups until they reached the dividend. This transition happened more often from the model of representing the trial number of composite-filled groups than it did from the representation that began with the empty groups. This evidence supported the analysis that something is distinctly different about these two models: The former indicates a more developed ability to simultaneously consider the four units of 1 as also one unit of 4 because the students who used this model appeared to more easily transition to a model that required more sophisticated unitizing.

**Models and Strategies Highlighting the Development of Three Layers of Unitizing**

As students began to be able to consider three layers of units simultaneously, I saw more use of number symbols and less use of concrete objects that required counting. Students who had constructed this level of a unitizing structure were able to use a variety of new models to solve a multiplication problem, including a repeated addition model, a representation of only the groups, a rudimentary ratio table, a representation of the groups with the objects in subgroups, a representation of the groups with the composite number symbol, the new whole model. The first three models were generally used with repeated addition or skip counting strategies, both of which were limited in terms of developing multiplicative thinking because they use consecutively addition, adding composites onto each new subtotal. The latter four models were used more multiplicatively, particularly in the junior grades, with doubling and splitting supporting the understanding of the distributive property.

In division contexts, students who could unitize three layers consecutively utilized the models of dealing out objects in consecutive groups until the dividend was reached as well as representing only the groups when utilized with repeated subtraction. In division, I also saw students represent only the groups to add up and use the rudimentary ratio table.

The three layers of unitizing are slightly different for division because one of the required layers is that the dividend contain smaller units, even though the number of those units is unknown. In multiplication, the number of units of units is known, but it is unclear what the unit total will be. This added complication of units right from the very beginning of division could have accounted for the reduced number of correct responses on division problems in the early primary grades.

**Models and Strategies Highlighting the Development of Four Layers of Unitizing**

The strategies already mentioned that occur alongside the new models for multiplication include the use of doubling; regrouping; and splitting or decomposing the composite, as is done when using the distributive property, building from known facts, and using partial products, all require four layers of unitizing to be considered simultaneously. Only a couple of students extended their use of the rudimentary ratio table to a truer form of ratio table when they used known facts to build to the dividend. Most students with this level of unitizing structure used either a form of symbolic notation or a previously described model such as the new whole, albeit with a more complex calculation strategy such as doubling.

The transition to this level of unitizing was not as clear as the proposed theoretical development of unitizing might falsely indicate. Although the doubling strategy does require four layers of units, those layers might not necessarily have to be simultaneously considered throughout the calculation. This lack of required simultaneity could have accounted for some of the errors in the use of this strategy for multiplication because some children were unable to track that each double accounted for two groups and proceeded to mistakenly calculate double the total number or more. Likewise, the use of splitting strategies resulted in a large variety in the required layers of simultaneous units and the length of time this simultaneity could be sustained. For example, when calculating 6 x 24, some students used only multiplicative strategies for the groups of 20 but added the groups of 4 back on to the subtotal for the 20s one at a time.

## Variations in the Development of Unitizing

### Gradual Development of Simultaneity

Some students were able to track the number of groups simultaneously when solving multiplication or division problems. This tracking of groups indicated a shift in the students proficiency with a particular level of unitizing. For this reason, as seen in Figure 20, I used arrows to indicate the shifting of various division models if they occurred with the simultaneous tracking of groups. The ability to simultaneously track the groups in division or in multiplication developed gradually as students expanded their understanding of the unitizing structure. As students began to consider each new layer of units, there was an increase in the time needed to consider all the layers and simultaneity was reduced temporarily. With practice these new unitizing structures became more familiar and the strategies associated with them become easier to execute. As students need less time to consider each layer, simultaneity increased again until it was possible to simultaneously track groups. Alternatively students may have made a shift into a new model or strategy that required the addition of a layer to the unitizing structure.

### Size of the Unit

The size of the unit could have affected the number of layers of units students were capable of utilizing. When students solved 4 × 10, they used models that required a more complex unitizing structure in earlier interviews than they did when solving 3 × 4. In the former, emergence of the model, where they represented only the groups to support their calculation in the spring of Grade 1, was noted. This model was not seen at all for

3 × 4 and did not emerge until the fall of Grade 2, when students were asked to calculate 6 × 4. The development of unitizing requires that students be able to consider individual units inside other units simultaneously. Children have a lot of experience doing this with certain numbers, for example, 2 (e.g., two legs or two arms per person); 5 (e.g., five fingers is one hand); and 10 (e.g., 10 fingers on both hands). Perhaps it is for this reason that children can use this model so early when working with 10s. Another possibility is that this particular cohort of students had the opportunity to develop their number sense around 10 because of the content and pedagogical knowledge of their teachers with respect to the reform-oriented instruction used in their classrooms.

**Observations of a Common Error**

Once the children were able to conceive of the problem and use the two numbers in the problems for different purposes, one common error that the students made at the early primary level was to have the number of groups match the number of objects inside the groups. For 6 × 4, they would create either four groups of 4 or six groups of 6. Only latching on to one of the numbers and using the same number in two different ways might give further insight into the order of development of the structure of unitizing. I believe that for the students who made this error, the process of using the two numbers for different purposes and holding each number and its purpose simultaneously in their heads was too great, so they had to let something go, which was the second number. The two different functions, as well as one of the numbers, were maintained. I did not code the data specifically on errors, but this was a common error, even among some of the stronger students. Many times, the students self-corrected when the interviewer repeated the question.

## Critical Models and Strategies in the Development of a Unitizing Structure

### Multiplication

When solving multiplication problems, the students began by modelling out each object in each group and using unitary counting to determine the total. As the students developed a beginning unitizing structure, they began to use one of two models, meaning that they either modelled one group that they iterated several times or used tallies, counters, or fingers to model only the groups to track their counting. These latter two models supported the development of rhythmic counting, an important skill that appears to help children solidify the beginning unitizing structure. Often in addition to rhythmic counting, the students began to count after the first set or more sophisticatedly after the first doublet. I theorize that the students' use of the doublet, followed by rhythmic counting, was a steppingstone to the next level of unitizing that accommodated three layers of units because the use of the doublet demonstrated full simultaneity of two layers of units. The falling back to rhythmic counting showed that the students could not maintain that simultaneity throughout the whole solution.

The next critical model appeared to be one of two similar representations: the new whole or representing the groups with number symbols (see Figure 21). Both models appeared to support the development of two clusters of strategies that required more multiplicative thinking, doubling and regrouping strategies, as well and decomposing or splitting strategies. These strategies can support the development of a broader and deeper understanding of the structure of number although they are very different from one another, with the latter cluster leading to more multiplicative solutions.

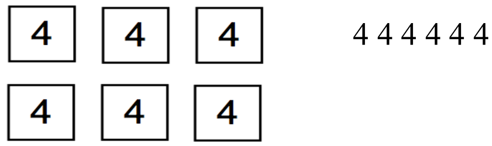



*Figure 21.* Graphical representation of the model representing the groups with number symbols and the model the new whole.

An interesting though little used model was one in which the students actually modelled out the objects in the groups in subgroups (see Figure 22). This model is interesting in that it enabled the students to use a far more sophisticated strategy and unitizing structure because it facilitated the structure supporting the construction of the distributive property. It is unclear whether highlighting this strategy in the classroom might encourage more students to construct the multiplicative layers of a unitizing structure.

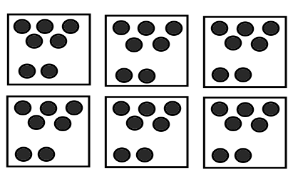


*Figure 22.* Graphical representation of the model representing the groups with the objects in subgroups.

**Division**

When solving quotative division problems, students used one of three underpinning models, an adding up model, a repeated subtraction model, or a partitioning model. In the early primary grades, the latter was the most commonly used underpinning model to solve the problem, but once the children began to use numerals in their calculations, they rarely used the partitioning model. Instead, most of the children used adding up models, which then developed much like that of unitizing in multiplication contexts. The repeated subtraction model also was used infrequently, despite the fact that it required fewer simultaneous layers of unitizing than some of the more sophisticated adding up models. Perhaps this low frequency reflected the increased difficulty of subtraction itself in comparison to addition, particularly with a larger and less friendly number like 16.

One of the major differences in division contexts was that the solution to the problem required students to determine the number of groups. If they used an adding up model, they needed to track the total as well as the number of groups. Consequently, I theorized that simultaneity, although part of a completed unitizing structure and critical to multiplicative thinking, had gradations of depth along each child's trajectory. It is possible that the ability to track simultaneously is a skill that could be increasingly developed in the classroom setting to help children develop a unitizing structure.

## Conclusion

The strategies and models provided connected but different pictures of the levels of unitizing that the students were utilizing. In some cases, a particular model, such as representing one group, seemed to give rise to new strategies such as rhythmic counting; at other times, it seemed as though a new strategy such as skip counting gave rise to a model, as was the case with the rudimentary ratio table. In the multiplication context, more complex levels of unitizing were achieved with the models of the new whole or representing the groups with the composite number symbol than were achieved with the more classic representation of repeated addition. In the division context, more complex levels of unitizing structures were demonstrated with the adding-up models than were demonstrated through either the repeated-subtraction model or the partitioning model of division.

Teachers interested in helping students to develop a sophisticated unitizing structure of numbers and researchers interested in examining its development need to recognize the differences among the various strategies and models in terms of the levels of unitizing that students will use to solve multiplication and division problems. In particular, having an awareness of the use of contexts to develop rhythmic counting, supported by constraints on the problem to facilitate students to model only one group or model just the groups might support the initial shift to two layers of unitizing by early primary students. For example a possible context might be figuring out cookies needed for people sitting around a table. This context may encourage rhythmic counting because if the places are set around the table, the model provided is one of just the groups. This may promote rhythmic counting on each plate. A constraint on the problem could be that only enough manipulatives to make one group could be provided. This would force students who want to count to count the one group over and over which will help develop a rhythmic counting rhythm.

When solving division problems it is critical for teachers and researchers to watch the student solutions. The variety of mathematical understanding cannot be captured and analysed accurately with a paper and pencil artifact as the levels of simultaneity and of layers of units cannot be determined. There were many instances of identical artifacts from students with a wide range of the development of a unitizing structure.

As students move into the junior grades, having a focus on doubling and regrouping strategies will help them to develop their unitizing structure, but it will not necessarily carry them as far as developing splitting strategies. Regrouping strategies enable students to gain a deeper understanding of number structures, but they also appear to be associated with more additive thinking by students.

Doubling or regrouping strategies often were paired with the use of additive strategies such as repeated addition of the new composite and not with multiplicative strategies such as multiplication of the new composite. On the other hand, splitting strategies were found in conjunction with a variety of strategies ranging from additive to multiplicative, indicating its flexibility to support a progression from additive to multiplicative thinking. Reform teachers and researchers interested in understanding the difference between these two strategy types should ensure that the new whole model is used in classrooms and interviews in lieu of the more classic repeated addition model because the new whole appears to lend itself equally to both types of unitizing structures. In addition, allowing the continued use of paper-and-pencil modelling, when numbers can be deconstructed, might encourage the expansion of the unitizing structure in this way because the students in this study generally used only doubling strategies when they worked mentally. Number selection is equally important in eliciting more advanced unitizing structures. Beginning with number problems that use 10s in the early primary grades might support the development of more complex unitizing structures that could then be challenged to extend to other numbers (e.g., 5, 2, 3).

## Considerations for Future Research

The examination of the proposed trajectory for the development of a unitizing structure by analyzing individual student trajectories over time would further illuminate the development of unitizing and provide insight into the accuracy of this developmental trajectory for the progression of the development of the unitizing structure.

The analysis of additional problems, including problems that use a composite unit of 10 and problems that are posed as ratio and proportion problems, could lead to the formation of a trajectory of development of unitizing structures. In addition, a reanalysis of the multiplication code with codes for simultaneously tracking the groups could shed light on the theory that although simultaneity is part of a complete understanding of the unitizing structure, there are gradations within each level of unitizing as simultaneity grows.

Future unitizing research should include an instructional program designed to support the development of more sophisticated unitizing structures. Examining the effect of explicit practice in rhythmic counting and reunitizing games and contexts that encourage students to split the composite into pieces that are easier to use as well as double or regroup them into larger composites would shed light on whether the practice of these strategies could support the development of more sophisticated forms of unitizing.

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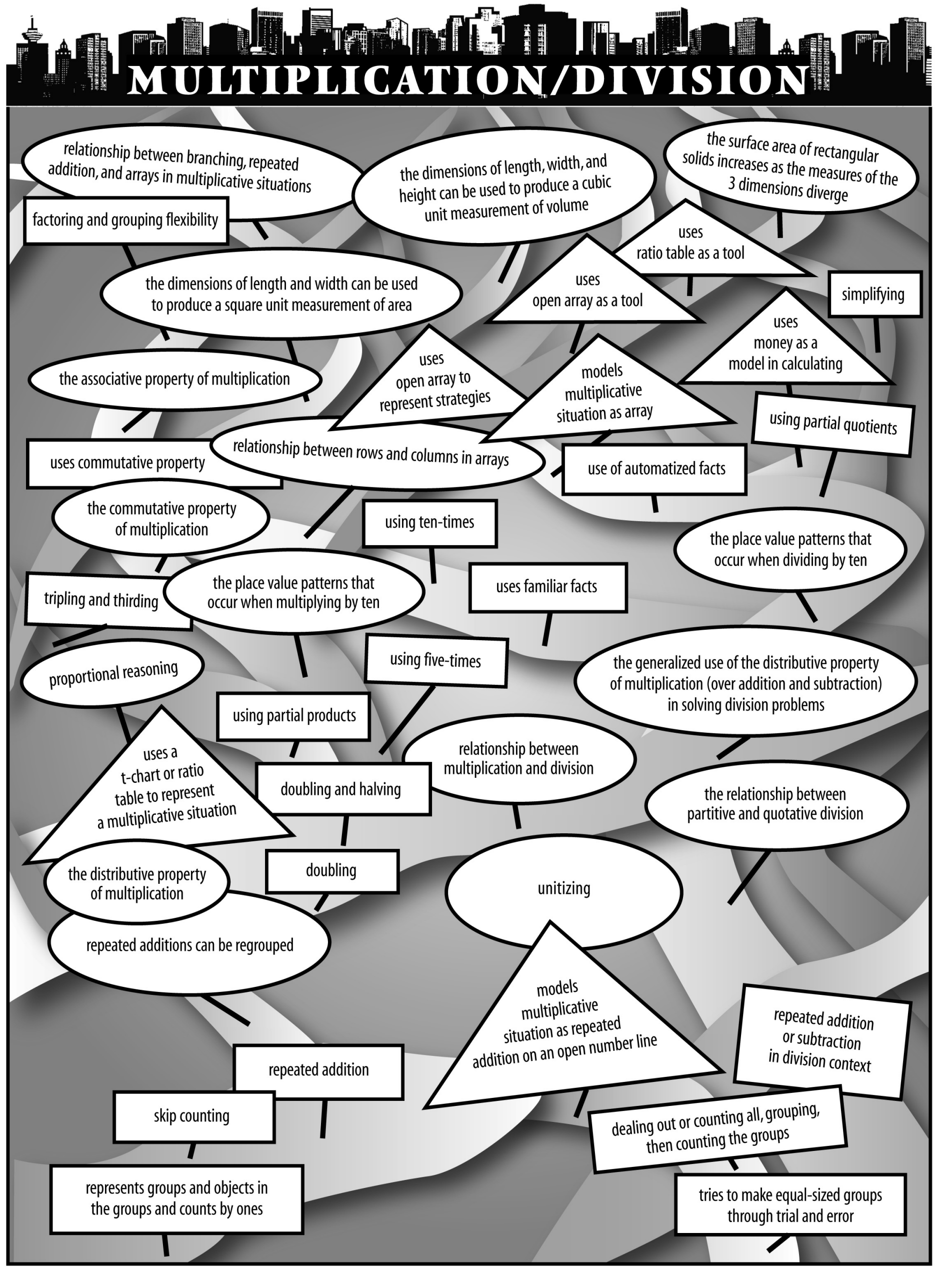
# APPENDICES

# Appendix A: Landscape of Learning




*The landscape of learning: multiplication and division on the horizon showing landmark strategies (rectangles), big ideas (ovals), and models (triangles).*

16 INVESTIGATING MULTIPLICATION AND DIVISION

Landscape of Learning (Fosnot, 2007b)

| # | Code | Problem | Problem Version 2 | 1F | 1S | 2F | 2S | 3F | 3S | 4F | 4S | 5I |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 3x4 (6 x 4) | Robin has 3 packages of gum. There are 4 pieces of gum in each package. How many pieces of gum does Robin have all together? | Robin has 6 packages of gum. There are 4 pieces of gum in each package. How many pieces of gum does Robin have all together? | 3 x 4 | | 6 x 4 | | | | | | |
| 11 | 15/3 or 21/3 quotative | Tad had 15 guppies. He put 3 guppies in each jar. How many jars did Tad put guppies in? | Tad had 21 guppies. He put 3 guppies in each jar. How many jars did Tad put guppies in? | | | 21 | | | | | | |
| 12 | 4x10 | Sandy has 4 packages of candy. There are 10 pieces of candy in each package. How many pieces of candy does Sandy have all together? | Corrine has 4 packages of baseball cards. There are 10 baseball cards in each package. How many baseball cards does Corrine have all together? | | | | | | | | | |
| 13 | 32/10 (36/10) (54/10) | Susan had 32 marbles. She put 10 marbles in each bag. How many bags did Susan fill with 10 marbles? Did she have any marbles left over? How many? | Ms. Suarez has 36 brownies to serve to her guests. She put 10 brownies on each plate. How many plates did she fill with 10 brownies. (AFTER SOLUTION ASK) Did she have any brownies left over? How many? | | | 36 / 10 | 54 / 10 | | | | | |
| 16 | 20/4 or 24/4 | Mr. Gomez had 20 cupcakes. He put the cupcakes into 4 boxes so that there were the same number of cupcakes in each box. How many cupcakes did Mr. Gomez put in each box? | 24 children signed up to play baseball. The coaches divided the children into 4 teams with the same number on each team. How many children were on each team? | 20/ 4 | | 24 / 4 | | | | | | |
| 28 | 19/5 | 19 children are going to the circus. 5 children can ride in each car. How many cars will be needed to get all 19 children to the circus? | 20 children are going on a picnic. 6 children can ride in each car. How many cars will be needed to get all 20 children to the picnic area? | | | 20 | | | | | | |
| 31 | 3:8 ?:12 | Three candies cost 8 cents how much do 12 candies cost? | | | | | | | | | | |
| 35 | 18/3 (21/3) quotative | Ms Bird had 18 flowers . She planted 3 flowers in each pot. How many pots of plants did Ms. Bird plant? | Aliza has 21 fish. She put 3 fish in each bowl. How many bowls did she use? | | | 21 | | | | | | |
| 40 | 24/4 part 42/3 | 24 children signed up to play baseball. The coaches divided the children up into 4 teams with the same number of children on each team. How many children were on each team? | Josie has 42 stickers she wants to paste on three pages of her album, How many stickers did she past on each page if she pasted all of them and did so equally? | | | | | | | | | |
| 53 | 6X24 | Sandy has 6 boxes of candy. There are 24 pieces in each box. How much candy does she have all together? | Sandy has 12 boxes of candy. There are 24 pieces in each box. How much candy does she have all together? | | | | | 6x24 | | | | 1 2 |
| 59 | 6X7 [mentally] | Josh has 6 fish bowls. There are 7 fish in each bowl. How many fish does Josh have? | | | | | | | | | | |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 64 | 604/10 | A bricklayer was laying bricks to build a wall. He had 604 bricks to lay. He laid 10 in each row. How many rows could he make? | | | | | | | | | | |
| 65 | 64/16 quotative | The teacher was packing pencils into boxes of 16. She has 64 pencils. How many boxes can she make? | | | | | | | | | | |
| 66 | 98/7 | The florist has a shipment of 98 flowers. He is making them into bouquets of 7 flowers. How many can he make? | | | | | | | | | | |
| 67 | 432/15 | There are 432 children going to the zoo. Each small bus holds 15 children. How many buses will be needed to take the children to the zoo? | | | | | | | | | | |
| 70 | 1:2:3 | Frankie has 3 fish, small, medium and large. The medium fish eats twice as much as the small. The large fish eats three times as much as the small. IF the small fish gets 1 pellet how much do the other fish fish get? If the fish grow and now the medium fish eats 4 pellets how much do the other two fish eat each? | | | | | | | | | | |
| 71 | 4 x 7 | John went shopping and bought 4 new pairs of pants and 7 new shirts. How many different outfits can he make using the new clothes he bought? | | | | | | | | | | |
| 72 | 12 x 18 | Plums in a grocery store display are arranged in 12 rows. There are 18 plums in each row. How many plums are there? How do you know? | | | | | | | | | | |
| 73 | 225/15 | A skyscraper has 225 windows on one side arranged in rows and columns. If there are 15 columns of windows, how many rows are there? | | | | | | | | | | |
| 74 | AxBxC=24 | The Laura Secord store is making new boxes to sell 24 chocolates in. The boxes must have more than one layer of chocolates. Draw a picture of how many chocolates can be in each layer. Give at least two different solutions. | | | | | | | | | | |

# Appendix C: Selected Questions From Full Instrument and Possible Strategies in Students' Solution Methods

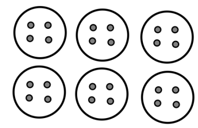
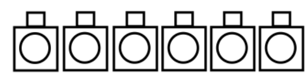
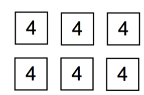
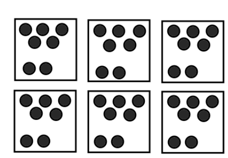
| Question Pairs | Calculation | Sample Wording | Possible Strategies | Type and Grade |
|---|---|---|---|---|
| **#9** | 3 x 4 or 6 x 4 | Robin has 3 packages of gum. There are 4 pieces of gum in each package. How many pieces of gum does Robin have all together? | Represents groups and objects in the groups and counts by ones, Skip counting, Repeated addition, Doubling, Using partial products, Using familiar facts, Use of automatized facts | Multiplication Multiple groups 1, 2 |
| **#53** | 6 x 24 or 12 x 24 | Sandy has 6 boxes of candy. There are 24 pieces in each box. How much candy does she have all together? | | 3, 4, 5 |
| **#11** | 15/3 or 21/3 | Tad had 15 guppies. He put 3 guppies in each jar. How many jars did Tad put guppies in? | Dealing out or counting all, then counting the groups, Skip counting, Repeated addition or subtraction in a division context, Using the ten structure, Doubling, Using ten-times, Using partial products, Using partial quotients, Using familiar facts, Use of automatized facts | Quotative Division Multiple groups 1, 2, 3 |
| **#65** | 64/16 | The teacher was packing pencils into boxes of 16. She has 64 pencils. How many boxes can she make? | | 4, 5 |
| **#12** | 4 x10 | Sandy has 4 packages of candy. There are 10 pieces of candy in each package. How many pieces of candy does Sandy have all together? | Represents groups and objects in the groups and counts by ones, Skip counting, Repeated addition, Using the ten structure, Doubling, Doubling and halving, Using partial products, Using familiar facts, Use of automatized facts | Multiplication Multiple groups 1, 2, 3 |
| #59 | 6 x 7 (mentally) | Josh has 6 fish bowls. There are 7 fish in each bowl. How many fish does Josh have? | Skip counting, Repeated addition, Doubling, Doubling and halving, Using partial products, Using familiar facts, Use of automatized facts | Multiplication Multiple groups 3, 4 |

*Note.* Grade designations with an *f* indicate that the item was administered only in the fall of that year. Grade designations with an *s* indicate that the item was only administered in the spring of that year. Computations in brackets are associated with slight changes in the instrument for different grades.

# Appendix D: A Priori Codes

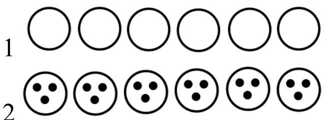
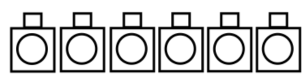
| | |
|---|---|
| Code type | a priori codes taken from the literature for initial coding |
| Strategies | Represents groups and objects in the groups and counts by ones; tries to make equal-sized groups through trial and error; dealing out or counting all, grouping, then counting the groups; skip counting; repeated addition; repeated addition or subtraction in division context; using the ten structure; doubling; doubling and halving; using partial products; using five-times; uses familiar facts; use of automatized facts; and, using partial quotients. |
| Models | models groups (i.e., concrete materials; tallies; fingers); models as repeated addition on an open numberline; models multiplicative situation as array; uses an open array; uses a t-chart or ratio table; uses money model in calculating |
| Additional Codes | correct; incorrect; incorrect but close; correct with help; multiplication context; division context |

## Appendix E: Models and Strategies Used by Students When Solving Multiplication Problems

| Name | Explanation of model | What that model looks like with 6 x 4 (6 packs of gum, 4 pieces in each pack) |
|---|---|---|
| Models | | |
| Represents groups and objects in the groups | With this model the student uses counters or tallies. One for each object and groups them into groups to help them keep track of the groups. |  |
| Represents 1 group and the objects in that group | With this model the student uses counters, tallies or fingers to represent one group and they use that representation to help them tally the total. |  |
| Represents just the groups with fingers, tallies or counters. | Here a student will skip count or rhythmically count in their heads, but they will use a counter, finger or tally to mark each group so that they can track the number of groups as they count. |  |
| Rudimentary Ratio Table | This is similar to the model above, but instead of using a tally, counter or finger to keep track of the groups, the student writes down the subtotal to help them track. | 4, 8, 12, 16, 20, 24 |
| Models the groups with number symbol | With this model students mark out groups with rectangles or circles and write the composite numeral inside, or just adjacent to each group. |   |
| Repeated Addition | With this model students write out all iterations of the composite using numerals joined by an addition sign. | 4+4+4+4+4+4 |
| New Whole | When using this model students write out all the iterations of the composite numeral all together without addition symbols or shapes defining a group. The grouped numerals together form the new whole. | 4 4 4 4 4 4 |
| Model the groups with the objects in subgroups | When using this model students model out all the objects but they do so by grouping them in subgroups such as would be formed by decomposing or splitting the composite. This model was occasionally also done with numerals (see strategy of skip counting by a value less than composite below. | 6 x 7 used for this model  |

| Name | Explanation of model | What that model looks like with 6 x 4 (6 packs of gum, 4 pieces in each pack) |
|---|---|---|
| Jottings | A mental calculation with written work to help students keep track of some of the numbers during the calculation. | lots of options, but the model doesn't fit into another category |
| Strategies | | |
| Counting by ones | Students count objects, tallies and occasionally fingers by ones beginning from one. | 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 1, 17, 18, 19, 20, 21, 22, 23, 24 |
| Counting On after the first set | Students count objects, tallies or fingers but they start from after the first iteration of the composite. | 4,…, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 1, 17, 18, 19, 20, 21, 22, 23, 24 |
| Rhythmic Counting | Students use the composite to construct their counting rhythm. A small pause usually occurs at the end of each composite. Students can also begin after the first set (se counting on) but then continue with rhythmic counting. | 1, 2, 3, 4,…, 5, 6, 7, 8,…, 9, 10, 11, 12,…, 13, 14, 15, 16,…, 17, 18, 19, 20,…, 21, 22, 23, 24 |
| Starting with a Doublet | Both counting on and rhythmic counting can begin after the first doublet (the first two iterations of the composite; rhythmic counting depicted) | Two groups is 8,…, 9, 10, 11, 12,…, 13, 14, 15, 16,…, 17, 18, 19, 20,…, 21, 22, 23, 24 |
| Skip Counting | Students navigate the counting sequence by the composite. | 4, 8, 12, 16, 20, 24 |
| Skip counting by a value less than the composite | Students decompose the composite and use skip counting by a new value(s) to calculate the total (6 × 7 depicted). | $5^2$ $5^2$ $5^2$ $5^2$ $5^2$ $5^2$<br>5, 10, 15, 20, 25, 30, 32, 34, 36, 38, 40, 42 |
| Doubling | Students double the composite and then use the new doubled composite to determine the answer. They then can use repeated addition, or new whole to combine the new composites. The students have to track that each iteration of the double is two groups. | Two 4s are 8.<br>8 + 8 + 8 = 24 |
| Regrouping | Students group together three or more composites to make a new composite that is then iterated. The students have to keep track of how many groups each new composite represents. | Three 4s are 12.<br>12 + 12 = 24 |
| Partial Products | Students use the alternative algorithm labeled partial products by using two or more products to build to the total. | 4 × 4 = 16<br>4 × 2 = 8<br>16 + 8 = 24 |
| Repeated Doubling | Students repeatedly double until they cannot double anymore. They may have to adjust by adding or subtracting composites. | Two 4s are 8.<br>Two 8s are 16.<br>One more 4 is 20 and one more 4 is 24. |

## Appendix F: Models and Strategies Used by Students When Solving Division Problems

| Name | Explanation of the Model | What the model looks like with 15 ÷ 3 (15 fish, 3 fish in each bowl, how many bowls) |
|---|---|---|
| Adding up Models | | |
| Trial number of empty groups | With this model, the students first draw out shapes to identify each group. They guess at the number of groups they will need. Then they go back and fill in each group with the composite number of objects. Finally they tally the number of objects they have and then add or remove groups as required. | Step 1 <br>Step 2<br>Step 3<br><br> |
| Trial number of composite-filled groups | This model is similar to the previous one except that the students fill the trial number of groups with the objects as they go not as two separate steps. Again they check their total and then add or remove groups as required | Step 1<br><br>Step 2<br><br> |
| Deals out composite groups tracking subtotals | This model is similar to the previous model except that the students put out one group at a time and track the subtotals as they go so that they know exactly when they have to stop. | <br> |
| Represents just the groups with fingers, tallies or counters. | Here a student will skip count or rhythmically count in their heads, but they will use a counter, finger or tally to mark each group so that they can track the number of groups as they count. |  |
| Repeated addition | With this model students write out all iterations of the composite using numerals joined by an addition sign. | 4+4+4+4+4+4 |
| Symbolic Notation | The students use standard notation to represent their thinking, but not using repeated addition or a traditional algorithm. | 4 + 4 = 8 8 + 8 = 16<br>4 + 4 = 8 16 + 8 = 24<br>4 + 4 = 8 |
| Jottings | A mental calculation with written work to help students keep track of some of the numbers during the calculation. | lots of options, but the model doesn't fit into another category |
| Rudimentary Ratio Table | This is similar to the model where only the groups are represented, but instead of using a tally, counter or finger to keep track of the groups, the | 4, 8, 12, 16, 20, 24 |

| | | |
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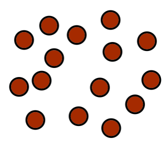
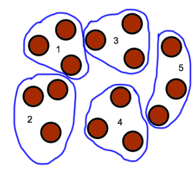
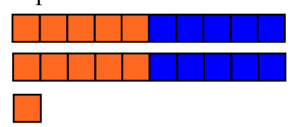
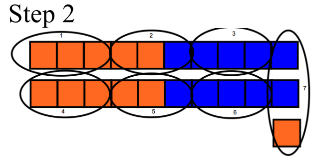
| | student writes down the subtotal to help them track. | |
| Subtraction Models<br>Repeated subtraction | Here students begin with the dividend and removes the composite calculating the new remaining minuend each time. | 15 – 3 = 12, 12 – 3 = 9, 9 – 3 = 6, 6 – 3 = 3, 3 – 3 = 0 |
| Partitioning Model<br>Represents total number of objects, groups by composite | With this model students represent the dividend with counters, tallies or fingers and then they group the objects with the composite number of objects in each group. | Step 1  Step 2  |
| Represents total number of objects using ten structure, groups by composite | This model is similar to the previous model except that the students use the ten structure to represent the dividend such as two ten sticks and one other cube to make 21. They may also use numerals such as 10, 10, 10, 10, 10, 10, 4 for 64. | Step 1  Step 2  |
| Strategies | | |
| Adding up model strategies are the same as the multiplication strategies. The composite is the divisor. When simultaneous tracking of groups occurs, the students have a double count where one number tracks the subtotal as with multiplication while the other tracks the number of groups totaled so far. | | |
| Repeated Subtraction | With this strategy the students begin with the dividend and repeatedly subtract the divisor using some method to track the number of subtractions (groups). With each subtraction they track the minuend that they use as their new total. | 15 – 3 = 12, 12 – 3 = 9, 9 – 3 = 6, 6 – 3 = 3, 3 – 3 = 0 |
| Rhythmically Counts back from dividend by composite | With this strategy the students begin with the dividend and then they use counters, drawings or fingers to support them to count back by the divisor (composite) | 15, 14, 13,…12, 11, 10,… 9, 8, 7, … 6, 5, 4, … 3, 2, 1 |

| | | |
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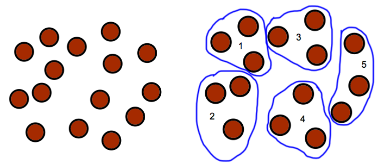
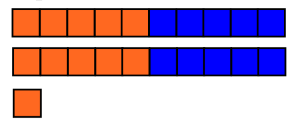
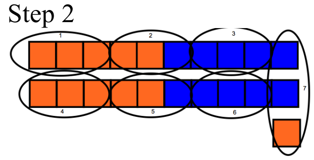
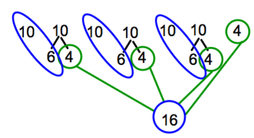
| | repeatedly. They also track the number of times they have removed the divisor. | |
| Counts by ones to the dividend, partitions groups | With this strategy students use manipulatives or drawings and they count by ones up to the dividend. They then group the manipulatives in groups of the divisor (composite) | Step 1 Step 2  |
| Builds to the dividend, makes groups | With this strategy, students use known facts to build to the dividend instead of counting by ones. | Step 1  Step 2  |
| Decomposes the dividend | With this strategy the dividend is decomposed and then restructured into groups the size of the composite. (64 ÷ 16 used to demonstrate this strategy) |   |